QUADRATURE OF THE CIRCLE

BOOK THREE:

PROPOSITIONS CONCERNED WITH CIRCLES.

THE ARGUMENTS.

This book is divided into four parts just as the members are divided.
The first part is concerned with the proportions of lines in circles.
The second on how angles and circular arcs may be compared with each other.
The third shows the mutual intersections and touching of circles.
The fourth is concerned with the power of lines in circles.

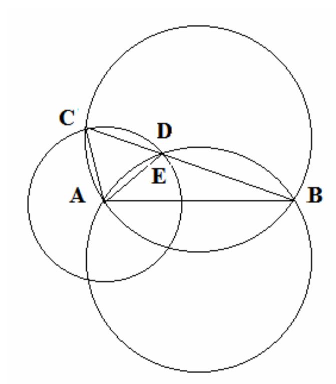
CIRCLES : PART ONE.

Lines in proportion in circles.

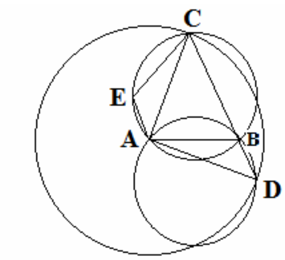
First Proposition.

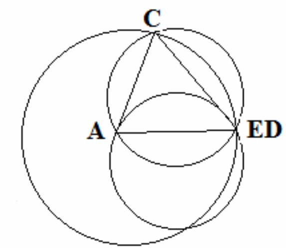
Equal circles will intersect each other in A and B; and with centre A, with the radius AC, a circle shall be described, crossing the equal circles in CD.
I say C, D, B, to be collinear points.

Demonstration.



In the first place the radius AC shall be smaller than AB, and CB may be drawn crossing the perimeter ADB at E, and the lines AE, AC may be joined. Since the angle ABC is common to each of the equal circles ADB, ACB , and the equal arcs of each of these AE, AC [§ 29. *bk.3*] sustain equal angles ; that is the right line AE is equal to AC, and the point E on the periphery of the circle ADC; but likewise by the construction E lies on the periphery of the circle ADB, therefore the point E is the same as the point D, and it passes through D and C; whereby the points C, D , B lie on the same line.





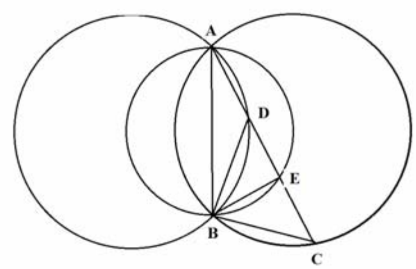
2. The radius AC shall be greater than the right line AB and the lines CB, BD may be joined : CE may be put in place equal to the right line AB ; and the lines AE, AD connected. Therefore since the line CE is put equal to the line AB, & CA itself equal to AD , and moreover by construction the circles ABC and ABD shall be equal to each other, hence the arc CE shall be equal to the arc AB [*ibid*], and the arc CEA equal to the arc ABD : from which the angle ABD is equal to the angle AEC [§ 26. *bk.3*]; but since the angle AEC together with the angle ABC is equal to two right angles [§ 21. *bk.3*], and hence the angle ABD together with the angle ABC shall be equal to two right angles; whereby the lines CB, BD are collinear. [The original Latin suggests the two lines point in the same direction.]

3. The radius AC shall be equal to the radius AB; it is apparent the point D lies at B; therefore, etc. Q.e.d.

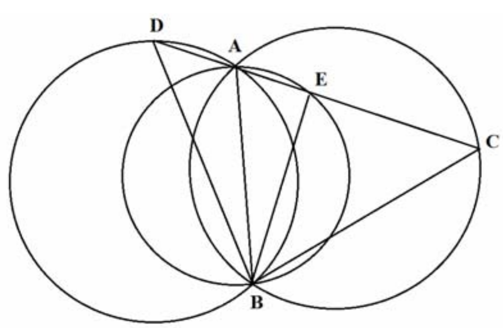
Proposition II.

Again two equal circles cross intersect at A & B, also the diameter of the circle AEB shall become AB, and some right line AD may be drawn, intersecting the perimeters of the equal circles at D and C, and the circle AEB at E. I say the right line DC to be bisected at E.

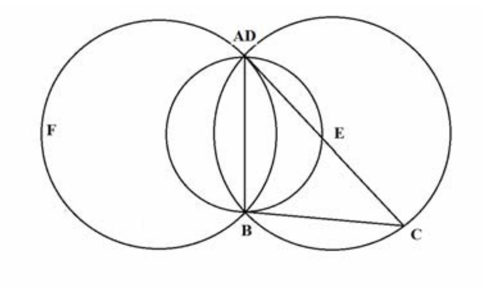
Demonstration.



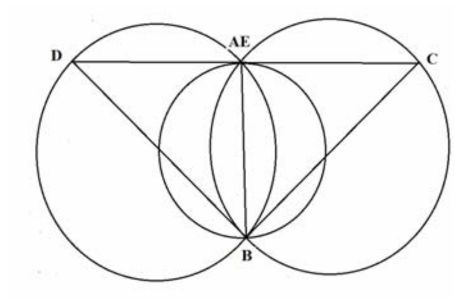
In the first case the points E & D may fall on the same side of the line AB ; the right lines DB, CB, EB may be drawn : Since the angle CAB, rests on the two arcs DB, CB of the equal circles, the subtended chords DB, CB will be equal and hence the angles EDB, ECB will be equal ; but the angles DEB, CER are right on account of the semicircle AEB, and truly on account of the common line EB of the triangles DBE, BEC, therefore the triangles DEB, CEB, are equal to each other and similar ; whereby CD is bisected in E.



In the second case A shall be the midpoint between D & E : Since the circles ACB, ADB, shall be equal, and the line AB common to each of the triangles DEB, EBC, the angles ADB, ACB, will be equal to each other; whereby since the angles at E also shall be right, DEB, EBC will be equal and similar to each other; and hence DE to be equal to the right line EC.



In the third case the line AC shall touch the circle AFB at A : also the point D shall coincide with A, therefore since AC touches the circle AFB, the angle CAB, is equal to the angle of the segment DFB [§ 32. *bk.3*], and hence to the angle ACB (since the segments AFB, ACB are equal) whereby also the angles BAC, BCA are equal : and since the angle AEB in the semicircle AEB is right, the triangles ABE, EBC will be similar and equal :from which the lines AE, EC are equal.



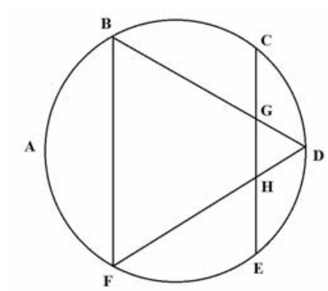
In the fourth case, the right line CD shall touch the circle of diameter AB at A, and thus the point E likewise shall coincide with the point A; therefore since DC shall be the tangent, [§ 18. *bk.3*] BAD, BAC will be right angles : moreover they are equal to the angles ADB, ACB, of equal segments : therefore the triangles ADB, ABC are similar and equal. Q.e.d.

PROPOSITION III.

With the circle divided into six equal parts by the points A, B, C, D, E, F and with BF and CE drawn, BD and FD may be put intersecting CE in G and H. I say the line CE to be trisected at G and H .

Demonstration.

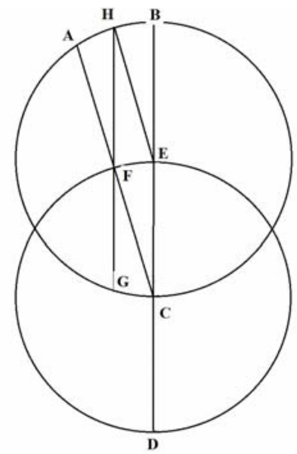
Indeed since the arcs BD, DF and FB may be put equal, BDF will be an equilateral triangle. From which since CE will be parallel to BF, it and GDH will be equilateral: but GD is equal to CG and HD is equal to HE [*Deducted from 109 & 111. Pappus book. 7*] ; therefore the lines CG, GH, HE are equal to each other, and CE is trisected at G and H. Q.e.d.



PROPOSITION IV

Two equal circles ABC, DEF may intersect each other by passing through their mutual centres C, E :truly some line HG may be put in place, parallel to the line BD acting *through each centre, crossing the perimeter at F ; and through F the line CFA. I say GF, FA, to be equal.*

Demonstration.



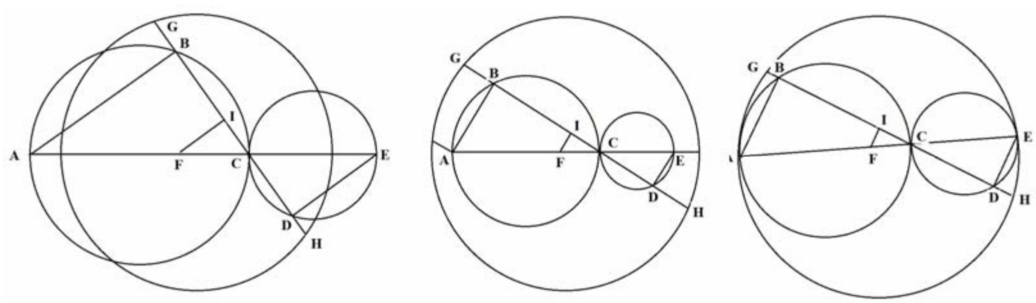
EH may be drawn ; since HG is parallel to BC, the angle FHE is equal to the angle HEB and since the arcs, HB, FE are equal also on account of the equality of the circles, the angle FCE is equal to the angle HEB, and thus to the angle FHE ; from which the parallelogram or Rhombus is CH: HF is equal to the line CE, that is, CF ; moreover the rectangle CFA is equal to the rectangle HFG [§ 35. *bk.3*] ; therefore the lines AF, GF also may be equated between themselves[§1.*bk6*]. Q.e.d.

Corollary.

Hence if follows that the lines HF, BE also can be equated to each other, since the line HF is equal to EC : and since HG is any line parallel to the diameter BC, it follows that all the parallel right lines BE, within the concavity ABC, and intercepted by the convex periphery SED, to be equal to each other, since each one may be equated to BE itself.

PROPOSITION V.

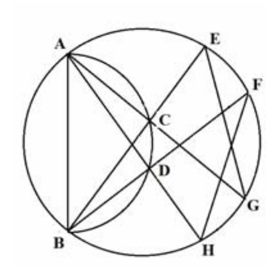
The circles ABC, CDE may touch each other externally at C ;truly the line AE acts through the centre of each, and is bisected at F, truly some circle may be described with centre F, and through C, the point of contact, the right line GCH may be put in place:
I say the lines GB, DH to be equal.



Demonstration.

AB and ED may be connected, and FI may be put parallel to these; this will be normal to GH, since the angle ABC shall be right ; and from which GH is bisected by I, and there is AF to FC, as BI to IC : and by interchanging there becomes AF to BI, as FC to IC. Then as EC to CF, thus DC is to CI, and by placing together and interchanging, so that FC to IC, thus EF to ID; but as FC is to IC, thus AF is to BI, therefore as AF to BI, thus EF to DI, and by interchanging as AF to EF, thus BI to DI. But from the hypothesis the lines AF, FE are equal to each other ; therefore DI, BI, also are equal to each other; if they may be taken away from the equalities IH, IG, the lines GB, DH remain equal to each other. Q.e.d.

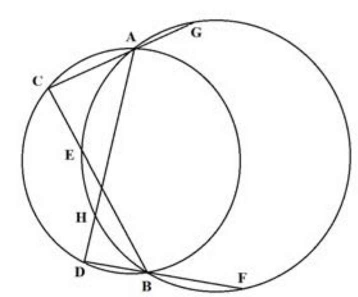
PROPOSITION VI.



Any two circles shall intersect each other in A and B; and with the points C and D assumed on the perimeter ACD and with the lines ACG, ADH: and BCE, BDF acting through those points.
I say the joined lines EG, FH to be equal.

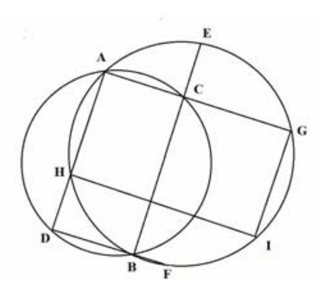
Demonstration.

In the first place, the arc ACB, cut off by the encircling arc AGB, shall contain each of the points C and D ; in which case since the angles CAD, CBD pertaining to the arc CD shall be equal [§ 27. *bk.3*], the arcs EF, GH [§ 29. *bk.3*] also shall be equal to each other; therefore with the common FG arc added , the arcs FH, EG and thus the lines FH, EG will be equal to each other [§ 29. *bk.3*] :



In the second place the arc ACB , contained by the encircling arc AFB, shall hold the points C and D; therefore since the angles CAD , CBD standing on the arc CD shall be equal, the remaining angles HAG, FBE also shall be equal ; and hence the arcs and lines EG, HF will be equal on taking away the common arc EH.

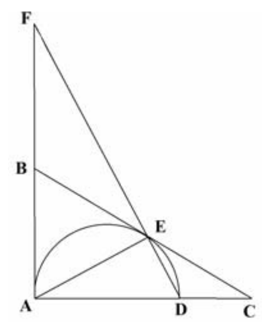
In the third case the point C shall be contained within the space of the circle AGB ; truly D shall be a point outside the arrangement. GI shall be drawn parallel to CB, and HI may be joined : since GI is parallel to CB, the angles ACB, AGI, are equal; and thence the angle AHI is equal to the angle ADB: since AHI , [§ 22. *bk.3*] with AGI , that is ACB is equal to two right angles, just as is the angle ACB with the angle ADB. From which the lines HI and DF shall be parallel, and thus the arcs HB, FI and consequently the lines HF, BI shall be equal; and whereby with HF joined, that is BI is equal to EG. Q.e.d.



PROPOSITION VII.

Let ABC be a right angled triangle, the sides AB, BC being tangents to the inscribed semicircle AED at A and E: and through E the right line DE may be put in place crossing AB at F.
I say the lines AB, BF to be equal.

Demonstration.

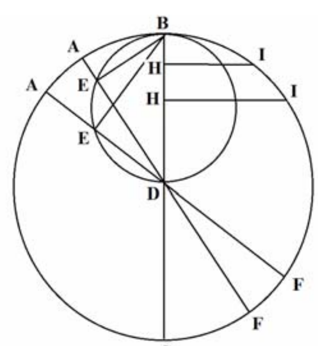


AE may be joined : therefore AED will be the right angle in a semicircle, & as the remainder AEF: which therefore is equal to the two angles EFA, EAF : but the angle EAF equals BEA, since AB, BE shall be tangents drawn from the same point; and thus equal; therefore the remaining angle AFE, is equal to the remaining angle BEF : whereby BF is equal to the right line BE, and this is equal to AB.
Q.e.d.

PROPOSITION VIII.

The circle DEB shall touch the circle ABC on the inside at B, passing through the centre D of the circle ABC, in addition with the right lines ADF drawn through D which cross over the circle DEB at E: with the right line DE made equal to DH, and with the right lines HI placed perpendicular to the diameter BC.
I say the lines EB, HI to be equal.

Demonstration.

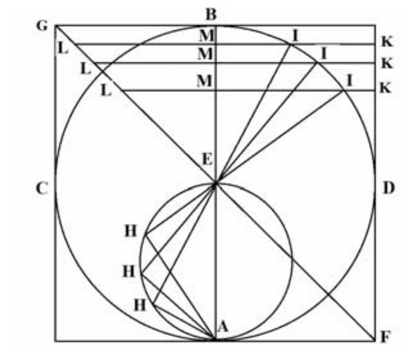


Since the lines DA are themselves equal to the line BD ; and just as, from the hypothesis, the lines HD are equal to the lines ED, the remainders HB shall be equal to the remainders EA; whereby the rectangles AEF, shall be equal to the rectangles BHC, *i.e.* the squares HI, to the squares EB: therefore EB, HI are equal. Q.e.d.

PROPOSITION IX.

Let the square GF be circumscribed for the circle ABC, of which the diameter shall be AB, and the centre E : and with the circle AHE described upon AE as diameter, and though the centre E, the lines HI may be put in place, and through I right lines parallel to AF, crossing FG, the diameter of the square in L, truly the right line FD in K.
I say HL, KL to be equal to each other.

Demonstration.

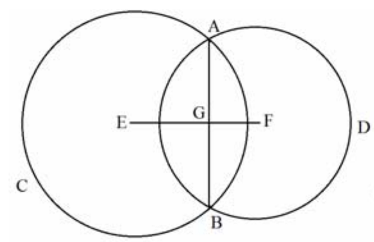


H and A may be connected; as well as the lines KL, which cut the line AB, the diameter in M: Since the lines LK are parallel to the tangent GB, the lines IM are normal to the diameter AB, [§18. *bk.3*] , and thus the angles IME are equal to the angles EHA ; but also the angles IEM are equal to the angles HEA, placed vertically opposite, and the lines EI are equal to the lines EA; therefore the triangles HEA shall be equal to the triangles IME : from which the lines HE shall be equal to the lines EM, and the lines HA equal to the lines MI. Again since the lines LM shall be parallel to GB, so that there will be GB to BE, thus LM to MB, but GB and BE are equal, therefore the LM are equal to EM also. Whereby since M K, that is EB, is equal to the lines El, and the lines LM to the lines ME, that is to HE, all from the demonstrations, the lines HI shall be equal to the lines LK; which was required to be shown.

PROPOSITION X.

So that if a right line may be drawn through the centres of circles that intersect, the other line joining the points of intersection.
I say that these two lines intersect orthogonally.

Demonstration.



Indeed the two circles shall be ABC, ADB , the centres of which E, F may be joined by the right line EF ; and the points of intersection A, B by the right line AB, crossing the line EF in G; it will be required to show the angles at G to be right, with EG drawn normally to AB, the section AB will be bisected at G; but the right line which is drawn from G, to the centre F in the circle ADB, bisecting AB, likewise stands orthogonally to AB ; therefore it is clear the lines AB, EF to be normal to each other. Q.e.d.

Corollary.

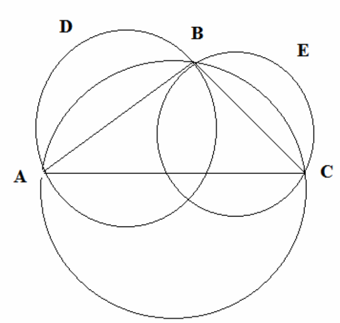
Hence it is apparent the line EF, which joins the centres of the intersecting circles , also to bisect the arc intercepted by the mutual peripheries.

ROPOSITION XI.

Let ABC be a triangle, on the base AC of which, the segment of some circle shall be described; it will be required to describe segments on the remaining sides of the triangle, similar to that, which has been described on the above base .

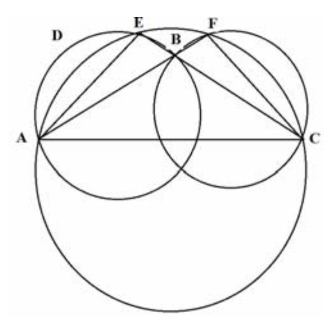
Construction & demonstration.

The first segment may be placed on the base, through the individual ends of the triangle ; moreover a circle may be described on BC, and another on BA, which may touch the right lines AB, BC at B.

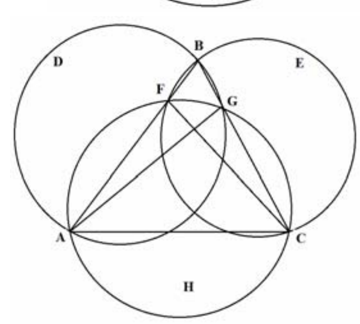


I say what was postulated has been done. Since the line BC touches the circle ADB at B ; the angle ABC is equal to the angle in the segment ADB ; in a similar manner the angle of the segment BEC [§ 32.*third*], is equal to the angle ABC ; whereby the angles of the segments AEB, ABC, BDC, are equal to each other ; and therefore are similar segments.

In the second case the vertex B of the triangle ABC may fall within the perimeter of the segment, to be constructed upon AC : with the sides AB, BC produced until they may meet the periphery of the circle AEC in F & E; circles may be described through AEB, BFC :and what is postulated will be performed.

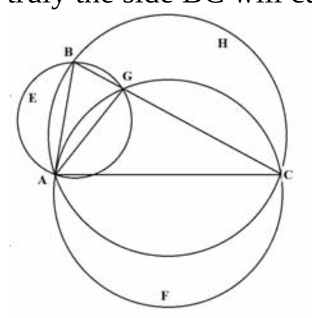


Indeed with AE, CF joined the equal angles AEC, AFC arise standing on the same arc, and hence the segments ADB, AEC will be similar : again since the angle AEC shall be equal to the angle AFC, that is BFC, and the segments AEC, BFC will be similar to each other. Whereby the three segments ADB, AEC, BFC are similar to each other.



In the third case B, the vertex of the triangle ABC shall be put in place beyond the encircling segment, which has been constructed upon the base AC ; and the perimeter shall intersect the sides of the triangle at the points F & G. Then circles are described through BFC, BGA. I say that what was desired has been done. The lines AG, FC are joined. Since the angles AFC, AGC are equal, and the supplementary angles BFC, BGA are equal, whereby the arcs ADB, BEC upon which they stand, shall be similar. Again since [the sum of] both the angle AGC as well as the angle AGB of the segment AHC [§ 22.*third*], is equal to two right angles; therefore with the common angle AGC removed: there will be the angles AGB, AHC remaining, and thus the angles of the remaining segments ADB, AGC equal, and AGC, BDA to be similar segments : but it has been shown the segment BEC to be similar to the segment ADB, therefore the three segments AGC, ADB, BEC are similar to each other.

In the fourth case the point B falling outside the segment of the base, BA may be put in place, the side of the triangle touching the circle AGC in A : truly the side BC will cut the same at G ; circles may be drawn through the points B, A, C, & A, B, G ; I say that these satisfy the desired outcome, and A, G may be joined. Since AB touches the circle AGC at A, the angle BAC is equal to the angle of the segment AFC [§ 32.*third*], from which BAC, AFC, and thus the remaining segments AGC, BHC are similar : further since the angle AGC both with the angle AGB, as well as with the angle of the segment AFC [§ 22. *bk.3*] , shall equal two right angles, with the common angle AGC removed, AGB, AFC will be equal angles ; and hence the angle of the segment AEB is equal to the angle AGC, and thus AEB, AGC are similar segments; therefore the truth of the proposition is agreed on.



Corollary.

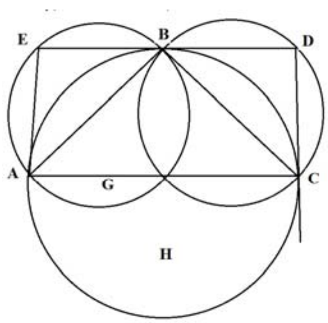
Hence it is apparent in the second case, if upon ABC with the sides AB, CB produced to fall at the common intersections E, F, the segments of the circles shall be similar to the segment of the triangle, if indeed circles are described through B and the points by which AEC and AFC cross the perimeter of the circle ; AEB, BFC will be similar segments to the segment ABC.

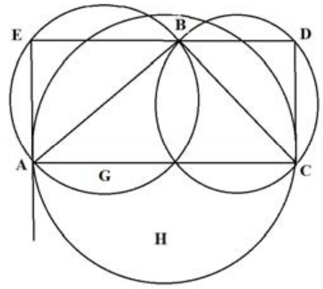
PROPOSITION XII.

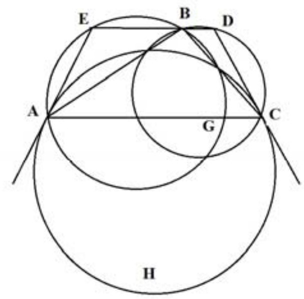
On the sides of triangle ABC similar segments of circles AEB, ABC, BDC shall be described; and the lines drawn from A & C shall be tangents to the circle ABC at A & C, but crossing the perimeters of the other circles at D & E.

I say that the points E,B, D to be collinear.

onstration.



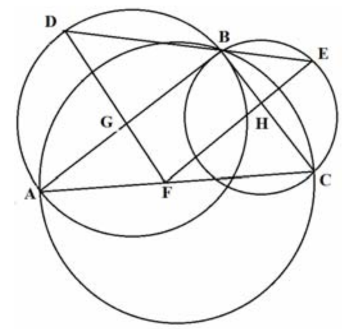




Since EA is a tangent to the circle ABC, the angle EAC [§ 32. *bk.3*] , is equal to the angle of the segment AHC, that is of the segment AGB which is similar to that; but the angle AEB, together with the angle of the segment AGB, is equal to two right angles [§ 22. *bk.3*] , therefore AEB , EAC are equal to two right angles; and thus EB, AC shall be parallel lines : it may be shown in the same way that AC, BD, to be parallel; whereby it may be agreed EB, BD, to be placed on the same line : q.e.d.

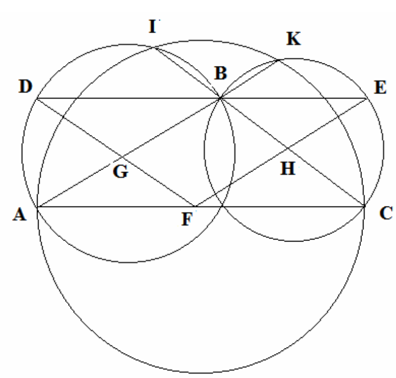
PROPOSITION XIII.

Upon the sides of triangle ABC, similar segments of the circles shall be described through the centres of which G, H, and from the centre F of the segment upon AC of the base of the described triangle, right lines are drawn crossing the perimeters at D & E.
I say the points D, B, E, to be established in a straight line.



Demonstration.

In the first case the vertex of the triangle ABC shall be on the perimeter of the circle which is described on the base AC, and DB, EB may be joined. Since FD acts through the centres F and G, it will bisect both the right line AB, [§10. *of this book*], as well as the arc ADB, also by the same account the radius FH bisects both the right line BC, as well as the arc BEC. Therefore since ADB, BEC shall be similar segments, the angles ABD, CBE shall be equal, standing on similar arcs: and whereby the angle ABD is equal to the angle BCE, since the arc EB is equal to the arc EC, but from the hypothesis, the angle ABC is equal to the angle BEC, therefore the angles ABD, ABC, CBE are equal to the three angles of the triangle BEC and thus equal to two right angles, whereby the lines DB, BE [§14. *first book*] are collinear.

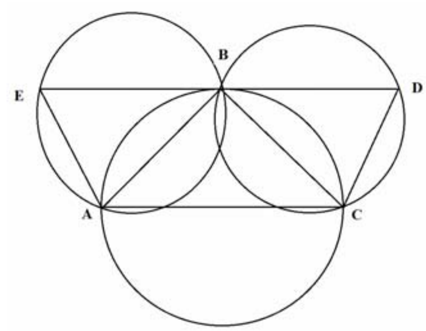


In the second case the apex of the triangle B, erected on the base of ABC, shall fall within the area of the segment AKC ; AB, CB shall be produced, then these will fall at the common intersections I and K of the circles [*Coroll. prop*.11, *of this book*] ; Because the lines FG, FH connect the centres of the circles, the arcs of the intersections of the same will be ADI, KEC bisected in D & E [10. *of this book*], from which the angles ABD and DBI, likewise KBE and CBE will be equal, but the angles ABI and CBK shall be vertically opposite to each other, and therefore the angle ABD, is equal to KBE and the angle IBD is equal to the angle EBC ; whereby the points D, B, E are collinear. Q.e.d.

PROPOSITION XIV.

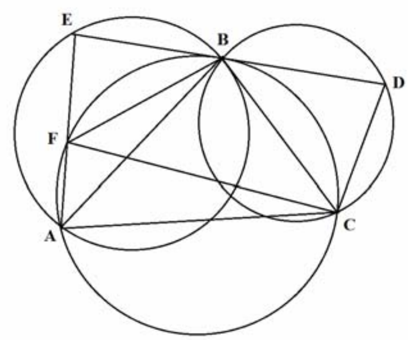
So that if the segments of similar circles were described on the sides of triangle ABC ; and if a certain right line ED, acts through the vertex B, it may intercept the perimeters of the circles AEB and BDC at D and E.

I say the lines drawn, between the various cases of the peripheries of the circles AEB, BDC and the furthest intercepts of the circle ABC, to be equal.

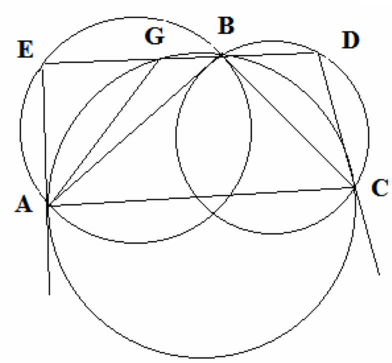


Demonstration.

In the first case ABC shall be an isosceles triangle ; and ED shall be a tangent to the circle ABC at B : and AE, CD may be joined ; Since there is a tangent at B, the angle EBA shall be equal to the angle ACB [§ 32. *section 3*] : for the same reason the angle CAB is equal to the angle CBD : from which since by hypothesis the angles BAC and BCA shall be equal, also the angles EBA, CBD are equal to each other, (on account of the similar segments AEB, CDB), and in addition the line AB is equal to the line BC; therefore the triangle AEV, is equal to the triangle CBD, and the line EB is equal to the line BD .

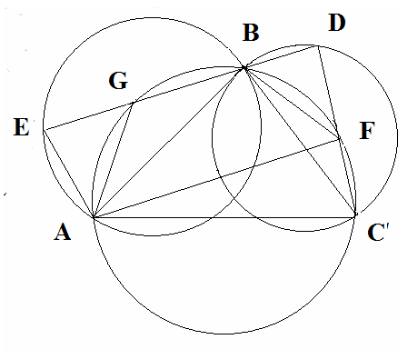


In the second case ED shall be a tangent, with the triangle ABC being scalene, I say ED to be bisected in B. AB shall be the side greater than the other BC, there may be put CF parallel to the tangent ED: and with AF joined, it will concur with DB produced to E. Since the lines CF, DE are parallel, the angle AFC is equal to the angle AED: but AFC [= ABC] can be put equal to an angle of the segment, AEB; therefore AF shall be concurrent with DB produced, on the perimeter of the circle AEB: Again since ED is a tangent, and the same



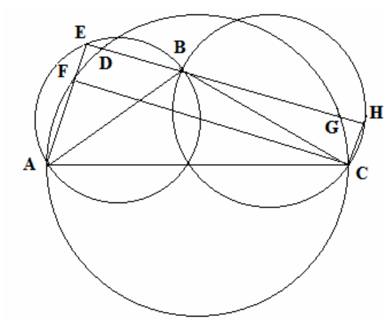
parallel to CF, the segments BF, BC shall be equal, and hence FB joined and BC also shall be equal to each other, from which the triangles FEB, BDC are similar and equal, and thus EB is equal to BD.

In the third place, the right line ED shall not be a tangent to the circle erected on the base AC, but intersects at some point G, and ED shall be parallel to AC. I say the lines EG, BD to be equal since ED, AC shall be parallel, and therefore the arcs AG, BC and the subtended chords to be equal: from which the angles GAC, BCA are equal, as the



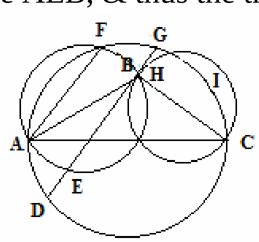
angles EGA, DBC from these to be put in place in turn ; on account of which the triangles EAG, BDC; and the right lines EG, BD are similar and equal to each other.

In the fourth place, so that if the right line ED, drawn through B crossing the perimeter ABC in G, shall not be parallel to AC, some line AF shall be drawn parallel to ED, and CF drawn, which may cross EB produced to D, and since ED, AF are parallel the arcs AG, BF are equal, and thus, the angle AFC is equal to the angle EDC : but AFC is equal to the angle of the segment BDC from its position [in the smaller circle], as D is on the perimeter of the circle BDC; and since ED,

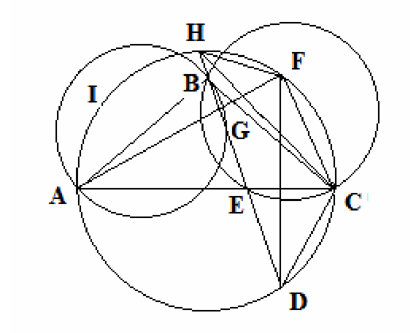


AF are parallel, the arcs AG, BF are equal, and thus the subtended chords AG, BF are equal, and thus the angles GAF, BFA also shall be equal, standing on equal arcs , and thus the angles EGA, DBF also are equal ; in addition the angles AEG, BDF are equal, on account of the similar segments ; therefore the triangles AEG, BDF are similar and equal; from which the lines EG, BD are equal.

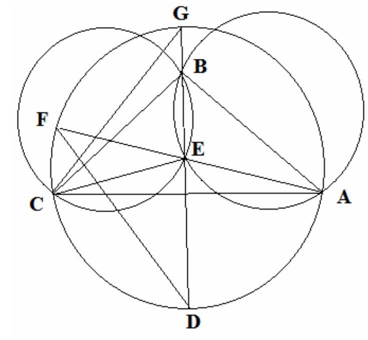
In the fifth case the vertex B of the triangle ABC shall lie within the space of the circle described upon the base: at first the right line EH passing through the peak B does not extend to meet the base of the triangle. Now CF shall be drawn parallel to EH: then it shall be appropriate for AF to be produced to E : it can be shown as before the point E to be in the perimeter of the circle AEB, & thus the triangles FED, GHC, and the sides ED, GH to be equal to each other.



In the following case the right line passes through the apex B, and crosses the base of the triangle: in which case it is required to show the line ED to be equal to the right line GH. Since EH drawn through B passes through the base of the triangle AC, either it shall be a tangent at B, (in which case it will be apparent for ED, GH, to be equal: since the triangles FHG, ADE may be easily shown to be equal from what

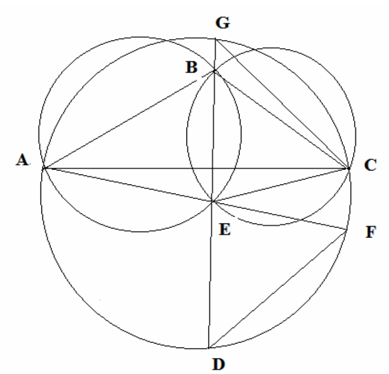


is given,) or each of the circles described above may be cut by the sides of the triangle ABC in G & E, but the circle constructed on the base in D, and at G on the circle HI; in which case the line AGF may be drawn from A through G, and the points EC, CF, FH, CD may be joined ; now the angle AGB together with the angle of the segment AIB [§ 22.*third book*] is equal to two right angles ; but the angle AIB is equal to the angle AFC from the hypothesis, and with the angle AGB equal to the angle EGF itself to AGB from being vertically opposite, therefore the angles AFC, EGF are equal to two right angles and thus BD, FC to be parallel, and the arcs HF, CD, subtending these to be equal ; from which the angles DHF, HDC are equal, standing on equal arcs : but the angle AGB is equal to the angle CEB, (since the remaining segments AGB, BEC shall be similar) and thus the remaining FGH, equal to the remaining CED; therefore the triangles FHG, CED are equal to each other; and the line HG to be equal to the line ED.



In the sixth case, so that if the right line GD acts through the vertex B, it shall pass through the common intersection E ; thus it may be shown that the right lines GE, ED to be equal : in the first place the segments shall be similar with greater semi-circles; and with the right line AEF drawn from A through E, the lines CE, CF, ED, CG may be joined: Therefore since similar segments are greater with greater semicircles, E will be beyond the line AC, if indeed it were possible, CEA should be one and the same with the line AC since therefore on account of the similitude of the segments, the angles BEA, BEC would be equal to each other: which cannot happen, since the segments shall be greater than semicircles, whereby the point E is not on the line AC. Therefore since the angle AFC, together with the angle ADC, shall be equal to two right angles, and thus with the angle ADC equal to the angle AEB (on account of the similar segments ADC, AEB), that is the angle DEF to the vertically opposite, the angle AFC together with the angle DEF will be equal to two right angles; from which FC, BD are parallel; and the arcs GF, DC, and thus the arcs GC, DE and the subtended chords of these shall be equal as before.

Truly if the similar segments were smaller semicircles: it will be shown as before, the point E, to be beyond the base line AC truly since the segments AFC, AEB are similar, the angles AEB, AFC are equal; but the angle AEB is equal to the vertically opposite angle DEF; therefore the angles DEF, AFC are equal, and the right lines BD, FC parallel ; and whereby the arcs GC, DF, and the subtended chords of these are equal, as before.



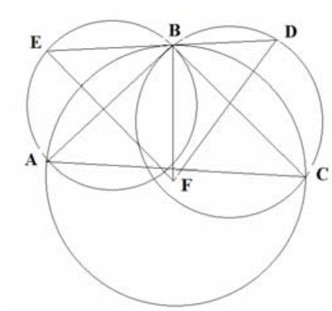
The remaining cases may be able to be established likewise when the vertex of the triangle ABC falls on the periphery or beyond the area of the circle which has been set up above the base AC *; but since all those have a common demonstration and constructions with the cases which we* *have explained, while without doubt the area itself may contain the vertex* B, *I consider it no longer necessary for the reader to be tired out* : *yet I forewarn from this, that while the vertex* B *of the triangle falls outside the segment of the base, it may exclude the cases, which we have advanced in the middle which order the line* BD *to be drawn on that account, so that it may not agree with the base of the triangle, which clearly is silly for this matter* [*i.e. a 'reducte ad absurdum' proof*].

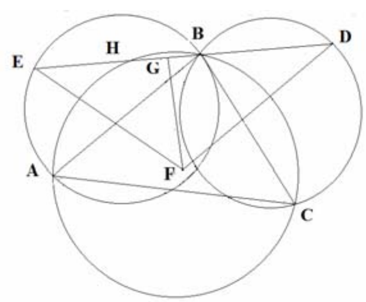
PROPOSITION XV.

If similar segments of the circles were constructed anew on the sides of the triangle ABC ; and through the vertex B the line ED may be drawn, and to its ends from the centre F of the circle described on the base, FE and FD may be drawn.

I say these to be equal to each other.

Demonstration





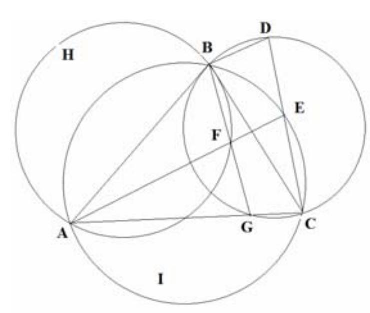
The right line ED shall touch the circle ABC at B; therefore BF shall be normal to ED [§ 18. *book 3*] :from which since the lines BE, BD, shall be equal from the preceding, also FE, FD shall be equal:

Now truly ED may intersect the circle ABC at a certain point H: and from F the right line FG may be drawn, perpendicular to ED: therefore HG, GB are equal [§3, *book 3*] ; but we have shown EH, BD [§14, *book 2*] also to be equal; therefore FE, FD are equal. Q.e.d.

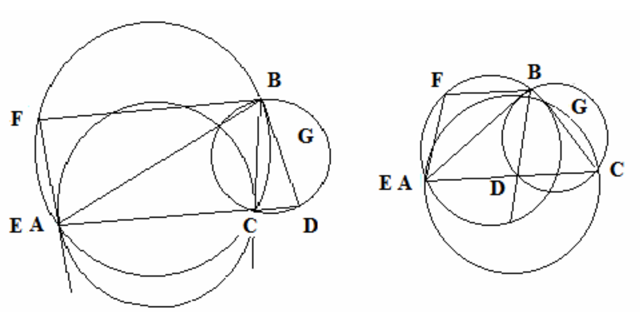
PROPOSITION XVI.

Similar segments of circles shall be put in place on the sides of triangle ABC, and with a right line CD drawn from C, to which the right line BF drawn from the vertex B of the triangle may be put parallel, crossing the circle AHB in F: I say ED, BF, to be equal lines.

