FIRST SECTION.

CONCERNING SIMPLE MICROSCOPES.

CHAPTER II

WITH TWO OR MORE CONVEX LENSES

SEPARATED BY SMALL CONSTANT DISTANCES

PROBLEM 1

56. *If a double lens shall be composed from two convex lenses, to construct a microscope of this kind for a given magnification, so that it may represent an object with clarity and distinction, however great it can be become.*

SOLUTION

Since here two nearby lenses are present joined to each other, from our general formulas the separation of these will be

$$\alpha + b = Aa\left(1 - \tfrac{1}{P}\right);$$

which since it must become a minimum, may be put $= \eta a$ with η denoting some fraction as small as the circumstances will permit, and hence we deduce

$$P = \tfrac{A}{A-\eta}$$

then, since each lens must be convex or its focal length positive, both this quantity

$$p = \mathfrak{A}a$$

as well as

$$q = -\tfrac{A}{P}a = -\left(A - \eta\right)a$$

must be positive and thus $\mathfrak{A} > 0$, but $A < 0$, which happens if $\mathfrak{A} > 1$. With this noted the magnification provided for us:

$$m = \tfrac{Ph}{a} = \tfrac{A}{A-\eta} \cdot \tfrac{h}{a},$$

from which the distance of the object from the first lens is defined :

$$a = \tfrac{A}{A-\eta} \cdot \tfrac{h}{m},$$

thus so that there shall become :

$$ma-h=\tfrac{\eta}{A-\eta}\cdot h;$$

then if the radius of the aperture of the first lens may be put $=x$, that of the second lens must be $\left(1-\tfrac{\eta}{A}\right)x$; from which there becomes, for the degree of clarity

$$y=\tfrac{hx}{ma} \text{ or } y=\left(1-\tfrac{\eta}{A}\right)x,$$

thus so that on account of $A<0$ the clear interval of the lens may be increased. Then for the apparent field we have found there

$$z=\tfrac{A-\eta}{\eta}\cdot\mathfrak{q}a\xi;$$

but here $\mathfrak{q}$ cannot be accepted greater, than the radius of the aperture of the second lens to become $=\left(1-\tfrac{\eta}{A}\right)x$, clearly which cannot be greater than the aperture; hence we deduce $\mathfrak{q}=\tfrac{4x}{Aa}$; from which it is concluded :

$$z=\tfrac{(A-\eta)x}{A\eta}.$$

For the place of the eye there is

$$O=\tfrac{\eta q}{(A-\eta)}\cdot\tfrac{h}{m};$$

which since it shall be negative, it will be required for the eye to be applied directly, and since the lenses themselves are close together, hence no colored margin will be need to be worried about.

Therefore now the radius of confusion must be considered chiefly, which is [Book I, Supp. VII.]

$$\tfrac{\mu m x^3}{4a^2h}\left(\tfrac{\lambda}{\mathfrak{A}^3}+\tfrac{\nu}{A\mathfrak{A}}-\tfrac{\lambda'}{A^3P}\right),$$

where the latter term will be positive on account of $A<0$ and thus this quantity will be greater than zero always ; whereby here the whole matter may be reduced to this, so that this same amount may be rendered a minimum, which can be done, since the letters A and $\mathfrak{A}$ at this stage shall be left to our choice. It is at once apparent as small a value as possible is required to be effected by the letters λ and λ', which must be attributed to be 1, and since the quantity P may differ little from unity, we may define the letter $\mathfrak{A}$ or A thus, so that this formula

$$\tfrac{1}{\mathfrak{A}^3}-\tfrac{1}{A^3}+\tfrac{\nu}{A\mathfrak{A}}$$

may become a minimum. But before we may differentiate that, we will consider the relation between $\mathfrak{A}$ and A more carefully, which can be expressed thus:

$$\tfrac{1}{\mathfrak{A}}=1+\tfrac{1}{A};$$

from which it will be clear at once to be

$$\frac{d\mathfrak{A}}{\mathfrak{A}^2} = \frac{dA}{A^2}$$

or

$$d\mathfrak{A} : dA = \mathfrak{A}^2 : A^2;$$

whereby, if that formula may be differentiated and put equal to zero, in place of the differentials $d\mathfrak{A}$ and dA it will be allowed to write the proportionals of these $\mathfrak{A}^2$ and A^2, from which the following equation results:

$$\frac{3}{\mathfrak{A}^2} - \frac{3}{A^2} + \frac{v}{\mathfrak{A}} + \frac{v}{A} = 0,$$

which clearly is resolved into these factors :

$$\left(v + \frac{3}{\mathfrak{A}} - \frac{3}{A}\right)\left(\frac{1}{\mathfrak{A}} + \frac{1}{A}\right) = 0,$$

thus so that either one or the other of these factors must be equal to zero; but the first factor equated to zero gives

$$v + 3 + \frac{3}{A} = \frac{3}{A} \text{ or } v + 3 = 0;$$

since which shall be unable to be done, we will equate the other factor to zero and we will find

$$1 + \frac{2}{A} = 0 \text{ or } A = -2 \text{ and } \mathfrak{A} = 2.$$

With which values substituted into our equation for removing the confusion we will have

$$\frac{\mu m x^3}{a^2 h}\left(\frac{1}{8} + \frac{1}{8P} - \frac{v}{4}\right) = \frac{1}{k^3}$$

or

$$\mu m x^3 \left(2 - 2v + \tfrac{1}{2}\eta\right) = \frac{8a^2 h}{k^3},$$

and because there becomes $a = \frac{2}{2+\eta} \cdot \frac{h}{m}$, there will be

$$x^3 = \frac{4 \cdot 2h^3}{(2+\eta)^2 \mu m^3 k^3 \left(2 - 2v + \frac{1}{2}\eta\right)}$$

and thus

$$x = \frac{4h}{km} \sqrt[3]{\frac{1}{(2+\eta)^2 \mu (4 - 4v + \eta)}};$$

with which value found for *x,* everything which pertains to the construction of the microscopes, are determined in the following manner.

Construction of this microscope

I. Distance of the object before the first lens

$$a = \frac{2}{2+\eta} \cdot \frac{h}{m}.$$

II. For the first lens the focal length is

$$p = 2a = \frac{4}{2+\eta} \cdot \frac{h}{m},$$

and since there is $\lambda = 1$,

$$\text{the radius} \begin{cases} \text{anterior} = \frac{p}{2\rho - \sigma} \\ \text{posterior} = \frac{p}{2\sigma - \rho}; \end{cases}$$

of which the radius of the aperture must be $= x$.

III. But the interval between the prior and posterior lens is assumed to be $= \eta a = \frac{1}{2}\eta p$, where η will be agreed to be assumed so small, as the proximity of the lenses demands, lest they may not touch each other.

IV. For the posterior lens the focal length is

$$q = (2+\eta)a = \left(1 + \tfrac{1}{2}\eta\right)p,$$

and since $\lambda' = 1$, the

$$\text{anterior radius} = \frac{q}{\rho} \quad \text{and} \quad \text{posterior radius} = \frac{q}{\sigma};$$

the aperture of which lens must be given, of which the radius $= \left(1 + \frac{1}{2}\eta\right)x$ and therefore a little greater than that of the first lens, which indeed in practice is not accustomed to be attended to, where the posterior lens is left the whole aperture.

V. The eye must be applied in contact to the posterior lens and then an area in the object will be discerned, the radius of which will be $z = \frac{2+\eta}{2\eta} \cdot x$, from which again it is understood as small a fraction must be assumed for η, as the circumstances permit.

VI. For the order of clarity we have found $y = \left(1 + \frac{1}{2}\eta\right)x$, from which for the manner set out above, by which we have measured the clarity, the measure of the clarity will be

$$=20y=(20+10\eta)x.$$

COROLLARY 1

57. So far from the preceding, these microscopes which depend on a single lens are to be preferred, as far as here the value of x produced will be greater than before ; but before we had found

$$x=\frac{h}{km}\sqrt[3]{\frac{1}{\mu}},$$

now truly

$$x=\frac{h}{km}\sqrt[3]{\frac{64}{(2+\eta)^2\mu(4-4v+\eta)}},$$

thus so that with η ignored the present value of x shall be to the preceding as $\sqrt[3]{\frac{4}{1-v}}$ to 1, which ratio on account of $v=\frac{1}{5}$ is reduced to around this value :

$\sqrt[3]{5}:1=1,70998:1=171:100$ approximately, or as 12 : 7.

COROLLARY 2

58. Hence therefore it is apparent a double lens of this kind brings a significant gain, since the order of the clarity may be almost twice as great and thus the latter inconvenience mentioned above § 55 now shall be notably diminished. Truly on the other hand the first inconvenience in the proximity of the object position here is increased by a very small amount, but so very small, just as if the difference shall be unnoticed, nor here also the limitation of the field to have suffered any delay.

SCHOLIUM 1

59. Therefore it is required to be considered especially from the determination of all these various kinds of telescopes, that a small fraction may be allowed to be assumed for η, as indeed, where it was enacted above with regard to the telescopes, we have reduced as far as $\frac{1}{50}$; moreover it is seen easily by no means useful to have such a diminution with very thin lenses, since in the account of the focal lengths from these small lenses in no way may so great a thinness be able to be given as with greater lenses. Indeed for which it will be observed, if the focal lengths of the two lenses were 50 in., generally nothing impedes, why a separation of these of one inch may not be put in place ; but if the focal lengths of the two lenticules shall be only $\frac{1}{10}$ in., certainly in no way can a separation of these $=\frac{1}{500}$ in. be put in place ; from which it may be seen with some doubt, whether a value of the letter η smaller than $\frac{1}{5}$ may be given here. Indeed in the case mentioned, where the focal lengths of the two lenticules $=\frac{1}{10}$ dig., it will be with difficulty to take

the trouble over these so fine, so that the separation of these may not exceed $\frac{1}{50}$ in., lest of course they may touch each other ; as we may set out the measure in the following example with more care.

EXAMPLE

60. Therefore there shall be $\eta = \frac{1}{5}$ and its glass shall be of the kind, for which the refraction is $n = 1,55$, but we may grant to the letter *k* the value as before $= 20$ and in the usual manner we may assume $h = 8$ in., from which we will obtain the following determinations for the construction of the microscope.

Construction of microscopes of this kind

I. Distance of the object before the lens $a = \frac{7,273}{m}$ in.

II. Focal length of the first lens $p = \frac{14,546}{m}$ in., from which its construction will be had thus:

$$\text{radius} \begin{cases} \text{of the anterior face } = \frac{-p}{1,2460} = -0,8025p = \frac{11,6742}{m} \text{ in.} \\ \text{of the posterior face} = \frac{p}{3,0641} = +0,32636p = \frac{4,7472}{m} \text{ in.} \end{cases}$$

III. Now the value of x is sought, which will be

$$x = \frac{8}{5m}\sqrt[3]{\frac{1}{0,9381 \times 3,2696 \times 4,84}} = \frac{0,6510}{m} \text{ in.}$$

and thus the radius of the aperture of this lens is obtained.

IV. Moreover the interval between this lens and the posterior lens

$$= \tfrac{1}{5}a = \frac{1,455}{m} \text{ in.}$$

V. Focal length of the posterior lens

$$q = \frac{16,001}{m} \text{ in.,}$$

from which its construction will be had thus :

$$\text{radius}\begin{cases}\text{of the anterior face} = \frac{q}{0,1907} = 5,2438q = \frac{83,9065}{m}\ \text{in.}\\ \text{of the posterior face} = \frac{q}{1,6274} = 0,61447q = \frac{9,8322}{m}\ \text{in.}\end{cases}$$

VI. For this lens it suffices to give the aperture a little greater than the preceding lens and the eye will be required to be applied directly to that.

VII. We gave found for the degree of clarity $y = \frac{0,716}{m}$ in.; from which, if the clarity may be measured as above , we will have the measure of clarity $20y = \frac{14,32}{m}$.

