§18.2. **Chapter Eighteen.** [p.46.]
From the given sides of a plane triangle: to find the area, altitude, diameters of the inscribed &
circumscribed circles, and any angle.

Before I approach this proposition, I should say a few things beforehand about right angled
triangles; as we are able with the help of these to judge more correctly about the truth
and certainty of these propositions which are to be presented.

Proclus [410 - 485], in the 47th proposition of the 1st book of Euclid, propounded two methods,
by means of which: with either leg given about the right angle, we can find the other leg and the
hypotenuse; just as in the works of Ramus [Pierre de Ramie, 1515 - 1572], we have the same
conclusions 5e, Book 12. Of these methods, one is attributed to Plato, the other to Pythagoras, who
were able to propound propositions in general, without distinguishing between cases involving odd
and even numbers, such as: If half the length of the side, &c. And, If the square of the length of
the first given side , &c ¹. However, I will propound three general propositions (the latter two
methods have emanated from the first of these), by means of which: from any given side adjacent
to a right angle, to find as many right angled triangles as you please. These propositions are of this
kind:

*If three straight lines are in continued proportion: the legs of a right angled triangle are:*

1. Half the difference of the extremes, and the mean; while the hypotenuse certainly is half the sum
   of the extremes², or

2. The legs are sum of the first & second [proportions]; the sum of the second and half the third;
   and the hypotenuse is the sum of the first and second and half the third³, or

---

§18.1. **Synopsis: Chapter Eighteen.**

Three variations of the Pythagorian Triplet are presented, based on 3 numbers in continued proportion – by
means of which right-angled triangles can be constructed, both for sides of lengths either integer or rational, and
hence any required triangle.

Subsequently, a number of well-known theorems on solving triangles are presented numerically: the use of
logarithms is shown to ease the burden of calculation of the length of an unknown side, the altitude, area, in-radius,
etc., of a triangle.
3. The legs are the difference of the first and second, the difference of the second and half the third; while the hypotenuse is the difference of the second and the sum of the first and half the third.

Let the numbers in continued proportion be 8, 12, 18. By the first proposition, the sides sought are 5, 12, 13; by the second proposition they are: 20, 21, 29 and 30, 16, 34; and by the third proposition, they are: 4, 3, 5 & 6, 8, 10. With the continued proportionals 18, 24, 32, the right angled triangles are 7.24.25 | 42.40.58 | 56.33.65 | 6.8.10 | 8.15.17.

The first of these propositions is a logical consequence of this following proposition. In the right angled triangle ABC: the leg BC is the mean proportional between, on either side, the sum BD and the difference BE, of the hypotenuse and the remaining leg AB; and conversely. The second and third propositions are demonstrated from the first. And all of these can easily be shown, as with straight lines geometrically, so also by expressing the measures of the lengths by numbers. Three numbers in continued proportion are found by division. And through logarithms, Axiom 3, Ch. 2. If any number divides the square of the other: the root of the number divided is the mean proportional between the divisor and the quotient. Let the given leg of the right angle be 12, of which the square is 144. Let the divisors be 2, 3, 4, 5, 6, 7, 8, 9, 10, 11. The quotients are 72, 48, 36, 28, 24, 20, 18, 16, 144, 13311. As you see here * [Table 18-1] The mean is everywhere 12, the extremes are the divisors and the quotients. The numbers in column D are half the sums, in column S half the differences. These columns give us 19 right angled triangles of which the given leg is 12, the remaining leg is had in column S, certainly with the hypotenuses from the

[p.47]
directly opposite side in column D. As A 12, 16, 20; B 12, 9, 15; C 12, 5, 13; E 12, 3\(\frac{1}{2}\), 12\(\frac{1}{2}\).

[See Table 18-1.]