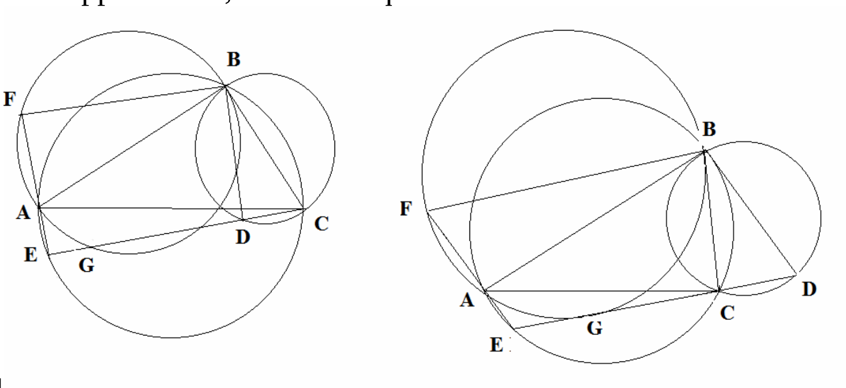
Demonstration.



In the first case the vertex B of the triangle shall be put in place on the perimeter of the circle ABC; truly the line DC shall be placed outside triangle ABC : the joined line BD shall be put parallel to AF, crossing the right line CD in E ; therefore the angle AEC is equal to the angle BDC, but the angle of the segment AEC is equal to the angle BDC, therefore the point E, is common to the intersection of the circle ABC and to the right line CD : and since the segments of the circles ABC, BDC will be similar, the angle AFB is equal to that which is contained in the segment AIC ; therefore the point F is common to the intersection of the circle AHB, and to the right line BF: therefore since BFED shall be a parallelogram, it is evident the lines ED, BF to be equal to each other. Which was required to be demonstrated.



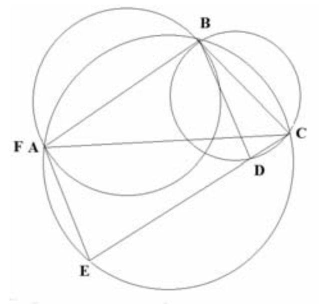
In the second case, thus the line CD is put in line with AC , and the right line BF put parallel to the base AC; AF shall be joined; and BD may be put parallel to AF crossing AC in D. Since AC, BF, and likewise AF, BD shall be parallel, the angle CAF (that is CDB in the second figure) together with the angle BFA (that is BDC in the first figure) shall be equal to two right angles, but the angle BFA, from the construction is equal to the angle in the segment CGB. Therefore the angle CDB, together with the angle of the segment CGB is equal to two right angles, whereby the point D is on the perimeter of the circle CGB and AC cuts the same line twice: whereby since FD shall be a parallelogram : it is apparent FB, ED to be equal

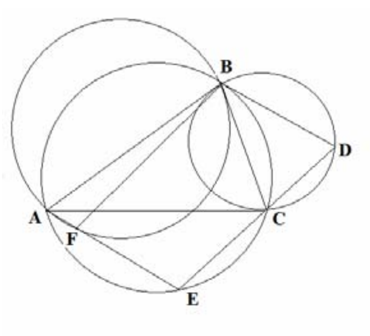


l ines.

In the third case, the right line CD may fall within the base of the triangle ABC, crossing the circle BCD at D and the circle ABC at E; thus still so that with BF drawn parallel to CD, it may fall above the line AB, I say that BF, DE to be lines equal to each other; for a line may be drawn from E through A, crossing BF at F and BD may be joined; therefore since the segments BDC, AEC shall be similar, the angle BDC is equal to the angle AEC, and thus BD, FE shall be parallel lines from which FD is a parallelogram and FB, DE equal sides. Again the angle EFB together with the angle FED, that is equal to the angle of the segment AGB, is equal to two right angles: whereby the point F, truly is on the periphery of the circle AFB in the second figure, since the angle BDC, that is ABC, together with the angle AEC, is equal to two right angles, BD and AE will be parallel : the rest as before.

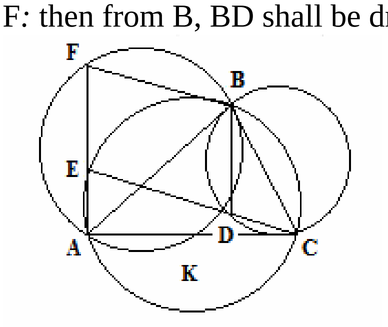
In the fourth case, BF shall be the same as the line AB; from C the right line CE may be drawn parallel to AB: and FE may be joined, the right line BD is sent from B, which shall be parallel to EF: crossing CE at D: it will be shown that the first point D to be the intersection of the circle BDC, and of the right lines BD, CE: whereby since FB, CE & BD, FE, shall be parallel, it will be apparent FB, ED to be equal.





In the fifth place, so that if the right lines DE, BF may be put in place below the right line AB , and BF indeed may cut the circle AFB in F: the right line AFE may be established, crossing CE at E : and AF parallel to BD; it is evident that the first point E stands on the perimeter of the circle AEC, just as the point D is on the perimeter of the circle BDC: from which since FB, DE, and FE, BD shall be parallel, it is evident FB, ED to be equal.

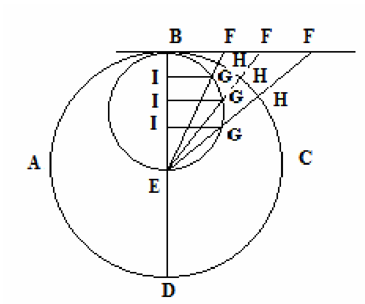
In the sixth place, finally the lines FB, DC shall be put in place above the base AC, & CD shall cross the circle ABC at E; and acting through E, the right line AE crosses FB at F*:* then from B, BD shall be drawn parallel to AF: AFB will be the angle in the segment



AFB: and the point D, common to the intersection of the right lines CD, BD and the perimeter BDC : since the angle BDC shall be equal to the angle of the segment AKC, which with the angle AEC, that is AFB, that is BDE, shall be equal to two right angles : it is apparent therefore the lines FB, DE, to be equal to each other. Q.e.d.

PROPOSITION XVII.

The two circles ABC, EBG shall touch each other on the inside at B ; and E shall be the centre of the greater circle ; then on putting BF to be the tangent, EF may be drawn, crossing the circle EBG at G: from which the right lines GI may be put normal to the diameter BD .
I say EI, EG, EH, EF to be four lines in continued proportion.



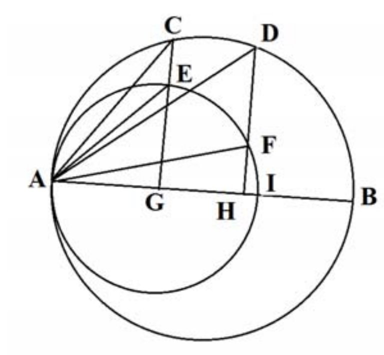
Demonstration.

Indeed, from the Elements, the continued proportions are EI, EG, EB or EH : but as EI to EG, thus EB or EH to EF; therefore EI, EG, EH, EF are in a continued ratio . Q.e.d.

PROPOSITION XVIII.

Likewise two circles ABC, AEF, may touch each other at A; of which the diameter of the greater AB may contain the centres, to which with CG, DH placed normally, which cross the perimeter AEI at E and F there may be placed AC, AD: and AE, AF.
I say AC to AD, to have the same ratio, which is found between AE, AF.

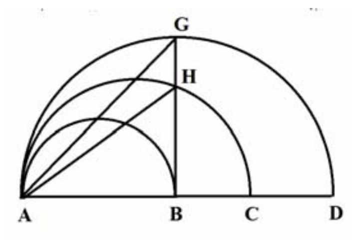
Demonstration.



Indeed the square AC, is to the square AD as the rectangle GAB to the rectangle HAB by the Elements, that is as the line GA [§1.*bk*6] to the line HA; but as GA to HA, thus the rectangle GAI to the rectangle HAI, that is the square AE, to the square AF ; therefore as the square AC to the square AD, thus the line AE is to the line AF, whereby as the line AC to the line AD, thus AE to AF. Q.e.d.

PROPOSITION XIX.

AB, AC, AD shall be continued proportionals of the diameters of the circles touching each other at the same point inside, and with BG erected from B crossing the perimeters at H and G, AG, AH may be joined.
I say AB, AH, AG to be in continued proportion.



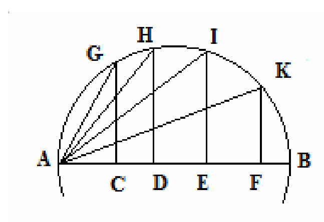
Demonstration.

Indeed AB, AH, AC are in continued proportion, just as also AB, AG, AD by the Elements, whereby AG is the mean between AH, AD; that also but from the hypothesis : AC is put to be the mean between AB, AD. Therefore AC, AG are equal lines ; and thus AB, AH, AG are in a continued ratio. Q.e.d.

PROPOSITION XX.

If the diameter AB were divided into the continued proportionals AC, AD, AE, AF, &c. and CG, DH, EI, FK may be put normally to the diameter. I say the joined lines AG, AH, AI, AK, also to be in continued proportions.

Demonstration.

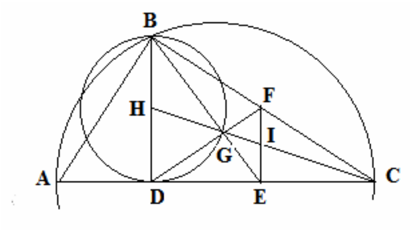


Indeed from the Elements it will be apparent the squares AG, AH, AI, AK between themselves are as the rectangles BAC, BAD, BAE, BAF ; but these rectangles are as AC, AD, AE, AF ; therefore the squares AG, AH, AI, AK are as the lines AC, AD, AE, AF: which since they may be put in continued proportion, it is evident the squares AG, AH, AI, AK, as well as the lines, to be in continued proportions. Q.e.d.

PROPOSITION XXI.

In the semicircle ABC the right line BD may be placed perpendicular to the diameter AC, by which it may be bisected at H; with centre H, and with the radius HB, the circle BGD may be described, and there may be put HC, indeed crossing the perimeter of the circle BGD at G, with the right line BG, and with the diameter AC at E.

I say the three lines AD, DE, EC, to be continued in the same ratio.



Demonstration.

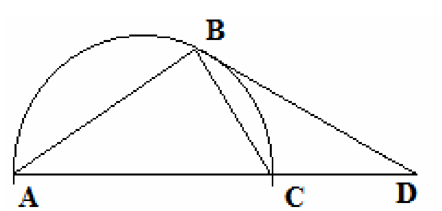
EF may be put parallel to BD, and DG, GF, AB may be joined, since EF is parallel to BD, which is bisected at H, EI will be equal to IF: from which so that EI to BH, that is FI ad HD, thus EG to GB, that is IG to GH, and on interchanging so that FI to IG, thus DH to HG, whereby since the angles DHG, FIG shall be equal from the contained proportional sides (on account of HD, FI being parallel), the triangles FIG, HDG shall be similar : just as GIE, BHG, likewise FEG, BDC: therefore the points D, G, F are collinear, as is apparent from the Elements. Further since the triangles BDG, DGE are similar, BD will be to DE as BG to DG: that is EG to GF: but as EG to GF thus DE is to EF since DGE, GEF shall be similar triangles, therefore BD is to DE as DE to EF; whereby the lines BD, DE, EF are proportional and the rectangle BDFE to be equal to the square DE, moreover the triangles ADB and EFC also are similar: therefore as AD to DB, thus FE to EC; from which the rectangle ADEC is equal to the rectangle BDEF, that is to the square DE ; therefore AD, DE, EC are continued proportionals. Q.e.d.

PROPOSITION XXII.

Let the triangle ABC be inscribed in a semicircle, of which the other side BC shall be equal to the radius.

I say that BC, AB, and the sum from AC, CB, to be three lengths in continued proportion.

Demonstration.



AD shall become three on two times CD: and BD may be joined : the arc AB is twice the arc BC, since CB is the side of a hexagon ; and therefore the angle

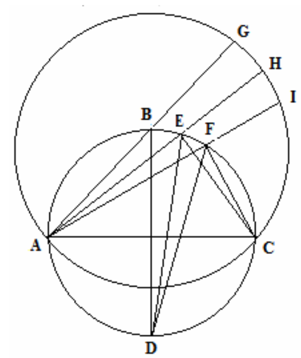
BCA, is twice the angle BAC: but the angle BCA, also is twice the angle BDA [§16. *bk*1] since BCD is isosceles by the construction; therefore the angle BDA is equal to the angle BAD and AB, BD sides equal to each other; from which the triangles BDC, BDA are similar : and as CD or BC is to DB, thus DB of AB is to AD, as AC is to CB. Q.e.d.

PROPOSITION XXIII.

The diameters of the circles AC, BD intersect each other at right angles ; and with these put in place AEC, AFC; ED, FD are joined.

I say DE, to DF, to be in the same ratio as the sum of the two AE, EC, to the sum AF, FC.

Demonstration.



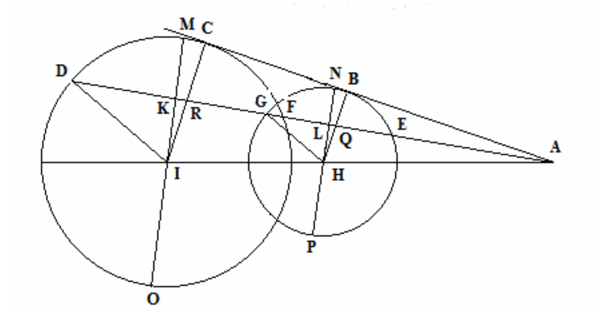
With centre B, radius AB, the circle ACG may be described, which the lines AB, AE, AF produced meet in G, H, I; Because AG, BD, are themselves diameters of their own circles, the angles AHG, AIG will be equal to the angles DEB, DFB, and also the angles BAE, EAF are equal to the angles BDE, EDF, because they stand on the same arcs; therefore the triangles GAH, HAI are similar to the triangles BDE, EDF; and the lines AH, AI proportional to the lines themselves ED, FD; whereby since AH, AI shall be equal to the lines [*Serenus bk.*2 *prop.*45] AEC, AFC, it is apparent the sum AE, EC to hold the same sum AF, FC as the right lines DE to DF.
Q.e.d.

PROPOSITION XXIV.

AB shall touch the unequal circles ;it is agreed with the line through each centre acting through A : from which there may be put AD crossing the circles at E, F, G, D.

I say AB to AC to be held in the same ratio, as prevails for AG to AD, or AE to AF.

Demonstration.



The diameters [radii] IC, HB may be drawn from the centres to the points of contact, likewise the other two diameters HN, IM, the normals to the right line DA , therefore since the tangents at B and C are right, the right lines HB, IC will be parallel.

Therefore as AB to AC, thus HB to IC, that is, as HQ to IR. Thence, since the angles IKR, HLQ are right, and IRK, HQL are equal angles (since HQ, IR shall be parallel,) ; the triangles IRK, HQL will be similar; and hence HL is to IK, as HQ to IR, that is: (as has been shown now) as HB to IC, that is (since the radii HN, IM, shall be equal to the radii HB, IC) as HN to IM.

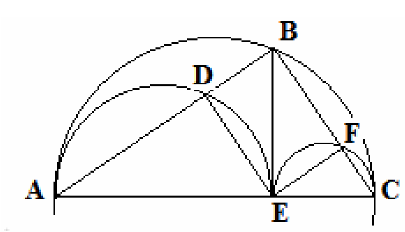
Therefore since HL is to IK, as HN is to IM, on interchanging and inverting there will be NH to PH, as MI to KI, and thus the square NH to the square LH, as the square MI to the square KI, but [§ 5. *bk*2] the square NH is equal to the rectangle NLP with the square LH, and the square MI to the rectangle MKO with the square KI; therefore the rectangle NLP with the square LH is to the square LH, as the rectangle MKO with the square KI to the square KI: therefore on dividing the rectangle NLP is to the square LH, as the rectangle MKO to the rectangle KI. And the rectangles NLP, MKO are equal to the squares GL, D K, so that GL, DK shall be normal to the diameters NP, MO ; therefore the square GL is to the square LH as the square DK to the square KI. Therefore the right line GL is to the right line LH, as the right line DK to the right line KI; therefore since also the angles GLH, DKI are equal to right angles from the construction, the triangles [6. *bk*6] GLH, DKI to be similar, and thus the angles HGL, IDK to be equal. Therefore GH and Dl are parallel. Therefore as AG is to AD, thus AH to AI, that is since HB and IC also shall be parallel, as AB to AC. Which was first shown in a similar manner, IF, HE to be parallel together, and thus to be as AH to AI, that is AB as to AC, thus AE to AF. Q.e.d.

PROPOSITION XXV.

AC shall be the diameter of the semicircle ABC divided in some manner at E: and with the semicircles AED, EFC described on AE, EC, EB is erected normally to the diameter, and AB, BC may be put in place, crossing the perimeters at D and F.

I say the ratio CF to DA, to be three times that, which CB has to AB.

Demonstration.



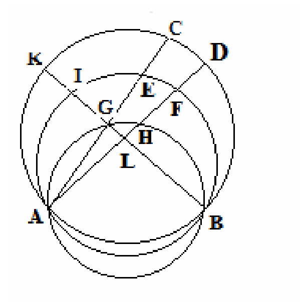
DE, EF may be joined. Since EB is normal to the diameter AC of the semicircle ABC : AC, CB, CE and likewise AC, AB, AE shall be continued proportionals ;

but as BC is to CE, thus EC is to CF (on account of the similar triangles EFC, BEC .) and as AB is to AE thus AE is to AD (on account of the similar triangles ABE, ADE) ; therefore both AC, CB, CE, CF, as well as AC, AB, AE, AD are continued proportionals : from which since the first of each of the series shall be the common AC, there will be [§ 27.*progressions*] CF to AD the fourth to the fourth, in the triplicate ratio CB to AB, second to second. Q.e.d.

PROPOSITION XXVI.

Three circles may cut each other at the points A, B, ad from A some right lines AC, AD drawn shall cross the perimeters at E, F, G, H:
I say that GE, EC, to be proportionals with the lines HF, FD.

Demonstration.



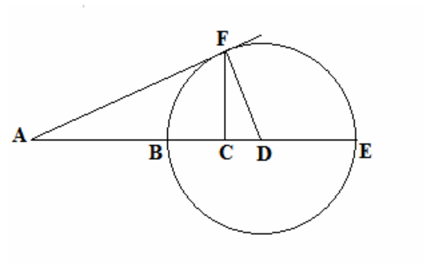
The right line BK acts through G, crossing AD at L and the perimeters at I and K, since the rectangles ALH, GLB shall be equal [§ 35. *bk*3]; just as the rectangles ILB, ALF; and KLB, ALD; will be reciprocal ratios of the sides [§ 14. *bk*6] ; that is, AL will be to LB, as GL to HL, and AL shall be to LB, as IL to LF, or KL to LD: whereby also as GL to LH, thus also IG to HF [§ 19. *bk*5], and LK to LD, or IK to FD; again since there shall become as AG to GB, thus IG to EG or KG to GC (on account of the equal rectangles AGE, IGB, likewise KGB, AGC) IG will be to GE, as KI to EC; whereby from the equality EC shall be to FD as GE to HF. Q.e.d.

PROPOSITION XXVII.

From the point A placed outside the circle, a line AD of this kind drawn to the centre of the same circle shall be divided into three continued proportions, the middle one of which shall be the radius BD, and the third CD: and from C, with the perpendicular CF drawn, and AF to be joined.
I say AF to be a tangent to the circle, and conversely.

Demonstration.

DF may be joined. Since DC to DB that is DF to DA, are proportionals about the common angle ADF, FCD, AFD will be similar triangles: from which the angle AFD is equal to the right angle FCD per hypothesis : whereby AF is a tangent to the circle. Which was the first [necessary condition].



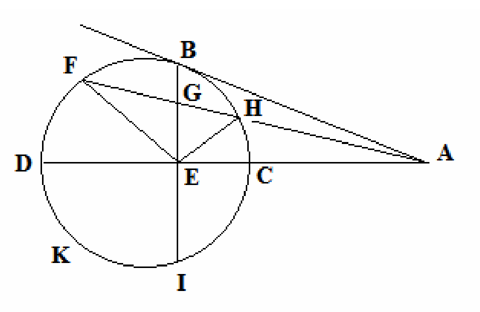
Now truly let AF be a tangent and FC be placed normal to the diameter AD. I say AD, BD, CD to be in continued proportions : for since AF shall be a tangent and FD the radius, the angle AFD shall be right and thus equal to the angle FCD, but the angle ADF is of the common triangles AFD, FCD, therefore these triangles are similar, and AD to DF, is as DB or DF to DC. Q.e.d.

This is demonstrated otherwise from Apollonius, Bk.1,Prop.35.

PROPOSITION XXVIII.

The *line AB shall be a tangent to the circle BCD : and with AD drawn through the centre BE may be put orthogonal to that, that a certain line AF may cut in G, crossing the circle at H and F:*

I say the square AB, to be equal to the rectangle FGH, together with the square GA



Demonstration

Indeed the square AB is equal to the sum of the squares BE and EA, that is to the rectangle BGI, together with the squares GE and EA, as is apparent from the Elements [*Bk*2 *prop*.5] : but the square GA, is equal to the sum of the two squares GE, EA, therefore the square AB is equal to BGI that is to the rectangle GFH together with the square GA. Q.e.d.

PROPOSITION XXIX.

With the same in place, if HE and FE may be joined ;

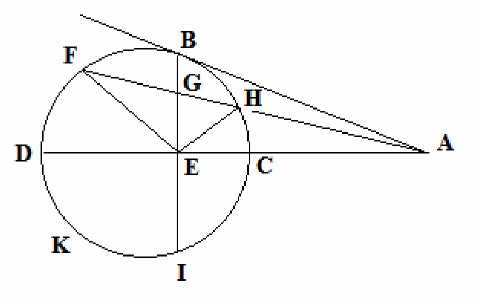
I say the three lines HE, BE, FE, to be in continued proportions :

Demonstration.

Indeed if HE is understood to be produced to K, EK will be equal to EF, as is deduced from the Elements, and from which the rectangle HEF is equal to the rectangle BEI, that is to the square BE. Q.e.d .

PROPOSITION XXX.

With the same in place: I say that AF to AH, to have the same ratio as FG to GH, or if AF may be divided at G and H, the extreme and mean ratio to be proportional.



Demonstration.

The rectangle FGH, together with the square GA, is equal to the square BA [§ 28. *of this*], but the square BA, [§ 35. *bk*3] is equal to the rectangle FAH, therefore the rectangle FGH together with the square GA is equal to the rectangle FAH : but the square AG, is equal to the rectangles [§2. *bk*2] AGH and GAH; therefore the rectangles FGH, AGH, GAH, are equal to the rectangle FAH : but they are also equal to the rectangles [§1. *bk*2] FGHA, GAH ; therefore with the common rectangle GAH removed, the rectangle FGHA remains, that is equal to the rectangles AGH, FGH, that is equal to the rectangle FAGH ; from which as FA to HA, thus FG to GH. Q.e.d.

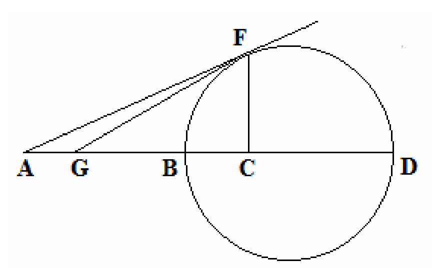
Scholium.

This proposition is proposed by Pappus book seven, proposition 154*; but since the outcome is seen to be applied to the case where the right line AD has been drawn through the centre of the circle BDC; and thus may be seen to exclude the case of the right line AF, it has pleased to show it to be really universal: So that the same is demonstrated in Apollonius book 3, proposition 37, but by a discussion different from the present.*

PROPOSITION XXXI.

From A the right line AD shall be drawn through the centre of the circle BDF, and AD to AB shall be the same ratio, as DC to CB: the CF to be erected normally, and AF may be joined.
I say AF shall touch the circle.

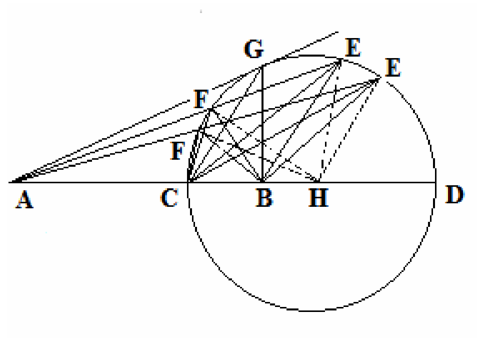
Demonstration.



For if AF may not touch, FG may be put in place to be touching and crossing AD at G ; therefore there will become by the preceding, DG to GB, as DC to CB; but from the hypothesis, as DC to CB, thus DA is to AB, therefore as DG to GB thus DA to AB, and on dividing, as DB to BG, thus DB to DA: which cannot happen ; since the points A and G are supposed to be different *:* whereby FG is not the tangent, but AF. Q.e.d.

PROPOSITION XXXII.

AB shall be divided at C, and some right line AE drawn from A, showing the angle EAB to be less than a right angle : It will be required to assign the points E and F on the line AE, to which right lines may be drawn from C and B, bisecting the angles AFB and AEB.



Construction & demonstration.

With AB produced to D, so that there may become thus AC to CB, as AD to DB: and on CD thus the circle CFD shall be described with diameter CD and centre H: moreover the right line AE will act either to cut the circle, be a tangent, or be neither: Therefore in the first place the circle will cut the AE at the points F and E ; I say that to be what they are required to do : for with BG erected perpendicularly to the diameter CD, AC, FC, FB, FH : CE, BE, HE, may be joined, as therefore so that AC to CB, thus AD is to DB, and BG shall be the normal, with the diameter CD: AG will be the right line touching the circle [§31. *of this book*] from which AH, CH, BH , are continued proportionals [§27. *of this book*]; but FH or EH, is equal to the mean CH ; therefore [§11. *progressions*] the angles AFB, AEB, are bisected by the right lines CF,CE.

But if AG may touch the circle CGD at G: the points GB, GC, GH; since AH, CH, BH are in continued proportion; and GH is equal to the mean line CH, [*ibid.*] the angle AGB will be bisected by the line CG.

Moreover it is evident if AEF does not meet the circle, the proposed matter ceases.

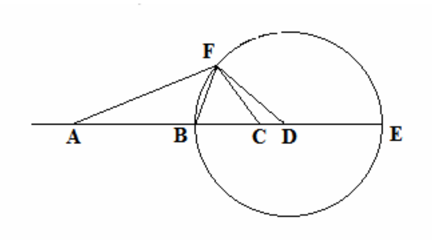
Corollary.

With the same recalled, and likewise : the problem is solved in the same manner, where it is desired to show these same points F & E on the line AE, to which with lines drawn from C & B ; AC, CB shall become proportional to the lines AF, FB, AE, EB : for it is found for the preceding points F & E from which lines are drawn to C & B, bisect the angles AFB, AEB, resolve the problem: for with the angles AFB, AEB, bisected by the right lines FC, EC; AF to FB , & AE to EB will be as AC to CB.

PROPOSITION XXXIII.

The diameter BE of the circle BFE shall be produced to some point A: cutting which circle there may be put AF : & from F, the right line CF is drawn, so that there shall be AF to FC, as AB to BC. I say that AB to BC to hold the same ratio as AE at EC.

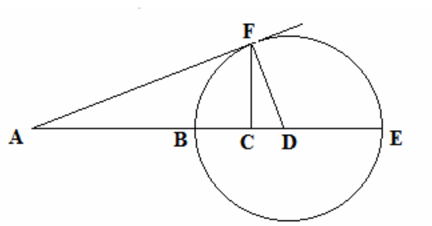
Demonstration.



DF, BF are joined: therefore since there shall be AF to FC as AB to BC, the angles AFB, BFC are equal; Again since DF shall be equal to DB, [*Euc*. *Bk*6. *Prop.3*] AD, BD, CD shall be in continued proportion ; from which AB is to BC, as [§11 & §1. *progressions*] AD to BD & since DE shall be a right line equal to DB, thus AE to EC will be as AB to BC [§5. *lines*]. Q.e.d.

PROPOSITION XXXIV.

AB shall be to BC, as AE to EC and BE shall be bisected at D. I say that AD, BD, CD to be in continued proportion.



Demonstration.

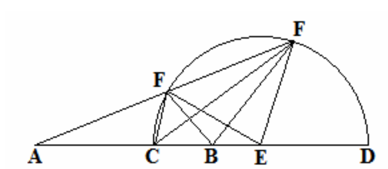
The circle is described with centre D and with the radius BD, & is erected CF normal to AE; meeting the circle at F. And DF, AF shall be drawn : therefore since AB shall be to BC, as AE to EC, the line AF will be touching the circle [§31. *of this book*], from which the lines AD, BD, CD [§27. *of this book*] are in continued proportion. Q.e.d.

This is to be found in Prop.4 and Prop.5 in the Book 1 on lines demonstrated otherwise .

PROPOSITION XXXV.

AB shall be a right line, drawn through the centre of the circle CFD ; so that moreover there may thus become AC to AD, as CB to BD, & with AF drawn cutting the circle in F: BF, CF shall be joined.

I say AC, CB, & AF, FB, to be proportional lines.



Demonstratio.

EF may be put in place from the centre E. Because there is put so that AD to AC, thus DB to CB; AE, CE, BE shall be lines in a continued ratio [§34.*of this part*]. Therefore since FE shall be equal to the middle CE, there will be AC to BC, as AF to FB [§11.*book on progressions*].

This is the converse of prop. 33.

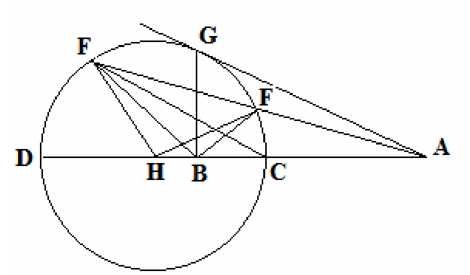
PROPOSITION XXXVI.

Some line AD shall be divided at C: and with the circle described on CD as diameter, a line AE may be drawn from A crossing the circle at F: moreover FB may be drawn, so that AF, FB shall be in proportion with the lines AC, CB, and from B with the normal erected which shall cross the circle at G.

I say that AG shall be a line tangent to [i.e. touching] the circle.

Demonstration.

Since there shall be AF to FB, as AC to CB, the angle AFB is bisected [*Euc*. *Bk*6. *Prop.3*] : truly since the right line FH, is equal to the right line CH, then AH, CH , BH shall be in continued proportion [*Book* 2. *Concerning Progressions*] : and from which GA touches the circle [§27. *of this Book*]. Q.e.d.



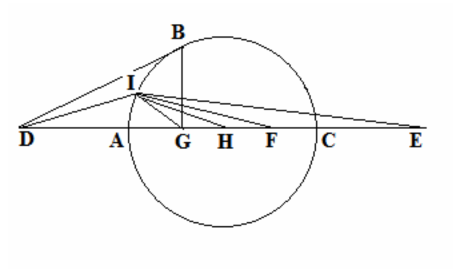
PROPOSITION XXXVII.

AC shall be the diameter of the circle ABC , and each end shall be produced to the points D & E, equally distant from the centre H, thence with DB touching the circle at B, and with BG sent normally to the diameter CA, CF shall become equal to AG: and with the right lines DI, IE drawn: IG, IF are joined.

I say DI, IE themselves are proportional to IG, IF.

Demonstration.

HI may be joined, since DB touches the circle & BG is placed normal to the diameter, DH, AH, GH will be in continued proportion [*as above*] & also as EH, CH, FH; but HI is equal to AH ; therefore as EC to CF thus EI to IF [§11.*Book2 Proportions*]; but as EC to CF, thus AD to AC, that is : DI to IG, whereby as EI to IF, thus DI to IG. On being interchanged and inverted, thus DI to IE as IG to IF.

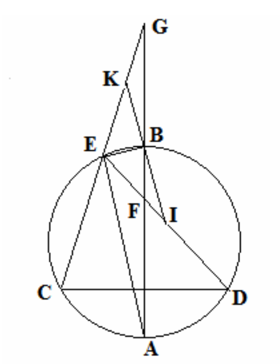


PROPOSITION XXXVIII.

The diameter AB of the circle ABC, shall cut some right line DE in F, and with the arc AC to be equal to the arc AD, the line CG may be drawn from C through E meeting the diameter in G.

I say AG to GB to have the same ratio as AF to FB.

Demonstration.



AE, EB may be joined , & IK may be drawn through B parallel to AE: since the arcs AC, AD are equal, the angles AEC, AED are equal also: truly since the angle AEB in the semicircle is right, and thus from the two angles AEC, GEB, with the equal angles AEC, AED taken away, the angles DEB, GEB remaining are equal and the angles EBK , EBI are right angles, since IK is parallel to AE, therefore the sides IB, KB are equal: whereby as AE to IB , there is AF to FB, thus AE is to KB, that is as AG to GB.

Q.e.d.

PROPOSITION XXXIX.

The circle ABC intersects another circle, having its centre on the perimeter ABC: and with the points of the intersection joined by the line BC, some line AE may be drawn from

the centre of the intersected circle A, meeting the perimeters at D & E, and truly the right line BC at F.
I say the lines AF, AD, AE to be in continued proportion.

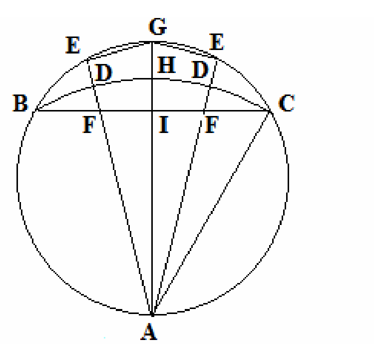
Demonstration.

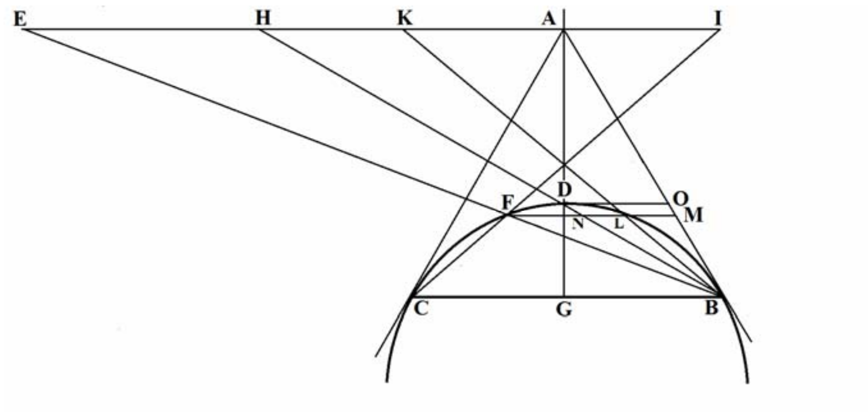
From A through the centre of the circle ABE the diameter AG may be drawn meeting the perimeters of the circles at G & H, truly the right line BC at I, and GE, AC may be joined: therefore since AG shall pass through the centres of the intersecting circles themselves, BC will be divided normally at I [§10.*of this Book*], from which the angle AIF is equal to the angle AEG drawn in the semicircle: but GAE is the common angle of each of the triangles AIF, AGE , therefore the triangles AIF, AGE are similar; whereby as AI to AF, thus AE to GA [$\frac{AI}{AF}=\frac{AE}{GA}$] : and thus the rectangle FAE [= FA.AE] is equal to the rectangle IAG [= IA.AG], that is to the square AC: [if $\frac{IA}{AC}=\frac{AC}{AG}$; then IA.AG = AC.AC], or [equally] to the square AD : therefore AF, AD, AE are continued proportionals. Q.e.d.

PROPOSITION XL

AB, AC shall be tangents to the circle BDC, and BC may be drawn, to which AE may be drawn parallel, and the diameter AG may be drawn from A, meeting the perimeter at D, and through which BG may be drawn from B; then for any point F assumed on the perimeter there may be drawn CFI, BFE.
I say the rectangle IAE to be equal to the square AH.

Demonstration.





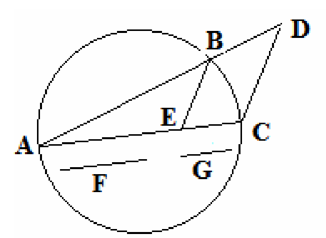
The arc CF may be taken equal to the arc BL, & with the right line BL drawn through L meeting AE at K : thus LF also joined will cut AB at M & HB at N: finally the tangent DO may be drawn. Since the arcs CF, BL may be put equal, LF will be parallel to BC and thus to the right line AH : truly since OD, OB are tangents drawn from the same point, OB & OD shall be equal and thus MB, MN also : and hence, since ML, MB, MF are proportionals, ML, MN, MF also are to be continued proportionals [*Euc*. *Bk*3. *Prop*.36]: but as ML to MN, thus AK to AH, & as MN to MF thus AH is to AE, therefore AK, AH, AE are proportionals : but AK is itself equal to AI (since BC is parallel to that, it shall be bisected by the diameter AG) therefore AI, AH, AE are in continued proportion. Q.e.d.

PROPOSITION XLI.

With the segment ABC of the circle given, and with the right line CD given, placed to AC in some manner; it shall be required to draw the right line AD, that may divide the arc ABC at B, according to the given ratio F to G.

Construction & demonstration.

The base of the segment AC shall be divided at E, following the given ratio F to G: thence EB may be put in place, which will be parallel to CD, ABD may be put in place, meeting the perimeter at B : I say what is desired follows from the Elements.

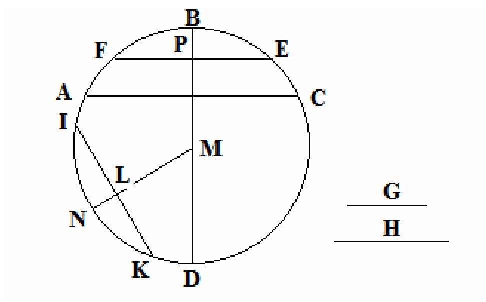


Here it is noteworthy, that the point C may be able to be assumed not only at the end of the right line AC, but also either within or outside the circle, in whatever part of AC produced.

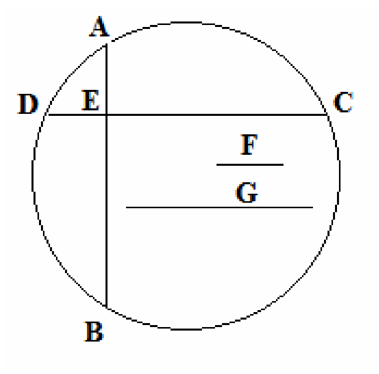
PROPOSITION XLII.

With the circle ABC given, and with the line AC, which shall not be greater than the diameter BD, and with the ratio G to H, it shall be necessary to inscribe another line FE in the circle, which will be equidistant from AC, so that as the ratio G, to H is had, also there may be had FE to AC.

Construction & demonstration.



The diameter BD may be had normal to AC, and it may become so that as H to G, thus AC to IK: which applied in the circle ABC, is bisected at the point L; and ML may be joined, for the right line ML may be placed equal to MP, and FPE, which shall be equidistant to AC; it is apparent that what is sought has been done; for the right lines FE, IK since they shall be equally distant from the centre, are equal to each other.

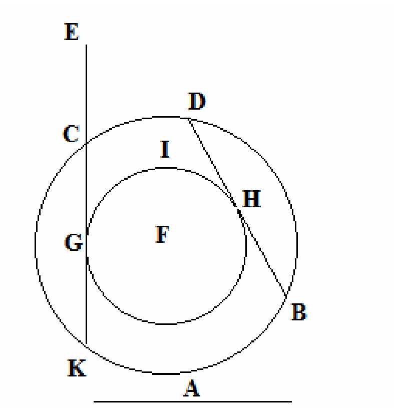


PROPOSITION XLIII.

The right line AB which shall not be a diameter, intersects the other right line CD at right angles at E, so that DE to EC may have the given ratio F to G.

The construction & demonstration of this you will find in our book on the ellipse : which here I do not judge requiring to be put in place, since clearly it shall be depending on the ellipse.

PROPOSITION XLIV.



From a given point outside the circle, a line to be sent into a circle which shall be equal to a given length; only, that shall not be greater than the diameter of the circle.

Construction & demonstration.

