TAUTOCHRONE CURVES IN A FLUID OFFERING RESISTENCE ACCORDING TO THE SQUARE OF THE SPEED.

L. Euler.

§1.

After *Huygens* first discovered that the cylcoid was a tautochronous curve *in vacuo* according to the hypothesis of uniform gravity; *Newton* and *Hermann* have also given tautochrones following the hypothesis of non-uniform gravity acting, and pulling towards some fixed point as centre. Moreover, they have considered the motion to arise in a vacuum, with no resistance. Truly pertaining to resisting media, *Newton* has also shown that the cycloid is a tautochrone in a medium for which the resistance is proportional to the speed; moreover, as far as any other kinds resisting media are concerned, there has been no progress made either in producing the curves themselves or in demonstrating possible tautochronism in them [The 3rd edition of the *Principia* that Euler refers to finally in §35 alters this view to include the type of resistance offered here. It may be of interest to the reader to observe that Johan. Bernoulli published a paper in the *Memoire de l'Acad. Roy. des Sciences* in 1730, also present in his *Opera Omnia*, T. III, p.173, with the title (in tra. from French): *Method for Finding Tautochrones in Media Resisting as the Square of the Speed*; in which Euler does not get a mention.]

§2. Indeed it is not difficult to find the curve in some resisting medium, upon which a weight falls in the same manner as along a given curve in a vacuum. At I understood it [initially], I looked for these curves in a medium with some hypothetical resistance, upon which a weight might fall as equally along a cycloid in a vacuum, which would then appear to me to be tautochrones in these resisting media : since the descent of bodies on these might be the same as of bodies along tautochrones *in vacuo*. And I gave this very property to the curves published in the Actis. Lips. 1726 [E001 in this series of translations: Euler's hypothesis was in error, as explained there in detail; essentially, the resistance depends on the speed, and not on the arc length as Euler had assumed for mathematical expediency]; and bodies descend in the same way along any of these curves in the medium with the relevant resistance applied, as those weights descend along a cycloid in a vacuum; on account of which I have called these curves tautochrones.

§3. Truly after more careful consideration of this matter, this has thus to be understood; as the whole [descent] curve to be traversed in the resistive medium may be considered to depend upon the descent from the beginning along on a curve assumed to be *in vacuo*. Whereby, if another starting point is taken on the given curve in the resistive medium, the other will produce the same descent time as before. From which it is also understood that if a curve is taken, upon which a body shall have equal times of descent in a resisting medium, and on a cycloid in vacuo, clearly from a given starting descent point; yet this resistive curve still does not have the property that a weight starting anywhere on its descent will arrive at the lowest point at the same time, and indeed the descent time does not agree with the descend time along the cycloid, except that which starts from a (unique) given point. [We understand the physical cause for these difficulties as due to damping forces, which have a component proportional to the velocity of a viscous nature, and another proportional to the square of the velocity that can be attributed to collisions between the fluid particles and the falling mass, both of which increases the period, and reduce the *vis viva* or kinetic energy of the mass. Newton in Book II of the *Principia*, §6 Third Edition, set out these ideas for a Huygens-type pendulum with resistance. See § 35 for details.]

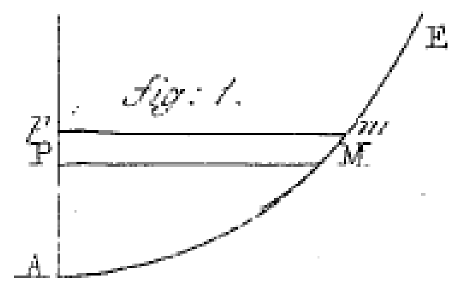
§ 4. The same will become more obvious if the reader should turn his attention and look at the equations given there [in E001], and the manner in which they have been elicited. Indeed, it will be grasped from these that up to this stage a certain letter [i. e. variable, which is the arc length] which is a kind of constant, is brought forwards, upon which the curve itself and the start of the descend depends [*i. e.* specifying the kind of resistance]. Therefore if it is desired to proceed otherwise, then constant appearing will be different, and hence the curve is acquired from another parameter, and different from the first. I discovered the inconvenience with this theory not long after [it was published], and this was also found by the celebrated *Hermann* in a dissertation concerning various kinds of motion pubished in the Actis 1727 [actually in *Comm.Ac.Sc.* in that same year], in which he establihed curves that can be found from the same principle that I have used, and which also are seen to be tautochrones. These curves are not nevertheless of the kind that I presented, on account of the reasons examined and explained by me here, and then he has investigated the descent times of these curves, and he has concluded that they are not constant for different starting points of the descent.

§ 5. I am therefore unwilling, as I had produced these curves for a number of cases from the principle of tautochronism, for another judgement to be brought about them, except that for any medium with the resistence given there by an equation with different values of a constant kept in mind, a whole family of curves can be shown upon which weights released from any point of the descent can reach the lowest point in equal times, and that motion is carried out in the same way that weights fall along a cycloid in a vacuum. [Thus, Euler realised that his curves were not physically meaningful, but yet they had a certain mathematical charm, as they gave rise to a family of curves defined according to his defunct hypothesis of tautochronism in a resistive medium.] Therefore since I had recognised that these curves were not suitable for the production of tautochrones, all at once I realised that not only was this truth for these curves, but also for others that I had communicated [to Johan. Bernoulli one presumes] and upon which we had puzzled over; as truly we may discover tautochrones in any medium with resistance. Therefore after I had tackled the problem in a number of ways, I have finally acquired tautochrones, but only for the one resistance put equal to the square of the velocity, and for which I have set down an explanation in this dissertation.

§ 6. Since recently I had uncovered a certain new method, with which from first principles not only the cycloid, but in addition to this I have found innumerable other tautochronous curves in the vacuum, and I have established this method to use for finding tautochronous curves in resisting media also. From the universal realibility of the method, if there are tautochrones in media with resistence, these ought to be found with the aid of this method. Moreover besides, from the extent that I have been effective here in this analysis, it has become apparent to me to be absolutley necessary, in order that the velocity of the body upon some curve in the resisting medium, for which it is desired to be a tautochrone, that the ascent or descent from some point must be expressed, not in algebraic terms but somehow in a transcendent manner [Euler has invented the exponential function with a negative argument for this purpose]. It shall truly be only in a vacuum, and in a medium that the resistance is kept according to the square of the velocity, that tautochrones exist even in fluids. Here I will show how to find these which are considered to offer resistance in the ratio of the square of the velocity.

§ 7. I have thus wished to set up the first problem, that recently I had used to solve the same question in the vacuum, in order that, whatever is done here for a given curve I shall be able to find another curve [for the ascent], which joined with the descent curve will lead to tautochronous oscillations. In truth that question was not permitted to tie itself in a knot, since planely it will be different from what was seen in the vacuum case. For to go about the thing in the same way that was followed in the vacuum case, it is expedient, as with two curves, that there are always two points given and two that can be found, in which the speeds of the oscillating bodies are equal, and the determination of these points should not depend on the velocities; but for which points if the speeds are equal for one oscillation, then likewise shall be equal in the other oscillation. That truly may not be the case for a fluid, here I will be content to determine that curve upon which the descending body always reached the lowest point in equal times.

§ 8. Let CMS be the curve sought referred to the vertical axis AP. The following property of this curve is required : that a body arranged on this curve in the fluid falls in equal times to arrive at the lowest point A, from any starting point on the curve. The motion can be made from E, and it is evident that a body falls due to the force of gravity, as long as its specific gravity is greater than that of the fluid, with a continued acceleration, and truly likewise to be delayed on account of the resistance of the fluid in the ratio of the square of the speed, while finally when it arrives at A it retains a certain speed that I can equate to that which it can acquire by sliding from the height b in a vacuum [*i. e.* without any resistance on the same curve]. In order that the body starting here from A with this speed can ascend again as far as E, it is required that the force of gravity, which before aided the motion, should be reversed ; and truly the resistive force, which before was adverse [up the curve], is now follows and aids the weight, which has to be done, in order that here the ascent is absolutely the same as the descent. Since it is better to consider the ascent, this is presented first.



§ 9. Therefore the body is placed to ascend from A with the velocity of the height b, thus in order that it may accelerate in the square ratio of the speed, it can again arrive at E. This body shall be a cylinder of height a with the axis along the direction of the motion ; although this shape will be less suitable for oscillating, yet, since the calculation will be simpler and easy than for other shaped that can be moved across, I want to retain this figure. Again, let the specific gravity of the body be to that of the fluid as m to n. The body ascending in this manner will arrive at M, where its speed will be equal to that generated by a body falling throuth a height v in a vacuum, [thus, $V = \sqrt{(2gv)}$ or $\sqrt{v}$ in Euler's usual way of expressing gravity]. The arc AM the body traverses is called s, and the abscissa AP, x. From the motion everything shall arrive at a nearby place, the body truly at m. The speed of the body at m arising from the height $v + dv$ and M$m = ds$ with P$p = dx$.

§ 10. Since the body is moving in the fluid, it is not trying to descend with all its weight, but by that excess of the weight over the weight of an equal volume of fluid [by Archimedes principle]. Therefore the [vertical] force acting on the body is to the true weight as $m - n$ to m. If therefore the force of gravity is called g, the force acting will be equal to $(m - n)g : m$. With the body rising through the element of length Mm, if only the force of gravity were acting then $dv = -dx$, as the body obviously is rising in a vacuum. Moreover, since the force acting is in the same ratio to the force of gravity as $m - n$ to m, the effect of this [apparent weight] force to the effect of that force [the true weight] will be as $-dx$ as $m - n$ to m. On account of which, with this force acting : $dv = -(m - n)dx : m$ here a negative sign is used, since the force of gravity is placed opposing the motion of the body. Therefore this equation is obtained : $dv = -(m - n)dx : m$, from which the motion of the body ought to be obtained, except for the acceleration added due to the resistance of the body.

§11. We can now see how large the resistance of the fluid shall be in the case of a cylinder with a velocity acquired from a height v in the motion towards its base. This force is equal to the force that the fluid with the same speed exerts on a cylinder at rest; truly this force is equal to the weight of a cylinder of fluid of this height v, and with the base equal to that of the oscillating cylinder. [Thus Euler has taken the resistive force to be in proportion the the height fallen, which is as the square of the speed. He now takes the ratio of the densities to give a resistance proportional to (density of fluid) $\times$ vel.^{2}] Therefore the weight of this cylinder of fluid is to the weight of this actual cylinder as nv to ma [as the base areas are the same]. Therefore these resistances have a force to the force of gravity as nv to ma. Moreover for the body ascending through Mm, if the weight should accelerate, and acting along the direction Mm, it will be $dv = ds$. Therefore the force of the resistance for each ratio is producing an effect and increasing the speed of the body, and it shall become as $dv = nvds : ma$; but if it should be slowed down, then it will be as $dv = -nvds : ma$.