<table>
<thead>
<tr>
<th>To be divided</th>
<th>DD</th>
<th>SS</th>
<th>D</th>
<th>S</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>72</td>
<td>74</td>
<td>70</td>
<td>37</td>
</tr>
<tr>
<td>3</td>
<td>48</td>
<td>51</td>
<td>45</td>
<td>253</td>
</tr>
<tr>
<td>4</td>
<td>36</td>
<td>40</td>
<td>32</td>
<td>20</td>
</tr>
<tr>
<td>5</td>
<td>28</td>
<td>38</td>
<td>238</td>
<td>11</td>
</tr>
<tr>
<td>6</td>
<td>24</td>
<td>20</td>
<td>15</td>
<td>9</td>
</tr>
<tr>
<td>7</td>
<td>20</td>
<td>27</td>
<td>13</td>
<td>6(\frac{1}{4})</td>
</tr>
<tr>
<td>8</td>
<td>18</td>
<td>26</td>
<td>10</td>
<td>5</td>
</tr>
<tr>
<td>9</td>
<td>16</td>
<td>25</td>
<td>7</td>
<td>5(\frac{1}{2})</td>
</tr>
<tr>
<td>10</td>
<td>14</td>
<td>24</td>
<td>44</td>
<td>22</td>
</tr>
<tr>
<td>11</td>
<td>13</td>
<td>24</td>
<td>122</td>
<td>22</td>
</tr>
<tr>
<td>14</td>
<td>10</td>
<td>24</td>
<td>122</td>
<td>22</td>
</tr>
<tr>
<td>15</td>
<td>94</td>
<td>24</td>
<td>123</td>
<td>27</td>
</tr>
<tr>
<td>20</td>
<td>72</td>
<td>272</td>
<td>122</td>
<td>64</td>
</tr>
<tr>
<td>25</td>
<td>576</td>
<td>3076</td>
<td>1924</td>
<td>962</td>
</tr>
<tr>
<td>30</td>
<td>48</td>
<td>348</td>
<td>252</td>
<td>174</td>
</tr>
<tr>
<td>35</td>
<td>44</td>
<td>394</td>
<td>3034</td>
<td>1540</td>
</tr>
<tr>
<td>40</td>
<td>36</td>
<td>436</td>
<td>366</td>
<td>182</td>
</tr>
<tr>
<td>45</td>
<td>32</td>
<td>482</td>
<td>412</td>
<td>241</td>
</tr>
<tr>
<td>50</td>
<td>288</td>
<td>5288</td>
<td>4712</td>
<td>2356</td>
</tr>
<tr>
<td>Divisors Quotients Sum of extremes Diff. of extremes Hypot. Legs</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>22(\frac{1}{2})</td>
<td>16</td>
<td>11(\frac{1}{2})</td>
<td>9</td>
<td>5</td>
</tr>
<tr>
<td>22(\frac{1}{2})</td>
<td>16</td>
<td>11(\frac{1}{2})</td>
<td>9</td>
<td>5</td>
</tr>
</tbody>
</table>
And if two right-angled triangles of this kind are placed thus, so that both sides have the common leg; we have an oblique-angled triangle of the given sides, of which the area can be found from the rational lengths. As [in Fig. 18-2] BAE, CAE constitute BAC, BAD; in the same way, OPX, OXN give PON, OPQ.

1. We find the area of a triangle of the given sides by the method of Ch. 9, Book 12 of Ramus. If from half of the combined lengths of the sides, the lengths of the sides are taken away one by one: the square root of the continued product of half perimeter and the factors from the subtractions is the area of the triangle\(^6\). Let the three sides be 20, 11, 13. [ \(\triangle\) ABD in Fig. 18-2 above, upper

---

\(\theta\)
figure], or from the combined sides 44, the semi-perimeter is 22; with the numbers 2, 11, 9 remaining from the subtraction of the sides from the semi-perimeter, the continued product emerges from these four numbers by multiplication. But the logarithm of this product is equal to the sum of the logarithms of the factors by Axiom 2, Ch. 2. Therefore the logarithms are taken which must be for the semi-perimeter and the remaining numbers. The sum of which is the logarithm of the continued product: & half the sum is the logarithm of the square root, or the area sought. For

\[
\text{Logarithms}
\]

<table>
<thead>
<tr>
<th>Semi-perimeter</th>
<th>22</th>
</tr>
</thead>
<tbody>
<tr>
<td>Numbers remaining</td>
<td>11</td>
</tr>
<tr>
<td>Product</td>
<td>3,63908,78711</td>
</tr>
<tr>
<td>Area</td>
<td>66,1,81954,39355</td>
</tr>
</tbody>
</table>

[p.48.]

Hence the area of the given triangle is 66 [Fig. 18-2]. We find the same size of area if we multiply the altitude of the triangle AE 12 by halve the base 5\text{\textsuperscript{1/2}}, because the rectangle from the whole base and altitude is double the area of the triangle by Prop. 41, Book 1 of Euclid.

2. By finding the logarithm of the area, we can find the altitude of the same triangle, or the perpendicular from any angle to the opposite side: if from the logarithm of the area is taken away the logarithm of half the side on which the perpendicular from the opposite angle is incident, the remainder is the logarithm of the altitude, by Axiom 2, Ch. 2.

It is indeed equal to the product of the base halved by the altitude, and therefore the logarithm of the area is equal to the logarithms of the base halved and the altitude.

\[
\text{The logarithm}
\]

| of the area 66 | 1,81954,39355 |
| of half the base 6\text{\textsuperscript{1/2}} | 0,81291,33566 |
| of the altitude 10\text{\textsuperscript{2/13}} | 1,00663,05789 |

[Table 18-4] As from the angle B with the base continued, the
perpendicular BM shall fall [Fig. 18-4], the logarithm of which is 1,00663,05789, certainly BM itself is nearly 101538461, or $10^{2/13}$.

We find the same altitude from geometry, thus\(^7\): the difference is taken of the squares of the sides BA, BC, comprising the angle ABC, from which the perpendicular falls on the base CA continued if necessary, as here. This difference 279 is divided by the base CA 13, the quotient $21^{6/13}$ will be CO, the sum of the base segments from the angles O and C, or A and C [if M lies within AC], to the point M, on which the perpendicular falls: from this sum CO, the given base CA 13 is taken away, which is the difference of the segments, leaves AO $8^{6/13}$ the double of the smaller segment AM $4^{3/13}$. Then by taking away the square of AM, $17^{152/169}$ from AB squared, 121 (or CM squared from CB squared) there remains $103^{17/169}$, the square of the right line required BM, $10^{2/13}$.

3. The radius of the inscribed circle is found thus: the logarithm of the semi-perimeter is taken away from the logarithm of the area; the logarithm of the radius required is left. As in the triangle with the sides 20,13,11.

