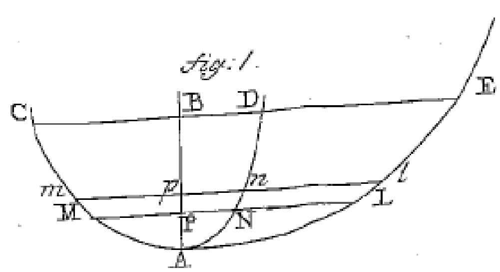
Concerning
INNUMERABLE TAUTOCHRONIC CURVES
IN VACUO.

Leonard Euler.

§1.

As many times as I have contemplated that conspicuous property of tautochronism that *Huygens* seized upon in the first place as a property of the cycloid, I have always wondered whether or not perhaps other curves had the same property. And this seemed more plausible to me, since I considered that *Huygens* himself had not arrived at the cycloid from the contemplation of tautochronism but rather it had been detected from a careful examination of the cycloid as a property found amongst others. *Newton* and indeed *Hermann*, who henceforth treated the same problem, elucidated the cycloid analytically, but they used a basic principle that is not sufficiently wide : that it is necessary for the accelerations by the different paths traversed to be in proportion. Indeed the accelerations can be determined in other ways, in order that the property of tautochronism may nevertheless be conserved. On account of suspecting that law, it seems apparent to me that besides the cycloid, one might perhaps encounter the same property of tautochronism amongst other curves.

§2. I was thinking about removing this doubt with the help of some natural principle from which, without any other principle taken from elsewhere, but only from tautochronic considerations, curves could be found with this special property. Therefore, for a long time I collected all the studies and works relevant to this investigation, until at last the prayer was answered for everything that I wished for and pursued. Moreover I have observed, when a question is proposed about finding a tautochronous curve, that there are always two involutes in the question to be distinguished the one from the other. One of which requires a curve of such a kind that a falling weight reaches the lowest point in the same time, where any point on the curve can be taken as the starting point of the descent. While the other above-mentioned curve, from the curves sought of this kind, is taken from all those for which the oscillations for the descent and ascent have the same common time, or are isochronous. The cycloid has been taken to be the only satisfactory curve for that consideration : to this truly besides the cycloid innumerable other curves have been found agreeable to me in my quest.

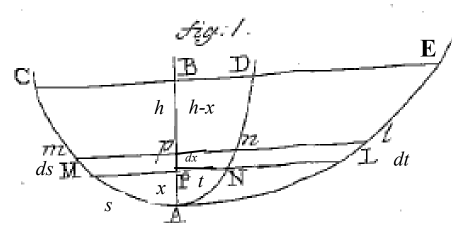


§3. Following this question, I propose only this much at first, in order that for a given curve, some curve AMC may be found joining the given curve AND in A, and of such a kind that a weight upon the curve CMAND composed from these curves is free to have all oscillations in equal times. Moreover, after I have reached this solution, these cases are to be investigated, for which these two curves constitute a single continual curve, and which are supported by the same equation. Curves of this kind have seemed to me to be most noteworthy, that are outstanding in producing the same effect as the cycloid, and those may be equally adapted for use in clocks. Besides, not without admiration, I have learned that algebraic curves can also give rise to these tautochronous curves, to which problems Analysts are always striving, with so much eagerness to reach solutions. Therefore all these, except for those I have overlooked, have been put together here for presentation, as besides the novelty of the method itself, those which are worthy are to be set forth by that analysis.

§4. Therefore let the given curve be AMC, and truly the sought curve AND, which have the common vertical axis AB. The weight can begin to descend from some point C, and it will ascend again in the other curve to the same height D, as the straight line CD is horizontal ; indeed we ignore all resistance. Therefore with this oscillation the body traverses the arc CAD, following the Galilean law, and in some place M it will have a speed which it acquired by slipping and falling through the height BP, with MPN truly drawn through M to the horizontal. But the curve is required to satisfy the tautochronic condition, in order that the time of the oscillation remains the same, and it shall retain the same quantity, wherever the point C may be taken. To take account of this, a principle expressing this time should thus be prepared, so that in this neither the line AB will be present, nor any other quantity depending on the position of the point C.

§5. The maximum speed of the body, provided that this oscillation is free, is at the lowest point A, and it corresponds to the height AD, from which it obviously arises. And this speed itself ought to have been established from the square root of this height AD, and in the same way, in the location M, the speed is as the square root of the height BP. On account of which, with the elements M*m* and N*n* taken with equal heights, the speed of the body will be as $\sqrt{\mathrm{BP}}$ provided that both describe the same height ; And the sum of the very small times in which these elements are traversed is $\frac{\mathrm{M}m+\mathrm{N}n}{\sqrt{\mathrm{BP}}}$. Therefore the time will be given from the integral of this, by which the arc MAN is completed, and with AP + AB put there, the time for the whole oscillation will be produced.

§6. Now the following are put in place [Fig. 1 is redrawn with some extra labels in place; the slight tilt in the diagram should be ignored.]: AB = h, AP = x; the arc AM = s and AN = t. Then the following are equal : Pp = dx; Mm = ds; Nn = dt and BP = $h - x$. Therefore the speed that the body has in traversing the elements M*m* and N*n*, will be $\sqrt{(h - x)}$. [Euler adopts the common strategy of the time of taking $2g = 1$, in some arb. units, for the acceleration of gravity g.] And besides, the time taken for which these elements are traversed is $\frac{ds+dt}{\sqrt{(h-x)}}$, or by putting $ds + dt = dv$, this will be $\frac{dv}{\sqrt{(h-x)}}$. The total time is given by the integral of this, by which the arc MAN is completed, if indeed some constant is added, in order that the variable $x = 0$ when the time itself vanishes [*i. e.* the trivial case of the mass or bead at rest at the bottom of the curve]. If $x = h$ is put into that integral, then the time for the whole oscillation is obtained [half the modern period]. On account of this principle, neither the letter h nor any other depending on it ought to be present. Hence the letter [*i. e.* variable]



v should be found in terms of x in order that this property [of independence] of the integral can be found.

§7. Let $dv = pdx : e$; I divide by [some constant] e, in order that homogeneity can be conserved, since the function p will then be known. And thus the differential for the sum of the times is equal to $pdx : e\sqrt{(b - x)}$ or $\frac{1}{e} pdx : \sqrt{(b-x)}$. Now, as it is clear from the preceding, it is necessary for $pdx : \sqrt{(b - x)}$ to be thus established in order that, if it is integrated, and some such constant is to be added, then the integral will made 0 if $x = 0$, and for $x = h$, the dependence on h is to be missing from the expression. It is necessary to determine p in order that these conditions are satisfied. The integral of this expression $pdx : \sqrt{(h - x)}$ is in agreement with, (as it must have some constant value), an expansion in terms of some number of simple terms ; for indeed it is allowed to resolve an expression of this kind as a series of irrational terms, [*i. e*. this is one way to evaluate the integral, which also allows the boundary conditions to be implemented]. It is therefore necessary, that each one of these terms is put in place with the quantity x, or of a function worthy of the positive exponent of this; so that the whole expression shall vanish if $x = 0$.

§8. Hence such individual terms will have the form gx^m, where g is to be given in terms of h also. Since truly for all these terms with x made equal to h, h ought to vanish or to have disappeared from the computation : if x is placed equal to h, the terms then have this form gh^m , from which in order that h may be eliminated, it is required that $g = nh^{-m}$ where n is not a function of h, but denotes the number that the force in the given quantity multiplies; truly this quantity can be divided by e in this way, so that n is hence able to be the only significant number. Here therefore by this reason the individual terms are of the form $nh^{-m}x^m$ [*i. e.* an expansion can be made in powers of x/h.]. Whereas the dimensions of h can remove the dimensions of x, it is seen that the integral itself ought to have no dimensions. Then it is also evident in the integral, except for h and x, that it is not required for other numbers and quantities to be kept; and thus it follows that the same holds in both the integral and in the differential cases. Wherefore p is not required to be a function of h, it must be a function of x, and p will be a power of x itself which shall be x^n.

§9. By this condition, the differential $pdx : \sqrt{(h - x)}$ has been transformed into $x^n dx : \sqrt{(h - x)}$. And it follows from the other prior condition, from which the integral is unable to have any dimensions, that n can hence be determined. Moreover it is required, concerning this, in order that the integral has no dimensions, that the dimensions are cancelled in the differential itself, with the element dx in place taking up one dimension; indeed it is apparent, that the differential without dx shall have the same number of dimensions as the integral. Truly the number of dimensions in our differential $x^n dx : e\sqrt{(h-x)}$ is $n + 1 - \frac{1}{2}$ or $n + \frac{1}{2}$, which therefore ought to be zero; thus it is required that $n = -\frac{1}{2}$. From which it emerges that

$p = x^{-\frac{1}{2}}$ or $1 : \sqrt{x}$, hence again the diff. is $dx : e\sqrt{x}$. Since e finally is the only assumption in order that homogeneity can be conserved, it may be written as :
$e = 1 : \sqrt{a}$; and hence $dv = dx\sqrt{(a : x)}$.