§ 12. If therefore only the force of gravity shall act by retarding the motion of the body, then by §10 it will be $dv = -(m - n)dx : m$, [We can interpret this equation from a modern point of view by regarding $mgdv$ as the change in potential energy for the body moving on a tautochrone in the vacuum, while $(m - n)gdx$ is the equivalent energy change along the curve in the fluid; where dv and dx are both vertical], and moreover, if only the accelerating force equal to the force of the resistance shall act, then by § 10 it will be $dv = -nvds : ma$. From which taken together, if both act alike, then this becomes :
$dv = nvds : ma - (m - n)dx : m$, or $madv + (m - n)adx - nvds = 0$. From which equation the motion of the body ought to be determined. [The interesting thing here is that all the differentials refer to lengths.] Moreover since in this equation v has a single dimension, it can be integrated. It can be reduced to this form : $dv - \frac{nvds}{ma} = \frac{-(m-n)dx}{m}$. This is multiplied by $c^{\frac{-nv}{ma}}$; where c indeed denotes the number of which the hyperbolic logarithm is one, and we have : $c^{\frac{-nv}{ma}}dv - \frac{nc^{\frac{-ns}{ma}}vds}{ma} = \frac{-(m-n)c^{\frac{-ns}{ma}}dx}{m}$. Of which the integral is the following : $c^{\frac{-nv}{ma}}v = C - \frac{(m-n)}{m}\int c^{\frac{-ns}{ma}}dx$.

§ 13. Putting $\frac{(m-n)}{m}\int c^{\frac{-ns}{ma}}dx = t$, [note that this '$t$' is a length and not a time!] since the integral of this can be found for a known curve, or even constructed. Moreover thus, if it can be done, a constant is put in the integration, in order that the integral is zero if x is equal to 0, from which the value of t to be determined can be arrived at. Therefore we have the following equation $c^{\frac{-ns}{ma}}v = C - t$. The value of this constant must be taken so that with $x = 0$, then $v = b$, such indeed as is to be put equal to the speed of the body at the point A; but with $x = 0$, both $s = 0$ and $t = 0$, and since $c^0 = 1$, then C = b. On account of which the following is found to establish and completely fit our equation: $v = c^{\frac{-ns}{ma}}(b-t)$. And hence it is understood too, where the velocity vanishes, or from the time the body was ascended in the curve, that is where $v = 0$, or $t = b$. Truly the speed itself of the body at M will be as $\sqrt{v}$ or as $c^{\frac{ns}{2ma}}\sqrt{(b-t)}$.

§ 14. Since now the speed of the body M can be found, the small interval of time to traverse the small arc Mm, which is as $\frac{ds}{\sqrt{v}}$, or in place of v from the above value by substitution, as $\dfrac{ds}{c^{\frac{ns}{2ma}}\sqrt{(b-t)}}$. This quantity will express a [differential] element of time. Therefore the integral of this should be obtained, as with the constant added which will make the time zero if either x or s or t is equal to zero, as I say, if t is made equal to b, in which case the time of ascent is found from the integral, then b which depends on the length of the arc described, completely disappears from the computation. In order that this can now happen, as I have found it necessary to show elsewhere that the whole expression for the element of time must have zero dimension [in E012]. Therefore what kind of function s of t itself should be sought ? Since it is not possible to allow b to enter into the computation in the expression for s, but only t, it is therefore seen that $\frac{ds}{c^{(\frac{ns}{2ma})}} = \frac{dt\sqrt{e}}{\sqrt{t}}$, and thus the element of time will be as $\frac{dt\sqrt{e}}{\sqrt{(bt-tt)}}$. Hence the length of the isochronous pendulum *in vacuo* is 2e. [Note that the motion is compared with that for a particle on a cycloid in a vacuum, *c.f.* §9, E012, where the element of arc length is $dv = dx\sqrt{(a:x)}$, where x is the vertical variable rather than t, and dv is the arc length.]

§ 15. For the determination of the curve, in order that it shall be made tautochronous, this equation has arisen : $ds : c^{\frac{ns}{2ma}} = dt\sqrt{e} : \sqrt{t}$, from which the nature of the curve can be determined. This equation on integration gives this : $C - \frac{2ma}{n}c^{\frac{-ns}{2ma}} = 2\sqrt{et}$; as the term t = 0, shall make $s = 0$, it is necessary that C = $\frac{2ma}{n}$; Therefore this equation is found for the curve sought : $\frac{ma}{n}(1 - c^{\frac{-ns}{2ma}}) = \sqrt{et}$, and with the square taken this becomes : $\frac{mmaa}{nn}(1 - c^{\frac{-ns}{2ma}})^2 = et$. Which on differentation again gives $\frac{ma}{n}(1 - c^{\frac{-ns}{2ma}})c^{\frac{-ns}{2ma}}ds = edt$ or $mac^{\frac{-ns}{2ma}}ds - mac^{\frac{-ns}{2ma}}ds = nedt$. Truly, the time is given by $t = \frac{m-n}{m}\int c^{\frac{-ns}{ma}}dx$, hence $dt = \frac{m-n}{m}c^{\frac{-ns}{ma}}dx$. Whereby with t removed, this equation between s et

x is obtained: $mmac^{\frac{-ns}{2ma}}ds - mmac^{\frac{-ns}{2ma}}ds = (m-n)nec^{\frac{-ns}{2ma}}dx.$ Which on multiplication by $c^{\frac{ns}{ma}}$ will become: $mmac^{\frac{ns}{2ma}}ds - mmads = (m-n)nedx.$

§ 16. The differential equation has been found again from the integration; the integration truly gives,
$\frac{2m^3aa}{n}ac^{\frac{ns}{2ma}}ds - mmas - \frac{2m^3aa}{n} = (m-n)nex,$ after the constant $\frac{2m^3aa}{n}$ has been removed. Which it is more convenient to write in this way :
$c^{\frac{ns}{2ma}} = \frac{ns}{2ma} + 1 + \frac{(m-n)nnex}{2m^3aa} = \frac{m^2aas+2m^3aa+(m-n)n^2ex}{2m^3aa}.$ Indeed this equation will suffice for the construction of the curve; but it is more convenient to be free from the exponentials. On this account, logarithms are taken, and it becomes :
$\frac{ns}{2ma} = l(m^2aas + 2m^3aa + (m-n)n^2ex) - l(2m^3aa).$ Hence on differentiation this equation is obtained : $\frac{nds}{2ma} = \frac{m^2nads+(m-n)n^2edx}{m^2nas+2m^3aa+(m-n)n^2ex}$ and from this by re-ordering :
$m^2n^2asds + (m-n)n^3exds = 2(m-n)mn^2aedx.$ Which on division by nn sets out the following final equation for the curve sought : $m^2asds + (m-n)nexds = 2(m-n)maedx.$

§ 17. If thus the curve AME has the property that it can be written as $m^2asds + (m-n)nexds = 2(m-n)maedx$ then that will be a tautochrone in this sense, that a cylindrical body of length a above that is always falling in the same time to arrive at the lowest point A, from wherever the descent started. If a sphere is put in place of the cylinder, having the same specific gravity and diameter a, it will be required to write in place of a in the equation $\frac{4}{3}a$, [This is a scaling factor, which seems to be used to give a sphere of a certain volume that can be matched by the cylinder, for which the base area has not been assigned, as it cancels in the ratio of the cylindrical masses used initially.] and the equation is obtained for the motion of the sphere, the diameter of which is a :
$4m^2asds + 3(m-n)nexds = 8(m-n)maedx$. If the length of the isochronous pendulum to be oscillating *in vacuo* is called f, then the length $e = \frac{1}{2}f$; and hence the equation will result : $8m^2asds + 3(m-n)nfxds = 8(m-n)mafdx$. Now it is fitting to apply this equation to some special case.

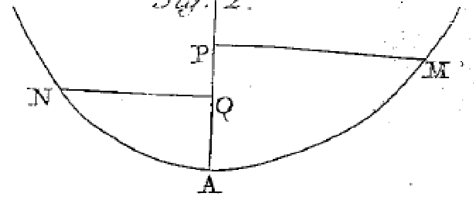
§ 18. We can put the density of the fluid to vanish, from which the motion of the body will become in a vacuum; and the density will be $n = 0$. Therefore with this in place, the following term of the equation will vanish : $3(m-n)nfxds$, and then the equation for the tautochrone *in vacuo* will be produced : $8m^2asds = 8(m-n)mafdx$. Which, as $n = 0$, on division by $8m^2a$ is reduced to $sds = fdx$. Truly this on integration is $ss = 2fx$, the equation for a cycloid, of which the diameter of the generating circle is $\frac{1}{2}f$. Which agrees absolutely with these equations which have been demonstrated for the tautochronism of cycloids. If therefore the equation found for a tautochronous curve in a fluid is reduced to a vacuum, the letter a denoting the diameter of the oscillating sphere has gone from the equation ; and therefore the tautochrone *in vacuo* does not depend on the size and shape of the oscillating body. But in a fluid, the size, shape, and specific gravity of the body are used in determining the tautochrone.

§ 19. The curve that we have found, serves as a tautochrone for the descent of the body, but from it the tautochrone can be found which is considered for the ascent of the body in the same fluid. Indeed the body is placed on the curve AME to ascend with an initial speed as before, generated from the height b; it will have [acting on it] both the force of gravity and the resistance of the fluid acting in the opposite sense. On which account, since this equation has been found for the descent above : $madv + (m - n)adx - nvds = 0$, where the force of the resistance, as the thing I have considered there was the acceleration ; in this case for ascending bodies the negative sign prefixed to the term $nvds$, which is attached to the force of the resistance of the fluid, must be changed into a plus sign. From which term the equation is obtained : $madv + (m - n)adx + nvds = 0$. From which the ascent of the same body, which before was placed to descend, will be determined.

§ 20. It is evident that this equation from above can be derived for the descent, but in that s becomes negative. Wherefore, it is not a necessity to find the tautochrone governing the ascent, as I may progress in the same manner as for the descent, but merely in the equation for the descending tautochrone found by replacing s with $-s$. This indeed will transform that into the tautochrone adapted for the ascent. If therefore the ascending body were a sphere of diameter a, and the specific gravity of this to that of the fluid are in the ratio m to n, the following equation is had for the tautochrone: $8m^2asds - 3(m - n)nhxdx + nvds = 8(m - n)mahdx$. Where I have put h in place of f, it is seen that the times of ascent ought not to be equal to the times of descent.