<table>
<thead>
<tr>
<th>Logarithm of the area found</th>
<th>1,81954,39355</th>
<th>66</th>
</tr>
</thead>
<tbody>
<tr>
<td>semi-perimeter</td>
<td>1,34242,26808</td>
<td>22</td>
</tr>
<tr>
<td>radius of inscribed circle for the triangle</td>
<td>0,47712,12547</td>
<td></td>
</tr>
</tbody>
</table>

[Table 18-5]

[Figure 18-5]

I assert that 3 is the radius of the inscribed circle.

Indeed the product of the radius and the semi-perimeter is equal to the area of the given triangle.

Because with straight lines drawn from the centre F to the angles A, B, & C, the given triangle is divided into 3 triangles: of which the altitudes are equal to the radius of the circle, but the bases are the sides of the given triangle [Fig. 18-5: FA is not drawn]. And with the common altitude FG taken by the half of each of the sides, that is by the semi-perimeter, the products are the areas of the component triangles AFC, AFB, and BFC [note: AFC is written again instead of BFC in the original, and P is omitted from the diagram]. That is, the area of the given triangle.
We will find the radius geometrically, thus: the right line CD is drawn perpendicular to the opposite line BA [Figure 18-5]. BD, 16. & CD, 12 are found. Then BFE is drawn bisecting the angle CBD. The centre F lies on the bisecting line, by Prop. 4, Book 4, Euclid. Thirdly, DE is found: which by Prop. 3, Book 6, Euclid, thus DE itself has the ratio to EC, as DB is to BC. & therefore, as the sum of the sides CB, BD 36, to BD 16; thus the total CD 12, to ED $5^{1/3}$. Fourthly, the line BP is sought, from the angle B, to the point of contact P; on which the radius FP falls, perpendicular to the line BA. But BP is 9. For with all the triangles, if whatever line AC 13 is taken away from the semi-perimeter 22, 9 remains, the distance of the opposite angle B from the points of contact G and P $^8$. Finally, by the rule of proportions, I seek PF. As BD 16 to DE $5^{1/3}$; so BP 9 to PF 3.

But we can find AD [in Fig. 18-5], either by that which preceded in the second section of this chapter, or just as easily by this method. With centre C and radius CA the semi-circle O E A H is described, & by Prop. 37, Book 3, and Prop. 16, Book 6 Euclid; AB 11, BH 7, BO 33, BE 21 are lengths in proportion. And by subtraction BA 11, AE will be 10, & AD $^9$.

It is possible also to find the inscribed radius of the circle by logarithms, thus. By taking away the lengths of the given lines from the semi-perimeter, the logarithms of the remainders are taken, & from the sum is taken away the logarithm of the semi-perimeter: the logarithm of the remaining half is of the radius required$^{10}$.

Thus, we can explain this as follows: The triangle ABC are constructed with the given sides AB, 20. BC, 11. AC, 13. & AB, AC, are produced to F & G, thus as AF 22 is equal to the semi-
perimeter and the right line AG, & the angles AFD, AGD are right; & with the centre D, radius DF the arc GVF is described touching the lines produced to F and G. The same arc by necessity touches the right line CB at the point V: with CV and VB are equal by the construction to CG, BF. (For if CB cuts the periphery, GVF is larger than CG & BF: if it does not touch the periphery then it is less than CG & BF.) Then with the centre E, the circle NOR is inscribed to the given triangle. AN & AR are 11; BN & BO 9; while CO, CR, BV & BF are 2. And ENB, BFD are similar triangles, because the angles at N and F are right, & both NBO & OBF are equal two right angles by Prop.13, Book 1, Euclid. & NBE, FDB taken together are worth the half of two right angles: and therefore the angles FBD, NEB, likewise the angles NBE and BDF are equal. Therefore, EN, NB: BF, FD are in proportion, and the rectangles EN by FD and NB by BF, equal. But also FA, NA: DF, EN are in proportion; and as DF to EN, thus the rectangle DF, EN to the square EN; & therefore as FA to NA, thus the rectangle EN, FD to the square EN $^{11}$. 

Therefore if the logarithm FA 22 is taken away from the logarithms of NA 11 & rectangle NB,BF; the logarithm of the square EN is left: of which half is the logarithm of the radius EN. The same comes about if the complement of the first logarithm is taken, etc. As we showed in Chapter 15.

<table>
<thead>
<tr>
<th>Proportions</th>
<th>Logarithms</th>
</tr>
</thead>
<tbody>
<tr>
<td>AF - - - - - - - 22</td>
<td>Complementary arithmetic 8,65757,73192</td>
</tr>
<tr>
<td>AN - - - - - - - 11</td>
<td>1,04139,26852</td>
</tr>
<tr>
<td>Oblong 18</td>
<td>NB 9 0,95424,25094</td>
</tr>
<tr>
<td>BF 2</td>
<td>0,30102,99957</td>
</tr>
<tr>
<td>Square</td>
<td>EN 9 10,95424,25095 $^{[10 \rightarrow 0]}$</td>
</tr>
<tr>
<td>EN 3</td>
<td>0,47712,12547</td>
</tr>
</tbody>
</table>

Hence undoubtedly, 198 by the continued product from these remainders is equal to the product from the semi-perimeter 22 in the square of the radius EN 9 by Prop.19, Book 7, Euclid. And if the semi-perimeter 22 is multiplied with these, 4356 is the continued product from AF, AN, NB, BF, which is equal to the continued product from AF, AF, EN, EN, the product from the square of AF into the square of EN$^{12}$. And if a square is multiplied by a square, the product is a square: and the
square root produced is a factor from the same square. And therefore 66 the root of the continued product of the semi-perimeter and the subtracted lengths, is the product of the semi-perimeter into the radius of the circle: which is always equal to the area of the triangle.