VIII. But for the apparent area its radius is deduced $z = \frac{8,580}{m}$ in.

SCHOLIUM 2

61. Here therefore a special prerogative consists in this before simple lenses, that a notably greater clarity may be shown ; but if we do not wish for greater clarity and thus we may attribute a smaller aperture to our lenses , as long as we may perceive a greater distinction, which quantity certainly is valued none the less. Since here we may be able to follow with this significant convenience for two lenses, it is easily understood to be able to be obtained with much more convenience for triple lenses, which can be increased even further for a quadruple lens. Here clearly I am discussing convex lenses, in as much as these may be joined together, thus just as though they may imitate a single lens used in the microscope ; indeed if we may wish to make use of concave lenses, the confusion may be removed completely, thus so that then it be able concede so great an aperture with lenses, as great as the figure of these permits; which argument we will treat more carefully in the following chapter.

PROBLEM 2

62. *If a microscope may be determined by three convex lenses placed close to each other, to determine its construction, so that for a given magnification and with a given order of distinction it may represent with clarity the largest object for which it can be done.*

SOLUTION

Since here three lenses occur, the two intervals between these are expressed thus: the first

$$\alpha + b = Aa\left(1 - \frac{1}{P}\right)$$

and the second

$$\beta + c = -\frac{AB}{P} \cdot a\left(1 - \frac{1}{Q}\right);$$

which since they must be minima, each may be put $=\eta a$; from which there is deduced:

$$\frac{1}{P}=1-\frac{\eta}{A},\quad P=\frac{A}{A-\eta},$$

then

$$\frac{1}{Q}=1+\frac{P\eta}{AB}\quad \text{and thus}\quad Q=\frac{AB}{AB+P\eta}=\frac{(A-\eta)B}{(A-\eta)B+\eta}.$$

Again since all three lenses must be convex or the focal lengths of these *p, q ,r* positive, we obtain these conditions:

$$p=\mathfrak{A}a>0,\quad q=-\frac{A\mathfrak{B}}{P}\cdot a>0,\quad r=\frac{AB}{PQ}>0;$$

from which it is apparent at first $\mathfrak{A}$ must be positive; but nothing is defined about *A* at this stage .
But the two latter focal lengths q and r may be considered, and since there shall be $\frac{q}{r}=-\frac{\mathfrak{B}Q}{B}$, $-\frac{\mathfrak{B}}{B}=\mathfrak{B}-1$ must become a positive quantity and thus $\mathfrak{B}>1$ and hence $\mathfrak{B}$ negative, from which it is evident $A<0$ and hence $\mathfrak{A}>1$.
So that it may apply to the colored margin, since these three lenses may pretend to be a single lens, nothing thus is required to be feared. Whereby we may consider the radius of confusion

$$\frac{\mu m x^3}{a^2 h}\left(\frac{\lambda}{\mathfrak{A}^3}+\frac{\nu}{A\mathfrak{A}}-\frac{\nu}{A^3P}\left(\frac{\lambda'}{\mathfrak{B}^3}+\frac{\nu}{B\mathfrak{B}}\right)+\frac{\lambda''}{A^3B^3PQ}\right)=\frac{1}{k^3},$$

in which all the affected terms of the letters λ are positive; from which it required to be effected, so that the minimum value of this formula may be agreed on, or perhaps may not differ very much from the minimum value, which finally $\lambda,\ \lambda',\ \lambda''$ may be given the value of the first letter, which is one, and since the letters *P, Q* shall differ little from unity, also unity may be written in place of these which leads the following formula to a minimum :

$$\frac{1}{\mathfrak{A}^3}+\frac{\nu}{A\mathfrak{A}}-\frac{1}{A^3}\left(\frac{1}{\mathfrak{B}^3}+\frac{\nu}{B\mathfrak{B}}-\frac{1}{B^3}\right).$$

Clearly the question now is reduced to this, so that both the letters *A* and *B* may be defined thus, so that the value of this formula may become a minimum. Therefore at first we will consider only the letter $\mathfrak{B}$, and it is evident must be taken thus, so that the formula

$$\frac{1}{\mathfrak{B}^3}+\frac{\nu}{B\mathfrak{B}}-\frac{1}{B^3}$$

may become a minimum, which since it shall be similar to the preceding problem, in the same manner there will be found $\mathfrak{B}=2$ and $B=-2$. Therefore with these values taken our formula will emerge

$$\frac{1}{\mathfrak{A}^3}+\frac{v}{A\mathfrak{A}}-\frac{1}{4A^3}+\frac{v}{4A^3};$$

from which the letters A and $\mathfrak{A}$ remain to be investigated from the nature of the minima, and since there shall be, as we have observed before,

$$\frac{1}{\mathfrak{A}}=1+\frac{1}{A}$$

and hence will give this differential equation

$$d\mathfrak{A}:dA=\mathfrak{A}^2:A^2,$$

which is resolved easily into these factors:

$$(v+3)(1+\tfrac{3}{2A})(1+\tfrac{1}{2A})=0,$$

which can be done in a twofold manner; in the first place, if $A=-\frac{3}{2}$ and thus $\mathfrak{A}=3$, then truly also, if $A=-\frac{1}{2}$ and hence $\mathfrak{A}=-1$; from which it is understood only the first solution can be used. Whereby for the solution of our problem we may put

$$\mathfrak{A}=3,\quad A=-\tfrac{3}{2}\quad\text{and}\quad \mathfrak{B}=2,\quad B=-2$$

and there will become :

$$1.\quad P=\frac{3}{3+2\eta},\quad Q=\frac{3+2\eta}{3+3\eta},$$

2. Truly also $p=3a,\ q=\frac{3}{P}\cdot a=(3+2\eta)a,\ r=\frac{3}{PQ}\cdot a=3(1+\eta)a.$

Moreover the formula for the magnification given above becomes here

$$m=PQ\cdot\frac{h}{a}=\frac{1}{1+\eta}\cdot\frac{h}{a},$$

from which we deduce

$$a=PQ\cdot\frac{h}{m}=\frac{1}{1+\eta}\cdot\frac{h}{m}.$$

From which observed we will consider again the equation for the confusion, which with all these values substituted will adopt this form:

$$\frac{(1+\eta)^2\mu m^3x^3}{h^3}\left(\frac{1}{27}\left(1+\frac{1}{P}+\frac{1}{PQ}\right)-\frac{2v}{27}\left(3+\frac{1}{P}\right)\right)=\frac{1}{k^3},$$

which again is reduced to this form :

$$\frac{(1+\eta)^2 \mu m^3 x^3}{27h^3}\left(3+\tfrac{5}{3}\eta-4\nu\left(2+\tfrac{1}{3}\eta\right)\right)=\frac{1}{k^3},$$

thus so that there shall become:

$$\frac{kmx}{3h}\sqrt[3]{(1+\eta)^2\,\mu\left(3+\tfrac{5}{3}\eta-4\nu\left(2+\tfrac{1}{3}\eta\right)\right)}=1,$$

from which the value of x is deduced easily, which provides the radius of the aperture of the first lens. Just as with care it will be allowed to find the two remaining apertures, as the form of these permits. Hence moreover there will be had for the clarity

$$y=\frac{hx}{ma}=(1+\eta)x$$

and hence the measure of the clarity $=20y$, if indeed x may be expressed in inches.

Finally it remains, so that we may determine the area more accurately the observed within the object viewed, the radius of which defined above z is defined in general thus:

$$z=\frac{\mathfrak{q}+\mathfrak{r}}{ma-h}\cdot ah\xi=Ma\xi$$

on putting

$$M=\frac{\mathfrak{q}+\mathfrak{r}}{ma-h}\cdot h;$$

moreover there will be

$$ma-h=-\frac{\eta}{1+\eta}\cdot h$$

and thus

$$M=-\frac{(\mathfrak{q}+\mathfrak{r})(1+\eta)}{\eta},$$

thus so that now $\mathfrak{q}$ and $\mathfrak{r}$ must be viewed as negative; but now the first fundamental equation must be adjoined

$$\mathfrak{B}\mathfrak{q}=(P-1)M=\frac{-2\eta}{3+2\eta}\cdot M$$

and hence

$$\mathfrak{q}=-\frac{\eta}{3+2\eta}\cdot M;$$

where if the value may be substituted in place of M, there will become

$$\mathfrak{q}=\frac{1+\eta}{2+\eta}\cdot\mathfrak{r};$$

so that it extends to the letter $\mathfrak{r}$, in place of that there may be taken unity, if both sides of the final lens may be taken equally convex ; but since it may be able to be taken almost plano-convex , its aperture will be reduced to a half, thus so that there may be put in place $\mathfrak{r}=-\frac{1}{2}$; from which there becomes

$$\mathfrak{q} = -\frac{(1+\eta)}{2(2+\eta)}$$

and hence

$$M = \frac{(3+2\eta)(1+\eta)}{2\eta(2+\eta)};$$

whereby on taking $\xi = \frac{1}{4}$ the radius of the area viewed will be found :

$$z = \frac{(3+2\eta)(1+\eta)}{8\eta(2+\eta)} \cdot a = \frac{3+2\eta}{8\eta(2+\eta)} \cdot \frac{h}{m},$$

which field is so great, so that from that no ratio shall be had to be overcome. But then there must be the radii of the two posterior lenses

$$\text{second radius} = \frac{1+\eta}{8(2+\eta)} \cdot q = \frac{3+2\eta}{8(2+\eta)} \cdot \frac{h}{m},$$
$$\text{third radius} = \frac{1}{8} r = \frac{3}{8} \cdot \frac{h}{m},$$

if indeed these values shall be greater than these, which the clarity postulates, indeed which are for the second

$$= \frac{x}{P} = \frac{3+2\eta}{3} \cdot x,$$

and for the third

$$= \frac{x}{PQ} = (1+\eta)x.$$

SCHOLIUM 1

63. It is seen to be necessary here for an objection occurring to be met, because in a particular part of this solution not the same formula has been used, by which the radius of confusion is expressed, which will have led us to the minimum value, but another formula, from which it may be able to satisfy the discrepancy well enough, especially if as we have done before, only with the radius of confusion not to be led to an absolute minimum value and thus it can elicit a lesser value than that which we have found, if anyone who may wish to undertake the labour, to be able to elicit the same formula holding the letters *P* and *Q* requiring to be treated following the method of maxima or minima; then clearly other values for the letters *A* and *B* may be going to be found with our small discrepancies, which certainly may become involved with the most troublesome formulas, so that these neither may not be worth the effort of being set out, nor may it be able to hope for the most skillful execution. But now we ourselves will be able to be content with this value found, which certainly now is small enough, even if it shall not be the absolute minimum, indeed if thence we arrive at microscopes of this kind, which by far are required to be preferred for common simple microscopes, since they achieve a much greater order of clarity, evidently for a given distinction, thus so that, if

we may wish to reduce the clarity a little by restricting the aperture of the first lens by a small amount, then we may be able to follow by gaining the maximum distinction.

[By *clarity* Euler means the light intensity passing through the sample, while *distinction* is a measure of the amount of detail present in the image viewed ; clearly these effects work in opposition: the smaller the aperture, the less the clarity or light intensity, but the greater the distinction, as the *confusion* is reduced, using Euler's term describing some of the ways rays arising from different parts of the object can pass through the same part of the image.]

COROLLARY 1

64. Since the focal length of the first lens shall be

$$p = 3a = \frac{3}{1+\eta}\cdot\frac{h}{m}$$

and the numbers $\mathfrak{A} = 3$ and $\lambda = 1$, and for this lens

$$\begin{cases} \text{anterior radius} = \frac{p}{\sigma - \mathfrak{A}(\sigma-\rho)} \\ \text{posterior radius} = \frac{p}{\rho + \mathfrak{A}(\sigma-\rho)} \end{cases}$$

and thus there becomes

$$\begin{cases} \text{anterior radius} \quad = \frac{p}{3\rho - 2\sigma} \\ \text{posterior radius} \ = \frac{p}{3\sigma - 2\rho}. \end{cases}$$

COROLLARY 2

65. In a similar manner, since the focal length of the second lens shall be

$$q = (3+2\eta)a = \frac{3+2\eta}{1+\eta}\cdot\frac{h}{m}$$

and the numbers $\mathfrak{B} = 2$ and $\lambda' = 1$, there will become

$$\begin{cases} \text{anterior radius} = \frac{q}{2\rho - \sigma} \\ \text{posterior radius} \ = \frac{q}{2\sigma - \rho}. \end{cases}$$

For the third lens, on account of

$$r = 3(1+\eta)a = \tfrac{3h}{m}, \quad \mathfrak{C} = 1 \ \text{ and } \ \lambda'' = 1$$

there will become

$$\begin{cases} \text{anterior radius} = \tfrac{r}{\rho} \\ \text{posterior radius} = \tfrac{r}{\sigma}. \end{cases}$$

COROLLARY 3

66. So that the intervals between these three lenses may be reached, these are assumed equal to each other and now each is found to be $= \eta a = \frac{\eta}{1+\eta} \cdot \frac{h}{m}$, while clearly the object may be located before the first lens for a distance $a = \frac{1}{1+\eta} \cdot \frac{h}{m}$, which distance therefore is a little smaller than in case of a simple or double lens.