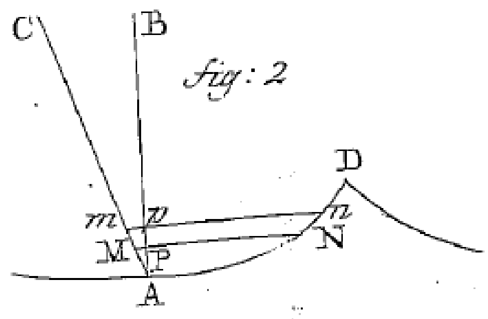
§10. Truly $dv = ds + dt$, whereby $ds + dt = dx\sqrt{(a:x)}$, the integral of this is $s + t = 2\sqrt{ax}$.
[Note that this is a purely geometrical quantity, and one can presume that it is the necessary and sufficient condition for a curve to be a tautochrone. Thus, Euler procedes to find suitable curves as tautochrones; he does not need to find the periods of the oscillations, as he compares these curves with the corresponding cycloid, for which the period of the associated pendulum is well-known.]
Therefore the sum of the arcs AM + AN is always in the ratio of the square root of the sagitta AP. Hence another curve ALE can be constructed, such that with MN and *mn* produced in L and *l*, the arc AL = AM + *AN*. Also *Ll* = M*m* + N*n* = *ds* + *dt* = *dv*. Hence AL = $v = 2\sqrt{ax}$, and thus $vv = 4ax$. From which it is observed that the curve ALE is a cycloid, of which the diameter of the generating circle is *a*.
[In modern terms, we have the arc length *s* related to the *y* coordinate by $s^2 = 8ay$; for such an inverted cycloid, where *t* is the angle the generating circle of radius *a* (note the change!) rolls through, and with A as the origin of coordinates. If the coordinates of the curve are given by : $x = a(t + \sin t)$, and $y = a(1 + \cos t) = 2a\cos^2 t/2$; then $dx = a(1 + \cos t)dt$, $dy = -a\sin t.dt$; from which $ds^2 = dx^2 + dy^2$; leading to $ds = 2a\sin(t/2).dt$; this can be integrated directly to give $s = s_0 - 4a\cos(t/2) = 4a\sqrt{(y/2a)} = 2a\sqrt{(x/a)}$ in Euler's notation.]
The body can descend on this cycloid from the point E and equal to the height of C or D; the velocity of this at L will be as $\sqrt{(h - x)}$. Hence the small time to pass through *Ll* is $dv : \sqrt{(h - x)}$. That which is equal to the sum of the times to pass through the elements M*m* and N*n*. Whereby the whole descent time through ELA is equal to the sum of the times to pass through the arcs CA and DA. Therefore the oscillation through CAD takes the same time as half the oscillation of a pendulum of length 2*a*, or the whole oscillation of a pendulum of length $\frac{1}{2}a$.
[Recall that the inverted and the associated upright cycloids are synchronous, with the period of the pendulum corresponding to the inverted cycloid, for which the bob travels along the evolute in s.h.m., (which is the other curve in this case) satisfying the usual period relation with length.]

§11. Now from these it is readily apparent, how from one given curve AC, it should be possible to find another curve AD [still on Fig. 1]. Let the horizontal coordinate [Euler calls such lines 'applied' or 'connected' to the point] of the sought curve AD be PN = z; then $Nn = dt = \sqrt{dx^2 + dz^2}$. Hence we have:
$ds + \sqrt{(dx^2 + dz^2)} = dx\sqrt{(a:x)}$. From which $\sqrt{(dx^2 + dz^2)} = dx\sqrt{(a:x)} - ds$.
And hence $dz = \sqrt{(adx^2 : x + ds^2 - dx^2 - 2dsdx\sqrt{(a:x)}}$. [*] Since the curve AMC shall be given, *ds* is given in terms of *x* and *dx*; therefore put $ds = pdx$. We then have :
$dz = dx\sqrt{(a:x + p^2 - 1 - 2p\sqrt{(a:x)})}$. [**]
[This is another basic equation in this work, to be used extensively.]
The equation for which, with *p* put in terms of *x* will express the nature of the curve sought AND. Hence it is understood, since *a* is independent of the curve, that thus as it is possible to be taken as it pleases, any of an infinite number of curves that can be found to

put in place of the sought curve AND, which with AMC joined presents a tautochronous curve. Nevertheless the cases should be noted that may occur, for which, if a is taken with some smaller quantity the curve sought will become imaginary.

§12. Let the right line AC with the given curve [Fig. 2] make some angle BAC with the vertical AB, then $ds = ndx$, with n denoting the number agreeing with that angle; hence $p = n$, whereby

$dz = dx\sqrt{(a:x+nn-1-2n\sqrt{(a:x)})}$.



Which equation gives in the first case n = 1, for which the line AC is vertical and coincident with AB. Here the equation shall be :

$dz = dx\sqrt{(a:x-2\sqrt{(a:x)})}$, putting $2\sqrt{ax} = q$, then $x = qq : 4a$, and $dx = qdq : 2a$; hence

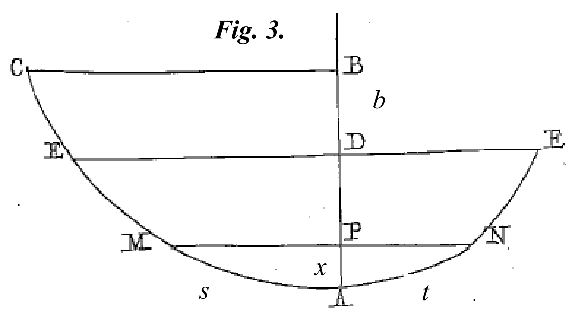
$dz = \frac{qdq}{2a}\sqrt{(\frac{4aa}{qq}-\frac{4a}{q})} = dq\sqrt{(\frac{a-q}{a})}$. Therefore $z = C - \frac{2(a-q)\sqrt{(a-q)}}{3\sqrt{a}} = C - \frac{2(a-2\sqrt{ax})\sqrt{(a-2\sqrt{ax})}}{3\sqrt{a}}$.

Since z will be equal to 0 if $x = 0$, it is required that $C = \frac{2a}{3}$, and hence

$z = \frac{2a\sqrt{a}-2(a-2\sqrt{ax})\sqrt{(a-2\sqrt{ax})}}{3\sqrt{a}}$. Which is the equation for a curve of the fourth order; here x is never able to be greater than $\frac{a}{4}$.

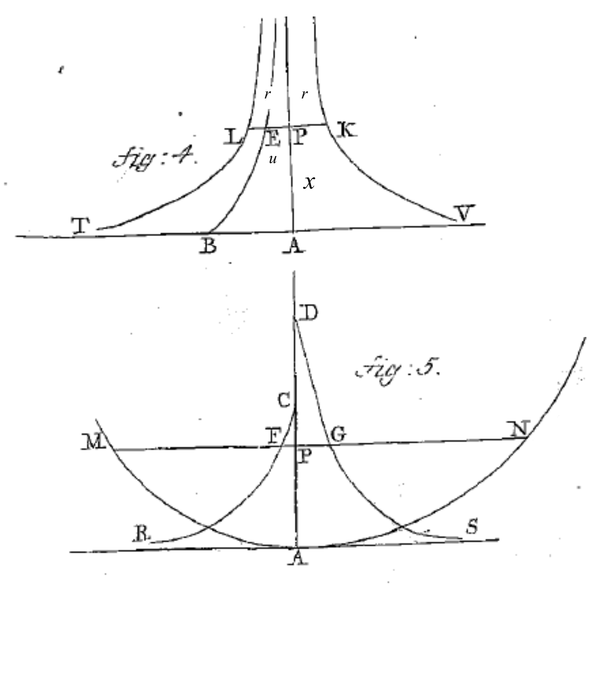
§13. If the first curve is a semi-cycloid AMC [Fig. 3], the diameter of the generating circle of which is AB = b ; then from what has been said, AP is equal to x, and AM to s, then $ss = 4bx$, and hence $s = 2\sqrt{bx}$. The other curve sought is ANE in which AN = t, and it is required that $s + t = 2\sqrt{ax}$, hence we have $t = 2\sqrt{ax} - 2\sqrt{bx}$. It may be said that $\sqrt{a} - \sqrt{b} = \sqrt{c}$; and $t = 2\sqrt{cx}$. Thus the other curve ANE is also a cycloid, and that for which any diameter c can be taken for the diameter. The oscillations truly are of the same time as half the oscillation of a pendulum the length of which is $2a$, or of the whole time if the length were $\frac{1}{2}a$. Now $\sqrt{a} = \sqrt{c} + \sqrt{b}$, hence $a = b + 2\sqrt{bc} + c$. Truly it is to be noted that in the larger cycloid AMC the starting point for the descent cannot be above E, where ED produced cuts the other curve is to be taken as the end point; otherwise the ascending body moves up along the curve AE beyond E, and the oscillation is nowhere terminated on the curve.

Fig. 3.



§14. We may consider the case, where both sides of the curve are equal to each other. Therefore $s = t$. Whereby since $s + t = 2\sqrt{ax}$; then $2s = 2\sqrt{ax}$; or $s = \sqrt{ax}$. From which it is recognised that each curve is a cycloid, and no other curve besides the cycloid are considered to be satisfactory [up to this stage]: Indeed our method has demonstrated above how I can solve the most general problem. As here $s = t$ has been put in place, thus any equation between s and t can be taken, and hence two curves are to be given, in order that the ascending and descending arcs have the same relation between them. For, if two curves are sought satisfying the problem CA and DA, in order that $s : t$ = AM : AN = $m : n$ always, then $mt = ns$, and $t = ns : m$. Hence $s + t = (ms + ns) : m = 2\surd ax$, thus expressed as $s = \frac{2m}{m+n}\sqrt{ax}$. Hence it is observed, that if the curve AC is a semi-cycloid of diameter $\frac{m^2 a}{(m+n)^2}$, then the other AND is also a semi-cycloid of diameter $\frac{n^2 a}{(m+n)^2}$ [*i. e.* on putting the factor within the square root].

