# FOR SOME GIVEN CURVE, IT IS REQUIRED TO FIND ANOTHER CURVE JOINED IN A CERTAIN WAY WITH THAT GIVEN, WHICH IS SUITABLE FOR PRODUCING A TAUTOCHRONE CURVE.

***By L. Euler*.**

§ I.

For more than four years I have been considering tautochronous curves suitable for *Master Sulli's* clock, see Comm. A. 1727 [E006 in this series of translations]. A certain other kind of oscillation comes to mind, that is confined by two curves, and is such that with one of these curves given, then it is always possible to find the other. On which account the question of tautochronism is indeterminate, and admits to an infinitude of solutions. This appears to be an elegant property to me, and is seen perhaps to have a certain use in practise ; as well as the solution itself, which is similar to the solution of problems involving reciprocal trajectories, and on that account a hitherto special manner of solution is required that is not much used in the solution of problems; these considerations urged me on, so that once again I could propound my most suitable method of solving problems of this kind, and that I could treat adapted to this case.

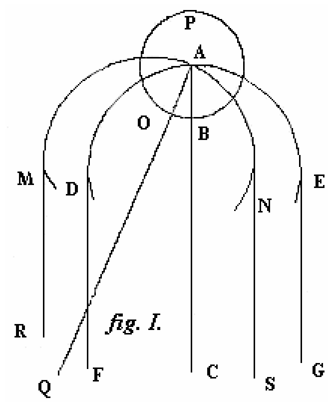



*fig. I.*

§. 2. But the present problem has its origin in a certain general kind of oscillation, which it is convenient to set out here before everything else. I consider the pulley ABP moving about the axis passing through the centre A (*Fig. I. Tab. V*). To this pulley two curved plates AD and AE have been attached, with threads placed around them, which plates always end where the tangents are vertical, as at D and E, and from which hang vertically weights at rest [lit. 'destitute of the force of inertia'], so that these may be moved with the aid of a force. Let the situation FDAEG be the state of equilibrium, and from A there is drawn the vertical ABC, which is conceived to be moving together with the pulley about A. Thus in order that the machine is then at rest since the line ABC is vertical. Therefore from this the ratio of the distances of the points D and E from the line AC can now be determined, if the two weights applied at F and G are given.

§. 3. The machine can be turned aside from a state of rest into some position RMANS, so that the line which before was the vertical AC arrives at the position AQ, with the angle CAQ traversed. Hence the threads MR and NS, acted on by the given forces touch the plates at M and N, as they are vertical. From this it is evident that the influence of the force F turning the pulley moved to R is increased, with the other, now moved from G to S is diminished [These are really the torques that have changed]. From which it follows that the machine in the position RMANS by no means is able to remain at rest, but it always tends towards the position FDAEG, into which it was put at equilibrium. Therefore by itself, as it is not impeded, it is carried to the equilibrium situation; and by that motion, since it is continuously driven by an acceleration towards that place, it is therefore unable to be at rest there, but from that place runs off in the opposite direction, and thus performs perpetual oscillations.

§. 4. With the machine described that I have proposed, this is the problem. *How are the plates AD and AE to be curved, so that isochronous oscillations are produced, or which all are completed in the same time.* It is clear that the present problem arising from these is with regard to a [kind of] number which are called indeterminates, since two curves AD and AE are to be determined, of which only one is determined from the condition of the problem. Therefore the problem from such is that from one of the curves given, it is possible to find the other. Then without doubt there is the case in which these two curves are equal to each other and similar. On account of which I divide this question into two parts, from which a twofold solution arises. In the first place I certainly set out the method, *in which with one or the curves given the other can be found;* and next, *how they can elicit the case, in which both curves are equal to each other and similar.*

§. 5. Towards solving these problems it is agreed to bring the mind to bear on two homologous points of the curves AD and AE, which I take to be those at which the plates are tangents at the same time, as D and E, likewise the points M and N. This is the mutual relation of these points, as with the curves set out owing to the common axis AQ, that the tangents at these points are parallel, as MR and NS. In that state of equilibrium FDAEG itself, these tangents are parallel to the axis, but in other positions they are not the same, and the tangents make an angle with the axis ; from which the *angle of declination* CAQ, [in Fig. I] or by which the position of the machine CAQ from the state of equilibrium can be measured, and the angles are indeed equal to each other. For since NS is parallel to AC, the angle that SN makes with the axis QA produced, is equal to the angle CAQ.

§. 6. (*Fig. II. Tab. V*) In order that I can treat this with more care, let AM and AN be the two curves sought, the common axis of which is AC. In these two homologous points M and N are taken, the tangents are MR and NS parallel to each other. The applied lines MP and NQ are drawn, and the sum of the angles PMR + QNS is equal to two right angles. Or if the normals MT and NA are drawn to the curves, then the angles PMT and QNV are equal to each other. But because the subnormals PT and QV are directed along opposite directions, one of the angles PMT and QNV is the negative of the other. Again since the angles of the tangents with the axis AC are equal to the angles PMT and QNV, it follows that the angle of declination of the machine is equal to either of the angles contained by the normals and the applied lines.

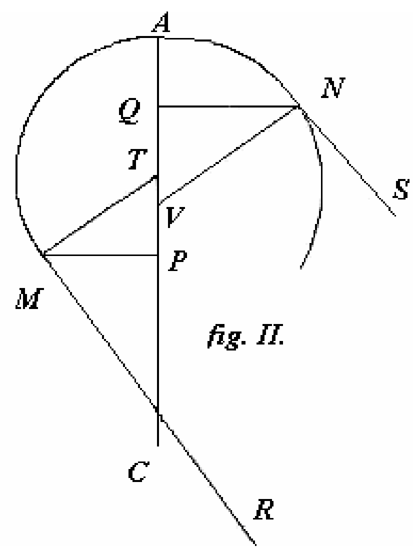


fig. II.

§. 7. (*Fig. III. Tab. V*) With these in place I turn my attention to the motion of the machine. The pulley performs oscillations ABO, and then it tends towards the natural state, it arrives at the situation RMANS, with the axis at AQ, distant from the vertical AC by the angle CAQ, or BAO. There is present at the point O the velocity arising from the height *v*, by which the circumference of the wheel has turned. Let the weight of the pulley be equal to P, and the vis viva, which is present in the wheel, if that should be everywhere of uniform thickness, is equal to $\frac{1}{2}Pv$.

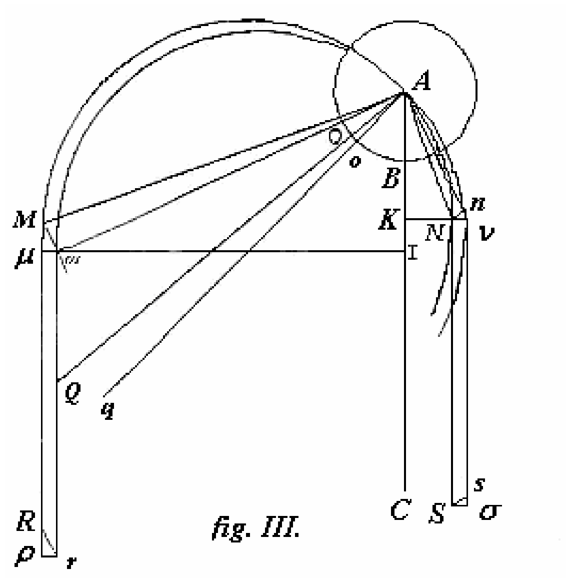


fig. III.

For if all the parts of the pulley are moving with the same velocity acquired from the height *v*, then the vis viva = *Pv*. Moreover since the parts which are closer to the centre are moving slower there, the vis viva is made smaller, as is made clear by computation.
[By 'weight' we take Euler to mean mass in this instance, and in the *Mechanica*, the speed squared of a point mass in uniform gravity is taken as corresponding to the height fallen *v*; hence, the vis viva [twice the kinetic energy] of the pulley with moment of inertia I, corresponding to twice the kinetic energy, is given by
$I\omega^2 = \frac{1}{2}MR^2 \times \frac{V^2}{R^2} = \frac{Mv}{2} = \frac{Pv}{2}$;]

§. 8. In a moment of time the machine can progress to the situation *rm*A*ns*, and with the axis AQ at A*q*, in order that the angle of declination is diminished by the angle OA*o*. Therefore the point M arrives at *m*, and N at *n*, and the angle OA*o* = MA*m* =NA*n*, on account of the uniform motion of the whole machine. Meanwhile the force applied at R has fallen to *r*, while the other at S has risen to *s*; and on account of MR = *mr*, the short line R*r* is parallel and equal to the short line M*m*. And likewise S*s* is parallel and equal to the element N*n*. The horizontal lines $r\rho$ and $S\sigma$ are drawn, and also the lines $m\mu, N\nu$, which are produced as far as to the vertical AC at I and K. With this passage through O*o* the velocity of the circumforence of the pulley is increased, so that now it is equal to the velocity acquired from the height $v + dv$. Thus the vis viva of the pulley in the present situation is equal to $\frac{1}{2}P(v + dv)$.