BD is put in place in the circle BCD equal to the given right line A ; then with the same centre F another circle may be described, tangent to BD at H : Then moreover

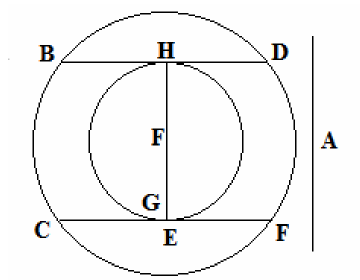
EG may be put in place from the given point E [of length A], tangent to the same circle at G; truly meeting the circle BDC at K & C. I say that CK to be sought; since indeed HB, GK shall be equal to each other, they shall stand at the same distance from the centre F (since they are tangents to the same circle GIH;) : and therefore we shall have put CK equal to the right line A: which was required to be established. [There were typos in the original proof.]

PROPOSITION XLV.

The same to arise with the given point placed within the area of the circle. But it will be necessary in this case for the given A not to be smaller than that right line, which passes through E for a diameter, the normal also shall pass through E.

Demonstration.

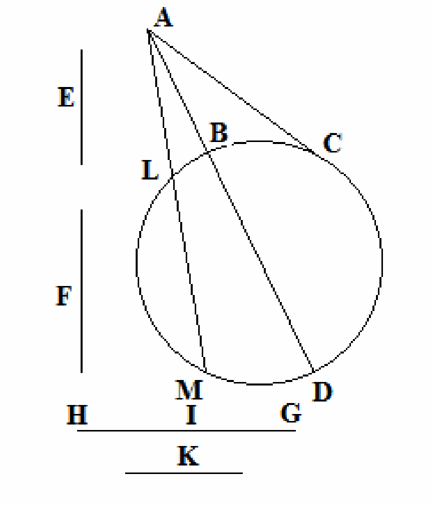
With A given as before, it may be applied in the circle, equal to BD : and the circle is described with the centre F tangent to BD at H, that will pass through the given point E, or will lie below that : indeed since by the hypothesis the normal also drawn through E to the diameter, shall not be able to be greater than the given line A, that is, DB: it is clear EF and the diameter, which the right line drawn through E shall cut orthogonally, cannot be greater than the diameter, which the line DB given equal to A will divide orthogonally; so that the circle described for the radius FH, either passes through E, or it lies below ; in each case it may be treated by E touching the radius of the circle described by FH, it is apparent in that case to be equal to the right line DB, that is, to the given A: whereby we have done what was demanded,



PROPOSITION XLVI.

From the point A given outside the circle to draw a line which may be divided by the perimeter in a given ratio: but it will be required that given ratio cannot be greater than that which is found between the parts of the line which is drawn from the given point through the centre of the circle.

Construction & demonstration.

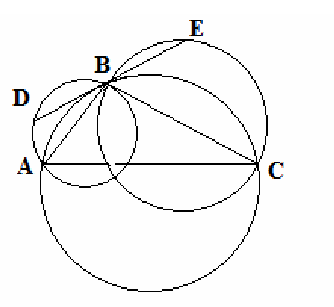


The given point shall be A, & the given circle BCD: also the given ratio shall be E to F, which shall be smaller than the ratio AB to BD, of the parts of the diameter: with the right lines E, F may be made equal to GH & GI, & GI indeed itself E: and with K extended the mean between IG, GH, from A there may be put the tangent AC, & so that K to GI thus there may become AC to AL : from the given lengths to be apparent that AL be smaller than AC, & not smaller than AB, and thus the point L to be on the perimeter of the circle : therefore the line AM may be drawn from A through L crossing the circle at L & M, I say AM to satisfy what is required; for since the rectangle LAM becomes equal to the square AC & the rectangle IGH to the square K, & there may be put moreover LA to AC as IG to the square K, it follows that GI, K, GH shall be proportional lines in the same ratio as AL, AC, AM ; from which so that first GI to the excess HI, that is, by construction as E to F, thus the fourth AL to the excess LM; we may deduce therefore from the given point, &c. Which was required to be established.

PROPOSITION XLVII.

The two circles ABD, BEC intersect each other at B. It shall be required to draw the right line DBE through B, which right line BD may be put in place equal to the right line BE.

Construction & demonstration.

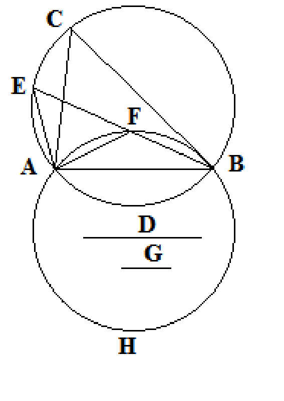


AB, CB shall be put to be tangents to the circles BEC, ABD at B ; with the perimeters crossed at A & C : henceforth the circle ABC shall be described through A, B, C, which shall have the tangent DE at B : I say what was postulated has been shown, indeed the segments ADB, BEC, ABC to be similar to each other [11. *above*] and therefore the tangent DE is bisected at B [14. *above*]: therefore we may put the right line through B, &c. Q.E.F.

PROPOSITION XLVIII.

The two circles ABC, AHB intersect each other at A & B as above. It shall be required to draw a right line BEF from B, so that the intercept EF from the given perimeter shall be equal to D , [*for the given ratio D:G*].

Construction & demonstration.



The tangent BC of the circle AHB at B may be put in place; it will be required that the given line D shall not be greater than BC ; with the points AC joined, it shall be the case that CB shall be to AB, thus as D to G. Then G itself, may be made equal to AF

[*i.e.* $CB : AB = D : G = AF : G$],

& with BF joined, the line BF may be extended to E: I say that what was sought has been done.

Indeed the line AE may be drawn, because the angle AFB together with the angle of the segment AHB, that is with the angle ABC (on account of the tangent CB) as well as with the angle AFE, is equal to two right angles;

$$[\, A\hat{F}B + A\hat{H}B = A\hat{F}B + A\hat{B}C = A\hat{F}B + A\hat{F}E = 2\text{rt.ang.} \,]$$

with the removal of the common angle AFB, the equal angles AFE & ABC will remain; but the angles ACB, AEB are, equal standing on the same arc:

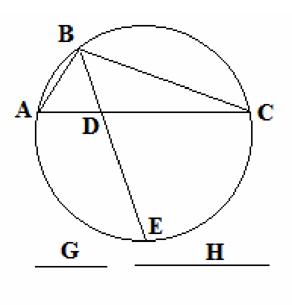
$$[\ \therefore\ A\hat{B}C = A\hat{F}E \quad \& \quad A\hat{C}B = A\hat{E}B\ ]$$

therefore the triangles AEF, ACB are similar triangles ; from which AF is to FE as AB to BC, that is by the construction as G to D: & by interchanging so that as AF to G, thus EF to D:

$$[\, AF : FE = AB : BC = G : D \,]$$

whereby, since AF & G may be made equal, then EF & D will be equal lines. Therefore we have completed what was demanded.

PROPOSITION XLIX.



In the given segment of the circle ABC, from A, & from C two lines to be inclined, themselves arranged crosswise at the perimeter: which may contain the ratio G to H between themselves.

Construction & demonstration.

The segment with the subtended right line AC to be divided at D, following the ratio G to H; the arc AEC may be bisected at B, & from E through D, the right line EDB may be put in place, AB, BC may be joined: I say AB, BC to be sought quantities. For since the angles ABE, EBC standing with equal arcs shall be equal, AB to BC will be as AD to DC [§ 6. *bk.3*], that is by the construction as G to H.

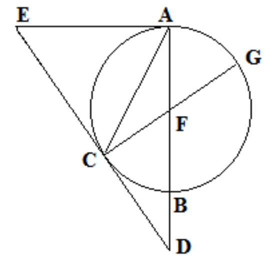
CIRCLES: THE SECOND PART

Concerned with the preparation of angles and circular arcs.

PROPOSITION L.

The right line AE shall be a tangent to the circle ABC at A, the diameter of which AB, on which some point E may be assumed E, and indeed EC may be put in place to be a tangent to the circle at C, but crossing the diameter AB extended to D, and they may be joined by CA.

I say the angle CEA to be twice the angle CAD.



Demonstration.

The diameter CG may be drawn through the centre F : Since ED is a tangent to the circle at C, & CG is a diameter, the angle GCD shall be right [§ 8. *bk.3*], and thus equal to the angle EAD; but the angle EDA is common to the triangles CFD, EDA; therefore the angle CFD is equal to the angle AED ; but the angle CFD is twice the angle CAD [§ 20. *bk.3*], & therefore the angle AED is double of the same CAD. Q.e.d.

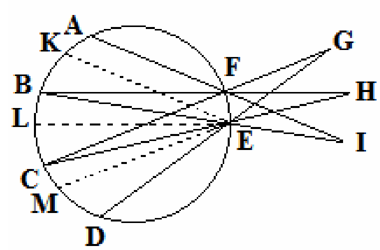
PROPOSITION LI.

With the three equal arcs AB, BC, CD, assumed in the perimeter ABC, the right lines AF, BF, CF, & BE, CE, DE may be drawn through two certain points F,E, and the right lines meeting at G, H, I.

I say the angles I, H, G, to be equal between themselves.

Demonstration.

The right lines EK, EL, EM may be drawn from the point E, which shall be equidistant to the lines AF, BF, CF; and thus the angles KEB, LEC, MED, will be equal to the angles I, H, G; truly since the arcs AB, BC, CD are placed equal, & the arcs AK, BL, CM are equal to the arc EF, & thus between themselves, also the remaining arcs KB, LC, MD, & the angles KEB, LEC, MED are equal to these in place: & whereby the angles I, H, G, are equal between themselves. Q.e.d.

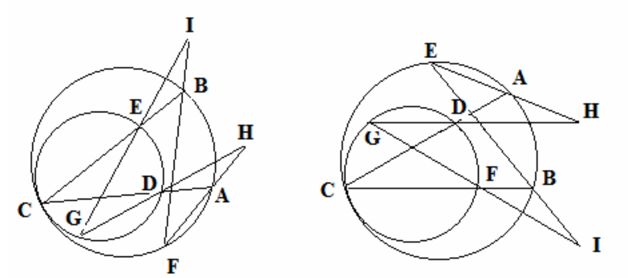


PROPOSITION LII.

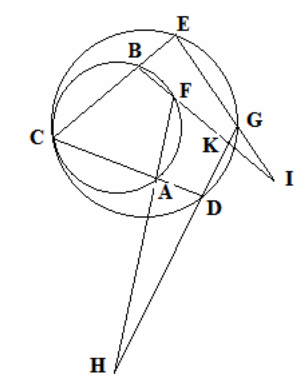
The two circle ABC, DEC shall touch each other internally at the point C: from which with CA, CB drawn, the points G, F of the individual arcs may be taken, from which the right lines GE, GD, FB, FA may be put in place.

I say that if these two lines may meet, the angles GIF, FHG to be equal.

Demonstration.



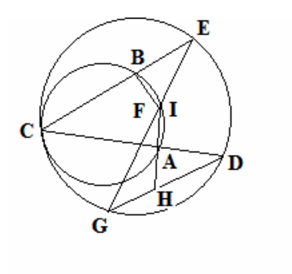
In the first place the points G & F may be put in place either above or below & with the angle ACB put in place:



Since the angles CBF, CAF standing on the arc CF are equal [§ 21. *bk.3*], as for the same reason, the angles CEG, CDG, that is IEB, HDA put vertically opposite, the angle IEB, EBI taken likewise, will be equal to the angles HDA, DAH; & to the third EIB, the third DHA will be equal.

So that if each of the points F & G may be contained by the angle ACB, we will demonstrate this assertion as follows : since the angles [§ 22. *bk.3*] ACB, AFB, shall be equal to two right angles, & as with the angles ECD, EGD: therefore with the common angle ECD removed, there will remain AFB, with the angle EGD , and hence the remaining AFI with the remaining DGI are equal ; moreover for the vertically opposite equal angles FKH , GKI, therefore the remaining angles FIG, FHG are equal also.

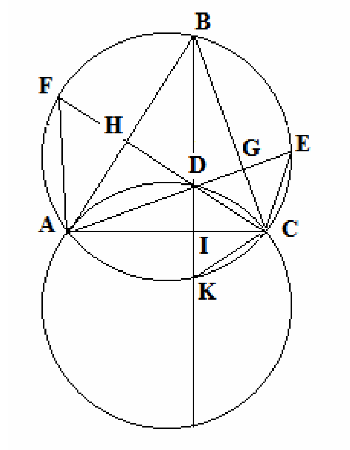
Now one or other of the points, for example F, may fall within, & the other outside the angle ACB: I say again the angles H, I to be equal to each other, since indeed the angle FBC both with the angle FAC, as well as with the angle EBF shall be equal to two right angles [§ 22. *bk.3*], with the common FBC removed, the angle EBI is equal to the angle FAC, that is DAH, but the angles CEG & CDG stand on the same equal arcs, therefore the remaining angle I, is equal to the remaining angle H. Q.e.d.



PROPOSITION LIII.

T*he normals may be drawn from the vertices of the triangle ABC, to the opposite sides. I say those normals themselves intersect at the same point.*

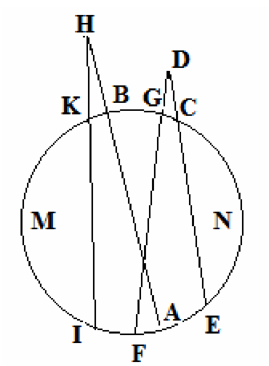
Demonstration.



A circle may circumscribe the triangle ABC, & through the points AC another circle ADC may be described, equal to the circle ABC. And BI produced will cross the perimeters of the circles at D & K, & indeed through D from A & C, the lines CDF, ADE are introduced and AF, CE, CK may be joined ; and thus AFC, AEC will be equal angles supported by the arc AC ; truly since the angle EAC is common to each of the equal circles, also the arcs DC, CE are equal, as are the subtended chords of these; on account of the same reasoning, the lines AF, AD also are equal. Again since for the angle DKC, that is KDC (on account of the equal circles) the two internal angles DCB & DBC are equal, the two arcs FB, KC that is the arc DC will be equal to the two arcs BE, EC; and thus with the equal arcs KC, CE taken away, the remaining arcs FB, BE will be equal ; whereby the angles FCB, BCE also are equal; & just as the angles FAB, BAD, from which with the lines AF, AD, & thus the angles AFD, ADF shall be equal, and the remaining angles FHA, DHA also will be equal to each other, & in the same manner the angle at G to be right ; whereby the normals BI, CH, AG, themselves to be apparent to cross at the same point. Q.f.d.

[The original diagram is not drawn correctly.]

PROPOSITION LIV.



In the given circle ABC, from a given point D beyond that, the lines DCE, DGF shall be drawn bearing the arcs FE, GC. It will be required for another point H to be designated outside the circle, to send two right lines into the circle, which two arcs by which they may intercept the circle, will be equal to the arcs FE, GC.

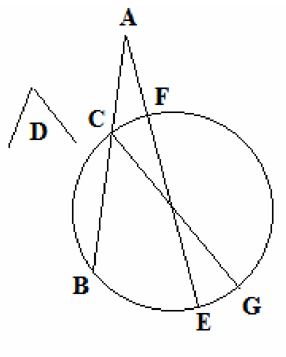
Construction & demonstration.

The lines HI, HA may be drawn from H, so that [§44, of this section] KI, BA, intercepted by the circle, shall be equal to the right lines CE, GF, what was demanded will be done, since indeed AB, FG shall be equal lines, the arcs GEF, AIB shall be equal, & as the arcs CNE, KMI shall be equal, on account of the equal lines IK, CE; therefore with the equal arcs KMI, CNE removed, the arcs KB, IA will remain, equal to the arcs GC, FE. Therefore we have dismissed the lines from the point H, &c. Q.e.d.

PROPOSITION LV.

From the point A outside the circle, with the line AB dropped through the same; it shall be required to draw another AE, which shall remove CF, BE, which assumed likewise, may contain the equal angle D.

Construction & demonstration.



The angle D shall be made equal to BCG : & from A there may be put in place AE, which shall show the right line FE, equal to the line CG [§44, of this section] ; and what is required will be performed : for since CG, FE shall be equal lines, the arc FBE, is equal to the arc CBG; so that with the removal of the common arc CB, the arc BG will remain, equal to the two arcs BE, CF; whereby the angle D, is equal to the angle, which shall be contained by the two arcs CF, BE taken at the same time. Therefore we have completed what was demanded.

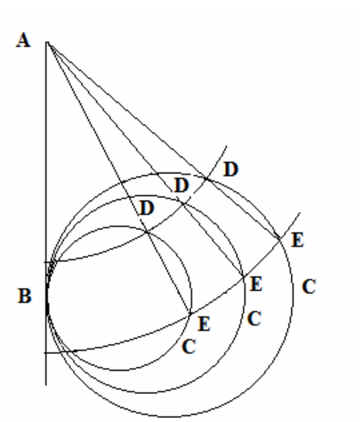
CIRCLES: PART THREE

Concerned with the mutual intersection and contact of circles.

PROPOSITION LVI.

The right line AB may touch the circles BDC themselves at the same point of contact B, and with the centre A, some circular interval may be described, crossing the perimeter of the tangential circles themselves at D, and the right lines ADE may be put in place.

I say the right lines DE to be equal to each other, as if they were to be for the same circle.

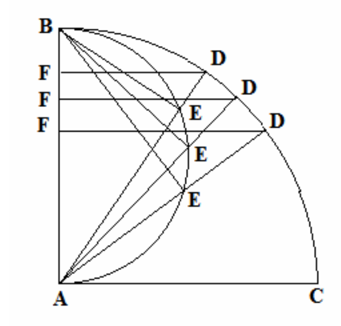


Demonstration.

Indeed since AB shall be the tangent, the rectangles DAE will be equal to the square AB, and thus equal amongst themselves ; therefore the lines DA, DB, DE, are in continued proportion; but the first DA are equal to each other & AB the common mean proportionals, therefore also the differences of the first and third proportional DE, will be equal to each other. Q.E.D.

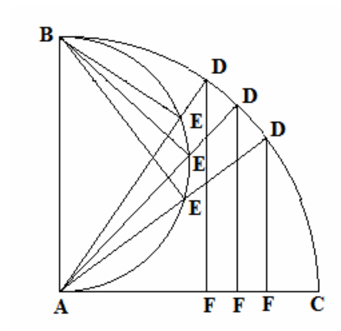
PROPOSITION LVII.

Upon AB for the radius of the quadrant of the circle ABC, the circle AEB shall be described; and with the right lines AED drawn, crossing the circle at EE, through EE the normals DF to the radius AB may be put in place.
I say AE to be equal to the right lines AF.



Demonstration.

The points BE may be joined : since the lines AD shall be equal to the lines AB , & the angles AEB, AFD right ; & moreover the angles FAE shall be common ; the triangles BEA will be equal & similar to the triangles AFD: from which the remaining sides AF, AE with equal angles subtended are equal.

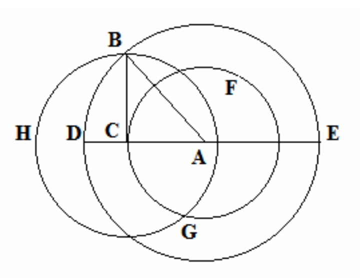


So that if the right lines DF may be dropped normally to the base AC, thus the proposition may be shown : since the lines DF shall be parallel to AB, the angles BAD, ADF, therefore shall be equal: & hence of which the angles AEB AFD, shall be right, the triangles ABE, ADF are similar & equal, & the sides subtended with equal angles. Q.e.d.

PROPOSITION LVIII.

Through the centre A of the two parallel circles BDE, CFG with the diameter DCE in place, CB is drawn tangent to the lesser circle from the point where it shall be intersected by the diameter.
I say the radius of the circle BC described to be equal to the ring intercepted by the two circumferences.

Demonstration.



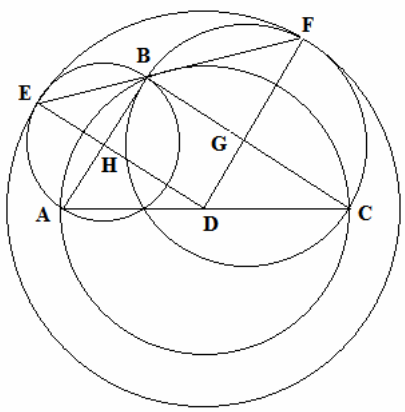
AB shall be drawn; so that the square AB shall be to the two squares BC, CA, thus as the circle DBE [§ 12. *bk*.2], to the circles HBG, CFG; but the square AB is equal to the squares AC, CB; and the circle DBE is equal to the circles HBG, CFG taken together; therefore with the common circle CFG removed, the annulus will remain from the perimeters of the equidistant circles intercepted, equal to the circle BHG: q.f.d.

$$AB^2:\left(BC^2+CA^2\right)=\odot DBE:\left(\odot HBG+\odot CFG\right)$$
but $AB^2=BC^2+CA^2$
$$\therefore \odot DBE=\odot HBG+\odot CFG$$
and the annulus $=\odot DBE-\odot CFG=\odot HBG.$

PROPOSITION LIX.

A circle may be super scribed on the triangle ABC, with the similar segments of the circles put in place from the similar sides, and from the centre of the circle, which is constructed on the base, a circle may be described which shall touch each of the circles AEB, BFC. & I say that this circle will touch the third circle also.

Demonstration.

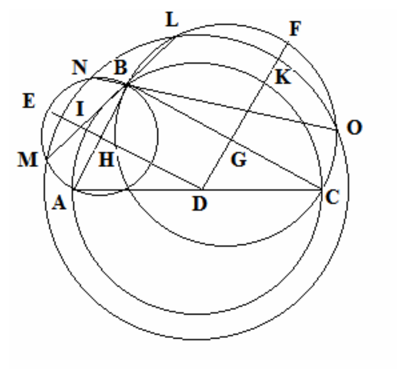


The right lines DE & DF may be drawn from D through the centres H & G of the circles AEB, BFC : & the circle described with the radius ED shall touch the circle AEB in E; therefore the right line DHE passing through each centre D & H will meet each circle at the point of contact E [§11.*tertii*], truly since the similar segments described upon the triangles, with the sides joined together [§11. *this section*] BF will be in the direction of EB itself ; from which since DE, DF ; [§13. *this section*] shall be equal lines, F will be a common point of the peripheries of the circles EF, BFC & since the same DF may join the centres D & G of each , it is apparent the circle EF, [*Schol. to* §13. *this section*] to touch the circle BFC at F. Q.e.d.

PROPOSITION LX.

With the same in place: if the circle described with the centre D will have intersected each of the circles AEB, BFC :
& I say that it will cut the other circle; & the segments made from each will be similar.

Demonstration.



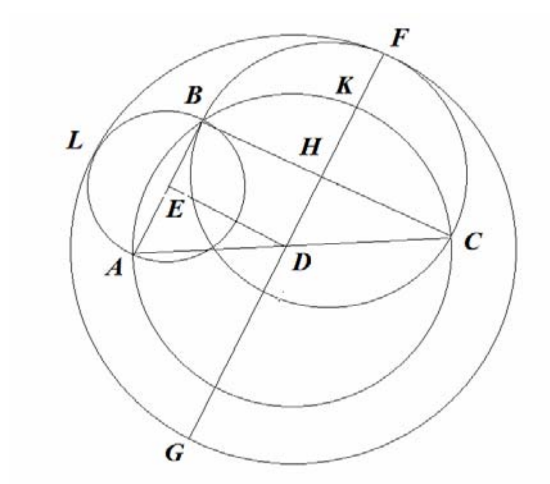
The circle described with centre D, may intersect the circle AEB at M & N: therefore it will fall below E & will cut the line ED at I; truly since the right lines DE, DF [§15. *this section*] are equal, it is clear the circle described by the radius DI, also to fall below F, & thus to cut the circle BFC at L & O. Q.e.d. first.

With this in place I say the segments MEN, LFO cut off to be similar to each other, with MB drawn may cross the circle LFO at L, it has been shown from proposition [§15. *this section*], DL joined, to be equal to the right line DM, whereby since the point M is on the perimeter of the circle NLO, & the point L will be on the same perimeter, it may be shown in the same manner, the line OB joined to occur on the common intersection with the circles MEB, MLO at N, from which since with the angle NBM standing vertically opposite the angle LBO, shall be equal; hence the arcs MEN, LFO will be similar. Q.e.d.

PROPOSlTlON LXI.

The semicircles ALB, AKC, BFC are described on the sides of the triangle ABC, which shall touch the circle described with the centre D, at F & L.
I say FG, the diameter of the circle touching each, to be equal to the diameter of the circles ALB, BFC.

Demonstration.



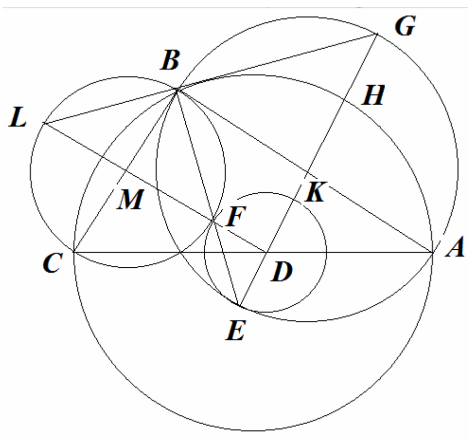
The right line FG may be placed through the centres D & H, and the centres D & E may be joined; because the similar segments to be described on the sides of the triangle are semicircles, & the diameters AB, BC are to be divided equally by DH, DE in the circle ABC, the angles DHB, DEB will be right, from which AB, DH are parallel & HE to be a parallelogram, and thus BE, HD, likewise BH, ED are equal, whereby since FG shall be twice DF, that is twice FH & HD, that is BH, BE, then FG will be equal to the diameters AB, BC. Q.e.d.

PROPOSITION LXII.

Again similar segments shall be constructed on the individual sides of the triangle ABC, and with the centre D the circle KFE will be described on the base, which will touch the circle AGB inside at E.

I say also that it will touch the other circle BLC at F.

Demonstration.



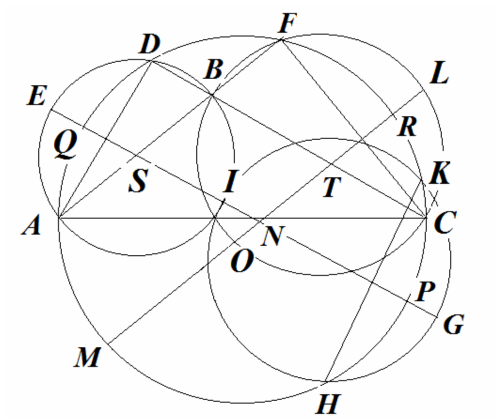
The right lines DG, DL may be put in place from D through the centres of the circles AGB, BLC : & DG indeed produced will cross the perimeter of the circle AGB at the point of contact E ; truly DL at some point F of the circle BLC, and BF, FE may be joined, therefore since DL, DG shall pass through the centres of the circles AGB, BLC, LB, GB shall be drawn in line[§.13 *this section*]: truly since LMF is the diameter of the circle BLC, the angle LBF & thus the angle GBF will be right, and hence BF extends to E. Again since cum LD, DG shall be equal lines, & also the angles BLD, BGD will be equal ; but the angles have been shown at B to be right, therefore the remaining angle BED is equal to the remaining angle LFB, that is EFD : & from which the equal lines are DE, DF & the point F in common with the perimeter of the circle KFE : & since MD, the line drawn through F, shall touch the centres D & M, it shall be clear the circle KFE [§.13 *this section*] touches the circle BLC at F. Q.e.d.

PROPOSITION LXIII.

The similar segments shall be constructed on the triangle with the sides AEB, AD, FC, BLC & with the lines DA, FC, drawn from the intersection of the points D, F, to the points A, C, KH may be prepared in the circle equal to FC, & parallel to the right line DA : then the circle IKGH may be described equal to the circle BLCO, & passing through the points K, H.

This circle IKGH I say to touch the circle ADI.

Demonstration.

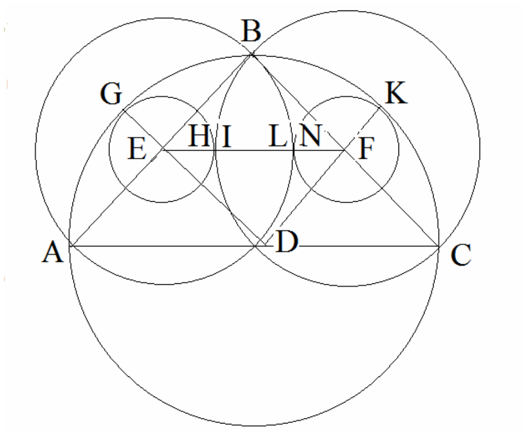


The centres of the three circles AEB, ADFC, BLC shall be S, N, T. Through the centres S, N the right line QSINPG may be drawn, crossing the circle AEB in I, & the circle KGH in G : the right line LRTNO truly may be put through the centres T, N crossing the circle BLC at O. Since QG joining the centres N, S, is normal to AD [§10. *this section*], and hence HK is parallel to AD; therefore the centre of the circle KGH also lies on the line QG, now the point I may be considered as the intersection of the circle AEB and of the right line QG. Since NR, is equal to NP, & RL [*Apparent from the Elements.*] equals PG, NL is equal to NG. Therefore indeed it has been shown here in §.63, also ON equals IN, therefore OL is equal to IG, which certainly has been placed between the point G, & the point I, where the circle AEB has cut the right line QG. Whereby since the diameter KGH shall be equal ot the right line OL, the diameter of the circle BLCO, also will be equal to the right line IG; therefore the circle KGH passes through I. And also we have shown above the centre of this circle to be on the right line QG to be passing through the centre of the circle AEB; therefore the circle KGH touches the circle AEBC. Q.e.d.

PROPOSITION LXIV.

The semicircles ABL, BIC, AGC may be described on the sides of the triangle ABC, the centres of which shall be D, E, F, moreover the circles with centres E or F, may be constructed HG, KN, which shall touch the circle AGC at G & K.
I say that these same circles, also touch the remaining circles at I & L.

Demonstration.

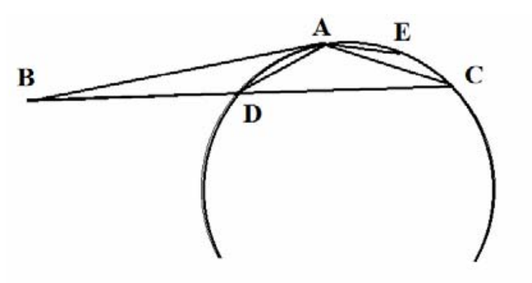


FE, DEG, DF, may be drawn, since the diameters AB, CB are bisected at the centres E & F, EF will be parallel to AC, therefore as AB to EB, thus AC to EF; whereby since AB shall be twice EB, also AC shall be twice EF. Therefore AD shall be half of AC, that is GD will be equal to EF; similarly since the three diameters AC, AB, BC are bisected at the centres D, E, F it is evident [Part 6, §.2] ED, BC, & DF, AB to be parallel; therefore EBFD is a parallelogram; therefore FB that is FI, is equal to DE. Whereby since the whole FB to the total GD, & the part FI to the part DE shall be equal, also the remainder IE will be equal to the remainder GE, that is the diameter of the circle GE, or EH; whereby the point I, is common to the two peripheries GH & BIC or the points H & I are entirely the same point. From which, since EF drawn through the centres of the circles shall pass through the point I, it is evident there to become a point of contact, the demonstration will proceed with the same agreed on, concerned with the other circle described with centre F.

PROPOSITION LXV.

In the triangle ABC, BD, BA, BC shall be in continued proportion: & a circle may be described through the points A, D, C.
I say that circle to touch the right line AB.

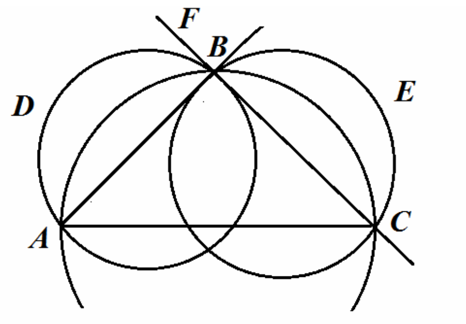
Demonstration.



Since the points ADC are on the perimeter of a circle, therefore if the right line AB may not touch the circle, it may cross the same at some other point E: therefore the rectangle ABE will be equal to the rectangle DBC, that is, by the hypothesis, equal to the square BA, which is absurd [*from the Elements*]: whereby BA will not cross the circle, except at A.

PROPOSITION LXVI.

If similar segments of circles were described on the sides of the triangle ABC, I say the sides AB, CB of the triangle produced are tangents to the segments ADB, BEC at B.



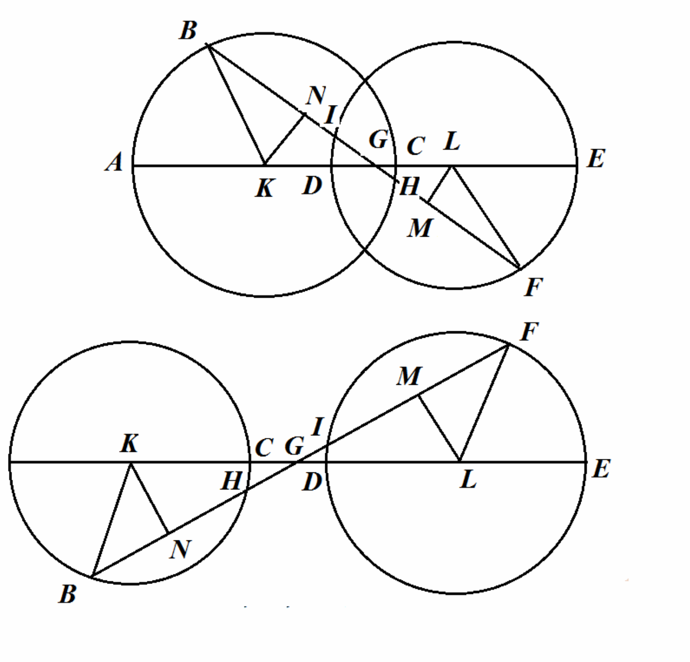
Demonstration.

Since both the angle ABC, as well as the residual angle ABF, described together on the same circle, is equal to two right angles ; therefore, the residual angle of the segment is equal to the angle FBA : truly since ADB, ABC are similar and thus the angles of these are equal, the angle FBA together with the angle of the segment ADB to equal two right angles : whereby the angle ABF, shall be equal to the angle of the residual arc of the circle ADB : from which FC is a tangent at B, as is well known from the Elements. It is shown in the same manner AB to be a tangent to the circle BEC at B. Q.e.d.

Here is the converse of this proposition.

PROPOSITION LXVII.

With the two circles ABC, DEF, it shall be required to show the point G, by which the lines drawn may divide the circle into similar parts.



Construction & Demonstration.

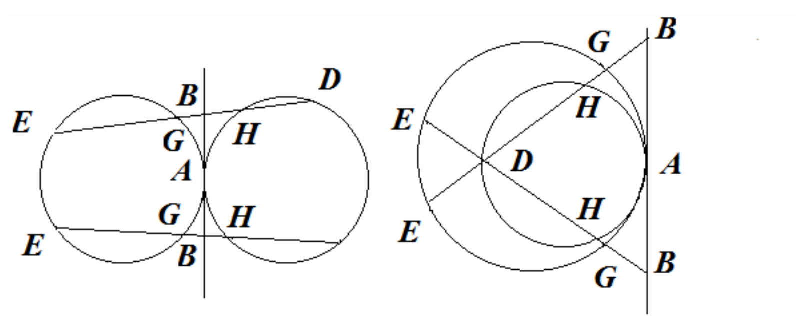
With the right line drawn through each centre K & L, AE may be divided at G, so that the ratio AG to GE, shall be the same as with the ratio AC to DE. I say the point G to be what is required. For in whatever manner BGF may be drawn crossing the perimeter at I & H: since AG to GE, has the same ratio as AC to DE, from the construction there will become AC to CG, as DE to DG ad hence AG to GK as EG to GL, & GK to GL as KA to LE, that is BK to LF. Therefore with the perpendiculars KN, LM dropped to the line FB, the triangle KGN will be similar to the triangle GML, whereby as KG to GL, that is AK to EL, that is AC to DE , thus is KN to LM; therefore BH is to IF, as AC to DE, since they stand in the same proportion from the centres as the diameters of the circles are between themselves. Whereby they subtend similar segments & and they divide the circles similarly. Therefore we have completed that which was required.

CIRCLES: PART FOUR

Concerned with the powers of lines in circles.

PROPOSITION LXVIII.

Two circles shall touch each other at the point A, through which tangential line AB, with some line EBD present cutting the perimeters at E, G, H, D.
I say the rectangle GBE, to be equal to the rectangle HBD.



Demonstration.

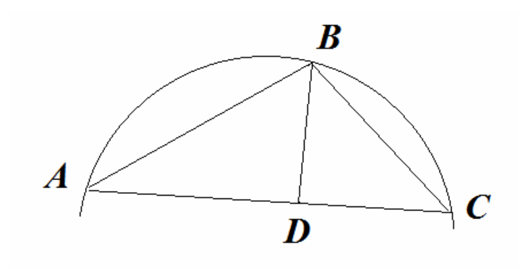
This is evident at once, since the rectangle HBD shall be equal to the square of the tangent line AB.

PROPOSITION LXIX.

The triangle ABC shall be inscribed in the segment of the circle ABC, from the vertex B of which, the normal BD shall be dropped to AC.
I say the parallelogram ABC with the angle ABC, to be equal to the rectangle AC,BD.

Demonstration.

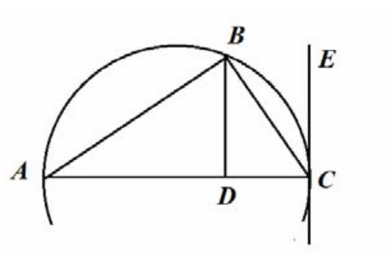
The parallelogram ABC is twice the triangle ABC [*by the Elements*] ; but the rectangle AC, BD is twice the same, since it shall have the same base AC, & the height BD; therefore the parallelogram ABC with the angle ABC, is equal to the rectangle ACBD. Q.e.d.



PROPOSITION LXX.

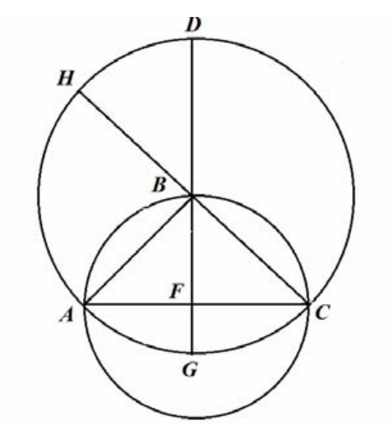
Again if the triangle ABC were inscribed in the segment, from the vertex of which BD shall be dropped, parallel to the tangent CE.
I say the rectangle ABC to be equal ot the rectangle ACBD.

Demonstration.



For since EC, BD shall be parallel, the angles DBC, BCE shall be equal, and the angles DBC, BAC to be equal on account of the tangent EC ; therefore the triangles DBC, ABC are similar. From which as AC to AB thus BC is to BD [Part 6, §.17]: therefore to be apparent the rectangles ABBC, & ACBD to be equal.

PROPOSITION LXXI

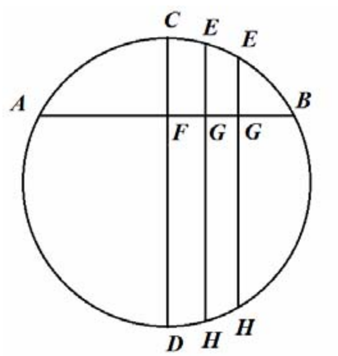


Two circles shall intersect each other thus, so that one may pass through the centre of the other, as with AC drawn, which point drawn from the point of intersection, shall be the diameter of the circle passing through the centre of the other circle; to which the normal DBG is erected through B, taken to be on the perimeter ABC, crossing the perimeter ADC at D & G, & AC at F:
I say the rectangle DBG to be equal to the rectangle ACBF.
[The original diagram has been corrected.]

Demonstration.

With the right line CBH drawn, since the circle ABC may pass through the centre of the other circle, HB will be equal to AB [*Serenus, book.*2, §.45.] From which the rectangle ABC is equal to the rectangle HBC, but HBC is equal to the rectangle DBG ; & therefore DBG is equal to ABC, that is [§.69 here] equal to the rectangle ACBF. Q.e.d.

