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Concerning
An Infinite Number Of Curves
Of The Same Kind.
Or
A Method For Finding
Equations For An Infinite Number Of Curves
Of The Same Kind.

Leonhard Euler.

§1.

Here I call curves of the same kinds such curves that are only different from each other because of the length of a certain constant line, which by assuming some or other values determines these curves. The constant line has been called the modulus by the Celebrated Hermann, and by others the parameter; but since the name parameter can create ambiguity I will retained the name module [History has not treated Euler well in this regard, as the name parameter is now used ; in this translation we use the term parameter, with which the reader is familiar]. Thus a parameter is a constant invariable line, by which each one of an infinitude of curves is determined ; moreover it has different values and thus it is a variable, if it is to refer to different curves. Thus if in the equation $y^2 = ax$, a is taken for the parameter, from the variability of a innumerable parabolas can arise placed on the same axis and having a common vertical axis.

§2. Therefore an infinite number of curves of the same kind are all expressed by the same equation, as the parameter is given successive values, to which we always assign the letter a. Indeed if all and every successive values are attributed to this parameter, the equation continues to give all the curves that are contained in the one equation. With Hermann, we will call this a parameter equation; in which therefore, in addition to the other constants which have the same value for all the curves, the quantities that belong to any curve are the parameter a and two pertaining variables, and either the abscissa and the applied line [the coordinates z and y], the abscissa and the length of the curve, or the abscissa and the area of the curve, etc., are of this kind as postulated by the problem to be solved.

§3. Therefore, there are the variable quantities x and z, which enter into the parameter equation with the parameter a. It is evident, if an algebraic equation is given between x, z, and a for a single curve in which a is considered as constant, then the same is likewise a parameter equation, or to pertain to all curves, only if a becomes a variable in this way. But if no algebraic equation can be given between x and z, it is with difficulty that a parameter equation can be found. For if $z = \int Pdx$, where P is given in terms of a, z, and x in some manner or $dz = Pdx$, in which equation a is considered as a constant; it is understood that a parameter equation is to be had, if the integral equation $dz = Pdx$ is differentiated anew, also with a made variable. But when an integration cannot be performed, a method of this kind is needed, in which the differential equation which is produced, if the integral is differentiated anew, also with a made variable, can be found.

§4. Indeed, the equation $dz = Pdx$ suffices for the curves to be constructed and understood. For, with the parameter given a certain value *a,* the equation $dz = Pdx$ can be constructed, from which with this done, one of an infinite number of the curves is obtained, and in the same way others can be found with different values put in place of *a*. But if for these curves certain points must be assigned, as demanded by some problem; such an equation as $z = \int Pdx$ is not sufficient, but an equation is required free from the sums of the signs in which if it is not algebraic, also belongs the differentials of *a*. Therefore for a given differential equation or a single curve $dz = Pdx$ in which *a* is considered as a constant, it is required to look for a differential equation, in which *a* is a variable, and this is an equation of the parameter. Now this parametric differential equation is sometimes of the first order, sometimes of the second or higher, sometimes also even deeper cannot be found.

§5. Therefore the method which I discuss is one from which the parameter can be found from the differential equation $dz = Pdx$, in which *a* is constant ; first I put *P* to be a function of *a* and *x* only, as $\int Pdx$ may be possible to be established by quadrature [*i.e.* expanding as a series and integrating term by term]. Therefore we have $z = \int Pdx$, in the integration of this Pdx, *a* has to be taken as a constant. Now with the differential of $\int Pdx$ if *a* is treated as a variable also ; where the equation of the parameter can be obtained by putting it equal to *dz*. Moreover, the differential of $\int Pdx$ has itself this form: $Pdx + Qda$, and hence $dz = Pdx + Qda$ is the equation of the parameter, if only the value of *Q* can become known.

§6. Moreover the following theorem is of use in finding the value of Q. *The magnitude A composed in some manner from the two variables t and u, if it is differentiated with t put constant, and this differential is differentiated anew with u put constant and t varying, and the same results if in the inverse order, A first is differentiated with u put constant and this differential is differentiated anew with t put constant and u to be varying.* As let $A = \sqrt{(t^2 + u^2)}$ be differentiated with *t* made constant, and there is obtained : $\frac{nudu}{\sqrt{(t^2+nu^2)}}$. This is differentiated anew with *u* put constant and there is produced : $\frac{-ntudtdu}{(t^2+nu^2)^{\frac{3}{2}}}$. Now in the inverse order, $\sqrt{(t^2 + u^2)}$ is differentiated with *u* constant, and this is the differential $\frac{tdt}{\sqrt{(t^2+nu^2)}}$, which differentiated anew with *t* constant gives $\frac{-ntudtdu}{(t^2+nu^2)^{\frac{3}{2}}}$, that agrees with the first result found. And agreement is determined in the comparisons of any other examples. [We now know that continuity is involved before coming to this conclusion.]

§7. Moreover although the truth of the demonstration of this theorem may be easily completed, yet I would like to add the following demonstration arising from this differentiation. Since *A* is a function of *t* and *u*, *A* goes to *B* if $t + dt$ is put in place of *t* ;

but with $u+du$ in place of u A goes to C. Moreover with $t+dt$ in place of t and $u+du$ in place of u at the same time, A is moved to D. From these it is evident, if in B, $u+du$ is written in place of u to arrive at D; and in a like manner if in C, $t+dt$ is put in place of t, this also leads to the production of D. From these presented, if A is differentiated with t held constant, $C-A$ is produced, for with $u+du$ put in place of u, A goes to C, moreover the differential is $C-A$. If again in $C-A$, $t+dt$ is put in place of t, $D-B$ is produced, whereby the differential is $D-B-C+A$. Now in the inverse order, $t+dt$ in place of t in A gives B, and thus the differential of A with only t variable gives $B-A$. This differential with $u+du$ put in place of u, becomes $D-C$, whereby the differential of this is $D-B-C+A$, in which it agrees with the differential found from the first operation. Q.E.D.

§8. Moreover this theorem can be put to use to find the value of Q in this manner. Since P and Q are functions of a and x, then $dP=Adx+Bda$ and $dQ=Cdx+Dda$, and since z is equal to $\int Pdx$, it also is a function of a and x, and moreover we can put $dz=Pdx+Qda$. Now following the theorem, z is differentiated with x put constant, and this differential Qda is again differentiated with a put constant giving $Cdadx$. From the other operation, the differential of z first with a constant is Pdx, and now the differential with x constant is $Bdadx$. Whereby on the strength of the theorem, $Cdadx$ and $Bdadx$ must be equal, from which $C=B$. Moreover B is given from P; for the differential of P with x put constant divided by da gives B. Therefore since $dQ=Bdx+Dda$, then $Q=\int Bdx$, if a is considered constant in these equations.

§9. We have therefore from these, $dz=Pdx+da\int Bdx$, from $dP=Adx+Bda$. Therefore if it is possible to integrate Bdx, the desired equation for the parameter is obtained. But if it is not possible to be integrated, then this equation is as equally useless as the first $z=\int Pdx$, for each involves the integration of a differential in which a must be considered as constant, which is contrary to the nature of the parameter equation, clearly in which a must be equal to a variable as well as x and z.

§10. Moreover when Bdx does not admit to an integration: yet the equation found as useless is not to be disregarded. For if the integration of Bdx follows from $\int Pdx$ the equation of the parameter can be shown. For if $\int Bdx=\alpha\int Pdx+K$ with the function K of a and x arising algebraically, then on this account

$$\int Pdx=z,\ \int Bdx=\alpha z+K \text{ and } dz=Pdx+\alpha zda+Kda,$$

which actually is the [differential] equation of the parameter. Therefore as often as either Bdx itself can be integrated, or the integration of Pdx is to be deduced, the equation of the parameter is obtained, which is a differential equation of the first order. But if Pdx is integrable, indeed there is no need for this : as $z=\int Pdx$ is at the same time the equation of the parameter.

§11. But if $\int Bdx$ can neither be shown to be algebraic nor able to be reduced to the form $\int Pdx$, it has to be considered, whether $\int Bdx$ can be reduced to the integral of another differential, in which *a* is not present. For with such an integral in which *a* is not present, then it does not disturb the parametric equation if the differential can be taken as it pleases. And by the same rule, if $\int Pdx$ can be reduced to another integral, which does not contain *a*, nor indeed is this a help in the determination of Q, but $z = \int Pdx$ gives the parameter equation at once, as if it becomes $\int Pdx = b\int Kdx$ with *b* given for *a* and *K* [is a function of] *x* only, and the equation of the parameter is given by $z = b\int Kdx$ or $dz = \frac{zdb}{b} + Kbdx$.

