§24.1. *Synopsis: Chapter Twenty -Four.*

Before proving this theorem, four Lemmas are presented:

The First Lemma L1 shows how to find the lengths of the diagonals of a cyclic quadrilateral given the lengths of the sides, and the ratio of the lengths of the diagonals is first found :

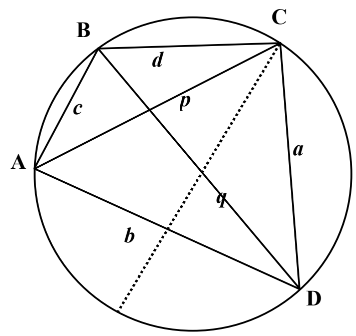


Figure 24-1

If a cyclic quadrilateral has sides of length a, b, c, d in that order; and the diagonals p and q intersecting within the circle have lengths p, q, as shown in Fig 24-1, then $p/q = (ad + bc)/(ab + dc)$.

A well-known theorem of Ptolemy states that the product of the diagonals $pq = (ac + bd)$: *hence,* $p^2 = (ac + bd)(ad + bc)/(ab + dc)$; *while* $q^2 = (ac + bd)(ab + dc)/(ad + bc)$, *from which the lengths of p and q follow. Lemmas 2, 3, and 4 are concerned with solving proportionalities involved with the main theorem which then follows :*

L2. Equality of sums or differences of proportions with ratios:

if $a/b = c/d$, *then* $(b \pm a)/(d \pm c) = a/c$; *e.g.* 1^{st} *difference*/3^{rd} *difference* = 1^{st} *proportion*/3^{rd} *proportion.*

L3: Finding the proportions $a/b = c/d$ *from the given* 1^{st}, 2^{nd}, *and* 3^{rd} *differences* $b - a$, $c - b$, *and* $d - c$.

L4: Finding the proportions $a/b = c/d$ *from the given* 3^{rd} *and* 4^{th} *differences* $c - b$, $d - a$ *and the given* 3^{rd} *to* 1^{st} *ratio,* c/a.

The Main Theorem: To form a cyclic quadrilateral from four given straight lines, of which three taken together are greater in length than the remaining one.

§24.2. ***Chapter Twenty Four.***[p.63.]

A cyclic quadrilateral may be constructed from four given straight lines, of which three taken together are longer than the one remaining .

Let the sides of the quadrilateral of length 2, 3, 4, and 5 be given ; and in order that we can specify the positions of the sides, let it be in some ordered manner: AB, 2; BC, 3; CD, 4; DA, 5. But first the diagonals AC, BD are found, which intersect in the point E. This First Lemma is shown as it shall prove useful.

First Lemma.

Any diagonal segment AE is to the remaining diagonal BD, as the product from the sides with the common ending AB, AD 10 of the same segment to 26, the sum of the factors from the same sides AB, 2 and AD, 5 with the common ending of the sides BC, 3 and DC, 4 [i.e. AE/BD = AB.AD/(AB.BC + AD.DC). See Fig. 23-2][1].

For the opposite triangles AEB, DEC are similar, and likewise BEC, AED. Therefore AE is to BE, as 5 to 3, and AE is to ED, as 2 to 4. And *if the same numbers are multiplied, the products multiplied are proportional.* 5 & 3 by 2 make 10 & 6 , & 2 & 4 by 5 make 10 & 20.
[p.64.] [Thus, AE/BE = 5/3 = 10/6; AE/ED = 2/4 = 10/20]. Therefore AE is to BD, as the product 10 from the sides AB, AD adjoined to the segment AE: to the sum of the factors from the sides joining the ends of the same diagonal line, 26, AB by BC, 6: and AD by DC, 20 [Thus, (BE + ED)/AE = BD/AE = 26/10; or AE/BD = 10/26].

For the same reason, CE is to BD, as 12 to 26 ; & the total of AC to BD, as 22 to 26.