EXAMPLE

67. All there lenses may be prepared from common glass, for which there is $n = 1,55$, then truly there may be put in place $\eta = \frac{1}{5}$ and there may be assumed $k = 20$ and $h = 8$ in. and hence for the quantity x requiring to be determined this equation thus will be had

$$\tfrac{5mx}{6} \sqrt[3]{0,9381 \times 1,44 \times 1,4105} = 1,$$

which worked out gives $x = \frac{0,96796}{m}$ in., from which the following arises :

Construction of microscopes of this kind

I. Distance of the object before the first lens

$$a = \tfrac{20}{3m} = \tfrac{6,666}{m} \text{ in.}$$

II. For the first lens, of which the focal length $= p = \frac{20}{m}$, there will become

$$\text{radius} \begin{cases} \text{of the anterioris face } = -\tfrac{p}{2,6827} = -0,37276p = -\tfrac{7,4552}{m} \text{ in.} \\ \text{of the posterioris face} = \ +\tfrac{4,5008}{p} = 0,22218p = +\tfrac{4,4436}{m} \text{ in.,} \end{cases}$$

for which lens an aperture may be attributed, of which the radius

$$x = \tfrac{0,96795}{m} \text{ in.}$$

III. While for the distance

$$\eta a = \frac{4}{3m} \text{ in.} = \frac{1,3333}{m} \text{ in.}$$

the second lens shall be located, of which the focal length is

$$q = \frac{22,666}{m} \text{ in.},$$

and there will become

$$\text{radius}\begin{cases}\text{of the anterior face} & = -\frac{q}{1,2460} = -0,80257q = -\frac{18,1915}{m} \text{ in.}\\ \text{of the posterioris face} & = \frac{q}{3,0641} = +0,32636q = +\frac{7,3973}{m} \text{ in.}\end{cases}$$

of which the radius of the aperture shall be $\frac{1,55}{m}$ in.
and the distance to the following lens as before $= \frac{1,3333}{m}$ in.

IV. For the third lens, of which the focal length is $r = \frac{24}{m}$ in.
there will become

$$\text{radius}\begin{cases}\text{of the anterioris face} & = \frac{r}{0,1907} = 5,2438r = \frac{124,851}{m} \text{in.}\\ \text{of the posterioris face} & = \frac{r}{1,6274} = 0,61447r = \frac{14,7473}{m} \text{in.};\end{cases}$$

of which the aperture can be so great, so that is radius shall be $= \frac{6}{m}$ in., and the eye may be applied directly to this lens.

V. Then truly, since there shall be

$$y = \frac{6}{5}x = \frac{1,1615}{m} \text{ in.},$$

the measure of the clarity shall be $= -\frac{23,230}{m}$, which is greater by half than the preceding case.

VI. Finally, for the area of the object viewed, we will have its radius

$$z = \frac{85}{11m} = \frac{7,7273}{m}.$$

SCHOLIUM 2

68. Lest anyone may wonder why the area observed in this case to be so much greater than for the previous case, it will be observed in the preceding case the given aperture of the eyepiece lens not be given of greater aperture, than the order of clarity demands;

which therefore we have arranged, so that at this stage an exceedingly small aperture must be attributed to the first lens and thus in this case it was agreed both lenses to have no greater effect than that same aperture may demand, evidently so that the thickness of these therefore may be rendered smaller. But in the present case the matter will be resolved otherwise further, since now the first lens may acquire almost as great an aperture as its figure permits ; from which these lenses by necessity must have only a discus shape, of which the arc of the more curved face may contain thirty degrees of arc ; from which also the two remaining lenses will be able to be given a much greater aperture. Truly here generally it is required to observe in the preceding case also the apparent field now is about to become so great, so that everything may be able to be contained there. But since here the aperture of the first lens has been found to be so great, that a greater figure may not be allowed, it may be seen to be useless to pursue this further to four lenses, since we may be able to arrive at a calculation for putting in place a greater value for x ; truly on account of this same reason this same investigation will be of the greatest interest; indeed since so far we have not attributed a greater value to the letter k than twenty, from which an exceedingly moderate grade of the distinction may arise, now we will be able to assume a much greater value of this letter, whereby it may surely lead to the highest level in the perfection for these microscopes, and that without any detriment of the clarity. Indeed whatever have been observed for common microscopes at this stage, always some degrees of confusion were present ; from which, if microscopes of this kind may be produced now, which may represent objects much more distinctly, from observation they will reveal much to us, which were until now unknown, thus so that no longer shall a much greater magnification be desired with so much enthusiasm.

PROBLEM 3

69. *If a microscope may depend on four convex lenses and connected very close to each other, to investigate its construction, so that for a given magnification and with a given degree of distinction, an object may be represented with the greatest clarity which can happen.*

SOLUTION

Since here four equations occur, these three intervals are expressed thus:

$$\text{First } = Aa\left(1-\tfrac{1}{P}\right),$$
$$\text{second } = -\tfrac{AB}{P}\cdot a\left(1-\tfrac{1}{Q}\right),$$
$$\text{third} = \tfrac{ABC}{PQ}\cdot a\left(1-\tfrac{1}{R}\right);$$

[Recall that the first interval is equal to the first image distance α plus the second object distance b, where the ratio of these is expressed by $\alpha : b = -P$, since in this case a magnified virtual image is formed , hence we have in addition, $\alpha = Aa$, so that the interval $\alpha + b = Aa - \frac{\alpha}{P} = Aa - \frac{Aa}{P} = Aa\left(1 - \frac{1}{P}\right)$, and so on for the other intervals.]

which since they must be minima, we may put each interval $= \eta a$, from which we will obtain

$$\frac{1}{P} = 1 - \frac{\eta}{A} \text{ or } P = \frac{A}{A-\eta};$$

then

$$\frac{1}{Q} = 1 + \frac{P\eta}{AB} \text{ or } Q = \frac{(A-\eta)B}{(A-\eta)B+\eta};$$

truly the third

$$\frac{1}{R} = 1 - \frac{\eta}{(A-\eta)BC+C\eta} \text{ or } R = \frac{(A-\eta)BC+C\eta}{(A-\eta)BC+C\eta-\eta}$$

or

$$R = \frac{ABC-\eta(B-1)C}{ABC-\eta(B-1)C-\eta}.$$

Now since all four lenses must be convex or the focal lengths of these p, q, r, s positive, we arrive at these conditions:

$$p = \mathfrak{A}a > 0, \quad q = -\frac{A\mathfrak{B}}{P} \cdot a > 0, \quad r = \frac{ABC\mathfrak{C}}{PQ} \cdot a > 0, \quad s = -\frac{ABC}{PQR} \cdot a > 0;$$

from which we deduce initially

$$\frac{r}{s} = -\frac{R\mathfrak{C}}{C} > 0,$$

thus, so that

$$-\frac{\mathfrak{C}}{C} = \mathfrak{C} - 1$$

must become positive, from which there becomes $\mathfrak{C} > 1$, hence C is negative and thus AB positive.
[Recall that for this convex lens,
$\frac{1}{c} + \frac{1}{\gamma} = \frac{1}{r}$, and $\frac{1}{c} + \frac{1}{Cc} = \frac{1}{\mathfrak{C}c}$, giving $1 + \frac{1}{C} = \frac{1}{\mathfrak{C}}$, $C\mathfrak{C} + \mathfrak{C} = C$, or $\mathfrak{C} + \frac{\mathfrak{C}}{C} = 1$, from which the result follows, etc.]
Then since there shall be

$$\frac{q}{r} = -\frac{\mathfrak{B}Q}{B\mathfrak{C}},$$

there will become

$$-\frac{\mathfrak{B}}{B} = \mathfrak{B} - 1 > 0;$$

from which there becomes $\mathfrak{B} > 1$ and hence $B < 0$ and likewise $A < 0$; finally since there shall be

$$\frac{p}{q} = -\frac{\mathfrak{A}P}{A\mathfrak{B}},$$

also there must become

$$-\tfrac{\mathfrak{A}}{A}=\mathfrak{A}-1>0;$$

hence

$$\mathfrak{A}>1 \text{ and } A<0.$$

As far as it concerns the colored margin, regarding that as there is no need to be concerned, as we have noted above. Whereby for the radius of confusion we may consider this equation:

$$\frac{\mu m x^3}{a^2 h}\left(\frac{\lambda}{\mathfrak{A}^3}+\frac{v}{A\mathfrak{A}}-\frac{v}{A^3 P}\left(\frac{\lambda'}{\mathfrak{B}^3}+\frac{v}{B\mathfrak{B}}\right)+\frac{1}{A^3B^3PQ}\left(\frac{\lambda''}{\mathfrak{C}^3}+\frac{v}{C\mathfrak{C}}\right)-\frac{\lambda'''}{A^3B^3C^3PQR}\right)=\frac{1}{k^3},$$

in which all the terms of the affected letters λ are positive; from which the minimum value of this formula or perhaps a minimum not differing much from the minimum must be agreed on ; which in the end the first of the letters $\lambda,\ \lambda',\ \lambda'',\ \lambda'''$ may be attributed a minimum value $=1$, and since P, Q, R may differ little from unity, in place of these there may be written unity and the following formula reduced to a minimum :

$$\tfrac{1}{\mathfrak{A}^3}+\tfrac{v}{A\mathfrak{A}}-\tfrac{1}{A^3}\cdot W$$

with there becoming

$$W=\tfrac{1}{\mathfrak{B}^3}+\tfrac{v}{B\mathfrak{B}}-\tfrac{1}{B^3}\left(\tfrac{1}{\mathfrak{C}^3}+\tfrac{v}{C\mathfrak{C}}-\tfrac{1}{C^3}\right);$$

where evidently this formula W generally to be similar to that formula, as was required to be rendered a minimum in the preceding problem, only with this difference, in order that the letters B and C to be used here before they were A and B. Whereby now we know, in order that this formula W may become a minimum, there must be taken

$$\mathfrak{B}=3,\ \ B=-\tfrac{3}{2},\ \ \mathfrak{C}=2 \text{ and } C=-2;$$

with which values substituted the formula W will become

$$W=\tfrac{1}{9}-\tfrac{8v}{27}.$$

On account of which, our formula being reduced to a minimum will become

$$\tfrac{1}{\mathfrak{A}^3}+\tfrac{v}{A\mathfrak{A}}-\tfrac{1}{A^3}\left(\tfrac{1}{9}-\tfrac{8v}{27}\right),$$

which differentiated on account of $d\mathfrak{A}:dA=\mathfrak{A}^3:A^3$ produces

$$\tfrac{3}{\mathfrak{A}^2}+\tfrac{v}{A\mathfrak{A}}+\tfrac{v}{\mathfrak{A}}-\tfrac{3}{A^3}\left(\tfrac{1}{9}-\tfrac{8v}{27}\right)=0,$$

which again is reduced to this form:

$$3\left(\frac{1}{\mathfrak{A}^2}-\frac{1}{9A^2}\right)+v\left(\frac{1}{\mathfrak{A}}+\frac{1}{A}+\frac{8v}{9A^2}\right)=0;$$

in which if in place of $\frac{1}{\mathfrak{A}}$ its value $1+\frac{1}{A}$ may be written, and there will be produced

$$(v+3)\left(1+\frac{2}{A}+\frac{8}{9A^2}\right)=0;$$

which factor is resolved further into these two factors :

$$\left(1+\frac{4}{3A}\right)\left(1+\frac{2}{3A}\right),$$

of which the first equated to zero gives

$$A=-\frac{4}{3} \text{ and thus } \mathfrak{A}=4,$$

truly the following :

$$A=-\frac{3}{2} \text{ and hence } \mathfrak{A}=-2;$$

which therefore does not agree with our principles. On account of which the following values are assumed to be found for the minimum value required to be obtained :