§15. Since $s + t$ ought to be $2\surd ax$ in order that both sides of the curve present tautochronous oscillations, isochronous to a pendulum having a length $\frac{1}{2}a$ [Recall that both sides of the curve are covered in the time for half an oscillation of the pendulum (or a quarter of the modern idea of period), hence the length of the string is shortened from $2a$ or twice the diameter, to $a/2$ by a factor of 4, or to the radius of the generating circle]; Let $s = \surd ax + v$, and $t = \surd ax - v$. Therefore in this way two satisfactory curves can be found. Therefore $ds = \frac{adx}{2\sqrt{ax}} + dv$, and $dt = \frac{adx}{2\sqrt{ax}} - dv$. Putting $dv = udx$: we have the elements of arc length : $ds = \frac{adx}{2\sqrt{ax}} + udx$, and $dt = \frac{adx}{2\sqrt{ax}} - udx$. Whereby if y and z denote the 'y - coordinates' of either curve, then [in * §11] $dy = dx\sqrt{(\frac{a}{4x} + \frac{au}{\sqrt{ax}} + uu - 1)}$. And $dz = dx\sqrt{(\frac{a}{4x} - \frac{au}{\sqrt{ax}} + uu - 1)}$. If some function of x is substituted here in place of u ; two equations are obtained for curves satisfying the problem. Here it should be observed, if we put $a = 4b$ then $dz = dx\sqrt{(\frac{b}{x} - \frac{2bu}{\sqrt{bx}} + uu - 1)}$. Which equation agrees with the equation of § 11, $dz = dx\sqrt{(\frac{a}{x} + pp - 1 - \frac{2ap}{\sqrt{ax}})}$ if $b = a$, and $u = p$. From which it is understood that the [t arc] curve $dz = dx\sqrt{(\frac{a}{4x} - \frac{au}{\sqrt{ax}} + uu - 1)}$, also with that [$s$ arc] curve $ds = udx$ or $dy = dx\surd(uu - 1)$ constitute adjoining tautochronous curves, with oscillations absolved in the same time as a pendulum of length $\frac{1}{2}b$ or $\frac{1}{8}a$.

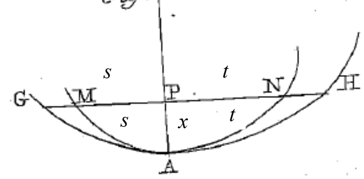


§ 16. Some curve BE is set up on the axis AP [this is an example of the theory in §15 : essentially, the area under one curve is the ordinate of the second curve which is the tautochronic curve that can then be drawn; Fig. 4], in which AP is put equal to x, PE shall be equal to u. Then the cubic hyperbola VKLT is drawn, of which the y coordinate PK or PL, if called r, will satisfy

$4xr^2 = a$, with a certain line taken for unity, it will be PK or PL = $\sqrt{(a:4x)}$. Then two new curves RF and SG are set up, in which there shall be
PF = $\sqrt{(LE^2 - 1)}$; and PG = $\sqrt{(KE^2 - 1)}$.
[For which $LE^2 = (\sqrt{\frac{a}{4x}} - u)^2 = \frac{a}{4x} + u^2 - u\sqrt{\frac{a}{x}}$, and similarly for KE^2]
Hence $PF = \sqrt{(\frac{a}{4x} - \frac{au}{\sqrt{ax}} + uu - 1)}$, and $PG = \sqrt{(\frac{a}{4x} + \frac{au}{\sqrt{ax}} + uu - 1)}$. From which the products PM by 1is taken equal to the area APFR : and PN by 1 taken equal to the area APGS. They will be,
since [PM × 1 =] $APFR = \int dx\sqrt{(\frac{a}{4x} - \frac{au}{\sqrt{ax}} + uu - 1)}$, and
[PN × 1 =] $APGS = \int dx\sqrt{(\frac{a}{4x} - \frac{au}{\sqrt{ax}} + uu - 1)}$, PM = z and PN $= y$ [are hence the ordinates of the required tautochronic curves MA and NA], and therefore the curves MA and NA joined at A demonstrate a tautochronous curve.

§17. It is evident how from these demonstrated, that for some given curve it may be possible to find another suitable curve in order that a tautochronous curve can be produced. Now, I have decided to investigate these cases, for which both these curves thus, as is becoming, are joined to make the same continuous curve; as both innumerable cycloids and other curves are obtained, which have the same outstanding application for clocks. Let MAN be a curve of this kind [Fig. 6] placed around the vertical axis AP. As before, AP is called x, the arc AM, s, and the other arc AN, t, and it is required that $s + t = 2\sqrt{ax}$. The other curve GAH is set up such that the y coordinates of this curve, PG and PH shall be equal to the arcs AM and AN respectively. Hence PG = s, and PH = t, and that will be a property of the curve GAH, in order that $s + t = 2\sqrt{ax}$. It is evident, if the curve GAH is given, the other curve MAN can be set up from that, and if that curve is continuous, then this will be the kind of curve sought. Here therefore the question has been simplified, in order that the curve GAH can be found, which shall be continuous, and that can have the property such that GP + PH = $2\sqrt{a}$.AP.

Fig: 6.



§ 18. Hence for a singular abscissa [or x value] AP on the curve GAH [Fig. 6] there correspond two y-coordinates GP and PN, of which one is negative if the other is positive. Hence there ought to be an equation between any x and y coordinates, [Note : *Abscissa* and *applicata* according to Euler : thus, our notation was not in full use at this time.] in order that the letter denoting the y ordinate for the single x abscissa has two suitable values in agreement with the question. In order that this can be brought about more easily, I assume a new variable v, [not to be confused with the total arc length aready defined], moreover thus with v made positive, the point G can be found ; truly with v made negative, the point H can then be found. We will therefore consider x, as a function of v itself, and of s. But the function signifying s will give t when negative, since PN falls on the other half of the axis, if v should become $-v$.

§ 19. Since the abscissa AP can stay the same for the points G and H on either side, it is required that as for x for v thus be determined, in order that it can stay the same when v

is changed into -*v*. Or *x* ought to be an even function of *v* : such a function shall be P, and *x* = P. The *y* coordinate PG is put as *s* = Q + R, with Q denoting an odd function, and R is truly an even function of *v*. In this formula Q + R put -*v* in place of *v*, and that will be changed into - Q + R; as this agrees with these remarks concerning even and odd functions, which I presented in the dissertation on reciprocal trajectories [E005]. With -*v* put in place if *v*, the point H is given , since - Q + R expresses the y coordinate PH. Which moreover, since it must fall in the other part, it will have the negative value - Q + R. Therefore the absolute magnitude for the y coordinate PH or *t* is Q - R, [essentially taking the modulus], thus it is found that *t* = Q - R. And truly it is the case that *s* = Q + R, and *x* = P.

§ 20. From the terms of the problem, this has the property, as has been shown in §17 that $s + t = 2\sqrt{ax}$. Whereby since *s* = Q + R, *t* = Q - R, and *x* = P, hence $2Q = 2\sqrt{a}P$ or QQ = *a*P, hence P = QQ : *a*. This has to be sought, or this value of P itself found, and furthermore, following which P must be an even function of *x,* or do we find that they incompatible amongst themselves ? If indeed they are in disagreement, then nothing can be elicited from the proposition. But these are not in disagreement : for, since Q is an odd function, the square of this will be an even function, again the division by *a* has no effect on this, it is evident here that P is to be put equal to the even function. Hence P = QQ : *a*. From these the curve GAH can be found. Indeed AP or *x* is taken equal to QQ : *a* , and PG or *s* = Q + R, where some odd function is taken in place of Q, in place of R truly some even function of v can be substituted. Since *x* = QQ : *a* then $Q = \sqrt{ax}$, and hence $s = R + \sqrt{ax}$. Here R can be taken as some even function of Q or $\sqrt{ax}$, or only of $\sqrt{x}$ itself.

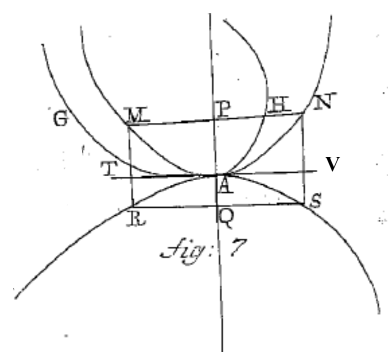
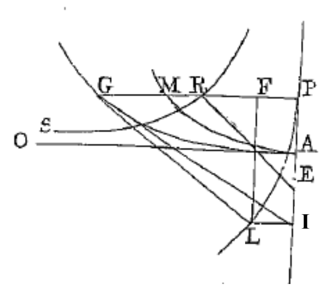


Fig: 7

§ 21. Now the [geometrical] construction of the curves is easily elicited from our carefully established preparations. Around the vertical axis AP [see Fig. 7] the parabola MAN is set up, the parameter of which is equal to *a*. Hence the ordinate MN is drawn at right angles, and it is given by $PM = \sqrt{ax}$, if AP = *x*. Below this parabola around the same axis some curve QAS is drawn, the axis of which is likewise the diameter [it is also symmetric]. The vertical lines MR and NS are drawn, cutting the horizontal line through A in T and V. It follows that $AT = \sqrt{ax}$, and $AV = -\sqrt{ax}$; but TR and SV are equal. Which, since they are positioned at the same place, they are an even function of the line AI which is $\sqrt{ax}$, whereby IR will express the function R [I is an unmarked point on the vertical axis, see Fig. 8]. Then the new curve GAH can be constructed, the *y* coordinate of which will be PG = PM + TR, it will be either from PH on account of the law of continuity = PN - SV or PN - TR. Whereby $PG = R + \sqrt{ax}$. Thus it follows that GAH is the same curve which is sought.

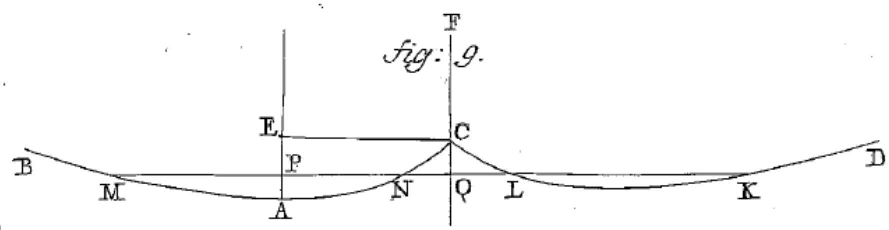
Fig: 8


§ 22. Hence in this way an infinite number of curves can be found, not indeed tautochronous curves, but such that from which tautochronous curves are able to be constructed. Let the curve AG be constructed in the preceding way [see Fig. 8], if then another curve can be made, in order that the arc of this curve AM everywhere shall be equal to the corresponding y coordinate PG, then this curve will be a tautochrone (§17). Truly from the given curve AG, the required curve AM can be constructed in the following way. The line GI is drawn tangent at G, crossing the axis produced in I. With centre G, the arc of a circle is described with radius GP, which the horizontal drawn from I cuts in L. The points G and L are joined, and from P some arbitrary length PE on the axes is taken, but everywhere the same. Then from E the line ER is drawn parallel to LG itself, cutting the y coordinate PG in R. Through all the points determined in this way that cross the curve the point R is determined for the curve SR, and which generally have the horizontal asymptote AO. Hence the curve AM can be constructed, such that rectang. PM.PE is equal to the area OAPRS. This curve AM is a tautochrone. Indeed the arc AM = PG. [WOW!]