[*i.e.* EC/BD = 12/26; and (AE + EC)/BD = 22/26.]

AE, EB	5, 3	10, 6 prop.
AE, ED	2, 4	10, 20 prop.
AE, BD		10, 26 prop.
CE, EB	4, 2	12, 6 prop.
CE, ED	3, 5	12, 20 prop.
CE, BD		12, 26 prop.
AE + EC, ED		10 + 12, 26 prop.

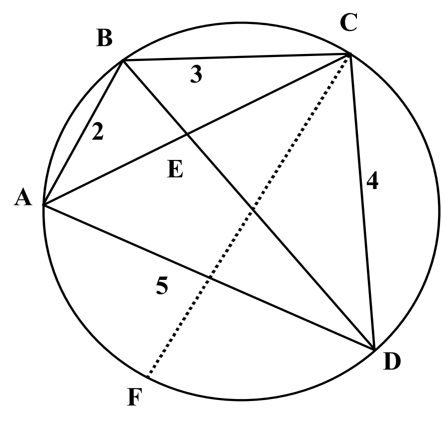


Figure 24-2.

Hence these consequences are deduced:
The diagonals are sums of products from adjoining proportional sides.

AB by AD,	10	BA by BC,	6
CB by CD,	12	DA by DC,	20
AC	22	BD	26
		sum of proportionals	

And with the products from the opposite sides BC by AD, 15: & AB by CD, 8: taken at the same time equal to the product of the diagonals; (by prop. 54, book 1, Pitiscus, & Ptolemy, book 1, Mathematicae Syntaxis).[2] With the numbers given, the product of AC by BD is 23. But given the rectangle AC,BD 23; & with the ratio of the sides AC to BD as 11 to 13: the sides themselves are given: AC, $\ell.19\,{}^{6}/_{13}$; BD, $\ell.27\,{}^{2}/_{11}$.

For the rectangle formed from the sides is the mean proportional between the square of the sides. As AC to BD, thus the square of AC, to AC by BD: & AC by BD, to the square[3] of BD [$i.e. \frac{AC}{BD} = \frac{AC^2}{AC.BD} = \frac{AC.BD}{BD^2}$: this particular form is adopted to find AC^2 as both *AC.BD* and *AC/BD* are known; and similarly for BD^2].

proportions	11	proportions	Sq. AC	
	13		AC by BD, 23	
	AC		Sq. BD	
	BD			
				Logarithms
proportions	11	proportions	13 comp. ar.	888605665
	13		11	104139269
	23		23	136172784
	$27\,{}^{2}/_{11}$ Sq. BD		$19\,{}^{6}/_{13}$ Sq. AC	128917718

[Table 24-2]

Therefore there are the triangles ABC, ADC of given sides: & with any triangle ABC to be inscribed with the given sides, the diameter of the circle CF is given by section 4, Chapter 18, which for this circle is $\ell.27\,{}^{42}/_{120}$. See Fr. Vièta at the end of *Pseudomesolabii* & Johannes Praetorius *Problems*.

OTHERWISE, In order that this may happen, [i.e. the result established] three other Lemmas are to be shown.

Second Lemma

If the lengths of four lines are to be in proportion, then : *The differences (or the sum) of the first & second proportions, is to the difference (or sum) of the third & fourth proportions, as the first to the third proportion; & conversely*, by Prop. 19, Book 5, *Euclid* [4].

	2			7		Difference
4		6	14		21	Proportionals
	10			35		Sum

[Table 24-3]

Third Lemma.

[p.65.] With four proportionalities: *If the difference of the first and second proportions, the second & third, and the third & fourth are given; then the proportions themselves are given.*

Let the proportionalities be A, B: C, D; & the differences 2, 8, 7. The difference of A & C is 10, to equal the sum of the differences 2, 8 given: & the ratio of A to C, as 2 is to 7, by the second Lemma *. [Thus, C - A = 10, and A/C = 2/7 = 4/14]. The difference 5 is taken between 2 & 7. For this itself has, to 2 (or 7), as the difference 10 between A & C, to A 4 (or C 14) [5].