§12. But if all this comes to nothing then the position is indicated that a parameter differential equation of the first order cannot be given. On account of which an order in higher differentials must be sought. To this end I differentiate the equation $dz = Pdx + da\int Bdx$ anew. Moreover I place $dB = Edx + Fda$, with which done the differential of $\int Bdx$ is given by $Bdx + da\int Fdx$. Therefore with the differentiation carried out and in place of $\int Bdx$ with the value of this from the same equation surely $\frac{dz}{da} - \frac{Pdx}{da}$ put in place, there is obtained

$ddz = Pddx + dPdx + \frac{dzdda}{da} - \frac{Pdxdda}{da} + Bdadx + da^2\int Fdx$. Therefore

$\int Fdx = \frac{ddz}{da^2} - \frac{dzdda}{da^3} - \frac{dzddx}{da^2} - \frac{dPdx}{da^2} + \frac{Pdxdda}{da^3} - \frac{Bdx}{da}$. But since $\int Bdx = \frac{dz}{da} - \frac{Pdx}{da}$ and $\int Pdx = z$, if $\int Fdx$ can be reduced to the integrals $\int Bdx$ and $\int Pdx$ or if the expression itself can be integrated, the parameter equation is obtained, which is a differential equation of the second order. As if $\int Fdx = \alpha\int Bdx + \beta\int Pdx + K$, with α and β some given constants in terms of *a*, and also *K* for constant *a* and *x*, this equation of the parameter is given :

$\frac{daddz - dzdda - Pdaddx + Pdxdda - dPdadx}{da^3} - \frac{Bdx}{da} = \frac{adz - \alpha Pdx}{da} + \beta z + K$. But *B* and *F* are easily found for a given *P*.

§13. And if moreover as rarely happens, $\int Fdx$ either no longer contains *a*, or it can be reduced to another form, in which *a* is not present, the equation found of the second order differentials can be obtained for the proper parameter. But if all these do not yet succeed, then there is still this differentiation to be put in place, in which the differential of $\int Fdx$ becomes $Fdx + da\int Hdx$, by putting $dF = Gdx + Hda$. With which done it is apparent either $\int Hdx$ can be found, or the integration depends on the preceding $\int Fdx$, or $\int Bdx$ and $\int Pdx$, or it is possible for *a* to be eliminated from under the integration sign. Because of this if it should be concealed still, a parameter equation of the parameter of the third order in the differentials is obtained; but if now *a* is not in place, the differentiation is to be continued in the same way until the integral signs are eliminated.

§14. I move from these general premises to special ones, and the cases are to be explained, in which the function *P* is determined in some manner. Therefore let *P* be a function of *x* only, clearly not involving *a*, which I designate by the letter *X*, and hence $dz = Xdx$, which indeed is an equation that does not contain *a*, that is seen to pertain to a single curves, unless one is allowed to add a constant of integration, and it is able to become $z = \int Xdx + na$ or on differentiating, $dz = Xdx + nda$, which is truly the equation of the parameter. The same equation is produced, if I differentiate *X* with *x* kept constant according to the rule, and hence $B = 0$ is produced and $\int Bdx$ is equal to the constant *n*, therefore the equation of the parameter arises, $dz = Xdx + nda$, which rather takes the place of the integral $z = \int Xdx + na$.

§15. Now if we let $P = AX$, with the presence of the function *A* of *a*, and *X* still is a function only of *x*. Therefore since $z = \int Pdx$ becomes $z = \int AXdx$ or since in the integration *a* must be considered as a constant, $z = A\int Xdx$. From which equation or from the differential of this, the equation for the parameter sought becomes : $Adz - zdA = A^2 Xdx$. Indeed in place of *A* since it is a function of a only, it is able to be the equation of the parameter *a* : for in place of a parameter any function of this can be used as a rule for the same parameter.

§16. Let $P = A + X$ with the letters A and X retaining the same values as before. Hence there becomes $dz = Adx + Xdx$ and $z = Ax + \int Xdx$, which is now the equation of the parameter ; since the parameter A is not involved under the sign of the summation. Moreover, if $\int Xdx$ offends, the differential equation $dz = Adx + xdA + Xdx$ of the parameter can be obtained.

§17. By similar reasoning it is allowed to find the parameter equation, if $P = AX + BY + CZ$ etc. where *A, B, C* are any functions of the parameter *a*, and *X*, *Y, Z* are any functions of *x* and of constants except *a*. And for which as $dz = AXdx + BYdx + CZdx$ the equation becomes $z = A\int Xdx + B\int Ydx + C\int Zdx$, which likewise is the equation of the parameter, with the parameter *a* never found past the sign of the summation.

§18. Let $P = (A + X)^n$ or $z = \int dx(A + X)^n$. The differential of *P* with *x* placed constant is $n(A + X)^{n-1} dA$, that which divided by *da* gives the above value B, see §8. Therefore the differential equation becomes

$$dz = (A + X)^n dx + ndA\int (A + X)^{n-1} dx \text{, or } \int dx(A + X)^{n-1} = \frac{dz - (A+X)^n dx}{ndA}.$$

Therefore since $\int dx(A + X)^n = z$, if these two integrals are related to each other, or also if $\int dx(A+X)^{n-1}$ can be expressed algebraically, then the equation sought has been found. If neither appertains to the other, then another differentiation is to be put in place. Moreover the differential of $\int dx(A+X)^{n-1}$ is :

$$dx(A+X)^{n-1}+(n-1)dA\int(A+X)^{n-2}dx=\text{Diff.}\frac{dz-(A+X)^{n}dx}{ndA}$$

And thus there arises : $\int dx(A+X)^{n-2}=\frac{1}{(n-1)dA}\text{Diff.}\frac{dz-(A+X)^{n}dx}{ndA}-\frac{dx(A+X)^{n-1}}{(n-1)dA}$.

Whereby it is to be apparent whither or not $\int dx(A+X)^{n-2}$ can be integrated or to be reduced to an integral earlier.

§19. If n is a positive whole number then the equation of the parameter is algebraic. For $(A+X)^{n}$ can be resolved into a finite number of terms, of which each multiplied by dx can be integrated, thus as the parameter a does not enter under the sign of the summation. Moreover, this is the equation of the parameter :

$z=A^{n}x+\frac{n}{1}A^{n-1}\int Xdx+\frac{n.n-1}{1.2}A^{n-2}\int X^{2}dx$ etc. Therefore there remains to be examined for which cases if n is not a positive whole number, that have the above mentioned conditions in place.

§20. In the first case, let $X=bx^{m}$, where b can also depend on a ; hence $z=\int(A+bx^{m})^{n}dx$. [The original has x rather than z as the subject.] In the first place this formula is integrable , if $m=\frac{1}{i}$ with i designating some positive whole number : and then also if $m=\frac{-1}{n+i}$. Therefore in these cases the equation of the parameter is algebraic. But if $m=-\frac{1}{n}$, where b is unable to depend on a, for that indeed the equation does not admit to integration, but the following :

$dz=(A+bx^{\frac{-1}{n}})^{n}dx+ndA\int dx(A+bx^{\frac{-1}{n}})^{n-1}$ avoids the integral, and makes the equation of the parameter a differential of the first kind.

§21. Moreover, not only for whatever value of m is given, and a differential equation of the parameter can be found of the first degree, but also if we should have

$z=\int x^{m}dx(A+bx^{k})^{n}$. For this becomes

$dz=x^{m}dx(A+bx^{k})^{n}+ndA\int x^{m}dx(A+bx^{k})^{n-1}$. But

$\int x^{m}dx(A+bx^{k})^{n}=\frac{x^{m+1}(A+bx^{k})^{n}}{m+nk+1}+\frac{nkA}{m+nk+1}\int x^{m}dx(A+bx^{k})^{n-1}$, or

$\int x^{m}dx(A+bx^{k})^{n-1}=\frac{(m+nk+1)}{nkA}-\frac{x^{m+1}(A+bx^{k})^{n}}{m+nk+1}$. Consequently this equation of the parameter is produced : $Akdz=(A+bx^{k})^{n}(Akx^{m}dx-x^{m+1}dA)+(m+nk+1)zdA$. In a similar manner, the equation of the parameter can be found, if

$z=B\int x^{m}dx(A+bx^{k})^{n}$, for in order that the difference can be produced in this way one must write $\frac{z}{B}$ in place of *z*, and $\frac{Bdz-zdB}{B^2}$ in place of *dz,* if indeed *B* also depends on *a*.