PROPOSITION LXXII.



AB cuts the diameter CD of the circle ABC, normally at F, and some right lines EH may be put in place, crossing AB at G. I say, the rectangle EGH together with the square FG, to equal the rectangle EGH together with the square FG.

Demonstration.

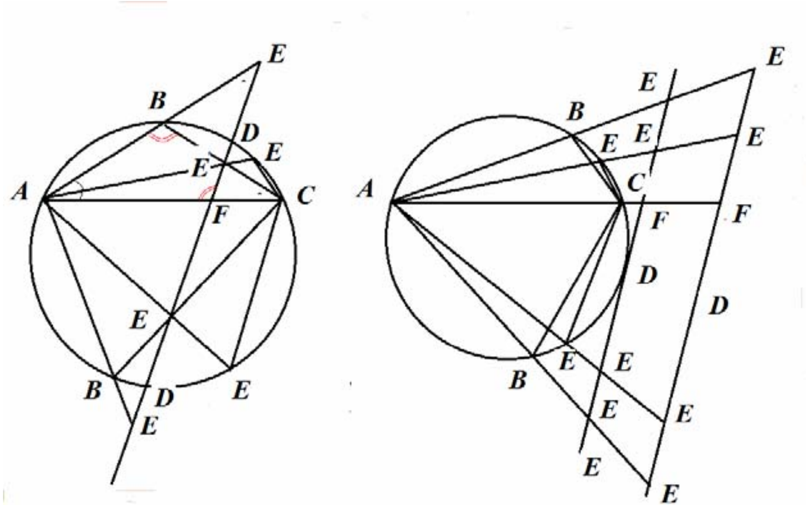
Since AB is normal to the diameter CD, it will be bisected at F ; & since it is not bisected at G, AGB will be the rectangle together with the square FG, [Part 2, §.5] equal to the square AF.
But the rectangle AGB is equal to the rectangle EGH [Part 3, §.35] ; therefore with the square FG added, EGH will be the rectangle together with the square FG , equal to the square AF. Therefore the rectangle EGH, together with the square FG, is equal to the rectangle EGH together with the square FG. Q.e.d.

$$\textit{i.e.}\ \mathrm{AGB} = \mathrm{AG.GB} = \left(\mathrm{AF+FG}\right)\left(\mathrm{AF} - \mathrm{FG}\right) = \mathrm{AF}^2 - \mathrm{FG}^2 = \mathrm{EG.GH} = \mathrm{EGH}$$

PROPOSITION LXXIII.

ABC shall be the segment of a circle, the subtended chord of which AC, shall be intersected by the right line ED, on crossing AC at F, so that the angle AFE shall be equal to the angle of the segment ABC: then more right lines AE drawn from A shall cross the perimeter at B: & at E of the line ED.
I say the rectangles EAB, to be equal to each other.

Demonstration.



Because the angles AFE, ABC are equal & with the common angle BAC, therefore the similar triangles AFE, ABC emerge, from which so that as AF to AE, there shall be AB ad AC. Therefore the rectangle CAF is equal to the rectangle EAB [Part 6, §.17]. Whereby the rectangles EAB, are equal to each other. Q.f.d.

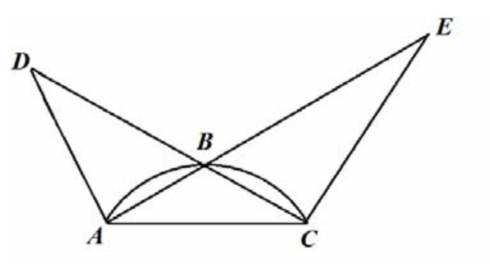
PROPOSITION LXXIV.

With the sides of the triangle ABC inscribed in the segment ABC produced, the triangles may be shown ADC, CAE having the individual angles DAC, ACE equal to the angle of the segment.

I say the square AC, to be equal to the rectangle ADCE.

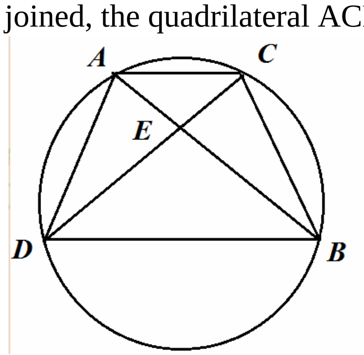
Demonstration.

Because the angle ABC is equal to the angle DAC, & ACB the common angle of the triangles ABC, ADC; ABC, ADC will be similar triangles. In the same way it may be shown the triangles ABC, AEC to be similar : & therefore the triangle ADC is similar to the triangle AEC, & the sides DA, AC , CE to be in proportion : from which the square AC is equal to the rectangle DACE. Q.e.d.



PROPOSITION LXXV.

The lines AB, CD, cross each other at right angles in the circle ABC . With the ends joined, the quadrilateral ACBD arises.



I say the squares AC, BD, to be equal to the squares AD, CB.

Demonstration.

For since the angle at E shall be right, the squares AD, CB, are equal to the squares AE, ED, CE, EB; but from the same also the squares AC, DB are equal ; therefore the squares AC, BD, are equal to the squares AD, CB.

PROPOSITION LXXVI.

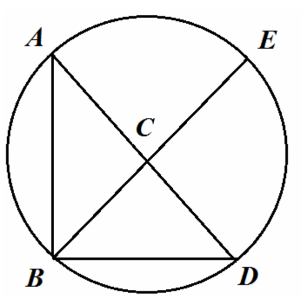
With the same in place,
I say the rectangles ADCB, ACDB taken together, to be double the figure of the quadrilateral ACBDA.

Demonstration.

The two rectangles ADCB, ACDB are equal to the rectangle ABCD, but the rectangles AEC, AED, BEC, & BED are equal to the rectangle ABCD which taken together [*To be apparent from the Elements*] is twice the figure ACBDA, & therefore the rectangles ADCB, ACDB are equal to the rectangles AEC, AED, BEC, BED & thus twice the figure ACBDA. Q.e.d.

PROPOSITION LXXVII.

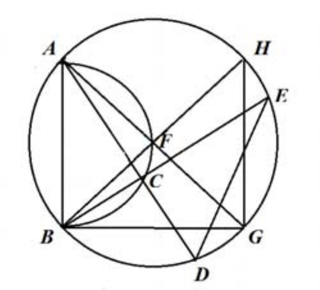
Again the lines AD, BE intersect each other at right angle at C.
I say the four parts of the square AC, CD, CB, CE, taken likewise, to be equal to the square on the diameter.



Demonstration.

Initially another of the lines, for example AD, shall pass through the centre of the circle, and AB, BD may be joined ; therefore, since the angles ACB, DCB shall be right, the squares AB, BD, will be equal to the squares AC, CB, & CB, CD or CE, CD, (since BE is bisected at C from the division of the diameter AD) but the square AD, is equal to the squares AB, BD too, since ABD thus shall be the right angle of the semicircle, therefore also the square AD, is equal to the four squares AC, CB, & CB, that is CE, CD : which was required to be shown initially.

So that if neither of the lines AD, BE may pass through the centre, with AB joined, the semicircle ACB may be described, which will pass through the point C, since the angle ACB may be put as right; from A, truly the diameter AG may be drawn intersecting with the circle ACB at F, through which there may be put BFH : and HG, ED may be joined, therefore the lines HG, ED will be equal to each other and thus the squares will be equal to each other [This part,§18.] : truly since the squares AB, HG, are equal to the squares HF, FG, AF, FB, that is to the square on the diameter AG as we have shown earlier; & therefore the squares ED, AB, are equal to the square on the diameter AG ; but the squares ED, AB, are equal to the squares EC,

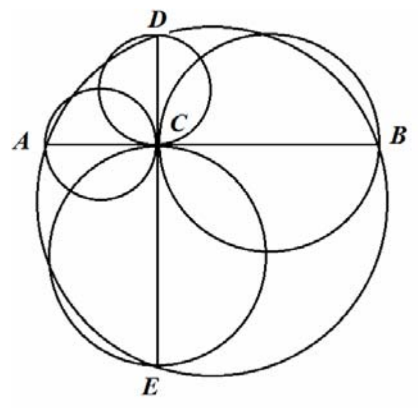


CD, & AC, CB, & therefore the squares EC, CD, AC, CB, are equal to the squares. Q.f.d.

PROPOSITION LXXVIII.

Any two right lines AB, DE, intersect each other anew at C, at right angles in the circle ABD, & circles may be described on the above parts.

I say that those likewise taken to be equal to the circle ABD.



Demonstration.

The circles have the same ration to each other as described by the squares of the diameters [Part 2, §12.] : moreover we have shown in the preceding proposition the squares AC, CD, CB, CE, to be equal to the diameter of the circle ADB ; & therefore the above circles AC, CD, CB, CE, described are equal to the circle ADB. Q.f.d.

PROPOSITION LXXIX.

Some triangles AEC, ADC shall be describe in the semicircle ABC, and with the tangent line HK acting, which is parallel to the diameter AC; EI, DK may be drawn normal to the tangent HK.

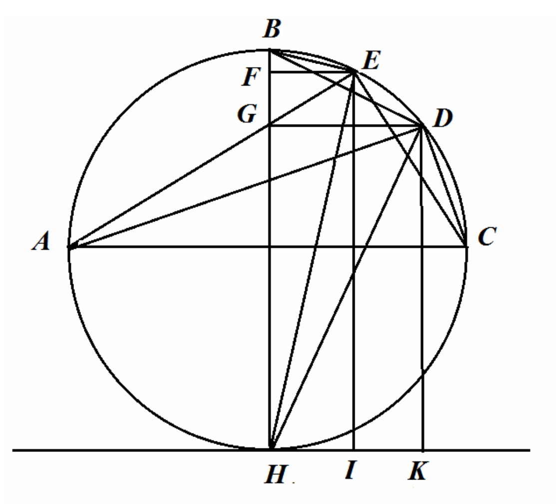
I say the square composed from AE, EC, to the square composed from AD, DC, to have that same ration as contained by EI to DK.

Demonstration.

After the diameter HB, EF, DG may be drawn normally to that and EH, DH, EB, DB may be joined, because FE stands normally to the right line BH; FH, HE, HB will be lines in continuous proportion, as thence is apparent from the *Elements*, for the rectangle FHB to be equal to the square HE

$$\left[\begin{array}{l} i.e.\ \mathrm{FHB}=\mathrm{FH.HB}=\mathrm{HE.HE} \\ \quad \therefore \mathrm{FH}:\mathrm{HE}=\mathrm{HE}:\mathrm{HB},\ etc. \end{array}\right];$$

and for the same reason the square HD, is equal to the rectangle GHB; whereby so that FH to GH, that is EI to DK, thus the square EH to the square DH : Again, since as there shall be HE thus HD, thus the



square composed from [§14, *this section*] AE, EC, to that composed from AD, DC, will be as the square HE, to the square HD, thus the square composed from AE, EC, to the square composed from AD, DC. Q.f.d.

PROPOSITION LXXX.

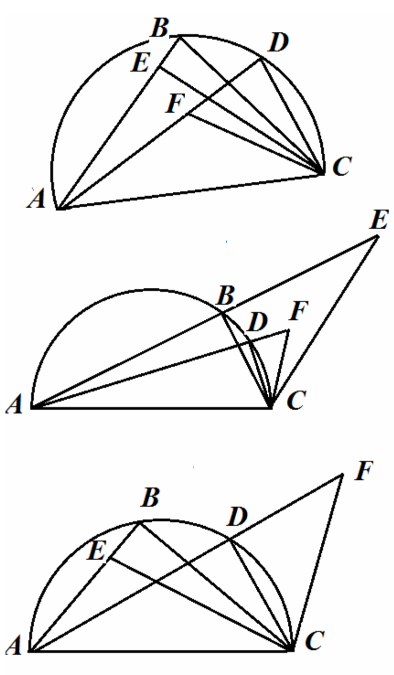
Two triangles ABC, ADC shall be inscribed in the semicircle ABC; and from C, to the opposite sides CE, CF, may be drawn in some manner at right angles.

I say if within the area of the circle with right angles occurring at AB, AD ; at E, F, the squares AE, EC, together with the rectangle AEB taken twice, to be equal to the squares AF, FC together with the rectangle APD taken twice.

Truly if they may happen outside the circle, I say the squares AE, EC, with less the rectangle AEB taken twice, to be equal to the squares AF, FC less AFD taken twice;

But if moreover the one may fall within the arc of the circle, the other outside, I say the squares AE, EC & with the rectangle AEB taken twice, to be equal to the squares AF, FC less the rectangle AFD taken twice.

Demonstration.



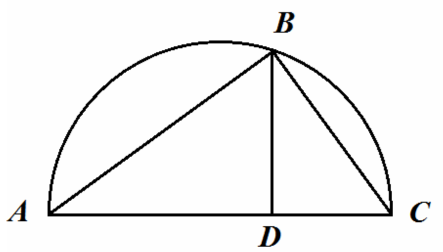
If indeed CE, CF may fall within the area of the semicircle, they will constitute the obtuse angles AEC, AFC, with ABC, ADC may become right, from which by the *Elements* the squares AE, EC, with the rectangle AEB, taken twice, will become equal to the square AC; but the squares AF, FC together with the rectangle AFD taken twice are equal to the same square AC ; therefore the squares AE, EC with the rectangle AEB taken twice are equal to AF, FC together with the rectangle AFC taken twice. Which was the first part.

Truly if CE, CF may fall outside the semicircle; it shall be apparent the angles AEC, AFC to be acute: whereby both the squares AE, EC less the rectangle AES taken twice, will be equal to the square AC, as well as the squares AF, FC less the rectangle AFD taken twice: from which the truth of the second part is evident.

The demonstration of the third part is clear from what has been said before; therefore, &c. Q.e.d.

PROPOSITION LXXXI.

The triangle ABC may be inscribed in the semicircle ABC, from the vertex of which the perpendicular BD may be sent to the diameter.



I say the rectangle ADC, & the rectangle ABC; and finally the square AC to be in continued proportion.

Demonstration.

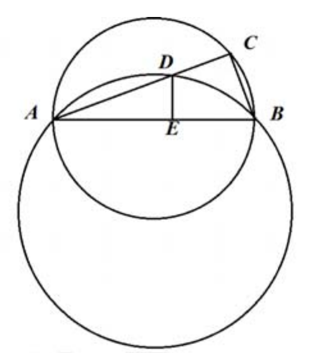
The square AC is to the rectangle ABC, that is the rectangle ACBD as [§1. *part six*.] AC ad DB : but the rectangle ACBD is to the square BD [§ 35.*part three*] that is the rectangle ADC, as AC to BD, therefore they will continue in the same ratio AC squared, the rectangle ABC, together with the rectangle ADC. Q.e.d.

PROPOSITION LXXXII.

The circle ADB may be described through the ends of the diameter AB of the circle ABC, and with AC put in place AC, which shall cross the perimeter ADB at D, DE may be dropped normally to the diameter AB, and DB, CB may be joined.

I say the rectangle from AB, DE, to the rectangle ADB, to have that ratio which is between the right lines CB, DB.

Demonstration.



Because the angle ACB in the semicircle is right and thus equal to the angle AED ; & the angle DAE the common angle of the triangles AED, ACB , the triangles ADE, ACB will be similar, from which as AB to CB thus AD to DE, & the rectangle ABDE is equal to the rectangle ADCB: but the rectangle ADCB is to the rectangle ADB as CB to DH, & therefore the rectangle ABDE to the rectangle ADB as CB to DE. Q.e.d.

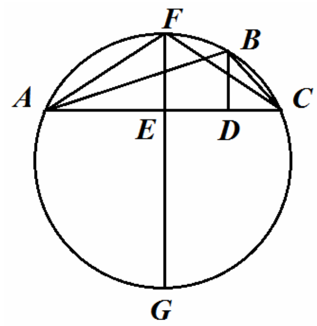
PROPOSITION LXXXIII.

AC may cross the diameter FG of the circle ABC at E at right angles, as it will cut BD normally at D, and AB, BC may be joined.

I say the rectangle ABC to prevail in that same ratio to the rectangle ACBD, as the diameter FG to the line AC.

Demonstration.

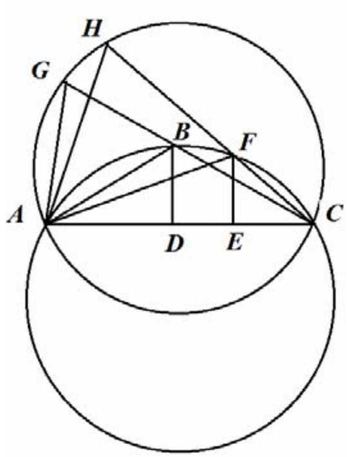
AFC may be joined because the line AC is bisected at E and thus has been divided at right angles, so that AF, FC will be equal lines, & the rectangle AFC to be equal to the square AF, that is [§.47, *first part &* §.5, *second part*.] equal to the rectangle EFG: since FE, FA, FG are proportionals. But the rectangle EFG is to the rectangle CFE as FG to AC; & therefore the rectangle AFC to the rectangle ACFE that is [§.69, *of this*] to the parallelogram AFC, as FG to AC; because truly the rectangle AFC is to the rectangle ABC as the AFC parallelogram at the angle AFC, to the parallelogram ABC, at the same angle ABC (of which it may be had composed from the same ratios) there will become on permuting, the rectangle ABC to the parallelogram ABC, that is for the rectangle ACBD as the rectangle AFC to the parallelogram AFC that is [*ibid*] to the rectangle ACFE, that is, from the demonstration, as FG to AC. Q.e.d.



PROPOSITION LXXXIV.

ADC will cut the circle ABC at some points AC, and AG may be joined ; CEH, CBG, may be put in place crossing the perimeters of the circles at B, E, G, H and AB, AE may be joined.
I say the rectangle ABC to hold that same ratio to the rectangle AEC, as the rectangle GBC maintains to the rectangle HEC.

Demonstration.



AG, AH may be joined : since the angles ABC, AFC of the same segment are equal, they will be equal to each other & the remaining [exterior] angles ABG, AFH also will be equal to each other : & from which, since AGC, AHC shall be equal to emerge standing on the same arc , AGB, AHF will be similar triangles : & AF to AH, as AB to AG is moreover the ratio of the rectangle ABC, to the rectangle AFC composed from the ratio AB to AF, that is GB to HF, from BC to GC, & from these same ratios also composed from the ratio of the triangle GBC to HFC; therefore as the rectangle ABC to AFC, thus the rectangle GBC to HFC. Q.e.d.

First Corollary.

Hence consequently, with the normals BD, FE dropped, the rectangle GBC shall be to the rectangle HFC, as the right line BD to the right line EF, indeed as the rectangle ABC to the rectangle AFC, thus as GBC to the rectangle HFC; but ABC to AFC, the rectangle is, as the parallelogram ABC to the parallelogram at the angle ABC, to the parallelogram AFC at the angle AFC (since they have been established from the same ratio) that is, as the rectangle ACBD [§.69. *here*] to the rectangle ACFE; therefore as the rectangle ACBD to the rectangle on ACFE, thus the rectangle GBC to the rectangle HFC: whereby as BD to EF, thus the rectangle GBC to the rectangle HFC. Q.f.d.

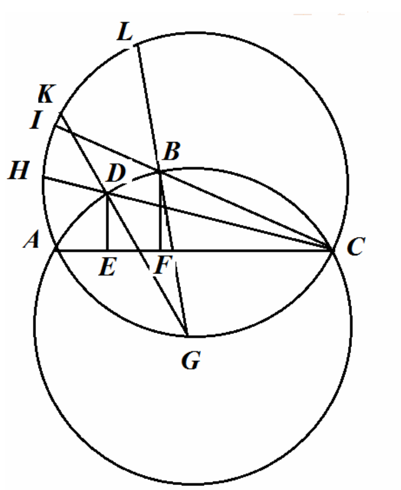
Second Corollary.

With the same figure remaining, it is clear AG to be to AH as AB to AF, or as GB to HF, since the triangles shown AGB, AHF are similar to each other: which thus I wished especially to put in place, since it is required to be assumed a number of time later.

PROPOSITION LXXXV.

The perimeter of the circle AGC shall pass through the centre of the circle ABC, and AC may be joined, some normals BF, DE may be put in place to AC: then from the centre G, the right lines GBL, GDK may be acting through B & D.

I say DE , BF to be proportional to the lines KD, LB .



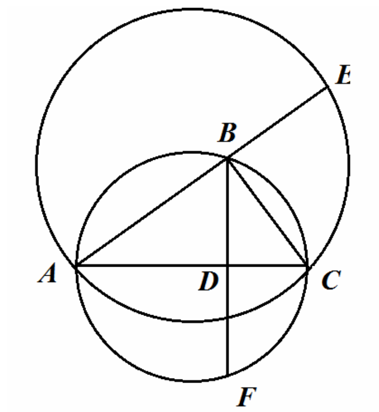
Demonstration.

The right lines CDH, CBI may be put in place through D & B so that as DE to BF, thus the rectangle HDC [*by the preceding coroll.*] to the rectangle IBC, that is KDG to LBG: but also as KD to LB, thus the rectangle KDG to the rectangle LBG, since DG, BG shall be equal lines ; therefore as DE to BF; thus KD to LB. Q.f.d.

PROPOSITION LXXXVI.

Again the circle ABC shall occur, the diameter of which shall be AC, the centre of the circle AEC, and with the right line ABE drawn from A, the line BDF may be put normal to AC.
I say the lines AC, AE, & put together with AC & BF, to be in continued proportion.

Demonstration.

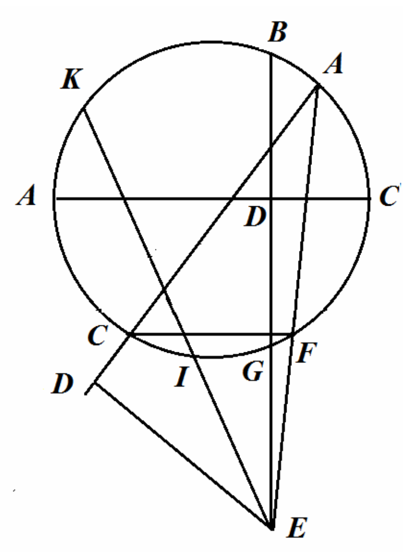


The square AE is equal to the squares AB, BE & the rectangle ABE taken twice [§.4. *of part* 2]: but the square BE is equal to the square BC [*Serenus book* 2. *prop*.45], & the rectangle ABE taken twice is equal to the rectangle ABC (that is ACBD [§.69. *this section*]) taken twice, that is, for the rectangle ASBF taken once ; therefore the square AE, is equal to the squares AB, BC, that is to the square AC & the rectangle AC,BF, taken once. But AC, with the square together with the rectangle ACBF, is equal to the rectangle on AC & composed from AC, BF ; therefore the square AE, is equal to the rectangle on AC & put together with ACBF. From which the lines AC, AE & with the lines from ACBF are in continued proportion. Q.f.d.

PROPOSITION LXXXVII.

With the point E taken outside the area of the circle ABC, the normal ED may be dropped to the diameter AC, and some other line EIK may be put in place.
I say if the common intersection point D of the lines AC, DE may fall within the circle, so that the square ED may surpass the rectangle IEK by the rectangle ADC: truly if D, may fall outside, I say that ED may be deficient to the rectangle IEK, by the rectangle ADC.

Demonstration.



DE produced, may cross the perimeter of the circle at B: the square DE is equal to the squares DG, GE together with the rectangle DGE taken twice, that is with rectangle BGE taken once, but the rectangle BGE together with the square GE is equal to the rectangle BEG, therefore the square DE is equal to the square DG, (that is to the rectangle CDA), together with the rectangle GEB, that is IEK.

So that if D outside the area of the circle may meet with the diameter AC produced, from E the right line EA may be drawn, crossing with the perimeter of the circle at F, and CF may be joined: and thus the triangles ADE & CAF shall be similar & DA, AE to be in proportion with the sides AF, AC: & from which the rectangles CAD, FAE, are equal to each other; truly since the square AE is equal to the rectangles AEF, EAF,

$$\left[\begin{array}{l} \textit{i.e.}\ \mathrm{AE.AE} = \left(\mathrm{AF+EF}\right).\left(\mathrm{AF+EF}\right) = \mathrm{AF.AF+EF.EF+2AF.EF} \\ \&\ \mathrm{AEF+EAF} = \mathrm{AE.EF+EA.AF} = \left(\mathrm{AF+FE}\right).\mathrm{EF}+\left(\mathrm{AF+FE}\right).\mathrm{AF} \\ \qquad\qquad = \mathrm{2AF.EF+EF.EF+AF.AF},\ \textit{etc.} \end{array}\right]$$

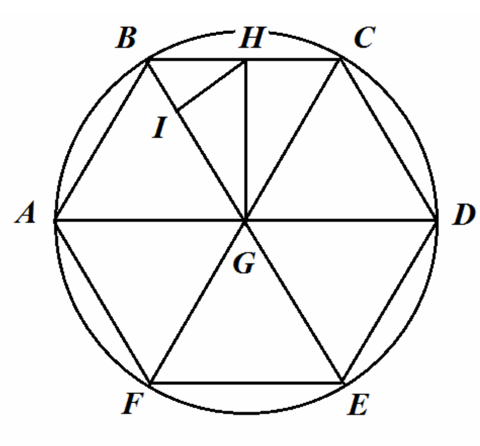
and the squares AD, DE are equal to the same square AE ; moreover the square AD is equal to the rectangles ADC, DAC, the rectangles AEF, EAF, will be equal to the rectangles ADC, DAC together with the square DE, but it is clear the rectangle DAC, to be equal to the rectangle FAE, whereby the remainders also are equal to each other, that is the rectangle CDA increased by the square DE, is equal to the rectangle FEA, that is IEK. Q.e.d.

PROPOSITION LXXXVIII.

Some regular polygon may be inscribed in the circle ABC; and with GH drawn from the centre G, GH may stand normally to the side BC ; from H there may be put HI, at right angles to BG.

I say, to obtain the account of the whole polygon, to be taken for as many squares of BG as the polygon has squares, as well that ratio that the right line HI has to BG.

Demonstration.



The right lines AG, BG, CG, &c. shall be drawn from the centre G to the angles of the polygon. Because the sides of the regular polygon are equal, the individual triangles AGB, BGC, CGD, &c. will be equal to each other and hence twice the triangles BHG, since BC shall be divided into two equal parts at H: but the double of the triangle BHG, is the rectangle BHG [*i.e.* BH.HG and not drawn in the figure], that is double BGH [§.70, *of this section*], therefore the triangle BCG is equal to the rectangle BGH; but the rectangle BGHI is to the square BG, as HI to BG; therefore the triangle BCG also, is to the square BG, as HI to BG: truly since the same may be demonstrated in the same manner concerning the individual triangles of the polygon, it is apparent the ratio of the whole polygon to the square of just as many lines BG taken as the number of sides of the polygon, to have that same proportion as the line HI to the right line BG. Q.f.d.

[Recall: the area of a regular polygon is given by the formula :

$$\tfrac{1}{2}\times perimeter\times apothem=\tfrac{1}{2}\times p\times GH\ .]$$

PROPOSITION LXXXIX.

With the same in place,

I say twice the polygon to the square of the line which shall be equal to the perimeter of the polygon , to have that ratio as the line HG to the perimeter line of the polygonal .

Demonstration.

Indeed the whole polygon is equal to a triangle having the base equal to the whole length of the perimeter [*by the Elements*], truly with the altitude HG ; therefore twice the polygon shall be equal to that triangle, taken twice, that is, by having the base equal to the perimeter of the polygon & HG the altitude; but this square of the line equal to the whole perimeter of the polygon prevails in that proportion, as the line HG to the line equal to the total perimeter; therefore the polygon taken twice to the square of the line, which shall be equal to the square of the perimeter, has that same ratio, as the line HG, to the right line of the total perimeter of the polygon: q.e.d.

Corollary.

It is not seen to follow from that proposition by omitting that which we have said in the book concerned with geometric progressions, the circle taken twice, to the square of its periphery, to preserve that ratio, so that the radius to the perimeter of the circle but taken once to the square of the perimeter of the circle, to contain that proportion, as the fourth part of the diameter to the perimeter of the circle, and thus the truth of the proposition of Archimedes concerned with the ratio of the circle to the rectangle & and hence equally to be demonstrated otherwise.

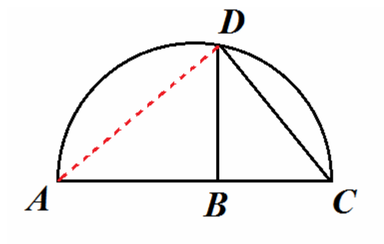
$$\left[\textit{i.e.}\text{ algebraically, }\ 2\pi r^2:4\pi^2r^2=r:2\pi r=1:2\pi\ \text{ and }\ \pi r^2:4\pi^2r^2=\tfrac{1}{4}2r:2\pi r=1:4\pi.\right]$$

PROPOSITION XC.

The right line BD stands at right angles to the diameter of the circle ABC, and DC may be joined.

I say the rectangle ABD, to the rectangle ACD, to have the triple of that ratio, that prevails for the line DB to DC.

Demonstration.



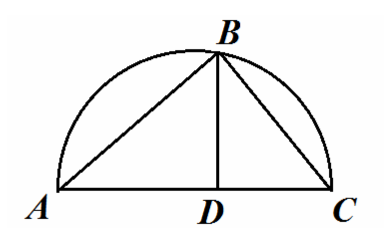
Since DB is normal to the diameter AC, from the Elements, the right lines CB , BD, BA, $\left[1:r:r^2\right]$ likewise CB, CD, CA are in continued proportion $\left[1:s:s^2\right]$; from which since the first term BC shall be common to each series, the ratio [§.17. *progressions*.] of the third parts shall be AC to AB $\left[s^2:r^2\right]$ double of that with the second CD, to the second DB, $\left[s:r\right]$ but the ratio of the rectangle ABD, to the rectangle ACD has been composed from the ratio AC to AB $\left[s^2:r^2\right]$, that is double DC to DB $\left[s:r\right]$, & from the ratio DC to DB $\left[s:r\right]$, therefore is appears the ratio of the rectangle ABD to the rectangle ACD $\left[s^3:r^3\right]$, to be the triple of the line DC to DB $\left[s:r\right]$. Q.f.d.

PROPOSITION XCI.

The triangle ABC may be inscribed in the semicircle ; from the vertex of which the normal BD may be dropped to the base.

I say the rectangle DAB, to the rectangle DCB, to have three times that ratio, than the line AB to BC.

Demonstration.



Indeed since BD shall be normal to the diameter AC, once more the three AC, AB, AD, & AC, CB, CD shall be in a continued ratio: truly since they have the common first term AC, AD to DC, shall be in the triple ratio AB to BC, but the rectangle DAB to the rectangle DCB has the ratio composed from the ratio DA to DC, that is double the ratio AB to BC & from AB to BC, therefore the rectangle DAB, to DCB has triple the ratio AB to BC. Q.e.d.

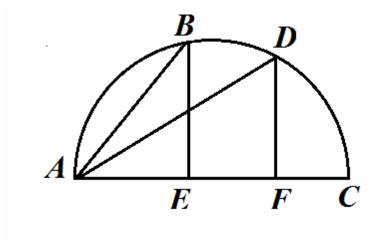
PROPOSITION XCII.

The normals EB, FD shall meet the diameter AC of the semicircle ABC at E & F and AB, AD shall be joined.

I say the rectangle EAB, to the rectangle FAD, to have the triple ratio of the right lines AB to AD.

Demonstration.

The line AE to AF has the duplicate ratio AB ad AD, since both AC, AB, AE, as well as



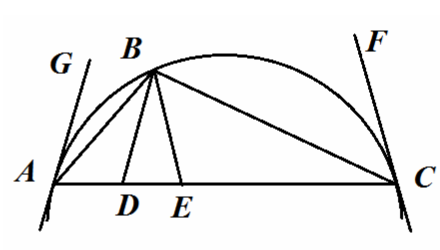
AC, AD, AF shall be continued proportionals, and they shall have the common first A, since from which the ratio of the rectangle EAB to the rectangle FAD shall be composed from the ratio AE to AF, & AB to AD, it is evident the rectangle EAB to the rectangle FAD to have the triple ratio of that which is had by AB to AD. Q.f.d.

PROPOSITION XCIII.

The triangle ABC may be inscribed in any segment ABC it pleases, and with the tangents drawn AG, CF, from the vertex B of the triangle the two line BD, BE parallel to the tangents may be put in place.

I say the rectangle DAB to the rectangle ECB, to be in the continued triple ratio of that, which AB has to BC.

Demonstration.



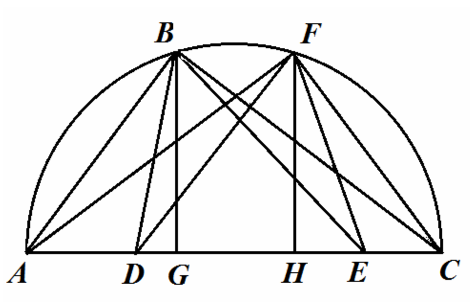
Since AG shall be a tangent, the angle GAB that is : ABD (on account of AG, BD being parallel) is equal to the angle ACB [§.16. *part three*] : & since FC also shall be a tangent to the circle, the angle FCB that is EBC (on account of the parallel lines EB, FC) is equal to the angle BAC: therefore the triangles ABD, BEC are similar to the triangle ABC: so that as AC to AB, there shall be AB to AD: & as AC to CB, thus CB to CE ; therefore the two series of continued proportionals have the common first term AC; from which AD to EC, the third to the third [§.17. *progressions*] have the second ratio AB to BC of the second to the second : moreover the ratio of the rectangle DAB to the rectangle ECB, is composed from the ratio AD and EC, & AB to CB, therefore the rectangle DAB to the rectangle is in the triple ratio of the rectangle DAB to the rectangle ECB .Q.e.d.

PROPOSITION XCIV.

The diameter of the circle ABC shall be cut by the points at D & E ; equally removed from the centre ; from which two certain points B, F of the perimeter the right lines may be drawn DB, DF, EB, EF.

I say the [sum of the] squares DB, BE to be equal to the [sum of the] squares DF, FE taken likewise.

Demonstration.



The lines AB, BC, AF, FC, may be joined and from B & F the perpendiculars BG, FH to the diameter AC, which fall initially between D & E; therefore the angles ADF, CEF shall be greater than right angles ; from which the square FC [§.12. *of the second part here*] will exceed the square FE, by the square CE together with the rectangle CEH taken twice,

$$i.e.\left[\begin{array}{l} FC^2 = FH^2 + HC^2; FE^2 = FH^2 + HE^2 \\ \therefore FC^2 - FE^2 = HC^2 - HE^2 = (HE+EC)^2 - HE^2 = 2HE.EC + EC^2 \end{array}\right]$$

& the square AF exceeds the square DF by the square AD, together with the rectangle ADH taken twice; but the rectangle CEH, taken twice, together with the rectangle ADH [*By the Elements*] taken twice, is equal to the rectangle CED taken twice;

$$i.e. \ [2CE.EH + 2AD.DH = 2CE.EH + 2CE.DH = 2CE.ED]$$

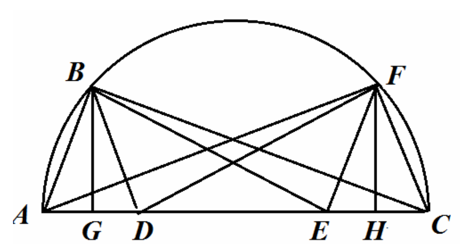
therefore with the equal squares EC, AD added, there will become with CED, the rectangle taken twice, that is the rectangle ECA taken once, [which is] the excess by which the squares AF, FC, exceeds the two squares DF, FE. In the same manner it may be shown the two squares AB, BC to exceed the two squares DB, BE, by the rectangle DAC that is ECA; therefore since the squares AF, FC to exceed the two squares AB, BE, & the excess DAC, ECA, over the squares DF, FE, DB, BE, also shall be equal, with these removed the squares DB, BE remain, equal to the squares AF, FE.

$$i.e.\left[\begin{array}{l} AF^2=FH^2+AH^2;FD^2=DH^2+HF^2 \\ \therefore AF^2-FD^2=AH^2-DH^2=\left(AD+DH\right)^2-DH^2=2AD.DH+AD^2 \end{array}\right]$$

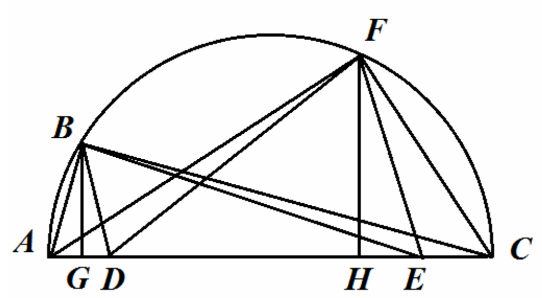
$$\left[EC^2+AD^2+2CE.ED=EC\left(EC+AD+DE\right)+CE.ED=EC.AC+CE.ED=ECA+CED\right]$$

$$\begin{array}{l} FC^2-FE^2=2HE.EC+EC^2 \\ AF^2-FD^2=2AD.DH+AD^2 \\ \therefore FC^2-FE^2+AF^2-FD^2=2HE.EC+EC^2+2AD.DH+AD^2 \\ =AD^2+EC^2+2EC.ED=2EC^2+2EC.ED=2EC\left(EC+ED\right)=2EC.CD=2ECD. \end{array}$$

$$\therefore AF^2+FC^2-DF^2-FE^2=EC.CA=ECA$$



In the second case, the normals sent from B & F may fall between the points AD, EC; indeed BG between A & D; FH truly between E & C, since the angle ADF shall be greater than a right angle, the square AF, will exceed the squares AD, DF by the rectangle ADH, taken twice, [§12, *second part*.] that is by the rectangle ADE, together with the rectangle ADEH taken twice: truly since the angle FEC is less than a right angle, the square FC is smaller than the square EF [§.13. *second part*,], EC by the rectangle CEH, that is ADEH taken twice; therefore the rectangle ADE taken twice is the excess by which the two squares AF, FC exceed the four squares AD, DF, CE, EF ; therefore with the squares AD, CE taken away, there remains the rectangle ADE taken twice by which the squares AF, FC, exceed the squares DF, EF; by the same reasoning it is shown the squares DB, BE, to be surpassed by the squares AB, BC, with the rectangle CED taken twice; therefore since the squares AB, BC, shall be equal to the squares AF, FC, & the rectangle ADE (the excess of the squares AF, FC, over the squares DF, FE) shall be equal to the rectangle CED, (for the excess by which the squares AB, BC surpass the squares DB, BE) with the excesses removed, the squares DF, FE remain, equal to the squares DB, BE.



In the third place, the one interval of the normals FH, may be contained within DE, the other truly BG within AD : it will be shown as before, the rectangle CED taken twice, to be removed with the equal squares AD, EC to be equal to the excess by which the squares CB, BA will surpass the squares EB, BD; likewise the squares DF, FE, the larger squares from the squares AF, FC, with the rectangle ADE taken twice removed from the squares AD, EC: therefore since the totals shall be equal & the excesses in addition shall be equal, also the remaining squares DF, FE, will be equal to the remaining squares DB, BE. Q.f.d.

Scholium.

It may be possible & more convenient to demonstrate this proposition more easily, but I have determined to leave this for some other person with enthusiasm.

End of Book Three.

QUADRATURAE
CIRCULI

LIBER TERTlUS
DE
CIRCULIS.

ARGUMENTUM.

Liber hic omnis in quatuor partes veluti membra dividitur.
Prima de linearum incirculis agit proportione.
Secundo angulos & arcus circulares inter se comparat.
Tertia circulorum mutuas intersectiones & contactus exhibet.
Quarta linearum in circulis potentiam contemplatur.

CIRCULORUM

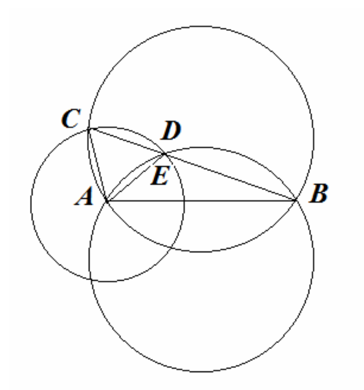
PARS PRIMA.