	2		8		7		Given Differences		5			10		Given Differences
A,		B:		C,		D	Proportionals	2		7:	A,		C	Proportionals
4		6		14		21								

[Table 24-4]

Therefore the proportions are A, 4; B, 6; C, 14; D, 21.

*To be noted:[A modification to *L*2] *If the proportionalities from the first to the last are increasing or decreasing, the sum of the first and second differences is taken* [i.e. the differences are added as above]. *Otherwise* [in order to avoid negative signs], *if the second is increasing and the third decreasing, or vice-versa: the smaller difference is taken from the larger, & the part remaining is the difference between the first and the third. As*

	5		11		2			3	6		Given Differences	3 , 2: 6, C. 4. Prop.
A,		B.		C,		D	5, 2:		A, C		Proportionals	3 , 5: 6, A. 10. Prop.
10		15		4		6						

[Table 24-5]

Fourth Lemma.

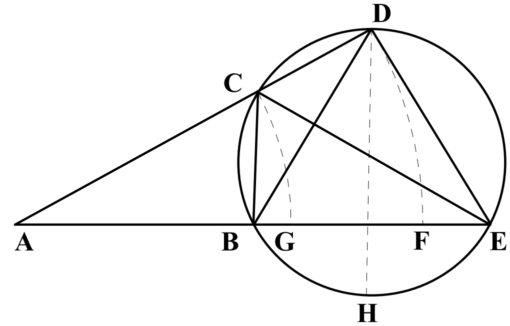
With four proportionalities: *If the differences of the proportions are given of the first and fourth, and of the second and third, together with the ratio of the first to the third proportion, then the proportions themselves are found.*

Let the differences be given: of the first and the fourth 23, of the second and the third 12, & let the ratio of the first to the third be as 2 to 9. [The remaining differences between neighbouring proportions are found initially.] First of all, the difference of the mean proportions * [*i.e.* C - B] 12 is taken from the difference of the extremes [*i.e.* D - A] 23; there remains 11, the sum of the difference of the first and the second, and of the third and fourth proportionals [i.e. (B-A) + (D-C) = (D-A) - (C-B)] . Then the terms of the given ratio between the first and the third are added : the sum of the terms is to the preceding term [*i.e.* 11]; as the sum of the differences to the preceding difference ; [from (B-A)/A = (D-C)/C we have (B-A)/(D-C) = A/C and subsequently ((B-A)+(D-C))/(D-C) = (A+C)/C : hence 11/(D-C) = 11/9 gives (D-C) = 9, and (B-A) = 2][6]. Therefore the first, second, third, and fourth differences are now given. But the ratio of the second and the third proportions is given already. Therefore, by the third Lemma, the first of the proportions and the rest can be worked out. [The proportional numbers are A 4; B 6: C 18; D 27].

**With everything increasing as in the diagram, or decreasing.*

[*Finally, The Main Theorem: To construct a cyclic quadrilateral from four given straight lines, of which three taken together are greater in length than the remaining one.*]

Let the trapezium of unequal sides be BCDE: BC, 6; CD, 9; DE, 12; and EB, 15; of which the opposite sides CD, BE are continued, meeting in the point A. And the lines AB, AC: AD,AE are in proportion: by Prop.36, Book 3, *Euclid*. And the triangles ABC and



[Figure 24-3]

ADE are similar: by Prop.6, Book 6, *Euclid*. [Briggs now sets about finding all the differences associated with the ratio AB/AC = AD/AE, which is equivalent to *a/b = c/d*]. And therefore BC, 6, and DE, 12 are proportional to AB as the first and AD as the third proportions, [i.e. AB/AD = BC/DE = 6/12]. Also CD, 9, the difference of the second and third proportions [2nd difference], and BE,15 the difference of the first and fourth proportions[4th difference], are given.

