E 281 : A SPECIAL KIND OF ALGORITHM

1. The consideration of continued fractions, the most productive use of which I have shown now a number of times through the whole of analysis, has led me to signed quantities formed in a certain way, the nature of which is prepared thus, so that a singular algorithm may be required. Therefore since the greater part of the analysis shall depend for the most part on an algorithm used to accommodate certain quantities, not without merit will it be allowed to presume that this unusual algorithm may be going to become more than just a little useful in the future uses of analysis, if indeed it may be allowed to be developed carefully, even if I may consider that it not to be taken for granted to the extent that it may deserve to be compared with these present algorithms.

2. Moreover I have deduced how these quantities are to be acted on to consititute this algorithm. If the continued fraction may be had

$$a+\cfrac{1}{b+\cfrac{1}{c+\cfrac{1}{d}}},$$

the value of which shall require to be investigated from the numbers *a, b, c, d* , as far as the fractions are formed with the signs for the operations assumed, in the following manner:

$$\begin{array}{ccccc} & a & b & c & d \\ \frac{1}{0}, & \frac{a}{1}, & \frac{ab+1}{b}, & \frac{abc+c+a}{bc+1}, & \frac{abcd+cd+ad+ab+1}{bcd+d+b}. \end{array}$$

$$\left[a+\cfrac{1}{b+\cfrac{1}{c+\cfrac{1}{d}}} = a+\cfrac{1}{b+\cfrac{d}{cd+1}} = a+\cfrac{1}{\cfrac{bcd+b+d}{cd+1}} = a+\frac{cd+1}{bcd+b+d} = \frac{abcd+ab+ad+cd+1}{bcd+b+d} \right]$$

Evidently in the first place the fraction $\frac{1}{0}$ is obtained always, the second $\frac{a}{1}$, the numerator of which is the first index number *a*, truly the denominator unity. And for the following fractions both the numerator as well as the denominator is found, if the final of the preceding terms may be multiplied by the index number written above and for that to be added to the penultimate product.

3. But it is agreed the last of these fractions to be equal to the continued fraction proposed, moreover the preceding ones to approach very near to that particular value itself, so that no contained fraction may be able to be shown without more numerators, which may then be able to approach closer to that value. And from this principle this problem was originally resolved, as discussed by Wallis at one time; from which other

fractions may be sought either from a greater or smaller number of constant terms, which shall differ very little from the proposed number, so that small differences planely may be unable to be shown, unless we are willing to use more terms.

4. But these other uses, for which continued fractions supply the needs, I note, with this being overlooked in the first place, for that series of fractions both the numerators as well as the denominators follow the same law of the progression formed from the indices, and able to be formed separately. For in each series either the numerator or the denominator, whatever term multiplied by the index and by the preceding term being increased to produce the following term. Moreover, the final number of the upper series is composed from all four indices *a, b, c, d,* the previous one only from the three indices *a, b, c,* the one before that from the two indices *a* and *b.* But the lower numbers planely do not involve the first number *a*, but are formed equally from the remaining *b, c, d* by the same law.

5. Therefore since the ratio of the formation from the indices both for the numerators as well as from the denominators is the same and thence the number formed from the given indices becomes known, these same numbers, in as much as the have been formed from the indices, I am going to consider here and to treat their algorithm. But for whatever indices proposed, and however many *a, b, c, d* , shall become, I will denote the number formed from these (a, *b, c, d*) in this manner, and therefore with the expansion put in place

$$(a,\ b,\ c,\ d) = abcd + cd + ad + ab + 1$$

and in a similar manner for the denominators, with the first index *a* being omitted:

$$(b,\ c,\ d) = bcd + d + b.$$

6. Therefore this definition may be held in place between the signs ( ), between which the indices are written in order from left to right ; and the indices included by means of this closure will denote in the following the number formed from these same indices. Thus by increasing from the most simple cases we will have

$$\begin{aligned}
(a) &= a,\\
(a,b) &= ab + 1,\\
(a,\ b,\ c) &= abc + c + a,\\
(a,b,c,d) &= abcd + cd + ad + ab + 1,\\
(a,\ b,\ c,\ d,\ e) &= abcde + cde + ade + abe + abc + e + c + a
\end{aligned}$$

etc.,

from which progression to be apparent unity to hold the place of this sign (), evidently if no index shall be present.

7. In which way these expressions with the number of the index increased shall be required to be increased further, will be clear at oncefrom the law of formation, by which

any further expression is composed from the two previous preceding expressions . Evidently there becomes :

$$\begin{aligned}
(a,\ b) &= b(a)+1 = b(a)+(\ ),\\
(a,\ b,\ c) &= c(a,\ b)+(a),\\
(a,\ b,\ c,\ d) &= d(a,\ b,\ c)+(a,\ b),\\
(a,\ b,\ c,\ d,\ e) &= e(a,\ b,\ c,\ d)+(a,\ b,\ c).
\end{aligned}$$

In general therefore there will be had

$$(a,\ b,\ c\ ...\ p,\ q,\ r)=r(a,\ b,\ c\ ...\ p,\ q)+(a,\ b,\ c\ ...\ p),$$

which connexion must be considered as a corollary of the definition of the numbers which we may observe here.

8. In the establishment of these values, as have been shown before in § 6, a ratio is perceived from a more difficult way of composition. Moreover these also may be represented in this manner:

$$\begin{aligned}
(a) &= a(1),\\
(a,\ b) &= ab\left(1+\frac{1}{ab}\right),\\
(a,\ b,\ c) &= abc\left(1+\frac{1}{ab}+\frac{1}{bc}\right),\\
(a,\ b,\ c,\ d) &= abcd\left(1+\frac{1}{ab}+\frac{1}{bc}+\frac{1}{cd}+\frac{1}{abcd}\right),\\
(a,\ b,\ c,\ d,\ e) &= abcde\left(1+\frac{1}{ab}+\frac{1}{bc}+\frac{1}{cd}+\frac{1}{de}+\frac{1}{abcd}+\frac{1}{abde}+\frac{1}{bcde}\right)
\end{aligned}$$

etc.

Moreover in these denominators made from the first from two neighbouring indices, then truly with the product from two of these factors, which involve no common index, besides the products follow from three, four etc. combinations, which involve no common index; from which the account of the composition shall now be evident.

9. Now from this expansion it is evident, if the indices may be placed in the reverse order, planely the same numbers thence are formed. Evidently there will become

$$\begin{aligned}
&(a,\ b) &&=(b,\ a),\\
&(a,\ b,\ c) &&=(c,\ b,\ a),\\
&(a,\ b,\ c,\ d) &&=(d,\ c,\ b,\ a),\\
&(a,\ b,\ c,\ d,\ e) &&=(e,\ d,\ c,\ b,\ a)\\
&&&\text{etc.}
\end{aligned}$$

Therefore provided the order of the indices may be given, whether it shall be forwards or backwards, it is likewise the same; since in each way the same number thence formed is obtained.