4. The diameter of the circum-circle is found thus. The logarithm of the perpendicular from any vertex to the opposite side is taken away from the logarithms of the sides of the comprised angle, the remaining term is the logarithm of the diameter sought. For in any triangle ABC, the two sides AC, CB are the mean proportionals between the perpendicular CD, and the diameter of the circum-circle CE. Because the angles DAC, BEC are equal by Prop. 22, Book 3, & Prop. 14, Book 1, Euclid, & the angles CDB, CBE are right: therefore the remaining angles DCA, BCE are equal & triangles DCA, BCE are similar & therefore DC, CA: BC, CE are in proportion. Or if the interior perpendicular seems to be better suited, the proportionals will be FA, AB: AC, CE. [Using \( \triangle ABF \) and \( \triangle CAE \)].

\[
\begin{array}{ccc}
\text{Proportions} & \text{CD} & \text{12 Compl. Log} \\
\text{CA} & \text{13 Logarithm} & 8.92081,87539 \\
\text{CB} & \text{20 Logarithm} & 1,11394,33523 \\
\text{CE} & \text{2166666666} & (1)1,33579,21019 \\
\hline
\text{CE Diameter} & 21^{1/3}; \text{see page 37 concerning complementary arithmetic.} & \text{the whole sum.} \\
\end{array}
\]

5 If we wish to know any angle, the side opposite the same is taken away from the semi-perimeter: the remainder is the segment of one or other of the legs between the same angle & the point in which the circumference of the inscribed circle touches the line. But this segment is to the radius of the inscribed circle, as the radius of the circle to the tangent of half the
angle sought. As in triangle ABC: by taking the line CA 13 from the semi-perimeter 22, BG or BP 9 is left. I assert.

\[
\begin{array}{c|c|c|c}
\text{Proport.-ions} & \text{BF} & 9\text{ compl. arith.} & \text{Logarithms} \\
\text{Radius} & 100000000000 & 10000000000 & \\
\text{Tangent} & \text{GBF} & 333333333333 & 1\text{9522878744} & 18:26:06 \\
\end{array}
\]

Therefore the angle GBF is 18:26:06. & the whole angle GBA 36:52:12. By the same method the remaining angles at C and A can be found.

\[
\begin{array}{c|c|c|c}
\text{Proport.-ions} & \text{CG} & 11\text{ compl. arith.} & \text{Logarithms} \\
\text{Radius} & 100000000000 & 10000000000 & \\
\text{Tangent} & \text{GCF} & 272727272727 & 1\text{9435728569} & 15:15:18 \\
\end{array}
\]

We can also find any angle otherwise, through the altitude of the triangle found, by section 4 of this chapter, [Figure 18-8]. It is by means of the perpendicular from any angle to the opposite side. For as the common leg of the vertical angle & the angle sought, to the perpendicular of the same vertical angle ; so the total sine to the sine of the angle sought. For in the same triangle, if the angle CBA is required: they are

\[
\begin{array}{c|c|c|c}
\text{Proport.-ion} & \text{CB common leg} & 20\text{ compl. arith.} & \text{Logarithms} \\
\text{perpendicular} & 12 & 1079181264 & \\
\text{Sine Total} & 10000000000 & 10000000000 & \\
\text{Sine CBA} & 600000000 & 1\text{9778151250} & 36:52:12 \\
\end{array}
\]

Or, with the base BD made continuous, (in proportion) to the side DC: so the same radius of the circle, to the tangent of the angle required DBC.
Or, with the base BD made continuous, (in proportion) to the line BC: so the radius of the circle, to the secant of the angle DBC sought.

Or, for seeking the opposite angle: as the diameter CE of the circumscribed circle to CA, so the radius of the in-circle, to the sine of the angle sought.

But since any angle and its complement of two right angles have the same sine, as 63:0 & 117:0, then in order that we can be certain that the angle sought is acute or obtuse, it is noted that the two angles of any triangle opposite the smaller sides are always acute: but the third angle, that the largest side subtends, can sometimes be acute, or sometimes right or obtuse. Therefore in order that we know what kind of angle the largest side is placed opposite, the largest sum and difference is taken of the sides, and the mean (or the smallest) logarithm of these: and if the half of these logarithms is equal to the logarithm of the third side, the maximum angle of the opposite side is right. But if the logarithm of the third side is more than half, the angle will be acute: but if less, obtuse. [A result that we would now relate to the cosine rule]. As in the triangles CAB, DAB, EAB, [Fig. 18-10] let the sides CA, DA, EA, be of 10 parts: AB, 8: BC, 7: BD, 6: BF, 5: GB is the sum of
the sides from the angle A, 18: BF the difference of the same sides 2, half the sum of the logarithms 077815125 is equal to the logarithm of the side BD, & therefore BD is by Ch. 17 the mean proportional between GB & BF, & the angle ABD is right, by Prop. 13, Book 6, Euclid. But the logarithm of the line CB is more than half the sum of the logarithms, & ABC is acute. On the other hand, ABE is obtuse, & the logarithm of the line

BE is less than half of the same.