§22. Moreover with the determinations of this kind of *P* dispatched, clearly which are less widely known, I proceed to others which are used much more often. These determinations are continued here for that certain proposed property of the function, whereby the function holds a number of the same dimensions everywhere [*i. e.* in all its terms] as the variable. For such functions allow a certain special method of differentiation. For let *u* be a function of zero dimensions of *a* and $x : \frac{a}{x}, \frac{\sqrt{(a^2-x^2)}}{a}$, and other similar ones are of this kind, in which the dimension of the numbers *a* and *x* in the denominator is equal to the dimension of the numbers the numerator.
[It may clarify matters to work an example as we go here. Let $U(x,a) = x^2 + ax + a^2$ be a homogeneous function in *x* and *a*; then on replacing *x* by λx; and *a* by λa where λ is some constant, then $U(\lambda x, \lambda a) = \lambda^2(x^2 + ax + a^2),$ and hence we say that this particular homogeneous function is of degree 2, equal to the power of λ. On the other hand, $u(x,a) = \frac{U(x,a)}{x^2} = 1 + \frac{a}{x} + \frac{a^2}{x^2}$ is a homogeneous function of degree zero, for $u(\lambda x, \lambda a) = \frac{U(\lambda x, \lambda a)}{\lambda^2 x^2} = 1 + \frac{a}{x} + \frac{a^2}{x^2} = \frac{U(x,a)}{x^2} = u(x,a)$, as the degree of λ is zero.]
Moreover such a function *u* on differentiation gives $Rdx + Sda$; I say that $Rx + Sa = 0$. For if in the function *u* we put $x = ay$, all the *a* are themselves cancelled and no other letter [*i.e.* variable] remains in the equation except *y* and constants. On this account, after substitution no other letter except *dy* is found. Moreover since $x = ay$ then $dx = ady + yda$, and thus $du = Rady + Ryda + Sda$. Hence the equation becomes $Ry + S = 0$, [as the coefficient of *da* is now zero, and hence $Rx / a + S = 0$ or $Rx + Sa = 0$.]

§23. Now if *u* is a [homogeneous] function of dimensions *m* of *a* and *x*, and $du = Rdx + Sda;$ then $\frac{u}{x^m}$ is a [homogeneous] function of *a* and *x* of dimension zero. Therefore $\frac{u}{x^m}$ is differentiated and there is produced
$\frac{xdu-mudx}{x^{m+1}}$ or $\frac{Rxdx-mudx+Sxda.}{x^{m+1}}$ For as it is the differential of a [homogeneous] function of zero degree , then $Rx^2 - mux + Sax = 0,$ or $Rx + Sa = mu.$ Whereby if *u* is a homogeneous function of *m* dimensions of *a* and *x*; and we put $du = Rdx + Sda;$ then $Rx + Sa = mu$ and thus $du = Rdx + \frac{da}{a}(mu - Rx)$ or $adu = Radx - Rxda + muda.$
[Writing in the form:
$$d\left(\frac{u}{x^m}\right) = \frac{du}{x^m} - \frac{mu}{x^{m+1}}dx = \frac{Rdx+Sda}{x^m} - \frac{mu}{x^{m+1}}dx = \left(\frac{R}{x^m} - \frac{mu}{x^{m+1}}\right)dx + \frac{Sda}{x^m};$$
hence $\left(\frac{R}{x^m} - \frac{mu}{x^{m+1}}\right)x + \frac{Sa}{x^m} = 0,$ from the previous section, or
$$Rx + Sa = mu,$$
the required result.]

§24. With these put in place $dz = Pdx$ or $z = \int Pdx$, let P be a function of n dimensions in *a* and *x*, therefore *z* is such a function of of dimension $n + 1$. Whereby on putting $dz = Pdx + Qda$, then we have $Px + Qa = (n+1)z.$ From which the value of Q substituted gives the parameter equation : $dz = Pdx + \frac{da}{a}((n+1)z - Px)$ or $adz - (n+1)zda = Padx - Pxda$. Which is a differential equation of the first order only. Moreover since generally $Q = \int Bdx$, then in this case $(n+1)\int Pdx = a\int Bdx + Px.$ From which it is evident that in this case the integral $\int Bdx$ can always be reduced to $\int Pdx$.

§25. Likewise the equation of the parameter comes about by considering *P* alone. For by putting $dP = Adx + Bda$, there becomes $nP = Ax + Ba.$ But if we let $dz = Pdx + da\int Bdx$, then $dz = Pdx + \frac{da}{a}\int (nPdx - Axdx)$ in which integration *a* is kept constant. Therefore we have $\int nPdx = nz,$ and $\int Axdx = Px - \int Pdx$ as $\int Adx = P$ [on integrating *Adx* by parts]. Thus we have $dz = Pdx + \frac{da}{a}(n+1)z - Px$, which clearly agrees with the preceding.

§26. With *P* retaining it value of *n* dimensions. Let $z = \int APXdx$, where *A* is a function of *a* and *X* of *x* only. Therefore we have the equation: $\frac{z}{A} = \int PXdx$. By putting $dP = Adx + Bda,$ (in which the letter *A* is not to be confused with the other which is a function of *a* only[Thus, the *A* in the formula *APX* is not the same as that in $dP = Adx + Bda$]) there results [from the theorem in §23] $nP = Ax + Ba.$ Therefore the differential of *PX* with *x* put constant is *BXda*. Consequently the differential is obtained $d.\frac{z}{A} = PXdx + da\int BXdx = PXdx + \frac{da}{a}\int (nPXdx - AXxdx).$ Hence, $\int nPXdx = \frac{nz}{A}$ and $\int AXxdx = PXx - \int PXdx - \int PxdX.$ [on differentiation by parts again, where the integral of *A* w.r.t *x* giving *P*, and subsequently *X* and *x* are diff. in turn] Whereby the total derivative becomes $d.\frac{z}{A} = PXdx - \frac{PXxda}{a} + \frac{(n+1)zda}{Aa} + \frac{da}{a}\int PxdX.$ Therefore unless $\int PxdX$ is able to be reduced to $\int PXdx$ or clearly to be integrated, a differential equation of the first order is unable to be given.

§27. But if the equation should be given by $z = R\int Pdx$, with *R* being some algebraic function of *a* and *x* with *z* constant, but *P* is a function of *a* and *x* of dimension *n*. Since we have $\frac{z}{R} = \int Pdx$ then [from the previous sections] $d.\frac{z}{R} = Pdx + \frac{da}{a}(\frac{(n+1)z}{R} - Px) = \frac{Rdz - zdR}{R^2}$ or $Radz - zadR - (n+1)Rzda = PR^2adx - PR^2xda.$ Moreover it is held in general, that just as often as $z=\int Pdx$ can be reduced to a parameter equation, so also can $z=R\int Pdx$ be reduced to a parameter equation. Indeed no other distinction is present, except that it is z in the one case, and it must be $\frac{z}{R}$ in the other. Whereby if R is either an algebraic quantity or such a transcending function that its differential of the variable a can also be put in place outside the summation, then the parameter equation can be found through what has been given already. On account of which in such cases that follow, even if this rule extends more widely, then it is permitted to omit these, [on the understanding that they still apply.]

§28. We can put $z=\int(P+Q)dx,$ or $z=\int Pdx+\int Qdx$, and P is a function of a and x of dimension $n-1$, and also Q is now a function of the same a and x of dimension $m-1$. Therefore since the differential of $\int Pdx$ is $\frac{P(adx-xda)}{a}+\frac{da}{a}\int nPdx$, [from §24, where it is shown with a change of order, that $adz-nzda=P(adx-xda)$] and the differential of $\int Qdx$ is $\frac{Q(adx-xda)}{a}+\frac{da}{a}\int mQdx$; then $dz=\frac{(P+Q)(adx-xda)}{a}+\frac{da}{a}(n\int Pdx+m\int Qdx).$ Putting $\frac{adz-(P+Q)(adx-xda)}{da}=u,$ and there becomes $u=n\int Pdx+m\int Qdx.$ If u again is differentiated, [as we have returned to a relation of the original kind], then $du=\frac{(nP+mQ)(adx-xda)}{da}+\frac{da}{a}(n^2\int Pdx+m^2\int Qdx).$ Therefore on putting $\frac{adu-(nP+mQ)(adx-xda)}{da}=t,$ then $t=n^2\int Pdx+m^2\int Qdx.$ Now with the integrals $\int Pdx$ and $\int Qdx$ eliminated from the these three equations of z, u and t, this equation $mnz-(m+n)u+t=0$ is produced. Which equation is the sought parameter equation, if values to be assumed are substituted in place of u and t.