From these given proportions and differences, the first proportion AB and the second proportion AC can be found: GF equal to the line CD, 9 is taken from the line BE, 15; then BG & FE are left: which together make 6; and BG is the difference between the first & the second proportions [1st difference], & FE the difference between the third and fourth proportions [3rd difference: All the differences (in order 1 - 4) BG, CD, FE, and BE are now shown in Fig. 24-3; while only CD, 12 and BE, 15 are yet known].

Therefore AB, AD: BG, FE are proportionals, by the first Lemma. That is, as I have shown before BC, DE : BG, FE are proportionals:

[for (AC - AB)/AB = (AE - AD)/AD, i.e. BG/AB = FE/AD, or BG/FE = AB/AD = BC/DE.]

And by the second Lemma, as the sum BC, DE 18; is to BC, 6: so the sum BG, FE, 6, is to BG, 2 [for, (BC + DE)/BC = (BG + FE)/BG, or 18/6 = 6/BG giving BG = 2].

Therefore with the first difference BG, 2 given; and the second and the third differences CD, 9 , FE, 4. By the second Lemma, as the ratio of the first to the third proportions AB to AD is as 6 to 12, or as BC to DE, the difference of the larger term is as 6 to 12, thus with the difference 11 [= BG + CD] of the first and third proportions AB, AD; to AD the third proportion, 22. [*i.e*. AB/AD = BC/DE = 6/12; hence (AD - AB)/AD = (DE - BC)/DE, or BF/AD = 6/12, i.e. (CD + BG)/AD = 6/12, or 11/AD = 6/12, giving AD, 22 and by the addition of FE, 4, AE = 26]. Therefore the proportions are AB, 11; AC, 13; AD, 22; AE, 26.

Now the diagonals BD, CE are found thus [from Lemma one]. The products of the opposite inscribed sides BC by DE, 72 & CD by BE, 135: are taken together to equal to the product of the diagonals, by Prop. 54, Book 1, *Pitiscus*. Therefore the product of the diagonals is 207. But the diagonals themselves are proportional to the lines AB, 11: AC, 13. Because the triangles ABD and ACE are equi-angular and similar; & therefore the [known] rectangle BD, CE is to the square BD, as 13 to 11: or, as AC to AB. BD therefore is $\ell.175\ ^{2}/_{13}$, & CE, $\ell.\ 244\ ^{7}/_{11}$. CBD is therefore the triangle of given sides 6, 9, $\ell.175\ ^{2}/_{13}$. And the perpendicular from the angle B to the side AC, $\ell.25\ ^{95}/_{169}$. So the diameter of the circumscribed circle, DH, is $\ell.\ 246\ ^{27}/_{40}$ [By the method prescribed in Chapter 18, section 4].

§24.3. *Notes on Chapter Twenty Four.*

[1] In terms of ratios, this lemma becomes:

AE/BD = (AB.AD)/(AB.BC + AD.DC).

Or, on inverting: BD/AE = BC/AD + DC/AB

For, from the similar triangles:

AB/CD = AE/ED = BE/BC; BC/AD = BE/AE = EC/ED; hence,

AE/ED + EC/ED = AC/ED = AB/CD + BC/AD , gives the ratio for one diagonal;

while BE/AE + ED/AE = BD/AE = BC/AD + CD/AB gives the ratio for the other diagonal: and the required form of the proposition on inverting.

These results can be easily arranged to give Briggs' First Lemma written in a more concise notation: If a cyclic quadrilateral has sides of length *a, b, c, d* in that order; and the diagonals intersecting within the circle have lengths *p*, *q*, then $p/q = (ad + bc)/(ab + dc)$.

[2] The second theorem quoted becomes $pq = (ac + bd)$.