<table>
<thead>
<tr>
<th>sum of the sides GB</th>
<th>18</th>
<th>Logarithms</th>
</tr>
</thead>
<tbody>
<tr>
<td>difference of the sides BF</td>
<td>2</td>
<td>0,30102,999</td>
</tr>
<tr>
<td>sum of the logarithms</td>
<td></td>
<td>1,55630,250</td>
</tr>
<tr>
<td>half of their sum</td>
<td></td>
<td>0,77815,125</td>
</tr>
<tr>
<td>BC</td>
<td>7</td>
<td>0,84509,804</td>
</tr>
<tr>
<td>BD</td>
<td>6</td>
<td>0,77815,125</td>
</tr>
<tr>
<td>BE</td>
<td>4</td>
<td>0,69897,000</td>
</tr>
</tbody>
</table>

[Table 18-13]

These sines, tangents and secants, with the aid of which the angles are found, are found in the works of Palatine, or elsewhere from a Canon of Triangles as you please. Or if we are satisfied with the logarithms themselves, the Canon of Triangles by Edm. Gunther recently published by us will make amends abundantly; or that which I have prepared, [which was posthumously published as the Trigonometria Britannica] & sometime I hope to be given the light by me; the same previous book will be better in both ways, both for the sines and tangents themselves, and for the logarithms of the same. However, the whole sine [i.e. the sine of 90°] is understood to be written with eleven places, of which the logarithm is 10 [comp. arith.] in the place of the characteristic, as we have shown in Ch. 4. Thus the rest of the sines are set out all the way to 5° 45 minutes. & the tangents from 45°, 0 min to 5°, 43 min, have 9 for the characteristic. Beyond this all the way to 0°, 35 min, the characteristic of these is 8. But for the secants as far as 84°, 16 min., & the tangents from 45°, 0 min to 84°, 18 min, the characteristic is 10. And beyond those limits, nearly as far as the end of the quadrant, 11.
§18.3.  

Notes on Chapter 18.

1  See Heath, Vol. 1, *Euclid's Elements*, e.g. (Dover) p. 104, for notes on the commentaries on *The Elements* by these authors mentioned by Briggs, and other early writers.

2  Briggs sets the lengths in continued proportion to be $a$, $b$, $c$, where $a/b = b/c$. The mean proportional is $b$, while $a$ and $c$ are the extremes. The triangle has sides of length $(c - a)/2$, $\sqrt{ac}$, and $(a + c)/2$; by writing $c$ as $b^2/a$ and removing fractions, it is an easy matter to show that this set of numbers is equivalent to the Pythagorean Triplet: $(b^2 - a^2, 2ab, b^2 + a^2)$, with $b > a$.

We may also note that Briggs favours using the Theorem of Pythagoras in the proportional form: 

$$(c - a)/b = b/(c + a),$$

where $(a, b, c)$ is a Pythagorean Triplet.

3  The triplet is now $a + b$, $b + c/2$, $a + b + c/2$, where $a$, $b$, $c$ are as above: in a similar manner, this set of lengths of sides is equivalent to the triplet $((a + b)^2 - a^2, 2a(a + b), (a + b)^2 + a^2)$.

4  The triplet is now $b - a$, $c/2 - b$, $a + c/2 - b$: This triplet being equivalent to the triplet: $(b - a)^2 - a^2, 2a(b - a), (b - a)^2 + a^2)$, where $b > a$.

There is hence nothing extra gained from propositions 2 and 3: there is obviously an indefinite number of equivalent ways of stating the Pythagorean Triplet. For, by starting from $(m^2 - n^2, 2mn, m^2 + n^2)$, $m > n$ as the basic triplet, and inserting simple sums and differences of $a$ and $b$, other triplets can be produced. E.g. letting $m = a + b$ and $n = b$ gives eventually $b + c$, $b + a/2$, $a/2 + b + c$, etc.

5  The proportionals are allowed to increase or decrease, so that 32, 24, 18 is used to construct the 3rd and 5th triplets from the second set of ratios, while 18, 12, and 8 are used in the first to generate the extra triangles.

6  From the half-perimeter $s$, where $2s = (a + b + c)$ is taken the length of side $a$ to give the factor $(s - a)$: the square root of the factors by continuation from the half perimeter and the subtractions of the 3 sides will be the area of the triangle. This is Heron’s formula $A = \sqrt{s(s - a)(s - b)(s - c)}$. 

This follows readily, Fig. 18-4, from \(OC\cdot CA = (CM + OM)(CM - OM)\)
\[= (BC^2 - BM^2) - (AB^2 - BM^2) = BC^2 - AB^2, \text{ etc.}\]

In modern standard notation, where \(a, b,\) and \(c\) are the lengths of the sides of the triangle opposite the angles \(A, B,\) and \(C,\) if the lengths of the tangents from \(A, B,\) and \(C\) to the in-circle are \(x, y,\) and \(z,\) where \(2s = a + b + c\) then \(s = x + y + z\) and \(x = s - a, y = s - b,\) and \(z = s - c.\)

According to Euclid, \(BA\cdot BE = BO\cdot BH\) (Prop. 37, Book 3), where \(BE\) is the unknown. The other proposition (Prop. 16, Book 6), relating the products of the diagonals to the sum of the products of the opposite sides of a cyclic quadrilateral, does not seem to have been used here.