§29. In a similar manner, if $z=\int(P+Q+R)dx,$ and P a function of $n-1$, Q a function of $m-1$ and R a function of $k-1$ dimensions of a and x. Put

$$u=\frac{adz-(P+Q+R)(adx-xda)}{da}$$

and $t=\frac{adu-(nP+mQ+kR)(adx-xda)}{da}$, and $s=\frac{adt-(n^2P+m^2Q+k^2R)(adx-xda)}{da}$. With which done, this parameter equation is produced :

$$kmnz-(km+kn+mn)u+(k+m+n)t-s=0.$$

§30. Again let $z=\int(P+Q)^k dx,$ where P is a function of dimension n, and Q is a function of dimension m, both of a and x. Therefore, when $dP=Adx+Bda$ and $dQ=Cdx+Dda$, then [from the theorem in §23 applied to P and Q in turn], $nP=Ax+Ba$ and $mQ=Cx+Da$. Moreover, the differential of $(P+Q)^k$ with x placed constant and divided by da is $k(B+D)(P+Q)^{k-1}$. On this account, $dz=(P+Q)^k dx+\frac{kda}{a}\int(P+Q)^{k-1}(Ba+Da)dx.$ [The second term is just the differential under the integral w.r.t. a, with extra a's for convenience.] Moreover since $Ba = nP - Ax,$ also $Da = mQ - Cx,$ $Adx = dP$ and $Cdx = dQ$ as a is constant in this integration, we have on substitution [recall that as a rule for a function of two variables, a term differentiated w.r.t. x and multiplied by dx has a constant, and vice versa; for these homogeneous functions, there is no difference apart from the labels]:

$dz = (P+Q)^k dx + \frac{kda}{a}\int (P+Q)^{k-1}(nPdx + mQdx - xdP - xdQ),$ or

$dz = \frac{(P+Q)^k (adx - xda)}{a} + \frac{da}{a}\int (P+Q)^{k-1}((nk+1)Pdx + (mk+1)Qdx).$ [on taking and adding $\frac{xda}{a}(P+Q)^k$ from and to the first and second terms]

Putting $\frac{adz - (P+Q)^k (adx - xda) - zda}{kda} = u$ then we have

$u = \int (nPdx + mQdx)(P+Q)^{k-1}.$ Whereby if the integral

$\int (nPdx + mQdx)(P+Q)^{k-1}$ depends on the integral $\int (P+Q)^k dx$ a differential equation of the first order is obtained for the parameter; but with a smaller power if the differentiation should be continued. Moreover it becomes

$du = (nPdx + mQdx)(P+Q)^{k-1} + \frac{uda}{a} - \frac{da}{a}(nP + mQ)(P+Q)^{k-1}x +$

$\frac{da}{a}\int (kn^2P^2dx + (2kmn + n^2 - 2mn + m^2)PQdx + km^2Qdx)(P+Q)^{k-2}.$

[There appears to be a 'spare' integral sign at the end of this formula in the original text.]

And on putting $t = \frac{adu - uda - (nP + mQ)(P+Q)^{k-1}(adx - xda)}{da}$, there arises

$t = \int (kn^2P^2dx + (2kmn + n^2 - 2mn + m^2)PQdx + km^2Qdx)(P+Q)^{k-2}$

§31. Therefore when it is evident from the three integrals whether or not these depend in turn on each other, and if it is the case that they do not, then an algebraic equation is obtained between t, u and z, which gives, with assumed values substituted in place of t and u, a parameter differential equation of the second order. Moreover when it is observed to be easier in special cases, or where they do depend on each other in turn, then it is agreed that they can be reduced to other forms. Therefore when

$z = \int (P+Q)^k dx,$ then $u = mz + (n-m)\int (P+Q)^{k-1}Pdx,$ and

$t = (2km + n - m)u - (km^2 - m^2 + mn)z + (n-m)^2(k-1)\int (P+Q)^{k-2}P^2dx.$

Therefore it is to be asked whether $\int (P+Q)^{k-2}P^2dx$ can be reduced to these:

$\int (P+Q)^{k-1}Pdx$ and $\int (P+Q)^k dx$. Or even

$\int (P+Q)^{k-2}P^2dx = \alpha\int (P+Q)^{k-1}Pdx + \beta\int (P+Q)^k Pdx + V$ with V designating some algebraic quantity given in terms of a and x, and α and β are coefficients composed from more constants and a.

§32. Therefore let $V = T(P+Q)^{k-1}$ and the differential of this with *a* placed constant is $V = dT(P+Q)^{k-1} + (k-1)(TdP + TdQ)(P+Q)^{k-1}$. Hence the following equation is produced :

$$P^2dx = \alpha P^2dx + \alpha PQdx + \beta P^2dx + 2\beta PQdx + \beta Q^2dx + PdT + QdT + (k-1)TdP + (k-1)TdQ,$$

which can be divided by *dx*. But T should thus be accepted, as the corresponding terms can be made to cancel by taking appropriate values for α and β .

§33. If *z* is not completely determined by $\int Pdx$, but by the quantity $\int Qdz$, with *Q* given everywhere in terms of *a* and *z*, and *P* by *a* and *x*; this equation is obtained : $Qdz = Pdx$ in which the indeterminates *x* and *z* are in turn separated from each other. Now the equation of the parameter can be found in this manner : Since $Qdx = Pdx$ can be differentiated and each member to be put in place also with the variable *a* with the aid of $dP = Adx + Bda$ and $dQ = Cdz + Dda$. Hence there arises :
$Qdz + da\int Ddz = Pdx + da\int Bdx$ or $Qdz = Pdx + da(\int Bdx - \int Ddz)$. Which equation, if $\int Bdx$ and $\int Ddz$ can be eliminated, gives the required parameter equation.

§34. Let *P* be a function of $m-1$ dimensions of *a* and *x*, and *Q* a function of $n-1$ dimensions of *a* and *z*. With these put in place :
Diff.$\int Pdx = \frac{mda\int Pdx + P(adx - xda)}{a}$, and Diff.$\int Qdz = \frac{nda\int Qdz + Q(adz - zda)}{a}$. From which this equation arises : $(m\text{-}n)\int Pdx = \frac{Q(adz-zda)}{da} - \frac{P(adz-zda)}{da}$ as $\int Pdx = \int Qdz$.
Whereby if it should be that $m = n$, then $Qadz - Qzda = Padx - Pxda$, which is the parameter equation, or $\frac{da}{a} = \frac{Qdz - Pdx}{Qz - Px}$.

§35. Now if *m* and *n* are not equal, the differential equation of the parameter is of the second order. For since $(m\text{-}n)\int Pdx = \frac{Q(adx-zda)}{da} - \frac{P(adz-zda)}{da}$ then

$$\text{Diff.}\frac{Q(adz-zda)}{da} - \frac{P(adx-xda)}{da} = \frac{m(m\text{-}n)da\int Pdx}{a} + \frac{(m\text{-}n)P(adx-xda)}{a}$$
$$= \frac{mQ(adz-zda) - nP(adx-xda)}{a}.$$

Which is the equation of the parameter sought.

§36. If in the proposed equation $dz + Pdx = 0$ the indeterminates cannot be separated from each other, thus so that *P* is a function involving *x*, *z* and *a* ; then the equation should be multiplied by a certain quantity R, in which the terms $Rdz + PRdx$ is to be considered as the differential of some integral *S*. Thus we have :
$dS = Rdz + PRdx = 0$, and thus *S* = Const. But in order to find the quantity *R*, let $dP = Adx + Bdz$ and $dR = Ddx + Edz$, where we take *a* as a constant for the moment. With these in place, we have $d.PR = (DP + AR)dx + (EP + BR)dz$, on account of which we have $D = EP + BR$. But as $D = \frac{dR - Edz}{dx}$ this becomes

$Edz + EPdx + BRdx = dR.$ Now since $dz + Pdx = 0,$ it is found that $dR = BRdx,$ and $lR = \int Bdx.$ Now *B* is known from the given *P*, and since *B* depends on *z* and *x*, *Bdx* should be integrated with the help of the equation $dz + Pdx = 0$, if indeed it can be done. Thus let $\int Bdx = K,$ then $R = e^{K}$ with $le = 1$.

