(pp. 361 - 363) *Translated and annotated by Ian Bruce.*

The construction of isochronous curves in media with different forms of resistance, by Leonhard Euler of Basle.

It has been observed amongst geometers that the ordinary cycloid is an isochronous or tautochronous curve. The force of gravity is considered to act uniformly towards the centre of the earth, which is taken to be at an infinite distance. In a medium with resistance, which for simplicity is taken in proportion to the speed, the isochrone is again a cycloid. This has been shown by Newton, that most outstanding of men, in his *Principia, Book II, Prop. 26*. Moreover, I marvel greatly that no one has yet considered hypotheses for isochrones in media with other forms of resistance: for the two forms considered so far are hypothetical. It may well be of merit to examine in greater depth these other outstanding media in the science of the motion of bodies in resisting media. In the matters I have investigated here, it has given me pleasure to find the tautochronous curves for several different kinds of resisting media, with the centre of the gravitational force at an infinite distance and exerting a uniform attraction. I communicate this work to the literary community in order that it can be subjected to careful scrutiny.

Therefore I shall relate the most general method of this kind for the construction of an isochronous line : the medium can offer resistance in the ratio of some function of the speed, and according to this wide-ranging hypothesis, I can thus construct the isochrone.

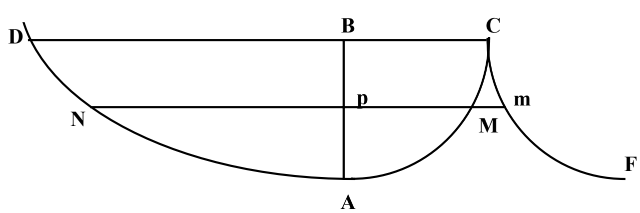


Figure 1.

Let AB be a vertical line, or a normal to the horizontal plane. Upon this line, for some given axis, a curve AND of such a kind can be described, in order that the abscissa AP, x, is given, and PN the coordinate z normal to this line. Now, Z is a certain function of the arc length, which has the same ratio to the arc length z as the resistance has itself to the speed, considered as a function, so that one can say: $dx = zdz + Zdz$. [1] On constructing this curve AND , another curve AMC is constructed on the same axis AB, such that with NP produced to M, an arc of the curve AM is equal to the connecting line PN [2]. This curve CMA is an isochrone, a body on this curve descending of course by the force of gravity alone, and the body will fall from some point M, always arriving at the point A in an equal interval of time, [the isochrone relating times to traverse arcs on different curves, while the tautochrone relates times to traverse arcs on the same curve) the tautochrone property].

It follows from this construction that these curves occasionally have a cusp at C, and with the curves returning to the region from whence they came. The cusp is formed when the element [derivative] of PM vanishes. This element is equal to $\sqrt{(dz^2 - dx^2)}$, from AM = PN = z. But since $dx = zdz + Zdz$, $\sqrt{(dz^2 - dx^2)} = dz\sqrt{(1 - (z + Z)^2)}$, which should be equal to zero, if z + Z = 1, therefore the cusp is present at the point where the sum z + Z is equal to one.

Hence this continued curve keeps the same shape as the cycloid; for it has an infinite number of cusps, and the parts of the curves between the cusps are all similar and equal to each other. For PM produced to m is by the rule applied successively: PN = AMC - Cm, but PN = AM and hence Cm = CM. Hence the sections of the curve

AMC, F*m*C, from the same cusp C are established equal and similar. Hence from the construction, point F corresponds to point A, and a similar and equal part is added on, similarly placed to AMC. Thus too, for the continued curve CMA, the diameter is AB, and therefore the curve has the same form in the plane as the ordinary cycloid.

We can apply this general construction to particular cases: putting the resistance to zero results in $Z = 0$, and hence $dx = zdz$, i. e. $ax = zz$, and hence the curve AND is the parabola of Apollonius. Thus AMC is an isochrone of the ordinary cycloid, as Huygens has now demonstrated.