§ 23. I can accomplish this result analytically. Since x ought to be an even function of v itself, moreover above Q is equal to $\sqrt{ax}$, it is necessary that $\sqrt{ax}$ is an odd function of v itself, but by placing $\sqrt{ax} = v$, it becomes $x = v^2 : a$ an even function, as required. Therefore we have this equation from §20: $s = \mathrm{R} + v$, where R denotes an even function of v. And thus $ds = d\mathrm{R} + dv$, $d\mathrm{R} = \mathrm{V}dv$, where it is necessary that V shall be an odd function of v. Whereby $ds = dv(1 + \mathrm{V})$, and thus $ds^2 = dv(1 + 2\mathrm{V} + \mathrm{VV}) = dx^2 + dy^2$. Moreover since
$x = v^2 : a$, $dx = 2vdv : a$, and $dx^2 = 4v^2dv^2 : a^2$. Consequently
$dy^2 = dv^2(1 + 2\mathrm{V} + \mathrm{VV} - 4v^2 : a^2)$ and $dy = \frac{dv}{a}\sqrt{(a^2 + 2a^2\mathrm{V} + a^2\mathrm{V}^2 - 4v^2)}$. I have been unable to make this equation rational in any way, by substituting legitimate values in place of V, so that obviously V can be put equal to an odd function of v. On this account I am unaware of other cases that are thus able to be elicited, which allow integration, except that which I am about to show here.

§ 24. Put $a\mathrm{V}$ equal to $2v$ which can be done, in order that the terms $a^2\mathrm{V}^2$ and $4v^2$ cancel each other out; this will give $dy = \frac{dv}{a}\sqrt{(aa + 4av)}$, which equation admits to integration, since v is of the first degree. The integral of this is the equation :
$y = \frac{C+(a+4v)\sqrt{(a+4v)}}{6\sqrt{a}} = \frac{C+(a+4\sqrt{ax})\sqrt{(a+4\sqrt{ax})}}{6\sqrt{a}}$ as $v = \sqrt{ax}$. In order that y may vanish, with x put equal to 0, it is required that $\mathrm{C} = -a\sqrt{a}$; therefore
$y = \frac{-a\sqrt{a}+(a+4\sqrt{ax})\sqrt{(a+4\sqrt{ax})}}{6\sqrt{a}}$ or $6y\sqrt{a} + a\sqrt{a} = (a + 4\sqrt{ax})\sqrt{(a + 4\sqrt{ax})}$. Hence we have
$a + 4\sqrt{ax} = (6y\sqrt{a} + a\sqrt{a})^{\frac{2}{3}} = \sqrt[3]{(36ayy + 12aay + a^3)}$. On account of which with the cube of both sides taken, this gives $12aa\sqrt{ax} + 48aax + 64ax\sqrt{ax} = 36ayy + 12aay$ or $12ax + (3a + 16x)\sqrt{ax} = 9yy + 3ay$. Which completely reduced to rational terms gives the fourth order equation: $81y^4 + 54ay^3 - 216axyy - 256ax^3 + 9aayy - 72aaxy + 48aaxx - 9a^3x = 0$.

§ 25. We therefore have an algebraic curve of the fourth order, and which like the cycloid

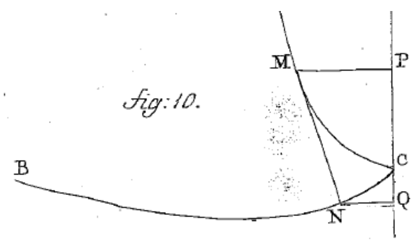


is suited for making all the equal time oscillations. It will therefore be worth the effort to describe this here in a little more detail. Let AE be the axis; our curve will have that form BACD, and the said AP is equal to *y*, that will be the equation that we found in the preceding §. Moreover the time, for which some oscillation through MAN is resolves, will be equal to the time for the oscillation of a regular pendulum, the length of which is $\frac{1}{2}a$. That curve from the other part of the axis AE is to be noted, in which C is the point of reversion; truly the point C can be found by taking AE = $\frac{1}{16}a$, and the *y* coordinate EC = $\frac{1}{6}a$. Again in the curve produced thus the motion is at C, as the arc CD is equal and similar to the arc CAB. On account of which if from C the vertical line CF is drawn, that will be orthogonal to the diameter of the curve. Truly oscillations can be set up as well in the other part BAC.

§ 26. Since therefore CF is the diameter of this curve, we can look for the equation relative to this diameter. Obviously CQ = *t*, and QM = *z*, then AP = *x* = AE - CQ = $\frac{1}{16}a$ - *t,* and PM = *y* = QM - CE = *z* - $\frac{1}{6}a$, with these values substituted in place of *x* and *y* in the equations in § 24, the following equation will result,
$81z^4+216atzz+256at^3-18aazz=0$, or, which is more suitable so that the properties of this curve can be found, or this $t=\frac{aa-(\sqrt[3]{36azz}-a)^2}{16a}$ or $z=\pm(a\pm\sqrt{(aa-16at)})^{\frac{2}{3}}:6\sqrt{a}.$ Thus it can be seen that there are four values for the remaining *t*, and thus in this way, if both signs are + then we have the point M; If the first sign is - and the other + then we have the point K : If the first is + and the second is taken - we have the point N; and if then both have a - sign in place, we obtain the point L.

§ 27. From this equation it is seen that this curve is integrable. Putting $a\pm\sqrt{(aa-16at)}=p$, then $t=\frac{2ap-pp}{16a}$ and $z=\frac{p\sqrt{a}}{6\sqrt{a}}$. Hence $dt=\frac{dp}{8}-\frac{pdp}{8a}$. And thus $zdt=\frac{pdp\sqrt{p}}{48\sqrt{a}}-\frac{ppdp\sqrt{p}}{48a\sqrt{a}}$. Which integration gives $\int zdt=\frac{pp\sqrt{p}}{120\sqrt{a}}-\frac{p^3\sqrt{p}}{168a\sqrt{a}}$. Which amount expresses the area contained between the abscissa, the y coordinate, and the curve. Hence it is agreed from the curve found that it is rectifiable. Since
$z=\frac{p\sqrt{p}}{6\sqrt{a}}$ then $dz=\frac{dp\sqrt{p}}{4\sqrt{a}}$; and hence $dz^2=\frac{pdp^2}{16a}$.
Truly $dt^2=\frac{dp^2}{64}-\frac{pdp^2}{32a}+\frac{p^2dp^2}{64aa}$. Whereby $dt^2+dz^2=\frac{dp^2}{64}+\frac{pdp^2}{32a}+\frac{p^2dp^2}{64aa}$, and hence $\sqrt{dt^2+dz^2}=\frac{dp}{8}+\frac{pdp}{8a}$. Consequently $\int\sqrt{dt^2+dz^2}=\frac{2ap+pp}{16a}$. Which expression gives either the arc CN or CAM or also the negatives of these CL or CLK.

§ 28. With the area and the length of this curve found ; there remains to be found what special use this curve may have for clocks, as we can investigate the radius of osculation [*i. e*. curvature], and from that the evolute of this curve can be found, for which the oscillations are resolved for a pendulum set up on this curve. The radius of osculation will indeed be, for constant *dz,* $\frac{(dt^2+dz^2)^{\frac{3}{2}}}{dzddt}$, the value of which can be found from above. For in fact $(dt^2+dz^2)^{\frac{3}{2}}=\frac{(a+p)^3}{512a^3}$, and $dz=\frac{dp\sqrt{p}}{4\sqrt{a}}$, hence since *dz* is placed constant, then this gives $ddz=0=\frac{2pddp+dp^2}{8\sqrt{ap}}$, hence we have $ddp=\frac{-dp^2}{2p}$. And hence since $dt=\frac{adp-pdp}{8a}$, this gives $ddt=\frac{addp-dp^2-pddp}{8a}=\frac{-adp^2-pdp^2}{16ap}$. From these values substituted in the formula $\frac{(dt^2+dz^2)^{\frac{3}{2}}}{dzddt}$, there arises the radius of osculation $\frac{-(a+p)^2\sqrt{ap}}{8aa}$, the - sign indicates the radius of osculation and the diameter to be diverging from each other.



§ 29. Since the radius of osculation is known, it is easy to find the evolute of our tautochronous curve. Let CNB be the tautochrone [Fig. 10]. There remains [the coordinates of a point on the curve from §27] CQ = $t=\frac{2ap-pp}{16a}$, QN = $z=\frac{p\sqrt{p}}{6\sqrt{a}}$. Let the radius osculation be $\frac{(a+p)^2\sqrt{ap}}{8a}$, which touches the sought evolute CM in M. From M the coordinate line MP is sent to the axis , and the coordinates shall be CP = *x* and PM = *y*. These coordinates may be found from the known relation between the coordinates CQ and QN. With this calculation omitted here as it is easy, we have $x=\frac{2ap+5pp}{16a}$ and $y=\frac{(3aa-3pp+4ap)\sqrt{ap}}{24aa}$, with the help of this equation the evolute CM can now be described through an infinite number of points. Moreover if we wish to eliminate *p*, in order that the above equation will be between *x* and *y*, *p* from the first equation is found in terms of *a* and *x*, which value if it can then be substituted in the other equation, gives rise to the following equation :

$576ayy-\frac{37632}{125}axx-\frac{32160}{625}a^2x+\frac{529}{3125}a^3=(\frac{2304}{125}xx+\frac{8608}{625}ax+\frac{529}{3125}a^2)\sqrt{(a^2+80ax)}$. Which equation, if truly it is reduced to rational quantities, will be of the fifth power.