§. 9. Therefore from the angle OA*o*, the vis viva of the pulley has increased by the element $\frac{1}{2}Pdv,$ which increment owes its increase to the forces applied at R and S. On account of which I will investigate by how much in this short interval of time the vis viva shall have increased from these forces. Let the weight of the force at R be equivalent to R , and the weight of the force at S be equivalent to S. It is evident that with the weight falling through $R\rho$, there is generated a vis viva R. $R\rho$ or $R.M\mu$. But now on account of the force S rising through $\sigma s$ there has been a vis viva destroyed S. $\sigma s$ or $S.nv.$ With these joined together, there has been a vis viva generated equal to $R.M\mu - S.nv.$ For this force [here meaning work or energy in modern terms] generated must be equal to the increment itself observed $\frac{1}{2}Pdv.$ According to which this equation is acquired : $\frac{1}{2}Pdv = R.M\mu - S.nv.$

[In modern terms, we say that the increase in the kinetic energy is equal to the work done : thus we see that in Euler's day, the fundamental quantities such as force or weight, velocity, and distance were understood; however, other derived concepts such as work and energy in its different forms were not fully understood, and were replaced by the rather controversial *vis viva* of Leibniz, equal to $mV^2$. In addition, the relation between distance fallen from rest *v* and speed *V* for unit mass under unit gravity was given by $v = V^2$ for a given body, while the change in the vis viva in general could be written as force or weight by distance traversed. The amazing thing to come out of this was that Euler and his colleagues usually got the correct results, and the vis viva was to linger on into the 19th century as a legitimate means of calculation.]

§. 10. On account of the similar triangles M*m* $\mu$ , *m*AI, and N*nv*, ANK, it follows that

$$\mathrm{M}\mu = m\mathrm{I.M}m : \mathrm{A}m \text{ and } nv = \mathrm{NK} : \mathrm{N}n : \mathrm{AN}.$$

Moreover, then

$$\mathrm{M}\mu = m\mathrm{I.O}o : \mathrm{AO}; \text{and } n\mathrm{v} = \mathrm{NK.O}o : \mathrm{AO}.$$

Hence :

$$\mathrm{M}\mu = m\mathrm{I.O}o : \mathrm{AO}; \text{and } nv = \mathrm{NK.O}o : \mathrm{AO}.$$

On this account, the equation arises :

$$\tfrac{1}{2}\mathrm{P.AO.}dv = \mathrm{R.}m\mathrm{I.O}o - \mathrm{S.NK.O}o.$$

Now

$$\mathrm{O}o : \mathrm{AO} = \text{angle OA}o,$$

hence

$$\tfrac{1}{2}\mathrm{P.}dv = (\mathrm{R.}m\mathrm{I} - \mathrm{S.NK})\mathrm{OA}o.$$

But here the angle OA*o* is an element of the angle BAO, which is equal to the angle, that the connected line of the curve AM or AN makes between the normal drawn to the curve with the axis AQ. Therefore the element of this angle is equal to the element OA*o*. From which it is evident that the element *dv* can be expressed in terms of quantities from the given curve, and neither are the lines from each curve to be mixed together. Thus from the given curves of the laminas in this way the motion of the pulley can be found, and from that the oscillations, from which it is possible to judge whether they will be isochrones, or otherwise.

§. 11. Now we can investigate that condition, by which the oscillations of the pulley must be isochronous. We take a point on the circumference of the pulley, that lies at the lowest place with the machine at rest, that which is the point O. Hence it is necessary, and in order that this point makes isochronous oscillations, that wherever the motion might have started, it arrives in equal times at the lowest point B. According to this it is necessary that the accelerations of the point O towards B are as the paths to be described [i. e. essentially s.h.m.] while they are coming to B, that is, as the angles OAB. From which it follows that *dv* is as BO.O*o*, Moreover I put

$$adv = \mathrm{BO.O}o.$$

§. 12. With this value substituted in place of *dv* in the above equation, this equation appears :

$$\tfrac{1}{2}\mathrm{P.AO.BO.O}o = \mathrm{R.}a.m\mathrm{I.O}o - \mathrm{S.}a.\mathrm{NK.O}o.$$

This is divided by O*o*, and multiplied by 2, and there arises :

$$\mathrm{P.AO.BO} = 2\mathrm{R.}a.m\mathrm{I} - 2\mathrm{S.}a.\mathrm{NK}.$$

Which clearly is free from differential quantities. The constant is changed a little, and it becomes :

$$\mathrm{P.}b.\mathrm{BAO} = \mathrm{R.}m\mathrm{I} - \mathrm{S.NK}.$$

Moreover the angle BAO can be shown on the curves AM and AN themselves, and with the quantities related to these expressed. From which it follows that the equation found is sufficient to solve the problem.
[Thus, as with s.h.m., the period is independent of the amplitude, and the acceleration is proportional to the negative displacement; hence the angle is used in place of the acceleration.]

§. 13. Because the first differential equation shall be found [of such a kind that], with it divided by the differential quantity it is reduced to an integrated equation, showing that it is possible to find an algebraic equation immediately in the following much shorter and easier manner without differentiation. The moment of the force R required to turn the pulley is R.*m*I, and the moment of the other force S is S.NK. Since this moment is contrary to that moment and smaller, the pulley is acted on by a strength [torque] from O to B :
R.*m*I – S.NK, and this is proportional to the acceleraton, which, in order that the oscillations are isochronous, must be as in the way to be described, that is as BO, either BAO, or also as P.*b*.BAO to which it can be put equal, and it becomes as above :

$$\mathrm{P.}b.\mathrm{BAO} = \mathrm{R.}m\mathrm{I} - \mathrm{S.NK}.$$

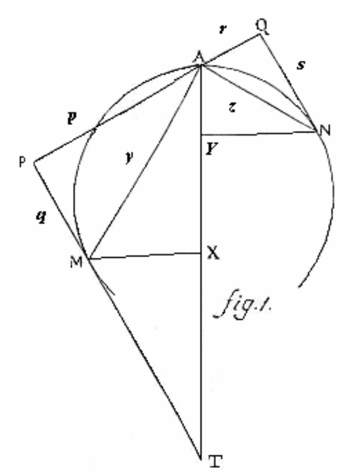


*fig. 1.*

§. 14. Now *m*I and NK are the perpendiculars from A to the tangents; and BAO is equal to the angle that the tangents make with the axis. Therefore with this fact applied to the following diagram (*Fig. I, Tab. VI*; italic labels added), this equation is found :

$$\mathrm{P}.b.\mathrm{ATM} = \mathrm{R.AP} - \mathrm{S.AQ}.$$

The differential of this is taken :

$$\mathrm{P}.b.d\mathrm{ATM} = \mathrm{R}.d\mathrm{AP} - \mathrm{S}.d\mathrm{AQ}.$$

Now *d*.ATM is equal to the angle that two nearby elements of the curve make with each other, and therefore this is equal to the angle intercepted between the two infinitely close normals, which is obtained from the element of the curve divided by the radius of osculation ; which must be the quotient in each curve on account of the parallel tangents , now the latter must be the negative of the former. Hence on that account I put :

$$\mathrm{AM} = y,\ \mathrm{AP} = p,\ \mathrm{PM} = \sqrt{(yy - pp)} = q,$$

and

$$AN = z, AQ = r, QN = \sqrt{(zz - rr)} = s.$$

[Note that $p$ and $q$ are the pedal coordinates of the point M with the pole A, and with the polar length $y$, etc. ]
The radius of osculation at M is there equal to $ydy:dp$, and then here at N it is equal to $zdz:dr$.
[See Fig. 52 ammended, Book I, Part 5a, Prop. 75, *Mechanica* in this series of translations for a simple derivation of this result.]
Then the element of the one curve is equal to $ydy:q$; and in the other it is equal to $zdz:s$ ; [as the increment of the tangential angle is $dp/q$, etc.]
hence in order that the element of the angle ATM shall be in that curve equal to $dp:q$, and from that, it is equal to $dr:s$.
Now it is necessary that $dp:q = -dr:s$.

§. 15. With these values substituted, this equation comes about :

$$\mathrm{P}bdp : q = Rdp - Sdr.$$

Now we have :

$$-dr : s = dp : q,$$

from which

$$dr = -sdp : q,$$

with which substituted this equation results ;

$$\mathrm{P}b.dp : q = Rdp + Ss.dp : q$$

it can be divided by $dp$, with which done, and by multiplying by $q$, this equation is obtained :

$$\mathrm{P}b = Rq + Ss.$$

From which this property of the curve sought follows, that the sum

$$\mathrm{R.PM} + \mathrm{S.QN}$$

is always constant with the tangents taken parallel. Hence these two equations are found :

$$\mathrm{P}b = Rq + Ss \text{ and } sdp + qdr = 0.$$

From which joined together the problem is easily satified.