§ 21. Therefore since I may have found a tautochrone both for descent and ascent ; if these are joined at the lowest points, they will represent a tautochrone serving at the same time for ascending and descending. Let AM be the tautochrone for descent, and the other AN for ascent; it is observed, if the body is always taken to descend on the curve AM, so that it can rise to another point on the curve AN, then these oscillations are going to be completely equal in times, wherever the starts of the descents in AM are taken. If therefore AP were x and AM s, then $8m^2asds + 3(m - n)nfxds = 8(m - n)mafdx$, and for the other curve, if we call AQ = u, and AN = t, then it will be $8m^2atdt - 3(m - n)nhudt = 8(m - n)mahdu$. Truly the time of one oscillation is equal to the times of the two half oscillations of two pendulums *in vacuo*, of which the length of one is f, and of the other h, or of one whole oscillation of a pendulum of which the length is $\frac{f+h+2\sqrt{fh}}{4}$. [From adding and squaring the appropriate $\sqrt{l}$ terms.]

Fig. 2.



§ 22. If we take $f = h$, then the two curves AM and AN are parts of the same continues curve : One can understand from that, since then, if u is put in place of x and in place of s, since in the other curve the arcs will be negative, $-t$, that equation pertaining to the descent is changed into the one serving for the ascent. Hence the curve MA from the one part is continued in the othe part as the curve AN; this whole curve MAN has the property that the motion is resolved for equal times of oscillations for a sphere of diameter a, and of specifec gravity m set up in the above fluid of specific gravity n. Truly the descent must be made along the curve MA, and the ascent along AN, except perhaps these curves have this property, that if with the descent along NA and the ascent to be made along AM, all the whole oscillations as before should be tautochrones.

§ 23. The equation of the exponentials in § 16 only differs from that, which we deduced in § 21, since there a is that which here is $\frac{4}{3}a$, and e, which here is $\frac{1}{2}f$. If therefore in that equation is put $\frac{4}{3}a$ in place of a, and $\frac{1}{2}f$ in place of e, then

$64m^3aac^{\frac{3ns}{8ma}} - 64m^3aa - 24m^2nas = 9(m-n)n^2fx$ is obtained, which equation agrees with that which was found to serve for the descent in § 21,
$8m^2asds + 3(m-n)nfxds = 8(m-n)mafdx$. But for the other equation pertaining to the ascent, $8m^2atdt - 3(m-n)nhudt = 8(m-n)mahdu$, this equation corresponds :
$64m^3aac^{\frac{-3ns}{8ma}} - 64m^3aa + 24m^2nat = 9(m-n)n^2fhu$. These exponential equations exponentiales are sufficient for the curves to be constructed, the coordinates of which will be x and s; and u and t, from which henceforth these tautochronous curves can be constructed.

§ 24. Since c shall be the number of which the hyperbolic logarithm is 1, then
$c^z = 1+\frac{z}{1}+\frac{z^2}{1.2}+\frac{z^3}{1.2.3}+\frac{z^4}{1.2.3.4}etc.$ This is $c^{\frac{3ns}{8ma}}$ on account of the ratio or I can say, $\frac{3n}{8m}=k$:
$c^{\frac{ks}{a}} = 1+\frac{ks}{a.1}+\frac{k^2ss}{a^2.1.2}+\frac{k^3s^3}{a^3.1.2.3}+\frac{k^4s^4}{a^4.1.2.3.4}etc.$, thus

$$64m^3a^2c^{\frac{ks}{a}} = 64m^3a^2+\frac{64m^3a\,ks}{1}+\frac{64m^3k^2ss}{1.2}+\frac{64m^3k^3s^3}{1.2.3}+\frac{64m^3k^4s^4}{1.2.3.4}etc.$$

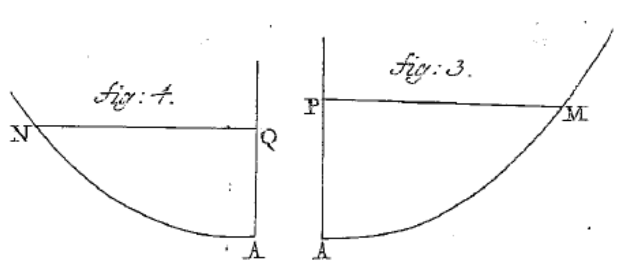
Therefore the above equation of exponentials, which on account of $\frac{3n}{8m}=k$ and hence $3n = 8km$, is changed into :

$64m^3a^2c^{\frac{ks}{a}} - 64m^3a^2 - 64km^3as = 64(1-\frac{8}{3}k)k^2m^3fx$, or

$a^2c^{\frac{ks}{a}} - aa - kas = (1-\frac{8}{3}k)k^2fx$, is reduced to the following constant from an infinite number of terms $\frac{k^2ss}{1.2}+\frac{k^3s^3}{a.1.2.3}+\frac{k^4s^4}{a^2.1.2.3.4}etc. = (1-\frac{8}{3}k)k^2fx$, which divised by kk gives

$\frac{ss}{1.2}+\frac{ks^3}{a.1.2.3}+\frac{k^2s^4}{a^2.1.2.3.4}etc. = (1-\frac{8}{3}k)fx$ and likewise for the ascent it will be :
$\frac{ss}{1.2}-\frac{ks^3}{a.1.2.3}+\frac{k^2s^4}{a^2.1.2.3.4}etc. = (1-\frac{8}{3}k)hu.$

§ 25. From these equations it is gathered, both the curve for the descent and for the ascent depart in cycloids, if $k:a$ is indefinitely small ; truly $k=\frac{3n}{8m}$; Hence these curves will be cycloids if $3n:8ma$ were made into vanishing quantities. That it is possible to happen in two ways; First if $n:m = 0$, that is, if the density of the fluid is zero, from which case the motion is in a vacuum. The other case is if $a=\infty$ or if the oscillating sphere were infinitely large for any ratio of the described arc s. Hence it will happen, that the tautochrone will also be a cycloid. Again, and thus it can be concluded, by how much more or



less shall be the fraction $3n : 8ma$ or how much $n : ma$ that will be more or less how much the tautochrone differs from a cycoid. From which it can be seen by how much more or less a given sphere oscillating in some fluid following a cycloidal path will depart from being a tautochrone.

§ 26. Now I shall carefully consider, what kind of figure the tautochrones found ought to have, and first that which is relevant to the descending curve. Let AMB be such a curve upon the axis AP as with the said abscissa AP, x, the y coordinates espress PM. For this equation, this equation is applicable :
$8m^2asds + 3(m - n)nfxds = 8(m - n)mafdx$, or this
$64m^3aa\,c^{\frac{3ns}{8ma}} - 64m^3aa - 24m^2nas = 9(m - n)n^2fx$. From this equation it is apparent that this curve nowhere has a turning point in the other direction, but to be uniformly traced out, liks a parabola, as it progresses to infinity. Moreover, the curve thus formed, the arcs of which are equal to the corresponding y coordinates PM, will have a point of reversion there where $ds = dx$. This is truly the case there, when
$8m^2as + 3(m - n)nfx = 8(m - n)maf$. Which with the equation of the exponential taken together gives the point of contrary inflection to be in that place, where
$s = \frac{8ma}{3n} l \frac{8m^2a + 3(m-n)nf}{8mma} = \frac{8ma}{3n} l(1 + \frac{3(m-n)nf}{8mma})$.

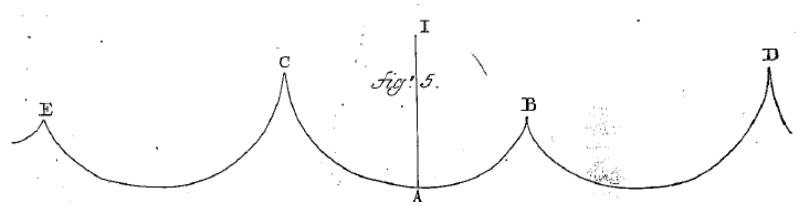
§ 27. Since $8m^2as + 3(m - n)nfx = 8(m - n)maf$ then $x = \frac{8ma}{3n} - \frac{8m^2as}{3(m-n)nf}$. But it has been found that $s = \frac{8ma}{3n} l(1 + \frac{3(m-n)nf}{8mma})$. On account of which the point of reversion will be at a height x from the innermost point A, and this gives $x = \frac{8ma}{3n} - \frac{64m^2na}{9(m-n)n^2} l(1 + \frac{3(m-n)nf}{8mma})$. I change the logarithm into a series, as it will make it easier to judge the position of the point of reversion. Indeed it is the case that : $l(1 + \frac{3(m-n)nf}{8mma}) =$
$\frac{3(m-n)nf}{8mma} - \frac{9(m-n)^2n^2ff}{2.64m^4aa} + \frac{27(m-n)^3n^3f^3}{3.512m^6a^3} - \frac{81(m-n)^4n^4f^4}{4.4096m^6a^6} etc$,
hence $\frac{64m^2na}{9(m-n)n^2f} l(1 + \frac{3(m-n)nf}{8m^2a}) =$
$\frac{8ma}{3n} - \frac{3(m-n)f}{2m} + \frac{3(m-n)^2nff}{3.8m^3a} - \frac{9(m-n)^3n^2f^3}{4.64m^5aa} + \frac{27(m-n)^4n^3f^4}{5.512m^7a^3} etc.$ Consequently it is found that
$x = \frac{(m-n)f}{2m} - \frac{3(m-n)^2nff}{3.8m^3a} + \frac{9(m-n)^3n^2f^3}{4.64m^5aa} - \frac{27(m-n)^4n^3f^4}{5.512m^7a^3} etc.$ Since this series has that property, as it is noted from the logarithms, in order that the sum of this shall be less than the first term, it can be seen that the smaller the fraction $\frac{nf}{ma}$, that the more the series converges, and hence the point of reversion is situated higher.

§ 28. The curve for the ascent shall be AN, in which, with AQ = u and QN = t, it will be $8mmatdt - 3(m - n)nhudt = 8(m - n)mahdu$, or
$64m^3aa\,c^{\frac{-3ns}{8ma}} - 64m^3aa + 24m^2nat = 9(m - n)nnhu$.
Truly this curve does not have a turning point, but is extended uniformly to infinity, not truly as before as a kind of parabola, but nearly as a hyperbola. Indeed it will diverge more from the axis than that. If the tautochrone serving for the ascent is to be constructed from this curve, it is required that the curve is described for the same axis, the arcs of which shall be equal to the y-coordinates QN. The point of reversion of this tautochrone

will be obtained, if we take $u = \frac{(m-n)h}{2m} + \frac{3(m-n)^2 nhh}{3.8m^3 a} + \frac{9(m-n)^3 n^2 h^3}{4.64m^5 aa} etc$, therefore the situation is always higher, than in the curve for the descent, and these are certainly the case, as the curve goes to infinity, or none are present, which will arise if $3(m - n)nh$ is equal to or greater than $8mma$.