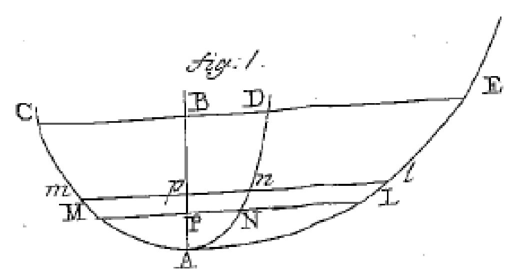
§ 30. Finally this curve should not be disregarded with silence, this tautochronous curve is truly the same, as we found joining with the right line to the vertical (§ 12). In that indeed only the equations are different, since there the parameter *a* squared will be greater than here. Since therefore the length of the isochronous pendulum for our tautochronous curve is $\frac{1}{2}a$, if this will be the same curve joining the vertical AE, then the length of the isochronous pendulum will be $\frac{1}{8}a$. Therefore the times of the oscillations in the curve MAN are twice as large as the as the oscillations through PAN. Whereby if in both places with the weight slipping until it may ascend as far as N, *t*MA + *t*AN = 2*t*PA + 2*t*AN. Consequently *t*MA - *t*AN = 2*t*PA. Therefore the difference of the times of descent through the arcs MA and NA is equal to twice the descent times through the vertical AP.

De
INNUMERABILIBUS CURVIS
TAUTOCHRONIS IN VACUO.
Auct. Leonh. Eulero.
§1.

Quoties ego insignem tautochronismi proprietatem, quam *Hugenius* primus in cylcoide inesse deprehendit, contemplatus sum, semper dubitabam, an praeter cycloidem aliae curvae eandem forte habeant proprietatem. Hocque mihi eo probabilius videbatur, quod ipsum *Hugenium* non ex tautochronismi contemplatione ad cycloidem pervenisse intelligebam: sed potius cycliodis proprietates scrutantem hanc ipsum inter alias detexisse. *Newtonus* quidem atque *Hermannus*, qui deinceps eandem rem tractarunt, analytice cycloidem elicverunt, sed usi sunt principio non satis late patente hoc; accelerationes viis percurrendis esse opertere proportionales. Aliis enim modis accelerationes possunt determinari, ut tautochronismus nihilominus conservetur. Quampbrem mihi jure suspicare visus sum, praeter cycloidem in alias fortasse curvas eandem tautochronismi proprietatem competere.

§2. Ad hanc dubitationem tollendam genuina opus esse methodo censebam, qua sine ullo principio aliunde assumto ex sola tautochronisme consideratione curvae hac proprietate praeditae ervi possent. Diu igitur omne studium operamque in hanc investigationem contuli, donec tandem voti compos factus, quicquid desiderabam, sum consecutus. Animadverti autem, cum de curva tautochrona invenienda quaestio proponitur, duas omnino quaestiones bene a se invicem distinguendas in ea esse involutas. Quarum altera hujusmodi curvam requirit, in qua grave descendens aequalibus temporibus ad punctum infimum perveniat, ubicunque sumatur initium descensus. Altera vero in ejusmodi curvis inquirendis est occupata, super quibus integrae oscillatines ex descensu et ascensu constantes omnes sint isochronae. Illi quidem quaestioni solam cycloidem satisfacere deprehendi: huic vero praeter cycloidem innumerabiles aliae convenire mihi inventae sunt.

§3. Posteriorem hanc quaestionem primum hoc modo proposui, ut data curva quacunque AMC inveniatur curva ei in A jungenda AND ejusmodi, ut gravesuper composita ex iis curvae CMAND oscillans omnes oscillationes absolvat aequalibus temporibus. Postquam autem hujus solutonem sum adeptus, eos investigavi casus, quibus hae duae cuvrae unam constituant lineam continuam, atque eadem contineantur aequatione. Hujusmodi mihi curvae admodum notatu dignae visae sunt, eo quod eundem quem cyclois, praestent effectum et aeque ac illa ad horologia accommodari possint. Praeterea non sine admiratone cognoui in his curvis tautochronis curvas etiam algebraicas contineri, ad quas Analystae in problematis solvendis tanto

Fig. 1.


semper studio pervenire nituntur. Haec igitur omnia eo, quo ipse sum assecutus modo , hic proponere constitui tam propter ipsius methodi novitatem, quam eorum, quae ex ea prodierunt, dignitatem.

§4. Sit igitur curva data AMC, quaesita vero AND,quae communem habeant axem verticalem AB. Incipiat grave descensum ex puncto quocunque C, ascendet id rursus in altera curva ad eandem altitudinem D, ita ut recta CD sit horizontalis; animum enim ab omni resistentia abstrahimus. Hac ergo oscillatione percurrit corpus arcum CAD, secundum legem Galieanam et in quovis loco M habebit celeritatem, quam lapsu ex altitudine BP acquirit, ducta nempe per M horizontali MPN. Tautochronismi autem conditio requirit, ut tempus hujus oscillationis sit constans, retineatque eandem quantitatem, ubicunque accipiatur punctum C. Quamobrem formula hoc tempus exprimens ita esse debebit comparita , ut in ea neque linea AB insit neque quaequam alia quantitas a loco puncti C pondens.

§5. Maxima celeritas corporis, dum hanc oscillationem absoluit, est in puncto infimo A. atque respondet altitudini AD, quippe ex qua est genita. Haecque celeritas ipsa debet exponi radice quadrata ex hac altitudine AD, et simili modo in loco M celeritas est ut radix quadrata ex altitudine BP. Quamobrem sumtis elementis M*m* et N*n* aequae altis, erit corporis celeritas dum utrumque describit eadem atque ut √BP; Et tempusculorum, quibus haec elementa percurruntur, summa est $\frac{\mathrm{M}m+\mathrm{N}n}{\sqrt{\mathrm{BP}}}$. Hujus ergo onterale dabit tempus, quo arcus MAN absolvitur, et posito in eo AP + AB, prodibit tempus integrae oscillitionis.

§6. Ponantur nunc AB = h, AP = x; arcus AM = s et AN = t. Erit Pp = dx; Mm = ds; Nn = dt et BP = h - x. Cereritas ergo, quam habet corpus elementa Mm et Nn percurrens, erit $\sqrt{(h-x)}$. Et propterea tempus, quo haec elementa absolvuntur, est $\frac{ds+dt}{\sqrt{(h-x)}}$, seu posito $ds + dt = dv$, erit id $\frac{dv}{\sqrt{(h-x)}}$. Cujus integrale dabit tempus, quo arcus MAN absolvitur, siquidem tanta constans adjicitur, ut facto $x = 0$ ipsum tempus evanescat. In illo integrali deinde, si ponatur $x = h$, habebitur tempus totius oscillationis. Quamobrem in eo neque litera h neque alia ab ea pendens inesse debet. Inveniri ergo debet litera v in x ut integrale hanc obtineat proprietatem.

§7. Fiat $dv = pdx : e$; per e divido, ut homogeneitas conservari possit, cum cognita fuerit functio p. Est itaque differentiale summae temporum = $pdx : \sqrt{(b - x)}$ sive $\frac{1}{e} pdx : \sqrt{(b-x)}$. Jam, ut ex praecedentibus elucet, oportet $pdx : \sqrt{(b - x)}$ ita esse constitutum , ut, si integretur talisque constans addatur, quae faciat integrale = 0, si $x = 0$ factoque $x = h$, tum h penitus ex expressione excedat. Hisque conditonibus ut satis fiat, oportet determinare p. Consistat integrale hujus $pdx : \sqrt{(b - x)}$ debita constante auctum quotcunque terminis simplicibus; nam et irrationalia in series hujusmodi terminorum resolvere licet. Necesse est igitur, ut unusquisque horum terminorum quantitate x seu dignitate ejus exponentis affirmative sit affectus; ea propter ut tota expressio evanescat, si fiat $x = 0$.

§8. Singuli ergo termini talem habebunt formam gx^m, ubi g etiam in h dari ponitur. Cum vero in hisce omnibus facto $x = h$, h debeat evanescere seu ex computo egredi : fiat $x = h$, termini hanc habebunt formam gh^m , ex qua ut h eliminetur, oportet sit $g = nh^{-m}$ ubi n ipsa h non sit affectum, sed devotet numerum quem vis in quantitatem datam ductum;

hance veo quantitatem in e complecti licet, ut ergo n solum numerum significare possit. Hac ergo ratione singuli termini erunt $nh^{-m}x^{m}$ .Ubi cum dimensiones ipsius h destruant dimensiones ipsius x, perspicuum est integrale nullam dimensionem habere debere. Deinde id quoque manifestum est in integrali praeter h et x, et numeros alias quantitates contineri non oportere; unde sequitur idem et in differentiali locum habere. Quapropter p cum ab h affici nequeat, in meris x dari debebit, eritque p potentia ipsius x quae sit x^{n}.

§9. Ex hac conditone differentiale $pdx:\surd(h-x)$ transmutatum est in $x^{n}dx:\surd(h-x)$. Accedat altera atque prior conditio, qua integrale nullam habere debet dimensionem, ut inde n determinetur. Requiritur autem ad id, ut integrale nullius sit dimensionis, ut et in differentiali dimensiones sese destruant elemento dx unam dimensionem implere posito; manifestum enim est, semper differentiale tot habere dimensiones, quot integrale. Numerus vero dimensionum in nostro differentiali $x^{n}:e\sqrt{(h-x)}$ est $n+1-\frac{1}{2}$ seu $n+\frac{1}{2}$, qui ergo debet aequari nihilo; unde habetur $n=-\frac{1}{2}$. Ex quo emergit $p=x^{-\frac{1}{2}}$ seu $1:\sqrt{x}$, hinc porro erit $dv=dx:e\sqrt{x}$. Quia e eum in finem tantummodo erat assumtum ut homogeneitas conservetur, fiat $e=1:\sqrt{a}$; eritque $dv=dx\sqrt{(a:x)}$.