§. 16. We can apply these equations to the axis AT. The applied lines MX and NY are drawn; let AX = $x$, XM = $y$, and AY = $v$; YN = $z$. Then we have

$$\mathrm{PM} = q = \frac{xdx+ydy}{\sqrt{(dx^2+dy^2)}} \text{ and } \mathrm{QN} = s = \frac{vdv+zdz}{\sqrt{(dv^2+dz^2)}}.$$

And on account of the tangents PM and QN being parallel to each other, then

$$dx : dy = dv : -dz, \text{or } dz = -dvdy : dx.$$

Now the above equation

$$\mathrm{P}b = Rq + Ss$$

is transformed into this :

$$\mathrm{P}b = \mathrm{R}(xdx + ydy) : \sqrt{(dx^2 + dy^2)} + \mathrm{S}(vdv + zdz) : \sqrt{(dv^2 + dz^2)}$$

on substituting $-dvdy : dx$ in place of $dz$, then

$$\mathrm{P}b = \mathrm{R}(xdx + ydy) : \sqrt{(dx^2 + dy^2)} + \mathrm{S}(vdx - zdy) : \sqrt{(dx^2 + dy^2)}.$$

Hence there is elicited :

$$z = vdx : dy + (\mathrm{R}(xdx + ydy) \text{ -} \mathrm{P}b\sqrt{(dx^2 + dy^2)}) : \mathrm{S}dy.$$

Calling $dx : dy = \xi$ : and

$$(\mathrm{R}(xdx + ydy) \text{ -} \mathrm{P}b\sqrt{(dx^2 + dy^2)}) : \mathrm{S}dy = \mathrm{B}.$$

then $z = \xi v + \mathrm{B}$;
hence

$$dz = \xi dv + vd\xi + d\mathrm{B} = -dvdy : dx = -dv : \xi .$$

Consequently

$$dv + \xi\xi dv + v\xi d\xi + \xi d\mathrm{B} = 0;$$

which divided by $\sqrt{(1+\xi\xi)}$ then becomes :

$$dv\sqrt{(1+\xi\xi)} + v\xi d\xi : \sqrt{(1+\xi\xi)} + \xi d\mathrm{B} : \sqrt{(1+\xi\xi)} = 0 .$$

Which integrated gives :

$$v\sqrt{(1+\xi\xi)} + \int \xi d\mathrm{B} : \sqrt{(1+\xi\xi)} = C .$$

On account of which :

$$v = \frac{C - \int \xi d\mathrm{B} : \sqrt{(1+\xi\xi)}}{\sqrt{(1+\xi\xi)}} \text{ and } z = \frac{C\xi + B\sqrt{(1+\xi\xi)} - \xi\int \xi d\mathrm{B} : \sqrt{(1+\xi\xi)}}{\sqrt{(1+\xi\xi)}}.$$

§. 17. Moreover with this value due substituted in place of B, clearly

$$(\mathrm{R}(x\xi+y))-\mathrm{P}b\sqrt{(1+\xi\xi)}):\mathrm{S},\quad \xi\int\xi d\mathrm{B}:\sqrt{(1+\xi\xi)}=\mathrm{R}x\sqrt{(1+\xi\xi)}:\mathrm{S}-\frac{\mathrm{P}b}{\mathrm{S}}\int\frac{\xi\xi d\xi}{1+\xi\xi}.$$

Thus

$$v=\frac{\mathrm{CS}-\mathrm{R}x\sqrt{(1+\xi\xi)}+\mathrm{P}b\int\xi\xi d\xi:(1+\xi\xi)}{\mathrm{S}\sqrt{(1+\xi\xi)}}$$

and

$$z=\frac{\mathrm{CS}\xi+Ry\sqrt{(1+\xi\xi)}+\mathrm{P}b\int\xi\xi d\xi:(1+\xi\xi)}{\mathrm{S}\sqrt{(1+\xi\xi)}}.$$

Therefore with the one given curve AN, or from the equation between *v* and *z*, if in that in place of *v* and *z* the values found are substituted, there is had the equation between *y* and *x*, or the other curve. But if we attend carefully to the convenience of the construction, the curve AM is given or the equation between *x* and *y*; and there is found from some given point M, the homolog of this N, on account of AY and YN, which are easily found by quadrature. Now it is clear that from the one the other curve can be constucted with the help of the quadrature of the circle ; and thus it is not possible for each curve to be algebraic.

§. 18. If the equation is in place between the perpendicular to the tangent and the tangent itself, then the whole work is much easier expedited. For on putting AP = *p*, P*m* = *q*, and AQ = *r*, QN = *s*, then by §.15. P*b* = R*q* + S*s* and *sdp* + *qdr* = 0, thus there is elicited

$$s=(\mathrm{P}b-\mathrm{R}q):\mathrm{S},$$

and

$$dr=-sdp:q=-dp(\mathrm{P}b-\mathrm{R}q):\mathrm{S}q.$$

Therefore from the given equation between *p* and *q*, there is hence found the equation between *s* and *r*. Or more easily if the equation is in place between *r* and *s*, on substituting the values for *s* and *r* into *p* and *q*, then there is produced the equation between *p* and *q*. Hence from either of these curves , the other can easily be found.

§. 19. We illustrate this rule by several examples. Let one of the curves AN be a circle, the periphery of which passes through A, let the radius of this be equal to *a*. Then

$$rr-2ar+ss=0 \text{ or } r=a+\sqrt{(aa-ss)}$$

and hence :

$$dr=-sds:\sqrt{(aa-ss)}.$$

Put *dr* in place of $-dp(\mathrm{P}b-\mathrm{R}q):\mathrm{S}q$ and in place of *s*, $(\mathrm{P}b-\mathrm{R}q):\mathrm{S}$, and hence in place of *ds*, put $-\mathrm{R}dq:\mathrm{S}:$

and this equation arises :

$$dp = -Rqdq : \sqrt{(\mathrm{SS}aa - (\mathrm{P}b - \mathrm{R}q)^2)}.$$

Which equation expresses the nature of the other curve, and because it has been separated, it can be constructed by quadrature. Since homologous points are found by this method, it is possible to take some applied line with respect to the axis AT, and the other applied lines can be found also, with this observation, that the tangents to homologous points are only parallel in a single case ; hence it is evident that the same applied curves can be satisfied in an infinite number of ways.

§. 20. Let one curve AN again be a circle, having the centre at A , then $r$ is constant and $s = 0$, hence $q = \mathrm{P}b$: R, and thus is constant. Therefore on account of this property the curve sought arises from the evolute of a circle of such a radius that the radius of this is P$b$ : R. Now the manner in which this curve is attached here is that the centre of the generating circle is placed above the centre of the pulley A. What such curves must be can be gathered from the dissertation *Concerning the generation of certain new tautochrones* [*de novo quodam tautochr.genere* inserted in Comm. A. 1727 ; Communications of the St. Petersburg Acad. Sc.; E006 in this series of translations]. For since the other curve AN is a circle having the centre at A and the weight or force S, then it exercises the same effect in any position, so hence the other curve alone must produce the tautochronism. Which, since that is case cited, it is necessary that the curve sought arises, as there, from the evolute of a circle.

§. 21. I now progress to the case of another problem, and I investigate the case in which both curves are equal to each other and likewise attached around the axis. Hence let the same curves AM and AN be similarly placed around the axis AC. It is necessary to find such an equation between the coordinates AX, $x$ and XM, y, so that on putting $x$ = AY, $v$, in that makes $y$ = NY = $z$. So that this can be done I consider these two properties found : in the first place, with the normals MT and NV or $n$V drawn, let the angle
XMT = YNV = Y$n$V;
thus in order that
P$b$ = R.PM + S.QN.
But since the curves are equal and similarly attached, it is necessary that as R = S, that as well there is put P$b$ : R = 2$v$,

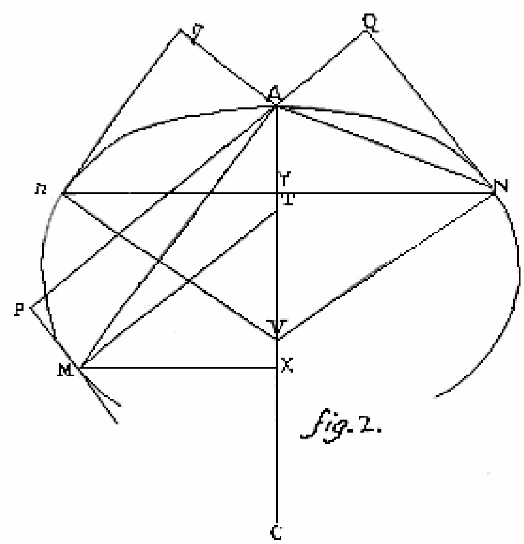


*fig. 2.*

and the one property is contained by this equation :
PM + QN = 2$c$,
and the other property is presented by :
PM + QN = 2$c$, or PM + $qn$ = 2$c$.