$$\mathfrak{A}=4,\ A=-\frac{4}{3},\ \mathfrak{B}=3,\ B=-\frac{3}{2}\ ,\ \mathfrak{C}=2,\ C=-2;$$

from which values found above there will be expressed :

$$P=\frac{4}{4+3\eta},\ Q=\frac{4+3\eta}{4+5\eta},\ R=\frac{4+5\eta}{4+6\eta}$$

and the focal lengths :

$$p=4a,\ q=(4+3\eta)a,\ r=(4+5\eta)a \text{ and } s=(4+6\eta)a.$$

Moreover the formula for the magnification found above becomes here :

$$m=PQR\cdot\frac{h}{a}=\frac{4}{4+6\eta}\cdot\frac{h}{a},$$

from which there is gathered

$$a=\frac{2}{2+3\eta}\cdot\frac{h}{m}.$$

Therefore with these values substituted the equation for the confusion being removed will lead to this form:

$$\frac{\mu\left(1+\frac{3}{2}\eta\right)^2 m^3x^3}{1\cdot 64h^3}\left(1+\frac{1}{P}+\frac{1}{PQ}+\frac{1}{PQR}-2v\left(6+\frac{3}{P}+\frac{1}{PQ}\right)\right)=\frac{1}{k^3},$$

which with the values substituted for P, Q, R will be changed into this form:

$$\frac{k^3m^3x^3}{64h^3}\cdot\mu\left(1+\frac{3}{2}\eta\right)^2\left(4+\frac{7}{2}\eta-v\left(20+7\eta\right)\right)=1,$$

thus so that there shall become

$$\frac{kmx}{4h}\sqrt[3]{\mu\left(1+\frac{3}{2}\eta\right)^2\left(4+\frac{7}{2}\eta-v\left(20+7\eta\right)\right)}=1;$$

from which the value of x is defined readily ; which will give the radius of this aperture, unless it may be produced greater, in order that the first lens may be able to allow a greater aperture ; but if a greater may be produced, then the value of k may be increased as far as to that, so that the first lens may be able to take this aperture, and thus it will be apparent, whenever these microscopes shall be going to be endowed with an order of distinction ; clearly the maximum apertures corresponding to the lenses defined for the value taken for x and then from this equation the value of k may be deduced. And thus in this manner with x defined for the order of clarity we will have

$$y=\frac{2+3\eta}{2}\cdot x$$

and hence the measure of the clarity

$$=20y=\left(20+30\eta\right)x.$$

Concerning the area viewed in the object here I define nothing, since surely it shall be a maximum, if indeed the maximum apertures may be attributed to the individual lenses, of which they are large.

COROLLARY 1

70. Since for the first lens the focal length shall be

$$p=4a=\frac{8}{2+3\eta}\cdot\frac{h}{m}$$

and the numbers $\mathfrak{A}=4$ and $\lambda=1$, the radius

$$\begin{cases}\text{of the anterior face} = \frac{p}{\sigma-4(\sigma-\rho)}\\ \text{of the posterior face} = \frac{p}{\rho+4(\sigma-\rho)}.\end{cases}$$

COROLLARY 2

71. The construction of the three remaining lenses will be the same as in the preceding problem ; evidently for the second the radius

$$\begin{cases}\text{of the anterior face} = \frac{q}{\sigma-3(\sigma-\rho)}\\ \text{of the posterior face} = \frac{q}{\rho+3(\sigma-\rho)}\end{cases}$$

with there being

$$q = \frac{2(4+3\eta)}{2+3\eta}\cdot\frac{h}{m}.$$

For the third lens the radius

$$\begin{cases}\text{of the anterior face} = \frac{r}{\sigma-2(\sigma-\rho)}\\ \text{of the posterior face} = \frac{r}{\rho+2(\sigma-\rho)}\end{cases}$$

with there being

$$r = \frac{2(4+5\eta)}{2+3\eta}\cdot\frac{h}{m}.$$

Finally for the fourth lens, the radius

$$\begin{cases}\text{of the anterior face} = \frac{s}{\rho}\\ \text{of the posterior face} = \frac{s}{\sigma}\end{cases}$$

with there being

$$s = \frac{2(4+6\eta)}{2+3\eta}\cdot\frac{h}{m} = \frac{4h}{m}.$$

COROLLARY 3

72. As it extends to the intervals, all are equal to each other, evidently $=\eta a$; therefore any of these will be

$$= \frac{2\eta}{2+3\eta}\cdot\frac{h}{m},$$

evidently because the object before the first lens must be put in place at the distance

$$a=\frac{2}{2+3\eta}\cdot\frac{h}{m}.$$

EXAMPLE

73. Therefore all four lenses may be put in place made from common glass, for which $n=1,55$; then truly again there may be established $\eta=\frac{1}{5}$ and $h=8$ in., but truly k at this stage we may leave indefinite; from which this equation will be had:

$$\frac{kmx}{32}\sqrt[3]{0,9381\times1,69\times(-0,2776)}=1,$$

where the $-$ sign does not disturb the calculation, since here the discussion shall be concerned with the absolute quantity; from which there is found

$$kx=\frac{42,0692}{m},$$

from which it will now be apparent, if there may be taken $k=20$, the value of x going to be produced to become exceedingly large ; whereby we will investigate x from the figure of the lens and then hence the value of k itself, so that we may know the order of the distinction more carefully.

And thus we will have the following

Construction of these microscopes:

I. Distance of the object before the lens

$$a=\frac{80}{13m}=\frac{6,1538}{m}\text{ in.}$$

II. For the first lens, of which the focal length

$$p=\frac{24,6152}{m}\text{ in.},$$

there will be the radius

$$\begin{cases}\text{of the anterior face } =-\frac{p}{4,1194}=-0,24275p=-\frac{5,9755}{m}\text{ in.}\\ \text{of the posterior face} =+\frac{p}{5,9175}=+0,16842p=+\frac{4,1457}{m}\text{ in.};\end{cases}$$

the radius of which aperture will be able to be taken $x=\frac{1,0364}{m}$ in.; then truly for the order of distinction there will be approximately $k=41$; whereby since the distinction shall follow the cube of k itself, here the distinction will be eight times greater than in the preceding cases, where $k = 20$. Then truly the distance to the following lens will be $= \frac{1,2308}{m}$ in.

III. For the second lens, of which the focal length $q = \frac{28,308}{m}$ in., the radius

$$\begin{cases} \text{of the anterior face} = -\frac{q}{2,6827} = -0,3727q = -\frac{10,552}{m} \text{ in.} \\ \text{of the posterior face} = +\frac{q}{4,5008} = +0,2222q = +\frac{6,2896}{m} \text{ in.}; \end{cases}$$

of which the front aperture can be taken a little greater.
The distance to the following lens is as before.

IV. For the fourth lens, of which the focal length is $r = \frac{30,77}{m}$ in., the radius

$$\begin{cases} \text{of the anterior face} = -\frac{r}{1,2460} = -0,8025r = -\frac{24,693}{m} \text{ in.} \\ \text{of the posterior face} = +\frac{r}{3,0641} = +0,3264r = +\frac{10,043}{m} \text{ in.}; \end{cases}$$

of which the aperture again is a little larger than the preceding and the interval to the following is as before.

V. For the fourth lens, of which the focal length is $s = \frac{32}{m}$ in., the radius

$$\begin{cases} \text{of the anterior face } = \frac{s}{0,1907} = 5,2438s = \frac{167,8016}{m} \text{ in.} \\ \text{of the posterior face} = +\frac{s}{1,6274} = 0,6145s = \frac{19,6630}{m} \text{ in.}; \end{cases}$$

of which the aperture anew shall be a little greater, and the eye is applied straight to that.

VI. For the order of the clarity is $y = 1,3x = \frac{1,3473}{m}$ in., from which the measure of the clarity shall be $\frac{26,946}{m}$, thus so that, unless we may wish to magnify by more than twenty six times, up to this stage we may be able to enjoy full clarity.

SCHOLIUM

74. Behold therefore a kind of simple microscopes, which is seen to merit the maximum attention, since without detriment they will represent objects much more distinctly, than generally is accustomed to happen. Yet meanwhile we are forced to admit these instruments cannot be applied to outstanding magnifications ; if indeed we may wish a magnification $m = 100$, indeed lenses up to this point will be able to be prepared

easily, but the distance of objects will become only $\frac{6}{100}$ in., which distance certainly shall be able to become extremely small, especially if the objects were not exceedingly smooth. But yet a magnification to 150 or 200 perhaps may be able to be pushed, if the greatest need may demand. But then we will prefer compound microscopes of this kind, which not only represent objects equally clearly and distinctly, but also they will allow a greater length of the objects. But since in this chapter we have considered only convex lenses, now also we will introduce concave lenses, from which thus it will be able to effect, so that confusion may vanish completely; but another inconvenience will upset the execution, while evidently there will be a need for exceedingly small lenses, just as we will see in the following chapter.

A NOTE

75. In this example it happens to be singularly noteworthy, that a formula for negative confusion will be produced ; but it is clear that would be going to be produced either greater or less, if another value were attributed to the letter η ; since also plainly this confusion will be reduced to zero, if η may be taken thus, so that this formula will be had:

$$4+\tfrac{7}{2}\eta-\nu(20+7\eta)=0;$$

from which for the case of the example there will follow

$$\eta=\tfrac{8}{7}\cdot\tfrac{5\nu-1}{1-2\nu}=\tfrac{1,3040}{3,7436}=0,34833;$$

this is just about true, if we may assume $\eta=\frac{1}{3}$. Besides truly this confusion will be able to be reduced to zero in another way, clearly if for the first lens the number λ shall not be equal to unity, but we may put $\lambda=1+\omega$; for then in the formula that first term 4 will be able to be increased by a small amount ω, thus so that $\omega-0,2776$ will be produced in the case of the example, from which there will become $\omega=0,2776$ and the first lens will be able to be constructed from the number $\lambda=1+\omega=1,2776$; the construction of the remaining lenses to remain the same. Therefore for the first lens this construction will be able to be substituted on account of $\tau\sqrt{(\lambda-1)}=0,4768$: the radius

$$\begin{cases}\text{of the anterior face}=\frac{-p}{4,1194-0,4768}=-\frac{p}{3,64266}\\ \text{of the posterior face}=\frac{p}{5,9375-0,4768}=\frac{p}{5,4607}\end{cases}$$

or

$$\text{the radius}\begin{cases}\text{of the anterior face}=-0,27453p=-\frac{6,758}{m}\text{ in.}\\ \text{of the posterior face}=+0,18312p=\frac{+4,508}{m}\text{ in.;}\end{cases}$$

so that if this lens may be substituted in place of the first lens, with all the remaining kept the same, from these microscopes at this stage a greater degree of perfection will be agreed on, especially since now the first lens allows a greater aperture.

SECTIO PRIMA.
DE
MICROSCOPIIS SIMPLICIBUS.

CAPUT II

DUABUS PLURIBUSVE LENTIBUS CONVEXIS

INTER SE PROXIME IUNCTIS CONSTANTIBUS

PROBLEMA 1

56. *Si lens duplicata ex duabus lentibus convexis sit composita, pro data multiplicatione cuiusmodi microscopium construere, quod obiecta, quantum fieri potest, clare et distincte repraesentet.*

SOLUTIO

Quoniam hic binae lentes sibi proxima iungendae occurrunt, ex formulis nostris generalibus earum intervallum erit

$$\alpha + b = Aa\left(1 - \tfrac{1}{P}\right);$$

quod cum debeat esse minimum, statuatur $= \eta a$ denotante η fractionem tam parvam, quam circumstantiae permittunt, atque hinc colligemus

$$P = \tfrac{A}{A-\eta}$$

deinde, quia utraque lens debet esse convexa seu utriusque distantia focalis positiva, tam haec quantitas

$$p = \mathfrak{A}a$$

quam ista

$$q = -\tfrac{A}{P}a = -\left(A-\eta\right)a$$

debet esse positiva ideoque $\mathfrak{A} > 0$, at $A < 0$, id quod fit, si $\mathfrak{A} > 1$. Hoc notato multiplicatio nobis praebet

$$m = \tfrac{Ph}{a} = \tfrac{A}{A-\eta} \cdot \tfrac{h}{a},$$

unde definitur distantia obiecti

$$a = \tfrac{A}{A-\eta} \cdot \tfrac{h}{m},$$

ita ut sit

$$ma-h=\tfrac{\eta}{A-\eta}\cdot h;$$

deinde si semidiameter aperturae primae lentis ponatur $=x$, secundae lentis debet esse $\left(1-\tfrac{\eta}{A}\right)x$; unde pro gradu claritatis fiat

$$y=\tfrac{hx}{ma}\text{ seu }y=\left(1-\tfrac{\eta}{A}\right)x,$$

ita ut ob $A<0$ lentium intervallum claritatem augeat. Deinde pro campo apparente ibi invenimus

$$z=\tfrac{A-\eta}{\eta}\cdot \mathfrak{q}a\xi;$$

at hic $\mathfrak{q}$ maius accipi nequit, quam ut semidiameter aperturae secundae lentis fiat $=\left(1-\tfrac{\eta}{A}\right)x$, quippe quae apertura maior esse nequit; hinc colligimus $\mathfrak{q}=\tfrac{4x}{Aa}$; unde concluditur

$$z=\tfrac{(A-\eta)x}{A\eta}.$$

Pro loco autem oculi est

$$O=\tfrac{\eta q}{(A-\eta)}\cdot\tfrac{h}{m};$$

quae cum sit negativa, oculum immediate adplicari oportet, et quia lentes sibi sunt proximae, hinc nullus margo coloratus erit metuendus.