Briggs makes use of the standard result \(A = rs = \sqrt[s]{(s - a)(s - b)(s - c)}.\)

Briggs is commenting that the similar triangles \(NEB\) and \(FBD\) give \(NE/BF = NB/FD,\)
or \(EN/FD = BF/NB,\) and as \(NE\) and \(FD\) are parallel, the triangles \(ANE\) and \(AFD\) are also similar, with equal ratios of sides. Finally, \(FA/NA = FD/EN = [BF\cdot NB/NE]/NE = (BF\cdot NB)/EN^2\)
\(= DF\cdot EN/EN^2).\) The reason for doing this is to produce via geometry the lengths \(FA, NA, NB,\) and \(BF\) that correspond to \(s\) and the differences \((s - a), (s - b),\) and \((s - c)\) respectively, used to find \(EN = r\) from \(s/(s - a) = (s - b)(s - c)/r^2\)

i.e. the factors are \(s - a, s - b,\) and \(s - c,\) where \((a, b, c) = (11, 13, 20); s = 22,\) and
\((x, y, z) = (11, 9, 2)\) are the lengths of the tangents; \(11.9.2\) being the product of half the perimeter into the square of the radius \(EN: r^2 s = (s - a)(s - b)(s - c),\) by Prop. 19, Book 7, Euclid.

The tangent was defined at the time as the ratio of \(z\) to \(R,\) where \(R = 10000000000\) in the case of Briggs' tables.

The sine was the ratio of \(x\) to \(R,\) while the secant was defined by the ratio \(R\) to \(y.\)

i.e., from the sine rule, \(BC/CD\) (or \(AB/AF\)) = \(\sin(\pi/2)/\sin(CBA)\) in Figure 18-8.
§18.4. Caput XVIII. [p. 46.]

_Datis trianguli plani lateribus: invenire Area, Altitudinem, Diametros circulorum inscripti & circumscripti, & quemlibet angulum._

Antequam hanc propositionem aggrederiam, liceat pausa de Triangulo rectangulo praefari; ut eius ope possimus, de veritate & certitudine eorum quae tradentur rectius iudicare.

Proclus ad quadragesimam septimam propositionem primi libri Euclidis, duos modos tradidit, quibus possimus, _dato altero crure circa rectam angulum, invenire crus reliquum & hypotenusam._ Horum unus tribuitur Platoni, alter Pythagorae, eosdem habemus apud Ramum tanquam consectaria 5.e.12.lib. qui sine distinctione numeri paris & impares, generalius proponi potuissent. _Si dimidius numeri pro crure, &c._ Et, _Si quadratus numeri pro crure primo dati, &c._ Ego vero tres tradam generales propositiones (ex quarum prima isti modi fluxerunt) per quas poterimus _dato quolibet latere circa rectum, invenire quolibet triangula rectangula._ Hae sunt huiusmodi.

_Si tres recta sunt continue proportionales._

1. _Crura; semissis differentia extremarum, & media; hypotenusa vero erit semissis summa extremarum._
2. _Crura; prima & secunda, secunda & semissis tertia; hypotenuse vero prima & secunda & semissis tertia._
3. _Crura; Differentia prima & secunda, Differentia secunda & semissis tertia; hypotenusa vero erit Differentia secunda & aggregati e prima & semisse tertia._

Ut sunto continue proportionales 8, 12, 18. per primam erunt latera quaesita 5, 12, 13; per secundum, 20, 21, & 29, & 30, 16, 34; per tertiam 4, 3, 5 & 6, 8, 10.

Continue proportionales 18, 24, 32, triangula rectangula erunt 7.24.25|42.40.58|56.33.65|6.8.10|8.15.17.

Harum prima est consectarium huius sequentis proportionis. In triangulo rectangulo ABC: crus utrumlibet BC est medium proportionale inter BD summam, & BE differentiam hypotenuseae AC, & reliquum cruris AB & contra. Secunda & tertia demonstrantur per primam, atque haec omnia, tam in lineis rectis, quam in numeris, linearum mensuras exprimentibus, ostendi poterunt facilimente. Numeri tres continue proportionales per divisionem inveniuntur. & per Logar.3.ax.c.2. Si numeros quilibet quadratum alterius divisorit: latus divisi est medium proportionale inter divisorem & quotum.

_Esto datum crus anguli recti 12, eius quadratum144. Sunto divisores 2, 3, 4, 5, 6, 7, 8, 9, 10, 11._

Quoti erunt 72,48, 36, 288, 24, 20⁴/₇, 18, 16, 144, 13⁹/₁₁. ut hic vides *Medius ubique est 12,* extremi sunt divisores &quoti. Columna _D_ sunt semisses summarum, columna _S_ semisses differentiarum, haec columnae dant nobis 19 triangula rectangula: quorum crus datum est 12, reliquum crur habetur in columna _S_, hypotenusa vero e regione in columna _D_. ut _A_ 12, 16, 20; _B_ 12, 9, 15; _C_ 12, 5, 13; _E_ 12, 3₅, 1₅₂₅."
Et si duo huiusmodi triangula rectangula ita locentur, ut id sit crus comune utrinque; habeimus triangulum obliquangulum datorum laterum, cuius area erit rationalis: \( BAE, CAE \) constituitur \( BAC, BAD \); eodem modo, \( OPX, OXN \) dabunt \( PON, OPQ \).