§ 29. I now progress to the construction of the curve itself, and I seek the equation between the orthogonal coordinates. AME shall be the tautochrone serving for the descent. Let AP = x, PM = y and the arc AM = s. The nature of this curve is expressed from § 24 for this equation: (Fig. 1 again)
$8mmasds + 3(m - n)nfxds = 8(m - n)mafdx$. ds is placed constant, and the equation is differentiated, giving $8mmads^2 + 3(m - n)nfdxds = 8(m - n)mafddx$. Make $ds = pdx$, then $dy = dx\surd(pp - 1)$; and truly $dds = 0 = pddx + dxdp$, whereby $ddx = -dxdp{:}p$. With which substituted in the equation, that will be changed into $8m^2appdx + 3(m - n)nfpdx + 8(m - n)mafdp{:}p = 0$. From which $dx = \frac{-8(m-n)mafdp}{8m^2ap^3+3(m-n)nfpp}$ is obtained. Wherefore in order that the curve can be constructed, with a third variable p accepted, the x is taken to be $x = -8(m-n)maf\int\frac{dp}{8m^2ap^3+3(m-n)nfpp}$. Thereupon by taking $y = dx\surd(pp - 1)$, giving $y = 8(m-n)maf\int\frac{-dp\sqrt{(pp-1)}}{8m^2ap^3+3(m-n)nfpp}$. And by this means the curve sought will be constructed.

§ 30. In a similar manner as the curve for the ascent was constructed, there is hardly a need for this, as in that construction is placed - a in place of a. Indeed in this manner, as from the general equations for the construction of the speed of the bodies in a resisting



medium it is allowed for the expression of the motion, the equatio serving for the descent can be changed into that which pertains to the ascent. Againm the radius of osculation of the curve at the point M will be $\frac{(m-n)fdy}{mc^{\frac{3ns}{8ma}}ds}$. Hence it is apparent that the radius of osculation at the lowest point A will be $\frac{m-n}{m}f$, to which at that point the length of the pendulum of equal period should be taken. And hence from the construction it is possible to put together such a shape that our curve may have. AB shall be the descending tautochrone, (Fig. 5) which will be continued with AC the ascending curve. Beyond B and C it can continue into D and E, in order that the arcs BD and ED are similar and equal to the arc BAC. And in this manner the curve is produced indefinitely.

§ 31. Now we must carefully consider what motion shall be found for such a body or sphere placed upon the tautochrone. We will consider one oscillation, for which the sphere has acquired a velocity at the lowest point by falling from the height b. It can be said as before, the height v is the generator of the velocity of the sphere in the descent to some point of the curve. From § 13 this is $v=c^{\frac{ns}{ma}}(b-t)$ where $t=\frac{m-n}{m}\int c^{\frac{-ns}{ma}}dx$. Here truly *a* designates the height of the cylinder oscillating, as hence if the sphere is introduced, then $\frac{4}{3}a$ in put in place of *a*, as has been shown in § 17 and $v=c^{\frac{3ns}{4ma}}(b-t)$, and $t=\frac{m-n}{m}\int c^{\frac{-3ns}{4ma}}dx$. Since it is by considering these equatons together that the nature of the curve can be expressed, which results in this equation : $64m^3aac^{\frac{-3ns}{8ma}} - 64n^3aa - 24m^2nas = 9(m-n)n^2fx$; or from the differential of this, as dx is had , $24m^2nac^{\frac{3ns}{8ma}}ds - 24m^2nads - 24m^2nas = 9(m-n)n^2fdx$, or $8m^2ac^{\frac{3ns}{8ma}}ds - 8m^2ads = 3(m-n)nfdx$.

§ 32. Therefore from the last equation: $\frac{m-n}{m}dx=\frac{8ma}{3nf}c^{\frac{3ns}{8ma}}ds-\frac{8ma}{3nf}ds.$ Hence, this becomes : $\frac{m-n}{m}c^{\frac{-3ns}{4ma}}=\frac{8ma}{3nf}c^{\frac{-3nt}{8ma}}ds-\frac{8ma}{3nf}c^{\frac{-3ns}{4ma}}ds.$ Which integrated gives : $t=C-\frac{64m^2aa}{9nnf}c^{\frac{-3ns}{8ma}}+\frac{32m^2aa}{9nnf}c^{\frac{-3ns}{4ma}}$. The constant C added thus must be determined, as by putting $s = 0$, and by making $t = 0$, as is required by § 13, hence $C=\frac{32m^2ea}{9nnf}$. On account of which, since that expression can be avoided, the square will be $t=\frac{32m^2aa}{9nnf}(1-c^{\frac{-3ns}{8ma}})^2=\dfrac{32m^2aa}{9n^2fc^{\frac{3ns}{4ma}}}(c^{\frac{3ns}{8ma}}-1)^2$. But from the exponential equation for the curve , we have $c^{\frac{3ns}{8ma}}-1=\frac{24m^2nas+9(m-n)n^2fx}{64m^2aa}$. And hence also, $t=\dfrac{(8m^2as+3(m-n)nfx)^2}{128m^4aafc^{-\frac{3ns}{4ma}}}$.

From these it is found : $v=\dfrac{32m^2aa}{9n^2fc^{\frac{3ns}{4ma}}}(c^{\frac{3ns}{8ma}}-1)^2=bc^{\frac{3ns}{4ma}}-\dfrac{32m^2aa}{9n^2f}(c^{\frac{3ns}{8ma}}-1)^2$. Or also by this means: $v=bc^{\frac{3ns}{4ma}}\dfrac{(8m^2as+3(m-n)nfx)^2}{128aaf}$.

§ 33. This expression for the speeds will give the place on the descent curve, for which the speed of the sphere has a maximum value; and indeed for points other than the minimum. Truly there will be a point there, where $dv = 0$. Whereby since it will found that $v=bc^{\frac{3ns}{4ma}}-\dfrac{32m^2aa}{9n^2f}(c^{\frac{3ns}{8ma}}-1)^2$, then $dv=\frac{3nb}{4ma}c^{\frac{3ns}{4ma}}ds-\dfrac{8ma}{3nf}(c^{\frac{3ns}{8ma}}-1)c^{\frac{3ns}{8ma}}ds$. Hence $dv = 0$ if $\frac{3nb}{4ma}c^{\frac{3ns}{4ma}}=\dfrac{8ma}{3nf}c^{\frac{3ns}{8ma}}-\dfrac{8ma}{3nf}$, or if $c^{\frac{3ns}{4ma}}=\dfrac{32m^2a^2}{32m^2a^2-9n^2bf}$. Hence it can be deduced: $s=\frac{8ma}{3n}l\dfrac{32m^2a^2}{32m^2a^2-9n^2bf}$ or $s=\frac{8ma}{3n}l(1-\dfrac{9n^2bf}{32m^2a^2}$. From which it can be gathered that the arc *s* for that is the larger for which the product *bf* shall be greater, as a^2, with the rest of the parts. Again from the final velocity, which is as $\sqrt{b}$, the whole descent arc is found by making $v = 0$. In which case :
$c^{\frac{3ns}{8ma}}\sqrt{b}=\frac{4ma}{3n}(c^{\frac{3ns}{8ma}}-1)\sqrt{\frac{2}{f}}$ or $3nc^{\frac{3ns}{8ma}}\sqrt{\frac{1}{2}bf}=4mac^{\frac{3ns}{8ma}}-4ma$, hence

$c^{\frac{3ns}{8ma}} = \frac{4ma}{4ma-3n\sqrt{\frac{1}{2}bf}}$. Therefore the whole descent arc will be $= \frac{8ma}{3n} l(1-\frac{3n\sqrt{\frac{1}{2}bf}}{4ma})$.

§ 34. Hence, if the body with the descent speed acquired should ascend in the other part of the same curve, (indeed, serving the part of the ascent curve) the whole arc for the descent is found $= \frac{8ma}{3n} l(1+\frac{3n\sqrt{\frac{1}{2}bf}}{4ma})$. If these logarithms are resolved in series, the arc for the descent is obtained $= 2\sqrt{(\frac{1}{2}bf)} + \frac{3n(\frac{1}{2}bf)}{2.2ma} + \frac{9nn(\frac{1}{2}bf)^{\frac{3}{2}}}{3.8m^2aa} + \frac{27n^3(\frac{1}{2}bf)^2}{4.32m^3a^3}$ *etc.* In a similar manner, the ascent arc will be : $= 2\sqrt{(\frac{1}{2}bf)} - \frac{3n(\frac{1}{2}bf)}{2.2ma} + \frac{9n^2(\frac{1}{2}bf)^{\frac{3}{2}}}{3.8m^2a^2} - \frac{27n^3(\frac{1}{2}bf)^2}{4.32m^3a^3}$ *etc.* From which it is observed that the arc for the ascent is less than the arc for the descent. If $\frac{nbf}{ma}$ were made extremily small, the first terms of these two series would be assumed to be sufficient, and then the difference between the arcs of descent and ascent will be $\frac{3nbf}{4ma}$. Since the sum of these will be $2\sqrt{2bf}$. Therefore the differences *q, p* are in the square ratio of the sums.

§ 35. This is therefore a tautochrone in a medium which resists the motions in the square ratio of the speeds. Truly for other forms of hypothetical resistance of the medium, for which the resistance of everything for a certain speed is fitted or proportional to a function , the tautochrones cannot be found by this method; it is not indeed by a fault of the method, as if that should not be universal, but with a defect in the analysis; as the velocity could not be expressed in the other hypothesies. But I persuaded that this hypothesis of the resistance only following the square of the speeds has a place in the nature of things. Though indeed it is agreed from experiments, a fluid besides this [dependence on the square of the velocity] exercises another resistance arising from its tenacity, for which the force has been seen to be independent of the velocity, yet *Newton* in *Princip. Phi.,* latest edition, pag. 274, considers rather that this resistive force is not completely independent of the velocity, and that there is also a component proportional to the velocity. [See, *e. g*. Cohen's translation of the *Principia*, for Newton's final Scholium on the matter on p. 678, (and p.189), to which Euler is referring; Newton finally asserts that the total resistance can depend on a sum of terms comprising a constant and terms proportional both to the velocity and to the velocity squared; i. e. $F_{fr} = a + bv + cv^2$]. From which it can be understood that the motive forces are reduced in the ratio of the distances traversed, and with the deductions from other ratios due to the nature of the resistance omitted, which movement would never come to rest, if the resistance were in proportion to the velocity [The case investigated long before by John Napier, when constructing the first logarithms, of an exponential decay in velocity with distance. See *e. g.* , this writer's article in the *AJP*, Feb. 2000, on *Napier's Logarithms*.], which finally are confirmed by experiment; if indeed [only] the above resistance is present, following which the motion loses motive force [*i. e.* energy] in the ratio of the distances described, a tautochrone can be displayed; and that is easily formed from what has been found. Indeed, all that is required is to put the quantity $x + gs$ in place of x in the equation for our tautochrone, where the letter g, which has come from the size of this resistance due to tenacity or friction, should be determined. From which factor the sought tautochrone may be found.