§10. Erat vero $dv = ds + dt$, quare $ds+dt=dx\sqrt{(a:x)}$, cujus integrale est $s+t=2\sqrt{ax}$. Erit igitur summa arcuum AM + AN semper in ratione subduplicata sagittae AP. Construatur ergo alia curva ALE, talis ut productis MN, mn in L et l sit arcus AL = AM + AN. Eritque Ll = Mm + Nn = $ds + dt = dv$. Unde AL = $v = 2\sqrt{ax}$, adeoque $vv = 4ax$. Ex qua perspicuum est curvam ALE esse cycloidem, cujus circuli generatoris diameter est a. Descendat corpus in hac cycloide ex puncto E aeque alto ac C vel D; erit velocitas ejus in L ut $\surd(h-x)$. Ergo tempusculum per Ll est $dv:\surd(h-x)$. Id quod igitur aequale est summae tempusculorum per elementa Mm, Nn. Quare totum tempus descensus per ELA, aequale erit summae temporum per arcus CA et DA. Oscillatio ergo per CAD contemporanea est dimidiae oscillationi penduli longitudinis 2a, seu integrae oscillationi penduli long. $\frac{1}{2}a$.

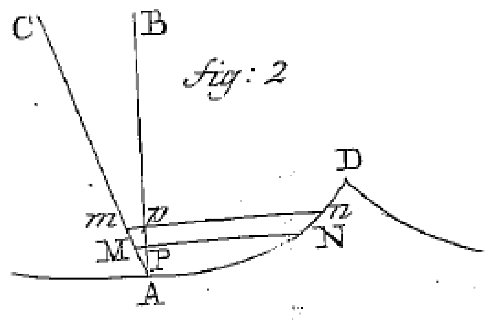
§11. Ex his jam facile apparet, quomodo data altera curva AC inveniri debeat altera AD. Sit quaesitae AD applicata PN = z; erit $\mathrm{N}n=dt=\sqrt{dx^{2}+dz^{2}}$. Erit igitur

$$ds+\sqrt{(dx^{2}+dz^{2})}=dx\sqrt{(a:x)}.$$

Unde $\sqrt{(dx^{2}+dz^{2})}=dx\sqrt{(a:x)}-ds$

. Denique $dz=\sqrt{(adx^{2}:x+ds^{2}-dx^{2}-2dsdx\sqrt{(a:x)}}$. Cum curva AMC sit data, dabitur ds in x et dx; ponatur igitur $ds = pdx$. Erit $dz=dx\sqrt{(a:x+p^{2}-1-2p\sqrt{(a:x)})}$. Quae aequatio, cum p in x dari ponatur, exprimet naturam curvae AND quaesitae. Hinc intelligitur, cum a non a curva pendeat, et ideo pro lubitu accipi possit, infinitas inveniri posse curvas loco quaesitae AND,quae cum AMC junctae tautochronas praebeant. Notandum tamen accidere casus, quibus, si a quantitate quadam minor accipiatur curva quaesita fiat immaginaria.

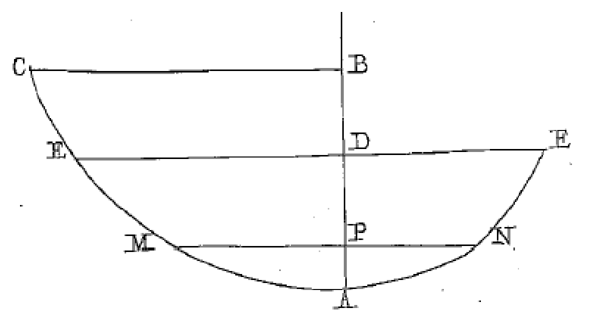
§12. Sit curva data AC linea rect, cum verticali AB anglum quemcunque BAC constituens, erit $ds = ndx$, n denotante numerum ei angulo convenientem, unde $p = n$, quare $dz = dx\sqrt{(a:x+nn-x-2n\sqrt{(a:x)})}$. Quae aequatio integrationem admittit I casu n = 1, quo recta AC sit verticalis inciditque in AB. Hic sit $dz = dx\sqrt{(a:x-2\sqrt{(a:x)})}$, fiat $2\sqrt{ax} = q$, erit $x = qq:4a$, et $dx = qdq:2a$; ergo $dx = \frac{qdq}{2a}\sqrt{(\frac{4aa}{qq}-\frac{4a}{q})} = dq\sqrt{(\frac{a-q}{a})}$.



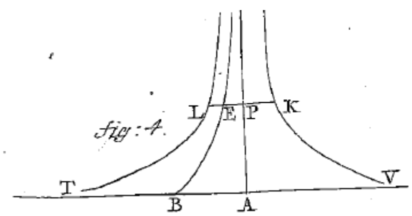
Est igitur $z = C - \frac{2*(a-q\sqrt{(a-q)})}{3\sqrt{a}} = C - \frac{2(a-2\sqrt{ax})\sqrt{(a-2\sqrt{ax})}}{3\sqrt{a}}$. Ut z fiat = 0 si x = 0, oportet sit C $= \frac{2a}{3}$, adeoque est $z = \frac{2a\sqrt{a}-2(a-2\sqrt{ax})\sqrt{(a-2\sqrt{ax})}}{3\sqrt{a}}$. Quae est aquatio ad curvam quarti ordinis; Hic x nunquam $\frac{a}{4}$ superare potest.

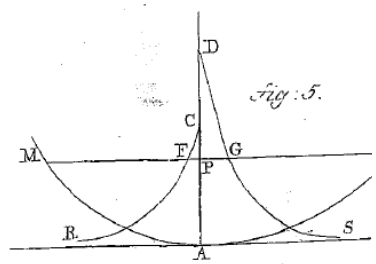
§13. Si curva altera AMC fuerit semicyclois, cujus diameter circuli generatoris AB = b. Erit dictis AP, x, AM, s, tum $ss = 4bx$, ergo $s = 2\sqrt{bx}$. Sit altera curva quaesita ANE in qua AN = t, oportet sit $s + t = 2\sqrt{ax}$, unde habebitur $t = 2\sqrt{ax} - 2\sqrt{bx}$. Dicatur $\sqrt{a} - \sqrt{b} = \sqrt{c}$; $t = 2\sqrt{cx}$. Est itaque altera curva ANE etiam cyclois idque quaecunque : ejus enim diameter c pro lubitu potest accipi. Oscillationes vero cotemporaneae sunt dimidiae oscillationi penduli, cujus longitudo est 2a, vel integrae si longitudo fuerit $\frac{1}{2}a$. Est $\sqrt{a}$ + = $\sqrt{c} + \sqrt{b}$, unde $a = b + 2\sqrt{bc} + c$. Notandum vero in cycloide majori AMC initium descensus non supra punctum E, ubi ED producta secat, esse accipiendum; alioquin enim corpus ascendens in curva AE ultra E ascenderet, et oscillatio nusquam terminaretur.

§14. Quaeramus casus, quibus ambae curvae sint inter se aequales. Erit igitur $s = t$. Quare cum sit $s + t = 2\sqrt{ax}$; erit $2s = 2\sqrt{ax}$; seu $s = \sqrt{ax}$. Ex quo cognoscitur, utramque curvam esse cycloidem, neque alias hoc sensu satisfacere curvas praeter cycloidem : Supra enim demonstratum est nostra methodo problema propositum generalissime solvi. Quemadmodum hic positum erat $s = t$, sic quaecunque aequatio inter s et t potest accipi, et deinde duae curvae dari, ut arcus ascensus et descensus eam habeant inter se relationem. Ut, si quaerantur duae curvae problemati satisfacientes CA, DA, ut sit semper AM : AN = $m : n$, erit $mt = ns$, et $t = ns : m$. Ergo $s + t = (ms + ns) : 2\sqrt{ax}$, unde efficitur $s = \frac{2m}{m+n}\sqrt{ax}$. Perspicuum ergo est, curvam AC esse semicycloidem diametri $\frac{m^2a}{(m+n)^2}$, et alteram AND quoque semicycloidem diametri $\frac{n^2a}{(m+n)^2}$.



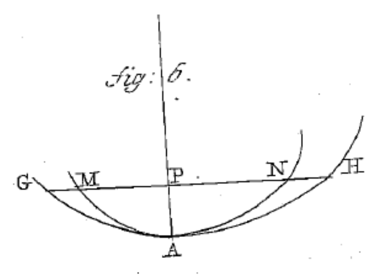
§15. Cum esse debeat $s + t = 2\sqrt{ax}$, ut ambrae curvae praebeant tautochronam oscillationes isochronas penduli longitudinis $\frac{1}{2}a$ habentem; Sit $s = \sqrt{ax} + v$, et $t = \sqrt{ax} - v$. Hoc igitur modo duae curvae invenientur satisfacientes. Erit itaque $dx = \frac{adx}{2\sqrt{ax}} + dv$, et $dt = \frac{adx}{2\sqrt{ax}} - dv$. Ponatur $dv = udx$: habebitur $ds = \frac{adx}{2\sqrt{ax}} + udx$, et $dt = \frac{adx}{2\sqrt{ax}} - udx$. Quare si y illius et z hujus curvae denotent applicatas, erit $dy = dx\sqrt{(\frac{a}{4x} + \frac{au}{\sqrt{ax}} + uu - 1)}$. Atque $dz = dx\sqrt{(\frac{a}{4x} - \frac{au}{\sqrt{ax}} + uu - 1)}$. Hic si loco u substituatur quaecunque functio ipsius x; habentur duae aequationes pro curvis problemati satisfacientibus. Observandum hic, si ponatur $a = 4b$ fore $dz = dx\sqrt{(\frac{b}{x} - \frac{2bu}{\sqrt{bx}} + uu - 1)}$. Quae aequatio convenit cum aequatione § 11, $dz = dx\sqrt{(\frac{a}{x} + pp - 1 - \frac{2ap}{\sqrt{ax}})}$ si sit $b = a$, et $u = p$. Ex quo intelligitur curvam $dz = dx\sqrt{(\frac{a}{4x} - \frac{au}{\sqrt{ax}} + uu - 1)}$, etiam cum hac $ds = udx$ seu $dy = dx\sqrt{(uu - 1)}$ conjunctam constituere tautochronam oscillationes absoluentem eodem tempore, quo pendulum longitudinis $\frac{1}{2}b$ seu $\frac{1}{8}a$.