§. 22. Up to this point, therefore, the problem has been reduced in order that such a curve A$n$M can be found, so that, if two points M and $n$ are taken on that curve, from which the normals MT and $n$V drawn, with the y-coordinates MX, $n$Y, or which make angles that cross each other at right angles, the one is the negative of the other, then with the tangents MP and *nq* drawn, and to these are sent from A the perpendiculars AP and A*q*, in order that, as I say, the sum PM + *nq* is constant or equal to $2c$. On account of which I look for the equation between the tangent PM and the angle XMT, or a quantity depending on these, from which with the angle made negative, that avoids being negative too, such that, if in place of the angle XMT, or instead of this quantity there is put the negative of this arising, then PM is changed into another which with PM makes the sum 2*c*. [See the following section.]

§. 23. Since the sine of any angle, on making the angle negative, becomes itself negative, in place of the angle I take the sine of this; therefore let the sine of the angle XMT = $\xi$, and the tangents PM = *q*, an equation is required between $\xi$ and *q*, in which on putting $-\xi$ in place of $\xi$, *q* changes into *s*, so that $q+s=2c$. Then for this reason I put $q = c + \mathrm{Q}$, with Q designating some odd function of $\xi$, for on making $\xi$ negative, Q turns into – Q, so that hence $s = c - \mathrm{Q}$, and therefore the sum $q + s$ is equal to 2*c*, as was required.

§. 24. Moreover, with the equation found between *q* and $\xi$, it is necessary from these to find the equation between the coordinates AX and XM or between *x* and *y*; which is effected in the following way. If AX = *x*, and XM = *y*, then

the sine of the angle $\mathrm{XMT} = dy:\sqrt{(dx^2+dy^2)}$,

and the tangent $\mathrm{M}P = (xdx+ydy):\sqrt{(dx^2+dy^2)}$.

And thus we have :

$$\xi = dy:\sqrt{(dx^2+dy^2)}$$

and

$$q = (xdx+ydy):\sqrt{(dx^2+dy^2)}.$$

Therefore with these values substituted in the equation between $\xi$ and *q*, a new equation results between *x* and *y*, from which such a curve can be recognised. Now putting $dy = pdx$, then

$$\xi = p:\sqrt{(1+p^2)}$$

thus it is apparent that $\xi$ becomes negative, if *p* is made negative. On account of which instead of Q some odd function of *p* must be put in place, and it is

$$q = (xdx+ydy):\sqrt{(dx^2+dy^2)} = c+Q,$$

or

$$(x+py):\sqrt{(1+pp)} = c+Q.$$

§. 25. Therefore the equation between *x* and *y* for the curve sought must be of such a kind that on putting
*dy : dx = p*, then

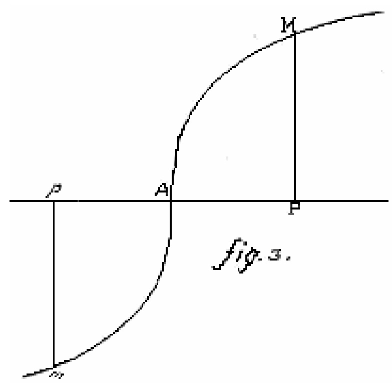
$$(x+py):\sqrt{(1+pp)}=c+Q$$

with Q denoting some odd function of *p*, or by $Q\sqrt{(1+pp)}$ , on account of $\sqrt{(1+pp)}$ being an even function of *p*, an odd function shall be written in place $Q\sqrt{(1+pp)}$ or Q alone; and then

$$x+py=c\sqrt{(1+pp)}+Q\,.$$

Therefore with the functions of *p* being substituted in place of Q, and then in place of *p* by the value *dy : dx*, the equations are obtained, in which nothing is present except *x* and *y* with their differentials and constants.

§. 26. The relation that Q and *p* have between each other can best be expressed by the curve MA*m*, which has similar and equal arcs around the point A, placed in contrary directions, as is plain from inspection. For if through the point A, which as it is the centre of the figure, the right line P*p* is drawn, and as this is considered to be the axis, on which the applied lines PM [*y*-coordinates] are sent. I say that with the abscissa AP expressing *p*, the applied line PM can designate Q. For with AP taken negative, as A*p*, the applied line *pm* is equal to the applied line PM, and is the negative of this, in order hence that the quantity expressing the applied line PM is an odd function of *p*, and on that account is able to show Q.




§. 27. Now from this given curve it is asked how to find and to be able to construct the curve sought constrained by *x* and *y* ; or how from given *p* and Q, the equation between *x* and *y* can be found and constructed. Which is shown to be done in the following manner. Since

$$x+py=c\sqrt{(1+pp)}+Q\,,$$

then

$$dx+pdy+ydp=cpdp:\sqrt{(1+pp)}+dQ.$$

Now since *dy = pdx*, then *dx = dy:p;* and hence

$$dy(1+pp)+ydp=cppdp:\sqrt{(1+pp)}+pdQ.$$

This is divided by $\sqrt{(1+pp)}$ and there is obtained :

$$dy\sqrt{(1+pp)}+ydp:\sqrt{(1+pp)}=cppdp:(1+pp)+pdQ:\sqrt{(1+pp)}.$$

Which integrated gives :

$$y\sqrt{(1+pp)} = c\int ppdp : (1+pp) + \int pdQ : \sqrt{(1+pp)}.$$

On account of which, then:

$$y = \frac{cp - \int dp : (1+pp) + \int pdQ : \sqrt{(1+pp)}}{\sqrt{(1+pp)}}.$$

Now since

$$x = Q + c\sqrt{(1+pp)} - py,$$

then

$$x = \frac{cp + cp\int dp : (1+pp) + Q\sqrt{(1+pp)} - p\int pdQ : \sqrt{(1+pp)}}{\sqrt{(1+pp)}}.$$

From which both $x$ and $y$ are agreed upon, and with $p$ and Q to be given separated, and consequently the curve sought can be constructed from the given curve MA*m*.

§. 28. So that we can have examples of tautochronous curves of this kind, I put Q = $ap$, then

$$x + py = c\sqrt{(1+pp)} + ap,$$

thus, on making $y - a = z$, this becomes

$$xx + 2pxz + ppzz = cc + ccpp,$$

hence

$$pp = \frac{2pxz + xx - cc}{cc - zz}$$

consequently

$$p = \frac{xz \pm c\sqrt{(xx + zz - cc)}}{cc - zz} = dy : dx = dz : dx,$$

or

$$ccdz - zzdz - xzdx = cdx\sqrt{(xx + zz - cc)}.$$

This equation, although reduced to rationality, can neither be separated nor integrated, which I consider to be unusual, since that curve can nevertheless be constructed. For it is given by :

$$y = \frac{cp - a\sqrt{(1+pp)} - c\int dp : (1+pp)}{\sqrt{(1+pp)}}$$

or

$$z = \frac{cp - c\int dp : (1+pp)}{\sqrt{(1+pp)}}$$

and

$$x = \frac{c + cp\int dp : (1+pp)}{\sqrt{(1+pp)}}.$$

Hence the construction has been done with the equation found, though it cannot be separated or integrated by the customary ways of reduction. Without doubt, indeed by performing the construction of this equation, many other outstanding constructions from other inseparable equations will be able to be found.

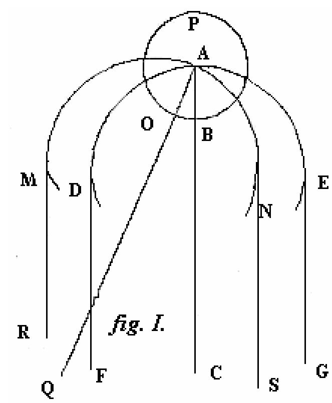
§. 29. This manner in which more difficult equations can be separating has been brought together in a note by the most celebrated *Hermann* in Book II of our works, inserted on p. 188, [I. Hermann. *Concerning the construction of differential equations of the first order by the method of separation of the indeterminates*. Comment.acad.sc.Petrop. 2, (1727), 1729. pp. 188-199.] on this method by which so difficult equations can be solved by separation : he expounds on the method for an infinitude of differential equations to be constructed, that the Cel. man has seized upon, with the help of equations similar to those from which our equations have been deduced, and that he calls Canonical, and to have come upon equations of this kind. Therefore by this method all equations which are satisfied by our problem can be constructed, and for which perhaps any are clearly seen to be inseparable. And indeed *Hermann*, with equation §. 28, that I communicated to him, separated it at once with the aid of his method, and he had found the same construction that I have noted here a little later. And thus it cannot be doubted, that the celebrated man shall soon be giving a separation of many equations that up to the present have been inseparable.