CURVA TAUTOCHRONA IN FLUIDO RESISTENTIAM FACIENTE SECUNDUM QUADRATA CELERITATUM.

Auct. Leonh. Eulero.

§1.

Postquam *Hugenius* primum invenisset cylcoidem esse curvam Tautochronam in vacuo et Hypothesi gravitatis uniformis; *Newtonus* atque *Hermannus* dederunt quoque Tautochronas pro hypothesi gravitatis diformiter agentis et tendentis ad punctum quodcunque fixum tanquam centrum. Posuerunt autem motum fieri in vacuo, neque ullam pati resistentiam. Quod vero ad media resistentia attinet, *Neutonus* etiam demonstravit cycloidem esse tautochronam in medio in celeratum ratione resistante; ad alia autem resistentia media neque ipse neque quisquam alius est progressus, ut, quae curvae in iis tautochronismum producant ostenderent.

§2. Non quidem est difficile in medio quocunque resistente invenire curvam, super qua grave eadem modo descendat, quo super data curva in vacuo. Id, cum intellexissem eas quaesivi in quacunque resistentiae hypothesi curvas, super quibus grave aequaliter descendat ac super cycloide in vacuo, quae mihi curvae tum tautochronae in mediis his resistentibus esse videbantur, eo quod corporis super iis descensus aequalis esset descensui corporis super Tautochrona curva in vacuo. Atque hanc ipsam proprietatem eae habent curvae quas in Actis. Lips. Ao. 1726. dedi, et corpora super quaque earum in medio, ad quod ea pertinet, resistente collocata eodem descendunt modo, quo super cycloide in vacuo; quamobrem eas etiam Tautochronarum nomine appellavi.

§3. Rem vero hanc postea accuratius perpendens, eam ita habere deprehendi; ut tota curva in medio resistente percurrenda ab initio descensus super curva in vacuo percurrenda assumpto pendeat. Quare si in curva data aliud ponatur descensus initium ipsa curva in medio resistente similem descensum producens alia erit. Ex quo intelligitur etiam si habeatur curva, super qua corpus in medio resistente aequalem habeat descensum, ac super cycloide in vacuo, initio descensus videlicet dato; tamen hanc nondum eam habere proprietatem, ut grave ubicunque descensum inchoaverit eodem tempore ad punctum infinum perveniat, etenim descensus non cum descensu in cycloide congruet, nisi is ex dato puncto incipiatur.

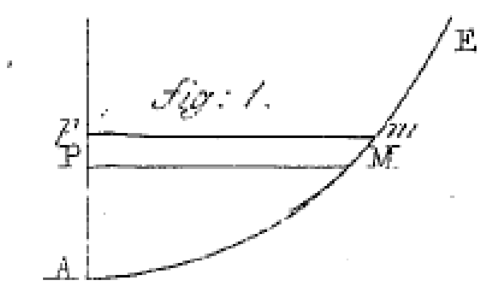
§ 4. Idem attendenti uberius palam fiet, si inspiciat aequationes ibi datas, et modum, quo erutae fuerunt. Deprehendet enim in iis adhuc literam, quae constantis speciem prae se fert; quae vero re ipsa ac initio descensus pendet. Id ergo si aliter voluerit assumere ea apparens constans alia erit et idcirco curva quasi alium parametrum acquiret, et a priori diversa euadet. Hoc incommodum non diu post ipse animadverti, et praeterea Celeb. *Hermannus* in dissertatione de motibus variatis Actis Anni 1727 inserta, in qua ostendit, curvas, quae ex eodem principio, quo ipse usus sum, inveniantur, quaeque tautochronae esse videantur, hujusmodi tamen non esse, tum ob rationes a me quoque perspectas hicque expositas, tum tempus descensus ipsum investigavit, idque constans non esse pro variis descensus initiis reperit.

§ 5. Nolo igitur, quanquam eas ipse principio in tautochronarum numero habui, aliud judicium de iis ferri, nisi quod pro quolibet medio resistente aequatio ibi data ob constantem memoratam variabilem ponendam totam exhibeat familiam curvarum, super quibus gravia ex debito cuique puncto descensum incipientia aequalia tempore ad punctum infimum perveniant, idque eodem modo, quo in vacuo super cycloide. Cum igitur cognovissem curvas has ad tautochronismum producendum non esse aptas, statim non solum ego verum etiam alii, cum quibus communicaveram in id incubvimus, ut veras tautochronas in medio quocunque resistente inveniremus. Postquam igitur rem multis modis tentassem, potitus sum tandem tautochrona, sed in unica tantum resistentiae hypothesi juxta velocitatis quadrata, quam in hac dissertatione exponere constitui.

§ 6. Cum nuper novam quandam detexissem methodum, qua a priori nonsolum cycloidem, sed insuper infinitas adhuc alias inveni curvas tautochronas in vacuo, ea quoque ad tautochronas in mediis resistentibus inveniendas uti institui; methodi universalitate fretus, si quae sint tautochronae in mediis resistentibus, eas ope hujus methodi inveniri debere. Quantum autem adhuc hac in re efficere potui, prorsus mihi necesse esse visum est, ut velocitatas corporis super curva quacunque in eo resistenti medio. pro quo tautochrona desideratur, descendentis vel ascendentis in puncto quocunque possit exprimi, non quidem algebraice, sed transendenter quomodocunque. Id vero cum nonnisi in vacuo, et in medio resistentiam in ratione duplicata celeritatum faciente praestare in potestate sit, tautochronam saltem in fluidis, quia haec in ratione duplicata celeritatum resistere putantur, hic invenire docebo.

§ 7. Volui primum problema ita instituere, ut nuper eandem quaestionem invacuo tractavi, ut, quemadmodum ibi factum est, data curva quacunque invenirem aliam, quae cum ea conjuncta tautochronismum oscillationibus inducat. Verum istam quaestionem nondum enodare licuit, cum plane dissimilis sit ejus quae ad vacuum spectat. Nam ad rem eodem modo, quo in vacuo seci, expediendam opus est, ut in duabus curvis data et quaesita duo semper puncta dari queant, in quibus corporis oscillantis celeritates sunt aequales, atque ut eorum determinatio non ab ipsa velocitate pendeat; sed quibus in punctis una oscillatione celeritates aequales fuerunt, ibidem in aliis oscilllationibus sint aequales. Id vero cum in fluido fieri nequeat, contentus hic ero eam determinasse curvam, super qua corpus descendens aequali semper tempore ad punctum idfimum pertingat.

§ 8. Sit CMS curva quaesita ad axem AP verticalem relata. Hanc ejus esse oportet proprietatem, ut corpus super ea in fluido collocata descendens aequalibus temporibus ad punctum infimum A perveniat, ubicunque descensum adorsum fuerit. Fiat is ex E, perspicuum est, corpus descendendo a vi gravitatis, quatenus ejus gravitas specifica major est illa, quam habet fluidum, continuo accelerati, simul vero propter resistentiam fluidi continuo in ratione duplicata celeritatum retardari, donec tandem in A retineat certam celeritatem, quam ponam in vacuo acquiri posse lapsu ex altitudine b. Ut corpus ex A hac celeritate rursus in E usque ascendere possit, oportet gravitatis vim, quae ante promovebat, adversam; resistentia vero vim, quae ante adversa erat, nunc secundam et promoventem pondere, quo fiet, ut hic ascensus prorsus similis sit descensui. Quia magis juvat ascensum considerare, hoc praemittere oportuit.



§ 9. Ascendere ergo ponatur corpus ex A velocitate altitudinis b, ita ut acceleretur in ratione duplicata celeritatum, perveniet id rursus ad E. Sit hoc corpus cylinder altitudinis *a* secundum axis directionem motus; etsi haec figura minus idonea sit ad oscillandum, tamen, quia calculus sit simplicior, facileque ad alias figuras tranferri potest, hanc figuram retinere volui. Sit porro gravitas specifica corporis ad eam fluidi ut *m* ad *n*. Pervenerit corpus hoc modo ascendens ad M, ubi ejus celeritas sit tanta quanta ex altitudine *v* in vacuo generatur. Dicatur arcus percursus AM, *s*, et abscissa AP, *x*. Momento perveniant omnia in situm proximum corpus nempe in *m*. Erit celeritas corporis in *m* genitae ex altitudine $v + dv$ aequalis atque M*m* = *ds* et P*p* = *dx*.

§ 10. Quia corpus in fluido versatur, non tot suo pondere descendere conatur, sed excessu sui ipsius ponderis super pondus aequalis voluminis fluidi. Vis ergo corpus sollicitans est ad verum ejus pondus ut $m - n$ ad *m*. Si igitur vis gravitatis dicatur *g*, erit vis haec sollicitans = $(m - n)g : m$. Ascensdente corpore per elementum M*m*, si vis gravitatis ipsa g ageret foret $dv = -dx$, si nimirum corpus in vacuo ascenderet. Quia autem vis sollicitans est ad vim gravitatis ut $m - n$ ad *m*, erit effectus illius ad hujus effectum $- dx$ ut $m - n$ ad *m*. Quamobrem agente vi hae sollicitante, erit $dv = - (m - n)dx : m$ signum hic negatiuum obtinet, quia vis gravitatis contraria ponitur motui corporis. Haec igitur haberetur aequatio $dv = - (m - n)dx : m$, ex qua motus corporis determinari deberet, nisi acceleratio, quae resistiae fluidi aequalis est, accederet.