10. Hence therefore by inverting the §7 formulas it follows in this way to become :

$$\begin{aligned}
&(a,\ b) &&= a(b)+1,\\
&(a,\ b,\ c) &&= a(b,\ c)+(c),\\
&(a,\ b,\ c,\ d) &&= a(b,\ c,\ d)+(c,\ d),\\
&(a,\ b,\ c,\ d,\ e) &&= a(b,\ c,\ d,\ e)+(c,\ d,\ e),
\end{aligned}$$

and in general there will be for however many indices

$$(a,\ b,\ c,\ d\ \text{ etc.}) = a(b,\ c,\ d\ \text{ etc.})+(c,\ d\ \text{ etc.}).$$

11. Therefore if there may be put

$$\begin{aligned}
(a,\ b,\ c,\ d,\ e\ \text{etc.}) &= A,\\
(b,\ c,\ d,\ e\ \text{etc.}) &= B,\\
(c,\ d,\ e\ \text{etc.}) &= C,\\
(d,\ e\ \text{etc.}) &= D,\\
(e\ \text{etc.}) &= E\\
&\text{etc.},
\end{aligned}$$

we will have these equalities :

$$\begin{aligned}
A &= aB+C \quad \text{or} \quad \tfrac{A}{B}=a+\tfrac{C}{B},\\
B &= bC+D \quad \text{or} \quad \tfrac{B}{C}=b+\tfrac{D}{C},\\
C &= cD+E \quad \text{or} \quad \tfrac{C}{D}=c+\tfrac{E}{D},\\
&\text{etc.} \qquad\qquad \text{etc.}
\end{aligned}$$

12. Therefore since there shall be:

$$\tfrac{C}{B}=\frac{1}{b+\tfrac{D}{C}},\quad \tfrac{D}{C}=\frac{1}{c+\tfrac{E}{D}},\quad \tfrac{E}{D}=\frac{1}{d+\tfrac{F}{E}}\quad \text{etc.}$$

there will become with these values substituted :

$$\frac{A}{B}=a+\frac{C}{B}=a+\frac{1}{b+\frac{D}{C}}=a+\cfrac{1}{b+\cfrac{1}{c+\frac{E}{D}}}=a+\cfrac{1}{b+\cfrac{1}{c+\cfrac{1}{d+\frac{F}{E}}}}.$$

From which, if e shall be the final index, thus so that there shall be $E=e$ and $F=1$, and there will become:

$$\frac{A}{B}=\frac{(a,\ b,\ c,\ d,\ e)}{(b,\ c,\ d,\ e)}=a+\cfrac{1}{b+\cfrac{1}{c+\cfrac{1}{d+\frac{1}{e}}}}$$

and thus it is apparent, how by numbers of this kind the values of continued fractions may be able to be expressed conveniently.

13. Therefore if the number of indices were infinite, also the continued fraction will extend to infinity and its value will be

$$=\frac{(a,\ b,\ c,\ d,\ e\ \text{etc.})}{(b,\ c,\ d,\ e\ \text{etc.})}.$$

But in turn the properties of continued fractions known to us will become evident from the significantly affected indices of this form , which it will be worth the effort to establish with care. Therefore the continued fraction shall be proposed, either extending to infinity or terminated

$$a+\cfrac{1}{b+\cfrac{1}{c+\cfrac{1}{d+\cfrac{1}{e+\cfrac{1}{f+\text{etc.}}}}}}$$

the value of which will be indicated by the *V*, and with all the indices assumed *a, b, c, d, e, f* etc. there will be, as we have shown,

$$V=\frac{(a,\ b,\ c,\ d,\ e\ \text{etc.})}{(b,\ c,\ d,\ e\ \text{etc.})}.$$

14. If certain of these indices may become infinitely great, this will be able to be omitted in this expression with all the following, and the value of the continued fraction is expressed only by the value of the preceding continued fractions.

Thus if there shall be $b=\infty$, there will be

$$V=\frac{(a)}{1};$$

if there shall be $c=\infty$, there will be

$$V=\frac{(a,b)}{(b)};$$

if there shall be $d=\infty$, there will be

$$V=\frac{(a,b,c)}{(b,c)};$$

if there shall be $e=\infty$, there will be

$$V=\frac{(a,b,c,d)}{(b,c,d)}.$$

Therefore as in this case the continued fraction is broken off, thus also the value V does not implicate the indices, except those which precede the infinite index.

15. But if no index is infinitely great, these values themselves continually approach closer to the true value of V. Evidently if there were

$$V=\frac{(a,b,c,d,e \text{ etc.})}{(b,c,d,e \text{ etc.})},$$

the continued fractions set out in the following series

$$\frac{(a)}{1},\ \frac{(a,b)}{(b)},\ \frac{(a,b,c)}{(b,c)},\ \frac{(a,b,c,d)}{(b,c,d)},\ \frac{(a,b,c,d,e)}{(b,c,d,e)} \text{ etc.}$$

continually approach closer to the value V and the final of these at last will show the true value, if indeed the indices *a, b, c, d* etc. were numbers greater than unity. Indeed the first $\frac{a}{1}$; will be able to differ notably from V, but the second will approach closer, the third closer to this and thus so on again, until at last the final shall be going to be the true value V.

16. Therefore it is necessary, that the differences between two contiguous fractions of this kind shall become continually smaller ; so that which may be seen more clearly, we will investigate these differences, which will be

$$\frac{(a)}{1}-\frac{(a,b)}{(b)}=\frac{(a)(b)-1(a,b)}{1(b)},$$

$$\frac{(a,b)}{(b)}-\frac{(a,\ b,\ c)}{(b,c)}=\frac{(a,b)(b,c)-(b)(a,b,c)}{(b)(b,c)},$$

$$\frac{(a,b,c)}{(b,c)}-\frac{(a,\ b,\ c,d)}{(b,c,d)}=\frac{(a,b,c)(b,c,d)-(b,c)(a,b,c,d)}{(b,c)(b,c,d)}$$

$$\frac{(a,\ b,\ c,d)}{(b,c,d)}-\frac{(a,\ b,\ c,d,e)}{(b,c,d,e)}=\frac{(a,b,c,d)(b,c,d,e)-(b,c,d)(a,b,c,d,e)}{(b,c,d)(b,c,d,e)}$$

etc.

17. From the differences of these denominators, which are established from two factors, I observe at first these factors to be prime numbers between themselves, which indeed is shown clear enough from the preceeding. Indeed since for the final denominator above $(b,\ c,\ d)(b,\ c,\ d,\ e)$ there shall be

$$(b,\ c,\ d,\ e)=e(b,\ c,\ d)+(b,c),$$

there will be

$$\frac{(b,\ c,\ d,\ e)}{(b,\ c,\ d)}=e+\frac{(b,\ c)}{(b,\ c,\ d)}\ ,$$

from which the factors (*b, c, d*) and (*b, c, d, e*) will not have a common divisor, unless likewise where it shall be a common divisor of the numbers (*b, c*) and (*b, c, d*); truly on account of the same ratio the common divisor of these numbers is not given, unless which likewise shall be a common divisor of these (*b*) and (*b, c*) and finally of these 1 and *b;* which since they may have no common divisor, nor will they have these and therefore they will be thus numbers prime between themselfes. Hence truly also it is understood the numbers (*a, b, c, d* etc.) and (*b, c, d* etc.)to be prime between themselves.