De linearum in Circulis proportione.

Propositio Prima.

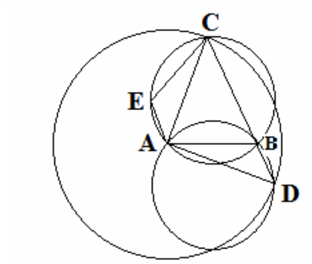
Aequales circuli sese intersecent in A & B; centroque A, intervallo AC, circulus describatur, occurrens aequalibus circulis in CD.

Dico C, D, B, puncta esse in directum.

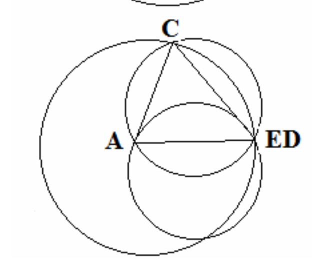


Demonstratio.

Sit primo radius AC, minor AB, ducaturque CB, occurrens ADB, perimetro in E, iunganturque AE, AC. Quoniam angulus ABC , ACB utrique circulorum aequalium ADB, ACB communis est, erunt arcus [*29, tertii*] AE, AC, illorumque subtensae aequales ; hoc est recta AE, aequalis AC, & E punctum in peripheria circuli ADC; sed item E, per constructionem est perimetro circuli ADB, igitur E punctum cum D, idem est, transitque per D, & C; quare in directum sunt puncta C, D , B.



2. Radius AC maior sit recta AB iungantur CB, BD : & AB rectae, aequalis applicetur CE, iunganturque; AE, AD. Quoniam igitur CE linea, aequalis ponitur rectae AB linea, aequalis ponitur rectae AB, & CA aequalis ipsi AD, sint autem & circuli ABC, ABD per construct. inter se aequalis, erit arcus [*ibid*] CE aequalis arcum AB , & arcus CEA, aequalis arcum ABD : unde angulus ABD, aequalis [26.tertii] angulo AEC. sed angulus AEC una cum angulo [21.tertii] ABC duobus rectis est aequalis, igitur & angulus ABD, cum angulo ABC duobus rectis aequatur. quare CB, BD lineae in directum sunt.

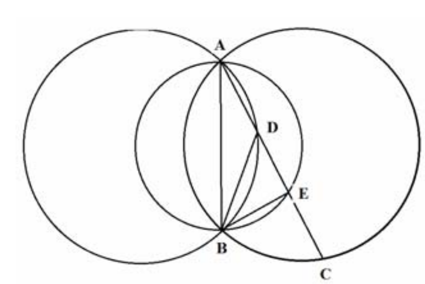


3. Radius AC, aequalis sit radio AB patet punctum D, incidere in B; igitur, &c. Quod fuit demonstrandum.

Propositio II

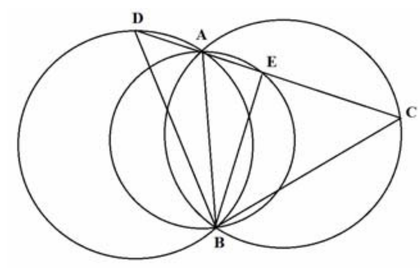
Occurrant sibi denuo aequales duo circuli in A & B. circuli quoque AEB diameter fit AB, ducaturque rectque quaevis AD, occurrens perimetris circulorum aequalium in D, & C, & circulo AEB in E. Dico in E bifariam dividi rectam DC.

Demonstratio.



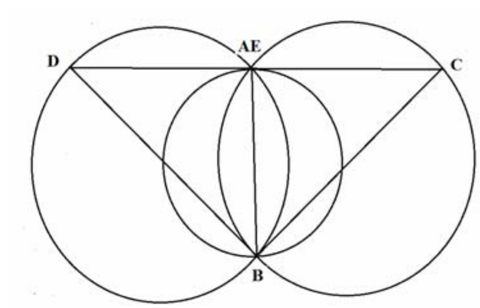
Cadant primo puncta E & D; ad eandem partem lineae AB ; ducantur rectae DB, CB, EB: Quoniam angulus CAB, duobus arcubus DB, CB circulorum aequalium insistit, aequales erunt subtensem DB, CB ac proinde anguli EDB, ECB aequales ; sunt autem anguli DEB, CER, recti ob AEB semicirculum, & EB verique triangulorum DBE, BEC communis, triangula igituur DEB, CEB, aequalia sunt inter se, & similia ; quare & CD in E bifariam divisa.

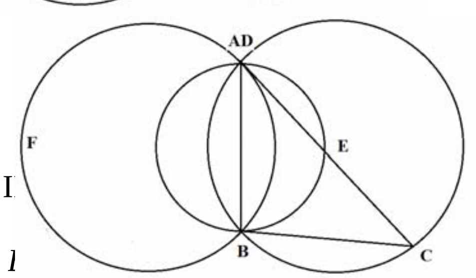
Secundo sit A punctum medium inter D & E : Cum circuli ACB, ADB, sint aequales, & AB linea, utrique communis triangulo DEB, EBC, erunt anguli ADB, ACB, inter se aequales; quare cum anguli quoque ad E recti sint, erunt triangula DEB, EBC inter se aequalia & similia; ac proindc DE rectae EC aequalis.



Tertio linea AC contingat circulum AFB in A : punctum quoque D idem sit cum A, quia igitur AC contingit circulum AFB, erit angulus CAB, aequalis [32.tertii] angulo segmenti DFB, ac proinde angulo ACB (cum AFB, ACB segmenta sunt aequalia) quare aequales quoque sunt anguli BAC, BCA : & quia angulus AEB rectus est in semicirculo AEB, erunt similia triangula & aequalia ABE, EBC : unde & AE, EC aequales sunt lineae.

Quarto. Contingat recta CD, circulum diametri AB in A, adeoque & punctum E, idem sit cum puncto A, cum igitur DC sit contingens, [*18.tertii*] erunt BAD, BAC anguli recti : sunt autem aequales anguli ADB, ACB, aequalium segmentorum ; igitur triangula ADB, ABC, similia sunt & aequalia. Quod fuit demonstrandum.



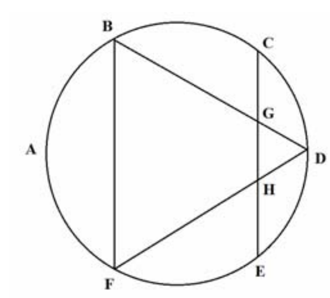


PROPOSITIO I

Diviso circulo in sex partes aequales, punctis A, l ponantur BD, FD, occurrences CE in G & H. Dico CE lineam in G & H triſariam esse divisam.

Demonstratio.

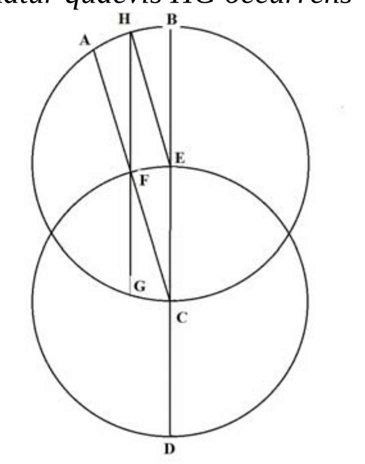
Cum enim arcus BD, DF, FB ponantur aequales, erit BDF triangulum aequilaterum. Unde cum CE



aequidistet BF, erit & GDH aequilaterum: est autem GD aequalis [*Deductae ex 109 & 111. Pappa lib.7*] CG & HD aequalis HE; igitur CG, GH, HE lineae sunt se aequales, & CE in G & H trifarium divisa.
Quod erat demonstrandum.

PROPOSITIO IV

Secent se invicem aequales duo circuli ABC, DEF per mutua centra C, E, transeuntees : lineae vero BD, per utriusque centrum actae, parallela ponatur quaevis HG occurrens perimetro in F ; & per F linea CFA. Dico GF, FA, aequales esse.



Demonstratio.

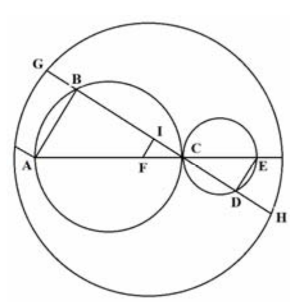
Ducatur EH quoniam HG, aequidistat BC erit angulus FHE aequalis angulo HEB & quia arcus, HB, FE ob circulum aequalitatem aequales quoque sunt, erit angulus FCE aequalis angulo HEB, adeoque angulo FHE ; unde parallelogrammum vel Rhombus est CH: & HF lineae aequalis CE, id est CF; est autem rectangulo CFA, aequale HFG rectangulum [*35.tertii*] ; igitur AF, GF lineae [*1.sexii*] quoque inter se aequantur.
Quod erat demonstrandum.

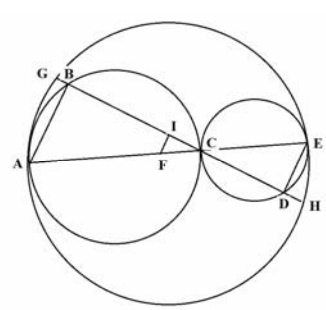
Corollarium.

Hinc sequitur HF, BE lineas quoque inter se aequari, cum HF linea aequatur ipsi EC : & quia HG est quaecumque aequidistans diametro BC, sequitur parallelas omnes rectae BE, cava circuli ABC, & convexa FCD peripheria interceptas, esse inter se aequales, cum singulae aequentur ipsi BE.

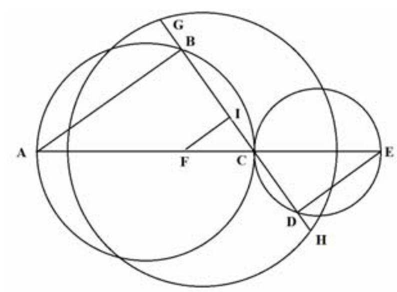
PROPOSITIO V.

Contingant sese circuli ABC, CDE exterius in C ; linea vero AE per utrusque centrum acta, ac divisa bifariam in F, describat quivis circulus centre F, & per C, punctum contactus, recta ponatur GCH:
Dico GB, DH aequales esse lineas.



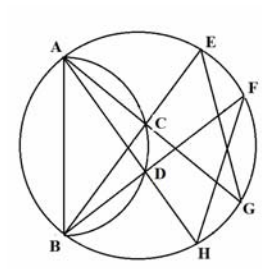


Demonstratio.



Jungantur AB, ED, illisque; aequidistans, ponatur FI, erit haec normalis ad GH, cum ABC angulus rectus sit ; undc & GH in I divisa est bifariam, estque AF ad FC, ut BI ad IC : & permutando AF ad BI, ut FC ad IC. Deinde ut EC ad CF, ita DC est ad CI, & componendo, permutando, ut FC ad IC, ita EF ad ID; sed ut FC ad IC, sic AF ad BI, igitur ut AF ad BI, ita EF ad DI, & permutando ut AF ad EF, ita BI ad DI sunt autem lineae AF, FE ex hypothesi aequales inter se ; ergo DI, BI, quoque inter se aequantur; quae si demantur ab aequalibus IH, IG, manent residuae GB, DH, inter se aequales. Quod fuit demonstrandum.

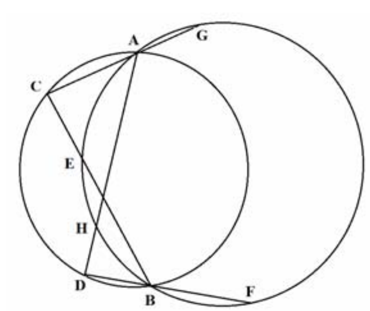
PROPOSITIO VI.



Intersecent sese quivis duo circuli in A & B; assumptisque in ACD perimetro punctis C, D, agantur per illa lineae ACG, ADH: & BCE, BDF.

Dico iunctas EG, FH aequales esse.

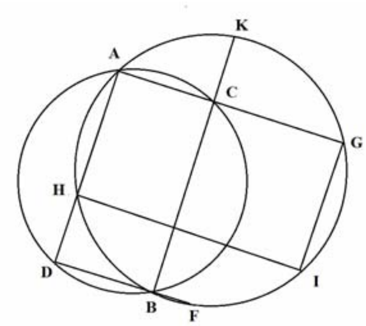
Demonstratio.



Primo, arcus ACB ambitu AGB interceptus, utrumque punctorum C & D , contineat ; quo casu cum anguli CAD, CBD arcum CD insistentes [*27.tertii*] aequentur, erunt EF, GH arcus [*26.tertii*] quoque inter se aequales; addito igitur communi arcu FG, erunt FH, EG arcus adeoque & lineae aequales [*29.tertii*] :

Secundo arcus ACB , extra AFB ambitum ; contentus , puncta C & D obtineat cum igitur anguli CAD , CBD arcum CD insistentes sint aequales, erunt quoque anguli HAG, FBE reliqui aequales ; ac proinde arcus FGE, FGH, adeoque iunctae EG, HF aequales.

Tertio punctum C intra AGB circuli spatium contineantur; D vero punctum extra collocatum sit. Ducatur GI aequidistans CB, iunganturque HI: quoniam GI, CB aequidistant, anguli ACB, AGI, aequales sunt; unde & angulus AHI aequalis est: angulo ADB: quia AHI , [*22.tertii*] cum AGI , hoc est ACB duobus rectis aequalis est, sicut est angulus ACB cum ADB. Unde aequidistantes sunt HI & DF, adeoque & arcus HB, FI & consequenter HF, BI aequales; quare &



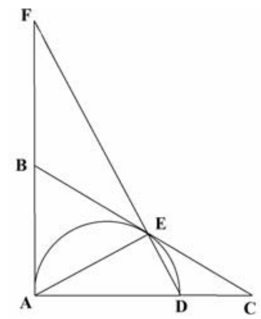
iuncta HF id est BI ipsi EG aequatur. Quod erat demonstrandum.

PROPOSITIO VII.

Esto ABC triangulo rectangulo, semicirculus inscriptus AED contingens AB, BC latera in A & E: &: per E ponatur recta DE occurrens AB, in F.
Dico AB, BF lineas, aequales esse.

Demonstratio.

Iungantur AE : erit igitur angulus AED in semicirculo rectus, uti & reliquus AEF: qui proinde aequalis est duobus angulis EFA, EAF : est autem angulo EAF aequalis BEA., cum AB, BE sint contingentes ex eodem puncto eductae; adoque aequales; reliquus igitur angulus AFE, reliquo BEF aequalis est : quare BF rectae BE, hoc est AB aequalis est.
Quod erat demonstrandum.

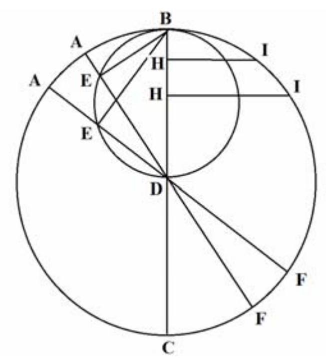


PROPOSITIO VIII.

Circulum ABC contingat intus DEB, in B. transiens per D centrum circuli ABC, ductis insuper per D rectis ADF quae circulo DEB occurrant in E: fiant DE rectis aequales DH, & perpendiculuares ponantur HI ad diametrum BC.
Dico EB, HI lineas, aequales esse.

Demonstratio.

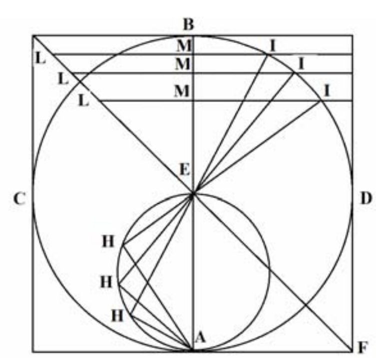
Quoniam BD, lineae ipsi DA sunt aequales; veluti & HD, rectis ED ex hypothesi, erunt
residuae HB, residuis EA aequales. quare AEF rectangula , rectangulis BHC aequalia, hoc est: quadrata HI, quadritis EB: aequales igitur sunt EB, HI. Quod demonstrandum fuit.



PROPOSITIO IX.

Esto circulo ABC, cuius diameter AB, & centrum E circumscriptum quadratum GF: & super AE ut diametro, descripto circulo AHE, per centrum E, lineae ponantur HI, & per I parallelae rectae AF, occurrentes FG, diametro quadrati in L, rectae vero FD in K.
Dico HL, KL lineas inter se aequales esse.

Demonstratio.



Coniungantur H, A ; & rectae KL, secent AB , diametrum in M: Quoniam LK lineae , tangenti GB aequidistant, erunt IM, [*18.tertii*] normales ad diametrum AB, adeoque anguli IME, angulis EHA aequales; sunt autem & IEM anguli, aequales angulis HEA, ad verticem positis, & EI lineae, aequales rectae EA; triangula igitur HEA, triangulis IME sint aequaliae : unde HE, lineis EM, & HA, rectis MI equales. Rursum cum LM, lineae GB aequidistent erit ut GB ad BE, ita LM ad MB sunt autem GB, BE aeqales, ergo & LM, EM aequales quoque sunt. Quare cum M K, hoc est EB, lineis EI, & rectae LM, ipsis ME, id est HE, ex demonstratis sint aequales, erunt HI lineae, aequales rectis L K; quod fuit demonstrandum.

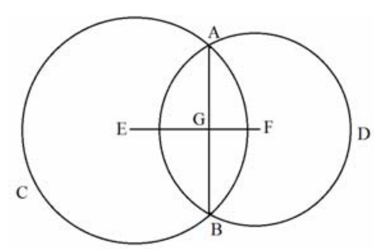
PROPOSITIO X.

Quod si per circulorum sesesecantium centra, recta ducta sit, quam altera intersecet, sectionum punctacentra coniungens.

Dico duas illas sese orthogonaliter decussare.

Demonstratio.

Sint enim circuli duo ABC, ADB quorum centra E,F,coniungat EF recta; & puncta GV intersectionum recta AB, occurrens EF lineae D in G, oportet ostendere angulos ad G rectos esse ducta EG normaliter ad AB, secta erit AB bifariam in G; sed recta quae ex G ducitur, ad F centrum, dividens AB bifariam in circulo ADB, eidem quoque AB orthogonaliter insistit ; patet igitur lineas AB, EF sibi invicem normales esse. Quod fuit demonstrandum.

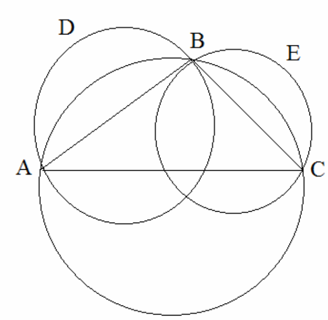


Corollarium.

Hinc patet, EF lineam, quae circulorum sese intersecandum centracentra coniungit, bifariam quoque dividere arcus; mutuis peripheriis interceptos.

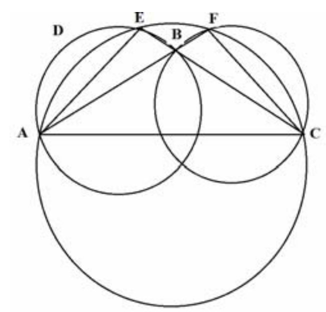
PROPOSITIO XI.

Esto ABC triangulum, super cuius basi AC, descriptum sit quodlibet circuli segmentum; oportet super reliquis trianguli lateribus, segmenta describere, similia illi, quod super base descriptu est segmento.

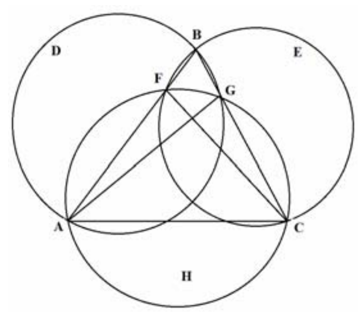


Constructio & demonstratio.

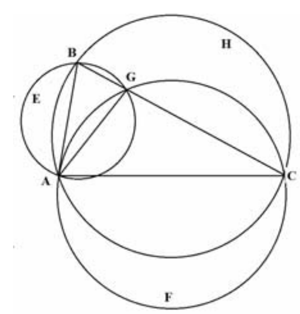
Transeat primo segmentum superbasi positum, per singula trianguli extrema ;describatur autem circulus per BC, uti & per BA, qui rectas AB, BC contingat in B. Dico factum quod postulatur. Quoniam BC linea contingit circulum ADB in B ; erit angulo ABC, aequalis angulus segmenti ADB ; idem modo angulus segmenti BEC [*32.tertii*], aequlis est angulo ABC ; quare segmentorum anguli AEB, ABC, BDC, aequales sunt inter se ; & similia proinde segmenta.



Secundo B vertex trianguli ABC cadat infra perimetrum segmemi, super AC exstructi : productis lateribus AB, BC donec occurrant peripheriae circuli AEC in F & E, describatur circuli per AEB, BFC : eritque peractum quod postulator : iunctis enim AE, CF exsurgens anguli AEC, AFC eidem arcum insistentes aequales, ac proinde ADB, AEC segmenta similia erunt : rursum cum angulus AEC sit aequalis angulo AFC, id est BFC erunt & segmenta AEC, BFC inter se similia.



Tertia B vertex trianguli ABC, extra segmenti ambitum constitutus sit, quod super basi extructum est ; occurratque perimeter trianguli lateribus in punctis F, & G. Tum per BFC, circuli describantur. Dico factum esse quod petitur. Iungantur AG, FC. Quoniam anguli AFC, AGC aequales sunt, erunt & BFC, BGA reliqui aequales, quare & arcus ADB, BEC quibus insistunt, similar sunt. Rursus cum angulus AGC tam cum angulo [*22.tertii*] segmenti AHC, quam cum AGB angulo, duobus rectis aequetur, dempto communi angulo AGC : erunt anguli AGB, AHC, adeoque & residuorum segmentorum anguli ADB, AGC, aequales & AGC, BDA, segmenta similia : est autem BEC segmentum ostensum simile segmento ADB, tria igitur segmenta AGC, ADB, BEC sunt inter se similia.



Quarto B punctum cadens extra segmentum baseos, constituat BA, trianguli latus contingens circulum AGC in AE latus vero BC, eundem secet in G ; per puncta B, A, C, & A, B, G, circuli describantur ; dico illos satisfacere petitioni iungantur A, G. Quoniam AB contingit circulum AGC in A, erit angulo [*32.tertii*] BAC, aequalis angulus segmenti AFC, unde BAC, AFC, ideoque reliqua AGC, BHC segmenta sunt similia : ulterius cum angulus AGB tam cum angulo AGB, quam cum angulo segmenti AFC [*22.tertii*], duobus rectis sit aequalis, dempto communi angulo AGC, erunt AGB, AFC aequales anguli ; unde & angulus segmenti AEB, aequatur angulo AGC, adeoque AEB, AGC segmenta sunt similia; igitur constat veritas propositionis.

Corollarium.

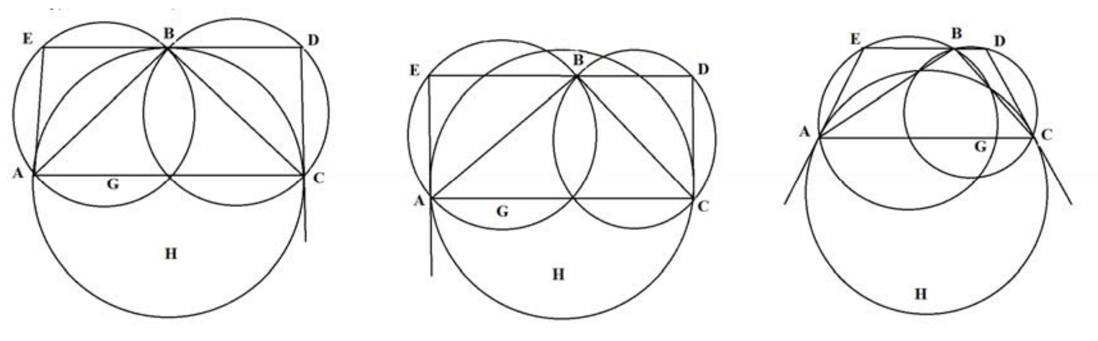
Hinc patet in secundo casu, si super ABC, trianguli lateribus similia descripta sint circulorum segmenta rectas AB, CB productas cadere in communes intersectiones E,F: si enim per B & puncta quibus AEC perimetro occurrunt circuli describantur erunt AEB, BFC segmenta similia segmento ABC.

PROPOSITIO XII.

Super ABC trianguli lateribus descripta sint AEB, ABC, BDC similia circulorum segmenta; ductaque lineae ex A & C circulum ABC, contingant in A, & C, peripheriis autem occurrant in D, &E.

Dico E,B, D puncta esse in directum posita.

Demonstratio.

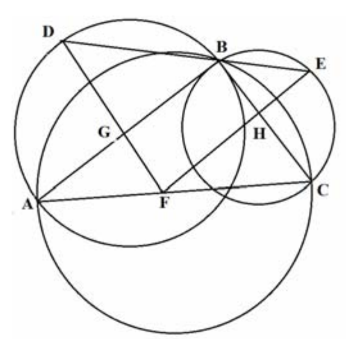


Quoniam EA, contingit circulum ABC erit angulo [32.*tertii*] EAC, aequalis angulus segmenti AHC, hoc est segmenti AGB quod illi simile est; sed angulus AEB, una cum angulo segmenti AGB; duobus rectis est aequalis [*22.tertii*], igitur AEB , EAC anguli duobus rectis sunt aequales; adeoque EB, AC lineae parallelae : eodem modo ostenduntur AC, BD, aequidistantes esse; quare constat EB, BD, in directum esse constitutas : quod erat demonstrandum.

PROPOSITIO XIII.

Super lateribus trianguli ABC, similia circulorum segmenta descripta sint per quorum centra G, H ,ex centro F, segmenti super AC basi trianguli descripti educantur rectae occurrentes perimetris in D & E.
Dico puncta D, B, E, in directum constituta esse.

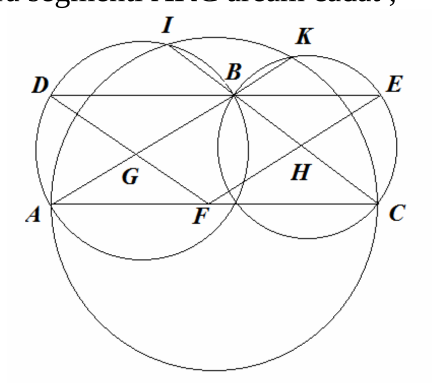
Demonstratio.



Primo vertex triangui ABC in perimetro sit circuli, qui super basi AC, describitur iunganturque DB, EB. Quoniam FD, per centra F & G, acta est, secabit bifariam, tum [10.huius] recta AB, tum, arcum ADB, eadem quoque ratione radius FH, dividet bifariam cum

BC rectam, cum HEC arcum. Igitur cum ADB, BEC similia sint segmenta anguli ABD, CBE similibus arcubus insistentes, aequales sunt : quare & angulus ABK aequalis est angulo BCE, quia EB arcus aequalis est arcum EC, est autem angulus ABC aequales angulo BEC ex hypothesi, igitur anguli ABD, ABC, CBE aequales sunt tribus angulis trianguli BEC adeoque duobus rectis aequales, quare lineae DB, BE [14.primi] sunt in directum.

Secundo B apex trianguli, super basi ABC erecti, intra segmenti AKC aream cadat ; producantur AB, CB, cadet illae in comunes [*Coroll. prop.undec.huius*] circulo tum intersectiones I & K ; Quoniam FG, FH lineae centra coniungunt circulorum sese intersectantium erunt arcus ADI, KEC in D & E bifariam [10.huius] divisi, unde ABD, DBI anguli , item KBE, CBE sunt aequales, sed & anguli ABI, CBK ad verticem oppositi quoque inter se aequantur, igitur & angulus ABD, ipsi KBE & IBD, angulo EBC aequalis; quare,in directum sunt D, B,E puncta. Quod erat demonstrandum.



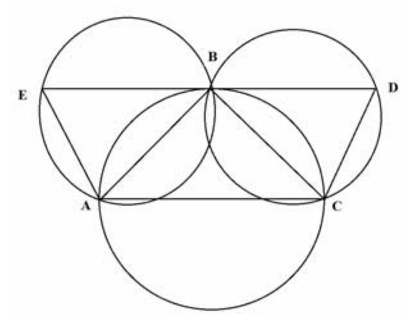
PROPOSITIO XIV.

Quod si super ABC triangli lateribus segmenta circulorumm similia, descripta fuerint ; & recta quaedam ED, per B, verticem acta, Circulorum AEB, BDC, perimetris occurrat in D, & E.

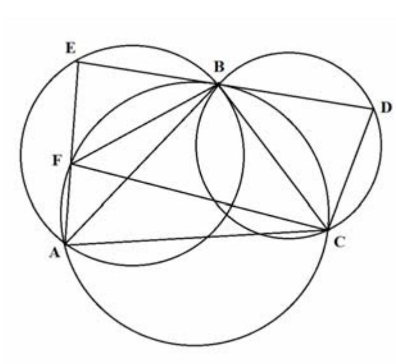
Dico lineas inter cauas circulorum AEB, BDC peripherias & extimam circuli ABC interceptas, aequales esse.

Demonstratio.

Sit primo triangulum isoscelium ABC; & ED contingat circulum ABC in B : iunganturque AE, CD; Quoniam B est contingens, erit angulo EBA, aequales angulus ACB [32.*tertii*] : eadem ratione angulus CAB aequalis est angulo CBD : unde cum anguli BAC, BCA per hypothesim sint aequales, erunt quoque anguli EBA, CBD inter se aequales, (ob AEB, CDB segmenta simila), insuper & AB, linea aequalis lineae BC; igitur triangulum AEV, triangulo CBD & EB latus, lateri BD est aequale.

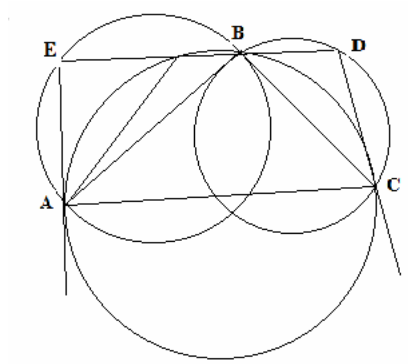


Secundo ED pontur contingens, triangulo ABC existente Scalene, dico ED in B bifariam secari. sit AB latus, altero BC maius, & ED contingenti, ponantur natura aequidistans CF: iunctaque AF, concurrat cum DB, producta in E. quoniam parallelae sunt CF, DE, erit angulus AFC angulo AED aequalis: sed AFC aequalis ponitur angulo segmenti

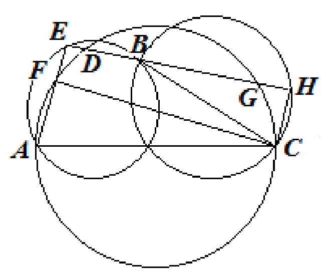


AEB; igitur concursus AF, cum DB producta, sit in perimetro circuli AEB: Rursum quia contingens est ED, eidemque aequidistat CF, erunt segmenta BF, BC aequalia, ac proinde iunctae FB, BC quoque inter se aequales, unde similia sunt, & aequalia triangula FEB, BDC, ideoque EB aequalis BD.

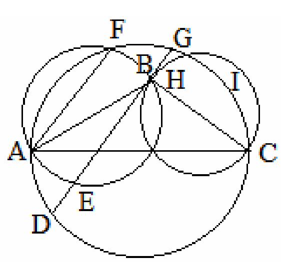
Tertio, recta ED circulum non contingat super basi AC erectum , sed intersecet in puncto quidam G. sitque ED aequidistans AC. dico EG, BD lineas aequari cum enim ED, AC aequidistent, erunt, AG, BC arcus adeoque & subtensae aequales: unde & anguli GAC, BCA aequales sunt, uti & anguli illis alternatim positi EGA, DBC; quocirca similia sunt & aequalia triangula EAG, BDC; & recta EG , ipsi BD aequalis.



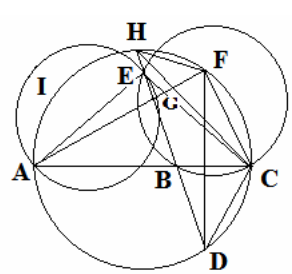
Quarto, quod si recta ED per B ducta occurrens perimetro ABC in G, non aequidistet AC, ducatur AF parallela ED, & ducta CF, occurrat EB, productet in D, quoniam parallelae sunt ED, AF, erit AFC angulus angulo EDC aequalis: sed AC aequatur angulo segmenti BDC ex positione, igitur D in perimetro est: Circuli BDC; & quia aequidistant ED, AF erunt arcus AG, BF, aequales, adeoque & subtensae AG, BF cumque aequales sint anguli GAF, BFA, aequalibus arcubus insistentes, erunt & anguli EGA, DBF quoque aequales ; sunt insuper aequales anguli AEG, BDF, ob segmenta similia; igitur triangula similia sunt & aequalia AEG, BDF; unde & lineae EG, BD, aequales.



Quinto cadat B vertex trianguli ABC intra aream circuli supra basim trianguli descripti: & primo recta EH per apice B transiens basi trianguli non occurrat. ducatur CF aequidistans EH, donec conveniat cum AF producta in E. ostendetur uti prius punctum E in perimetro esse circuli AEB, & FED, GHC triangula, adeoque & latera ED, GH inter se aequari. Secundo recta per apice B acta, occurrat basi trianguli : quo casu demonstrandum est linem ED, rectae GH aequalem esse.

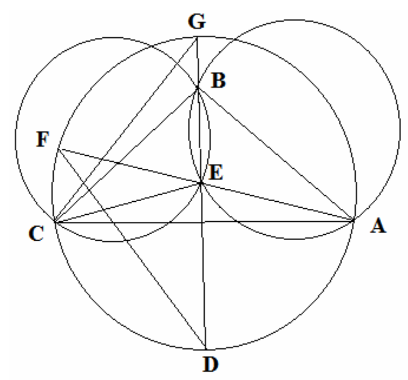


Quoniam EH per B ducta occurrit basi trianguli AC; vel alterum circulorum contingit in B, (quo casu apparet ED, GH, aequales esse: cum FHG, ADE triangula facile ex datis ostendantur aequalia.) vel utrumque circulorum super lateribus trianguli ABC descriptorum secat in G, & E, circulum autem super basi factum in D, & HI; quo casu ducatur ex A per G, linea AGF, iungaturque puncta EC,CF, FH, CD; angulus AGB una cum angulo segmenti AIB [22.*tertii*] aequalis est duobus rectis; sed angulo AIB, aequalis est angulus AFC ex hypothesi, & angulo AGB aequalis angulus EGF ipsi AGB ad verticem positus, igitur anguli AFC, EGF, sunt duobus rectis aequales adeoque BD,



FC parallelae, & arcus HF, CDE, eorumque subtensae aequales; unde & anguli DHF, HDC aequalibus arcubus insistentes, aequales sunt: est autem angulus AGB ,aequalis angulo CEB, (cum AGB, BEC, segmenta reliqua sint similia) adeoque &. reliquus FGH, aequalis reliquo CED; igitur triangula FHG, CED, sunt inter se aequalia; & HG latus aequale lateri ED.

Sexto, quod si recta GD per apicem acta, per communem intersectionem E transeat; sic ostendetur aequales esse rectas GE, ED : sint primo segmenta similia, semi-circulis maiora; ductaque ex A per E, recta AEF, iungantur CE, CF, ED, CG: Quoniam igitur segmenta similia, semicirculis sunt maiora, erit E extra lineam AC. si enim fieri possit, sit CEA una eademque linea cum AC cum ergo ob similitudinem segmentorum, anguli BEA, BEC sint inter se aequales, erunt etiam recti: quod fieri non potest, cum segmenta sint semicirculis maiora, quare punctum E non est in linea AC. Cum igitur angulus AFC, una cum angulo segmento ADC, duobus rectis sit aequalis, sitque angulo ADC, aequalis angulus AEB (ob ADC, AEB segmenta similia), id est angulus DEF ad verticem oppositus, erit angulus AFC una cum angulo DEF, duobus rectis aequalis; unde FC, BD lineae aequidistant; & arcus GF, DC, adeoque & arcus GC, DE eorumque subtensae aequales, &c ut prius.



Si vero segmenta similia minora fuerint semicirculis: demonstrabitur ut prius, punctum E, esse extra lineam baseos AC quia vero AFC, AEB segmenta sunt similia, erunt anguli AEB, AFC aequales; sed angulo AEB aequalis est angulus ad verticem DEF; igitur anguli DEF, AFC sunt aequales, & BD, FC rectae aequidistantes ; quare & arcus GC, DF, eorumque subtensae aequales sunt: &c. uti prius.

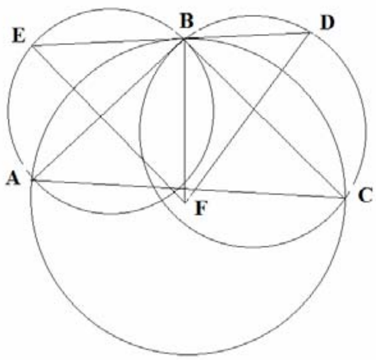
Reliquum esset casus eosdem explicare cum vertex trianguli ABC cadit in peripheriam vel extra aream circuli qui supra basim AC *constitutus est; sed quia omnes illi, demonstrationes communes habent & constructiones cum casibus quos explicuimus, dum nimirum vertex* B, *ipsa area continetur, lectorem non ulterius defatigandum censeo* : *hoc tamen praemoneo, ut dum* B *vertex trianguli, cadit extra segmentum baseos, casus excludat, quos in medium protulmus qui* BD *lineam iubent earatione duciione duci, ut cum basi trianguli non conveniat, quos ineptos esse huic materiae manifestum est*

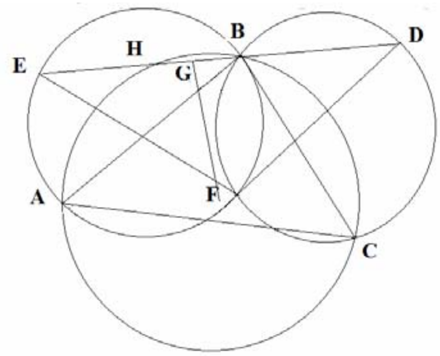
PROPOSITIO XV.

Si denuo super ABC trianguli lateribus segmenta circulorum similia constructa fuerint; ac per B verticem ponatur ED, ad cuius extremata ex centro F circuli super basi descripti, ducantur FE, FD.

Dico eas inter se aequales esse.

Demonstratio





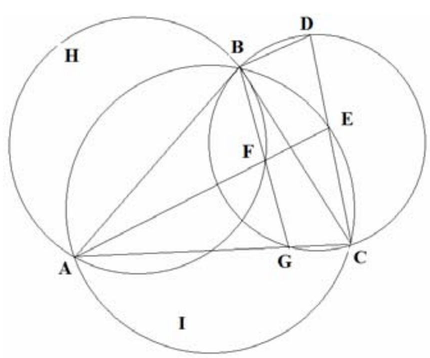
Contingat recta ED, circulum ABC in B; erit igitur BF normalis [18.*tertii*] ipsi ED : unde cum lineae BE, BD, per praecedentem aequales sint , erunt etiam aequales FE, FD:
Iam vero ED, circulum ABC secet in puncto quodam H: ducaturque ex F recta FG, perpendicularis ad ED: igitur HG, GB aequales sunt [3.*tertii*] ; ostendimus autem EH, BD [14.*binius*] quoque aequales esse ; igitur FE, FD aequales sunt, Quod fuit demonstrandum.

PROPOSITIO XVI.

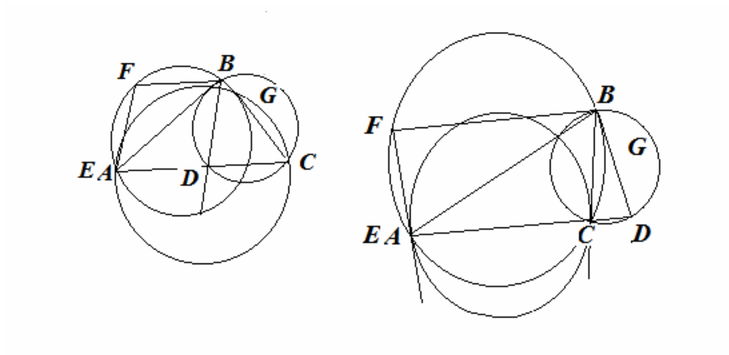
Super ABC, trianguli lateribus, segmenta circulorum similia constituta sint, & ex C recta educta CD; cum ex B vertice trianguli parallela ponatur BF, occurrens circulo AHB in F: Dico ED , B F, aequales esse lineas.