Nunc igitur potissimum considerari debet semidiameter confusionis, quae est [Lib. I, Supplem. VII.]

$$\tfrac{\mu mx^3}{4a^2h}\left(\tfrac{\lambda}{\mathfrak{A}^3}+\tfrac{\nu}{A\mathfrak{A}}-\tfrac{\lambda'}{A^3P}\right),$$

ubi posterius membrum ob $A<0$ erit positivum ideoque haec quantitas semper maior nihilo; quamobrem hic totum negotium eo redit, ut ista quantitas reddatur minima, id quod fieri potest, cum litterae A et $\mathfrak{A}$ adhuc arbitrio nostro sint relictae. Ad hoc efficiendum statim patet litteris λ et λ' minimum valorem, quem capere possunt, qui est 1, tribui debere, et cum quantitas P parum ab unitate differat, litteram $\mathfrak{A}$ vel A ita definiamus, ut haec formula

$$\tfrac{1}{\mathfrak{A}^3}-\tfrac{1}{A^3}+\tfrac{\nu}{A\mathfrak{A}}$$

fiat minimum. Ante quam autem eam differentiemus, relationem inter $\mathfrak{A}$ et A attentius consideremus, quae ita exprimi potest:

$$\tfrac{1}{\mathfrak{A}}=1+\tfrac{1}{A};$$

unde statim liquet esse

$$\frac{d\mathfrak{A}}{\mathfrak{A}^2}=\frac{dA}{A^2}$$

seu

$$d\mathfrak{A}:dA=\mathfrak{A}^2:A^2;$$

quare, si illa formula differentietur et nihilo aequalis ponatur, loco differentialium $d\mathfrak{A}$ et dA scribere licebit eorum proportionalia $\mathfrak{A}^2$ et A^2, ex quo sequens aequatio resultat:

$$\frac{3}{\mathfrak{A}^2}-\frac{3}{A^2}+\frac{v}{\mathfrak{A}}+\frac{v}{A}=0,$$

quae manifesto in hos factores resolvitur:

$$\left(v+\frac{3}{\mathfrak{A}}-\frac{3}{A}\right)\left(\frac{1}{\mathfrak{A}}+\frac{1}{A}\right)=0,$$

ita ut vel unus vel alter horum factorum debeat esse nihilo aequalis; prior autem factor nihilo aequatus dat

$$v+3+\frac{3}{A}=\frac{3}{A} \text{ seu } v+3=0;$$

quod cum fieri nequeat, alterum factorem nihilo aequemus et inveniemus

$$1+\frac{2}{A}=0 \text{ seu } A=-2 \text{ et } \mathfrak{A}=2.$$

Quibus valoribus in aequatione nostra pro confusione tollenda substitutis habebimus

$$\frac{\mu m x^3}{a^2 h}\left(\frac{1}{8}+\frac{1}{8P}-\frac{v}{4}\right)=\frac{1}{k^3}$$

seu

$$\mu m x^3\left(2-2v+\tfrac{1}{2}\eta\right)=\frac{8a^2h}{k^3},$$

et quia est $a=\frac{2}{2+\eta}\cdot\frac{h}{m}$, erit

$$x^3=\frac{4\cdot 2h^3}{(2+\eta)^2\mu m^3k^3\left(2-2v+\frac{1}{2}\eta\right)}$$

ideoque

$$x=\frac{4h}{km}\sqrt[3]{\frac{1}{(2+\eta)^2\mu(4-4v+\eta)}};$$

quo valore pro x invento omnia, quae ad constructionem microscopii pertinent, determinantur sequenti modo.

Constructio huius microscopii

I. Distantia obiecti ante lentem priorem

$$a = \frac{2}{2+\eta} \cdot \frac{h}{m}.$$

II. Pro lente priore est distantia focalis

$$p = 2a = \frac{4}{2+\eta} \cdot \frac{h}{m},$$

et quia est $\lambda = 1$, erit

$$\text{radius}\begin{cases}\text{anterior} = \frac{p}{2\rho-\sigma}\\ \text{posterior} = \frac{p}{2\sigma-\rho};\end{cases}$$

cuius aperturae semidiameter debet esse $= x$.

III. Intervallum autem inter lentem priorem et posteriorem sumtum est $= \eta a = \frac{1}{2}\eta p$, ubi η tam parvum assumi conveniet, quam proximitas lentium, ne se mutuo tangant, postulat.

IV. Pro lente posteriore distantia focalis est

$$q = (2+\eta)a = \left(1+\tfrac{1}{2}\eta\right)p,,$$

et quia est $\lambda' = 1$, fiet eius radius

$$\text{anterior} = \frac{q}{\rho} \quad \text{et} \quad \text{posterior} = \frac{q}{\sigma};$$

cui lenti apertura dari debet, cuius semidiameter $= \left(1+\frac{1}{2}\eta\right)x$ ideoque tantillo maior quam ea primae lentis, quod quidem in praxi non solet attendi, ubi posterior lens tota aperta relinquitur.

V. Lenti posteriori oculus immediate debet adplicari et tum cernet in obiecto spatium, cuius semidiameter erit $z = \frac{2+\eta}{2\eta} \cdot x$, unde intelligitur iterum pro η tam parvam fractionem sumi debere, quam circumstantiae permittunt.

VI. Pro gradu claritatis invenimus $y = \left(1+\frac{1}{2}\eta\right)x$, unde pro modo supra exposito, quo claritatem mensuramus, erit mensura claritatis

$$= 20y = \left(20+10\eta\right)x.$$

COROLLARIUM 1

57. Eatenus haec microscopia praecedentibus, quae lente simplici constant, sunt anteferenda, quatenus hic valor ipsius x hic maior prodit quam ante; ante autem inveneramus

$$x = \frac{h}{km}\sqrt[3]{\frac{1}{\mu}},$$

nunc vera

$$x = \frac{h}{km}\sqrt[3]{\frac{64}{(2+\eta)^2\mu(4-4v+\eta)}},$$

ita ut neglecto η praesens valor ipsius x sit ad praecedentem uti $\sqrt[3]{\frac{4}{1-v}}$, ad 1, quae ratio ob $v=\frac{1}{5}$ circiter reducitur ad hanc:

$$\sqrt[3]{5}:1 = 1,70998:1 = 171:100 \text{ proxime}$$

sive uti 12 : 7.

COROLLARIUM 2

58. Hinc ergo patet huiusmodi lentem duplicatam insigne lucrum afferre, cum gradum claritatis fere duplo maiorem largiatur sicque posterius incommodum supra § 55 memoratum iam notabiliter sit imminutum. Contra vero prius incommodum in proximitate obiecti situm hic aliquantillum augetur, sed tam parum, ut differentia sit quasi insensibilis, neque etiam limitatio campi hic ullam moram facesset.

SCHOLION 1

59. Pro omnimoda igitur horum telescopiorum determinatione inprimis perpendendum est, quam parvam fractionem pro η assumere liceat, quam quidem, ubi supra de telescopiis agebatur, usque ad $\frac{1}{50}$ imminuimus; facile autem perspicitur in tam exiguis lenticulis tantam diminutionem neutiquam locum habere posse, cum ratione distantiae focalis his lenticulis nullo modo tanta tenuitas dari possit quam maioribus lentibus. Cuilibet enim perspicuum erit, si distantia focalis duarum lentium fuerit 50 dig., nihil omnino impedire, quominus earum distantia unius digiti statuatur; at si duarum lenticularum distantia focalis tantum sit $\frac{1}{10}$ dig., nullo certe modo earum intervallum $=\frac{1}{500}$ dig. statui poterit; unde merito dubitandum videtur, num hic litterae η minor valor quam $\frac{1}{5}$ tribui possit. Casu enim modo allato, quo binarum lenticularum distantia focalis $=\frac{1}{10}$ dig., difficile erit eas tam graciles elaborare, ut earum intervallum non excedere debeat $\frac{1}{50}$ dig., ne scilicet se mutuo tangant; quam mensuram in sequenti exemplo adcuratius evolvamus.

EXEMPLUM

60. Sit igitur $\eta=\frac{1}{5}$ et vitrum eius sit speciei, pro qua refractio est $n=1,55$, litterae autem *k* tribuamus ut ante valorem $=20$ et more solito sumamus $h=8$ dig., unde pro constructione microscopii sequentes nanciscemur determinationes.

Constructio huiusmodi microscopiorum

I. Distantia obiecti ante lentem $a=\frac{7,273}{m}$ dig.
II. Distantia focalis lentis prioris $p=\frac{14,546}{m}$ dig.,
unde eius constructio ita se habebit:

$$\text{radius faciei}\begin{cases}\text{anterioris}=\frac{-p}{1,2460}=-0,8025p=\frac{11,6742}{m}\text{dig.}\\ \text{posterioris}=\frac{p}{3,0641}=+0,32636p=\frac{4,7472}{m}\text{dig.}\end{cases}$$

III. Nunc quaeratur valor ipsius *x,* qui erit

$$x=\frac{8}{5m}\sqrt[3]{\frac{1}{0,9381\times3,2696\times4,84}}=\frac{0,6510}{m}\text{ dig.}$$

sicque habetur semidiameter aperturae huius lentis.

IV. Intervallum autem inter hanc lentem et posteriorem

$$=\frac{1}{5}a=\frac{1,455}{m}\text{ dig.}$$

V. Lentis posterioris distantia focalis

$$q=\frac{16,001}{m}\text{ dig.,}$$

unde eius constructio ita se habebit:

$$\text{radius faciei}\begin{cases}\text{anterioris}=\frac{q}{0,1907}=5,2438q=\frac{83,9065}{m}\text{dig.}\\ \text{posterioris}=\frac{q}{1,6274}=0,61447q=\frac{9,8322}{m}\text{dig.}\end{cases}$$

VI. Huic lenti sufficit aperturam dare tantillo maiorem quam praecedentem eique oculum immediate adplicari oportet.

VII. Pro gradu claritatis invenimus $y=\frac{0,716}{m}$ dig.; unde, si claritas ut supra mensuretur, habebitur mensura claritatis $20y=\frac{14,32}{m}$.

VIII. Pro spatio autem apparente colligitur eius semidiameter $z = \frac{8,580}{m}$ dig.

SCHOLION 2

61. Hic igitur praecipua praerogativa prae lentibus simplicibus in hoc consistit, quod claritas notabiliter maior exhibeatur; sin autem non desideremus maiorem claritatem ideoque aperturam nostris lentibus minorem tribuamus, tanto maiorem distinctionem percipiemus, quod commodum certe non minus est aestimandum. Cum hic hoc insigne commodum duplicatione lentis simus assecuti, facile intelligitur triplicatione lentis multo maius commodum obtineri posse, quod lentem adeo quadruplicando adhuc ulterius augeri poterit. Hic scilicet loquor de lentibus convexis, quatenus eae sibi proxima iunctae, ita ut quasi unicam lentem mentiantur. in microscopio adhibentur; si enim etiam lentes concavas usurpare velimus, confusio plane tolli posset, ita ut tunc lentibus tanta apertura concedi posset, quantam earum figura admittit; quod argumentum in sequente capite adcuratius pertractabimus.