# QUOMODO DATA QUACUNQUE CURVA INVENIRI OPORTEAT ALIAM, QUAE CUM DATA QUODAMMODO IUNCTA AD TAUTOCHRONISMUM PRODUCENDUM SIT IDONEA.

*Auct. L. Eulero.*

§ I.

Meditanti mihi anti quadriennium de curva tautochrona ad Horologium Domini *Sullii* accommodata, vid. Comm. A. 1727. Occurrit aliud quoddam oscillationum genus, quod duabus continetur curvis, et tale est, ut data earum altera, altera semper inveniri possit. Quamobrem quaestio de hoc tautochronismo indeterminata est, et infinitas admittit solutiones. Quae proprietas, quae mihi elegans esse, et usum quendam fortasse in praxi habere posse visa est; praetereque ipsa solutio, quae problematis traiectoriarum reciprocarum similes est solutionis, et propter id peculiarem hucusque non multum usitatum solvendi modum requirit, me impulerunt, ut meam methodum ad huiusmodi problemata solvenda admodum idoneam iterum proponerem. huicque casui accommodatam traderem.




§. 2. Originem autem duxit praesens problema ex certo quodam oscillandi genere, quod hic ante omnia exponere convenit. Concipio trochleam ABP circa axem per centrum eius A transeuntem mobilem (*Fig. I. Tab. V*). Huic trochleae duae affirmatae sint laminae incurvatae AD, AE, filis circumductae, quae laminas semper deserant, ubi tangentes sunt verticales, ut in D et E, unde verticaliter dependeant tracta a potentiis vi inertiae destitutis, ne vi opus sit ad eas movendas. Sit situs FDAEG status aequilibrii, ducatur ex A verticalis ABC, quae una cum trochlea circa A moveri concipiatur; Ita ut machina tum sit in quiete, cum recta ABC fuerit verticalis. Ex hoc igitur iam ratio distantiarum punctorum D et E a recta AC determinatur, si datae fuerint potentiae in F et G applicatae.

§. 3. Detorqueatur machina ex statu quietis in situm quemvis RMANS, ut recta quae ante fuerat verticalis AC, in situm AQ perveniat, angulo CAQ percurso. Tangent ergo fila MR, NS, a datis potentiis sollicitata laminas in M et N, ut sint verticala. Ex hoc perspicuum est vim potentiae F in R translatae ad vertendam trochleam esse auctam, altera, vero G in S translatam esse diminutam. Ex quibus consequitur, machinam in RMANS positam nequaquam in quiete permanere posse, sed ad situm FDAEG, in quo ponitur aequilibrium, perpetuo tendere. Reipsa igitur, cum nihil impediat, in situm aequilibrii feretur; idque motu, quia continuo versus eum pellitur, accelerato : eo igitur situ quiescere non poterit, sed ex eo in contrariam plagam excurret, et ita perpetuo oscillationes peraget.

§. 4. Machina hac descripta, hoc est, quod mihi proposui, problema. *Quomodo laminae AD, AE sint incurvandae, ut oscillationes redantur isochoronae, seu ut* *omnes eodem tempore absolvantur.* Manifestum est, problema praesens ex eorum esse numero, quae indeterminata appellantur, propter duas curvas AD et AE determinandas, quarum una tantummodo ex conditione problematis determinatur. Problema igitur tale est, ut data harum curvarum altera, altera inveniri possit. Erunt deinde sine dubio casus, quibus hae duae curvae sunt inter se aequales et similes. Quamobrem hanc quaestionem bipartitam propono, unde duplex nascitur solutio. Primo nempe methodum tradam, *qua data altera curvarum altera inveniri queat;* deinceps, *quomodo ii eruendi sint casus, quibus ambae curvae sint inter se aequales et similes.*

§. 5. Ad solutionem harum quaestionum animum attendere convenit ad duo puncto curvarum AD et AE homologa, quae mihi sunt ea, in quibus simul laminae a filis tanguntur, ut D et E, item M et N. Horum punctorum haec est mutua relatio, ut dispositis curvis ad communem debitumque axem AQ, tangentes in iis punctis sint parallelae ut MR, NS. In ipso statu aequilibrii FDAEG sunt eae tangentes ipsi axi parallelae, in aliis positionibus non item, sed faciunt cum axe angulum; ex quo *angulus declinationis* CAQ, seu qui metitur distantiam situs machinae CAQ a statu quietis cognosci poterit, sunt enim inter se aequlaes. Quia enim NS parallela est AC, erit angulus, quem SN producta cum axe QA constituit, aequalis angulo CAQ.

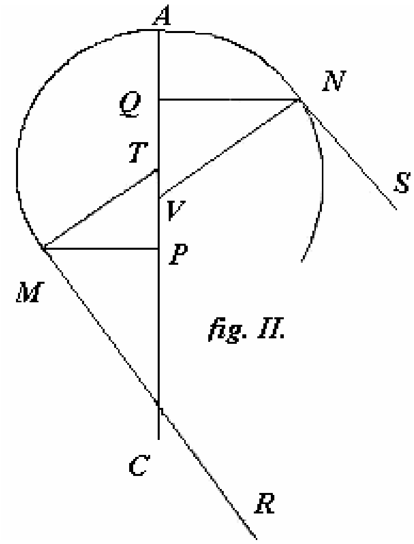
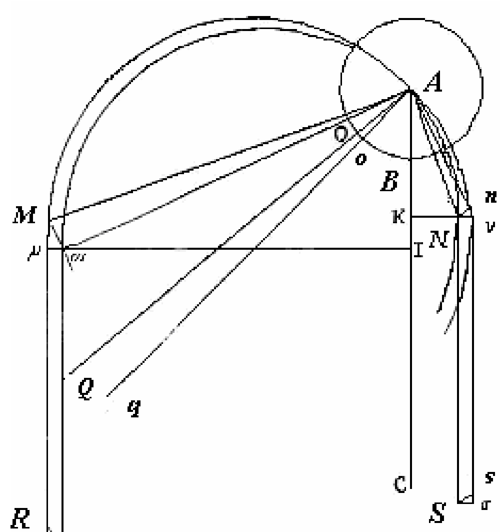


fig. II.

§. 6. (*Fig. II. Tab. V*) Ut hoc diligentius persequar, sint AM et AN duae curvae quaesitae, quarum axis communis sit AC. Capiantur in iis duo puncta homologa M et N, erunt tangentes MR, NS inter se parallelae. Ducantur applicatae MP, NQ, erit summa angulorum PMR + QNS aequalis duobus rectis. Sive ducantur normales MT, NA in curvas, erunt anguli PMT, QNV inter se aequales. Sed quia subnormales PT, QV, ad contrarias plagas diriguntur, erit angulorum PMT, QNV, alter alterius negativus. Porro cum anguli tangentium cum axe AC sint aequales angulis PMT, QNV, sequitur angulum declinationis machinae aequalem esse alterutri angulorum normalibus et applicatis contentorum.




§. 7. (*Fig. III. Tab. V*) His expositis ad ipsius machinae motum me converto. Peragat trochlea ABO oscillationes, et dum ad statum naturalem tendit, pervenerit in situm RMANS, ex axis in AQ, distans a verticali AC angulo CAQ, seu BAO. Insit in puncto O velocitas ex altitudine *v* oriunda, qua circumferentia trochleae convertitur. Sit pondus trochleae = P, erit vis viva, quae inest in trochlea, si ea fuerit ubique aequaliter crassa, = $\frac{1}{2}Pv$. Nam si omnes trochleae partes eadem velocitate ex alt. *v* acquisita moverentur, tum foret vis viva = *Pv*. Cum autem partes, quo centro sint propinquiores, eo tardius moveantur, vis viva dimidio sit minor, ut computum instituenti liquebit.

§. 8. Progrediatur puncto temporis machina in situm *rm*A*ns*, axisque AQ in A*q*, ut ergo angulo OA*o* angulus declinatinis deminuatur. Punctum igitur M perveniet in *m*, et N in *n*, eritque ang. OA*o* = MA*m* =NA*n* propter uniformem totius machinae motum angularem. Interim potentia in R applicata descendit in *r*, at altera in S ascendit in *s*; eritque ob MR = *mr*, lineola R*r* parallela et aequalis lineola M*m*. Et simili S*s* parallela erit et aequalis elemento N*n*. Ducantur horizontales, $r\rho$, $S\sigma$ nec non $m\mu, N\nu$, quae producantur in I et K ad verticalem AC usque. Transitu hoc per O*o* aucta erit velocitas circumferentiae trochleae, ut sit nunc = veloc. ex alt. $v+dv$ acquisitae. Unde vis viva trochleae in praesenti situ est = $\frac{1}{2}P(v+dv)$.