18. Therefore these cannot be minor differences, as if the numerators may be changed into unity, either positive or negative, which the examples truly indicate. Therefore the same will be agreed to be shown from the nature of the same numerators formed by the indices. Indeed for the first numerator, since there shall be

$$(a,\ b)=b(a)+1$$

by § 7, there will become

$$(a)(b)-1(a,b)=ab-b(a)-1\ =-1.$$

Then truly for the second on account of

$$(b,c)=c(b)+1\ \text{ and }\ (a,b,c)=c(a,b)+(a)$$

there will be

$$(a,b)(b,c)-(b)(a,b,c)=(a,b)c(b)+(a,b)-(b)c(a,b)-(b)(a),$$

which terms on account of $(a,b)c(b)-(b)c(a,b)$ cancelling each other will be changed into

$$(a,b)-(b)(a)+1,$$

thus so that the second numerator

$$(a,b)(b,c)-(b)(a,b,\ c)=+1.$$

19. Just as this second numerator has been reduced to the first negative obtained, thus the third can be shown to be equal to the second taken to be negative.
Now since

$$(b,\ c,\ d)=d(b,\ c)+(b)$$

and

$$(a,\ b,\ c,\ d)=d(a,\ b,\ c)+(a,\ b),$$

there will become

$$\begin{aligned}&(a,\ b,\ c)\ (b,\ c,\ d)-(b,\ c)\ (a,\ b,\ c,\ d)\\&=(a,\ b,\ c)d(b,\ c)+(a,\ b,c)(b)-(b,\ c)d(a,\ b,\ c)(b,\ c)(a,\ b).\end{aligned}$$

Therefore this expression becomes

$$(a,\ b)(b,\ c)+(b)(a,\ b,\ c)=1,$$

which is the second numerator taken negative. Moreover in the same manner the fourth numerator will be equal to the third taken negative and in gereral with any of the following taken negative.

20. Hence therefore we follow with the following noteworthy sequence reductions:

$$(a)(b)-1(a,\ b)=-1,$$
$$(a,\ b)(b,\ c)(b)(a,\ b,\ c)=+1,$$
$$(a,\ b,\ c)(b,\ c,\ d)-(b,\ c)(a,\ b,\ c,\ d)=-1,$$
$$(a,\ b,\ c,\ d)(b,\ c,\ d,\ e)-(b,\ c,\ d)(a,\ b,\ c,\ d,\ e)=+1$$

and in general

$$(a,\ b,\ c,\ d....m)(b,\ c,\ d....m,\ n)-(b,\ c,\ d....m)(a,\ b,\ c,\ d....m,\ n)=+1,$$

where $+1$ prevails, if in the first bracket the number of indices were even, otherwise truly -1.

21. Therefore these differences set out above will be :

$$\frac{(a)}{1}-\frac{(a,\ b)}{(b)}=-\frac{1}{1(b)},$$

$$\frac{(a,b)}{(b)}-\frac{(a,\ b,\ c)}{(b,\ c)}=+\frac{1}{(b)(b,\ c)},$$

$$\frac{(a,\ b,\ c)}{(b,\ c)}-\frac{(a,\ b,\ c,\ d)}{(b,\ c,\ d)}=-\frac{1}{(b,\ c)(b,\ c,\ d)},$$

$$\frac{(a,\ b,\ c,\ d)}{(b,\ c,\ d)}-\frac{(a,\ b,\ c,\ d,\ e)}{(b,\ c,\ d,\ e)}=+\frac{1}{(b,\ c,\ d)(b,\ c,\ d,\ e)},$$

$$\frac{(a,\ b,\ c,\ d,\ e)}{(b,\ c,\ d,\ e)}-\frac{(a,\ b,\ c,\ d,\ e,\ f)}{(b,\ c,\ d,\ e,\ f)}=-\frac{1}{(b,\ c,\ d,\ e)(b,\ c,\ d,\ e,\ f)},$$

etc.

from which, since these differences shall not be smaller, these same fractions approach as close to each other as can be done.

22. Since from § 7, there shall become

$$(b,c)-1=c(b),\ (b,\ c,\ d)-(b)=d(b,\ c),\ (b,\ c,\ d,\ e)-(b,\ c)=e(b,\ c,\ d)\ \text{ etc.,}$$

with two of these differences being added in turn, there will become

$$\frac{(a)}{1}-\frac{(a,\ b,\ c)}{(b)}=-\frac{c}{1(b,\ c)},$$

$$\frac{(a,b)}{(b)}-\frac{(a,\ b,\ c,\ d)}{(b,\ c,\ d)}=+\frac{d}{(b)(b,\ c,\ d)},$$

$$\frac{(a,\ b,\ c)}{(b,\ c)}-\frac{(a,\ b,\ c,\ d,\ e)}{(b,\ c,\ d,\ e)}=-\frac{e}{(b,\ c)(b,\ c,\ d,\ e)},$$

$$\frac{(a,\ b,\ c,\ d)}{(b,\ c,\ d)}-\frac{(a,\ b,\ c,\ d,\ e,\ f)}{(b,\ c,\ d,\ e,\ f)}=+\frac{f}{(b,\ c,\ d)(b,\ c,\ d,\ e,\ f)},$$

etc.

and here there will be

$$\frac{(a)}{1}=a \quad \text{and} \quad \frac{(a,b)}{(b)}=a+\frac{1}{b},$$

from which the remaining formulas will be able to be shown concisely.

23. Therefore from these formulas in §21, we will have the following values of the continued fractions

$$\frac{(a)}{1}=a,$$

$$\frac{(a,\ b)}{(b)}=a+\frac{1}{1(b)},$$

$$\frac{(a,\ b,\ c)}{(b,c)}=a+\frac{1}{1(b)}-\frac{1}{(b)(b,\ c)},$$

$$\frac{(a,\ b,\ c)}{(b,\ c)}=a+\frac{1}{1(b)}-\frac{1}{(b)(b,\ c)},$$

$$\frac{(a,\ b,\ c,\ d)}{(b,\ c,\ d)}=a+\frac{1}{1(b)}-\frac{1}{(b)(b,\ c)}+\frac{1}{(b,\ c)(b,\ c,\ d)}$$

etc.,

from which there will be in general, if the indices may extend indefinitely,

$$\frac{(a,\ b,\ c,\ d,\ e\ \text{etc.})}{(b,\ c,\ d,\ e\ \text{etc.})}=a+\frac{1}{1(b)}-\frac{1}{(b)(b,\ c)}+\frac{1}{(b,\ c)(b,\ c,\ d)}-\frac{1}{(b,\ c,\ d)(b,\ c,\ d,\ e)}+\text{etc.}$$