1. Aream trianguli datorum laterum inveniemus per 9.c.12.lib.Rami. Si de dimidio collectorum laterum latera sigillatim subducantur: latus continue facti e dimidio & reliquis erit area trianguli. Sunto tria latera 20.11.13 perimeter, seu latera collecta 44, semiperimeter 22, numeri reliquii subductis lateribus e semiperimetro 2.11.9 continue factus provenit ex horum quatuor numerorum multiplicatione, huius autem facti Logarithmus aequatur Logarithmis facientium per 2.ax.2.c sumantur igitur Logarithmi qui semiperimetro & numeris reliquis debentur. horum summa erit Logarithmi qui semiperimetro & numeris reliquis debentur. horum summa erit Logarithmus continue facti; & semissis summa erit Logarithmus lateris seu area quaesitae. Ut

2. Invento areae Logarithmo, poterimus altitudinem eiusdem trianguli vel perpendicularem ab angelo quolibet in latus oppositum invenire: si e Logarithmo areae auferatur Logarithmus dimidiati

[p.48] Erit igitur area dati triangulii 66. Eandem area quantitatem inveniemus si altitudinem trianguli \( AE \) 12 multiplicemus per dimidiatam \( 5^1/2 \), quia rectangulum e tota base 7 altitudine est duplum trianguli per 41.p.1.lib.Euclid.

2. Invento areae Logarithmo, poterimus altitudinem eiusdem trianguli vel perpendicularem ab angelo quolibet in latus oppositum invenire: si e Logarithmo areae auferatur Logarithmus dimidiati
lateris, in quod incidit perpendicularis ab angulo opposito, reliquis erit Logarithmus altitudinis per 2.ax.2.cap.

Est enim aequalis facto e base dimidiata in altitudinem, & idcirco area Logarithmus, dimidiatae basis & altitudinis Logarithmus equabitur.

Logarithmus.

\[
\begin{array}{ccc}
\text{areae} & 66 & 1,181954,39355 \\
\text{dimidiatae} 6^{1/2} & 0,81291,33566 \\
\text{altitudinis} 10^{-2/13} & 1,00663,05789 & 10^{1538461}
\end{array}
\]

Ut ab angulo B, in basim continuatam, cadat perpendicularis BM, eius Logarithmus erit 1,00663,05789, ipsa vero BM erit 10^{1538461} proxime, vel 10^{2/13}.

Eandem altitudinem Geometrice inveniemus sic. Sumatur differentia quadratorum e lateribus BA, BC, comprehendentibus angulum ABC, a quo perpendicularis cadit in basim CA continuatam si opus fuerit ut hic, hanc differentiam 279 dividat basis CA 13, quotus 21^{6/13} erit CO, summa segmentorum basis ab angulis O, C, vel A, C ad punctum M, in quod incidit perpendicularis: ab hac summa CO, auferatur data basis CA 13 quae est differentia segmentorum, restabit AO 8^{6/13} dupla minoris segmenti AM 4^{3/13}: deinde ablato quadrato AM 17^{152/169} e quadrate AB 121 (vel quad. CM e quad. CB) restabit 10^{3/17} , quadratum rectae BM quaesitae, 10^{2/13}.


\[
\begin{array}{ccc}
\text{areae inventae Logarithmus} & 1,181954,39355 & 66 \\
\text{found semiperimetro} & 1,34242,26808 & 22 \\
\text{radij circuli triangulo inscripti} & 0,47712,12547
\end{array}
\]

Aio radium inscripti circuli FG esse 3. Est enim factus a radio in semiperimetre aequalis areae trianguh dati, quia ductis rectis a centro F ad angulos ACB, secabitur datum triangulum, in tria triangula: quorum altitudines aequantur radio circuli, bases autem sunt latera dati trianguh. & communis altitude FG ducta in semisses singulorum laterum, id est in semiperimetre, dabt areas triangulorum componentium AFC, AFB, AFC. id est aream dati trianguh.

Geometriche, radium sic inveniemus: ducatur recta CD perpendicularis lateri opposito BA, & quaeruntur BD, 16, & CD, 12. deinde ducatur BFE bisecans angulum CBD. centrum F in bisecante. per 4.p.4.lib.Eucl. tertio, [p.49.] quaerenda est DE, quae per 3.p.6.lib.Eucl. ita se habet ad EC, ut DB ad BC. & idcirco ut summa laterum CB, BD 36, ad BD 16; sic CD 12, ad ED 5^{1/3}. Quarto, quaeritur recta BP ab angulo B, ad punctum contactus P; in quod cadit radius FP, perpendicularis lateri BA. est autem BP is 9. nam in omnibus triangulis si latus quodlibet AC 13 auferatur e semiperimetro 22, restabit 9 distantia anguli oppositi B a punctus contactuum GP. Tandum per proportionis regulam quaero PF. ut BD 16 ad DE 5^{1/3}; sic BP 9 ad PF 3.