PROBLEMA 2

62. *Si microscopium constet tribus lentibus convexis proxime inter se iunctis, eius constructionem investigare, ut pro data multiplicatione et dato distinctionis gradu obiecta maxima, qua fieri potest, claritate repraesentet.*

SOLUTIO

Cum hic tres lentes occurrant, bina intervalla inter eas ita exprimuntur:
prius

$$\alpha + b = Aa\left(1 - \tfrac{1}{P}\right)$$

et posterius

$$\beta + c = -\tfrac{AB}{P} \cdot a\left(1 - \tfrac{1}{Q}\right);$$

quae cum esse debeant minima, utrumque statuatur ηa ; unde colligitur

$$\tfrac{1}{P} = 1 - \tfrac{\eta}{A}, \quad P = \tfrac{A}{A-\eta},$$

deinde

$$\tfrac{1}{Q} = 1 + \tfrac{P\eta}{AB} \quad \text{ideoque} \quad Q = \tfrac{AB}{AB+P\eta} = \tfrac{(A-\eta)B}{(A-\eta)B+\eta}.$$

Cum porro omnes tres lentes debeant esse convexae seu earum distantiae focales p, q, r positivae, adipiscimur has conditiones:

$$p = \mathfrak{A}a > 0, \quad q = -\tfrac{A\mathfrak{B}}{P} \cdot a > 0, \quad r = \tfrac{AB}{PQ} > 0;$$

unde primo patet esse debere $\mathfrak{A}$ positivum; circa A autem nihil adhuc definitur. Considerentur autem binae postremae distantiae focales q et r, et cum sit $\frac{q}{r} = -\frac{\mathfrak{B}Q}{B}$, debet esse $-\frac{\mathfrak{B}}{B} = \mathfrak{B} - 1$ quantitas positiva ideoque $\mathfrak{B} > 1$ et hinc $\mathfrak{B}$ negativum, unde manifestum fore $A < 0$ hincque $\mathfrak{A} > 1$.

Quod ad marginem coloratum attinet, quia hae tres lentes quasi unam lentem simplicem mentiuntur, nihil adeo erit metuendum. Quare aequationem pro semidiametro confusionis contemplemur

$$\frac{\mu m x^3}{a^2 h}\left(\frac{\lambda}{\mathfrak{A}^3} + \frac{v}{A\mathfrak{A}} - \frac{v}{A^3 P}\left(\frac{\lambda'}{\mathfrak{B}^3} + \frac{v}{B\mathfrak{B}}\right) + \frac{\lambda''}{A^3 B^3 PQ}\right) = \frac{1}{k^3},$$

in qua omnes termini litteris λ adfecti sunt positivi; unde efficiendum est, ut huic formulae minimus valor concilietur vel saltim valor a minimo non multum discrepans, quem in finem primo litteris λ, λ', λ'' valor minimus, qui est 1, tribuatur, et cum litterae P, Q parum ab unitate differant, earum loco quoque unitas scribatur et sequens formula ad minimum perducatur:

$$\frac{1}{\mathfrak{A}^3} + \frac{v}{A\mathfrak{A}} - \frac{1}{A^3}\left(\frac{1}{\mathfrak{B}^3} + \frac{v}{B\mathfrak{B}} - \frac{1}{B^3}\right).$$

Quaestio scilicet nunc eo redit, ut ambae litterae A et B ita definiantur, ut valor huius formulae fiat minimus. Consideremus igitur primo solam litteram $\mathfrak{B}$, et manifestum est eam ita accipi debere, ut formula

$$\frac{1}{\mathfrak{B}^3} + \frac{v}{B\mathfrak{B}} - \frac{1}{B^3}$$

fiat minima, quae cum similis sit formulae praecedentis problematis eodem modo reperietur $\mathfrak{B} = 2$ et $B = -2$. His igitur sumtis valoribus nostra formula evadet

$$\frac{1}{\mathfrak{A}^3} + \frac{v}{A\mathfrak{A}} - \frac{1}{4A^3} + \frac{v}{4A^3};$$

pro qua ex natura minimi litterae A et $\mathfrak{A}$ supersunt investigandae, et cum sit, ut ante observavimus,

$$\frac{1}{\mathfrak{A}} = 1 + \frac{1}{A}$$

hincque

$$d\mathfrak{A} : dA = \mathfrak{A}^2 : A^2$$

differentiatio dabit hanc aequationem, quae facile resolvitur in hos factores:

$$(v+3)(1+\tfrac{3}{2A})(1+\tfrac{1}{2A}) = 0,$$

id quod duplici modo fieri poterit; primo scilicet, si $A=-\frac{3}{2}$ ideoque $\mathfrak{A}=3$, tum vero etiam, si $A=-\frac{1}{2}$ hincque $\mathfrak{A}=-1$; ex quo intelligitur solam priorem solutionem locum habere. Quocirca pro solutione nostri problematis statuamus

$$\mathfrak{A}=3,\quad A=-\tfrac{3}{2}\ \text{ atque }\ \mathfrak{B}=2\ \text{ et }\ B=-2$$

fietque

$$1.\ P=\tfrac{3}{3+2\eta},\quad Q=\tfrac{3+2\eta}{3+3\eta},$$

2. vero etiam $p=3a,\ q=\frac{3}{P}\cdot a=(3+2\eta)a,\ r=\frac{3}{PQ}\cdot a=3(1+\eta)a.$

Formula autem pro multiplicatione supra data fit hic

$$m=PQ\cdot\tfrac{h}{a}=\tfrac{1}{1+\eta}\cdot\tfrac{h}{a},$$

unde deducimus

$$a=PQ\cdot\tfrac{h}{m}=\tfrac{1}{1+\eta}\cdot\tfrac{h}{m}.$$

Quibus notatis denuo consideremus aequationem pro confusione, quae his omnibus valoribus substitutis induet hanc formam:

$$\tfrac{(1+\eta)^2\mu m^3x^3}{h^3}\left(\tfrac{1}{27}\left(1+\tfrac{1}{P}+\tfrac{1}{PQ}\right)-\tfrac{2\nu}{27}\left(3+\tfrac{1}{P}\right)\right)=\tfrac{1}{k^3},$$

quae porro ad hanc formam reducitur:

$$\tfrac{(1+\eta)^2\mu m^3x^3}{27h^3}\left(3+\tfrac{5}{3}\eta-4\nu\left(2+\tfrac{1}{3}\eta\right)\right)=\tfrac{1}{k^3},$$

ita ut sit

$$\tfrac{kmx}{3h}\sqrt[3]{(1+\eta)^2\ \mu\left(3+\tfrac{5}{3}\eta-4\nu\left(2+\tfrac{1}{3}\eta\right)\right)}=1,$$

unde facile valor ipsius x colligitur, qui praebet semidiametrum aperturae primae lentis. Duas reliquas tuto tam apertas relinquere licet, quam earum forma permittit. Hinc autem pro gradu claritatis habebitur

$$y=\tfrac{hx}{ma}=(1+\eta)x$$

hincque mensura claritatis $=20y$, si scilicet x in digitis exprimatur.

Denique superest, ut spatium in obiecto visum accuratius determinemus, cuius semidiameter z supra in genere ita est definita:

$$z=\tfrac{\mathfrak{q}+\mathfrak{r}}{ma-h}\cdot ah\xi=Ma\xi$$

posito

$$M = \frac{\mathfrak{q}+\mathfrak{r}}{ma-h} \cdot h$$

erit autem

$$ma - h = -\frac{\eta}{1+\eta} \cdot h$$

ideoque

$$M = -\frac{(\mathfrak{q}+\mathfrak{r})(1+\eta)}{\eta},$$

ita ut nunc $\mathfrak{q}$ et $\mathfrak{r}$ ut negativae spectari debeant; nunc autem adiungi debet prima aequatio fundamentalis

$$\mathfrak{B}\mathfrak{q} = (P-1)M = \frac{-2\eta}{3+2\eta} \cdot M$$

hincque

$$\mathfrak{q} = -\frac{\eta}{3+2\eta} \cdot M;$$

ubi si loco M valor substituatur, erit

$$\mathfrak{q} = \frac{1+\eta}{2+\eta} \cdot \mathfrak{r} \;;$$

quod ad litteram $\mathfrak{r}$ attinet, eius loco unitas accipi posset, si ultima lens esset utrinque aequaliter convexa; cum autem hinc proditura sit fere convexo-plana, eius apertura ad dimidium reducetur, ita ut statui debeat $\mathfrak{r} = -\frac{1}{2}$; unde fit

$$\mathfrak{q} = -\frac{(1+\eta)}{2(2+\eta)}$$

hincque

$$M = \frac{(3+2\eta)(1+\eta)}{2\eta(2+\eta)};$$

quocirca sumto $\xi = \frac{1}{4}$ habebitur semidiameter spatii conspicui

$$z = \frac{(3+2\eta)(1+\eta)}{8\eta(2+\eta)} \cdot a = \frac{3+2\eta}{8\eta(2+\eta)} \cdot \frac{h}{m},$$

qui campus tantus est, ut de eo nemo rationem habeat conquerendi. Tum autem semidiametri aperturae binarum posteriorum lentium debent esse

$$\text{secundae} = \frac{1+\eta}{8(2+\eta)} \cdot q = \frac{3+2\eta}{8(2+\eta)} \cdot \frac{h}{m},$$
$$\text{tertiae} = \frac{1}{8} r = \frac{3}{8} \cdot \frac{h}{m},$$

siquidem hi valores sint maiores iis, quos claritas postulat, quippe qui sunt pro secunda

$$=\frac{x}{P}=\frac{3+2\eta}{3}\cdot x$$

et pro tertia

$$=\frac{x}{PQ}=(1+\eta)x.$$

SCHOLION 1

63. Obiectioni hic occurrendum necesse videtur, quod in praecipua huius solutionis parte non ipsam formulam, qua semidiameter confusionis exprimitur, ad minimum valorem perduxerimus, sed aliam formulam, quae ab illa satis notabiliter discrepare possit, praecipue si, ut ante fecimus, statuamus $\eta=\frac{1}{5}$, atque hoc quidem statim lubenter concedimus nos hoc modo semidiametro confusionis non absolute minimum valorem induxisse atque adeo minorem eo, quem invenimus, erui posse, si quis laborem suscipere vellet ipsam formulam litteras P et Q continentem secundum methodum maximorum et minimorum tractandi; tum scilicet pro litteris A et B alios valores a nostris aliquantillum discrepantes esset inventurus, qui certe molestissimis formulis forent implicati, ut neque operae pretium esset eos evolvere neque ab artifice perfectissima exsecutio sperari posset. Nos autem hic valore invento, qui certe iam satis est exiguus, etsi non sit omnium minimus, contenti esse poterimus, si quidem inde eiusmodi microscopia adipiscimur, quae vulgaribus simplicibus longissime sunt anteferenda, cum multo maiorem claritatis gradum largiantur, pro data scilicet distinctione, ita ut, si aliquid de claritate remittere voluerimus aperturam primae lentis aliquantillum restringendo, tum maximum lucrum in distinctione simus consecuturi.

COROLLARIUM 1

64. Cum pro prima lente sit distantia focalis

$$p=3a=\frac{3}{1+\eta}\cdot\frac{h}{m}$$

et numeri $\mathfrak{A}=3$ et $\lambda=1$, erit huius lentis

$$\text{radius}\begin{cases}\text{anterior}=\frac{p}{\sigma-\mathfrak{A}(\sigma-\rho)}\\ \text{posterior}\ =\frac{p}{\rho+\mathfrak{A}(\sigma-\rho)}\end{cases}$$

sicque erit

$$\text{radius}\begin{cases}\text{anterior}=\frac{p}{3\rho-2\sigma}\\ \text{posterior}\ =\frac{p}{3\sigma-2\rho}.\end{cases}$$

COROLLARIUM 2

65. Simili modo, cum pro lente secunda sit distantia focalis

$$q=(3+2\eta)a=\frac{3+2\eta}{1+\eta}\cdot\frac{h}{m}$$

et numeri $\mathfrak{B}=2$ et $\lambda'=1$, erit eius

$$\text{radius}\begin{cases}\text{anterior}=\frac{q}{2\rho-\sigma}\\ \text{posterior}=\frac{q}{2\sigma-\rho}.\end{cases}$$

Pro lente autem tertia ob

$$r=3(1+\eta)a=\frac{3h}{m},\quad \mathfrak{C}=1 \text{ et } \lambda''=1$$

erit eius

$$\text{radius}\begin{cases}\text{anterior}=\frac{r}{\rho}\\ \text{posterior}=\frac{r}{\sigma}.\end{cases}$$

COROLLARIUM 3

66. Quod ad intervalla inter has ternas lentes attinet, ea assumta inter se aequalia et nunc utrumque inventum est $=\eta a=\frac{\eta}{1+\eta}\cdot\frac{h}{m}$, dum scilicet obiectum ante lentem primam collocatur ad distantiam $a=\frac{1}{1+\eta}\cdot\frac{h}{m}$, quae distantia ergo aliquanto minor est quam casu lentis simplicis et duplicatae.