24. But we will obtain from the formulas in § 22 :

$$\frac{(a,\ b,\ c)}{(b,c)}=a+\frac{c}{1(b,c)},$$

$$\frac{(a,b,\ c,\ d,\ e)}{(b,\ c,\ d,\ e)}=a+\frac{c}{1(b,c)}+\frac{e}{(b,\ c)(b,\ c,\ d,\ e)},$$

from which generally

$$\frac{(a,\ b,\ c,\ d,\ e\ \text{etc.})}{(b,\ c,\ d,\ e\ \text{etc.})}=a+\frac{c}{1(b,c)}+\frac{e}{(b,\ c)(b,\ c,\ d,\ e)}+\frac{g}{(b,\ c,\ d,\ e)(b,\ c,\ d,\ e,\ f,\ g)}+\text{etc.}$$

Then truly also

$$\frac{(a,\ b,\ c,d)}{(b,\ c,\ d)}=a+\frac{1}{b}-\frac{d}{(b)(b,\ c,\ d)},$$

$$\frac{(a,\ b,\ c,\ d,\ e,\ f)}{(b,\ c,\ d,\ e,\ f)}=a+\frac{1}{b}-\frac{d}{(b)(b,\ c,\ d)}-\frac{f}{(b,\ c,\ d)(b,\ c,\ d,\ e,\ f)}$$

and thus generally:

$$\frac{(a,\ b,\ c,\ d,\ e \text{ etc.})}{(b,\ c,\ d,\ e \text{ etc.})}=a+\frac{1}{b}-\frac{d}{(b)(b,\ c,\ d)}-\frac{f}{(b,\ c,\ d)(b,\ c,\ d,\ e,\ f)}$$
$$-\frac{h}{(b,\ c,\ d,\ e,\ f)(b,\ c,\ d,\ e,\ f,\ g,\ h)}-\text{etc.}$$

25. But with these delegated, which are considered for series, since I have pursued these in more detail, we may consider these, which relate to the algorithm of these individual magnitudes. And indeed the similar formulas from these, which have been found in § 20, which will provide us from § 22, from which it is apparent to become:

$$(a)(b,\ c)-1(a,\ b,\ c)=-c,$$
$$(a,\ b)(b,\ c,\ d)-(b)(a,\ b,\ c,\ d)=+d,$$
$$(a,\ b,\ c)(b,\ c,\ d,\ e)-(b,\ c)(a,\ b,\ c,\ d,\ e)=-e,$$
$$(a,\ b,\ c,\ d)(b,\ c,\ d,\ e,\ f)-(b,\ c,\ d)(a,\ b,\ c,\ d,\ e,\ f)=+f$$

and thus generally :

$$(a,\ b.....l)(b.....l,\ m,\ n)-(b.....l)(a,\ b.....l,\ m,\ n)=\pm n,$$

where the $+$ sign prevails, in the index number in the first bracket shall be even, otherwise the negative sign $-$.

26. Moreover by these similar reductions there is understood to become :

$$(a)(b,\ c,\ d)-1(a,\ b,\ c,\ d)=-(c,d),$$
$$(a,\ b)(b,\ c,\ d,\ e)-(b)(a,\ b,\ c,\ d,\ e)=+(d,\ e),$$
$$(a,\ b,\ c)(b,\ c,\ d,\ e,\ f)-(b,\ c)(a,\ b,\ c,\ d,\ e,\ f)=-(e,\ f)$$

and generally:

$$(a,\ b\ ...\ k)(b...k,\ l,\ m,\ n)-(b...k)(a,\ b\ ...\ k,\ l,\ m,\ n)=\pm(m,\ n),$$

where the upper or lower sign prevails, just as the number of the index in the first bracket were even or odd.

27. Moreover the account of these formulas from the above conditions is easily derived. For if there may be put :

$$(a,\ b...k,\ l,\ m)(b...k,\ l,\ m,\ n)-(b...k,\ l,\ m)(a,\ b...k,\ l,\ m,\ n)=A,$$
$$(a,\ b\ ....\ k,\ l)(b\ ....\ k,\ l,\ m,\ n)-(b.....k,\ l)(a,\ b\ ...\ k,\ l,\ m,\ n)=B,$$
$$(a,\ b............k)(b\ ...k,\ l,\ m,\ n)-(b.......k)(a,\ b\ ...k,\ l,\ m,\ n)=C,$$

it is clear to be

$$A=mB+C\,.$$

But there is

$$A=\pm 1 \quad \text{and} \quad B=\mp n$$

therefore

$$C=\pm 1\pm mn=\pm(m,\ n),$$

where the ambiguity of the signs requiring to be held have been established above.

28. If the order of the indices may be inverted in these formulas, these will become

$$(a\ ....y)(a,\ b....y,\ z)-(a,\ b...y,\ z)(a,b...y)=0,$$
$$(a,\ b\ ....y)(b,\ c....y,\ z)-(a,\ b...y,\ z)(b,\ c...y)=\pm 1,$$
$$(a,\ b,\ c.......y)(c,\ d....y,\ z)-(a,\ b...y,\ z)(c,\ d...y)=\pm(a),$$
$$(a,\ b,\ c,\ d.....y)(d,\ e....y,\ z)-(a,\ b...y,\ z)(d,\ e...y)=\pm(a,\ b),$$
$$(a,\ b,\ c,\ d,\ e...y)(e,\ f....y,\ z)-(a,\ b...y,\ z)(e,\ f...y)=\pm(a,\ b,\ c),$$
$$(a,b...........y)(f,\ g...y,\ z)-(a,\ b...y,\ z)(f,\ g...y)=\pm(a,\ b,\ c,\ d),$$

where the superior signs prevail, if the number of the index in the second bracket were even, otherwise they lower signs prevail.

29. If this series of indices finally may be truncated into two parts, in a similar manner there will become :

$$(a\ ...\ x)(a\ ...\ z)-(a\ ...\ z)(a...\ x)=0,$$
$$(a\ ...\ x)(b\ ...z)-(a\ ...\ z)(b\ ...\ x)=\pm(z),$$
$$(a\ ...x)(c\ ...z)-(a\ ...\ z)(c\ ...\ x)=\pm(a)(z),$$
$$(a\ ...x)(d...\ z)-(a\ ...\ z)(d\ ...\ x)=\pm(a,b)(z),$$
$$(a\ ...x)(e\ ...\ z)-(a\ ...\ z)(e\ ...\ x)=\pm(a,b,c)(z)$$

and hence finally there will be deduced generally

$$(a...l,\ m,\ n...p)(n\ ...p,\ q,\ r...z)-(a...l,\ m,\ n...p,\ q,\ r...z)(n...p)$$
$$=\pm(a\ ...\ l)(r\ ...\ z).$$

SPECIMEN ALGORITHMI SINGULARIS

Commentatio 281 indicis ENESTROEMIANI
Novi commentarii academiae scientiarum Petropolitanae 9 (1762/3), 1764, p. 53-69

1. Consideratio fractionum continuarum, quarum usum uberrimum per totam Analysin iam aliquoties ostendi, deduxit me ad quantitates certo quodam modo ex indicibus formatas, quarum natura ita est comparata, ut singularem algorithmum requirat. Cum igitur summa Analyseos inventa maximam partem algorithmo ad certas quasdam quantitates accommodato innitantur, non immerito suspicari licet et hunc algorithmum singularem non exigui usus in Analysi esse futurum, si quidem diligentius excolatur, etiamsi ei tantum non tribuendum censeam, ut cum receptis algorithmis comparari mereatur.