Poterit etiam radius inscripti circuli inveniri per Logarithmos, sic. Ablatis lateribus datis e semiperimetro, sumantur Logarithmi reliquorum; & e summa auferatur Logarithmus semiperimetri: semissis reliquii erit Logarithmus radii quaesitis.


Idcirco si Logarithmus FA 22 auferatur e Logarithmis NA 11 & rectanguli NB,BF; restabit Logarithmus quadrati EN : cuius semissis erit Logarithmus radii EN. Idem eveniet si sumatur primi Logarithmi complementum &c. ut cap.15. ostendimus.

\[
\begin{array}{c|c|c}
\text{proport.} & \text{AF} & \text{Compl. Arithm.} \\
\hline
\text{AN} & 11 & 8,65757,73192 \\
\hline
\text{Oblongum} & 18 & 1,04139,26852 \\
\text{NB} & 9 & 0,95424,25094 \\
\text{BF} & 2 & 0,30102,99957 \\
\hline
\text{Quadratum} & 9 & 10,95424,25095 [10 → 0] \\
\text{EN} & 3 & 0,47712,12547
\end{array}
\]

Hinc manifestum est, 198 continue factum a reliquis 11.9.2 aequari facto a semiperimetro 22 in quadratum radii EN 9. per 19.p.7.l.Eucl. 7 si semiperimeter 22 hos aequales factos multiplicit, erit 4356 continue factus ab AF, AN, NB, BF, aequalis continue facto ab AF, AF, EN, EN, vel facto a quadrato AF in quadratum EN. & si quadratus quadratum multiplicit factus erit quadratus: & latus facti erit factus a lateribus eorumdem quadratorem. & idcirco 66 latus continue facti a semiperimetro & reliquis, erit factus a semiperimetro in radium circuli inscripti: qui semper aequat areae trianguli.

magis commoda videatur, erunt FA, AB: AC, CE proportionales.

5. Si angulum quemlibet scire velimus, auferatur latus eidem oppositum e semiperimetro: reliquum erit segmentum alterutrius cruris inter angulum eundem & punctum in quo peripheria circuli inscripti tangit latus. Erit autem hoc segmentum ad radium circuli inscripti, ut radius circuli ad tangentem semissis anguli quaesiti. ut in triangulo ABC : ablato latere CA 13 e semiperimetro 22, restabit BG vel BP 9. aio

![Diagram](image)


Poterimus etiam angulum quemlibet aliter invenire, per altitudinem trianguli inventam, per 2.sect. huius Cap. id est per perpendiculararem ab angulo quolibet in latus oppositum. Nam ut crus commune anguli verticalis & anguli quaesiti, ad perpendiculararem ab eodem angulo verticali; sic sinus totus ad sinum anguli quaesiti. Ut in eodem triangulo, si quaseratur angulus CBA: erunt pro-.port. CB crus commone 20 compl. arith. 8698970004
CD perpendicularis 12 1079181246
Sinus Totus 1000000000 1000000000
Sinus CBA 6000000000 19778151250 36:52:12

Vel, ut BD basis continuata, ad latus BC: sic radius circuli, ad secantem anguli quaesiti DBC.
Arithmetica Logarithmica

Vel, ut BD basis continuata, ad latus BC: sic radius circuli, ad secantem anguli quaesiti DBC.

\[
\begin{array}{l|ll|l}
\text{pro-} & \text{BD} & 16 & \text{compl. arith.} & 87958800173 \\
\text{port.} & \text{CD} & 12 & & 10791812460 \\
\text{Radius} & 10000000000 & & 100000000000 \\
\text{Tangens} & DBC & 7500000000 & 198750612633 & 36:52:12 \\
\end{array}
\]


\[
\begin{array}{l|ll|l}
\text{summa laterum} & GB & 18 & 1,25527,251 \\
\text{differentia laterum BF} & 2 & 0,30102,999 \\
\text{summa logarithrorum} & & 1,55630,250 \\
\text{semissis} & BC & 7 & 0,77815,125 \\
& BD & 6 & 0,77815,125 \\
& BE & 4 & 0,69897,000 \\
\end{array}
\]

Sinus illi & tangentes secantesque, quorum ope anguli inveniuntur, quaerendi sunt in Opere Palatino, vel alio qualibet Canone Triangulorum.vel si ipsis Logarithmis contenti sinus, Canon Triangulorum a M. Edm. Guntero nuper editus nobis abunde satisfaciet, vel ille quem paratum habeo, & aliquando me in lucem daturum spemo, utroque modo idem praestabit. sive per ipsos Sinus & Tangentes, sive per eorum Logarithmos. Cum autem sinus Totus undeim notis scribi intelligitur, eius Logarithmus loco characteristicae habebit 10, ut Capite 4 ostendimus. Sic reliqui
sinus usque ad sinum gr. 5. 45\(^{\prime}\). & Tangentes a gradibus 45. 0 ad 5.43, habent 9 pro
characteristica.infra vero usque ad 0.35, eorum characteristica est 8. secantium autem usque ad
84.16, & Tangentium a grad 45.0 ad 84.18, characteristica est 10. atque ultra hos terminos, sere
usque ad finem Quadrantis, 11.