§. 9. Absoluto ergo angulo OA*o*, vis viva trochleae aucta est elemento $\frac{1}{2}Pdv$, quod incrementum ortum suum debet potentiis in R et S applicatis. Quamobrem investigabo quanta hac temporis particula ab his potentiss genita sit vis viva. Sit pondus potentiae in R aequivalens = R pondus potentiae in S aequivalens = S. Perspicuum est a potent. R descensu per $R\rho$, genitam esse vim vivam R.$R\rho$ seu $R.M\mu$ . At vero a potentia S propter ascensum per $\sigma s$ destructa est vis viva seu $S.nv$. His coniunctis generata habebitur vis viva = $R.M\mu - S.nv$. Huic vi genitae, aequale esse debet incrementum reipsa deprehendum $\frac{1}{2}Pdv$. Quocirca haec acquiritur aequatio, $\frac{1}{2}Pdv = R.M\mu - S.nv$.

§. 10. Propter triangula similia M*m* $\mu$, *m*AI, et N*nv*, ANK, erit

$$\mathrm{M}\mu = m\mathrm{I.M}m : \mathrm{A}m \text{ et } nv = \mathrm{NK} : \mathrm{N}n : \mathrm{AN}.$$

Est autem

$$\mathrm{M}\mu = m\mathrm{I.O}o : \mathrm{AO}; \text{ et } n\mathrm{v} = \mathrm{NK.O}o : \mathrm{AO}.$$

Ergo

$$\mathrm{M}\mu = m\mathrm{I.O}o : \mathrm{AO}; \text{ et } nv = \mathrm{NK.O}o : \mathrm{AO}.$$

Propterea haec emergit aequatio

$$\frac{1}{2}\mathrm{P.AO.}dv = \mathrm{R.}m\mathrm{I.O}o - \mathrm{S.NK.O}o.$$

Est vero

$$\mathrm{O}o : \mathrm{AO} = \mathrm{ang.OA}o,$$

ergo

$$\frac{1}{2}\mathrm{P.}dv = (\mathrm{R.}m\mathrm{I} - \mathrm{S.NK})\mathrm{OA}o.$$

Hic autem angulus OA*o* est elementum anguli BAO, qui aequalis est angulo, quem applicata curvae AM vel AN in axem AQ ducta cum normali in curvam constituit.

Huius igitur anguli elementum aequatur elemento OA*o*. Ex quo apparet elementum *dv* in quantitatibus ex curvis datis exprimi, neque lineas utriusque curvae inter se esse permixtas. Datis itaque curvis laminarum hoc modo motus trochleae invenietur, et inde oscillationes, de quibus iudicari poterit, utrum sint isochronae, an secus.

§. 11. Consideremus nunc eam conditionem, qua oscillationes trochleae isochronae esse debent. Accipiamus in circumferentia trochleae punctum, quod in infimo loco stat machina quiescente, id quod est punctum O. Necesse ergo est, ut et hoc punctum oscillationes isochronas conficiat, seu ut aequalibus temporibus, ubicunque motum inchoaverit, ad infinum punctum B perveniat. Ad hoc oportet, ut accelerationes puncti O versus B sint ut viae describendae, donec ad B perveniant, id est, ut anguli OAB. Ex quo fluit fore *dv* ut BO.O*o*, poni autem adv = BO.O*o*.

§. 12. Valore hoc loco *dv* in superiore aequatione substituto, haec provenit aequatio

$$\tfrac{1}{2}\mathrm{P.AO.BO.O}o = \mathrm{R.}a.m\mathrm{I.O}o - \mathrm{S.}a.\mathrm{NK.O}o.$$

Dividitur per O*o*, et multiplicetur per 2, orietur

$$\mathrm{P.AO.BO} = 2\mathrm{R.}a.m\mathrm{I} - 2\mathrm{S.}a.\mathrm{NK}.$$

Quae a differentialibus quantitibus prorsus est libera. Mutetur paulum constans, eritque

$$\mathrm{P.}b.\mathrm{BAO} = \mathrm{R.}m\mathrm{I} - \mathrm{S.NK}.$$

Potest autem ang. BAO in ipsis curvis AM, AN exhiberi, et quantitatibus ad eas pertinentibus exprimi. Unde sequitur aequationem inventam sufficere ad problema solvendum.

§. 13. Quod aequatio primum differentialis inventa sit, ea vero divisione facta per quantitatem differentialem ad integralem sit reducta, id indicat sine differentialibus statim ad inventam algebraicam pervenire posse sequenti modo multum breviore et faciliore. Momentum potentiae R, ad trocleam convertendam est R.*m*I, alterius potentiae S, est S.NK. Quia hoc illi contrarium est atque minuit, trochlea ab O ad B sollicitatur a vi R.*m*I – S.NK, huicque vi proportionalis est acceleratio, quae, ut oscillationes sint isochronae, debet esse ut via describenda id est ut BO, vel BAO, vel quoque ut P.*b*.BAO cui aequalis poni potest,eritque ut supra

$$\mathrm{P.}b.\mathrm{BAO} = \mathrm{R.}m\mathrm{I} - \mathrm{S.NK}.$$

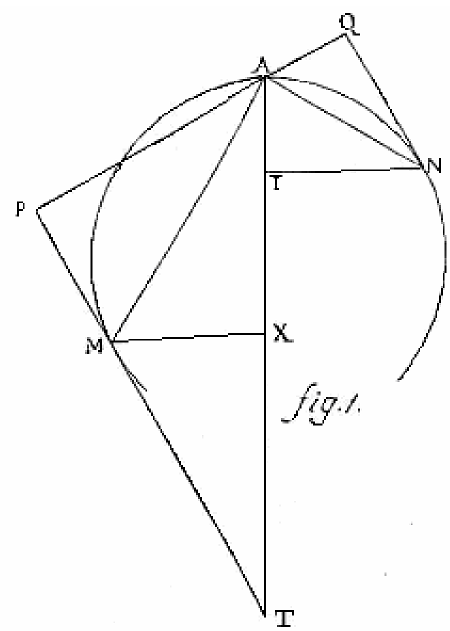


*fig. I.*

§. 14. Sunt vero mI, NK, perpendicula ex A in tangentes; et BAO aequatur angulo, quem tangentes cum axe constituunt. Facta igitur applicatione ad sequentem figuram (*Fig. I, Tab. VI*) reperitur haec aequatio

$$\mathrm{P.}b.\mathrm{ATM} = \mathrm{R.AP} - \mathrm{S.AQ}.$$

Sumatur eius differentialis

$$\mathrm{P}.b.d\mathrm{ATM} = \mathrm{R}.d\mathrm{AP} - \mathrm{S}.d\mathrm{AQ}.$$

Est vero *d*.ATM = angulo, quem duo elementa curvae proxima inter se constituunt, idcirco = angulo a duobus normalibus infinite propinquis interceptio, qui habetir elemento curvae per radium osculi divisio; qui quotus in utraque curva idem esse debet propter tangentes parallelas, in altera vero negativus esse debet eius, qui in altera accipitur. Hanc ob rem pono

$$\mathrm{AM} = y,\ \mathrm{AP} = p,\ \mathrm{PM} = \sqrt{(yy - pp)} = q,$$

atque

$$AN = z, AQ = r, QN = \sqrt{(zz - rr)} = s.$$

Erit radius osculi ibi = $ydy:dp$,
hic vero = $zdz:dr$.
Deinde elementum curvae in illa = $ydy:q$; in hac = $zdz:s$;
ut ergo elementum ang. ATM sit ex illa curva = $dp:q$, ex hac = $dr:s$.
Oportet vero esse $dp:q = -dr:s$.

§. 15. His valoribus substitutis, proveniet haec aequatio

$$\mathrm{P}bdp : q = Rdp - Sdr.$$

Est vero

$$-dr : s = dp : q,$$

unde

$$dr = -sdp : q,$$

quo substituto aequatio resultans

$$\mathrm{P}bdp : q = Rdp + Ssdp : q$$

dividi poterit per $dp$, quo facto, et multiplicato per $q$, habebitur ista aequatio

$$\mathrm{P}b = Rq + Ss.$$

Ex qua haec fluit proprietas curvarum quaesitarum, ut summa

$$\mathrm{R.PM} + \mathrm{S.QN}$$

semper sit constans sumtis tangentibus parallelis. Habentur ergo duae hae aequationes

$$\mathrm{P}b = Rq + Ss \text{ et } sdp + qdr = 0.$$

Ex quibus iunctis problemati facile satisfiet.