[3] That is, AC.AC: AC.BD :: AC.BD: BD.BD; or $AC^2/23 = AC/BD = {}^{11}/_{13}$, as AC.BD = 23, giving $AC^2 = 23 \times {}^{11}/_{13} = 19\ ^{6}/_{13}$; similarly $BD^2 = 23 \times {}^{13}/_{11} = 27^{2}/_{11}$.

[4] E.g., if $a/b = c/d$, then $(b \pm a)/a = (d \pm c)/c$ with several variations possible, where the quantities either all increase or decrease.

We may remember the second Lemma by the handy rule:- *first* sum/difference: *third* sum/difference as *first* proportional: *third* proportional. Where the latter *first* and *third* can be replaced by *second* and *fourth*.

[5] These rules can be justified algebraically: Briggs, however, has shortened the procedure considerably, and treats proportionals and their differences on the same footing, according to the second Lemma, which he applies to the proportional 2 : 7 :: A : C, where the differences are taken as 5 and C − A = 10. Thus immediately 2/5 = A/10, and 7/5 = C/10. B and D are then found from the differences.

[6] In modern notation, if $a/b = c/d$ are in proportion, then the second difference $(c - b)$ is 12, given; and the fourth difference $(d - a)$ is 23, and $a/c = {}^{2}/_{9}$, given. It follows that the difference of the extremes less the difference of the means: $(d - a) - (c - b) = (b - a) + (d - c)$ = 11, and $(b - a)/b = (d - c)/d$; or, $(b - a)/(d - c) = a : c$; (2nd Lemma).

Hence, $((b - a) + (d - c))/(d - c) = (a + c)/c$, or $11/(d - c) = {}^{2}/_{9} + 1$, and $(d - c)$ is 9, (the third difference), and $(d - c) + (c - b) = (d - b) = 21$. Again, $(d - a) - (b - a) = (d - b) = 21$; hence the first difference $(b - a) = 2$, and $c - a = 14$. Thus, all the differences have been found. As $a/c = 2/9 = 4/18$, then $a = 4$, $c = 18$; and subsequently, $b = 6$, and $d = 27$.

§24.4. **Caput XXIIII.** [p.63.]

Ex datis quatuor lineis rectis, quarum tres simul sumpta a sunt maiores reliqua, Quadrilaterum quod sit sit in Circulo constituere.

Sunto data latera, partium 2.3.4.5. & ut certi simus de laterum situ, fiat utcunque quod imperatur: sitque AB, 2. BC, 3. CD, 4. DA, 5.Imprimis autem quaerendae sunt Diagonij AC, BD quae intersecantur in puncto E. Quod ut commodius fiat demonstrandum est hoc Lemma.

Lemma.

Segmentum quodlibet AE est ad reliquam Diagonium BD,ut factus a lateribus eidem segmento conterminis AB, AD 10, ad 26 summam factorum a lateribus ijsdem AB, 2 . AD, 5 in sua contermina latera BC, 3 and DC, 4.

Sunt enim triangula opposita AEB, DEC similia. item BEC, AED. Et idcirco erit AE ad BE, ut 5 ad 3, & AE ad ED, ut 2 ad 4. Et si idem *numerus multiplicet facti sunt proportionaes multiplicatis.* 5 & 3 per 2 faciunt 10 & 6; &, 2 & 4 per 5 faciunt 10 & 20. [p.64.] Erit igitur AE ad BD, ut 10 factus a lateribus AB, AD conterminis segmento AE: ad 26 summam factorum ab ijsdem lateribus in sua latera contermina: AB in BC, 6: & AD in DC, 20. Eadem de causa, CE erit BD, ut 12 ad 26 ; & tota AC ad BD, ut 22 ad 26.

AE, EB	5, 3	10, 6 prop.
AE, ED	2, 4	10, 20 prop.
AE, BD		10, 26 prop.
CE, EB	4, 2	12, 6 prop.
CE, ED	3, 5	12, 20 prop.
CE, BD		12, 26 prop.
AE + EC, ED		10 + 12, 26 prop.