Demonstratio.

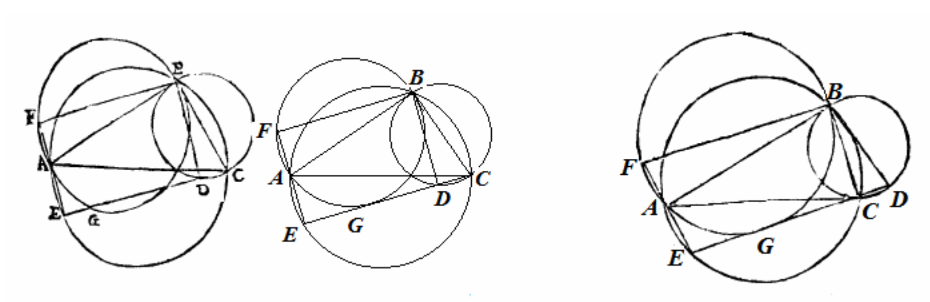
Sit primo vertex trianguli in perimerro circuli ABC constitutus; linea vero DC sit extra triangulum ABC : ponatur iunctae BD aequidistans AF , occurrens rectae CD in E ; igitur angulo BDC, aequalis est AEC , sed & BDC angulo aequatur angulus segmenti AEC, igitur punctum E, communis est intersectio circuli ABC & rectae CD : & quia segmenta



circulorum ABC, BDC similia erit angulus AFB aequalis illi qui segmento AIC continetur ; igitur & punctum F communis est intersectio circuli AHB, & recte BF: cum igitur parallelogrammum sit BFED, manifestum est ED, BF lineas inter se aequales esse. Quod oportuit demonstrare.

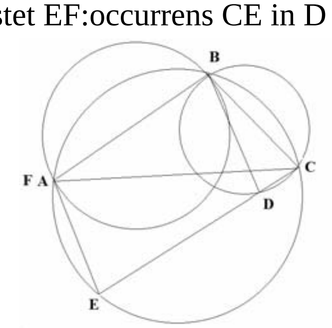


Secundo, CD linea in directum posita sic, cum AC , & recta BF aequidistans basi AC; iungantur AF; & BD ponatur aequidistans ipsi AF occurrens AC in D. Quoniam AC, BF, item AF, BD aequidistant, erit angulus CAF (id est CDB in Secundo figura) una cum angulo BFA (Id est BDC in prima figura) aequalis duobus rectis, sed angulo BFA, per constructonem est aequalis angulus segment CGB. Igitur CDB angulus, una cum angulo segmenti CGB duobus rectis est aequalis, quare punctum D est in perimetro circuli CGB & AC linea eidem bis occurrit: quare cum FD parallelogrammum sit: patet FB, ED lineas aequari.



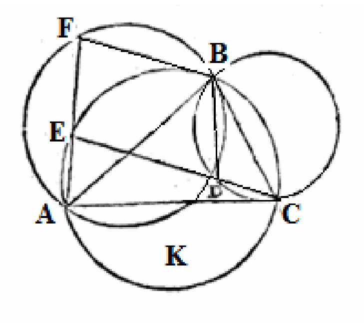
Tertio, recta CD cadat infra basim trianguli ABC, occurrens circulo BCD , in D ; & circulo ABC in E; ita tamen ut ducta BF parallela ipsi CD, cadat supra latus AB, dico BF, DE lineas esse inter se aequales ducatur enim ex E per A linea, occurrens BF in F iunganturque BD, cum igitur segmenta BDC, AEC similia sint, erit angulus BDC, aequalis angulo AEC, adeoque BD, FE linea aequidistantes unde FD parallelogrammum est & FB, DE latera aequalia.Rursum angulus EFB una cum angulo FED id est angulo segmenti AGB aequalis est duobus rectis: quare punctum F, est in peripheria circuli AFB in Secundo vero figura, quia angulus BDC id est ABC una cum angulo AEC, duobus rectis est aequalis, erunt BD, AE parallel : reliqua ut ante.

Quarto, sit BF linea eadem cum recta AB; ducatur ex C recta CE aequidistans AB: iunctaque FE, demittur ex B recta BD, quae aequidistet EF:occurrens CE in D : ostendetur ut prius punctum D esse communem intersectionem circuli BDC, & rectarum BD, CE:



quarc cum FB, CE & BD, FE, sint aequidistantes, patet FB, ED aequales esse.

Quinto, quod si rectae DE, BF infra latus AB constituantur , & BF quidem secet circulum AFB in F: ponatur recta AFE, occurrens CE in E : & AF aequidistans BD; ostendetur ut prius punctum E in perimetro AEC circuli consistere, uti & punctum D in perimetro circuli BD*C:* unde cum parallelae sint FB, DE, FE, BD, manifestum est, FB, ED esse aequales.



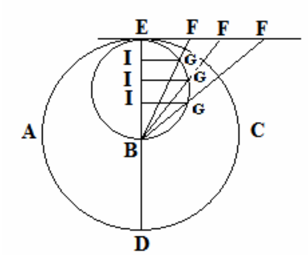
Sexto, tandem lineae FB, DC, supra basim AC constitutae sint, & CD occurrat circulo ABC in E; agaturque per E , recta AE, occurrens FB in F*:* deinde ex B ponatur BD parallela ipsi AF: erit angulus AFB in segmento AFB: & punctum D, communis intersecto rectarum CD, BD & perimetri BDC : cum angulus BDC aequalis sit angulo segmenti AKC, qui cum angulo AEC, hoc est AFB, hoc est BDE, duobus rectis aequalis: patet igitur lineas FB, DE, aequales esse inter se. Quod fuit demonstrandum.

PROPOSITIO XVII.

Contingant sese intus in B circuli duo ABC, EBG: sitque E centrum maioris; posita deinde BF contingente, ducantur EF, occurrences circula EBG in G: ex quibus normales ponantur ad BD diametrum, rectae GI.

Dico EI, EG, EH, EF esse quatuor in continua analogia.

Demonstratio.

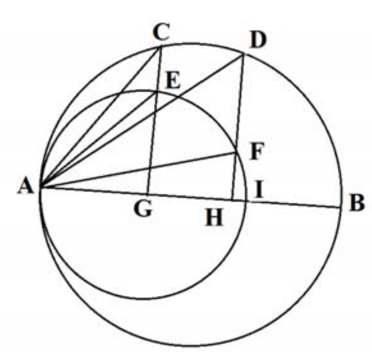


Sunt enim ex elementis continuae proportionales EI, EG, EB , hoc est EH: sed ut EI, ad EG, ita est EB, id est EH, ad EF sunt igitur in continua ratione EI, EG, EH, EF. Quod fuit demonstrandum.

PROPOSITIO XVIII.

Contingant sese item circuli duo ABC, AEF, in A; quorum centra contineat diameter maioris AB, ad quam positis normaliter CG, DH, quae occurrant perimetro AEI, in E & F collocentur AC, AD: & AE, AF.

Dico AC ad AD, eandem rationem habere, quae inter AE, AF, repetitur.



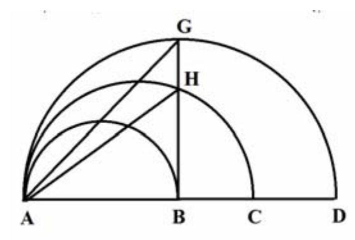
Demonstratio

Quadratum enim AC, ad AD quadratum est ut GAB rectangulum ad rectangulum HAB per elementa hoc est ut GA linea *1.sexti* ad lineam HA sed ut GA ad HA, sic GAI rectangulum ad rectangulum HAI, id est quadratum AE, ad AF quadratum; ergo ut quadratum AC, ad AD, ita est AE linea ad AF, quare ut AC linea ad AD lineam, ita AE, ad AF. Quod erat demonstrandum.

PROPOSITIO XIX.

Sint continuae proportionales AB, AC, AD, diametri circulorum sese intus in eodem puncto contingentium, erectaque ex B normaliter B G occurrente perimetris in H & G, iungantur AG, AH.

Dico AB, AH, A G, esse in continua analogia.



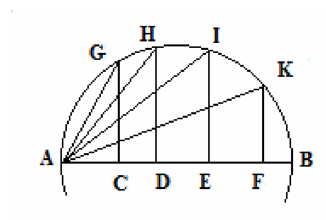
Demonstratio.

Sunt enim in continuata ratione AB, AH, AC veluti etiam AB, AG, AD per elementa, quare AG media est inter AH, AD; sed ex hypothesi ipsa quoque; AC media ponitur inter AB, AD. Igitur AC, AG lineae sunt aequales; adeoque in continua sunt ratione AB, AH, AG.

Quod fuit demonstrandum.

PROPOSITIO XX.

Si divisa fuerit diameter AB in continue proportionales AC, AD, AE, AF, &c. ponanturque normaliter ad diametrum, CG, DH, Ei, FK. Dico iunctas AG, AH, AI, AK, in continua quoque est analogia.



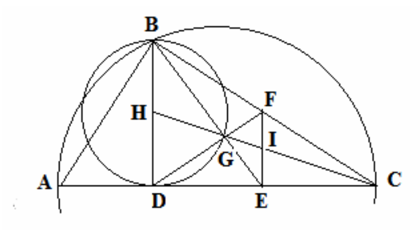
Demonstratio.

Sunt enim quadrata AG, AH, AI, AK inter se ut rectangula BAC, BAD, BAE, BAF ut ex elementis patet; sed rectangula illa sunt ut AC, AD, AE, AF ; igitur & quadrata AG, AH, AI, AK sunt ut lineae AC, AD, AE, AF: quae cum ponantur continuae proportionales, patet & quadrata AG, AH, AI, AK, adeoque & lineas, in continua esse analogia.Quod erat demonstrandum.

PROPOSITlO XXI.

In semicirculo ABC recta collocata sit BD perpendicularis ad diametrum AC, qua divisa bifariam in H; centro H, intervallo HB, describatur circulus BGD, ponanturque HC, quidem occurrens perimetro circuli BGD in G, recta vero BG, diametro AC in E.
Dico AD, DE, EC, tres lineas, eandem rationem continuare.

Demonstratio.

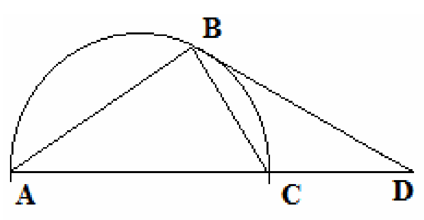


Ponatur EF, aequidistans BD, iunganturque DG, GF, AB, quoniam EF aequidistat BD quae in H divisa est bifariam, erit EI aequalis IF: unde ut EI ad BH, id est FI ad HD, sic EG ad GB, id est IG ad GH, & permutando ut FI ad IG, ita DH ad HG, quare cum anguli DHG, FIG lateribus proportionalibus contenti sint aequales (ob HD, FI parallelas), similia erunt triangula FIG, HDG: sicut & GIE, BHG, item FEG , BDC: sunt ergo in directum D, G, F, puncta, ut ex elementis patet.Ulterius quia similia sunt triangula BDG, DGE, erit BD ad DE ut BG, ad DG: id est EG ad GF: sed ut EG ad GF sic DE est ad EF cum DGE, GEF similia sint triangula, igitur BD est ad DE ut DE ad EF; quare sunt proportionales BD, DE, EF & rectangulum BDFE, quadrato DE aequale, sunt autem similia quoque triangula ADB, EFC igitur ut AD, ad DB, ita FE ad EC; unde rectangulo ADEC aequale est rectangulum BDEF, hoc est quadratum DE sunt igitur continuae proportionales AD, DE, EC: quod fuit demonstrandum.

PROPOSITIO XXII.

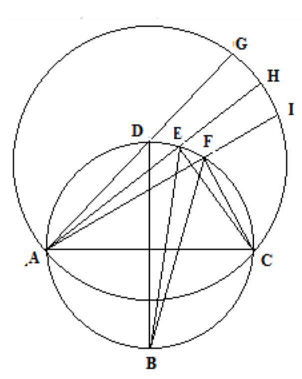
Semicirculo ABC inscriptum esto triangulum cuius alterum latus BC, semidiametro sit aequale.
Dico BC, AB, & aggregatum ex AC, CB, tres esse in continua ratione.

Demonstratio.



Fiat AD sesquialtera CD: iungaturque BD: arcus AB duplus est arcus BC, quia CB latus hexagoni est; igitur & angulus BCA, duplus anguli BAC: sed angulus BCA, quoque duplus anguli[16. primi] BDA cum sit BCD isosceles per constructionem; igitur angulus BDA aequalis est angulo BAD & AB, BD latera aequalia inter se; unde similia sunt triangula BDC, BDA & ut CD id est BC ad DB, ita DB hoc est AB ad AD, hoc est ad AC, CB. Quod fuit demonstrandum.

PROPOSITIO XXIII.



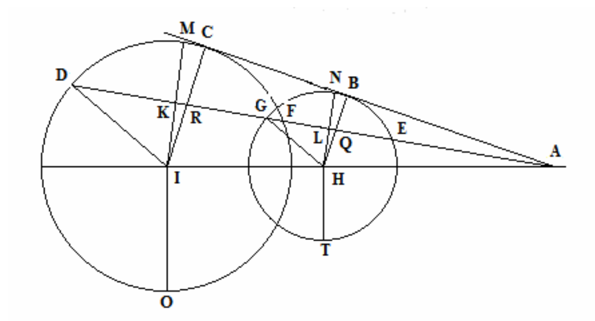
Intersecent sese ad rectos, diametri AC, BD; positisque AEC, AFC; iungantur ED, FD.
Dico DE, ad DF, eandem habere rationem quam aggregatum duarum AE, EC, ad aggregatum AF, FC.

Demonstratio.

Centro B, intervallo AB describatur circulos ACG, cui productae AB, AE, AF, occurrant in G, H, I; Quoniam AG, BD, circulorum suorum diametri sunt, anguli AHG, AIG aequales erunt angulis DEB, DFB, sed & anguli BAE, EAF aequales sunt angulis BDE, EDF, quod iisdem insistant arcubus; similia igitur sunt triangula GAH, HAI triangulis BDE, EDF; & AH, AI lineae proportionales ipsis ED, FD; quare cum sint aequales AH, AI, rectis [*Serenus l.*2 *prop.*45] AEC, AFC, patet aggregatum AE, EC, ad aggregatum AF, FC, eandem obtinere rationem,quam recta DE, ad DF.
Quod fuit demonstrandum.

PROPOSITIO XXIV.

Contingat AB inaequales circulos; conveniens cum recta per utriusque centrum acta in A : ex quo ponatur AD occurrens circulis in E, F, G,D.
Dico AB ad AC eandem continere rationem, quam obtinet AG ad AD, vel AE ad AF.



Demonstratio.

Ad puncta contactuum ex centris ducantur diametri IC, HB, itemque aliae binae diametri HN, IM, & ad rectam DA normales, quoniam igitur tanguli ad B & C recti sunt, paralleli erunt rectae: HB , IC. Ergo ut AB, ad AC, sic HB ad IC, hoc est HQ ad IR. Deinde, quia anguli, IKR, HQL, recti sunt, & IRK, HQL anguli (quod HQ, IR sint parallelae,) sunt aequales; triangula IRK, HQL, erunt similia; ac proinde HL est ad IK, ut HQ ad IR, hoc est: (sicut iam ostendi) ut HB ad IC, hoc est (quoniam HN, IM diametri, aequantur diametris HB, IC) ut HN ad IM. Quia igitur est: HL ad IK, ut HN ad IM, erit permutando ac invertendo NH ad LH, ut MI ad KI, adeoque quadratum NH ad quadratum LH, ut quadratum MI ad quadratum KI, sed [5. secundi] quadratum NH aequatur rectangulo NLP cuius quadrato LH, & quadratum MI, rectangulo MKO cum quadrato KI ergo rectangulum NLP, cum quadrato LH est ad quadratum LH, ut rectangulum MKO cum quadrato KI ad quadratum KI: ergo dividendo rectangulum NLP est ad quadratum

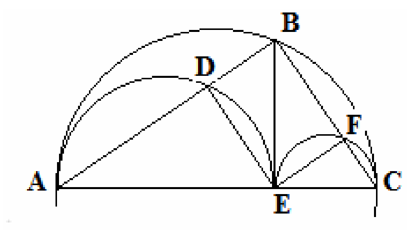
LH, ut rectangulum MKO ad quadratum KI. Atque rectangula NLP ,MKO aequantur quadratis GL, D K, quod GL, DK sint ad diametros NP, MO normales; ergo quadratum GL est ad quadratum LH ut quadratum DK ad quadratum KL. Ergo recta GL est ad rectam LH, ut recta DK ad rectam KI; quoniam igitur anguli quoque GLH, DKI, utpote recti ex constr. aequales sunt, erunt triangula [6. *sexti*] GLH, DKI similia, adeoque anguli HGL, IDK aequales. Ergo GH,Dl parallelae sunt. Ergo ut AG est ad AD, sic AH ad AI, hoc est quia HB, IC etiam sint parallelae, ut AB ad AC. Quod erat primum, simili ratione ostenduntur iunctae: IF, HE aequidistare, adeoque esse ut AH ad AI, id est AB ad AC, sic AE ad AF. Quod erat demonstrandum.

PROPOSITIO XXV.

Sit ABC semicirculi diameter AC divisa utcunque in E: descriptisque super AE, EC semicirculis ADE, EFC, erigitur EB normaliter ad diametrum, ponanturque AB, BC, occurrentes perimetris in D & F.

Dico rationem CF ad DA, triplitatam esse illius, quam habet CB, ad AB.

Demonstratio.



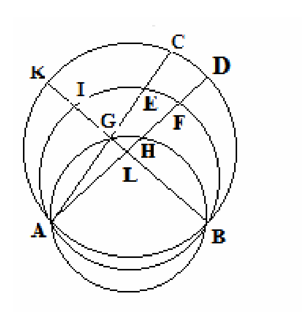
Iungantur DE, EF. Quoniam EB normalis est ad AC diametrum semicirculi ABC erunt AC, CB, CE, item AC, AB, AE continuae proportionales; sed ut BC ad CE sic EC ad CF (ob EFC, BEC triangula similia.) & ut AB ad AE sic AE ad AD (ob ABE, ADE triangula similia) igitur tam AC, CB, CE, CF, quam AC, AB, AE, AD sunt continuae proportionales : unde cum utrique seriei prima sit communis AC, erit [27.de progressionibus] CF ad AD quarta ad quartam, in triplitata ratione CB ad AB, secundae ad secundam. Quod erat demonstrandum.

PROPOSITIO XXVI.

Secent sese tres circuli in punctis A, B, & ex A quaevis rectae AC, AD,eductae perimetris occurrant in E, F, G, H:

Dico GE, EC, rectis HF, FD, esse proportionales.

Demonstratio.



Agatur recta BK per G, occurrens AD in L & perimetris in I & K, cum rectangula ALH, GLB, aequalia fint [35. *tertii*]; uti & rectangula ILB, ALF; & KLB, ALD; erunt rationes laterum reciprocae [14. *sexti*] ; hoc est, erit AL ad LB, ut GL ad HL, &. AL ad LB, ut IL ad LF, vel KL ad LD: quare etiam ut GL ad LH, ita IG ad HF [19. *quinti*], & LK ad LD, sive IK ad FD; rursum cum sit ut AG ad GB, ita IG ad EG vel KG ad GC (ob

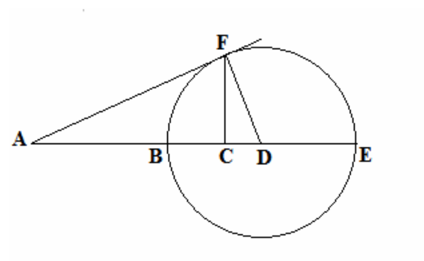
AGE, IGB, item KGB, AGC rectangulorum aequalitatem) erit IG ad GE, ut KI ad EC; quare ex aequo EC ad FD ut GE ad HF. Quod fuit demonstrandum.

PROPOSITIO XXVII.

E puncto A extra circulum posito, ducta ad centrum eiusdem linea AD divisa sit in tres continuo proportionales quarum media sit semidiameter BD, tertia CD: erectaque ex C perpendiculari CF, iungantur AF.

Dico AF contingentem esse & contra.

Demonstratio.



Ponatur DF. Quoniam DC, DB id est DF, DA, circum angulum communem ADF proportionales sunt, erunt FCD, AFD triangula similia: unde angulus AFD aequalis est angulo FCD per hypothesim recto : quare AF circulum cotingit. Quod erat primus.
Iam vero sit AF contingens & FC normaliter ad AD diametrum posita. Dico AD, BD, CD in continua esse analogia : cum enim AF sit contingens & FD diameter, erit angulus AFD rectus adeoque aequalis angulo FCD, est autem angulus ADF communis triangulis AFD, FCD, igitur triangula illa similia sunt, & AD ad DF, id est DB, ut DF ad DC. Quod erat demonstrandum.

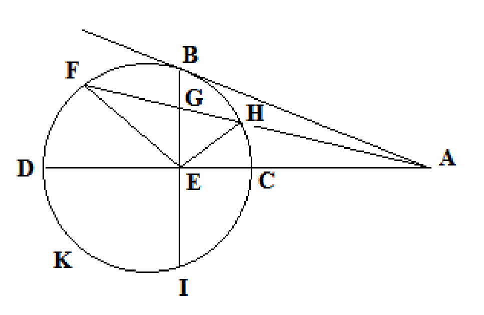
Est hac Apollonii l.prim.propos. 35.aliter demonstrata.

PROPOSITIO XXVIII.

Contingat AB linea circulum BCD : ductaque AD per centrum ponatur ad illam orthogonalis BE, quam in G secet quaedam AF, occurrens circulo in H,F:

Dico quadratum AB, aequari rectangulo FGH, una cum quadrato GA

Demonstratio



Est enim quadratum AB aequale quadratis BE, EA, hoc est rectangulo BGI, una cum quadratis GE, EA, ut ex elementis patet: est autem quadratum GA, aequale duobus GE, EA, ergo AB quadratum aequalitur BGI hoc est GFH rectangulo una cum quadrato GA. Quod fuit demonstratum.

PROPOSITIO XXIX.

Iisdem positis si iungantur HE, FE;
Dico lineas tres HE, BE, FE, in continua esse analogia :

Demonstratio.

Si enim HE intelligitur produci in K, erit EK aequalis ipsi EF, ut ex elementis educitur, unde & HEF rectangulum, aequale est rectangulo BEI, id est quadrato BE. Quod erat demonstrandum.

PROPOSITIO XXX.

Iisdem positis:
Dico AF ad AH, eandem habere rationem quam FG ad GH, sive AF in G & H divisam, extrema & media ratione proportionali.

Demonstratio.

Rectangulum FGH, una cum quadrato GA, aequatur BA [28. *huius*] quadrato, sed BA, quadrato aequale est [35. *tertii*] rectangulum FAH, igitur FGH rectangulum una cum GA quadrato aequatur FAH rectangulo: quadratum autem AG, aequale est [2. *secundi*] rectangulis AGH, GAH; rectangula igitur FGH, AGH, GAH, aequantur FAH rectangulo *:* rectangulo autem FAH, aequalia quoque sunt [1. *secundi*] rectangula FGHA, GAH ; ablato igitur communi rectangulo GAH, manet FGHA rectangulum, rectangulis AGH, FGH, id est rectangulo FAGH aequale; unde ut FA, ad HA, sic FG ad GH. Quod fuit demonstrandum.

Scholion.

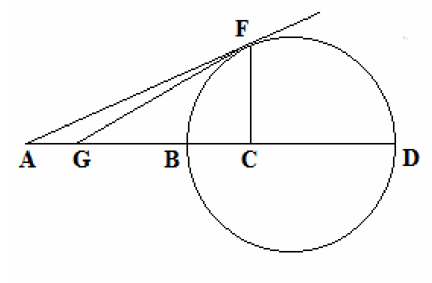
Haec propositio proponitur a Pappo libro septimo propositione 154*; sed quia soli casui videtur applicata quo recta AD, per centrum ducta est circuli BDC; adeoque & videatur casum rectae AF excludere, placuit ostendere universalem prorsus esse: Quod ipsam praesentat Apollonius lib. 3. propositione 37, sed diverso a praesentis discursu.*

PROPOSITIO XXXI.

Ex A ducta sit per centrum circuli BDF recta AD, sitque AD ad AB radio eadem, cum ratione DC, ad CB: erecta deinde normali CF, iungatur AF.
Dico AF contingere circulum.

Demonstratio.

*S*i enim non contingat, ponatur per F contingens FG occurrens AD in G ; erit ergo per praecedentem DG ad GB, ut DC ad CB; sed ex hypothesi ut DC ad CB, ita est DA ad AB, igitur ut DG ad GB sic DA ad AB, & dividendo ut DB ad BG, sic DB ad DA: quod fieri non potest ; cum puncta A & G supponantur diuersa *:* quare FG non est tangens: sed AF. Quod erat demonstrandum.

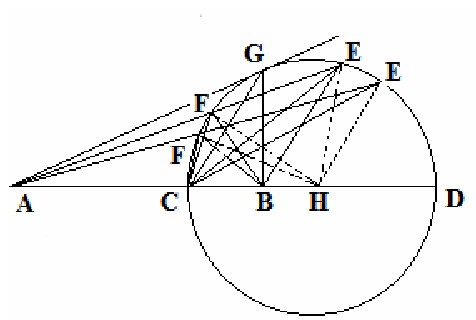


PROPOSlTIO XXXII.

Sit AB utcunque divisa in C, & ex A ducta quaevis AE, exhibens angulum EAB recto minorem : Oporteat in linea AE, puncta assignare E & F, a quibus ad C & B rectae eductae, angulos AFB, AEB bifariam secent.

Constructio & demonstratio.

Producta AB in D, fiat ut AC ad CB , ita AD ad DB: & super CD diametro descriptus sic circulus CFD: cuius centrum H: secabit autem ille vel continget rectam AD vel neutrum praestabit: Secet igitur primo AE lineam in F & E punctis ; Dico illa esse, quae desiderantur : erecta enim BG, perpendiculariter ad diameterum CD, iungantur AC, FC, FB, FH : CE, BE, HE, quoniam igitur ut AC ad CB, ita est AD ad DB, & normalis sit BG, diametro CD: erit AG recta contingens circulum [31. *huius*] unde & AH, CH, [27. *huius*] BH, continuae sunt proportionales; est autem FH vel EH, mediae CH, aequalis; ergo [11. *de progressionibus*] anguli AFB, AEB, per rectas CF,CE divisi sunt bifariam



Quod si circulum CGD contingat AG, in G: iungantur puncta GB, GC, GH; quoniam AH, CH, BH, sunt in continuata ratione; & GH linea mediae CH, aequalis est, [*ibid.*] erit AGB angulus bifarium divisus per rectum CG.

Manifest autem est si AEF recta circulo non occurrit, cessare materiam propositam.

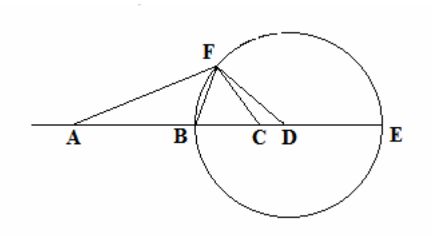
Corollarium.

In idem recidit, eodemque; modo soluitur problema, quo iisdem positis petuntur in AE linea cxhiberi puncta F & E, ad quae ductis ex C & B lineis; fiant AC, CB proportionales rectae AF, FB, AE, EB : inventa enim per praecedentem puncta F & E a quibus ad C & B ductae lineae, angulos AFB, AEB bifariam secant, problema soluunt: nam angulis AFB, AEB, divilis bifariam per rectas FC, EC, erunt AF ad FB , & AE ad EB ut AC ad CB.

PROPOSITIO XXXIII.

Sit BFE circuli diameter BE producta utcunque in A: ex qua secans ponatur AF : & ex F, recta CF, ut AF sit ad FC, sicut AB ad BC. Dico AB ad BC eandem rationem obtinere quam AE ad EC.

Demonstratio.

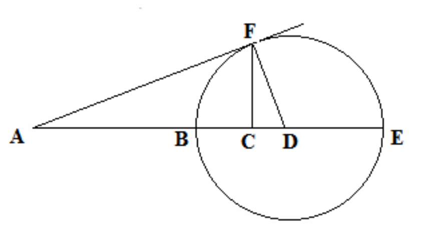


Iungantur DF, BF: cum igitur sit AF ad FC ut AB ad BC, erunt anguli AFB, BFC aequales; Rursum cum DF, aequalis sit DB, [3. *sexti*] erunt AD, BD, CD in continua analogia; unde AB est ad BC, ut [11&1. *de progressionibus*] AD ad BD & cum DE rectum fit aequalis DB, erit ut [5. *de lineis*] AB ad B C ita AE ad EC. Quod fuit demonstrandum.

PROPOSITIO XXXIV.

Sit AB ad BC, ut AE ad EC fitque BE divisa bifariam in D. Dico AD, BD, CD fore in continua analogia.

Demonstratio.



Describatur circulus centro D, intervallo BD, & erigitur CF normaliter ad AE; ocurrens circulo in F. Ducanturque DF, AF: cum igitur Sit AB ad BC, ut AE ad EC, erit linea [31. *huius*] AF contingens circulum, unde AD, BD, CD [27. *huius*] lineae in continua sunt analogia. Quod erat demonstradum.

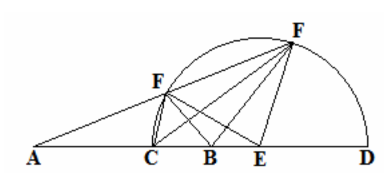
Invenies hanc in libro lineis propositione quarta vel quinta aliter demonstratam.

PROPOSITIO XXXV.

Sit AB recta, per centrum circuli CFD ducta; fiat autem ut AC ad AD, ita CB ad BD, & ducta AF secante circulum in F: iungantur BF, CF:
Dico AC, CB, & AF, FB, proportionales esse lineas.

Demonstratio.

Ex E centro ponatur EF. Quoniam ponitur ut AD ad AC, sic DB ad CB, erunt AE, CE, BE lineae [34.*huius*] in continua ratione. Igitur cum FE sit



aequalis mediae CE, est AC [11.*lib. de progressionibus*] ad BC, ut AF ad FB.

Est hac conversa trigesimae tertiae huius.

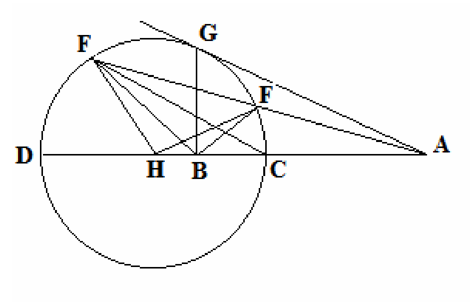
PROPOSITIO XXXVI.

Sit AD linea utcunque divisa in C: descriptoque super CD circulo, ponatur ex A linea AE occurrens circulo in F: ducatur autem FB, ut AF, FB proportionales sint, lineis AC, CB, & ex B erecta normalis occurrat circulo in G.

Dico AG lineam circulum contingere.

Demonstratio.

Cum sit AF ad FB, ut AC ad CB, erit angulus AFB divisus [3. *Sexti*] bifariam: quia vero linea FH, rectae CH aequalis est, erunt AH, CH , BH in continue ratione [11. *de progressionibus*] : unde & GA circulum contingit [27. *huius*]. Quod fuit demonstrandum.



PROPOSITIO XXXVII.

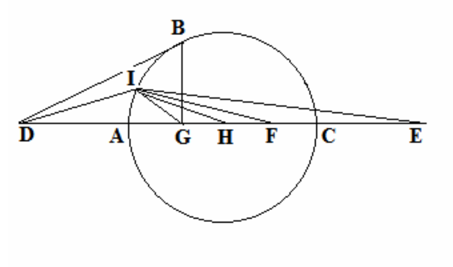
Circuli ABC diameter AC, utrimque producta sit in D & E puncta, aequaliter a centro H distantia, posita deinde DB contingente in B, demissaque normaliter BG, ad CA diametrum, fiat CF, aequalis AG: ductisque rectis DI, IE: iungantur IG, IF.

Dico DI, IE, ipsis IG, IF proportionales esse.

Demonstratio.

Iungatur HI, quoniam DB contingit circulum & BG normalis ponitur ad diametrum, erunt DH, AH, GH in continua analogia [*Ibid*] uti & EH, CH, FH; est autem HI, ipsi AH aequalis ; igitur ut EC ad CF ita EI ad IF [11. *de progressionibus*]; sed ut EC ad CF, ita

AD ad AC, hoc est :DI ad IG, quare ut EI ad IF, ita DI ad IG. & permutando invertendo DI ad IE ut IG ad lF.

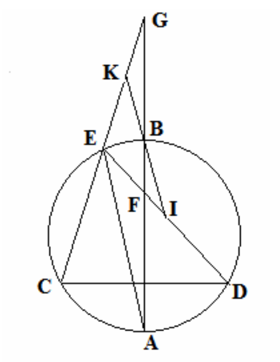


PROPOSITIO XXXVIII.

Circuli ABC diamertum AB, secet in F recta quaevis DE, sumptoque AC arcu aequali AD, ponatur ex C per E linea CG conueniens cum diametro in G.

Dico AG ad GB eandem habere rationem quam AF ad FB.

Demonstratio.



Iungantur AE, EB, & per B ponatur IK aequidistans ipsi AE: quoniam AC, AD arcus sunt aequales, erunt & anguli AEC, AED quoque aequales: quia vero angulus AEB in semicirculo rectus est, adeoque duobus AEC, GEB angulis aequalis, demptis AEC, AED aequalibus, aequales remanent anguli DEB, GEB sunt autem & anguli EBG , EBI recti, cum IK aequidistet AE, aequalia igitur sunt latera IB, KB: quare ut AE ad IB est B, id est AF ad FB, sic AE est ad KB, id est AG ad GB.

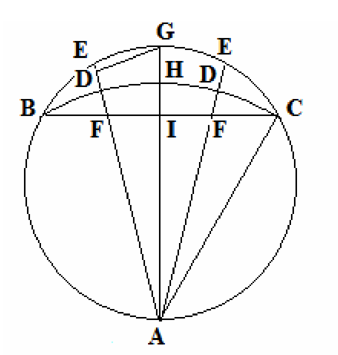
Quod erat demonstrandum.

PROPOSITIO XXXIX.

Circulum ABC intersecet alter, centrum habens in perimetro ABC: iunctisque intersectionum punctis BC, ponantur ex A centro circuli intersecantis, quaevis AE, occurrentes perimetris in D & E, rectae vero BC in F.

Dico lineas AF, AD, AE in continua esse analogia.

Demonstratio.



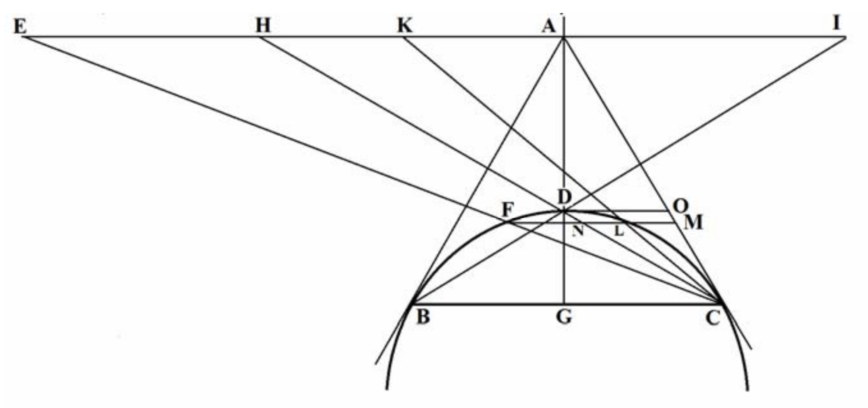
Ponatur ex A per centrum circuli ABE diameter AG occurrens perimetris in G & H, rectae vero BC in I, iunganturque GE, AC: cum igitur AG transeat per centra circulorum sese intersecantium, erit BC in I [10.*huius*] normaliter divisae, unde angulus AIF aequalis angulo AEG in semicirculo posito : est autem GAE angulus communis utrique triangulorum AIF, AGE , similia sunt igitur triangula AIF, AGE; quare ut AI, AF, sic AE ad GA: adeoque FAE rectangulum aequale, rectangulo IAG, id est quadrato AC, id est AD quadrato: proportionales igitur sunt AF, AD, AE. Quod erat demonstrandum.

PROPOSITIO XL

Sint AB, AC contingentes circulum BDC, ductaeque BÇ , ponatur aequidistans AE, & ex A diameter AG, occurrens perimetro in D per quod ex B, agatur BH; dein per quodlibet punctum in perimetro assumptum F ducantur CFI, BFE.

Dico rectangulum IAE quadrato AH aequale esse.

Demonstratio.



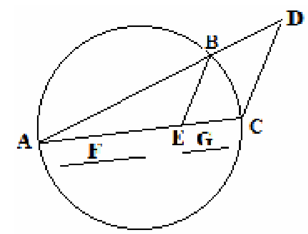
Sumatur arcum CF, aequalis BL, & per L ducta BL occurrat AE in K : LF quoque iuncta secet AB, contingentem in M & HB in N: denique ponatur DO contingens. Quoniam CF, BL arcus ponuntur aequales, erit LF ipsi BC adeoque & rectae AH aequidistans: quia vero OD, OB contingentes ex eodem educantur puncto aequales sunt OB, OD, adeoque & MB, MN: ac proinde quia ML, MB, MF sunt proportionales, erunt & ML, MN. MF quoque continuae: sed ut ML ad MN, sic AK ad AH, & ut MN ad MF sic AH est ad AE, proportionales igitur sunt AK, AH, AE: est autem ipsi AK aequalis AI (quia BC illi aequidistans, ab AG diametro divisa sit bifariam) igitur AI, AH, AE in continua sunt analogia. Quod erat demonstrandum.

PROPOSITIO XLI.

Dato segmento circuli ABC & recta CD, utcunque ad AC posita; oporteat rectam ducere AD, quam dividat arcus ABC in B, secundum datam rationem F ad G.

Constructio & demonstratio.

Dividatur AC basis segmenti in E, secundum datum rationem F ad G: erecta deinde EB, quae aequistet CD, occurrente perimetro in B, ponatur ABD: Dico factum quod petitur ex elementis.

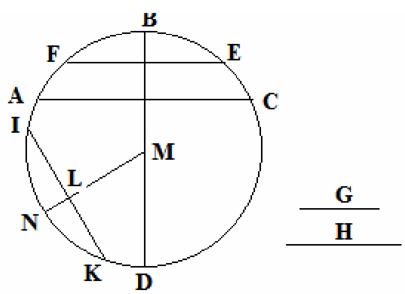


Hic notatu dignum est, quod punctum C, assumi possit non solum in termino rectae AC, sed vel intra, vel extra circulum, in quavis parte AC productae.

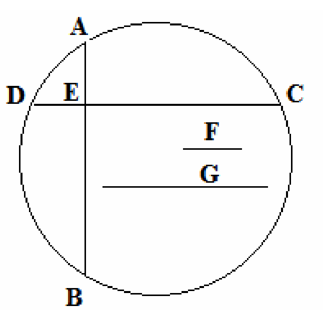
PROPOSITIO XLII.

Dato circulo ABC, linea AC, quae non sit maior diametro BD, & ratione G ad H, oporteat alteram FE, circulo inscribere, quae aequidistet AC, ut quam G, ad H, habet rationem, habeat quoque FE ad AC.

Constructio & demonstratio.



Ponatur BD diameter normaliter ad AC, fiatque ut H ad G, ita AC ad IK: quae in circule ABC applicata, dividatur bifariam in puncto L; iunctaque ML, fiat rectae ML aequalis MP, ponaturque ; FPE, quae aequidistet AC; patet factum quod quaeritur; nam rectae FE, IK cum sint aeque a centro remotae, inter se aequales sunt.