2. Sequenti autem modo ad eas quantitates, de quibus hic agere constitui, sum deductus. Si habeatur fractio continua

$$a+\cfrac{1}{b+\cfrac{1}{c+\cfrac{1}{d}}}$$

cuius valor sit investigandus, ex numeris *a, b, c, d* tanquam indicibus assumtis sequenti modo fractiones formantur:

$$\begin{array}{ccccc} a & b & c & d & \\ \frac{1}{0}, & \frac{a}{1}, & \frac{ab+1}{b}, & \frac{abc+c+a}{bc+1}, & \frac{abcd+cd+ad+ab+1}{bcd+d+b}. \end{array}$$

Primum scilicet locum obtinet semper fractio $\frac{1}{0}$, secundum $\frac{a}{1}$, cuius numerator est primus indicum *a,* denominator vero unitas. Sequentis cuiusque fractionis tam numerator quam denominator invenitur, si praecedentium ultimus per indicem supra scriptum multiplicetur et ad productum penultimus addatur.

3. Constat autem harum fractionum postremam ipsi fractioni continuae propositae esse aequalem, praecedentes autem tam prope ad hunc ipsum valorem accedere, ut nulla fractio numeris non maioribus contenta exhiberi queat, quae ad illum propius accedat. Atque ex hoc fonte problema illud a Wallisio olim tractatum facile resolvitur, quo proposita quacunque fractione ex ingentibus numeris constante aliae quaeruntur fractiones ex minoribus numeris constantes, quae tam parum a proposita discrepent, ut minus discrepantes exhiberi plane nequeant, nisi maiores numeros adhibere velimus.

4. Hoc autem aliisque usibus, quos fractiones continuae suppeditant, praetermissis hic inprimis observo in serie illa fractionum ex indicibus formatarum tam numeratores quam denominatores eandem progressionis legem sequi et seorsim efformari posse. In utraque enim serie, sive numeratorum sive denominatorum, quilibet terminus per indicem supra scriptum multiplicatus et termino antecedente auctus praebet terminum sequentem. Ultimus autem numerus superioris seriei componitur ex omnibus quatuor indicibus *a*, *b*, *c*, *d*, penultimus tantum ex tribus *a*, *b*, *c*, antepenultimus tantum ex duobus *a* et *b*. Inferiores autem numeri primum indicem *a* plane non involvunt, sed ex reliquis *b*, *c*, *d* aequali lege formantur.

5. Quoniam igitur ratio formationis ex indicibus tam pro numeratoribus quam pro denominatoribus est eadem ac datis indicibus numerus inde formatus innotescit, hos ipsos numeros, quatenus ex indicibus sunt formati, hic sum contemplaturus eorumque algorithmum traditurus. Propositis autem indicibus quibuscunque et quotcunque *a*, *b*, *c*, *d* numerum ex iis formatum hoc modo (a, *b*, *c*, *d*) denotabo eritque ergo evolutione instituta

$$\left(a,\ b,\ c,\ d\right) = abcd + cd + ad\ \ ab + 1$$

similique modo pro denominatoribus indicem primum *a* omittendo

$$\left(b,\ c,\ d\right) = bcd + d + b.$$

6. Haec ergo teneatur definitio signorum ( ), inter quae indices ordine a sinistra ad dextram scribere constitui; atque indices hoc modo clausulis inclusi in posterum denotabunt numerum ex istis indicibus formatum. Ita a simplicissimis casibus inchoando habebimus

$$\begin{aligned}
&(a) &&= a,\\
&(a,b) &&= ab + 1,\\
&(a,\ b,\ c) &&= abc + c + a,\\
&(a,b,c,d) &&= abcd + cd + ad + ab + 1,\\
&(a,\ b,\ c,\ d,\ e) &&= abcde + cde + ade + abe + abc + \ e + c + a
\end{aligned}$$

etc.,

ex qua progressione patet unitatem tenere locum huius signi (), si scilicet nullus adsit index.

7. Quemadmodum hae expressiones crescente indicum numero ulterius sint continuandae, ex formationis lege, qua quilibet ex duobus antecedentibus componitur, sponte liquet. Est scilicet

$$\begin{aligned}
(a,\ b) &= b(a)+1 = b(a)+(\ ),\\
(a,\ b,\ c) &= c(a,\ b)+(a),\\
(a,\ b,\ c,\ d) &= d(a,\ b,\ c)+(a,\ b),\\
(a,\ b,\ c,\ d,\ e) &= e(a,\ b,\ c,\ d)+(a,\ b,\ c).
\end{aligned}$$

In genere ergo habebitur

$$(a,\ b,\ c\ ...\ p,\ q,\ r) = r(a,\ b,\ c\ ...\ p,\ q)+(a,\ b,\ c\ ...\ p),$$

quae connexio tanquam corollarium definitionis numerorum, quos hic contemplamur, spectari debet.

8. In evolutione horum valorum, uti ante § 6 sunt exhibiti, difficulter ratio compositionis cernitur. Possunt autem ii quoque hoc modo repraesentari:

$$\begin{aligned}
(a) &= a(1),\\
(a,\ b) &= ab\left(1+\frac{1}{ab}\right),\\
(a,\ b,\ c) &= abc\left(1+\frac{1}{ab}+\frac{1}{bc}\right),\\
(a,\ b,\ c,\ d) &= abcd\left(1+\frac{1}{ab}+\frac{1}{bc}+\frac{1}{cd}+\frac{1}{abcd}\right),\\
(a,\ b,\ c,\ d,\ e) &= abcde\left(1+\frac{1}{ab}+\frac{1}{bc}+\frac{1}{cd}+\frac{1}{de}+\frac{1}{abcd}+\frac{1}{abde}+\frac{1}{bcde}\right)
\end{aligned}$$

etc.

In his autem denominatoribus occurrunt primo facta ex binis indicibus contiguis, tum vero producta ex binis illorum factorum, qui nullum indicem communem involvunt, tum sequentur producta ex ternis, quaternis etc. combinationibus, quae nullum implicant indicem communem; unde ratio compositionis iam fit perspicua.

9. Ex hac evolutione iam manifestum est, si indices ordine retrogrado disponantur, eosdem plane prodire numeros inde formatos. Erit scilicet

$$\begin{aligned}
(a,\ b) &= (b,\ a),\\
(a,\ b,\ c) &= (c,\ b,\ a),\\
(a,\ b,\ c,\ d) &= (d,\ c,\ b,\ a),\\
(a,\ b,\ c,\ d,\ e) &= (e,\ d,\ c,\ b,\ a)
\end{aligned}$$

etc.

Dummodo ergo ordo indicum detur, sive sit directus sive retrogradus, perinde est; utroque enim modo idem numerus inde formatus obtinetur.