§11. Videamus nunc, quanta sit resistentia fluidi in cylindrum velocitate alt. *v* motum basin suam obvertentem. Vis haec aequalis est vi, quam fluivum eadem celeritate motum in cylindrum quiescentem exereret; haec vero vis aequatur ponderi cylindri fluidi altitudinis *v*, et basis aequalis ei, quam habet ille cylinder oscillans. Est itaque pondus ejus cylindri fluidi ad pondus hujus ut *nv* ad *ma*. Vis igitur haec resistentiae se habet ad vim gravitatis quoque ut *nv* ad *ma*. Corpore autem ascendente per M*m*, si gravitas acceleraret, et secundum directionem Mm ageret, foret dv = ds. Vis ergo resistentiae pro ea ratione effectum edens et accelerans corpus motum, faciet ut sit $dv = nvds: ma$; si autem retardaret, foret $dv = -nvds: ma.$

§ 12. Si igitur sola vis gravitatis ageret retardando motum corporis, tum esset per §10 $dv = - (m - n)dx : m$, sin vero sola vis accelerans aequalis vi resistentiae ageret, tum esset per § 10 $dv = -nvds: ma$. Ex quibus colligitur, si utraque simul agat, tum esse $dv = nvds: ma - (m - n)dx : m$, seu $madv + (m - n)adx - nvds = 0$. Ex qua aequatione motus corporis determinari debet. Quia autem in hac aequatione *v* unicam habet dimensionem, ea integrari potest. Reducatur ad hanc formam $dv = -\frac{nvds}{ma} = \frac{-(m-n)dx}{m}$. Multiplicetur ea per $c^{\frac{-nv}{ma}}$; denotat vero c numerum , cujus logarithmus hyperbolicus est 1, habebitur $c^{\frac{-nv}{ma}}dv - \dfrac{nc^{\frac{-ns}{ma}}vds}{ma} = \dfrac{-(m-n)c^{\frac{-ns}{ma}}dx}{m}$. Cuius integralis est sequens $c^{\frac{-nv}{ma}}v = C - \frac{(m-n)}{m}\int c^{\frac{-ns}{ma}}dx.$

§ 13. Ponatur $\frac{(m-n)}{m}\int c^{\frac{-ns}{ma}}dx = t,$ cum ejus integrale ex curva cognita possit haberi, vel saltem construi. Ita autem, si fieri posset, integrari ponitur, ut ejus integrale fiat = 0, si ponatur $x = 0$, quo *t* determinatum valorem adipiscatur. Habemus ergo sequentem aequationem $c^{\frac{-ns}{ma}}v = C - t.$ Constans haec ita debet accipi, ut, posito $x = 0$, fiat $v = b$, talis enim ponitur esse celeritas corporis in puncto A, sed posito $x = 0$, erit et $s = 0$ et $t = 0$, unde quia $c^0 = 1$, oritur C = *b*. Quamobrem invenitur sequens ad institutum nostrum prorsus accommodata aequatio $v = c^{\frac{-ns}{ma}}(b-t)$. Et hinc quoque intelligitur, ubi velocitas evanescat, seu quousque corpus in curva sit ascensurum, ibi nimirum ubi est $v = 0$, seu t = b. Celerits vero ipsa corporis in M erit ut √v seu ut $c^{\frac{ns}{2ma}}\sqrt{(b-t)}$.

§ 14. Cum jam habeatur celeritas corporis M, erit tempusculum per arculum Mm, quod est ut $\frac{ds}{\sqrt{v}}$, seu loco v superiore valore substitutuo ut $\dfrac{ds}{c^{\frac{ns}{2ma}}\sqrt{(b-t)}}$. Id quod exprimit elementum temporis. Hujus ergo integrale ita debet esse comparatum, ut, ea adjecta constante, quae facit tempus = 0 si ponatur x vel t vel $s = 0$, ut inquam, si fiat $t = b$, quo in casu integrum obtinetur tempus ascensus, tum b quae a quantitate arcus descripti pendet prorsus ex computo abeat. Hoc ut fiat jam alibi demonstravi opertere, ut tota expressio elementi temporis nullam habeat dimensionem. Quaetitur ergo qualis s functio ipsius t esse debeat? Quia ad s exprimendum b in computum ingredi non potest, sed solum t, perspicuum est fore $\frac{ds}{c^{(\frac{ns}{2ma})}} = \frac{dt\sqrt{e}}{\sqrt{t}}$ et sic elementum temporis erit $\frac{dt\sqrt{e}}{\sqrt{(bt-tt)}}$. Ergo longitudo penduli isochroni in vacuo est $2e$.

§ 15. Ex determinatione curvae, ut fiat tautochrona, haec orta est aequatio $ds : c^{\frac{ns}{2ma}} = dt\sqrt{e} : \sqrt{t}$ ex qua natura curvae determinari debet. Aequatio ea integrata dat hanc $C - \frac{2ma}{n}c^{\frac{-ns}{2ma}} = 2\sqrt{et}$; ut facto t = 0, fiat s = 0, necesse est, ut sit C = $\frac{2ma}{n}$; Propterea haec invenitur aequatio pro curva quaesita, $\frac{ma}{n}(1-c^{\frac{-ns}{2ma}}) = \sqrt{et}$, et sumendis quadratis haec $\frac{mmaa}{nn}(1-c^{\frac{-ns}{2ma}})^2 = et$. Quae denuo differentiata dat $\frac{ma}{n}(1-c^{\frac{-ns}{2ma}})c^{\frac{-ns}{2ma}}ds = edt$ seu $mac^{\frac{-ns}{2ma}}ds - mac^{\frac{-ns}{2ma}}ds = nedt$. Est vero $t = \frac{m-n}{m}\int c^{\frac{-ns}{ma}}dx$, ergo $dt = \frac{m-n}{m}c^{\frac{-ns}{ma}}dx$. Quocirca ejecto t, habebitur aequatio inter s et x, haec $mmac^{\frac{-ns}{2ma}}ds - mmac^{\frac{-ns}{2ma}}ds = (m-n)nec^{\frac{-ns}{2ma}}dx$. Quae multiplicta per $c^{\frac{ns}{ma}}$ abit in hanc $mmac^{\frac{ns}{2ma}}ds - mmads = (m-n)nedx$.

§ 16. Aequatio differentialis inventa est iterum integrabilis; integrata vero dat, $\frac{2m^3aa}{n}ac^{\frac{ns}{2ma}}ds - mmas - \frac{2m^3aa}{n} = (m-n)nex$, postquam debita constans $\frac{2m^3aa}{n}$ ablata est. Quae magis accommadatur hoc modo $c^{\frac{ns}{2ma}} = \frac{ns}{2ma} + 1 + \frac{(m-n)nnex}{2m^3aa} = \frac{m^2aas+2m^3aa+(m-n)n^2ex}{2m^3aa}$. Haec quidem aequatio sufficeret ad curvam construendam ; sed commodior evadet liberata ab exponentialibus. Hanc ob rem sumantur logarithmi, eritque $\frac{ns}{2ma} = l(m^2aas + 2m^3aa + (m-n)n^2ex) - l(2m^3aa)$. Hinc differentiando acquiritur $\frac{nds}{2ma} = \frac{m^2nads+(m-n)n^2edx}{m^2nas+2m^3aa+(m-n)n^2ex}$ et ex hac ordinando $m^2n^2asds + (m-n)n^3exds = 2(m-n)mn^2aedx$. Quae divisa per nn praebet sequentem aequatonem finalem pro curva quaesita, $m^2asds + (m-n)nexds = 2(m-n)maedx$.

§ 17. Si itaque curva AME eam habuerit proprietatem, ut sit $m^2asds + (m-n)nexds = 2(m-n)maedx$ ea erit tautochrona hoc sensu, ut corpus cylindricum altitudinis a super es descendens eodem semper tempore ad punctum infimum A perveniat, ubicunque descensum inceperit. Si loco cylindri placuerit globum adhibere ejusdem gravitatis specificae et diametri a, oportebit loco a in aequatione scribere $\frac{4}{3}a$, habebiturque $4m^2asds+3(m-n)nexds=8(m-n)maedx$, pro motu globi, cujus diameter est a. Si longitudo penduli isochroni in vacuo oscillanti dictatur f, erit $e=\frac{1}{2}f$; et hinc resultabit aequatio $8m^2asds+3(m-n)nfxds=8(m-n)mafdx$. Hanc aequationem jam ad quemvis casum specialem accommodari licet.

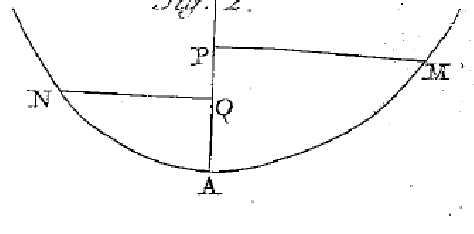
§ 18. Ponamus densitatem fluidi evanescere, quo motus corporus fiat in vacuo; erit n = 0. Hoc igitur posito aequationis terminus secundus $3(m-n)nfxds$ evanescit, et tunc pro tautochrona in vacuo prodibit aequatio $8m^2asds=8(m-n)mafdx$. Quae, cum sit n = 0, divisa per $8m^2a$ reducitur ad $sds = fdx$. Haec vero integrata est $ss = 2fx$, aequatio ad cycloidem, cujus circuli genitoris diameter est $\frac{1}{2}f$. Id quod prorsus congruit cum iis, quae de tautochronismo cycloidis demonstrata sunt. Si ergo aequatio inventa tautochronae in fluido ad vacuum reducitur, litera a diametrum globi oscillantis denotans exit ex aequatione; et tautochrona invacuo proinde a magnitudine et figura corporis oscillantis non pendet. Sed in fluido ad tautochronam determinandam et magnitudine et figura et gravitate specifica corporis oscillantis opus est.

§ 19. Curva, quam invenimus, tautochrona inservit descensui corporis, sed ex ea tautochrona, quae ad ascensum spectat in eodem fluido, inveniri poterit. Ponatur enim corpus in curva AME ascendere celeritate initiali, ut ante, ex altitudine b genita; habibit id et vim gravitatis, et resistentiam fluidi contrarias. Quamobrem, cum supra pro descensu haec inventa sit aequatio, $madv + (m - n)adx - nvds = 0$, ubi vis resistentiae, utr rem ibi consideravi, erat accelerans; hoc in casu corporis ascendentis signum - praefixum termino $nvds$, qui vim resistentiae fluidi exponit, mutari debet in +. Quo facto habebitur aequatio $madv + (m - n)adx + nvds = 0$. Ex qua ascensus ejusdem corporis, quod ante descendere positum est, determinabitur.

§ 20. Perspicuum est hanc aequationem ex superiore ad descensum spectante derivari posse, modo in illa fiat s negatiuum. Quocirca, ad tautochronam ascensui inservientem inveniendam non est necessarium, ut eodem, quo pro descensu, progrediar modo, sed tantum in aequatione pro tautochrona descensus inventa loco s poni poterit - s. Hoc enim ea transformabitur in tautochronam ad ascensum accommodatam. Si ergo corpus ascensdens fuerit globus diametri a, ejus gravitas specifica ad eam fluidi ut m ad n, habebitur pro tautochrona $8m^2asds - 3(m - n)nhxdx + nvds = 8(m - n)mahdx$. Ubi loco f posui h, ne tempora ascensus ex descensus aequalia esse debere videantur.

§ 21. Cum igitur curvam et descendente corpore et ascendente tautochronam invenerim; eae si in punctis infimis jungantur, repraesentabunt tautochronam ascensui et descensui simul inservientem. Sit AM tauto chrona pro descensu, altera AN pro ascensu; manifestum est, si corpus corpus semper

fig. 2.