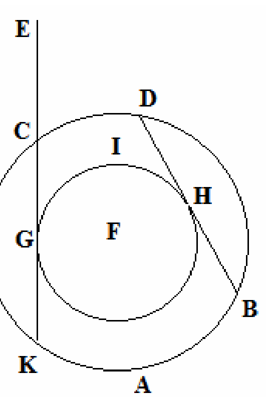


PROPOSITIO XLIII.

Rectam AB quae non sit diameter, altera CD intersecet ad angulos rectos in E, ut DE ad EC datam habeat rationem F ad G.

Constructio & demonstratio huius invenies in libro nostro de ellipsi: quas hic non ponendas iudico eo quod ab ellipsi plane sit dependens.

PROPOSITIO XLIV.



A Puncto extra circulum dato, lineam circula immittere quae datae fit aequalis; modo ea diametro circuli non sic maior.

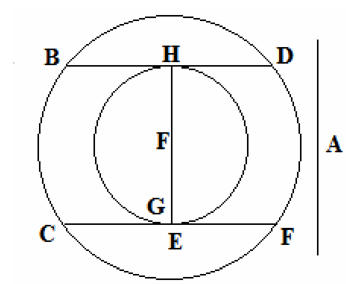
Constructio & demonstratio.

Applicetur datae rectae A in circulo BCD aequalis BD ; dein ex F centro describatur alter circellus, contingens BD, in H : Deinde tum ex E puncto dato ponatur EG, contingens eundem circellum in G; occurrens vero circulo ABDC in K & C. Dico CK, fore quaesitam; cum enim aequaliter DB, GK, distent a centro F (quia contingunt eundem circulum GIH;) aequales sunt inter se: ac proindc rectae A, aequalem posuimus GK: quod fuit praestandum.

PROPOSITIO XLV.

Idem praestare puncto intra aream circuli posito.oportet autem data A hoc casu non minorem esse recta illa, quae per E posita ad diametrum, per E quoque transeunteem normalis sit.

Demonstratio.

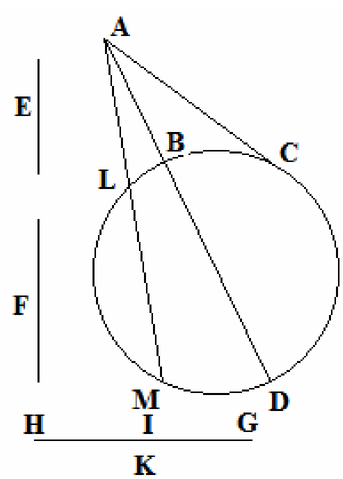


Data A ut prius ; accommodetur in circulo, aequalis BD : centroque F circulus describatur contingens BD in H, transibit ille per punctum datum E, vel infra illud cadet : cum enim ex hypothesi normalis per E ducta, ad diametrum, per E quoque ductam, maior esse non possit data linea, id est DB: patet & EF diametrum, quae rectam per E ductam secat orthogonaliter, maiorem non esse diametro, quae lineam DB datae A aequalem orthogonaliter divideret; unde circulus radio FH descriptus,vel transit per E, vel infra illud cadet; utroque casu agatur per E contingens circulum radio FH descriptum, patet illam aequari rectae DB, id est datae A: quare fecimus quod postulabatur,

PROPOSITIO XLVI.

A dato extra circulum puncto lineam educere quae in data ratione a perimetro dividatur: oportet autem rationem datam, maiorem non esse illa quae repetitur inter partes lineae quae a dato puncto per centrum ducitur.

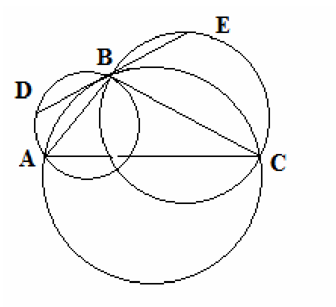
Constructio & demonstratio.



Sit datum punctum A & circulus BCD: data quoque ratio sit E ad F, quae minor sit rationis AB ad BD, partium diametri: fiat rectis E, F aequalis GH, & GI quidem ipsi E: intentaque K media inter I G, GH, ponatur ex A contingens AC, & ut K ad GI sic AC fiat ad AL : patet ex datis AL minorem esse AC, & non minorem AB, adeoque punctum L in perimetro esse circuli : ducatur igitur per L ex A linea AM occurrens circulo in L & M , dico AM satisfacere petitioni; cum enim LAM rectangulum aequale ut quadrato AC & IGH rectangulum quadrato K, ponatur autem & LA ad AC ut IG ad K, erunt GI, K, GH lineae proportionales eiusdem rationis cum AL, AC, AM ; unde ut prima GI ad excessum HI, id est per constructione ut E ad F, sic AL quarta ad excessum LM. eduximus igitur a dato puncto, &c. Quod erat faciendum.

PROPOSITIO XLVII.

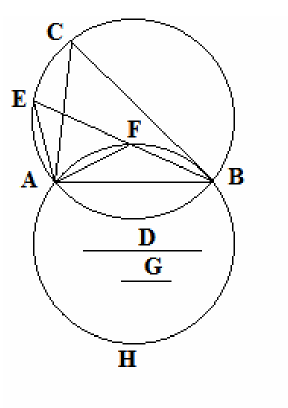
Intersecent sese invicem duo circuli ABD, BEC in B. Oporteat per B, rectam DBE ponere, quae BD rectae, aequalem BE constituat.



Constructio & demonstratio.

Ponantur AB, CB contingentes in B, circulos ABD, BEC; occurrentes perimetris in A & C : dein per A, B, C, circulus describatur ABC, quem in B contingat DE: dico factum quod postulatur ostensum est enim segmenta, ADB, BEC, ABC esse inter se similia [11. *huius*] similia ad eoque DE tangentem in B divisam [14. *huius*] bifariam: posuimus igitur per B rectam, &c. Quod erat faciendum.

PROPOSITIO XLVIII.



Secent invicem ut prius circuli duo ABC, AHB in A & B. Oporteat ex B rectam ducere BEF, ut EF perimetris intercepta sit datae D, aequalis.

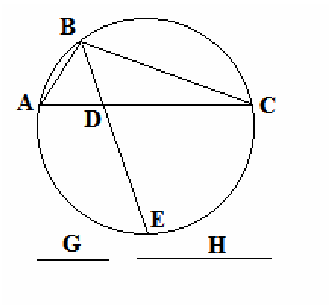
Constructio & demonstratio.

Constituatur BC contingens circulum AHB in B; oportet datam lineam D, hac contingente maiorem non esse ; iunctis AC fiat ut CB, ad AB, ita D, ad G. Dein ipsi G, fiat aequalis AF, & iuncta BF pertingat in E: dico factum esse quad petitur. Ducatur enim AE., quoniam angulus AFB tam cum angulo segmenti AHB, id est ABC angulo (ob CB tangentem) quam cum AFE, duobus rectis est aequalis; dempto communi AFB, remanebunt aequales anguli AFE, ABC; sunt autem & anguli ACB, AEB eidem insistentes arcum aequales: similia igitur sunt triangula AEF, ACB; unde AF est ad FE ut AB ad BC, id est per constructionem ut G ad D*:* & permutando ut AF ad G, sic EF ad D : quare cum AF, & G aequales ponantur, erunt & EF, D lineae aequales. Perfecimus igitur quod postulabatur.

PROPOSITIO XLIX.

In dato segmento circuli ABC,ex A, & C duas lineas inclinare, sese in perimetro decussantes: quae datam inter se rationem contineant G ad H.

Constructio & demonstratio.



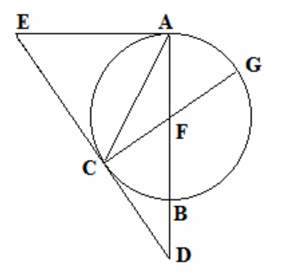
Divisa in D recta AC, segmentum subtendente, secundum rationem G, ad H; bisecetur arcus AEC in B, & per D ex E, recta ponatur EDB, iungantur AB, BC: Dico AB, BC esse quaesitas. Cum enim anguli ABE, EBC aequalibus arcubus insistentes aequales sint, erit AB ad BC ut AD ad DC [3. *sexti*], id est per constructionem ut G ad H.

CIRCULORUM PARS SECUNDO

De Angulorum & arcuum circularium comparatione.

PROPOSITIO L.

Circulum ABC, cuius diameter AB, contingat in A recta AE, in qua assumpto quovis puncto E, ponatur EC contingens quidem circulum in C, occurrens autem diametro AB protractae in D, iunganturque CA.
Dico angulum CEA duplum esse anguli CAD.



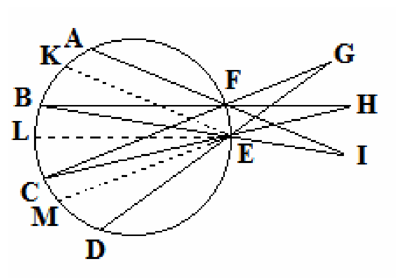
Demonstratio.

Per centrum F diameter ponatur CG : Quoniam ED circulum contingit in C, & CG diameter est, erit [18.*tertii*] angulus GCD rectus, adeoque angulo EAD aequalis; est autem angulus EDA communis triangulis CFD, EDA; igitur CFD angulo , angulus AED aequalis est; sed angulus [20.*tertii*] CFD duplus est anguli CAD, igitur & angulus AED, eiusdem CAD duplus est. Quod erat demonstrandum.

PROPOSITIO LI.

Assumptis in perimetro ABC, tribus arcubus aequalibus, AB, BC, CD, ducantur per duo quaedam puncta F,E, rectae AF, BF, CF, & BE, CE, DE, occurrantque in G, H, I.
Dico angulos I, H, G, inter se aequales esse.

Demonstratio.



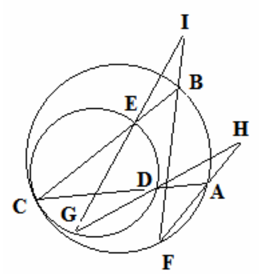
Ponantur ex puncto E , rectae EK, EL, EM quae aequidistent lineis AF, BF, CF; erunt itaque anguli KEB, LEC, MED, aequales angulis I, H, G; quia vero arcus AB, BC, CD ponuntur aequales, & AK, BL, CM arcus aequantur arcum EF, adeoque & inter se, reliqui

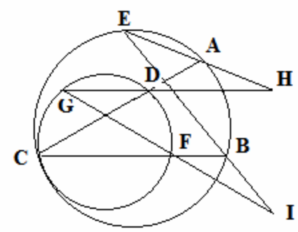
quoque arcus KB, LC, MD, & anguli KEB, LEC, MED illis insistentes aequales sunt: quare & anguli I, H, G, sunt inter se aequales.
Quod erat demonstrandum.

PROPOSITIO LII.

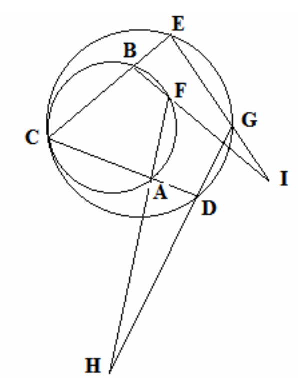
Contingant inuicem interius circuli duo ABC, DEC, in puncto C: ex quo eductis CA, CB, sumantur puncta G, F, in singulorum arcubus ex quibus rectae ponantur GE, GD, FB, FA.
Dico si lineae illae conveniant, angulos GIF, FHG aequales esse.

Demonstratio.



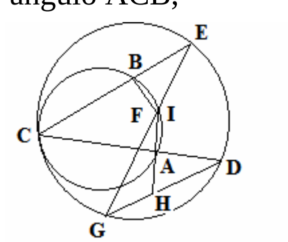


Sint primo G, & F, puncta vel supra vel infra & angulum ACB posita: Quoniam



anguli CBF, CAF, CF arcum insistentes aequales [21.*tertii*] sunt, uti ob eandem causam, anguli CEG, CDG. id est IEB, HDA ad verticem positi, erunt anguli IEB, EBI simul sumpti, aequales angulis HDA, DAH; quare & tertius EIB, tertio DHA aequalis erit.

Quod G verumque punctorum F, & G , angulo ACB, contineantur, hac ratione assertione demonstrabimus: cuius anguli [22.*tertii*] ACB, AFB, duobus rectis aequales sint, uti & anguli ECD, EGD: dempto igitur communi angulo ECD, manet AFB, angulo ECD , ac proinde reliquis AFI, reliquo DGI aequalis sunt autem ad verticem anguli FKH , GKI aequales, igitur reliqui FIG, FHG aequales quoque sunt.

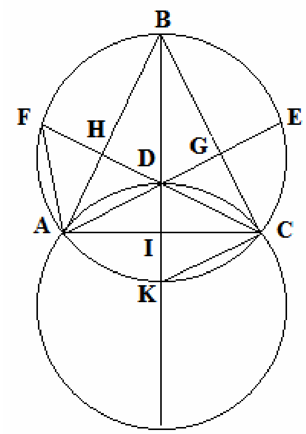


Cadat iam alterutrum punctorum, puta F, intra , & aliud extra angulum ACB: dico rursum angulos H, I esse inter se aequales, cum enim angulus FBC tam cum angulo FAC, quam cum EBF duobus rectis sit [*ibid*] aequalis, dempto communi FBC, erit angulus EBI aequalis angulo FAC id est DAH, sunt autem & anguli CEG, CDG eidem insistentes arcum aequales, igitur reliquus I, aequatur reliquo angulo H. Quod erat demonstrandum.

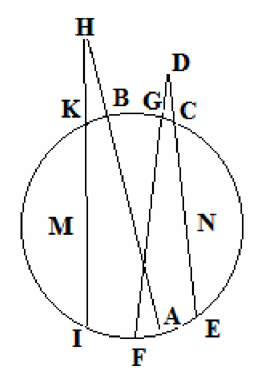
PROPOSITIO LIII.

Ductae sint normales a terminis trianguli ABC, ad opposita latera.
Dico illas sese in eodem puncto decussare.

Demonstratio.



Circumscribatur ABC triangula circulus, & per AC puncta alter circulus describatur ADC, aequalis circulo ABC. Productaque BI, occurrat perimetris in D, & K, & per D quidem ex A & C, agantur CDF, ADE, iunganturque; AF, CE, CK erunt itaque anguli AFC, AEC arcum AC insistentes aequales; quia vero angulus EAC utrique circulorum aequalium communis est, arcus quoque DC, CE aequales sunt, uti illorum subtensae; ob eandem rationem quoque lineae AF, AD sunt aequales. Rursum cum angulo DKC, id est KDC (ob circulos aequales) duo anguli interni DCB, DBC, aequales sint, erunt duo arcus FB, KC id est DC aequales duobus arcubus BE, EC; ablatis itaque aequalibus arcubus KC, CE, remanent quoque aequales arcus FB, BE; quare anguli FCB, BCE quoque sunt aequales; sicut & anguli FAB, BAD, unde cum AF, AD lineae, adeoque & anguli AFD, ADF aequales sint, erunt & reliqui FHA, DHA quoque inter se aequales, adeoque & recti eodem modo ostenditur angulos ad G positas, esse rectos; quare patet normales BI, CH, AG, sese in eodem puncto decussare. Quod fuit Demonstrandum.



PROPOSITIO LIV.

In datum circulum ABC, a dato extra eum puncto D, ductae sint DCE, DGF auferentes arcus FE, GC. Oportet ex alio puncto extra circulum assignato, H, duas rectas in circulum immittere, quae arcus duos intercipiant, duobus FE, GC arcubus aequales.

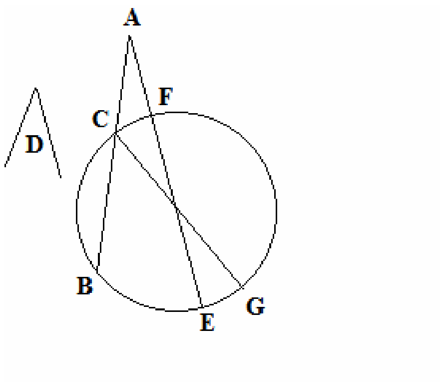
Constructio & demonstratio.

Ducantur ex H lineae HI, HA , ut [44.*huius*] KI, BA, circulo interceptae, aequales sint rectis CE, G F, factum erit quod fuit imperatum, cum enim AB, FG sint aequales lineae, erunt arcus GEF, AIB aequales , uti & arcus CNE, KMI, ob IK, CE aequales lineas; ablatis igitur aequalibus arcubus KMI, CNE, remanebunt arcus KB, IA, arcubus GC, FE aequales. demisimus igitur ex H puncto lineas, &c. Quod praestandum fuit.

PROPOSITIO LV.

A dato extra circulum puncto, demissa per eundem recta AB; oporteat alteram ducere AE, quae arcus auferat CF, BE, qui simul sumpti, angulum contineant aequalem dato D.

Constructio & demonstratio.



Fiat angulo D, aequalis BCG : & ex A ponatur AE, quae rectam FE, aequalem lineae CG [*ibid*] exhibeat; eritque peractum quod requiritur : cum enim CG, FE lineae aequentur, erit arcus FBE, aequalis arcum CBG; unde ablato communi CB, remanet arcus BG, aequalis duobus BE, CF; quare angulo D, aequalis est angulus, qui duobus arcubus CF, BE simul sumptis continetur perfecimus igitur quod petebatur.

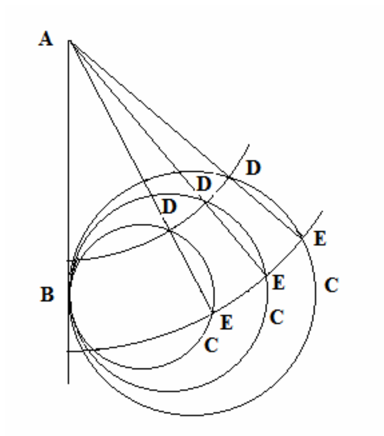
CIRCULORUM

PARS TERTIA

De mutua circulorum intersectione & contactu.

PROPOSITIO LVI.

Recta AB, contingat circulos BDC, sese in eodem B puncto, contingentes, centroque A, quovis intervallo circulus describatur, occurres perimetris circuloru sese contingentium in D, ponanturque rectae ADE.
Dico DE rectas inter se aequales esse, sive ad eundem esse circulum.

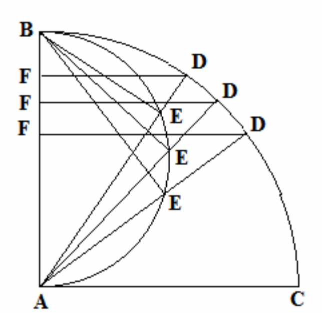


Demonstratio.

Cum enim AB sit contingens rectangula DAE, aequalia erunt quadrato AB, adeoque & inter se; igitur Iineae DA, DB, DE, in continua sunt analogia; sunt autem DA primae inter se aequales & AB media communis, igitur etiam primarum & tertiarum differentiae DE, aequales erunt. Quod fuit demonstrandum.

PROPOSITIO LVII.

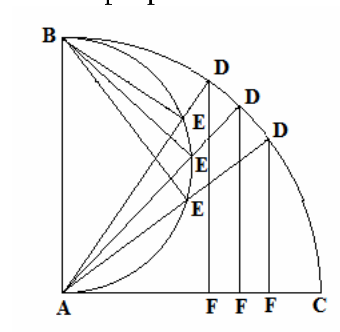
Super AB radio quadrantis circularis ABC, circulus descriptus sit AEB; ductisque rectis AED, occurrentibus circulo in EE, per EE ponantur DF normales ad radium AB.
Dico rectis AF, aequales esse AE.



Demonstratio.

Iungantur BE : cum AD aequales sint AB , & anguli AEB, AFD recti; sint autem & anguli FAE communes ; erunt BEA triangula aequalia & similia triangulis AFD: unde reliqua AF, AE latera aequalibus angulis subtensa sunt aequalia.

Quod si rectae DF demittantur normaliter ad basim AC, sic ostendetur propositum: cum DF sint aequidistantes AB, igitur anguli BAD, ADF, aequales sunt: proindeque; cuius & anguli AEB AFD, recti sint, similia sunt & aequalia triangula ABE, ADF, & latera aequalibus angulis subtensa. Quod fuit demonstrandum.

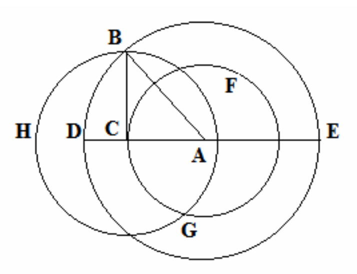


PROPOSITIO LVIII.

Per duorum circulorum parallelorum BDE, CFG centrum A acta diametro DCE, ponatur CB contingens minorem a puncto quo a diametro intersecatur.

Dico circulum radio BC descriptum aequalem esse annulo duabus circumferentiis intercepto.

Demonstratio.



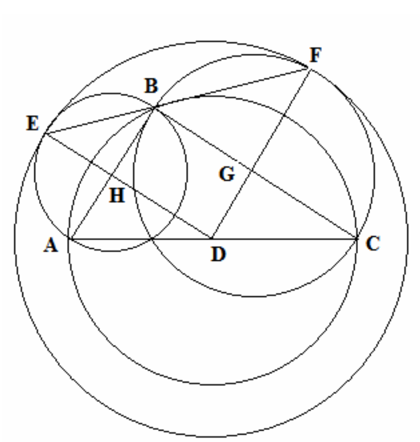
Ducatur AB; ut AB quadratum, ad duo quadrata BC, CA, ita circulus DBE [2. *duodecimi*.], ad circulos HBG, CFG; sed AB quadratum aequale est quadratis AC, CB. igitur & circulus DBE aequalis HBG, CFG, simul sumptis ablato igitur communi circulo CFG, remanebit annulus perimetris aequidistantium circulorum interceptus, circulo BHG aequalis: quod fuit demonstrandum.

PROPOSITIO LIX.

Super trianguli ABC, lateribus segmenta circulorum similia constituantur , centroque circuli,qui super basi exstructus est, circulus describatur qui alterutrum circulorum AEB, BFC contingat.

Dico quod & tertium quoque continget.

Demonstratio.



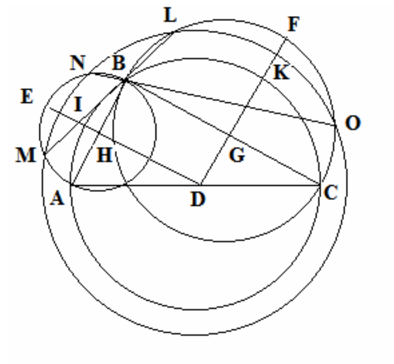
Ducantur ex D & G centra circulorum AEB, BFC rectae DE, DF: & circulus ED radio descriptus contingat circulum AEB in E; igitur recta DHE per centra D, & H transiens occurret utrique in [11.*tertii*] puncto contactus E, quia vero segmenta super trianguli lateribus descripta, similia sunt iuncta [11.*huius*] BF in directum erit ipsi EB ; unde cum DE ; [13.*huius*] DF lineae sint aequales, erit F punctum commune peripheriis circulorum EF, BFC & cum eadem DF centra DG utriusque centra coniungat, patet circulum EF, [*Schol.ad.* 13.*tertii*] in F contingere circulum BFC. Quod erat demonstrandum.

PROPOSITIO LX.

Iisdem positis: si circulus centro D descriptus alterutrum circulorum AEB, BFC secuerit:

Dico quod & alterum secabit; & secures ab utroque auferet similes.

Demonstratio.



Occurrat circulus centro D descriptus, circulo AEB in M & N: cadet igitur infra E & ED lineam secabit in I; quia vero rectae DE, DF [15.*huius*] aequales sunt, patet circulum radio DI descriptum , quoque infra F cadere, adeoque & circulum BFC secare in L & O. Quod erat primum.

Hoc posito dico secures MEN, LFO ablatas inter se esse similes, ducta MB occurrat circulo LFO in L, ostensum est propositione [15. *huius*], iunctam DL, aequari rectae DM, quare cum punctum M in perimetro est circuli NLO, erit & punctum L eadem perimetro, eadem ratione ostendetur, iunctam OB communi occurrere intersectioni circulo MEB, MLO in N, unde cum NBM angulo angulus LBO ad verticem insistens, aequalis sit; similes erunt arcus MEN, LFO. Quod erat demonstrandum.

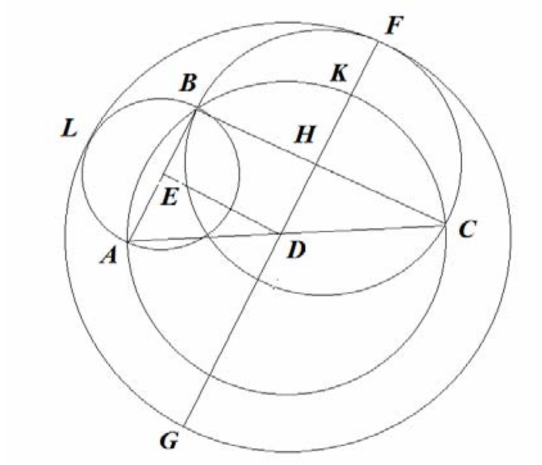
PROPOSlTlO LXI.

Super ABC trianguli lateribus semicirculi describantur ALB, AKC, BFC quos contingat circulus centro D, descriptus, in F & L.

Dico FG diametrum circuli utrumque contingentis, aequalem esse diametris circulorum ALB, BFC.

Demonstratio.

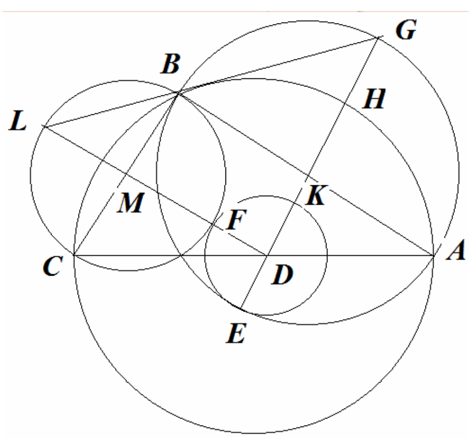
Per centra D & H ponatur recta FG, iunganturque centra D, E; quoniam segmenta similia super trianguli lateribus descripta, semicirculi sunt, & AB, BC a diametris DH, DE in circulo ABC bifariam divisae, erunt anguli DHB, DEB recti, unde AB, DH aequidistant & HE parallelogramum, adeoque BE, HD, item BH, ED linae aequales sunt, quare cum FG dupla sit, id est dupla duarum FH, HD, id est BH, BE, erit FG aequalis duabus diametris AB, BC. Quod erat demonstrandum.



PROPOSITIO LXII.

Iterum similia segmenta constructa sint super singulis lateribus trianguli ABC centroque D circuli super basi descripti, circulus describatur KFE,qui circulum AGB contingat interius in E.

Dico quod & alterum BLC continget in F.



Demonstratio.

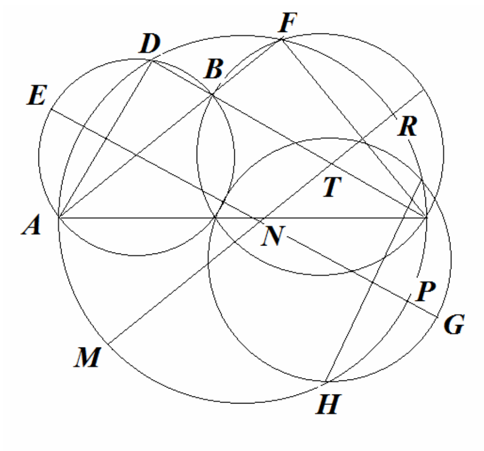
Per centra circulorum AGB, BLC; rectae ponantur ex D, DG, DL: & DG quidem producta occurret perimetro circuli AGB in puncto contactus E ; DL vero in aliquo puncto F circuli BLC, iunganturque BF, FE, cum igitur DL, DG transeant per centra circulorum AGB, BLC, erunt LB, GB ductae in directum [13 *huius*]: quia vero LMF diameter est circuli BLC, erit angulus LBF adeoque & GBF rectus, ac proindc BF pertinget in E. Rursum cum LD, DG lineae sint aequales, erunt & anguli quoque BLD, BGD aequales; sunt autem anguli ad B ostensi recti , reliquus igitur BED aequalis reliquo est LFB, id est EFD : unde & aequales lineae sunt DE, DF & punctum F perimetro circuli KFE commune: & cum MD, linea per F ducta, centra coniungat D & M , paret circulum KFE [13.*tertii*] in F contingere circulum BLC. Quod erat demonstrandum.

PROPOSITIO LXIII.

Super trianguli lateribus constructa sint segmenta similia AEB, AD, FC, BLC. & ab intersectionum punctis D, F, ad puncta A, C ductis lineis DA, F C, aptetur in circulo KH aequalis FC, & parallela rectae DA : describatur deinde circulus IKGH aequalis circulo

BLCO, & transiens per puncta K, H.
Dico hunc tangcre circulum ADI.

Demonstratio.



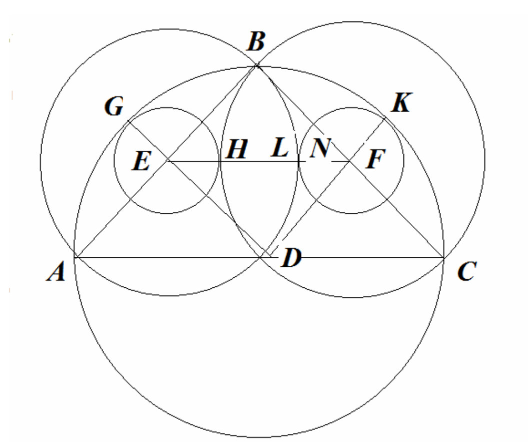
Trium circulorum AEB, ADFC, BLC centra sint S, N, T. per centra S, N ducatur recta QSINPG, occurrens circulo AEB in I, & circulo KGH in G : per centra vero T,N ponatur recta LRTNO occurrens circulo BLC in O. Quoniam QG iungit centra N, S, normalis [10.*huius*] est ad AD, ac proinde & ad HK ipsi AD parallelam. centrum igitur circuli KGH est quoque in linea QG. consideretur iam punctum I quatenus est intersectio circuli AEB ac rectae QG. Quoniam NR, est aequalis NP, & RL [*patet ex elementis*] aequalis PG, erit NL aequalis NG. Est vero per demonstrata 63 huius, etiam ON aequalis NP, ergo NL aequalis est IG, quae nempe posita est inter punctum G, & puntum I, in quo circulus AEB secabat rectum QG. Quare cum circuli KGH diameter aequalis fit rectae OL, diametro circuli BLCO, aequalis quoque erit rectae IG. circulus igitur KGH transit per I. Atqui etiam supra ostendimus centrum eius esse in recta QG transeunte per centrum circuli AEB; ergo circulus KGH tangit circulum AEBC. Quod erat demonstrandum,

PROPOSITIO LXIV.

Super ABC trianguli lateribus semicirculi describantur ABL, BIC, AGC, quorum centra sint D,E,F, centris autem F vel E, circuli construantur HG, KN, qui circulum AGC contingant in G & K .
Dico eosdem, quoque reliquos contingere in I, & L.

Demonstratio.

Ducantur FE, DEG, DF, quoniam diametri AB, CB bisectae sunt in centris E, & F, erit EF parallela AC, ergo ut AB ad EB sic AC ad EF quare cum AB dupla sit EB, erit AC dupla



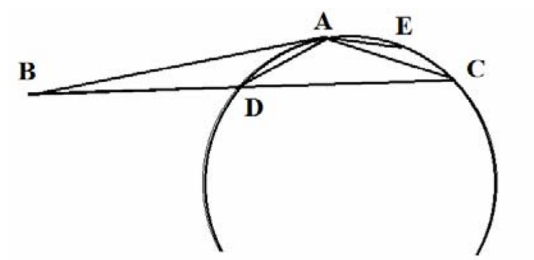
quoque EF. Ergo AD dimidia ipsius AC, hoc est GD, aequalis erit EF. similiter quoniam tres diametri AC, AB, BC bisecta sunt in centris D, E, F patet [22.*Sexti*] ED, BC, & DF, AB esse parallelas; parallelogrammum igitur est EBFD. ergo FB hoc est FI aequalis est DE. Quare cum tota FB toti GD, & pars F1, parti DE aequalis sit, residuum quoque IE aequale erit residuo GE, hoc est circuli diametro GE, sive EH. quare punctum I, commune est duabus peripheriis GH & BIC sive puncta H & I omnino eadem sunt inter se. Unde cum EF per centra circulorum ducta per I punctum transeat, manifestum est in eo contactum fieri, eodem pacto de altero circulo centro F descripto demonstratio procedet.

PROPOSITIO LXV.

In triangulo ABC, sint in continua analogia BD, BA, BC: & per puncta A, D, C describatur circulus.

Dico eum rectam AB contingere.

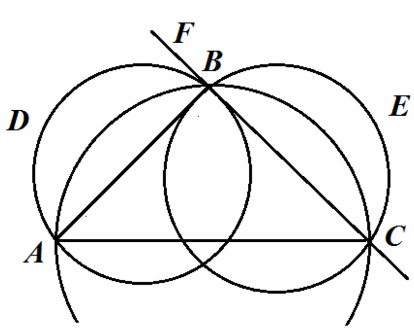
Demonstratio.



Quoniam ADC, puncta in perimero sunt circuli, igitur si non contingat recta AB, circulum, occurrat eidem in altero puncto E: igitur rectangulum ABE, aequale erit rectangulo DBC , hoc est per hypothesim quadrato BA, quod [*ex elementis*] est absurdum: quare non occurret BA, circulo nisi in A.

PROPOSITIO LXVI.

Si super ABC trianguli lateribus segmenta circulorum similia descripta fuerint, Dico AB, CB latera trianguli producta contingere in B segmenta ADB, BEC.

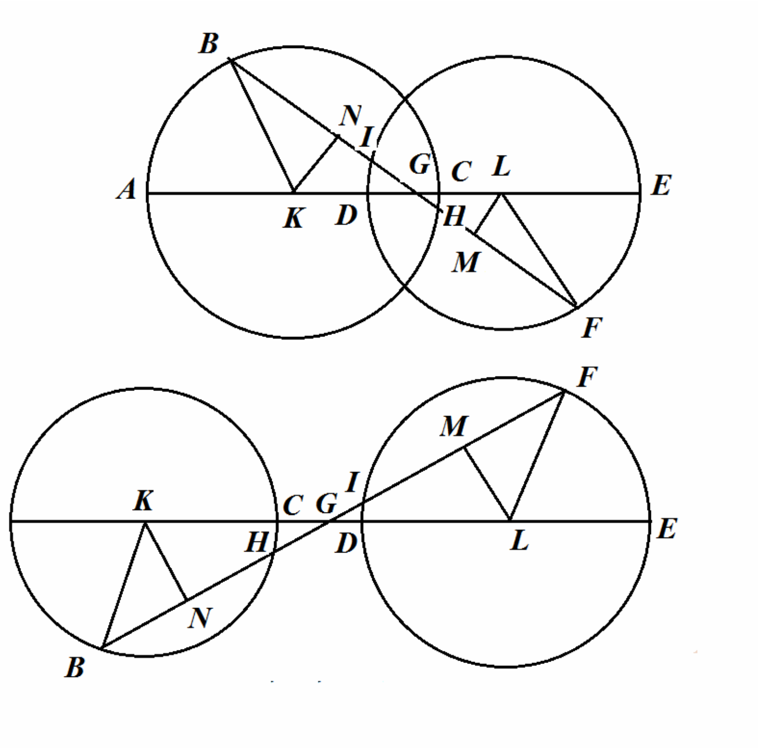


Demonstratio.
Angulus ABC tam cum angulo ABF quam angulo segmenti residui in eodem circulo

duobus rectis aequalis est; igitur angulus residui segmenti, angulo FBA aequalis est: quia vero ADB, ABC segmenta sunt similia adeoque & illorum anguli aequales, erit angulus FBA simul cum angulo segmenti ADB duobus rectis aequalis: quare angulus ABF, aequatur angulo residui arcus circuli ADB : unde FC eundem contingit in B, ut ex elementis patet. Eodem modo ostenditur AB contingerc in B circulum BEC. Quod erat demonstrandum.

Est haec conversa undecimae huius.

PROPOSITIO LXVII.



Datis duobus circulis ABC, DEF, oporteat exhibere punctum G, per quod lineae ductae dividant circulos in similes partes.

Constructio & Demonstratio.

Acta per utriusque centra KL recta AE, dividatur in G, ut ratio AG ad GE, sit eadem cum ut ratione AC ad DE. Dico punctum G esse quod quaeritur. Ducatur enim quaevis BGF occurrens perimetris in I & H: quoniam AG ad GE, eandem habet rationem quam AC ad DE, ex constructione erit AC ad CG, ut DE ad DG ac proinde AG ad GK ut EG ad GL, & GK ad GL ut KA ad LE, id est BK ad LF. Demissis igitur perpendicularibus KN, LM ad FB lineam, erit simile triangulum KGN, triangulo GML, quare ut KG ad GL, hoc est AK ad EL, id est AC ad DE , ita est KN , ad LM; igitur BH ad IF, est ut AC ad DE, cum eadem proportione distent a centris qua inter se sunt diametri circulorum. Quare similia segmenta subtendunt & circulos similiter dividunt. Perficimus igitur quod imperatum fuit.

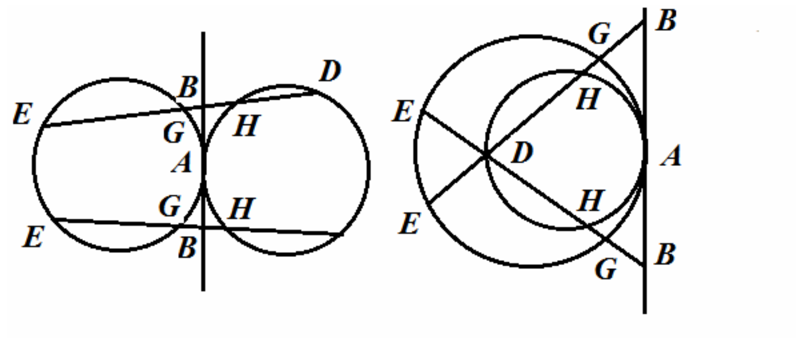
CIRCULORUM PARS QUARTA

De linearum in circulis potentia.

PROPOSITIO LXVIII.

Contingant sese circuli duo in A puncto, per quod acta contingens AB, occurrat cuiuis EBD secanti perimetros in E, G, H, D.

Dico GBE rectangulum, rectangulo HBD aequale esse.



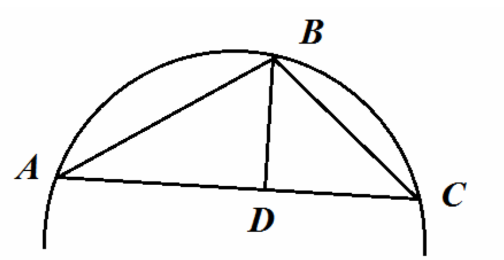
Demonstratio.

Patet, cum utrumque quadrato contingentis AB, aequale sit.

PROPOSITIO LXIX.

Segmento circuli ABC inscriprum sit triangulum ABC, a cuius vertice B, normalis demittatur BD.

Dico parallelogrammum ABC in angulo ABC, aequari rectangulo AC,BD.



Demonstratio.

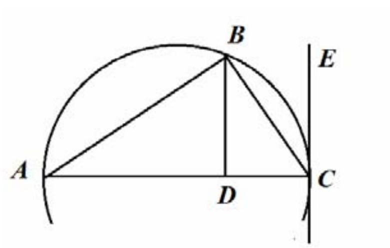
Parallelogrammum ABC duplum est [*per elementis*] trianguli ABC; sed & AC, BD rectangulum, eiusdem duplum est, cum basim habeat eandem AC, & BD altitudinem igitur parallelogrammum ABC in angulo ABC, aequale est recctangulo ACBD. Quod erat demonstrandum.

PROPOSITIO LXX.

Si rursum segmento ABC inscriptum fuerit triangulum, a cuius vertice demissa BD, aequidistet contingenti CE.