§37. Therefore since the equation is $dS = e^{K}dz + e^{K}Pdx = 0,$ in order to find the equation of the parameter, let $dK = Fdx + Gdz + Hda,$ and it becomes $de^{K} = e^{K}(Fdx + Gdz + Hda).$ Then the integral is taken of $e^{K}Hdz$ with *z* only made variable, and with *x* and *a* constants, with this done the equation of the parameter is $e^{K}dz + e^{K}Pdx + da\int e^{K}Hdz = 0,$ or dividing by e^{K} this equation arises : $dz + Pdx + e^{-K}da\int e^{K}Hdz = 0.$ Another equation of the parameter is found, by putting $dP = Adx + Bdz + Cda,$ for the differential of $e^{K}P$ with *x* and *z* constant here is $e^{K}(Cda + PHda)$. Then $e^{K}dx(C + PH)$ is integrated with only *x* variable, with which done the equation of the parameter is $dz + Pdx + e^{-K}da\int e^{K}dx(C + PH) = 0.$ But the parameter equations of this kind, unless $dz + Pdx = 0$ can be determined without R, are hardly of any use.

§38. Therefore we can consider special cases, in which the equation is given by $dz + Pdx = 0, P$ is a function of zero dimensions of *x* and *z*, and with the parameter *a* given by constants and not to be evaluated. Now an integrable formula can always be returned for $dz + Pdx$ if it is divided by $z + Px$, on account of which $S = \int\frac{dz+Pdx}{z+Px} = \text{Const.}$ Moreover, it is given by $\int\frac{dz+Pdx}{z+Px} = l(z + Px) - \int\frac{xdP}{z+Px}.$ Thereupon on putting $z = tx,$ the function *P* becomes a function of *t* only which is *T*. Whereby the equation becomes $S = l(z + Px) - \int\frac{dT}{t+T}$, which can be shown by quadrature [*i. e.* term by term integration of the series expansion].

§39. Therefore nothing else has to be done in order that the parameter equation can be found, unless $\int\frac{dz+Pdx}{z+Px}$ is differentiated for some variable parameter *a* put in place also. Therefore the equation $dP = Adx + Bdz + Cda$ is put in place, where $Ax + Bz = 0$ everywhere [by §23]. Now the coefficient of *dx* [in the integral] is differentiated, surely $\frac{P}{z+Px}$ with only *a* variable, the derivative of this is $\frac{Czda}{(z+Px)^2}$. Then $\frac{Czdz}{(z+Px)^2}$ is integrated with having x only for a variable, with which done the equation of the parameter sought is given : $dz + Pdx + (z + Px)da\int\frac{Czdx}{(z+Px)^2} = 0$. In a similar manner, from the coefficient of *dz* which is $\frac{1}{z+Px}$, this equation of the parameter is found: $dz + Pdx - (z + Px)da\int\frac{Cxdz}{(z+Px)^2} = 0$, in which integration only *z* is taken as a variable. Or also, even by this equation [from §38]:

$dz + Pdx = (z + Px)da\int \frac{Cdt}{(t+T)^2}$, [the original has D rather than C in the integral] in which C and T are only given in terms of t and a .

§40. I am not able to present here that generality of the homogenous equations, so called by the most celebrated *Johan. Bernoulli*, which are all contained by that equation $dz + Pdx = 0$, but I can add a solution. In as much as from the equation (§. 38) $l(z + Px) = \int \frac{dT}{t+T} = l(t + T) - \int \frac{dt}{t+T}$, where $t = \frac{z}{x}$ and $T = P$. There is therefore produced $lx + \int \frac{dt}{t+T} = 0$, or by adding a constant $l\frac{c}{x} = \int \frac{dt}{t+T}$. As if the proposed equation is $nxdz + dx\sqrt{(x^2 + z^2)} = 0$ then we have $P = \frac{\sqrt{(x^2+z^2)}}{nx}$, and on putting $z = tx$, it becomes $T = \frac{(1+tt)}{n}$, and thus $l\frac{c}{x} = \int \frac{ndt}{nt+\sqrt{(1+tt)}}$, putting $\sqrt{(1+tt)} = t + s$ then $t = \frac{1-ss}{2s}$ and $dt = \frac{-ds(1+ss)}{2ss}$. Whereby the equation becomes :

$$l\frac{c}{x} = \int \frac{-nds(1+ss)}{(n+1)s-(n-1)s^3} = \frac{-n}{n+1}ls + \frac{n^2}{n^2-1}l((n-1)s^2 - n - 1).$$

§41. Yet a special case in which the use of the calculation of §36 is apparent, shall be for the proposed equation $dz + pzdx - qdx = 0$, in which p and q are given in some way by a and x. Which equation taken together with that general equation $dz + Pdx = 0$ gives $P = pz - q$, from which $B = p$, and $lR = \int pdx$, or $R = e^{\int pdx}$. Therefore when $\int pdx$ can be designated by quadrature, the value of R is known, and thus the proposed equation multiplied by $e^{\int pdx}$ is integrable : therefore it becomes $e^{\int pdx}dz + e^{\int pdx}pzdx - e^{\int pdx}qdx = 0$, and the integral of $e^{\int pdx}z$ $= \int e^{\int pdx}qdx$ or $z = e^{-\int pdx}\int e^{\int pdx}qdx$. Thus the equation must be differentiated with a and x put as variables : $e^{-\int pdx}\int e^{\int pdx}qdx$, and the differential put equal to dz, with which done we have the equation of the parameter. Therefore with these in position, $dp = fdx + gda$ et $dq = hdx + ida$ the equation of the parameter is produced :

$$dz = -e^{-\int pdx}(pdx + da\int gdx)\int e^{\int pdx}qdx + qdx + e^{-\int pdx}da\int e^{\int pdx}(idx + qdx\int gdx),$$

or for the sake of brevity, on putting $\int e^{\int pdx}qdx = T$ then

$$dz = -e^{-\int pdx}Tpdx + qdx + e^{-\int pdx}da\int e^{\int pdx}idx - e^{-\int pdx}da\int Tgdx.$$

From which exercise it can be understood, how the parameter equation can be found in the most efficient manner, as the indeterminates in a proposed equation can be separated from each other.

DE
INFINITIS CURVIS
EIUSDEM GENERIS.
SEV
METHODUS INVENIENDI
AEQUATIONES PRO INFINITIS CURVIS
EIUSDEM GENERIS.
AUCTORE

Leonh. Eulero.

§1.

Curvas eiusdem generis hic voco tales curvas, quae a se invicem non differunt nisi ratione lineae cuiusdam constantis, quae alios atque alios valores assumens eas curvas determinat. Linea haec constans a Cel. Hermanno modulus est vocatus, ab aliis parameter; quia autem parametri nomen ambiguitatem creare potest, moduli vocabulum retinebo. Est itaque modulus linea constans et invariabilis, dum una infinitarum curvarum quaecunque determinatur; varios autem habet valores et ideo variabilis est, si ad diversas curvas refertur. Sic si in aequatione $y^2 = ax$ sumatur a pro modulo, ex variabilitate ipsius a innumerabiles oriuntur parabolae super eodem axe positae et communnem verticem habentes.

§2. Infinitae igitur curvae eiusdem generis omnes unica aequatione exprimuntur, quam modulus cui nobis semper litera a indicabitur, ingreditur. Huic enim modulo, si successive alii atque alii valores tribuantur, aequatio continuo alias dabit curvas, quae omnes in una aequatione continentur. Aequationem hanc modulum continentem cum Hermanno modularem vocabimus; in qua igitur praeter alias constantes et eiusdem valoris in omnibus curvis quantitates insunt modulus a et duae variabiles ad curvam quamlibet pertenentes, cuiusmodi sunt vel abscissa et applicata, vel abscissa et arcus curvae, vel area curvae et abscissa etc. prout problema solvendum postulat.

§3. Sint igitur quantitates variabiles x et z, quae cum modulo a aequationem modularem ingrediuntur. Perspicuum est, si detur aequatio algebraica inter x et z et a, pro unica curva, in qua a ut constans consideratur, eandem fore simul modularem, seu ad omnes curvas pertinere, si modo a fiat variabilis. At si inter x et z non poterit aequatio algebraica dari, difficile erit aequationem modularem invenire. Nam sit $z = \int Pdx$, ubi P in a, z, et x, quomodocunque detur, seu $dz = Pdx$, in qua aequatione a ut constans consideratur; intelligitur aequationem modularem haberi, si integralis aequationis $dz = Pdx$ denuo differentietur, posito etiam a variabili. Sed cum integrationem perficere non liceat, eiusmodi methodus desideratur, qua differentialis aequatio, quae prodiret, si integralis denuo differentietur posita etiam a variabili, inveniri possit.