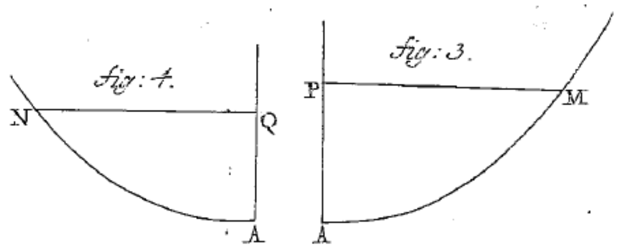
in curva AM descensum incipiat, ut altera punctum in curva AN ascendat, tum oscillationes has absolutum iri atqualibus temporibus, ubicunque initia descensus in AM assumantur. Si igitur AP fuerit x et AM s, erit $8m^2asds + 3(m - n)nfxds = 8(m - n)mafdx$, pro altera curva AN vero, si dicatur AQ = u, et AN = t, erit $8m^2atdt - 3(m - n)nhudt = 8(m - n)mahdu$. Tempus vero oscillationis unius aequale est duabus dimidiis oscillationibus duorum pendulorum in vacuo, quorum alterius longitudo est f, alterius h, seu uni integrae oscillationi penduli cujus long. = $\frac{f+h+2\sqrt{fh}}{4}$.

§ 22. Si fuerit $f = h$, erunt duae curvae AM, AN partes ejusdem curvae continuae: Id quod ex eo intelligi potest, quod tum, si loco x ponatur u et loco s, quia in altera curva arcus fiunt negativi, $-t$, aequatio illa ad descensum pertinens mutetur in hanc ascensui inservientem. Curva ergo MA ab altera parte continuatur in curva AN, et tota curva MAN hanc habet proprietatem ut globus diametri a, et gravitatis specificae m super in fluido gravitatis specificae n constituta motum aequalibus semper temporibus oscillationis absolvat. Descensus vero fieri debent in curva MA, et ascensus in AN, nisi forte eae curvae hanc insuper habeant proprietatem, ut et, si descensu in NA et ascensus in AM fierent, oscillationes totae omnes ut ante essent tautochronae.

§ 23. Aequatio exponentialis § 16 in eo solum differt ab ea, quam § 21 dedimus, quod ibi sit a id quod hic est $\frac{4}{3}a$, et e, quod hic $\frac{1}{2}f$. Si ergo in ea aequatione ponatur $\frac{4}{3}a$ loco a, et $\frac{1}{2}f$ loco e, habebitur $64m^3aac^{\frac{3ns}{8ma}} - 64m^3aa - 24m^2nas = 9(m - n)n^2fx$ quae aequatio convenit cum ea quae descensui § 21 inservire inventa, $8m^2asds + 3(m - n)nfxds = 8(m - n)mafdx$. At alteri aequationi ad ascensum pertinenti $8m^2atdt - 3(m - n)nhudt = 8(m - n)mahdu$, respondet haec $64m^3aac^{\frac{-3ns}{8ma}} - 64m^3aa + 24m^2nat = 9(m - n)n^2fhu$. Hae aequationes exponentiales sufficiunt ad curvas construendas , quarum coordinatae sint x et s; atque u et t, ex quibus deinceps ipsae curvae tautochronae construi poterunt.

§ 24. Cum c sit numerus cujus logarithmus hyperbolicus est 1, erit $c^z = 1 + \frac{z}{1} + \frac{z^2}{1.2} + \frac{z^3}{1.2.3} + \frac{z^4}{1.2.3.4} etc.$ Hanc ob rationem est $c^{\frac{3ns}{8ma}}$ seu dicto $\frac{3n}{8m} = k$,
$c^{\frac{ks}{a}} = 1 + \frac{ks}{a.1} + \frac{k^2ss}{a^2.1.2} + \frac{k^3s^3}{a^3.1.2.3} + \frac{k^4s^4}{a^4.1.2.3.4} etc.$ adeoque
$64m^3a^2c^{\frac{ks}{a}} = 64m^3a^2 + \frac{64m^3a\ ks}{1} + \frac{64m^3k^2ss}{1.2} + \frac{64m^3k^3s^3}{1.2.3} + \frac{64m^3k^4s^4}{1.2.3.4} etc.$ Aequatio igitur superior exponentialis, quae ob $\frac{3n}{8m} = k$ et inde $3n = 8km$, mutatur in
$64m^3a^2c^{\frac{ks}{a}} - 64m^3a^2 - 64km^3as = 64(1 - \frac{8}{3}k)k^2m^3fx$, seu in
$a^2c^{\frac{ks}{a}} - aa - kas = (1 - \frac{8}{3}k)k^2fx$, reducetur ad sequentem ex terminorum infinito numero constantem $\frac{k^2ss}{1.2} + \frac{k^3s^3}{a.1.2.3} + \frac{k^4s^4}{a^2.1.2.3.4} etc. = (1 - \frac{8}{3}k)k^2fx$, quae divisa per kk dat
$\frac{ss}{1.2} + \frac{ks^3}{a.1.2.3} + \frac{k^2s^4}{a^2.1.2.3.4} etc. = (1 - \frac{8}{3}k)fx$ similiter pro ascensu erit
$\frac{ss}{1.2} - \frac{ks^3}{a.1.2.3} + \frac{k^2s^4}{a^2.1.2.3.4} etc. = (1 - \frac{8}{3}k)hu.$

§ 25. Ex his aequationibus colligitur, curvam utramque et descensus et ascensus abire in cycloides, si $k:a$ fuerit infinite paruum; est vero $k = \frac{3n}{8m}$; Ergo eae curvae erunt cycloides si $3n:8ma$ fuerit quantitas evanescens. Id duplici modo evenire potest; Primo si $n:m = 0$, id est, si fluidi densitas nulla sit, quo casu motus in vacuo. Alter est casus, si $a = \infty$ seu si globus oscillans fuerit infinite magnus ratione videlicet arcuum descriptorum s. Id ergo acciderit, tautochrona quoque erit cyclois. Porro et id inde concluditur, quo major minorve sit fractio $3n : 8ma$ seu tantum $n : ma$ eo magis minusve tautochronas a cycloide discrepare. Ex quo, quanto magis minusve in quovis fluido datus globus secundum cycloidem oscillans a tautochronismo aberret, perspici poterit.



§ 26. Perpendam nunc, qualem tautochronae inventae figuram habere debeant, et primum ea, quae ad descensum pertinet. Sit AMB talis curva super axe AP ut dictis abscissis AP, x, applicatae PM exprimant s. Habebitur pro hac curva haec aequatio, $8m^2asds + 3(m - n)nfxds = 8(m - n)mafdx$, vel haec $64m^3aac^{\frac{3ns}{8ma}} - 64m^3aa - 24m^2nas = 9(m - n)n^2fx$. Ex hac aequatione apparet hanc curvam nusquam habere punctum flexus contrarii, sed uniformi tractu, ut parabolam, in infinitum progredi. Curva autem inde formata, cujus arcus sunt respondentibus applicatis PM aequales, ibi habebit punctum reversionis ubi $ds = dx$. Hoc vero ibi, ubi est $8m^2as + 3(m - n)nfx = 8(m - n)maf$. Quae cum exponentiali aequatione conjuncta dat punctum flexus contrarii esse in eo loco, ubi

$$s = \frac{8ma}{3n} l \frac{8m^2a + 3(m-n)nf}{8mma} = \frac{8ma}{3n} l(1 + \frac{3(m-n)nf}{8mma}).$$

§ 27. Cum sit $8m^2as + 3(m - n)nfx = 8(m - n)maf$ erit $x = \frac{8ma}{3n} - \frac{8m^2as}{3(m-n)nf}$. Sed inventum est $s = \frac{8ma}{3n} l(1 + \frac{3(m-n)nf}{8mma})$. Quamobrem punctum reversionis erit ad altitudinem x ab imo puncto A, estque $x = \frac{8ma}{3n} - \frac{64m^2na}{9(m-n)n^2} l(1 + \frac{3(m-n)nf}{8mma})$ Convertam logarithmum in seriem, ut facilius de loco puncti reversionis judicare liceat. Est vero $l(1 + \frac{3(m-n)nf}{8mma}) =$

$$\frac{3(m-n)nf}{8mma} - \frac{9(m-n)^2n^2ff}{2.64m^4aa} + \frac{27(m-n)^3n^3f^3}{3.512m^6a^3} - \frac{81(m-n)^4n^4f^4}{4.4096m^6a^6} etc$$

, ergo $\frac{64m^2na}{9(m-n)n^2f} l(1 + \frac{3(m-n)nf}{8m^2a}) =$

$$\frac{8ma}{3n} - \frac{3(m-n)f}{2m} + \frac{3(m-n)^2nff}{3.8m^3a} - \frac{9(m-n)^3n^2f^3}{4.64m^5aa} + \frac{27(m-n)^4n^3f^4}{5.512m^7a^3} etc.$$

Consequenter habebitur

$$x = \frac{(m-n)f}{2m} - \frac{3(m-n)^2nff}{3.8m^3a} + \frac{9(m-n)^3n^2f^3}{4.64m^5aa} - \frac{27(m-n)^4n^3f^4}{5.512m^7a^3} etc.$$

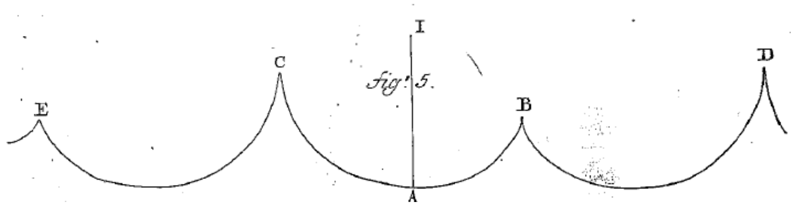
Quia haec series eam habet proprietatem, ut ex logarithmis notum est, ut summa ejus minor sit termino primo, manifestum est quo minor sit fractio $\frac{nf}{ma}$, eo magis eam convergere, et proinde eo esse punctum reversionis altius situm.

§ 28. Sit pro ascensu curva AN, in qua, dicta AQ = u sit QN = t, erit $8mmatdt - 3(m - n)nhudt = 8(m - n)mahdu$, seu

$64m^3aac^{\frac{-3ns}{8ma}} - 64m^3aa + 24m^2nat = 9(m-n)nnhu$. Neque vero haec curva habet punctum flexus contrarii, sed quoque in infinitum uniformiter protenditur, non vero ut prior, quemadmodum parabola, set fere ut hyperbola. Multo enim magis ab axe divergit quam illa. Si ex hac tautochrona ascensui inserviens construenda sit, oportet describere curvam ad eundem axem, cujus arcus sint applicatis QN aequales. Hujus tautochronae punctum reversionis habebitur, si capiatur $u = \frac{(m-n)h}{2m} + \frac{3(m-n)^2nhh}{3.8m^3a} + \frac{9(m-n)^3n^2h^3}{4.64m^5aa} etc$, semper ergo est altius situm, quam in curva pro descensu, et sunt prorsus casus, quam in infinitum excurrit, aut nullibi existit, id quod evenit si $3(m-n)nf$ est aequale vel majus quam $8mma$.