EXEMPLUM

67. Parentur omnes tres lentes ex vitro communi, pro quo est $n=1,55$, tum vero statuatur $\eta=\frac{1}{5}$ et sumatur $k=20$ et $h=8$ dig. atque hinc pro quantitate x determinanda habebitur ista aequatio

$$\frac{5mx}{6}\sqrt[3]{0,9381\times1,44\times1,4105}=1,$$

quae evo1uta dat $x=\frac{0,96796}{m}$ dig., unde sequens oritur

Constructio huiusmodi microscopiorum

I. Distantia obiecti ante lentem primam

$$a=\frac{20}{3m}=\frac{6,666}{m}\text{ dig.}$$

II. Pro lente prima, cuius distantia focalis $= p = \frac{20}{m}$, erit

$$\text{radius faciei}\begin{cases}\text{anterioris} = -\frac{p}{2,6827} = -0,37276p = -\frac{7,4552}{m}\text{ dig.}\\ \text{posterioris} = +\frac{4,5008}{p} = 0,22218p = +\frac{4,4436}{m}\text{dig.,}\end{cases}$$

cui lenti tribuatur apertura, cuius semidiameter

$$x = \frac{0,96795}{m}\text{ dig.}$$

III. Tum ad distantiam

$$\eta a = \frac{4}{3m}\text{ dig.} = \frac{1,3333}{m}\text{ dig.}$$

locetur lens secunda, cuius distantia focalis est

$$q = \frac{22,666}{m}\text{ dig.,}$$

eritque

$$\text{radius faciei}\begin{cases}\text{anterioris} = -\frac{q}{1,2460} = -0,80257q = -\frac{18,1915}{m}\text{dig.}\\ \text{posterioris} = \frac{q}{3,0641} = +0,32636q = +\frac{7,3973}{m}\text{dig.}\end{cases}$$

cuius aperturae semidiameter sit $\frac{1,55}{m}$dig.
et distantia ad lentem sequentem ut ante $= \frac{1,3333}{m}$ dig.

IV. Pro lente tertia, cuius distantia focalis est $r = \frac{24}{m}$ dig., erit

$$\text{radius faciei}\begin{cases}\text{anterioris} = \frac{r}{0,1907} = 5,2438r = \frac{124,851}{m}\text{dig.}\\ \text{posterioris} = \frac{r}{1,6274} = 0,61447r = \frac{14,7473}{m}\text{dig.;}\end{cases}$$

cuius apertura tanta esse potest, ut eius semidiameter sit $= \frac{6}{m}$dig., huicque lenti oculus immediate adplicetur.

V. Tum vero, cum sit

$$y = \frac{6}{5}x = \frac{1,1615}{m}\text{ dig.,}$$

erit mensura claritatis $= -\frac{23,230}{m}$, quae semisse maior est quam casu praecedente.

VI. Pro spatio denique obiecti conspicuo habebimus eius semidiametrum

$$z = \frac{85}{11m} = \frac{7,7273}{m}.$$

SCHOLION 2

68. Ne quisquam miretur hoc casu spatium conspicuum tanto maius esse inventum quam casu praecedente, is observet in casu antecedente lenti oculari non maiorem aperturam esse datam, quam gradus claritatis postulat; quod ideo fecimus, quod priori lenti adhuc exigua apertura tribui debebat ideoque hoc casu consultum erat ambas lentes non maiores efficere, quam ista apertura postularet, ut scilicet earum crassities eo minor redderetur. In praesente autem casu res longe aliter se habet, cum iam prima lens fere tantam aperturam requirat, quantam eius figura permittit; ex quo hae lentes necessario tantum discum habere debent, qui faciei curvioris arcum triginta graduum complectatur; ex quo etiam binis reliquis lentibus multo maior apertura tribui poterat. Verum hic in genere notandum est etiam casu praecedente campum apparentem iam tantum fore, ut quilibet de eo contentus esse possit. Quoniam autem hic primae lentis apertura fere iam tanta est inventa, ut eius figura maiorem non pateretur, inutile videri posset hanc investigationem ulterius ad quatuor lentes prosequi, quandoquidem calculum simili modo instituendo multo maiorem valorem pro x essemus adepturi; verum ob hanc ipsam causam ista investigatio maximi erit momenti; quia enim hactenus litterae k maiorem valorem non tribuimus quam viginti, unde admodum modicus distinctionis gradus nascitur, nunc huius litterae valorem multo maiorem assumere poterimus, quo his microscopiis summus certe perfectionis gradus inducetur idque sine ullo claritatis detrimento. Quaecunque enim adhuc per microscopia vulgaria sunt observata, semper haud exiguo confusionis gradu erant inquinata; ex quo, si eiusmodi microscopia nunc producantur, quae obiecta multo maiore distinctione repraesentent, ipsa observatione multa nobis patefacient, quae adhuc erant incognita, ita ut non amplius multo maior multiplicatio tanto studio sit desideranda.

PROBLEMA 3

69. *Si microscopium constet quatuor lentibus convexis et proxime inter se iunctis, eius constructionem investigare, ut pro data muliplicatione et dato distinctionis gradu obiecta maxima, qua fieri potest, claritate repraesentet.*

SOLUTIO

Cum hic quatuor lentes occurrant, tria inter eas intervalla ita exprimuntur:

$$\text{Primum } = Aa\left(1-\tfrac{1}{P}\right),$$
$$\text{secundum } = -\tfrac{AB}{P}\cdot a\left(1-\tfrac{1}{Q}\right),$$
$$\text{tertium} = \tfrac{ABC}{PQ}\cdot a\left(1-\tfrac{1}{R}\right);$$

quae cum esse debeant minima, quodlibet statuamus $=\eta a$, unde obtinebimus

$$\frac{1}{P}=1-\frac{\eta}{A} \quad \text{seu} \quad P=\frac{A}{A-\eta};$$

deinde

$$\frac{1}{Q}=1+\frac{P\eta}{AB} \quad \text{seu} \quad Q=\frac{(A-\eta)B}{(A-\eta)B+\eta};$$

tertio vero

$$\frac{1}{R}=1-\frac{\eta}{(A-\eta)BC+C\eta} \quad \text{seu} \quad R=\frac{(A-\eta)BC+C\eta}{(A-\eta)BC+C\eta-\eta}$$

seu

$$R=\frac{ABC-\eta(B-1)C}{ABC-\eta(B-1)C-\eta}.$$

Cum iam omnes quatuor lentes debeant esse convexae seu earum distantiae focales *p, q, r, s* positivae, has adipiscimur conditiones:

$$p=\mathfrak{A}a>0, \quad q=-\frac{A\mathfrak{B}}{P}\cdot a>0, \quad r=\frac{AB\mathfrak{C}}{PQ}\cdot a>0, \quad s=-\frac{ABC}{PQR}\cdot a>0;$$

unde primo colligimus

$$\frac{r}{s}=-\frac{R\mathfrak{C}}{C}>0,$$

ita ut esse debeat

$$-\frac{\mathfrak{C}}{C}=\mathfrak{C}-1$$

positivum, unde fit $\mathfrak{C}>1$, hinc C negativum ideoque AB positivum. Deinde cum sit

$$\frac{q}{r}=-\frac{\mathfrak{B}Q}{B\mathfrak{C}},$$

debebit esse

$$-\frac{\mathfrak{B}}{B}=\mathfrak{B}-1>0;$$

unde patet fore $\mathfrak{B}>1$ hincque $B<0$ simulque $A<0$; denique cum sit

$$\frac{p}{q}=-\frac{\mathfrak{A}P}{A\mathfrak{B}},$$

etiam esse debet

$$-\frac{\mathfrak{A}}{A}=\mathfrak{A}-1>0;$$

ergo

$$\mathfrak{A}>1 \quad \text{et} \quad A<0.$$

Quod ad marginem coloratum attinet, de eo non est opus ut simus solliciti, ut iam supra annotavimus. Quare pro semidiametro confusionis hanc contemplemur aequationem:

$$\frac{\mu m x^3}{a^2 h}\left(\frac{\lambda}{\mathfrak{A}^3}+\frac{\nu}{A\mathfrak{A}}-\frac{\nu}{A^3P}\left(\frac{\lambda'}{\mathfrak{B}^3}+\frac{\nu}{B\mathfrak{B}}\right)+\frac{1}{A^3B^3PQ}\left(\frac{\lambda''}{\mathfrak{C}^3}+\frac{\nu}{C\mathfrak{C}}\right)-\frac{\lambda'''}{A^3B^3C^3PQR}\right)=\frac{1}{k^3},$$

in qua omnes termini litteris λ adfecti sunt positivi; unde huic formulae valor minimus vel saltim a minimo non multum discrepans conciliari debet; quem in finem primo litteris

$\lambda, \lambda', \lambda'', \lambda'''$ valor minimus $=1$ tribuatur, cumque *P, Q, R* ab unitate parum differant, eorum loco scribatur unitas et sequens formula ad minimum perducatur:

$$\frac{1}{\mathfrak{A}^3}+\frac{v}{A\mathfrak{A}}-\frac{1}{A^3}\cdot W$$

existente

$$W=\frac{1}{\mathfrak{B}^3}+\frac{v}{B\mathfrak{B}}-\frac{1}{B^3}\left(\frac{1}{\mathfrak{C}^3}+\frac{v}{C\mathfrak{C}}-\frac{1}{C^3}\right);$$

ubi evidens est hanc formulam *W* omnino similem esse illi formulae, quam in praecedente problemate minimum reddi oportuit, hoc tantum discrimine, quod litterae *B* et *C* hic adhibitae ante erant *A* et *B*. Quare iam novimus, ut haec formula *W* fiat minima, capi debere

$$\mathfrak{B}=3,\quad B=-\tfrac{3}{2},\quad \mathfrak{C}=2 \quad \text{et} \quad C=-2;$$

quibus valoribus substitutis formula *W* fiet

$$W=\tfrac{1}{9}-\tfrac{8v}{27}.$$

Quocirca formula nostra ad minimum revocanda erit

$$\frac{1}{\mathfrak{A}^3}+\frac{v}{A\mathfrak{A}}-\frac{1}{A^3}\left(\frac{1}{9}-\frac{8v}{27}\right),$$

quae differentiata propter $d\mathfrak{A}:dA=\mathfrak{A}^3:A^3$ praebet

quae porro reducitur ad hanc formam:

$$\frac{3}{\mathfrak{A}^2}+\frac{v}{A\mathfrak{A}}+\frac{v}{\mathfrak{A}}-\frac{3}{A^3}\left(\frac{1}{9}-\frac{8v}{27}\right)=0,$$

quae porro reducitur ad hanc formam:

$$3\left(\frac{1}{\mathfrak{A}^2}-\frac{1}{9A^2}\right)+v\left(\frac{1}{\mathfrak{A}}+\frac{1}{A}+\frac{8v}{9A^2}\right)=0;$$

in qua si loco $\frac{1}{\mathfrak{A}}$ scribatur eius valor $1+\frac{1}{A}$, prodibit