§4. Ad construendas quidem et cognoscendas curvas aequatio $dz = Pdx$ sufficit. Nam, dato ipsi modulo a certo valore construentur aequatio $dz = Pdx$, quo facto habebitur una curvarum infinitarum, eodemque modo aliae reperientur aliis ponendis valoribus loco *a*. Sed si in his curvis certa puncta debeant assignari prout problema aliquod postulat; talis aequatio $z=\int Pdx$ non sufficit sed requirtur aequatio a signis summatoriis libera in qua si non est algebraica, etiam differentialia ipsius *a* insint. Ex data igitur aequatione differentiali pro unica curva $dz=Pdx$ in qua a ut constans consideratur, quaeri oportet aequationem differentialem, in qua et *a* sit variabilis, haecque erit modularis. Haec vero modularis interdum erit differentialis primi gradus, interdum secundi et altioris, interdum etiam penitus non poterit inveniri.

§5. Quo igitur methodum tradam, qua ex aequatione differentiali $dz=Pdx$, in qua *a* est constans, modularis possit invenire, quae *a* ut variabilem contineat; pono primo *P* esse functionem ipsarum *a* et *x* tantum, ut $\int Pdx$ saltem per quadraturus exhiberi possit. Erit igitur $z=\int Pdx$, in integratione ipsius Pdx, *a* pro constanti habita. Quaeritur nunc differentiale ipsius $\int Pdx$ si etiam a ut variabilis tractetur; quo invento ipsique *dz* aequali posito habebitur aequatio modularis. Differentiale autem ipsius $\int Pdx$ habebit hanc formam $Pdx+Qda$, eritque $dz=Pdx+Qda$ aequatio modularis, si modo valor ipsius *Q* esset cognitus.

§6. Ad inveniendum autem valorem ipsius Q sequens inservit theorema. *Quantitas A ex duabus variabilibus t et u utcunque composita, si differentietur posito t constante, hocque differentiale denuo differentietur positu u constans et t variabili, idem resultat ac si inverso ordine A primo differentietur positu u constante hocque differentiale denuo differentietur posito t constante et u variabili.* Ut sit $A=\sqrt{(t^2+u^2)}$, differentietur posito *t* constante, habibitur $\frac{nudu}{\sqrt{(t^2+nu^2)}}$. Hoc denuo differentietur posito *u* constante et prodibit $\frac{-ntudtdu}{(t^2+nu^2)^{\frac{3}{2}}}$. Iam ordine inverso differentietur $\sqrt{(t^2+u^2)}$ posito *u* constante, eritque differentiale $\frac{tdt}{\sqrt{(t^2+nu^2)}}$ posito *u* constante, eritque differentiale $\frac{tdt}{\sqrt{(t^2+nu^2)}}$, quod denuo differentiatum posito *t* constante dabit $\frac{-ntudtdu}{(t^2+nu^2)^{\frac{3}{2}}}$, id quod congruit cum prius invento. Atque similis convenientia in quibusque aliis exemplis cernetur.

§7. Quamvis autem huius theorematis veritatem exercitati facile perficiant, demonstrationem tamen sequentem adiiciam ex ipsius differentiationis natura petitam. Cum *A* sit functio ipsarum *t* et *u*, abeat *A* in *B* si loco *t* ponatur $t+dt$; at posito $u+du$ loco *u* abeat A in C. Posito autem simul $t+dt$ loco *t* et $u+du$ loco *u* mutetur A in D. Ex his perspicuum est, si in *B* scribatur $u+du$ loco *u* provenire *D*; similque modo si in *C* ponatur $t+dt$ loco *t* prodit urum quoque *D*. His praemissis, si differentietur *A* positio *t* constante prodibit $C-A$, nam posito $u+du$ loco *u* abit *A* in *C*, differentiale autem est $C-A$. Si porro in $C-A$ ponatur $t+dt$ loco t prodibit $D-B$, quare differentiale erit $D-B-C+A$. Inverso nunc ordine posito $t+dt$ loco *t* in *A* habebitur *B*, eritque differentiale ipsius A posito tantum *t* variabili $B-A$. Hoc differentiale posito $u + du$ loco u abit in $D - C$, quare eius differentiale erit $D - B - C + A$, in quod congruit cum differentiali priori operatione invento. Q.E.D.

§8. Istud autem theorema hoc modo inservit ad valorem ipsius Q inveniendum. Cum P et Q sint functiones ipsarum a et x, sit $dP = Adx + Bda$ et $dQ = Cdx + Dda$, atque z cum sit = $\int Pdx$, erit quoque functio ipsarum a et x, postitum autem est $dz = Pdx + Qda$. Iam secundum Theorema differentietur z posito x constante, eritque differentiale Qda hoc porro differentiatum posito a constante dabit $Cdadx$. Altera operatione differentiale ipsius z posito primo a constante est Pdx, huius vero differentiale posito x constante est $Bdadx$. Quare vi theorematis aequalia esse debent $Cdadx$ et $Bdadx$, ex quo sit $C = B$. Datur autem B ex P; differentiale enim ipsius P posito x constante divisum per da dat B. Cum igitur sit $dQ = Bdx + Dda$, erit $Q = \int Bdx$, si in hac integratione a ut constans consideretur.

§9. Ex his ergo habebitur $dz = Pdx + da\int Bdx$, existente $dP = Adx + Bda$. Si igitur Bdx integrari poterit, desiderata habitur aequatio modularis. At si integrari non potest aeque inutilis est haec aequatio ac prima $z = \int Pdx$, utraque enim involvit integrationem differentialis, in qua a ut constans debet considerari, id quod est contra naturam aequationis modularis, quippe in qua a aeque variabile esse debet ac x et z.

§10. Quando autem Bdx integrationem non admittit : non tamen aequatio inventa ut inutilis omnino est negligenda. Nam si integratio ipsius Bdx pendeat a $\int Pdx$ aequatio modularis poterit exhiberi. Si enim fuerit $\int Bdx = \alpha\int Pdx + K$ existente K functione ipsarum a et x algebraica, erit ob $\int Pdx = z, \int Bdx = \alpha z + K$ et $dz = Pdx + \alpha zda + Kda,$ quae aequatio revera erit modularis. Quoties igitur Bdx vel reipsa poterit integrari, vel ad integrationem ipsius Pdx deduci, aequatio habebitur modularis, quae erit differentialis primi gradus. At si Pdx est integrabile, ne hoc quidem opus est: sed $z = \int Pdx$ erit simul aequatio modularis.

§11. Si autem $\int Bdx$ neque algebraice exhiberi neque ad $\int Pdx$ reduci potest, dispiciendum est, num $\int Bdx$ ad integrationem alius differentialis, in qua a non inest, possit reduci. Tale enim integrale in qua a non inest non turbat aequationem modularem, cum si libuerit per differentiationem tolli possit. Atque eodem iure, si $\int Pdx$ reduci poterit ad aliud integrale, quod a non continet, nequidem hac ipsius Q determinatione opus est, sed $z = \int Pdx$ statim dat aequationem modularem, ut si sit $\int Pdx = b\int Kdx$ data b per a et K per x tantum, erit aequatio modularis $z = b\int Kdx$ seu $dz = \frac{zdb}{b} + Kbdx$.

§12. Si autem haec omnia nullum inveniant locum indicio est, aequationem modularem primi gradus differentialem non dari. Quamobrem in altioris gradus differentialibus quaeri debebit. Ad hoc differentio denuo aequationem $dz = Pdx + da\int Bdx$. Pono autem $dB = Edx + Fda$, quo facto erit ipsius $\int Bdx$ differentiale $Bdx + da\int Fdx$. Differentiatione igitur peracta et loco $\int Bdx$ eius valore ex eadem aequatione nemne $\frac{dz}{da} - \frac{Pdx}{da}$ posito, habebitur $ddz = Pddx + dPdx + \frac{dzdda}{da} - \frac{Pdxdda}{da} + Bdadx + da^2\int Fdx$. Erit igitur $\int Fdx = \frac{ddz}{da^2} - \frac{dzdda}{da^3} - \frac{dzddx}{da^2} - \frac{dPdx}{da^2} + \frac{Pdxdda}{da^3} - \frac{Bdx}{da}$. Cum autem sit $\int Bdx = \frac{dz}{da} - \frac{Pdx}{da}$ et $\int Pdx = z$, si $\int Fdx$ reduci poterit ad integralia $\int Bdx$ et $\int Pdx$ vel si reipsa duci poterit ad integrari, habebitur aequatio modularis, quae erit differentialis secundi gradus. Ut si fuerit $\int Fdx = \alpha\int Bdx + \beta\int Pdx + K$, datis α et β utcunque per a et constantes, et K per a et x constantes, erit aequatione modularis haec $\frac{daddz - dzdda - Pdaddx + Pdxdda - dPdadx}{da^3} - \frac{Bdx}{da} = \frac{adz - \alpha Pdx}{da} + \beta z + K$. At B et F ex dato P facile reperiuntur.