We can set the resistance proportional to the speed, then $Z = z$, and hence $dx = 2zdz$, i. e. $x = z^2$. Thus the curve AND is a parabola anew, and consequently isochrones of the cycloid result again, as Newton has shown as cited here. [Book II of the *Principia*, section 6].

Let the resistance of the medium be proportional to the square of the speed, and this hypothesis includes the situation for air, water, and nearly all fluids. In this case, $Z = az^2$, hence $dx = zdz + az^2dz$ and therefore $x = {}^1/_2\, z^2 + {}^1/_3 az^3$, from which curve, as I have explained above, it is possible to construct the isochrones. The quantity *a* has to take a greater value if the resistance is greater, but without doubt the amount of the resistance should always be assumed to proportional in this manner.

Thus by this general method here, an isochrone can easily be deduced for any conceivable hypothesis of the resistance, with gravity acting uniformly. Moreover, I have also a method for constructing isochrones for media with arbitrary resistance and a non-uniform force of gravity too. But I differ the analysis or demonstration of these and also that which I have proposed here to another suitable time; proposing in the meanwhile to the scientific community the following problem concerning the motion of bodies in resistive media, which does not differ much from the material discussed.

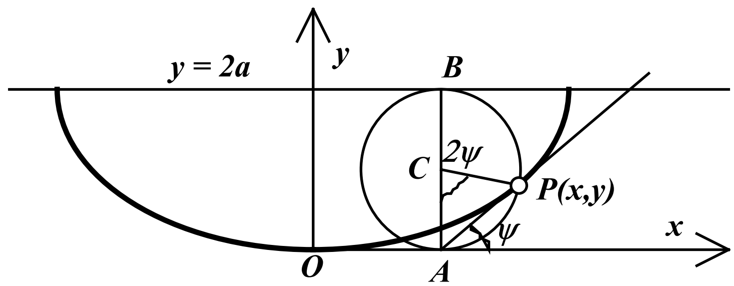
To find the line of quickest descent, or the brachystochrone, in some kind of resistive medium, at least for the case of uniform gravity.

P. S. I must make reference to the publication by that anonymous English gentleman [Pemberton] who posed the questions about reciprocal [i. e. orthogonal] trajectories some time ago to the celebrated Johan Bernoulli. I myself have devised a method for finding a general series of curves, with the exception of one unimportant curve still to be determined, for the trajectories problem to be satisfactorily resolved. All of which I will reveal in a year's time. Indeed the English gentleman did not set down the time [of descent] in his solutions of these reciprocating trajectories, as the celebrated Bernoulli has now found and published these, and he was able to show in their simplest form after the third order.

Notes :

1. There is a fundamental error in this equation, as reported in the Preface de l'Editeur on p. X of the *Commentationes Mechanicae*, Second Series Vol. 6 of Euler's *Opera Omnia*, as the resistance is a function of the *speed* of the particle at a point, rather than the arc length along the curve, which in turn depends on the point of departure along the curve; however, the latter is much easier to handle mathematically. It is thus not physically reasonable to make the assumption that the same resistance can be expressed as a function of the arc length and of the speed, which are taken to be in proportion.

There was, of course, rather compelling reasons why Euler went down this path. First, we should observe from a modern standpoint that there is nothing very unusual about the tautochrone result for the no resistance case, as the particle on the cycloidal wire executes s.h.m. with a period independent of the amplitude, as with other similar examples of such motion, the mass on a spring, electrical oscillations, etc. Thus, it would be unusual if the motion were not tautochronic. We give a brief outline of the relevant equations : The point P on the cycloid shown is generated by the circle of radius a rolling without slipping on the horizontal line y = 2a; the coordinates are :