§ 16. Constituatur super axe AP curva quaecunque BE, in qua posita AP = x sit PE = u. Tum describatur hyperbola cubicalis VKLT,cujus applicata PK vel PL si dicatur r, sit $4xr^2 = a$, recta quadam pro unitate accepta, erit PK vel PL $= \sqrt{(a:4x)}$. Deinde constituantur duae novae curvae RF, SG, in quibus sit

PF = $\sqrt{(\mathrm{LE}^2 - 1)}$; et PG = $\sqrt{(\mathrm{KE}^2 - 1)}$. Erit $\mathrm{PF} = \sqrt{(\frac{a}{4x} - \frac{au}{\sqrt{ax}} + uu - 1)}$, et $\mathrm{PG} = \sqrt{(\frac{a}{4x} + \frac{au}{\sqrt{ax}} + uu - 1)}$. Quibus factis accipiatur PM in 1 ducta aequalis areae APFR : et PN in 1 ducta aequalis areae APGS. Erunt cum sit $\mathrm{APFR} = \int dx\sqrt{(\frac{a}{4x} - \frac{au}{\sqrt{ax}} + uu - 1)}$, et $\mathrm{APGS} = \int dx\sqrt{(\frac{a}{4x} - \frac{au}{\sqrt{ax}} + uu - 1)}$, PM = z et PN $= y$, atque ea propter curvae MA et NA juncae in A exhibeunt curvam tautochronam.



§ 17. Ex hisce perspicuum est, quomodo data curva quacunque inveniri oporteat alteram tautochronismo producendo aptam. Nunc eos investigare statui casus, quibus ambae eae curvae ita, vt decet, junctae, eandem constituunt curvam continuam; ut et aliae curvae eaeque innumerae cycloidis similes habeantur, eundem effectum in horologiis praestantes. Sit MAN hujusmodi curva circa axem verticalem AP posita. Dicatur ut ante, AP, x, arcus AM, s, et alter AN, t, oportet esse $s + t = 2\sqrt{ax}$. Constituatur alia curva GAH talis, ut ejus applicatae PG, PH sint arcubus AM,AN respective aequales. Erit ergo PG = s, PH = t, eaque curvae GAH erit proprietas, ut sit $s + t = 2\sqrt{ax}$. Perspicuum est, si curva GAH fuerit data alteram MAN ex ea posse constui, atque si illa fuerit curva continua, et hanc quaesitam talem fore. Huc igitur quaesitio est reducta, ut inveniatur curva GAH, quae sit continua, eamque habeat proprietatem; ut GP + GH = $2\sqrt{a}$.AP.

§ 18. Respondent ergo in curva GAH singulis abscissis AP duae applicatae GP, PN, quarum altera est negativa, si altera affirmativa fuerit. Talis proinde aequatio inter abscissas et applicatas esse debet, ut litera applicatas denotans pro singulis abscissis duos habeat valores ad conditionem quaestionis accommodatos. Ut haec facilius efficiam, assumo novam indeterminatam v, ex qua una cum constantibus et abscissae et applicatae determinari debent; ita autem, ut, posita v affirmativa, inveniatur punctum G; posita vero v negativa, tunc punctum H inveniatur. Consideremus igitur x, tanquam functionem ipsius v, atque s. Functio autem s significans dabit t sed negative, quia PN ad alteram axis AP partem cadit, si v abeat in -v.

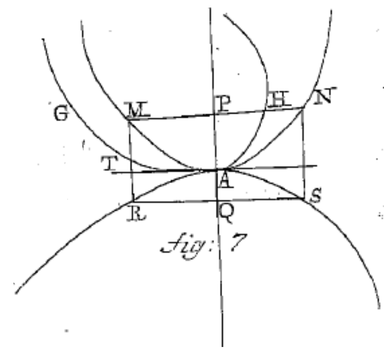


Fig: 7

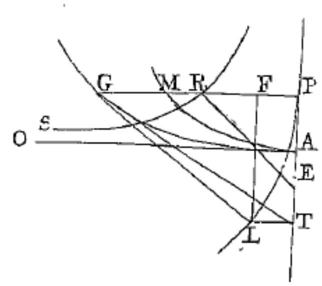
§ 19. Cum abscissa AP eadem maneat pro utroque puncto G et H, oportet ut ea x ita in v determinetur, ut eadem maneat transmutato v in -v. Sive x debet esse functio par ipsius v : sit talis functio P, erit x = P. Ponatur applicata PG, s = Q + R, denotantibus Q functionem imparem, R, vero parem ipsius v. Ponatur in hac formula Q + R loco v, -v, abibit ea in -Q + R; quemadmodum constat ex iis, quae de functionibus paribus et imparibus in dissertatione de trajectories reciprocis tradidi. Posito vero -v loco v, habebitur punctum H, quare - Q + R exprimet applicatam PH. Quae autem, cum in alteram partem cadere debeat, erit valor - Q + R negativus. Absoluta ergo applicatae PH seu t magnitudo erit Q - R, unde habetur t = Q - R. At vero est s = Q + R, et x = P.

§ 20. Ex conditione problematis haec habetur proprietas, ut sit $s + t = 2\sqrt{ax}$, ut in §17 ostensum est. Quare cum sit s = Q + R, t = Q - R, et x = P, erit 2Q = $2\sqrt{a\text{P}}$ seu QQ = aP, hinc P = QQ : a. Hic inquirendum est, an hic valor ipsius P inventus, et superior, secundum quem P debet esse functio par ipsius x inter se non repugnent ? Si enim repugnarent, nihil inde ad propositum elici posset. Non autem iis inter se repugnant : nam, quia Q est functio impar, ejus quadratum erit functio par; porro divisore a nihil ad haec faciente, perspicuum est hic P functione pari aequale poni. Est ergo P = QQ : a. Ex his curva GAH invenitur. Accipitur enim AP seu x = QQ : a, et PG seu s = Q + R, ubi loco Q quaecunque functio impar, loco R vero quaecunque par substitui potest ipsius v. Quia x = QQ : a erit Q = $\sqrt{ax}$, et idcirco s = R + $\sqrt{ax}$. Hic R potest accipi functio par ipsius Q seu $\sqrt{ax}$, sive duntaxat ipsius $\sqrt{x}$.

§ 21. Ex hisce facile elicitur curvarum nostro instituto inservientium constructio. Circa axem verticalem AP constituatur parabola MAN, cujus parameter = a. Ducta ergo ordinata ad axem orthogonali MN, erit, si sit AP = x, PM = $\sqrt{ax}$. Infra hanc parabolam circa eundem axem describatur curva quaecunque QAS, cujus axis AQ simul est diameter. Ducantur verticales MR, NS, horizontalem per A transeuntem secantes in T et V. Erit AT = $\sqrt{ax}$, et AV = -$\sqrt{ax}$; TR autem et SV erunt aequales. Quae, cum sint ad eandem plagam sitae, erunt functio par lineae AI quae est $\sqrt{ax}$, quare IR exprimet functionem R. Tum nova construatur curva GAH, cujus applicata PG sit = PM + TR, erit alter a PH ob legem continuitatis = PN - SV seu PN - TR. Quare erit PG = R + $\sqrt{ax}$. Unde sequitur curvam GAH eandem esse, quae quaeritur.

§ 22. Hoc ergo modo inveniuntur curvae infinitae, non quidem tautochronae, sed tales ex quibus tautochronae possunt construi. Sit curva AG praecedenti modo constructa, inde si alia AM construatur, ut ejus arcus AM ubique sit aequalis respondenti applicatae PG, erit haec curva tautochrona (§17). Ex data vero AG, requisita AM sequenti modo construetur. Ducatur recta in G tangens GI, occurrens axi producto in I. Centro G radio GP describatur arcus circuli PL, quem horizontalis ex I ducta secet in L. Jungatur GL, et a P in axe capiatur longitudo arbitraria PE, sed ubique eadem. Tum ex E ducatur linea ER parallela ipsi LG, secans applicatam PG in R. Per omnia hoc modo determinata puncta R transeat curva SR, quae pluremque assymtoton habebit horizontalem AO. Denique construatur curva AM talis, ut rectang. PM.PE aequale sit spatio OAPRS. Erit haec AM curva tautochrona. Est enim arcus AM = PG.

Fig. 8



§ 23. Rem analytice persequor. Cum x debeat esse functio par ipsius v, insuper autem sit Q = $\sqrt{ax}$, oportet sit $\sqrt{ax}$ functio impar ipsius v, pono $\sqrt{ax} = v$, erit $x = v^2 : a$ functio par, ut requiritur. Habemus igitur ex § 20 hanc aequationem $s = \mathrm{R} + v$, ubi R denotat functionem parem ipsius v. Erit itaque $ds = d\mathrm{R} + dv$, sit $d\mathrm{R} = \mathrm{V}dv$, necesse est, ut V sit functio ipsius v impar. Quare erit $ds = dv(1 + \mathrm{V})$, ideoque $ds^2 = dv(1 + 2\mathrm{V} + \mathrm{VV}) = dx^2 + dy^2$. Quoniam autem $x = v^2 : a$, erit $dx = 2vdv : a$, et $dx^2 = 4v^2dv^2 : a^2$. Consequenter $dy^2 = dv^2(1 + 2\mathrm{V} + \mathrm{VV} - 4v^2 : a^2)$ atque $dy = \frac{dv}{a}\sqrt{(a^2 + 2a^2\mathrm{V} + a^2\mathrm{V}^2 - 4v^2)}$. Hanc aequationem nullo modo rationalem efficere potui, substituendis loco V valoribus legitimis, ut nimirum V aequalis ponatur functioni impari ipsius v. Quamobrem nescio, an alii casus inde erui queant, quae integrationem admittunt, praeter eum, quem hic expositurus sum.