§ 29. Progredior nunc ad ipsius curvae constructionem et quaero aequationem inter coordinatas orthogonales. Sit AME tautochrona descensui inserviens. Sit AP = x PM = y et arcus AM = s. Hujus curvae natura exprimitur ex § 24 hac aequatione $8mmasds + 3(m-n)nfxds = 8(m-n)mafdx$. Ponatur ds constans, et differentietur aequatio, habetur $8mmads^2 + 3(m-n)nfdxds = 8(m-n)mafddx$. Fiat $ds = pdx$, erit $dy = dx\sqrt{(pp-1)}$; verum $dds = 0 = pddx + dxdp$, quare est $ddx = -dxdp:p$. Quibus in aequatione substitutis ea abibit in $8m^2appdx + 3(m-n)nfpdx + 8(m-n)mafdp:p = 0$. Ex qua obtinetur $dx = \frac{-8(m-n)mafdp}{8m^2ap^8+3(m-n)nfpp}$. Quocirca ad curvam construendam, accepta variabili tertia p, sumatur $x = -8(m-n)maf\int\frac{dp}{8m^2ap^8+3(m-n)nfpp}$. Deinde quia $dy = dx\sqrt{(pp-1)}$ capiatur $y = 8(m-n)maf\int\frac{-dp\sqrt{(pp-1)}}{8m^2ap^8+3(m-n)nfpp}$. Atque hoc modo curva quaesita erit constructa.

§ 30. Simili modo ut curva pro ascensu construatur, hoc tantum opus est, ut in illa constructine ponatru - a loco a. Hoc enim modo, ut ex aequationibus generalibus celeritatem corporum in medio resistente motorum exprimentibus videre licet, aequatio



descensui inserviens transmutatur in eam, quae ad ascensum pertinet. Porro radius osculi curvae in puncto M erit = $\frac{(m-n)fdy}{mc^{\frac{3ns}{8ma}}ds}$. Unde patet radium osculi in puncto infimo A esse = $\frac{m-n}{m}f$,cui in eo puncto longitudo penduli aequalis accipi debet. Denique ex constructione colligere licet, qualem figuram nostra curva habeat. Sit AB tautochrona descensus, quae continua erit cum AC curva ascensus. Ultra B et C continuatur in D et E, ita ut arcus BD, ED similes it aequales sint arcui BAC. Atque hoc modo in infinitum producitur

§ 31. Perpendamus nunc qualis corporis seu globi, ut positum est, super curva tautochrona inventa sit motus. Consideremus oscillationem unam, qua globus in puncto infimo habeat velocitatem ex altitudine b acquisitam. Dicatur, ut ante, altitudo genitrix velocitatis globi in puncto quocunque curvae descensus v. Erit ex § 13 $v = c^{\frac{ns}{ma}}(b-t)$ ubi est $t=\frac{m-n}{m}\int c^{\frac{-ns}{ma}}dx$. Hic vero *a* altitudinem cylindri oscillantis designat, ut ergo globus introducatur ponatur $\frac{4}{3}a$ loco a, prout § 17 factum est erit $v=c^{\frac{3ns}{4ma}}(b-t)$, et $t=\frac{m-n}{m}\int c^{\frac{-3ns}{4ma}}dx$. Cum his aequationibus ea quae naturam curvae exprimit est conjungenda, quae est haec $64m^3aac^{\frac{-3ns}{8ma}} - 64n^3aa - 24m^2nas = 9(m-n)n^2fx$; seu hujus differentialis, ut habeatur dx, $24m^2nac^{\frac{3ns}{8ma}}ds - 24m^2nads - 24m^2nas = 9(m-n)n^2fdx$, sive $8m^2ac^{\frac{3ns}{8ma}}ds - 8m^2ads = 3(m-n)nfdx$.

§ 32. Est igitur ex posteriore aequatione $\frac{m-n}{m}dx=\frac{8ma}{3nf}c^{\frac{3ns}{8ma}}ds-\frac{8ma}{3nf}ds.$ Unde erit $\frac{m-n}{m}c^{\frac{-3ns}{4ma}}=\frac{8ma}{3nf}c^{\frac{-3nt}{8ma}}ds-\frac{8ma}{3nf}c^{\frac{-3ns}{4ma}}ds.$ Quae integrata dat $t=C-\frac{64m^2aa}{9nnf}c^{\frac{-3ns}{8ma}}+\frac{32m^2aa}{9nnf}c^{\frac{-3ns}{4ma}}$. Constans C adjecta ita debet determinari, ut posito s = 0, fiat et t = 0, ut § 13 requirebatur, est igitur C = $\frac{32m^2ea}{9nnf}$. Quamobrem cum ea expressio evadat quadratum erit $t=\frac{32m^2aa}{9nnf}(1-c^{\frac{-3ns}{8ma}})^2=\frac{32m^2aa}{9n^2fc^{\frac{3ns}{4ma}}}(c^{\frac{3ns}{8ma}}-1)^2$. Sed ex aequatione exponentiali pro curva habetur $c^{\frac{3ns}{8ma}}-1=\frac{24m^2nas+9(m-n)n^2fx}{64m^2aa}$. Itaque erit quoque $t=\frac{(8m^2as+3(m-n)nfx)^2}{128m^4aafc^{-\frac{3ns}{4ma}}}$. Ex his reperitur

$$v=\frac{32m^2aa}{9n^2fc^{\frac{3ns}{4ma}}}(c^{\frac{3ns}{8ma}}-1)^2=bc^{\frac{3ns}{4ma}}-\frac{32m^2aa}{9n^2f}(c^{\frac{3ns}{8ma}}-1)^2.$$ Vel etiam hoc modo

$$v=bc^{\frac{3ns}{4ma}}\frac{(8m^2as+3(m-n)nfx)^2}{128aaf}.$$

§ 33. Expressio haec celeritatis dabit locum in curva descensus, quo velocitatem globus habet maximam; etenim ea non incidit in punctum infinum. Id vero punctum erit ibi, ubi $dv = 0$. Quare cum inventa sit $v=bc^{\frac{3ns}{4ma}}-\frac{32m^2aa}{9n^2f}(c^{\frac{3ns}{8ma}}-1)^2$, erit

$$dv=\frac{3nb}{4ma}c^{\frac{3ns}{4ma}}ds-\frac{8ma}{3nf}(c^{\frac{3ns}{8ma}}-1)c^{\frac{3ns}{8ma}}ds.$$ Ergo erit $dv = 0$; si

$$\frac{3nb}{4ma}c^{\frac{3ns}{4ma}}=\frac{8ma}{3nf}c^{\frac{3ns}{8ma}}-\frac{8ma}{3nf},$$ seu si $c^{\frac{3ns}{4ma}}=\frac{32m^2a^2}{32m^2a^2-9n^2bf}$. Unde deducitur

$$s=\frac{8ma}{3n}l\frac{32m^2a^2}{32m^2a^2-9n^2bf}$$ seu $s=\frac{8ma}{3n}l(1-\frac{9n^2bf}{32m^2a^2}$. Ex quo colligitur arcum *s* eo esse majorem quo factum *bf* majus fuerit, quam a^2, ceteris paribus. Porro ex velocitate finali, quae est ut $\sqrt{b}$, invenitur totus arcus descensus faciendo $v = 0$. Quo in casu erit $c^{\frac{3ns}{8ma}}\sqrt{b}=\frac{4ma}{3n}(c^{\frac{3ns}{8ma}}-1)\sqrt{\frac{2}{f}}$ seu $3nc^{\frac{3ns}{8ma}}\sqrt{\frac{1}{2}bf}=4mac^{\frac{3ns}{8ma}}-4ma$, unde

$c^{\frac{3ns}{8ma}} = \frac{4ma}{4ma-3n\sqrt{\frac{1}{2}bf}}$. Totus igitur arcus descensus erit $= \frac{8ma}{3n} l(1 - \frac{3n\sqrt{\frac{1}{2}bf}}{4ma})$.

§ 34. Deinceps, si corpus celeritate descensu acquisita in altera parte ejusdem curvae ascendat, (inservit enim ea pars ascensui) invenitur totus arcus descensus = $= \frac{8ma}{3n} l(1 + \frac{3n\sqrt{\frac{1}{2}bf}}{4ma})$. Si hi logarithmi in series resolvantur habebitur arcus descensus

$$= 2\sqrt{(\tfrac{1}{2}bf)} + \frac{3n(\frac{1}{2}bf)}{2.2ma} + \frac{9nn(\frac{1}{2}bf)^{\frac{3}{2}}}{3.8m^2aa} + \frac{27n^3(\frac{1}{2}bf)^2}{4.32m^3a^3} etc.$$ Simili modo erit arcus ascensus

$$= 2\sqrt{(\tfrac{1}{2}bf)} - \frac{3n(\frac{1}{2}bf)}{2.2ma} + \frac{9n^2(\frac{1}{2}bf)^{\frac{3}{2}}}{3.8m^2a^2} - \frac{27n^3(\frac{1}{2}bf)^2}{4.32m^3a^3} etc.$$ Ex quibus perspicuum est arcum ascensus esse minorem arcu descensus. Si $\frac{nbf}{ma}$ valde fuerit paruum, harum serierum duos terminos initiales tantum assumere sufficit, et tum differentia inter arcum descensus et ascensus erit $\frac{3nbf}{4ma}$. Cum eorum summa sit $2\sqrt{2bf}$. Sunt ergo differentiae *q, p* in ratione duplicata summarum.

§ 35. Haec est igitur tautochrona in medio, quod mobili resistit in ratione duplicata velocitarum. Pro aliis vero medii resistentis hypothesibus, quibus resistentia alii cuidam celeritatis dignitati aut functioni proportionalis ponitur, hac methodo tautochronae inveniri non possunt; non quidem vitio methodi, quasi ea universalis non esset, sed defectu analyis; quod in aliis hypothesibus velocitas non potest exprimi. Persuasus autem autem sum hanc solam resistentiae hypothesin secundum quadrata celeritatum in rerum natura locum habere. Quanquam enim ex experimentis constat, fluida aliam praeter hanc exercere resistentiam a tenacitate eorum ortam, quae velocitati proportionalis esse nonnullis visa est, tamen *Neutonus in Princip. Phi. Edit. novissima* pag. 274 potius existimat eam prorsus non a velocitate pendere, verum eam esse uniformem seu in ratione momentorum temporum. Qua sit ut vires vivae amissae sint, ut spatia percursa, id quod aliis rationibus ex natura hujus resistentiae deductis praetermissis ex eo intelligi potest, quod mobile, si resistentiae velocitatibus essent proportionales nunquam ad quietem perveniret, quod tamen tandem accidere experimenta confirmant; si vero insuper resistentia adsit, secundum quam mobile amittit de vi viva in ratione spatiorum descriptorum, tautochronam exhibere in promtu est; eaque facile ex inventa hac formari potest. Ponatur enim tantummodo in aequatione nostra tautochronae inventa loco x haec quantitas x + gs, ubi litera g, ex quantitate hujus resistentiae a tenacitate vel frictione orta determinari debet. Quo facto habebitur tautochrona quaesita.