$x = a(2\psi + \sin 2\psi); y = a(1-\cos 2\psi) = 2a\sin^2\psi.$

The angle ψ is the angle formed by the tangent to the curve at P, given by $dy/dx = \tan\psi$. It is readily shown that the arc length $s = 4a\sin\psi$, measured from O. It is also easy to show that a bead of unit mass sliding on this curve executes s.h.m., and that $d^2s/dt^2 = -(g/4a)s = -\omega^2 s$. A first integration of this equation gives the energy integral : $\dot{s}d\dot{s}/ds = -\omega^2 s$, i.e. $\frac{1}{2}\dot{s}^2 + \frac{1}{2}\omega^2 s^2 = \frac{1}{2}\omega^2\ s_0^2$, where $s_0 \leq 4a$. Hence, the first term gives the kinetic energy, while the second term becomes $2ag\sin^2\psi$ or gy, representing the potential energy; In Euler's notation, $g = 1/2$, $\omega^2 = 1/8a$. Hence, with $\dot{s} = z$, $zdz + \omega^2 sds = zdz + 1/8a.4a\sin\psi.4a\cos\psi\,d\psi = zdz + dy = 0$; hence $zdz = -dx$, and Euler has ignored the minus sign, as he reached his result without regard to energy considerations, and he has called our y coordinate x. One can see that s and z behave in the same manner in the frictionless equation, except for the constant term, and Euler had a habit of ignoring constants be setting them equal to one……. It was therefore expedient mathematically to simply add a term to the equation to represent the effect of resistance, but which was wrong physically. After the later publications of Newton and Herman, Euler amends his approach in E013, but only considers damping proportional to the square of the velocity.

2. Thus, the arc length of the inverted cycloid is synonymous with z : this is the length of string that has unwound from the conjugate upright cycloid in a Huygens type pendulum.

CONSTRUCTIO LINEARUM ISOCHRO-
-narum in mediae quocunque resistente, Autore
LEONDHADO EULERO,
Basileensi.

Notum est inter Geometras cycloidem ordinariam esse in medio non resistente isochronam sue tautochronam, vi gravitatis uniformiter versus centrum infinite distans tendente. In medio quoque pro simplici celeratum ratione resistente, isochronam esse eandem cycloidem, ostendit Vir summus, Newtonus in principiis sius Philosophiae Naturalis Lib. II Prop. 26. Oppido autem miror, neminem adhuc quicquam de isochronis in aliis medii resistentis hypothesibus, non imaginariis, quemadmodum sunt hae duae, meditatum fuisse; cum tamen haec egregia materia bene mereatur, quae in scientiae de motu corporum in medio resistente augmentum profundius examinetur. Ego, quae hac in re inveni, quasque feliciter detexi curvas tautochronas in medio quomodocunque resistente, centro virium infinite distante & uniformiter attrahente, hic cum communicabo, ut orbi literario-ansam praebeam, hanc materiam penitius perscrutandi.

Ut igitur modum generalissimum huiusmodi lineas isochronas construendi tradam: resistat medium in ratione cuiusvis functionis celeritatis, & pro hac amplissima hypothesi, sic construo isochronam.

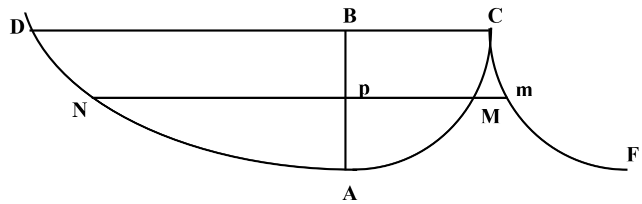


Figure 1.

Sit AB linea verticalis, seu normalis in planum horizontis; super hac, tanquam axe, describatur curva AND talis, ut, dictis abscissa AP x, huic applicata normali PN, z, & ipsius z functione quadam Z, quae eandem ad z habeat rationem, quam habet illa functio celeratis, secundum quam sit resistentia, ad ipsam celeritatem, ut inquam sit $dx = zdz + Zdz$. Hac curva facta, construatur alia AMC, super eodem axe AB, ut, producta NP in M, sit portio curvae AM aequalis applicatae PN. Erit haec curva CMA isochrona, corpus scilicet super ea, solo gravitatis nisu, descendens, in quocumque eius puncto M descensum adorsum fuerit, semper aequali tempore punctum A perveniet.