$$(v+3)\left(1+\frac{2}{A}+\frac{8}{9A^2}\right)=0;$$

qui posterior factor resolvitur in hos duos factores

$$\left(1+\tfrac{4}{3A}\right)\left(1+\tfrac{2}{3A}\right),$$

quorum prior nihilo aequatus dat

$$A=-\tfrac{4}{3}\ \text{ ideoque }\ \mathfrak{A}=4,$$

posterior vero

$$A=-\tfrac{3}{2}\ \text{ hincque }\ \mathfrak{A}=-2;$$

qui ergo nostro instituto non convenit. Quocirca pro valore minimo obtinendo sequentes nacti sumus valores:

$$\mathfrak{A}=4,\ A=-\tfrac{4}{3},\ \mathfrak{B}=3,\ B=-\tfrac{3}{2}\ ,\ \mathfrak{C}=2,\ C=-2;$$

ex quibus valores supra inventi ita exprimentur

$$P=\tfrac{4}{4+3\eta},\ Q=\tfrac{4+3\eta}{4+5\eta},\ R=\tfrac{4+5\eta}{4+6\eta}$$

et distantiae focales

$$p=4a,\ q=(4+3\eta)a,\ r=(4+5\eta)a\ \text{ et }\ s=(4+6\eta)a.$$

Formula autem pro multiplicatione supra data hic fit

$$m=PQR\cdot\tfrac{h}{a}=\tfrac{4}{4+6\eta}\cdot\tfrac{h}{a},$$

unde colligitur

$$a=\tfrac{2}{2+3\eta}\cdot\tfrac{h}{m}.$$

His igitur substitutis valoribus aequatio pro confusione tollenda hanc induet formam:

$$\tfrac{\mu\left(1+\frac{3}{2}\eta\right)^2 m^3x^3}{1\cdot 64h^3}\left(1+\tfrac{1}{P}+\tfrac{1}{PQ}+\tfrac{1}{PQR}-2\nu\left(6+\tfrac{3}{P}+\tfrac{1}{PQ}\right)\right)=\tfrac{1}{k^3},$$

quae loco *P, Q, R* valoribus substitutis abit in hanc formam:

$$\tfrac{k^3m^3x^3}{64h^3}\cdot\mu\left(1+\tfrac{3}{2}\eta\right)^2\left(4+\tfrac{7}{2}\eta-\nu\left(20+7\eta\right)\right)=1,$$

ita ut sit

$$\tfrac{kmx}{4h}\sqrt[3]{\mu\left(1+\tfrac{3}{2}\eta\right)^2\left(4+\tfrac{7}{2}\eta-\nu\left(20+7\eta\right)\right)}=1;$$

unde facile valor ipsius *x* definitur; qui nisi maior prodeat, quam ut prima lens tantam aperturam admittere possit, dabit huius aperturae semidiametrum; sin autem maior prodeat, tum valor *k* eo usque augeatur, quoad prima lens hanc aperturam capere possit, sicque patebit, quanto distinctionis gradu haec microscopia futura sint praedita; scilicet lentibus definitis pro *x* sumatur valor maximae aperturae respondens ac tum ex hac aequatione valor ipsius k eliciatur. Hoc itaque modo definito x pro gradu claritatis habebimus

$$y = \frac{2+3\eta}{2} \cdot x$$

hincque mensura claritatis

$$= 20y = (20 + 30\eta)x.$$

De spatio in obiecto conspicuo hic nihil definio, cum certe sit maximum, si quidem singulis lentibus maxima, cuius capaces sunt, apertura tribuatur.

COROLLARIUM 1

70. Cum pro prima lente sit distantia focalis

$$p = 4a = \frac{8}{2+3\eta} \cdot \frac{h}{m}$$

et numeri $\mathfrak{A} = 4$ et $\lambda = 1$, erit

$$\text{radius faciei} \begin{cases} \text{anterioris} = \frac{p}{\sigma - 4(\sigma - \rho)} \\ \text{posterioris} = \frac{p}{\rho + 4(\sigma - \rho)}. \end{cases}$$

COROLLARIUM 2

71. Reliquarum trium lentium constructio erit ut in problemate praecedente; pro secunda scilicet erit

$$\text{radius faciei} \begin{cases} \text{anterioris} = \frac{q}{\sigma - 3(\sigma - \rho)} \\ \text{posterioris} = \frac{q}{\rho + 3(\sigma - \rho)} \end{cases}$$

existente

$$q = \frac{2(4+3\eta)}{2+3\eta} \cdot \frac{h}{m}.$$

Pro tertia lente erit

$$\text{radius faciei} \begin{cases} \text{anterioris} = \frac{r}{\sigma - 2(\sigma - \rho)} \\ \text{posterioris} = \frac{r}{\rho + 2(\sigma - \rho)} \end{cases}$$

existente

$$r = \frac{2(4+5\eta)}{2+3\eta} \cdot \frac{h}{m}.$$

Pro quarta denique lente

$$\text{radius faciei}\begin{cases}\text{anterioris} = \frac{s}{\rho}\\ \text{posterioris} = \frac{s}{\sigma}\end{cases}$$

existente

$$s = \frac{2(4+6\eta)}{2+3\eta}\cdot\frac{h}{m} = \frac{4h}{m}.$$

COROLLARIUM 3

72. Quod ad intervalla attinet, omnia sunt inter se aequalia, scilicet $=\eta a$; eorum igitur quodlibet erit

$$= \frac{2\eta}{2+3\eta}\cdot\frac{h}{m},$$

quia scilicet obiectum ante primam lentem collocari debet ad distantiam

$$a = \frac{2}{2+3\eta}\cdot\frac{h}{m}.$$

EXEMPLUM

73. Ponantur omnes quatuor lentes ex vitro communi confectae, pro quo $n = 1,55$; tum vero statuatur iterum $\eta = \frac{1}{5}$ et $h = 8$ dig., at vero *k* adhuc indefinitum relinquamus; unde habebitur ista aequatio:

$$\frac{kmx}{32}\sqrt[3]{0,9381\times 1,69\times(-0,2776)} = 1,$$

ubi signum $-$ calculum non turbat, cum hic de quantitate absoluta sermo sit; unde reperitur

$$kx = \frac{42,0692}{m},$$

unde iam patet, si caperetur $k = 20$, valorem ipsius *x* proditurum esse nimis magnum; quare *x* ex figura lentium et tum hinc valorem ipsius *k* investigemus, ut gradum distinctionis adcuratius cognoscamus.

Habetur itaque sequens

Constructio horum microscopiorum

I. Obiecti ante lentem distantia

$$a = \frac{80}{13m} = \frac{6,1538}{m}\ \text{dig.}$$

II. Pro lente prima, cuius distantia focalis

$$p = \frac{24,6152}{m} \text{ dig.},$$

erit

$$\text{radius faciei} \begin{cases} \text{anterioris} = -\frac{p}{4,1194} = -0,24275p = -\frac{5,9755}{m} \text{ dig} \\ \text{posterioris} = +\frac{p}{5,9175} = +0,16842p = +\frac{4,1457}{m} \text{ dig.}; \end{cases}$$

cuius aperturae semidiameter sumi poterit $x = \frac{1,0364}{m}$ dig.; tum vero pro gradu distinctionis erit $k = 41$ circiter; quare cum distinctio sequatur cubum ipsius k, hic distinctio octies maior erit quam in casibus praecedentibus, ubi $k = 20$. Tum vero ad lentem sequentem erit distantia $= \frac{1,2308}{m}$ dig.

III. Pro secunda lente, cuius distantia focalis $q = \frac{28,308}{m}$ dig., erit

$$\text{radius faciei} \begin{cases} \text{anterioris} = -\frac{q}{2,6827} = -0,3727q = -\frac{10,552}{m} \text{ dig.} \\ \text{posterioris} = +\frac{q}{4,5008} = +0,2222q = +\frac{6,2896}{m} \text{ dig.}; \end{cases}$$

cuius apertura priore aliquanto maior sumi potest.

Distantia ad lentem sequentem est ut ante.

IV. Pro tertia lente, cuius distantia focalis est $r = \frac{30,77}{m}$ dig., erit

$$\text{radius faciei} \begin{cases} \text{anterioris} = -\frac{r}{1,2460} = -0,8025r = -\frac{24,693}{m} \text{ dig.} \\ \text{posterioris} = +\frac{r}{3,0641} = +0,3264r = +\frac{10,043}{m} \text{ dig.}; \end{cases}$$

cuius apertura iterum aliquanto maior quam praecedens et intervallum ad sequentem est ut ante.

V. Pro quarta lente, cuius distantia focalis est $s = \frac{32}{m}$ dig., erit

$$\text{radius faciei} \begin{cases} \text{anterioris} = \frac{s}{0,1907} = 5,2438s = \frac{167,8016}{m} \text{ dig.} \\ \text{posterioris} = +\frac{s}{1,6274} = 0,6145s = \frac{19,6630}{m} \text{ dig.}; \end{cases}$$

cuius apertura denuo aliquantillum maior, eique oculus immediate adplicatur.

VI. Pro quarta lente, cuius distantia focalis est $y = 1,3x = \frac{1,3473}{m}$ dig., unde mensura claritatis fit $\frac{26,946}{m}$, ita ut, nisi plus quam vicies sexies multiplicare velimus, adhuc plena claritate frui queamus.

SCHOLION

74. En ergo speciem microscopiorum simplicium, quae maximam attentionem mereri videtur, cum sine detrimento claritatis obiecta multo distinctius repraesentabunt, quam vulgo fieri solet. Interim tamen fateri cogimur haec instrumenta non ad praegrandes multiplicationes adplicari posse; si enim multiplicationem $m = 100$ desideremus, lentes quidem adhuc facile parari possent, sed distantia obiectorum fieret tantum $\frac{6}{100}$ dig., quae distantia utique nimis parva fieri posset, praecipue si obiecta non fuerint admodum laevia. Ceterum multiplicatio ad 150 vel 200 fortasse posset urgeri, si summa necessitas postularet. Deinceps autem eiusmodi microscopia composita proferemus, quae non solum aeque clare et distincte obiecta repraesentent, sed etiam maiorem elongationem obiectorum admittant. Quoniam autem in hoc capite lentes tantum convexas sumus contemplati, nunc etiam lentes concavas introducamus, quibus adeo effici poterit, ut confusio penitus evanescat; sed aliud incommodum exsecutionem turbabit, dum scilicet lentibus nimis exiguis erit opus, quemadmodum in capite sequente videbimus.

ANNOTATIO

75. In hoc exemplo singulari attentione dignum evenit, ut formula pro confusione prodierit negativa; evidens autem est eam sive maiorem sive minorem prodituram fuisse, si litterae η alius valor fuisset tributus; quin etiam haec confusio plane ad nihilum reduceretur, si η ita acciperetur, ut fieret haec formula:

$$4 + \tfrac{7}{2}\eta - v(20+7\eta) = 0;$$

unde pro casu exempli sequeretur

$$\eta = \tfrac{8}{7} \cdot \tfrac{5v-1}{1-2v} = \tfrac{1,3040}{3,7436} = 0,34833;$$

hoc est propemodum, si sumsissemus $\eta = \frac{1}{3}$. Praeterea vero alio modo haec confusio ad nihilum reduci posset, si scilicet pro prima lente numerum λ non unitati aequalem, sed $\lambda = 1 + \omega$ posuissemus; tum enim in formula illa primus terminus 4 particula ω augeri deberet, ita ut prodiret in casu exempli $\omega - 0,2776$, unde foret $\omega = 0,2776$ primaque lens ex numero $\lambda = 1+\omega = 1,2776$ construi deberet reliquarum lentium constructione eadem relicta. Pro prima igitur hac lente substitui poterit haec constructio ob $\tau\sqrt{(\lambda - 1)} = 0,4768$:

$$\text{radius faciei}\begin{cases}\text{anterioris} = \frac{-p}{4{,}1194-0{,}4768} = -\frac{p}{3{,}64266}\\ \text{posterioris} = \frac{p}{5{,}9375-0{,}4768} = \frac{p}{5{,}4607}\end{cases}$$

seu

$$\text{radius faciei}\begin{cases}\text{anterioris} = -0{,}27453p = -\frac{6{,}758}{m}\ \text{dig.}\\ \text{posterioris} = +0{,}18312p = \frac{+4{,}508}{m}\ \text{dig.};\end{cases}$$

quodsi ergo haec lens in exemplo allato loco primae lentis substituatur, reliquis omnibus servatis his microscopiis adhuc maior perfectionis gradus conciliabitur, praecipue cum iam prima lens maiorem aperturam admittat.