§13. Si $\int Fdx$ quod autem rarissime evenit vel non amplius in se contineat a, vel ad aliud possit reduci, in quo a non insit, aequatio inventa differentialis secundi gradus pro legitima modulari poterit haberi. Sed si haec omnia nondum succedant, adhuc differentiatio est instituenda, in qua differentiale ipsius $\int Fdx$ erit $Fdx + da\int Hdx$ posito $dF = Gdx + Hda$. Quo facto videndum est vel an $\int Hdx$ re ipsa possit exhiberi, vel an pendeat a praecedentibus $\int Fdx$, $\int Bdx$ et $\int Pdx$, vel an possit ex signo summatorio a eliminari. Quorum si quod obtiget, habebitur aequatio modularis differentialis tertii gradus; sin vero nullum locum habuerit, continuanda est differentiatio simili modo donec signa summatoria potuerint eliminari.

§14. His generalibus praemissis ad specialia accedo, casus evoluturus, quibus functio P quodammodo determinatur. Sit igitur P functio ipsius x tantum, a prorsus non involuens, quam littera X designabo, erit ergo $dz = Xdx$, quae quidem aequatio quia non continet a, ad unicam videtur curvam pertinere, neque integratione constantem addere liceat, poterit esse $z = \int Xdx + na$ seu differentiando $dz = Xdx + nda$, quae est vera aequatio modularis. Eadem aequatio prodiisset, si iuxta regulam X differentiassem posito x constante, unde prodit $B = 0$ et $\int Bdx = n$ constanti, orta igitur esset aequatio modularis $dz = Xdx + nda$ cuius loco potius integralis $z = \int Xdx + na$ usurpatur.

§15. Sit nunc $P = AX$, existente A functione ipsius a, et X ipsius x tantum. Cum igitur sit $z = \int Pdx$ erit $z = \int AXdx$ seu quia in integratione a ut constans debet considerari,

$z = A\int Xdx$. Quae aequatione seu eius differentialis $Adz - zdA = A^2 Xdx$ erit aequatio modularis quaesita. Loco A quidem cum sit functio ipsius a tantum, poni potest ipse modulus a : nam loco moduli eius functio quaecunque eodem iure pro modulo haberi potest.

§16. Sit $P = A + X$ litteris A et X eosdem ut ante retinentibus valores. Erit ergo $dz = Adx + Xdx$ atque $z = Ax + \int Xdx$, quae aequatio iam est modularis; quia modulus A non est in signo summatorio involutus. Si quem autem $\int Xdx$ offendat, differentialem aequationem $dz = Adx + xdA + Xdx$ pro modulari habere potest.

§17. Simili ratione modularem aequationem invenire licet, si fuerit $P = AX + BY + CZ$ etc. ubi A, B, C sunt functiones quaecunque ipsius moduli a, et X, Y, Z functiones quaecunque ipsius x et constantium excepta a. Namque ob $dz = AXdx + BYdx + CZdx$ erit $z = A\int Xdx + B\int Ydx + C\int Zdx$, quae simul est modularis, cum modulus a nusquam post signum summatorium reperiatur.

§18. Sit $P = (A + X)^n$ seu $z = \int dz(A + X)^n$. Differentiale ipsius P posito x constante est $n(A + X)^{n-1} dA$ id quod per da divisum dat superiorem valorem B vid. §8. Erit igitur $dz = (A + X)^n dx + ndA\int (A + X)^{n-1} dx$ seu $\int dx(A + X)^{n-1} = \frac{dz-(A+X)^n dx}{ndA}$.
Cum igitur sit $\int dx(A + X)^n = z$, si haec duo integralia a se invicem pendeant, vel $\int dx(A + X)^{n-1}$ algebraice etiam exprimi poterit, habebitur quod quaeritur. Si neutrum contingat denuo differentiatio est instituenda. Est autem differerntiale ipsius $\int dx(A + X)^{n-1} = dx(A + X)^{n-1} + (n-1)dA\int (A + X)^{n-2} dx = \text{Diff.}\frac{dz-(A+X)^n dx}{ndA}$
Erit itaque $\int dx(A + X)^{n-2} = \frac{1}{(n-1)dA}\text{Diff.}\frac{dz-(A+X)^n dx}{ndA} - \frac{dx(A+X)^{n-1}}{(n-1)dA}$. Quare videndum est an $\int dx(A + X)^{n-2}$ possit vel integrari vel ad priora integralis reduci.

§19. Si n fuerit numerus integer affirmatius aequatio modularis erit algebraica. Nam $(A + X)^n$ potest in terminos numero finitos resolvi, quorum quisque in dx ductus integrari potest, ita ut modulus a in signum summatorium non ingrediatur. Erit autem aequatio modularis haec $z = A^n x + \frac{n}{1} A^{n-1}\int Xdx + \frac{n.n-1}{1.2} A^{n-2}\int X^2 dx$ etc.
Examinandum igitur restat quibus casibus si n non fuerit numerus integer affirmatiuus, supra memoratae conditiones locum habeant.

§20. Sit primo $X = bx^m$, ubi b etiam ab a pendere potest; erit ergo $x = \int (A + bx^m)^n dx$. Haec formula primo ipsa est integrabilis, si $m = \frac{1}{i}$, designante i numerum quemcunque affirmatiuum integrum : deinde etiam si $m = \frac{-1}{n+i}$. His igitur casibus aequatio modularis fit algebraica. At si $m = -\frac{1}{n}$, ubi *b* ab *a* non pendere potest illa quidem aequatio integrationem non admittit sed sequens $dz = (A + bx^{\frac{-1}{n}})^n dx + ndA\int dx(A + bx^{\frac{-1}{n}})^{n-1}$ evadit integrabilis, fitque aequatio modularis differentialis primi casus.

§21. Non solum autem, quicunque valor ipsi *m* tribuatur aequatio modularis differentialis primi gradus haberi potest, sed etiam si fuerit $z = \int x^m dx(A + bx^k)^n$. Fiet enim $dz = x^m dx(A + bx^k)^n + ndA\int x^m dx(A + bx^k)^{n-1}$. Sed est $\int x^m dx(A + bx^k)^n = \frac{x^{m+1}(A+bx^k)^n}{m+nk+1} + \frac{nkA}{m+nk+1}\int x^m dx(A + bx^k)^{n-1}$, seu $\int x^m dx(A + bx^k)^{n-1} = \frac{(m+nk+1)}{nkA} - \frac{x^{m+1}(A+bx^k)^n}{m+nk+1}$. Consequenter habebitur aequatio modularis haec $Akdz = (A + bx^k)^n(Akx^m dx - x^{m+1}dA) + (m + nk + 1)zdA$. Simili modo modularis esset inventa, si fuisset $z = B\int x^m dx(A + bx^k)^n$ alia enim non prodisset differentia nisi quod loco *z* scribi debuisset $\frac{z}{B}$, et loco dz, $\frac{Bdz-zdB}{B^2}$ si quidem *B* ab *a* etiam pendeat.

§22. Missis autem huiusmodi litterae P determinationibus, quippe quae minus late patent, ad alias accedo, quae multo saepius usui esse possunt. Continentur hae determinationes ea functionis cuiusdam propositae proprietate, qua functio eundem ubique tenet dimensionum quantitatum variabilium numerum. Tales enim functiones peculiari modo differentiationem admittunt. Ut sit *u* functio nullius dimensionis ipsarum *a* et *x*, cuiusmodi funt $\frac{a}{x}, \frac{\sqrt{(a^2-x^2)}}{a}$ aliaeque similes, in quibus ipsarum *a* et *x* dimensionum numerus in denominatore aequalis est numerus dimensionum numeratoris. Det autem talis functio *u* differentiata $Rdx + Sda$; dico fore $Rx + Sa = 0$. Nam si in functione *u* ponatur $x = ay$, omnia *a* sese destruent et in ea praeter *y* et constantes nulla alia littera remanebit. Hancobrem in differentiali post substitutionem aliud differentiale praeter *dy* non reperietur. Cum autem sit $x = ay$ erit $dx = ady + yda$, ideoque $du = Rady + Ryda + Sda$. Debebit ergo esse $Ry + S = 0$.