§ 24. Ponatur $a\mathrm{V}$, id quod fieri potest; aequale $2v$, ut termini $a^2\mathrm{V}^2$ et $4v^2$ sese destruant; erit $dy = \frac{dv}{a}\sqrt{(aa + 2av)}$, quae aequatio integrationem admittit quia v unius tantum est dimensionis. Integralis ejus est haec aequatio

$$y = \frac{C + (a+4v)\sqrt{(a+4v)}}{6\sqrt{a}} = \frac{C + (a+4\sqrt{ax})\sqrt{(a+4\sqrt{ax})}}{6\sqrt{a}} \text{ ob } v = \sqrt{ax}$$

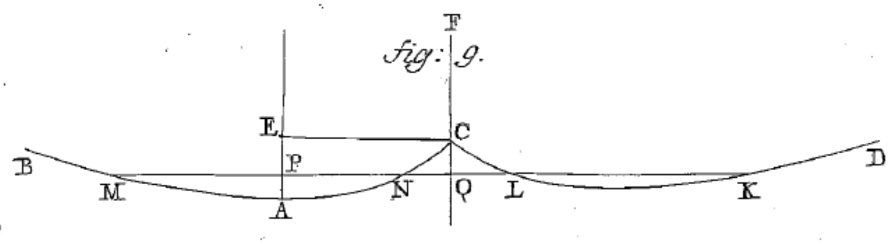
. Ut y evanescat, posito $x = 0$, oportet ut sit C = $-a\sqrt{a}$; erit igitur

$$y = \frac{-a\sqrt{a} + (a+4\sqrt{ax})\sqrt{(a+4\sqrt{ax})}}{6\sqrt{a}} \text{ seu } 6y\sqrt{a} + a\sqrt{a} = (a+4\sqrt{ax})\sqrt{(a+4\sqrt{ax})}$$

. Hinc habebitur

$$a + 4\sqrt{ax} = (6y\sqrt{a} + a\sqrt{a})^{\frac{2}{3}} = \sqrt[3]{(36ayy + 12aay + a^3)}.$$

Quamobrem sumtis utrinque cubis erit $12aa\sqrt{ax} + 48aax + 64ax\sqrt{ax} = 36ayy + 12aay$ seu $12ax + (3a + 16x)\sqrt{ax} = 9yy + 3ay$. Quae penitus ad rationalitatem reducta dabit hanc aequationem ordinis quarti : $81y^4 + 54ay^3 - 216axyy - 256ax^3 + 9aayy - 72aaxy + 48aaxx - 9a^3x = 0$.

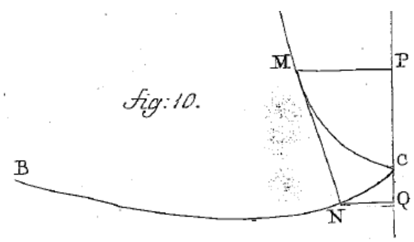


§ 25. Habemus ergo curvam algebraicam ordinis quartis, quae perinde atque cyclois ad oscillationes omnes aequitemporaneas faciendas est idonea. Eam igitur aliquanto accuratius hic describere operae pretium erit. Sit axis AE; habebit curva nostra hanc formam BACD, ejusque, dictis AP, *y*, ea erit aequatio, quam § praecedente invenimus. Tempus autem, quo oscillatio quaecunque per MAN absolvitur, aequale erit tempori oscillationis penduli ordinarii, cujus longitudo est $\frac{1}{2}a$. Notandum est, hanc curvam ab altera parte axis AE, in C habere punctum reversionis; Punctum vero C reperitur sumendo AE = $\frac{1}{16}a$, et applicatam EC = $\frac{1}{6}a$. Porro in C curva producta ita revertitur, ut sit arcus CD aequalis similisque arcui CAB. Quapropter si ex C ducatur verticalis CF, erit ea diameter curvae orthogonalis. Oscillationes vero in alterutra tantum parte BAC constitui debent.

§ 26. Cum igitur CF sit diameter hujus curvae, quaeramus aequationem ad hanc diametrum relatam. Sit nimirum CQ = *t*, et QM = *z*, erit AP = *x* = AE - CQ = $\frac{1}{16}a$ - *t,* et PM = *y* = QM - CE = *z* - $\frac{1}{6}a$, his valoribus loco *x* et *y* in aequationes in § 24 substitutis, sequens resultabit aequatio, $81z^4+216atzz+256at^3-18aazz=0$, sive, quae ad hujus curvae proprietates inveniendas magis est apta, haec $t=\frac{aa-(\sqrt[3]{36azz}-a)^2}{16a}$ seu $z=\pm(a\pm\sqrt{(aa-16at)})^{\frac{2}{3}}:6\sqrt{a}.$ Unde perspicuum est quatuor valores habere manente t, idque hoc modo, si ambo signa + valeant habetur punctum M; Si prius signum - et alterum + valeant habebitur punctum K : Si prius + et posterius - sumantur, punctum N; Si denique utrumque signum - locum habeat, obtinebitur punctum L.

§ 27. Ex hac aequatione perspicitur curvam hanc esse quadrabilem. Ponatur $a\pm\sqrt{(aa-16at)}=p$, erit $t=\frac{2ap-pp}{16a}$ et $z=\frac{p\sqrt{a}}{6\sqrt{a}}$. Ergo $dt=\frac{dp}{8}-\frac{pdp}{8a}$. Itaque $zdt=\frac{pdp\sqrt{p}}{48\sqrt{a}}-\frac{ppdp\sqrt{p}}{48a\sqrt{a}}$. Quod intergratum dabit $\int zdt=\frac{pp\sqrt{p}}{120\sqrt{a}}-\frac{p^3\sqrt{p}}{168a\sqrt{a}}$. Quae quantitas exprimit spatium inter abscissam, applicatam et curvam contentum. Constat deinde ex curvae invention eam esse rectificabilam. Quia $z=\frac{p\sqrt{p}}{6\sqrt{a}}$ erit $dz=\frac{dp\sqrt{p}}{4\sqrt{a}}$; unde $dz^2=\frac{pdp^2}{16a}$. Est vero $dt^2=\frac{dp^2}{64}-\frac{pdp^2}{32a}+\frac{p^2dp^2}{64aa}$. Quare $dt^2+dz^2=\frac{dp^2}{64}+\frac{pdp^2}{32a}+\frac{p^2dp^2}{64aa}$, et hinc $\sqrt{dt^2+dz^2}=\frac{dp}{8}+\frac{pdp}{8a}$. Consequenter $\int\sqrt{dt^2+dz^2}=\frac{2ap+pp}{16a}$. Quae expressio dat vel arcum CN vel CAM vel quoque eorum negativos CL vel CLK.

§ 28. Inventis area et longitudine hujus curvae; residuum est id, quod maxime ad usum ejus in horologiis pertinet, ut investigemus radium osculi, eoque invento curvae hujus evolutam, quo pendulum oscillationes in hac curva absoluens constitui queat. Radius osculi vero erit, posito *dz* constante $\frac{(dt^2+dz^2)^{\frac{3}{2}}}{dzddt}$, cujus valor ex superioribus invenietur. Namque est $(dt^2+dz^2)^{\frac{3}{2}}=\frac{(a+p)^3}{512a^3}$, atque $dz=\frac{dp\sqrt{p}}{4\sqrt{a}}$, hinc quia *dz* ponitur constans, erit $ddz=0=\frac{2pddp+dp^2}{8\sqrt{ap}}$, unde habetur $ddp=\frac{-dp^2}{2p}$. Denique quia $dt=\frac{adp-pdp}{8a}$, erit $ddt=\frac{addp-dp^2-pddp}{8a}=\frac{-adp^2-pdp^2}{16ap}$. His valoribus in formula $\frac{(dt^2+dz^2)^{\frac{3}{2}}}{dzddt}$ substitutis, orietur radius osculi = $\frac{-(a+p)^2\sqrt{ap}}{8aa}$, signum - indicat radium osculi et diametrum inter se divergere.



§ 29. Cum radius osculi sit cognitus, facile erit curvae nostrae tautochronae evolutam invenire. Sit CNB tautochrona. Maneant CQ = $t=\frac{2ap-pp}{16a}$, QN = $z=\frac{p\sqrt{p}}{6\sqrt{a}}$. Sit radius osculi = $\frac{(a+p)^2\sqrt{ap}}{8a}$, qui tanget in M evolutam quaesitam CM. Demittatur ex M in axem applicata MP, sintque CP = x, PM = y. Invenientur hae coordinatae ex relatione cognita coordinatarum CQ et QN. Calculo utpote facile hic omisso habebitur $x=\frac{2ap+5pp}{16a}$ et $y=\frac{(3aa-3pp+4ap)\sqrt{ap}}{24aa}$, harum aequationum ope evoluta CM per infinita puncta jam describi poterit. Si autem velimus p eliminare, ut aequatio inter x et y supersit, p ex priore aequatione invenitur in a et x, qui valor si deinde in altera substituatur,sequens emergit aequatio :

$$576ayy-\tfrac{37632}{125}axx-\tfrac{32160}{625}a^2x+\tfrac{529}{3125}a^3=(\tfrac{2304}{125}xx+\tfrac{8608}{625}ax+\tfrac{529}{3125}a^2)\sqrt{(a^2+80ax)}$$. Quae aequatio si prorsus ad rationalitatem reducatur, erit ordinis quinti.

§ 30. Id denique filentio praetereundum non est, hanc curvam tautochronam eandem esse prorsus, quam lineae rectae verticali jungendam invenimus (§ 12). In eo enim solo aequationes differunt, quod ibi parameter a quadruplo sit major quam hic. Quia igitur longitudo penduli isochroni pro nostra curva tautochrona est $\frac{1}{2}a$, erit si haec eadem curva cum verticali AE jungatur, longitudo penduli isochroni $\frac{1}{8}a$. Tempora ergo oscillationum in curva MAN duplo sunt majora quam oscillationum per PAN. Quare si in utroque lapsu grave ad N usque perveniat ascendendo, erit *t*MA + *t*AN =2*t*PA + 2*t*AN. Consequenter tMA - tAN = 2tPA. Differentia ergo temporum descensuum per arcus MA et NA aequatur duplo tempori descensus per verticalem AP.