Dico rectangulum ABC rectangulo ACBD aequale esse.

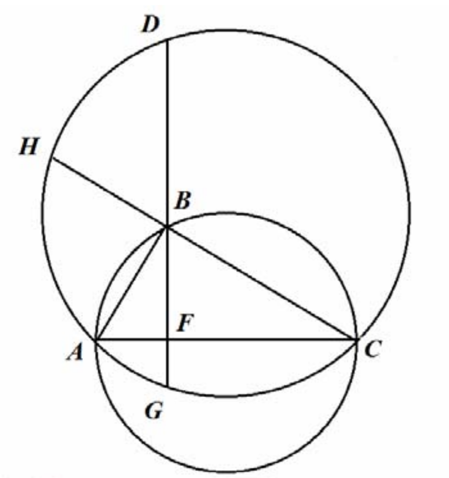
Demonstratio.



Cum enim EC, BD aequidistent, erunt anguli DBC, BCE, id est , anguli DBC, BAC ob EC tangem aequales;

similia igitur sunt triangula DBC, ABC. Unde ut AC ad AB ita BC est ad BD [17. *Sexti*.]: patet igitur AGBC, & ACBD rectangula esse aequalia.

PROPOSITIO LXXI

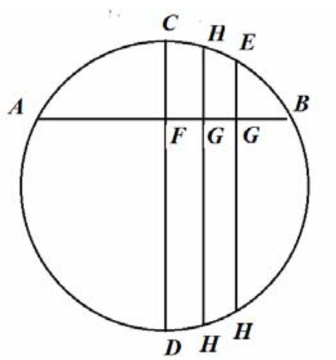


Secent invicem circuli duo quorum unus per centrum alterius transeat sic ut ducta AC quae puncta sectio centra coniungit, sit diameter circuli transeunteis per centrum alterius; ad quam erigitur per B, in perimetro ABC assumptum, normalis DBG, occurrens ADC perimetro in D & G, & AC in F:

Dico rectangulo DBG aequari ACBF rectangulum.

Demonstratio.

Ducta CBH, cum circulus ABC per alterius centrum transeat, erit [*Sereii lib.2.prop.45*] HB aequalis AB. Unde ABC rectangulum aequale est rectangulo HBC, sed HBC aequale est rectangulo DBG ; igitur & DBG aequatur AB id est [69.*huius*] ACBF rectangulo. Quod erat demonstrandum.



PROPOSITIO LXXII.

Diametrum circuli ABC, secet AB, normaliter in F, ponanturque quaevis EH, occurrentes AB in G.

Dico EGH rectangulum una cum FG quadrato, aequari EGH rectangulo una cum quadrato FG.

Demonstratio.

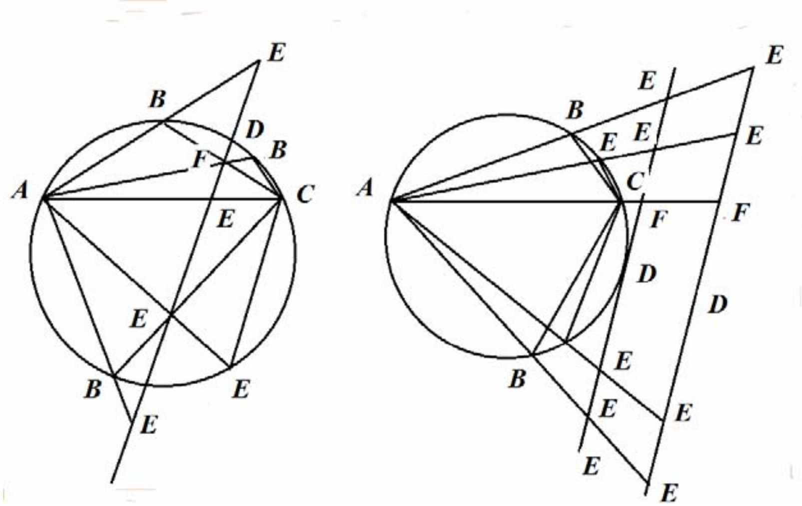
Quoniam AB normalis est ad diametrum CD erit bifariam divisa in F ; & quia non bifariam divisa est in G, erit AGB, rectangulum una cum quadrato FG, [*5.secundi*] aequale quadrato AF. Sed AGB rectangulo aequale est EGH [35.*tertii*.] rectangulum; addito igitur FG quadrato, erit EGH rectangulum una cum quadrato FG , quadrato AF aequale. Igitur & EGH rectangulum, una cum quadrato FG, aequale est EGH rectangulo una cum quadrato FG. Quod erat demonstrandum.

PROPOSITIO LXXIII.

Sit ABC segmentum circuli, cuius AC subtensa , secetur recta ED occurrente AC in F ut angulus AFE sit aequalis angulo segmenti ABC: ductae deinde ex A quotius rectae AE occurrant perimetro in B: & ED lineae in E.

Dico rectangula EAB, inter se esse aequalia.

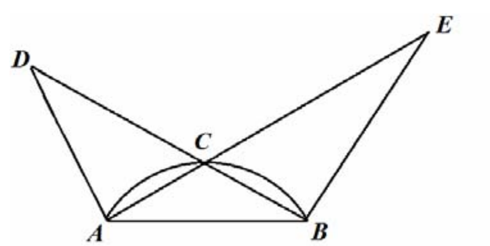
Demonstratio.



Quoniam aequales sunt anguli AFE, ABC, & communis angulus BAC, similia igitur existunt triangula AFE, ABC, unde ut AF ad AE, Sit AB ad AC. Rectangulum igitur CAF rectangulo EAB aequale est. Quare & rectangula EAB, inter se sunt aequale. Quod fuit demonstrandum.

PROPOSITIO LXXIV.

Segmento ABC inscripti trianguli ABC latera producta exhibeant triangula ADC, CAE habentia angulos DAC, ACE singulos angulo segmenti aequales.
Dico quadrato AC, aequari rectangulum ADCE.

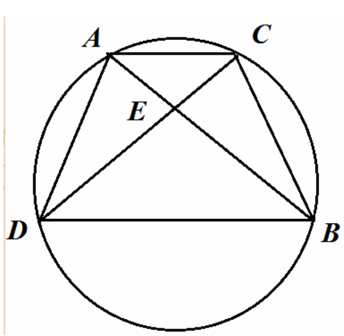


Demonstratio.

Quoniam angulus ABC aequalis est angulo DAC, & ACB communis triangulis ABC, ADC, similia erunt triangula ABC, ADC. Eodem modo similia ostendenter triangula ABC, AEC: igitur & ADC triangulum simile est triangulo AEC, & DA, AC , CE latera proportionalia: unde [*Ibid*] AC quadrata aequale est rectangulum DACE. Quod erat demonstrandum.

PROPOSITIO LXXV.

Occurrant invicem ad rectos in circulo ABC lineae AB, CD, in E. Iunctisque extremis, exsurgat quadrilaterum ACBD.
Dico quadratis AC, BD, aequari AD, CB, quadrata.



Demonstratio.

Cum enim anguli ad E recti sint, erunt quadrata AD, CB, aequalia quadratis AE, ED, CE, EB; sed iisdem aequalia quoque sunt quadrata AC, DB; igitur quadratis AC, BD, aequalia sunt AD, CB quadrata.

PROPOSITIO LXXVI.

Iisdem positis,
Dico rectangula ADCB, ACDB simul sumpta, dupla esse figurae quadrilaterae ACBDA.

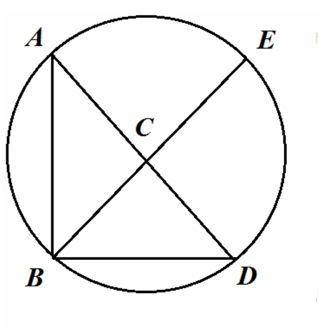
Demonstratio.

Rectangula duo ADCB, ACDB aequalia sunt [*Lud. a Ceul. & alii*] rectangulo ABCD, sed & ABCD rectangulo aequalia sunt rectangula AEC, AED, BEC,BED, quae simul sumpta [*Patet ex elementii.*] dupla sunt figurae ACBDA, igitur & rectangula ADCB, ACDB aeqalia sunt rectangulis AEC, AED, BEC, BED adeoque & dupla figurae ACBDA. Quod erat demonstrandum.

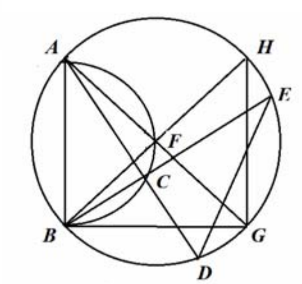
PROPOSITIO LXXVII.

Secent iterum sese ad rectos in circulo lineae AD, BE in C.
Dico quatuor quadrata partium AC, CD, CB, CE, simul sumpta, quadrato diametri esse aequalia.

Demonstratio.



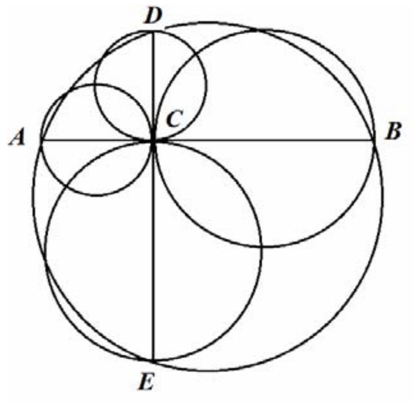
Transeat primo altera linearum, puta AD per centrum circuli, iunganturque AB, BD; cum igitur anguli, ACB, DCB recti sint,erunt quadrata AB, BD, aequalia quadratis AC, CB, & CB, CD sive CE, CD, (quia BE in C ab AD diametro divisa est bifariam) sed quadratum AD, quadratis AB, BD quoque aequale est, cum ABD sit angulus semicirculi adeoque rectus, igitur etiam quadratum AD, aequale est quatuor quadratis AC, CB, & CD, hoc est CE, CD : quod fuit primo demonstrandum.



Quod si neutra linearum AD, BE transeat per centrum, iuncta AB, describatur semicirculus ACB, qui per C punctum transibit, cum angulus ACB rectus ponatur; ex A, vero diameter ducatur AG occurrens circulo ACB in F, per quod collocetur BFH : iunganturque HG, ED, erunt igitur HG, ED lineae adeoque quadrata inter se aequalia [18. *huius*] : quia vero quadrata AB, HG, sunt aequalia quodratis HF, FG, AF, FB, id est quadrato diametri AG ut prius ostensum est ; igitur & quadrata ED, AB, quadrato diametri AG aequalia sunt ; sed quadrata ED, AB, aequalia sunt quadratis EC, CD, & AC, CB, igitur & quadrata EC, CD, AC, CB, quadrata diametri sunt aequalia. Quod fuit demonstrandum.

PROPOSITIO LXXVIII.

Secent sese denuo in C ad rectos duae quaevis AB, DE, in circulo ABD, & superpartibus, circuli describantur.
Dico illos simul sumptos aequales circulo ABD.



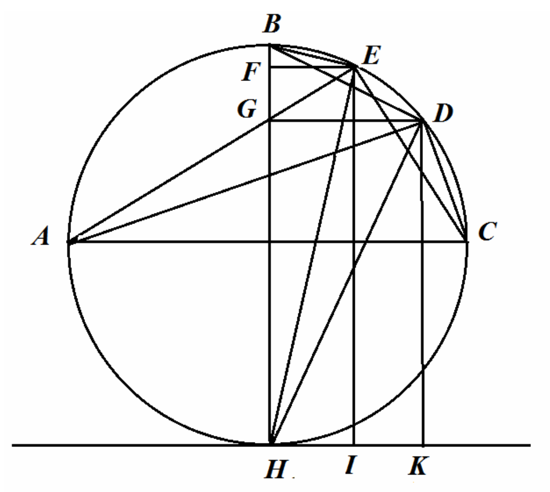
Demonstratio.

Circuli inter se eam rationem habent quam a diametris [2. *duodecimi*.] descripta quadrata: ostendimus autem praecedenti propositione quadrata AC, CD, CB, CE, aequari quadrato diametri circuli ADB ; igitur & circuli super AC, CD, CB, CE, describi aequales sunt circuli ADB. Quod fuit demonstrandum.

PROPOSITIO LXXIX.

Semicirculo ABC inscripta sint triangula quaecunque AEC, ADC, actaque contingente HK quae AC diametro aequidistet, ponantur EI, DK normales contingenti HK.
Dico quadratum compositae ex AE, EC, ad quadratum compositae, ex AD, DC eam habere rationem quam continet EI ad DK.

Demonstratio.



Post HB diametro ducantur ad illam normaliter EF, DG iunganturque EH, DH, EB, DB, quoniam FE normaliter insistit rectae BH, erunt FH, HE, HB, lineae in continua analogia, vt ex elementis patet unde rectangulo FHB, aequatur HE quadratum; eademque de causa quadrato HD, rectangulum GHB; quare ut FH ad GH, hoc est EI ad DK, Sic EH quadratum ad DH quadratum : Rursum cum sit ut HE ad HD, ita composita [14. *huius*] ex AE, EC, ad compositam ex AD, DC, erit ut quadratum HE, ad HD quadratum, sic quadratum compositae ex AE, EC, ad quadratum compositae ex AD, DC; Quod demonstrandum fuit.

PROPOSITIO LXXX.

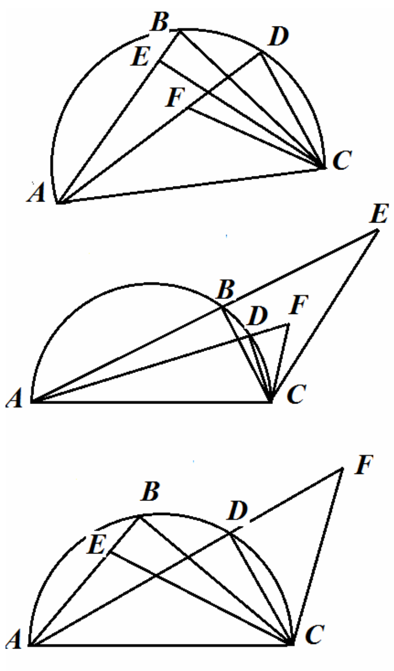
Semicirculo ABC inscripta sint triangula duo ABC, ADC; atque ex C, ad opposita latera recte ducantur CE, CF, utcunque.

Dico si intra aream circuli occurrant rectis AB, AD, in E, F, quadrata AE, EC, una cum rectangulo AEB bis sumpto, aequari quadratis AF, FC una cum rectangulo APD bis sumpto;

Si vero extra circulum occurrant, dico quadrata AE, EC, minus AEB rectangulo bis sumpto, aequari quadratis AF, FC minus AFD bis sumpto;

Quod si autem altera intra arcam circuli cadat, altera extra, dico quadrata AE, EC cum AEB & rectangulo bis sumpto, aequari quadratus AF, FC minus rectangulo AFD bis sumpto.

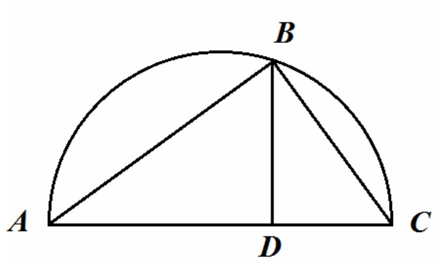
Demonstratio.



Si enim cadat CE, CF, intra aream semicirculi, constituent angulos obtusos, AEC, AFC, cum ABC, ADC recti fiat, unde per elementa quadrata AE, EC, cum rectangulo AEB, bis sumpto, quadrato AC aequalia erunt ; sed eidem AC quadrato aequalia sunt AF, FC, quadrata una cum rectangulo AFD bis sumpto; igitur quadrata AE, EC cum AEB rectangulo bis sumpto aequalia sunt AF, FC una cum rectangulo AFC bis sumpto. Quod erat primum.

Si vero CE, CF extra cadant; paret angulos AEC , AFC esse acutos : quare tam quadrata AE, EC minus rectangulo AES bis sumpto, aequabuntur quadrato AC, quam quadrata AF, FC minus AFD rectangulo bis sumpto: unde veritas secunde partis quoque manifesta est. Partis tertiae demonstratio ex ante dictis clare patet; igitur, &c. Quod erat demonstrandum.

PROPOSITIO LXXXI.



Semicirculo ABC triangulum inscribatur ABC, a cuius vertice ad diametrum demittatur perpendicularis BD.

Dico ADC rectangulum, & rectangulum ABC; denique quadratum AC in continua esse proportione.

Demonstratio.

Quadratum AC est ad rectangulum ABC, hoc est ACBD ut [1. *sexti*.] AC ad DB : sed U rectangulum ACBD est ad quadratum BD [35.*tertii*] hoc est rectangulum ADC, ut AC ad BD, igitur continuant eandem rationem AC quadratum, rectangulum ABC, una cum rectangulo ADC. Quod erat demonstrandum.

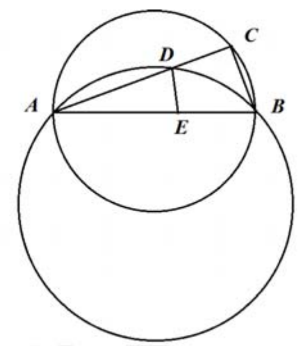
PROPOSITIO LXXXII.

Per extrema diametri AB circuli ABC, circulus describatur ADB, positaque AC, quae occurrat ADB perimetro in D, demittatur normaliter DE ad diametrum AB, iunganturque DB, CB.

Dico rectangulum ex AB, DE, ad rectangulum ADB, eam rationem habere quae est inter rectas CB, DB.

Demonstratio.

Quoniam angulus ACB in semicirculo rectus est adeoque aequalis angulo AED ; & AED, ACB triangulis communis angulus DAE, erunt ADE, ACB triangula similia, unde ut AB ad CB sic AD ad DE, & ABDE rectangulum aequale rectangulo ADCB: sed rectangulum ADCB est ad rectangulum ADB ut CB ad DH, igitur & rectangulum ABDE ad rectangulam ADB ut CB ad DE. Quod erat demonstrandum.



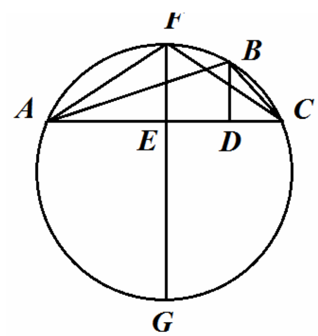
PROPOSITIO LXXXIII.

Occurrat circuli ABC diametro FG in E orthogona AC, quam in D secet normalililiter BD, iunganturque AB, BC.

Dico rectangulum ABC ad rectangulum ACBD rationem obtinere eandem, quam FG diameter ad AC lineam.

Demonstratio.

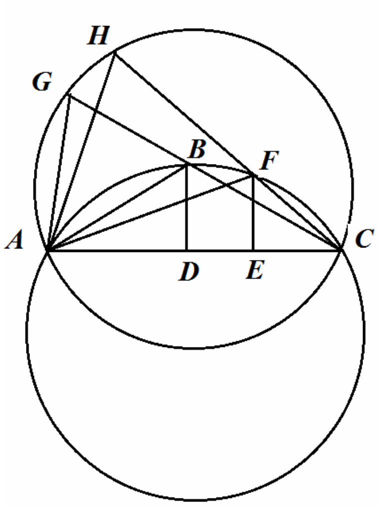
Iungantur AFC quoniam AC linea in E bifariam adeoque ad rectos est divisa, erunt AF, FC lineae aequales, & AFC rectangulum aequale quadrato AF id est [47.*primi &5. secundi*.] rectangulo EFG: quia FE, FA, FG sunt proportonales. Sed EFG rectangulum est ad rectangulum CFE ut FG ad AC; igitur & rectangulum AFC ad ACFE rectangulum id est [69.*huius*] parallelogramum AFC, ad FG ad AC; quia vero est AFC rectangulum ad rectangulum ABC ut AFC parallelogramum in angulo AFC, ad parallelogrammum ABC, in eodem angulo ABC (cuius ex iisdem rationes habeant compositas) erit permutando ABC rectangulum ad parallelogramum ABC, id est ad rectangulum ACBD ut AFC rectangulum ad Parallelogrammii AFC id est [*ibid*] ad ACFE rectangulum, id est ex demonstratis ut FG ad AC. Quod erat demonstrandum.



PROPOSITIO LXXXIV.

Secent se utcunque circuli ABC, ADC in punctis, AC, iunctaque AC ponantur CEH, CBG, occurrentes perimetris circulorum in B, E, G, iunganturque AB, AE.

Dico rectangulum ABC, ad AEC rectangulum eam rationem obtinere, quam GBC, ad rectangulum HEC.



Demonstratio.

Iungantur AG, AH : quoniam anguli ABC, AFC eiusdem segmenti aequales sunt, erunt & reliqui ABG, AFH quoque inter se aequales : unde cum & anguli AGC, AHC eide insistentes arcum sint aequales, erunt AGB, AHF triangula similia : & AF ad AH, ut AB ad AG. est autem ratio rectanguli ABC, ad AFC rectangulum composita ex ratione AB ad AF, hoc est GB ad HF, ex BC ad GC, & ex iisdem quoque composita ex ratio rectanguli GBC ad HFC; igitur ut rectangulum ABC ad AFC, sic GBC rectangulum ad HFC. Quod erat demonstrandum.

Corollarium primum.

Hinc consequens est demissis normalibus BD, FE esse GBC, rectangulum ad rectangulum HFC, ut est recta BD ad EF, est enim ut ABC rectangulum ad AFC sic GBC ad rectangulum HFC; sed ABC ad AFC, rectangulum est, ut ABC parallelogrammum in angulo ABC, ad parallelogrammum, AFC in angulo AFC (quia ex iisdem rationem habent compositam) hoc est ut rectangulum ACBD [69.*huius*] ad rectangulum ACFE; igitur ut rectangulum ACBD, ad rectangulum super ACFE, sic GBC rectangulum ad rect-angulum HFC: quare ut BD ad EF, sic GBC rectangulum ad rectangulum HFC. Quod fuit demonstrandum.

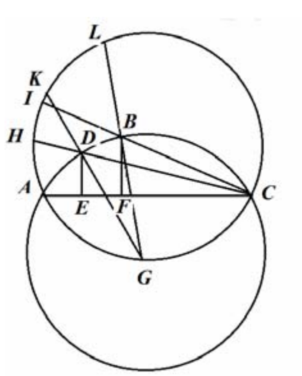
Corollarium secundum.

Eadem manente figura, patet AG esse ad AH ut AB ad AF,sive ut GB ad HF, quia AGB, AHF triangula ostensa sunt inter se similiae : quod speciatim ideo volui apponere, quia aliquoties postea assumendum est.

PROPOSITIO LXXXV.

Transeat per circuli ABC centrum, perimeter circuli AGC, iunctisique AC, ponantur quaecunque BF, DE normales ad AC: ex centro deinde G, per B & D, agantur rectae GBL, GDK.

Dico DE , BF lineis KD, LB esse proportionales.



Demonstratio.

Ponantur per D & B rectae CDH, CBI ut DE ad BF, sic HDC [*per coroll. praecedentis*] rectangulum ad IBC rectangulum, hoc est KDG ad LBG: sed est quoque ut KD ad LB, sic KDG rectangulum ad rectangulum LBG, cum DG, BG lineae sint aequales; igitur DE ad BF; sic KD ad LB. Quod fuit demonstrandum.

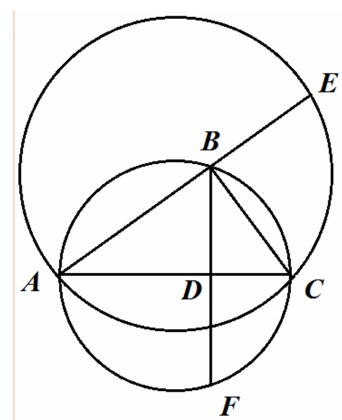
PROPOSITIO LXXXVI.

Occurrat iterum circulus ABC, cuius diameter AC, centra circuli AEC, ductaque ex A recta ABE, ponatur normalis BDF ad AC.

Dico lineas AC, AE, & compositam ex AC & BF, in continua esse analogia.

Demonstratio.

Quadratum AE, aequatur quadratis AB, BE & ABE rectangulo bis sumpto [4. *secundi*]: est autem quadratum BE aequale BC quadrato [*Serenus lib.2.prop.45*], & ABE rectangulum bis sumptum aequale ABC rectangulo (hoc est ACBD [69.*huius*]) bis sumpto, hoc est rectangulo ASBF semel sumpto ; igitur quadratum AE, aequlae est quadratis AB, BC, hoc est quadrato AC & rectangulo AC,BF, semel sumpto. Sed AC, quadrato una cum rectangulo ACBF, aequatur rectangulum super AC & composita ex AC, BF ; igitur quadratum AE, aequale est rectangulo super AC & composita ex ACBF. Unde lineae AC, AE & composita ex ACBF in continua sunt analogia. Quod fuit demonstrandum.



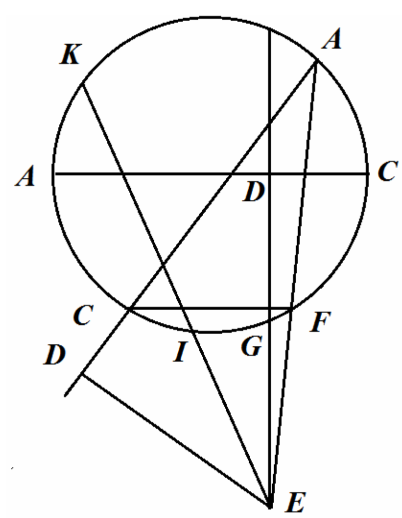
PROPOSITIO LXXXVII.

Extra aream circuli ABC, sumpto puncto E, demittatur ad diametertrum AC normalis ED, ponaturque quaevis alia EIK.

Dico si D punctum communis linearum AC, DE intersectio cadat intra circulum, quod ED quadratum superet rectangulum IEK rectangulo ADC: si vero D, extra cadat, dico quod ED quadratum deficiat a rectangulo IEK, rectangulo ADC.

Demonstratio.

Producta DE, circuli perimetro occurrat in B: quadratum DE aequale est quadratis DG, GE una cum DGE rectangulo bis sumpto id est rectangulo BGE semel sumpto, sed BGE rectangulum una cum quadrato GE aequatur rectangulo BEG, igitur quadratum DE aequale est quadrato DG, id est rectangulo CDA una cum rectangulo GEB id est IEK.

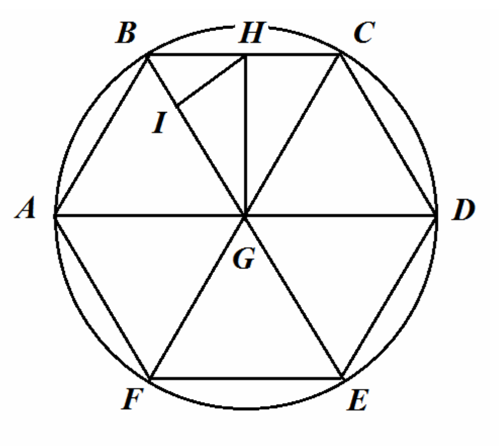


Quod si D extra circuli aream occurrat diametro AC productae, ducatur ex E recta EA, occurrens perimetro circuli in F, iunganturque CF: erunt itaque similia triangula ADE, & CAF, & DA, AE latera proportionalia lateribus AF, AD: unde & rectangula CAD, FAE, sunt inter se aequalia; quia vero quadratum AE aequatur rectangulis AEF, EAF, eidemque AE quadrato aequantur quadrata AD, DE ; quadratum autem AD, aequale est rectangulis ADC, DAC, erunt rectangula AEF, EAF, aequalia rectangulis ADC, DAC una cum quadrato DE, ostensum autem est rectangulum DAC, rectangulo FAE aequale, quare residua quoque inter se aequalia sunt, id est CDA rectangulum auctum quadrato DE, aequale rectangulo FEA, id est IEK. Quod erat demonstrandum.

PROPOSITIO LXXXVIII.

Circulo ABC, quodlibet polygonum inscribatur regulare; ductaque ex G centro, GH quae lateri BC normaliter insistat; ex H ponatur HI, ad rectos angulos ipsi BG.
Dico totum polygonum, ad BG quadratum toties sumptum, quot laterum est polygonum, eam rationem obtinere quam HI recta, ad BG.

Demonstratio.



Ducantur ex G centro ad angulos polygoni, rectae AG, BG, CG, &c. Quoniam polygoni regularis latera aequalia sunt, erunt singula triangula AGB, BGC, CGD, &c. inter se aequaliae ac proindc duplicia trianguli BHG, cum BC in H divisa sit bifariam: sed & BHG trianguli, duplum est rectangulum BHG, hoc est BGH [70.*huius*], igitur BCG triangulo aequale est BGHI rectangulum, est autem BGHI, rectangulum, ad quadratum BG, ut HI, ad BG; igitur etiam BCG triangulum, est ad BG quadratum, ut HI ad BG: quia vero idem de singulis polygoni triangulis eodem modo demonstratur, patet totum polygonum ad quadratum BG toties sumptum, quot laterum est polygonum eam proportionem habere quam HI linea, ad rectam BG. Quod fuit demonstrandum.

PROPOSITIO LXXXIX.

Iisdem positis,
Dico duplum polygoni ad quadratum lineae, quae polygoni perimetro fit aequalis, eam rationem habere, quam HG linea ad lineam polygoni perimetro aequalem.

Demonstratio.

Est enim totum polygonum, aequale triangulo basem habenti aequalem [*Per elemente*] lineae toti perimetro aequali, altitudmem vero HG; igitur duplum polygoni aequatur triangulo illi, bis sumpto, hoc est rectangulo basim habenti aequalem polygoni perimetro & HG altitudinem; sed rectangulum hoc ad quadratum lineae aequalis toti perimetro polygoni , eam obtinet proportionem, quam HG linea ad lineam aequalem toti perimetro; ergo polygonum bis sumptum ad quadratum lineae, quae perimetro sit aequalis, eam habet rationem, quam HG linea, ad rectam toti perimetro polygoni aequalem: quod erat demonstrandum.

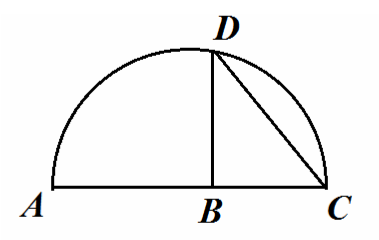
Corollarium.

Hoc loco non videtur omittendum sequi ex hac propositione per ea quae in libro de progressionibus Geometricis diximus, circulum bis sumptum, ad quadratum suae peripheriae, eam seruarc rationem, quam semidiameter ad perimetrum circulum autem semel sumptum ad quadratum perimetri circularis, eam proportionem continere, quam quarta pars diametri ad circuli perimetrum,atque adeo propositionis Archimedeae veritatem de comparatione circuli ad rectangulum & aequale aliter hinc posse demonstrari

PROPOSITIO XC.

Diametro circuli ABC insistat ad rectos angulos recta BD, iungaturque DC.
Dico rectangulum ABD, ad ACD rectangulum triplicatam eius habere rationem, quam obtinet linea DB ad DC lineam.

Demonstratio.



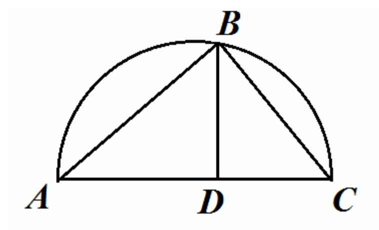
Quia DB normalis est ad diametrum AC rectae CB , BD, BA, item CB, CD, CA per elementa in continua sunt analogia; unde cum prima BC utrique seriei communis sit, erit [17.*Libri de progressionibus*.]ratio AC ad AB tertiae ad tertiam duplicata eius quam habet Secundo CD, ad DB Secundam, sed ratio rectanguli ABD, ad rectangulum ACD composita est ex ratione AC ad AB, hoc est duplicata DC, ad DB, & ex ratione DC, ad DB, paret igitur rationem ABD rectanguli ad rectangulum ACD, triplicatam esse lineae DC ad DB. Quod fuit demonstrandum.

PROPOSITIO XCI.

Semicirculo ABC triangulum inscribatur ABC; e cuius vcrtice ad basim demissa sit normalis BD.

Dico rectangulum DAB, ad DCB rectangulum, triplicatam rationem habere eius, quam linea AB ad BC.

Demonstratio.



Cum enim BD normalis sit ad diametrum AC, erunt denuo tres AC, AB, AD, & AC, CB, CD in continuata ratione: quia vero communem habent primam AC, erit AD ad DC, in triplicata ratione AB ad BC, sed DAB rectangulum ad DCB rectangulum rationem habet compositam ex ratione DA ad DC, id est duplicata AB ad BC & ex AB ad BC, igitur rectangulum DAB, ad DCB triplicatam habet rationem AB ad BC. Quod erat demonstrandum.

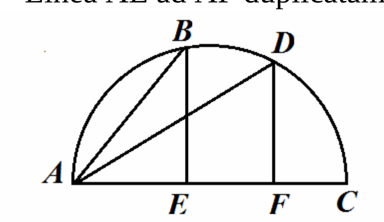
PROPOSITIO XCII.

Semicirculi ABC diametro AC in E & F occurrant normales EB, FD, iunganturque AB, AD.

Dico EAB rectangulum, ad FAD rectangulum, rationem habere triplicatam rectae AB, ad AD.

Demonstratio.

Linea AE ad AF duplicatam habet rationem AB ad AD, cum tam AC, AB,AE, quam AC, AD, AF continue sint proportionales, habeantque communem primam A, unde cum ratio rectanguli EAB ad FAD composita sit ex ratione AE ad AF, & AB ad AD, patet EAB rectangulum ad rectangulum FAD triplicatum habere rationem eius quam habet AB ad AD. Quod fuit demonstrandum.

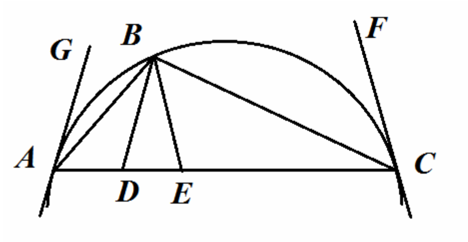


PROPOSITIO XCIII.

Segmenta cuiuis ABC triangulum inscribatur ABC, ductisque contingentibus AG, CF, ponantur ex B vertice trianguli duae BD, BE, contingentibus aequidistantes.

Dico rectangulum DAB, ad ECB, triplicatam continere rationem eius, quam habet AB ad BC.

Demonstratio.

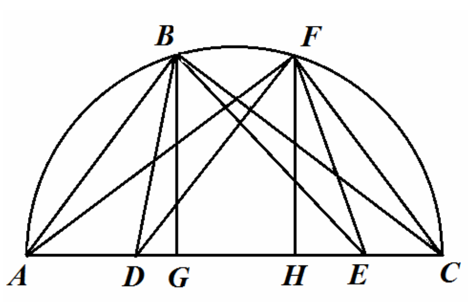


Cum AG sit contingens, erit angulo GAB id est: ABD (ob AG, BD parallelas) aequalis angulus [16.*Tertii*] ACB: &; quia FC quoque circulum contingit, angulo FCB id est EBC (ob EB, FC parallelas) aequalis est angulus BAC: triangula igitur ABD, BEC similia sunt triangulo ABC: unde ut AC ad AB, Sit AB ad AD: & ut AC ad CB, ita CB ad CE duae igitur continue proponionalium series communem habent primam AC; unde AD ad EC, tertia ad tertiam [17. *de progressionibus*] duplicatam habet rationem AB ad B C secundae ad secundam: ratio autem rectanguli DAB ad ECB, rectangulum, composita est ex ratione AD ad EC, & AB ad CB, triplicata igitur ratio est rectanguli DAB ad ECB rectangulum.Quod erat demonstrandum.

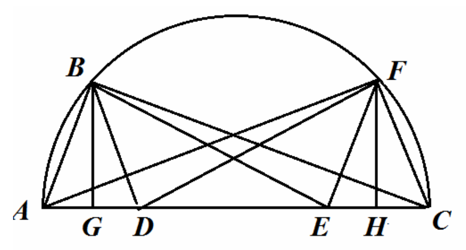
PROPOSITIO XCIV.

Circuli ABC diameter secetur in D & E, punctis; aequaliter a centro semotis ; ex quibus binae ad duo quaedam perimetri puncta B, F rectae deducantur DB, DF, EB, EF.
Dico quadrata DB, BE simul sumpta quadratis DF, FE esse aequalia.

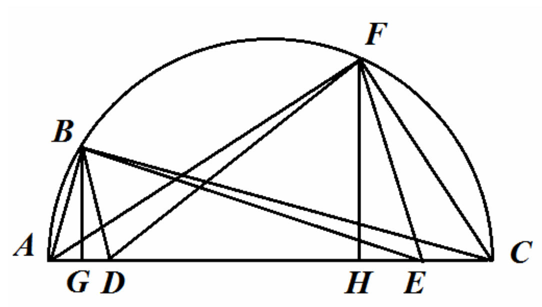
Demonstratio.



Iungantur AB, BC, AF, FC, ex B & F demittantur perpcndiculares BG, FH ad diametrum AC, quae cadant primo inter D & E; erunt igitur anguli ADF, CEF rectis maiores; unde quadratum [12. *secundi*] FC cxcedit quadratum FE, quadrato CE una cum rectangulo CEH bis sumpto, & quadratum IAF superat DF quadratum quadrato AD, una cum rectangulo ADH bis sumtum, est autem rectangulum CEH, bis sumptum, una cum rectangulo ADH [*Per elementii*] bis sumpto, aequale rectangulo CED bis sumpto; additis igitur quadratis aequalibus EC, AD, erit CED, rectangulum bis sumptum una cum quadratis AD, EC, hoc est rectangulum ECA semel sumptum, excess quo AF, FC, quadrata superant quadrata duo DF, FE. Eodem modo ostendetur quadrata duo AB, B C excedere quadrata duo DB, BE, rectangulo DAC id est: ECA; igitur cum quadrata AF, FC aequalia sint quadratis AB, BE, &excessus DAC, ECA, super quadratis DF, FE, DB, BE, aequales quoque sint, illis ablatis manent DB, BE quadrata, aequalia quadratis AF, FE.



Secundo normales ex B & F demisse cadant intra puncta AD, EC; BG quidem inter A & D; FH vero inter E & C, cum angulus ADF recto maior sit, quadratum AF, superat quadrata AD, D F, rectangulo ADH, bis sumpto, [12.*secundi*.] hoc est rectangulo ADE, una cum rectangulo ADEH bis sumptis: quia vero FEC angulus recto minor est, quadratum FC deficit a quadratis [13. *secundi*.] EF, EC rectangulo CEH hoc est ADEH bis sumpto; igitur ADE rectangulum bis sumptum est excesssus quo quadrata duo AF, FC superant quadrata quatuor AD, DF, CE, EF demptis igitur aequalibus quadratis AD, CE, remanet ADE rectangulum bis sumptum excessus quo AF, FC, quadrata, excedunt quadrata DF, EF; eadem ratione ostenditur quadrata DB, BE, superari a quadris AB, BC, rectangulo CED bis sumpto; igitur cum AB, BC, quadrata aequalia sint quadratis AF, FC, & ADE rectangulum (excessus quadratum AF, FC, super DF, FE quadratis) aequale sit CED rectangulo, (excessui quo AB, BC quadrata, superant DB, BE quadrata) demptis excessibus, remanent DF, FE quadrata, aequalia quadratis DB, BE.



Tertio normalium BG, FH, altera intra DE, altera vera intra AD, spatium contineatur: ostendetur ut prius, CED rectangulum bis sumptum, demptis aequalibus quadratis AD, EC aequale esse excessui quo CB, BA quadrata superant EB, BD; item DF, FE, quadrata superari a quadratis AF, FC, rectangulo ADE bis sumpto demptis quadratis AD, EC: igitur cum tota sint aequalia & excessus insuper aequales sint, residua quoque quadrata DF, FE, erunt residuis quadratis DB, BE aequalia. Quod fuit demonstrandum.

Scholion.

Posset aliter & commodius propositio haec demonstrari, ac facilius fortissis, sed aliam studio in aliquorum gratiam adhibere volui demonstrationem.

Libri Tertii Finis.