10. Hinc ergo sequitur fore formulas § 7 hoc modo invertendo

$$\begin{aligned}
(a,\ b) &= a(b)+1,\\
(a,\ b,\ c) &= a(b,\ c)+(c),\\
(a,\ b,\ c,\ d) &= a(b,\ c,\ d)+(c,\ d),\\
(a,\ b,\ c,\ d,\ e) &= a(b,\ c,\ d,\ e)+(c,\ d,\ e),
\end{aligned}$$

atque in genere erit pro quotcunque indicibus

$$(a,\ b,\ c,\ d\ \text{etc.}) = a(b,\ c,\ d\ \text{etc.})+(c,\ d\ \text{etc.}).$$

11. Si ergo ponatur

$$\begin{aligned}
(a,\ b,\ c,\ d,\ e\ \text{etc.}) &= A,\\
(b,\ c,\ d,\ e\ \text{etc.}) &= B,\\
(c,\ d,\ e\ \text{etc.}) &= C,\\
(d,\ e\ \text{etc.}) &= D,\\
(e\ \text{etc.}) &= E\\
&\text{etc.},
\end{aligned}$$

habebimus has aequalitates:

$$\begin{aligned}
A &= aB+C \quad \text{seu} \quad \tfrac{A}{B} = a+\tfrac{C}{B},\\
B &= bC+D \quad \text{seu} \quad \tfrac{B}{C} = b+\tfrac{D}{C},\\
C &= cD+E \quad \text{seu} \quad \tfrac{C}{D} = c+\tfrac{E}{D},\\
&\text{etc.} \qquad\qquad \text{etc.}
\end{aligned}$$

12. Cum igitur sit

$$\tfrac{C}{B} = \frac{1}{b+\tfrac{D}{C}},\quad \tfrac{D}{C} = \frac{1}{c+\tfrac{E}{D}},\quad \tfrac{E}{D} = \frac{1}{d+\tfrac{F}{E}}\quad \text{etc.}$$

erit his valoribus substituendis

$$\frac{A}{B} = a+\frac{C}{B} = a+\frac{1}{b+\frac{D}{C}} = a+\cfrac{1}{b+\cfrac{1}{c+\frac{E}{D}}} = a+\cfrac{1}{b+\cfrac{1}{c+\cfrac{1}{d+\frac{F}{E}}}}.$$

Unde, si e sit indicum ultimus, ita ut sit $E=e$ et $F=1$, erit

$$\frac{A}{B}=\frac{(a,\ b,\ c,\ d,\ e)}{(b,\ c,\ d,\ e)}=a+\cfrac{1}{b+\cfrac{1}{c+\cfrac{1}{d+\cfrac{1}{e}}}}$$

sicque patet, quemadmodum per huiusmodi numeros valores fractionum continuarum commode exprimi queant.

13. Si ergo indicum numerus fuerit infinitus, etiam fractio continua in infinitum excurret eiusque valor erit

$$=\frac{(a,\ b,\ c,\ d,\ e\ \text{etc.})}{(b,\ c,\ d,\ e\ \text{etc.})}.$$

Vicissim autem fractionum continuarum proprietates cognitae nobis insignes affectiones huiusmodi numerorum ex indicibus formatorum manifestabunt, quas diligentius evolvere operae erit pretium. Sit igitur fractio continua, sive in infinitum excurrens sive secus, proposita

$$a+\cfrac{1}{b+\cfrac{1}{c+\cfrac{1}{d+\cfrac{1}{e+\cfrac{1}{f+\text{etc.}}}}}}$$

cuius valor indicetur littera *V*, et sumendis omnibus indicibus *a, b, c, d, e, f* etc. erit, uti demonstravimus,

$$V=\frac{(a,\ b,\ c,\ d,\ e\ \text{etc.})}{(b,\ c,\ d,\ e\ \text{etc.})}.$$

14. Si quispiam horum indicum fiat infinite magnus, is in hac expressione cum omnibus sequentibus poterit omitti, et valor fractionis continuae tantum per indices, qui infinitum praecedunt, exprimetur.

Ita si sit $b=\infty$ erit

$$V=\frac{(a)}{1};$$

si sit $c=\infty$ erit

$$V=\frac{(a,b)}{(b)};$$

si sit $d=\infty$ erit

$$V=\frac{(a,b,c)}{(b,c)};$$

si sit $e=\infty$, erit

$$V=\frac{(a,b,c,d)}{(b,c,d)}.$$

Uti ergo his casibus fractio continua abrumpitur, ita etiam valor *V* alios indices non implicat, nisi qui indicem infinitum antecedunt.

15. Sin autem nullus indicum in infinitum excrescit, hi ipsi valores continuo propius ad verum valorem *V* accedunt. Scilicet si fuerit

$$V = \frac{(a,b,c,d,e \text{ etc.})}{(b,c,d,e \text{ etc.})},$$

fractiones in sequenti serie expositae

$$\frac{(a)}{1},\ \frac{(a,b)}{(b)},\ \frac{(a,b,c)}{(b,c)},\ \frac{(a,b,c,d)}{(b,c,d)},\ \frac{(a,b,c,d,e)}{(b,c,d,e)} \text{ etc.}$$

continuo propius ad valorem *V* accedunt earumque ultima demum eius valorem verum exhibebit, siquidem indices *a, b, c, d* etc. fuerint numeri unitate maiores. Prima quidem $\frac{a}{1}$ notabiliter ab *V* discrepare poterit, secunda autem propius accedet, tertia adhuc propius et ita porro, donec tandem ultima verum valorem *V* sit praebitura.

16. Necesse ergo est, ut differentiae inter binas huiusmodi fractiones contiguas continuo fiant minores; quod quo clarius perspiciatur, has differentias investigemus, quae erunt

$$\frac{(a)}{1} - \frac{(a,b)}{(b)} = \frac{(a)(b) - 1(a,b)}{1(b)},$$

$$\frac{(a,b)}{(b)} - \frac{(a,b,c)}{(b,c)} = \frac{(a,b)(b,c) - (b)(a,b,c)}{(b)(b,c)},$$

$$\frac{(a,b,c)}{(b,c)} - \frac{(a,b,c,d)}{(b,c,d)} = \frac{(a,b,c)(b,c,d) - (b,c)(a,b,c,d)}{(b,c)(b,c,d)}$$

$$\frac{(a,b,c,d)}{(b,c,d)} - \frac{(a,b,c,d,e)}{(b,c,d,e)} = \frac{(a,b,c,d)(b,c,d,e) - (b,c,d)(a,b,c,d,e)}{(b,c,d)(b,c,d,e)}$$

etc.

17. De harum differentiarum denominatoribus, qui ex binis factoribus sunt conflati, primum observo hos factores inter se esse numeros primos, quod quidem ex antecedentibus est satis manifestum. Cum enim pro denominatore
$(b,\ c,\ d)(b,\ c,\ d,\ e)$ sit

$$(b,\ c,\ d,\ e) = e(b,\ c,\ d) + (b,c),$$

erit

$$\frac{(b,\ c,\ d,\ e)}{(b,\ c,\ d)} = e + \frac{(b,\ c)}{(b,\ c,\ d)},$$

unde factores *(b, c, d)* et *(b, c, d, e)* communem divisorem non habebunt, nisi qui simul sit communis divisor numerorum (b, *c)* et *(b, c, d);* verum ob eandem rationem horum numerorum communis divisor non datur, nisi qui simul sit communis divisor horum *(b)* et (b, *c)* ac denique horum 1 et *b;* qui cum nullum habeant communem divisorem, neque illi habebunt eruntque propterea numeri [inter se] primi. Hinc vero etiam intelligitur numeros *(a, b, c, d* etc.) et *(b, c, d* etc.) esse inter se primos.