Hinc deducitur Consectarium:

Diagij, sunt aggregatis factorum a conterminis lateribus proportionales

.

AB in AD, 10		BA in BC, 6	
CB in CD, 12		DA in DC, 20	
AC	22	BD	26

aggregata proportionalia.

Et cum facti ab oppositis lateribus BC in AD, 15: & AB in CD, 8: simul sumpti, aequentur facto a Diagonijs; (per 54.p.libri 1. Pitisci, & Ptolm.lib.1, Mathematicae Syntaxis). erit factus ab AC in BD, 23. Datis autem rectangulo AC,BD 23; & ratione laterum AC ad BD ut 11 ad 13: dabuntur ipsa latera: AC, $\ell.19\ {}^{6}/_{13}$; BD, $\ell.27^{2}/_{11}$.

Est enim Rectangulum a lateribus comprehensum, medium proportionale inter Quadrata laterum. & ut AC ad BD, sic Qu. AC, ad AC in BD: & AC in BD, ad Quad. BD.

proportiones { 11, 13, AC, BD } proportiones { Qu. AC, AC by BD, 23, Qu. BD }

proportiones		proportiones		*Logarithmi.*
	11		13 comp. ar.	888605665
	13		11	104139269
	23		23	136172784
	$27^{2}/_{11}$ Qu. BD		$19^{6}/_{13}$ Qu. AC	128917718

Erunt igitur triangula ABC, ADC datorum laterum: & datis lateribus trianguli cuiuscunque inscripti ABC, dabitur diameter circuli CF. per 4.sect.18. quae in hoc circulo est $\ell.27\ {}^{42}/_{120}$. Vide Fr. Vietam ad finem Pseudomesolabii & Johannes Praetoriuj Problema.

ALITER,

Ut hoc fiat, alia tria Lemmata sunt demonstranda.

Lemma secundum.

Si quatuor rectae sint proportiones : *Differentia (vel summa) primat & secunda est ad differentium (vel summam) tertia & quartas;, ut prima ad tertium, & contra, per* 19.p.5.lib.Eucl.

[p. 65.]

Lemma Tertium.

In quatuor proportionalibus: *Si datae sint differentiae primae & secundae, secundae & tertiae, tertiae & quartae; dabuntur ipsa proportionales.* ut sint A, B: C, D; proportionales & differentiae 2, 8, 7. Erit differentia A & C 10, aequalis differentijs 2, 8 datis: & ratio A ad C, ut 2 ad 7, per

	2			7		Differentiae.
4		6	14		21	Proportionales.
	10			35		Summae.

secundum Lemma *. Sumatur 5 differentia 2 & 7. ea ita se habet, ad 2 (vel 7), ut 10 differentia A & C, ad A 4 (vel C 14). erit igitur A, 4. B, 6. C, 14. D, 21.

	2		8		7		Differentiae datae
A,		B:		C,		D	Proportionales
4		6		14		21	

	5			10		Differentiae datae
2		7:	A,		C	Proportionales

*Notandum: *Si proportionales a prima ad ultimam crescant vel decrescant, sumenda erit differentiarum primae & secundae summa. alias si crescente secunda et decrescat tertia, vel contra : auferenda erit minor differentia e maiore, & pars reliqua erit differentia primae & tertiae. ut*

	5		11		2	
A,		B.		C,		D
10		15		4		6

3	6	Diff. data	3 , 2: 6, C. 4. Prop.
5, 2:	A, C	Proport.	3 , 5: 6, A. 10. Prop.

Lemma quartum.