Ex hac constructione consequitur, curvas hasce habere alicubi in C cuspidem, & reverti in plagam, ex qua venerant, id quod accidere debet, ubi elementum applicare PM evanescit, elementum vero hoc aequatur $\surd dz^2 - dx^2$, ob AM = PN = z. sed quia $dx = zdz + Zdz$, erit $\surd dz^2 - dx^2 = dz \surd 1 - (z + Z)^2$ quod debet aequari ziphrae, erit $z + Z = 1$, ibi ergo est cuspis, ubi summa $z + Z$ aequatur unitati.

Haec ergo curvae eandem cum Cycloide habebit formam, habebit enim infinitos cuspides, & portiones cuspides constituentes omnes inter se similes & aequales, nam producta PM in m erit lege continuitatis PN = AMC - Cm, sed PN = AM ergo Cm = CM partes ergo AMC, FmC cuspidem C constituentes aequales erunt & similes. Proin ex constructione puncto F, puncto A respondenti, adnectetur portio similis, aequalis & similiter posita cum AMC. Sic quoque, curva CMA continuata, erit AB diameter, eandem ergo plane cum cycloide ordinaria habebit formam.

Ut applicemus hanc generalem constructionem ad hypotheses speciales; ponatur resistentia nulla, erit $Z = 0$ ergo $dx = zdz$, i. e. $ax = zz$, unde curva AND erit parabola Apolloniana. Erit ergo isochrona AMC cyclois ordinaria, uti Hugenius jam demonstravit.

Ponamus resistentiam celeritati proportionalem, erit $Z = z$, unde $dx = 2zdz$, i. e. $x = zz$, curva ergo AND erit denuo parabola, & consequenter isochrona rursus cyclois, quemadmodum Newtonus loco citato demonstravit.

Sit Resistentia ut quadratum celeratis, quae hypothesis locum habet in aere, aqua, omnibusque fere fluidis, erit $Z = azz$, unde $dx = zdz + azzdz$ ergo $x = {}^1/_2\, zz + {}^1/_3\, az^3$, ex qua curva, ut supra docui, construi poterit isochrona. Quantitas, a, eo major est accipienda, quo resistentia major est, semper nimirum proportionalis assumi debet quantitati resistentiae.

Et hac ratione pro quavis hypothesi resistentiarum excogitabili isochrona, methodo hac generali, facile deduci poterit, gravitate uniformiter agente. Pro aliis autem gravitatis hypothesibus, in medio quoque utcunque resistente, etiam possideo methodum construendi isochronas, quam autem, ut & eorum, quae hic proposui, analysin seu demonstrationem, in aliud opportunum tempus differo; proponens interim cultoribus scientiae hujus de motu corporum in mediis resistentibus problema sequens, quod ab hac materia haud multum abludit.

Invenire lineam celerrimi descensus, seu brachystochronam in medio quomodocunque restistente, saltem in hypothesi gravitatis uniformi.

P. S. Non possum, quin indicem Anonymo illi Angelo, cui jam per aliquantum temporis cum Celeb. Johanne Bernouilli circa trajectorias reciprocas res est, me adinvenisse methodum exquolibet linearum ordine, excepto secundo, ad minimum unam determinandi curvam problemati illi de trajectoriis reciprocis satisfacientem, quam, uno elapso anno, revelaturus ero; quo Anglo illi non defit tempus trajectoriam reciprocam, quam inveniendam ipsi proposuit Cel. Bernouilli, simplicissimam post illam tertii ordinis, quam invernire potuerit, publici indicandi.

Z z .