18. Differentiae ergo illae minores esse nequeunt, quam si numeratores in unitatem, sive affirmativam sive negativam, abeant, id quod re vera evenire exempla declarant. Conveniet ergo idem ex natura istorum numerorum per indices formatorum demonstrari. Pro primo quidem numeratore, cum sit

$$(a,\ b)=b(a)+1$$

per § 7, erit

$$(a)(b)-1(a,b)=ab-b(a)-1\ =-1.$$

Tum vero pro secundo ob

$$(b,c)=c(b)+1\ \text{ et }\ (a,b,c)=c(a,b)+(a)$$

erit

$$(a,b)(b,c)-(b)(a,b,c)=(a,b)c(b)+(a,b)-(b)c(a,\ b)-(b)(a),$$

quae propter terminos $(a,b)c(b)-(b)c(a,b)$ se tollentes abit in

$$(a,b)-(b)(a)+1,$$

ita ut sit secundus numerator

$$(a,b)(b,c)-(b)(a,b,\ c)=+1.$$

19. Quemadmodum hic numerator secundus ad primum negative sumtum est reductus, ita tertius ostendi potest secundo negative sumto esse aequalis.
Nam quia

$$(b,c,\ d)=d\,(b,c)+(b)$$

et

$$(a,\ b,\ c,\ d)=d\,(a,\ b,\ c)+(a,\ b),$$

erit

$$\begin{aligned}&(a,\ b,\ c)\ (b,\ c,\ d)-(b,\ c)\ (a,\ b,\ c,\ d)\\&=(a,\ b,\ c)d\,(b,\ c)+(a,\ b,c)(b)-(b,\ c)d\,(a,\ b,\ c)(b,\ c)(a,\ b).\end{aligned}$$

Haec ergo expressio transit in

$$(a,\ b)(b,\ c)+(b)(a,\ b,\ c)=1,$$

qui est numerator secundus negative sumtus. Eodem autem modo numerator quartus aequabitur tertio negative sumto et in genere quilibet sequens praecedenti negative sumto.

20. Hinc ergo consequimur sequentes reductiones non parum notatu dignas

$$(a)(b)-1(a,\ b)=-1,$$
$$(a,\ b)(b,\ c)(b)(a,\ b,\ c)=+1,$$
$$(a,\ b,\ c)(b,\ c,\ d)-(b,\ c)(a,\ b,\ c,\ d)=-1,$$
$$(a,\ b,\ c,\ d)(b,\ c,\ d,\ e)-(b,\ c,\ d)(a,\ b,\ c,\ d,\ e)=+1$$

et in genere

$$(a,\ b,\ c,\ d....m)(b,\ c,\ d....m,\ n)-(b,\ c,\ d....m)(a,\ b,\ c,\ d....m,\ n)=+1,$$

ubi +1 valet, si in primis vinculis numerus indicum fuerit par, contra vero -1.

21. Ipsae ergo differentiae supra expositae erunt

$$\frac{(a)}{1}-\frac{(a,\ b)}{(b)}=-\frac{1}{1(b)},$$
$$\frac{(a,b)}{(b)}-\frac{(a,\ b,\ c)}{(b,\ c)}=+\frac{1}{(b)(b,\ c)},$$
$$\frac{(a,\ b,\ c)}{(b,\ c)}-\frac{(a,\ b,\ c,\ d)}{(b,\ c,\ d)}=-\frac{1}{(b,\ c)(b,\ c,\ d)},$$
$$\frac{(a,\ b,\ c,\ d)}{(b,\ c,\ d)}-\frac{(a,\ b,\ c,\ d,\ e)}{(b,\ c,\ d,\ e)}=+\frac{1}{(b,\ c,\ d)(b,\ c,\ d,\ e)},$$
$$\frac{(a,\ b,\ c,\ d,\ e)}{(b,\ c,\ d,\ e)}-\frac{(a,\ b,\ c,\ d,\ e,\ f)}{(b,\ c,\ d,\ e,\ f)}=-\frac{1}{(b,\ c,\ d,\ e)(b,\ c,\ d,\ e,\ f)},$$

etc.

unde, cum hae differentiae minores esse nequeant, ipsae fractiones tam prope ad se invicem accedunt, quam fieri potest.

22. Cum sit ex § 7

$(b,c)-1=c(b),\ (b,\ c,\ d)-(b)=d(b,\ c),\ (b,\ c,\ d,\ e)-(b,\ c)=e(b,\ c,\ d)$ etc.,

erit binis illarum differentiarum addendis

$$\frac{(a)}{1}-\frac{(a,\ b,\ c)}{(b)}=-\frac{c}{1(b,\ c)},$$

$$\frac{(a,b)}{(b)}-\frac{(a,\ b,\ c,\ d)}{(b,\ c,\ d)}=+\frac{d}{(b)(b,\ c,\ d)},$$

$$\frac{(a,\ b,\ c)}{(b,\ c)}-\frac{(a,\ b,\ c,\ d,\ e)}{(b,\ c,\ d,\ e)}=-\frac{e}{(b,\ c)(b,\ c,\ d,\ e)},$$

$$\frac{(a,\ b,\ c,\ d)}{(b,\ c,\ d)}-\frac{(a,\ b,\ c,\ d,\ e,\ f)}{(b,\ c,\ d,\ e,\ f)}=+\frac{f}{(b,\ c,\ d)(b,\ c,\ d,\ e,\ f)},$$

etc.

eritque hic

$$\frac{(a)}{1}=a \quad \text{et} \quad \frac{(a,b)}{(b)}=a+\frac{1}{b},$$

unde reliquae formulae concinne poterunt exhiberi.