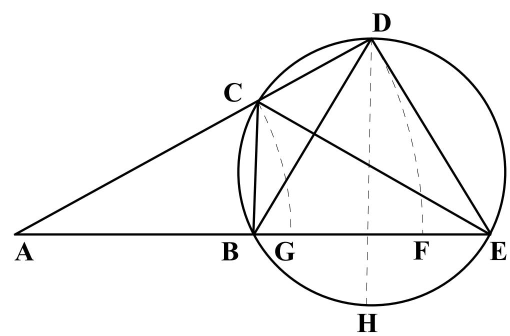
In quatuor proportionalibus: *Si data sint differentiae primae & quartae, secundae & tertiae; una cum ratione primae ad tertiam:ipsae proportionales inveniuntur.*

ut sunto datae differentiae, primae & quartae 23, secundae & tertiae 12, &sit ratio primae ad tertiam, ut 2 ad 9. imprimis differentia mediarum 12 * auferatur e differentia extremarum23;restabunt 11, summa differentiarum primae & secundae, tertiae & quartae. Deinde termini rationis datae inter primam & tertiam addantur : erit summa terminorum ad terminum antecedentem; ut summa differentiarum ad differentiam antecedentem.. Dabantur igitur differentiae, primae & secundae, tertiae & quartae. differentia autem secundae & tertiae antea dabatur. Idcirco per tertium Lemma, erit prima proportionalium, & reliquae notae.

**Crescentibus omnibus ut in diagramate: vel decrescentibus.*

Esto trapezium laterum inaequalium BCDE: BC, 6. CD, 9. DE, 12. EB, 15; cuius latera opposita CD, BE continuata, concurrent in puncto A. eruntque; rectae AB, AC: AD,AE proportionales: per 36.p.3.lib.Eucl. & triangula ABC, ADE similia: per 6.pr.6.lib.Eucl.

& idcirco BC, 6, DE, 12 proportionales rectis AB primae proportionalium, & AD tertiae. & sunt datae CD 9, differentia secundae & tertiae, & BE 15 differentia primae & quartae.



His datis quaerendae sunt AB prima & AC secunda. Auferatur GF aequalis rectae CD 9; e recta BE, 15; restabunt BG & FE: quae simul sumptae sunt 6; & est BG differentia primae & secundae, & FE differentia tertiae & quartae.

Sunt igitur AB, AD: BG, FE proportionales, per primum Lemma. id est, ut ante ostendi BC, DE : BG, FE sunt proportionales:

Et per tertium Lemma, ut summa BC, DE 18; ad BC, 6: sic summa BG, FE, 6, ad BG, 2 dantur igitur BG, 2 differentia primae & secundae, CD, 9 differentia secundae & tertiae, & FE, 4 differentia tertiae & quartae. Et per secundum Lemma, cum ratio primae ad tertiam sit ut 6 ad 12, vel ut BC ad DE, erit ut differentia 6 ad 12 terminum

[p.66.]

maiorem, sic, 11 differentia AB, AD; primae & tertiae, ad AD tertiae, 22. Erunt igitur AB, 11. AC, 13. AD, 22. AE, 26.

Diagononi autem BD, CE sic inveniuntur. facti ab oppositis inscriptis lateribus BC in DE, 72 & CD in BE, 135: simul sumpti aequantur facto a Diagonijs, per 54.p.1.lib.Pitisci. est igitur factus a Diagonijs 207. ipsae autem diagonij sunt proportionales rectis AB, 11: AC, 13. quia triangula ABD, ACE sunt aequiangula & similia;& idcirco rectangulum BD, CE est ad Quadratum BD,ut 13 ad 11: vel, ut AC ad AB. erit igitur BD $\ell.175\ ^{2}/_{13}$, & CE, $\ell.\ 244\ ^{7}/_{11}$. Est igitur CBD triangulum datorum laterum 6, 9, $\ell.175\ ^{2}/_{13}$. & perpendicularis ab angulo B in latus AC, $\ell.25\ ^{95}/_{169}$. diameter autem circuli circumscripti, DH, $\ell.\ 246\ ^{27}/_{40}$.