§. 16. Applicemus haec ad axem AT. Ducantur applicatae MX, NY ;sitque AX = $x$, XM = $y$, et AY = $v$; YN = $z$. Erit

$$\mathrm{PM} = q = \frac{xdx + ydy}{\sqrt{(dx^2 + dy^2)}} \text{ et } \mathrm{QN} = s = \frac{vdv + zdz}{\sqrt{(dv^2 + dz^2)}}.$$

Atque ob tangentes PM, QN inter se parallelas erit

$$dx : dy = dv : -dz, \text{seu } dz = -dvdy : dx.$$

Superior vero aequatio

$$\mathrm{P}b = Rq + Ss$$

transmutabitur in hanc

$$\mathrm{P}b = \mathrm{R}(xdx + ydy) : \sqrt{(dx^2 + dy^2)} + \mathrm{S}(vdv + zdz) : \sqrt{(dv^2 + dz^2)}$$

substituatur loco $dz, -dvdy : dx$; erit

$$\mathrm{P}b = \mathrm{R}(xdx + ydy) : \sqrt{(dx^2 + dy^2)} + \mathrm{S}(vdx - zdy) : \sqrt{(dx^2 + dy^2)}\ .$$

Unde elicietur

$$z = vdx : dy + (\mathrm{R}(xdx + ydy) \text{ -}\mathrm{P}b\sqrt{(dx^2 + dy^2)}) : \mathrm{S}dy.$$

Vocetur $dx : dy = \xi$ : et

$$(\mathrm{R}(xdx + ydy) \text{ -}\mathrm{P}b\sqrt{(dx^2 + dy^2)}) : \mathrm{S}dy = \mathrm{B}.$$

erit $z = \xi v + \mathrm{B}$;

unde

$$dz = \xi dv + vd\xi + d\mathrm{B} = -dvdy : dx = -dv : \xi\ .$$

Consequenter

$$dv + \xi\xi dv + v\xi d\xi + \xi d\mathrm{B} = 0;$$

dividatur per $\sqrt{(1+\xi\xi)}$ erit

$$dv\sqrt{(1+\xi\xi)} + v\xi d\xi : \sqrt{(1+\xi\xi)} + \xi d\mathrm{B} : \sqrt{(1+\xi\xi)} = 0\ .$$

Quae integrata dat

$$v\sqrt{(1+\xi\xi)} + \int \xi d\mathrm{B} : \sqrt{(1+\xi\xi)} = C\ .$$

Quocirco erit

$$v = \frac{C - \int \xi d\mathrm{B}:\sqrt{(1+\xi\xi)}}{\sqrt{(1+\xi\xi)}} \text{ ac } z = \frac{C\xi + B\sqrt{(1+\xi\xi)} - \xi\int \xi d\mathrm{B}:\sqrt{(1+\xi\xi)}}{\sqrt{(1+\xi\xi)}}\ .$$

§. 17. Est autem substituto loco B valore debito nempe

$$(\mathrm{R}(x\xi + y)) \text{ - } \mathrm{P}b\sqrt{(1+\xi\xi)}) : \mathrm{S}, \quad \xi\int \xi d\mathrm{B}:\sqrt{(1+\xi\xi)} = \mathrm{R}x\sqrt{(1+\xi\xi)} : \mathrm{S} - \frac{\mathrm{P}b}{\mathrm{S}}\int \frac{\xi\xi d\xi}{1+\xi\xi}.$$

Unde sit

$$v = \frac{\mathrm{CS} - \mathrm{R}x\sqrt{(1+\xi\xi)} + \mathrm{P}b\int \xi\xi d\xi:(1+\xi\xi)}{\mathrm{S}\sqrt{(1+\xi\xi)}}$$

atque

$$z = \frac{\mathrm{CS}\xi + Ry\sqrt{(1+\xi\xi)} + \mathrm{P}b\int \xi\xi d\xi:(1+\xi\xi)}{\mathrm{S}\sqrt{(1+\xi\xi)}}.$$

Dato ergo curva alterutra AN, seu aequatione inter *v* et *z*, si in ea loco *v* et *z* valores inventi substituantur, habebitur aequatio inter *y* et *x*, seu altera curva. Sed si ad commoditatem constructionis attendamus, detur curva AM seu aequatio inter *x* et *y*; et invenietur ex dato quovis puncto M, eius homologum N, propter AY et YN, quae per quadraturas facile determinantur. Perspicuum vero est alteram curvam ex altera constui ope quadraturae circuli ; et ideo utramque non posse esse algebraicam.

§. 18. Si habeatur pro alterutra curva aequatio inter perpendiculum in tangentem et ipsam, totum negotium multo facilius expedietur. Positis enim AP = $p$, P$m$ = $q$, et AQ = $r$, QN = $s$, erit per §. 15. P$b$ = R$q$ + S$s$ atque $sdp + qdr = 0$,
unde elicitur

$$s = (\mathrm{P}b - \mathrm{R}q):\mathrm{S},$$

atque

$$dr = -sdp : q = -dp(\mathrm{P}b - \mathrm{R}q):\mathrm{S}q.$$

Data ergo aequatione inter $p$, et $q$, inde invenietur aequatio inter $s$ et $r$. Seu facilius si habeatur aequatio inter $r$ et $s$, substituantur pro $s$ et $r$ valores in $p$ et $q$, prodibitque aequatio inter $p$ et $q$. Data ergo alterutra harum curvarum, altera facile reperitur.

§. 19. Illustremus hanc regulam exemplis nonnullis. Sit altera curva AN circulus, cuius peripheria per A transit, sit eius radius = $a$. Erit

$$rr - 2ar + ss = 0 \text{ seu } r = a + \sqrt{(aa - ss)}$$

indeque

$$dr = -sds : \sqrt{(aa - ss)} .$$

Ponatur loco $dr$, $-dp(\mathrm{P}b - \mathrm{R}q):\mathrm{S}q$ et loco $s$, $(\mathrm{P}b - \mathrm{R}q):\mathrm{S}$, ergo loco $ds$, $-\mathrm{R}dq : \mathrm{S}$, proveniet haec aequatio

$$dp = -\mathrm{R}qdq : \sqrt{(\mathrm{SS}aa - (\mathrm{P}b - \mathrm{R}q)^2)}.$$

Quae aequatio exprimit naturam alterius curvae, et quia est separata, per quadraturas construi potest. Quum hoc modo inveniantur puncta homologa, poterit altera utcunque applicari respectu axis AT, altera quoque applicari, hoc observato, ut tangentes in punctis homologis unico tantum in casu sint parallerae; perspicuum ergo est, easdem curvas infinitis modis applicatas satisfacere posse.

§. 20. Sit altera curva AN rursus circulus, centrum in A habens erit $r$ constans et $s = 0$, unde $q$ = P$b$: R, adeoque constans. Est igitur ob hanc proprietatem curva quaesita nata ex evolutione circuli, idque talis cuius radius est P$b$ : R. Modus vero huius applicandae hic est, ut centrum circuli generatoris ponatur super centro trochlochlea A. Quod haec curva talis esse debeat, ex Dissert. *de novo quodam tautochr.genere* in Comm. A. 1727. inserta colligi potest. Cum enim altera curva AN sit circulus centrum in A habens pondus seu potentia S, in quovis situ eundem exercit effectum, ut ergo

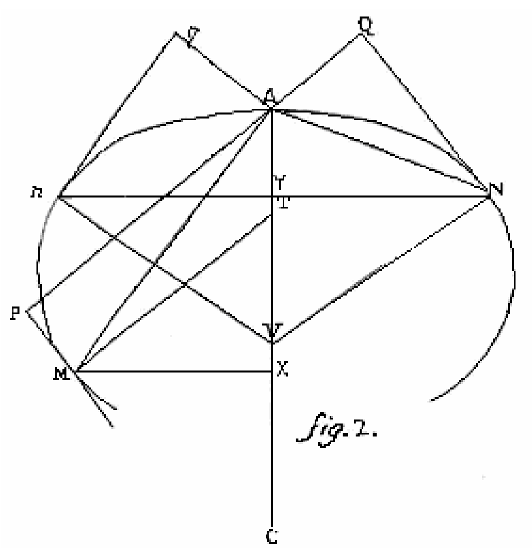


fig. 2.

altera curva sola tautochronismum producere debeat. Qui, cum sit ille casus citatus, necesse est, ut curva quaesita eadem sit, ut ibi, genita ex evolutione circuli.

§. 21. Progredior iam ad alterum problematis casum, et investigo casus quibus ambae curvae sint inter se eaedem et circa axem similiter applicatae. Sint ergo curvae AM, AN eaedem circa axem AC similiter positae. Oportet invenire talem aequationem inter coordinatas AX, *x* et XM, y, ut posito in ea *x* = AY, *v*, fiat *y* = NY = *z*. Ad hoc efficendum duas hasce proprietates inventas considero ; primo, ut ductis normalibus MT et NV seu *n*V, sit ang. XMT = YNV = Y*n*V; deinde ut sit P*b* = R.PM + S.QN. Sed quia curvae sunt aequales et similiter applicatae, necesse est ut sit R = S, ponatur autem Pb : R = 2*v*, erit altera proprietas hac aequatione contenta,
PM + QN = 2*c*, erit altera proprietas hac aequatione contenta, PM + QN = 2*c*, seu PM + *qn* = 2*c*.