23. Ex formulis ergo §21 habebimus sequentes fractionum continuarum valores

$$\frac{(a)}{1}=a,$$

$$\frac{(a,\ b)}{(b)}=a+\frac{1}{1(b)},$$

$$\frac{(a,\ b,\ c)}{(b,c)}=a+\frac{1}{1(b)}-\frac{1}{(b)(b,\ c)},$$

$$\frac{(a,\ b,\ c)}{(b,\ c)}=a+\frac{1}{1(b)}-\frac{1}{(b)(b,\ c)},$$

$$\frac{(a,\ b,\ c,\ d)}{(b,\ c,\ d)}=a+\frac{1}{1(b)}-\frac{1}{(b)(b,\ c)}+\frac{1}{(b,\ c)(b,\ c,\ d)}$$

etc.

unde in genere erit, si etiam indices in infinitum excurrant,

$$\frac{(a,\ b,\ c,\ d,\ e\ \text{ etc.})}{(b,\ c,\ d,\ e\ \text{ etc.})}=a+\frac{1}{1(b)}-\frac{1}{(b)(b,\ c)}+\frac{1}{(b,\ c)(b,\ c,\ d)}-\frac{1}{(b,\ c,\ d)(b,\ c,\ d,\ e)}+\text{etc.}$$

24. Ex formulis autem § 22 obtinebimus

$$\frac{(a,\ b,\ c)}{(b,c)}=a+\frac{c}{1(b,c)},$$

$$\frac{(a,b,\ c,\ d,\ e)}{(b,\ c,\ d,\ e)}=a+\frac{c}{1(b,c)}+\frac{e}{(b,\ c)(b,\ c,\ d,\ e)},$$

unde generaliter

$$\frac{(a,\ b,\ c,\ d,\ e\ \text{etc.})}{(b,\ c,\ d,\ e\ \text{etc.})}=a+\frac{c}{1(b,c)}+\frac{e}{(b,\ c)(b,\ c,\ d,\ e)}+\frac{g}{(b,\ c,\ d,\ e)(b,\ c,\ d,\ e,\ f,\ g)}+\text{etc.}$$

Tum vero etiam

$$\frac{(a,\ b,\ c,d)}{(b,\ c,\ d)}=a+\frac{1}{b}-\frac{d}{(b)(b,\ c,\ d)},$$

$$\frac{(a,\ b,\ c,\ d,\ e,\ f)}{(b,\ c,\ d,\ e,\ f)}=a+\frac{1}{b}-\frac{d}{(b)(b,\ c,\ d)}-\frac{f}{(b,\ c,\ d)(b,\ c,\ d,\ e,\ f)}$$

ideoque generaliter

$$\frac{(a,\ b,\ c,\ d,\ e\ \text{etc.})}{(b,\ c,\ d,\ e\ \text{etc.})}=a+\frac{1}{b}-\frac{d}{(b)(b,\ c,\ d)}-\frac{f}{(b,\ c,\ d)(b,\ c,\ d,\ e,\ f)}$$
$$-\frac{h}{(b,\ c,\ d,\ e,\ f)(b,\ c,\ d,\ e,\ f,\ g,\ h)}-\text{etc.}$$

25. Sed missis his, quae ad series spectant, quoniam ea iam fusius sum persecutus, perpendamus ea, quae ad singularem harum quantitatum algorithmum pertinent. Et formulas quidem iis similes, quae in § 20 sunt inventae, suppeditabit nobis § 22, ex quo patet esse

$$(a)(b,\ c)-1(a,\ b,\ c)=-c,$$
$$(a,\ b)(b,\ c,\ d)-(b)(a,\ b,\ c,\ d)=+d,$$
$$(a,\ b,\ c)(b,\ c,\ d,\ e)-(b,\ c)(a,\ b,\ c,\ d,\ e)=-e,$$
$$(a,\ b,\ c,\ d)(b,\ c,\ d,\ e,\ f)-(b,\ c,\ d)(a,\ b,\ c,\ d,\ e,\ f)=+f$$

ideoque generaliter

$$(a,\ b.....l)(b.....l,\ m,\ n)-(b.....l)(a,\ b.....l,\ m,\ n)=\pm n,$$

ubi signum + valet, si in primo vinculo numerus indicum sit par, contra signum $-$.

26. Per similes autem reductiones intelligitur fore

$$(a)(b,\ c,\ d)-1(a,\ b,\ c,\ d)=-(c,d),$$
$$(a,\ b)(b,\ c,\ d,\ e)-(b)(a,\ b,\ c,\ d,\ e)=+(d,\ e),$$
$$(a,\ b,\ c)(b,\ c,\ d,\ e,\ f)-(b,\ c)(a,\ b,\ c,\ d,\ e,\ f)=-(e,\ f)$$

et generaliter

$$(a,\ b\ ...\ k)(b...k,\ l,\ m,\ n)-(b...k)(a,\ b\ ...\ k,\ l,\ m,\ n)=\pm(m,\ n),$$

ubi signorum vel superius vel inferius valet, prout in primo vinculo numerus indicum fuerit vel par vel impar.

27. Ratio autem huius formulae ex supra repartis facile derivatur. Si enim ponatur

$$(a,\ b...k,\ l,\ m)(b...k,\ l,\ m,\ n)-(b...k,\ l,\ m)(a,\ b...k,\ l,\ m,\ n)=A,$$
$$(a,\ b\ ....\ k,\ l)(b\ ....\ k,\ l,\ m,\ n)-(b.....k,\ l)(a,\ b\ ...\ k,\ l,\ m,\ n)=B,$$
$$(a,\ b............k)(b\ ...k,\ l,\ m,\ n)-(b.......k)(a,\ b\ ...k,\ l,\ m,\ n)=C,$$

manifestum est esse

$$A=mB+C.$$

At est

$$A=\pm 1 \quad \text{et} \quad B=\mp n$$

ideoque

$$C=\pm 1\pm mn=\pm(m,\ n),$$

ubi de ambiguitate signorum tenenda sunt praecepta superiora.

28. Si ordo indicum in his formulis invertatur, eae fient

$$(a\ ....y)(a,\ b....y,\ z)-(a,\ b...y,\ z)(a,b...y)=0,$$
$$(a,\ b\ ....y)(b,\ c....y,\ z)-(a,\ b...y,\ z)(b,\ c...y)=\pm 1,$$
$$(a,\ b,\ c.......y)(c,\ d....y,\ z)-(a,\ b...y,\ z)(c,\ d...y)=\pm(a),$$
$$(a,\ b,\ c,\ d.....y)(d,\ e....y,\ z)-(a,\ b...y,\ z)(d,\ e...y)=\pm(a,\ b),$$
$$(a,\ b,\ c,\ d,\ e...y)(e,\ f....y,\ z)-(a,\ b...y,\ z)(e,\ f...y)=\pm(a,\ b,\ c),$$
$$(a,b...........y)(f,\ g...y,\ z)-(a,\ b...y,\ z)(f,\ g...y)=\pm(a,\ b,\ c,\ d),$$

ubi signa valent superiora, si numerus indicum in secundo vinculo fuerit par, contra autem valent inferiora.

29. Si haec indicum series in fine duobus truncetur, orietur simili modo

$$(a\ ...\ x)(a\ ...\ z)-(a...\ z)(a...\ x)=0,$$
$$(a\ ..\ .\ x)(b.\ .\ .z)-(a...z)(b\ ...\ x)=\pm(z),$$
$$(a\ .\ .\ .x)(c.\ .\ .z)-(a...z)(c.\ .\ .\ x)=\pm(a)(z),$$
$$(a.\ .\ .\ x)(d...\ z)-(a...\ z)(d...\ x)=\pm(a,b)(z),$$
$$(a...\ x)(e\ ...\ z)-(a...\ z)(e...\ x)=\pm(a,b,c)(z)$$

atque hinc tandem colligitur fore generaliter

$$(a...l,\ m,\ n...p)(n\ ...p,\ q,\ r...z)-(a...l,\ m,\ n...p,\ q,\ r...z)(n...p)$$
$$=\pm(a\ ...\ l)(r\ ...\ z).$$