§. 22. Huc ergo problema est reductum, ut inveniatuer curva A*n*M talis, ut, si in ea accipiantur curva duo puncto M et *n*, ex quibus normales MT et *n*V ductae cum applicatis MX, *n*Y angulos constitant, alterum alterius negatiuum, seu quae se mutuo ad angulos rectos interfecent; ductis deinde tangentibus MP et *nq*, in easque demissis ex A perpendiculis AP, A*q*, ut, inquam fit summa PM + *nq* constans seu aequalis, 2*c*. Quamobrem quaero aequationem inter tangentem PM, et angulum XMT, seu quantitatem independentem, quae facto angulo negativo, ipsa quoque negativa evadat, talem, ut, si loco anguli XMT, seu quantitatis vicem eius gerentis eius negativum ponatur, PM transmutetur in aliam quae, cum PM efficiat summam 2*c*.

§. 23. Cum sinus anguli cuiusvis, facto angulo negativo, abeat in sui negativum, loco anguli adhibeo eius sinum, sit igitur sinus anguli XMT = $\xi$, et tangens PM = *q*, requiritur aequatio inter $\xi$ et *q*, in qua, loco $\xi$ posito $-\xi$, *q* abeat in *s*, ut sit $q+s=2c$. Hanc ob rationem pono q = *c* + Q, designante Q functionem quamcunque imparem ipsius $\xi$, facto enim $\xi$ negativo et Q abibit in – Q, ut ergo sit *s* = *c* – Q, summa igitur *q* + *s* erit = 2*c*, ut requirebatur.

§. 24. Inventa autem aequatione inter *q* et $\xi$, oportet ex ea aequationem invenire inter coordinatas AX, XM seu inter *x* et *y*; id quod sequenti modo efficietur. Si AX = *x*, et XM = *y*, erit

sinus anguli $\quad \mathrm{XMT} = dy:\sqrt{(dx^2+dy^2)}$,

et tangens $\quad \mathrm{M}P = (xdx+ydy):\sqrt{(dx^2+dy^2)}$.

Erit itaque

$$\xi = dy:\sqrt{(dx^2+dy^2)}$$

et

$$q=(xdx+ydy):\sqrt{(dx^2+dy^2)}.$$

His igitur valoribus in aequatione inter $\xi$ et *q* substitutis, resultabit nova aequatione inter *x* et *y*, ex qua curva qualis sit cognoscetur. Ponatur vero *dy* = *pdx*, erit

$$\xi = p:\sqrt{(1+p^2)}$$

unde patet $\xi$ abire in negatiuum, si $p$ fiat negativum. Quamobrem loco Q poni potest functio quaecunque impar ipsius $p$, et erit

$$q = (xdx + ydy) : \sqrt{(dx^2 + dy^2)} = c + Q,$$

seu

$$(x + py) : \sqrt{(1 + pp)} = c + Q.$$

§. 25. Aequatio ergo inter $x$ et $y$ pro curva quaesita talis esse debet, ut posito $dy : dx = p$, sit

$$(x + py) : \sqrt{(1 + pp)} = c + Q$$

denotante Q functionem imparem ipsius $p$, sive cum et $Q\sqrt{(1 + pp)}$, ob $\sqrt{(1 + pp)}$ functionem parem ipsius $p$, sit functio impar, scribatur loco $Q\sqrt{(1 + pp)}$ solum Q; et erit

$$x + py = c\sqrt{(1 + pp)} + Q .$$

Substuteundis igitur loco Q determinatis functionibus ipsis $p$, et deinde loco $p$ eius valore $dy : dx$, habebuntur aequationes, quas non nisi $x$ et $y$ cum suis differentialibus et constantibus ingrediuntur.

§. 26. Relatio, quam Q et $p$ inter se habere, optime exprimetur curva MA$m$, quae circa punctum A habet arcus similes et aequales contrarie positos, ut ex inspectione palam est. Si enim per punctum A, quod tanquam centrum est figurae, ducatur recta P$p$, eaque tanquam axis consideretur, in quem demittantur applicatae PM. Dico abscissis AP exprimentibus $p$, applicatas PM designare posse Q. Nam sumta AP negativa ut A$p$, applicata $pm$ erit applicatae PM aequalis, eiusque negativa, ut ergo quantitas applicatam PM exprimens sit functio ipsius $p$ impar, et propterea Q exhibere possit.

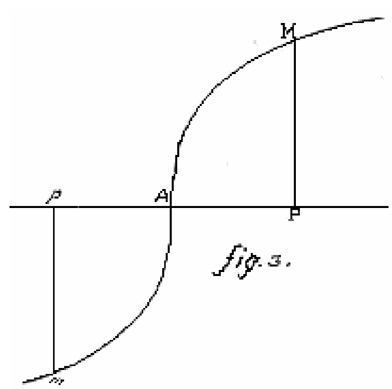


fig. 3.

§. 27. Quaeritur nunc quomodo ex curva hac data inveniri et construi possit curva quaesita coordinatis orthogonalibus $x$ et $y$ contenta; seu quomodo ex $p$ et Q datis inveniri et construi possint $x$ et $y$. Id quod sequenti modo efficietur. Cum sit

$$x + py = c\sqrt{(1 + pp)} + Q ,$$

erit

$$dx + pdy + ydp = cpdp : \sqrt{(1 + pp)} + dQ.$$

Quia vero est $dy = pdx$, erit $dx = dy:p$; unde

$$dy(1 + pp) + ydp = cppdp : \sqrt{(1 + pp)} + pdQ.$$

Dividatur per $\sqrt{(1 + pp)}$ et habebitur

$$dy\sqrt{(1+pp)} + ypdp : \sqrt{(1+pp)} = cppdp : (1+pp) + pdQ : \sqrt{(1+pp)}.$$

Quae integrata dat

$$y\sqrt{(1+pp)} = c\int ppdp:(1+pp) + \int pdQ : \sqrt{(1+pp)}.$$

Quamobrem erit

$$y = \frac{cp - \int dp : (1+pp) + \int pdQ : \sqrt{(1+pp)}}{\sqrt{(1+pp)}}.$$

Quoniam vero est

$$x = Q + c\sqrt{(1+pp)} - py,$$

erit

$$x = \frac{cp + cp\int dp:(1+pp) + Q\sqrt{(1+pp)} - p\int pdQ:\sqrt{(1+pp)}}{\sqrt{(1+pp)}}.$$

Ex quibus constat et *x* et *y*, in meris *p* et Q dari, et consequenter curvam quaesitam ex data MA*m* construi posse.

§. 28. Ut exempla habeamus huiusmode curvarum tautochronarum, pono Q = *ap*, erit

$$x + py = c\sqrt{(1+pp)} + ap,$$

unde, facto *y* – *a* = *z*, fiet

$$xx + 2pxz + ppzz = cc + ccpp,$$

ergo

$$pp = \frac{2pxz + xx - cc}{cc - zz}$$

consequenter

$$p = \frac{xz \pm c\sqrt{(xx + zz - cc)}}{cc - zz} = dy : dx = dz : dx,$$

sive

$$ccdz - zzdz - xzdx = cdx\sqrt{(xx + zz - cc)}.$$

Hanc aequationem, etsi ad rationalitatem reductam, nullo modo neque separare neque integrari potuit, id quod mihi magnopere mirum videtur, cum uti perspicuum est curva nihilominus constui possit. Est enim

$$y = \frac{cp - a\sqrt{(1+pp)} - c\int dp:(1+pp)}{\sqrt{(1+pp)}}$$

seu

$$z = \frac{cp - c\int dp:(1+pp)}{\sqrt{(1+pp)}}$$

et

$$x = \frac{c + cp\int dp:(1+pp)}{\sqrt{(1+pp)}}.$$

Aequatione ergo inventa construibilis est, quanquam consuetis modis reducendi nequaquam separari vel integrari possit. Non dubito, quin ex facie constructionis huius aequationis cognita multa praeclara de aliis aequationibus inseparabilibus inveniri queant.

§. 29. Modum hunc ad aequationes separatu tam difficiles perveniendi conferens cum schediasmate Celeb. *Hermanni* actorum Tomo II inserto, pag. 188, quo methodum tradit infinitas differentiales aequationes construendi, deprehendi Virum Celeb. ope similis aequationis ei, ex qua nostrae aequationes deducuntur, quam vocat Canonicam, ad eiusmodi aequationes pervenisse. Ea igitur methodo omnes aequationes, quae nostro problemati satisfaciunt, quaeque forte cuiquam prorsus inseparabiles videri queant, construi poterunt. Et sane *Hermannus*, cum aequationem §. 28, cum eo communicassem, eam statim ope methodi suae separavit, eandemque constructionem invenit, quam ego a posteriori cognitam hic apposui. Dubitari itaque nequit, quin Vir Celeb. plurium aequationum, quae adhuc inseparabiles habitae sunt, separationem sit daturus.