§23. Sin vero fuerit *u* functio *m* dimensionum ipsarum *a* et *x*, atque $du = Rdx + Sda;$ erit $\frac{u}{x^m}$ functio ipsarum *a* et *x* nullius dimensionis. Differentietur igitur $\frac{u}{x^m}$ et prodibit $\frac{xdu-mudx}{x^{m+1}}$ seu $\frac{Rxdx-mudx+Sxda.}{x^{m+1}}$ Quod cum sit differentiale functionis nullius dimensionis erit $Rx^2 - mux + Sax = 0,$ seu $Rx + Sa = mu.$ Quare si fuerit *u* functio *m* dimensionum ipsarum *a* et *x*; atque ponatur $du = Rdx + Sda;$ erit $Rx + Sa = mu$ ideoque $du = Rdx + \frac{da}{a}(mu - Rx)$ seu $adu = Radx - Rxda + muda.$

§24. His praemissis in $dz = Pdx$ seu $z = \int Pdx$ sit P functio n dimensionum ipsarum a et x, erit igitur z talis functio dimensionum $n + 1$. Quare si ponatur $dz = Pdx + Qda$, erit $Px + Qa = (n+1)z.$ Ex quo valor ipsius Q substitutus dabit aequationem modularem $dz = Pdx + \frac{da}{a}((n+1)z - Px)$ seu $adz - (n+1)zda = Padx - Pxda$. Quae tantum est differentialis primi gradus. Cum autem generaliter erat $Q = \int Bdx$, erit hoc casu $(n+1)\int Pdx = a\int Bdx + Px.$ Ex quo perspicitur hoc casu integrale $\int Bdx$ semper reduci ad $\int Pdx$.

§25. Eadem aequatio modularis proveniet ex consideratione solius P. Posito enim $dP = Adx + Bda$, erit $nP = Ax + Ba.$ Cum autem sit $dz = Pdx + da\int Bdx$, erit $dz = Pdx + \frac{da}{a}\int (nPdx - Axdx)$ in qua integratione a constans habetur. Erit igitur $\int nPdx = nz,$ et $\int Axdx = Px - \int Pdx$ ob $\int Adx = P.$ Habebitur itaque $dz = Pdx + \frac{da}{a}(n+1)z - Px$, id quod prorsus congruit cum praecedentibus.

§26. Retinente P suum valorem n dimensionum. Sit $z = \int APXdx$, ubi A sit functio ipsius a et X ipsius x tantum. Erit igitur $\frac{z}{A} = \int PXdx$. Posito $dP = Adx + Bda,$ (in quo littera A cum altera quae est functio ipsius a tantum non est confundenda) erit $nP = Ax + Ba.$ Ipsius PX differentiale igitur posito x constante erit $BXda$. Consequenter habebitur $d.\frac{z}{A} = PXdx + da\int BXdx = PXdx + \frac{da}{a}\int (nPXdx - AXxdx).$ Est vero $\int nPXdx = \frac{nz}{A}$ et $\int AXxdx = PXx - \int PXdx - \int PxdX.$ Quare fiet $d.\frac{z}{A} = PXdx - \frac{PXxda}{a} + \frac{(n+1)zda}{Aa} + \frac{da}{a}\int PxdX.$ Nisi igitur $\int PxdX$ reduci poterit ad $\int PXdx$ vel prorsus integrari, aequatio modularis differentialis primi gradus dari nequit.

§27. At si fuerit $z = R\int Pdx$, existente R functione quacunque algebraica ex a, x et etiam z constante, at P functione ipsarum a et x dimensionum n. Quia est $\frac{z}{R} = \int Pdx$ erit $d.\frac{z}{R} = Pdx + \frac{da}{a}(\frac{(n+1)z}{R} - Px) = \frac{Rdz - zdR}{R^2}$ seu $Radz - zadR - (n+1)Rzda = PR^2adx - PR^2xda.$ In universum autem teneatur, quoties $z = \int Pdx$ ad aequationem modularem reduci possit, toties etiam $z = R\int Pdx$ ad aequationem modularem reduci posse. Nullum aliud enim discrimen aderit, nisi quod in illo casu erat z, hoc casu debeat esse $\frac{z}{R}$. Quare si R fuerit vel quantitas algebraica, vel talis transcendens, ut eius differentiale posito etiam a variabili possit sine summatione exhiberi, aequatio modularis per praecepta data reperietur. Quamobrem in posterum tales casus, etiamsi latius pateant praetermittere licebit.

§28. Ponamus esse $z=\int(P+Q)dx,$ seu $z=\int Pdx+\int Qdx$ et P esse functionem ipsarum a et x dimensionum $n-1$, Q vero functionem earundem a et x dimensionum $m-1$. Cum igitur differentiale ipsius $\int Pdx$ sit $\frac{P(adx-xda)}{a}+\frac{da}{a}\int nPdx$ et differentiale ipsius $\int Qdx$ sit $\frac{Q(adx-xda)}{a}+\frac{da}{a}\int mQdx$; erit $dz=\frac{(P+Q)(adx-xda)}{a}+\frac{da}{a}(n\int Pdx+m\int Qdx).$ Ponatur $\frac{adz-(P+Q)(adx-xda)}{da}=u,$ eritque $u=n\int Pdx+m\int Qdx.$ Si igitur porro differentietur erit $du=\frac{(nP+mQ)(adx-xda)}{da}+\frac{da}{a}(n^2\int Pdx+m^2\int Qdx).$ Posito igitur $\frac{adu-(nP+mQ)(adx-xda)}{da}=t,$ erit $t=n^2\int Pdx+m^2\int Qdx.$ Eliminatis nunc ex his tribus aequationibus ipsarum $z,$ u et t integralibus $\int Pdx$ et $\int Qdx$, prodibit haec aequatio $mnz-(m+n)u+t=0.$ Quae aequatio, si loco u et t valores assumti substituantur, erit modularis quaesita.

§29. Simili modo si fuerit $z=\int(P+Q+R)dx,$ et P functio $n-1$, Q functio $m-1$ et R functio $k-1$ dimensionum ipsarum a et x. Ponatur $u=\frac{adz-(P+Q+R)(adx-xda)}{da}$ et $t=\frac{adu-(nP+mQ+kR)(adx-xda)}{da}$, et $s=\frac{adt-(n^2P+m^2Q+k^2R)(adx-xda)}{da}$. Quo facto erit aequatio modularis haec : $kmnz-(km+kn+mn)u+(k+m+n)t-s=0.$

§30. Sit porro $z=\int(P+Q)^k dx,$ ubi P sit functio n dimensionum, Q vero functio m dimensionum ipsarum a et x. Quando igitur est $dP=Adx+Bda$ et $dQ=Cdx+Dda$, erit $nP=Ax+Ba$ et $mQ=Cx+Da$. Differentiale autem ipsius $(P+Q)^k$ posito x constante divisum per da est $k(B+D)(P+Q)^{k-1}$. Hanc ob rem erit $dz=(P+Q)^k dx+\frac{kda}{a}\int(P+Q)^{k-1}(Ba+Da)dx.$ Cum autem sit $Ba=nP-Ax$ et $Da=mQ-Cx,$ et $Adx=dP$ et $Cdx=dQ$ ob a in hac integratione constans, erit $dz=(P+Q)^k dx+\frac{da}{a}\int(P+Q)^{k-1}(nPdx+mQdx-xdP-xdQ),$ seu $dz=\frac{(P+Q)^k(adx-xda)}{a}+\frac{da}{a}\int(P+Q)^{k-1}((nk+1)Pdx+(mk+1)Qdx).$ Ponatur $\frac{adz-(P+Q)^k(adx-xda)-zda}{kda}=u$ erit $u=\int(nPdx+mQdx)(P+Q)^{k-1}.$ Quare si integrale $\int(nPdx+mQdx)(P+Q)^{k-1}$ pendet ab integrali $\int(P+Q)^k dx$ habebitur aequatio modularis differentialis gradus primi; sin minus differentiatio est continuanda. Fit autem

$du = (nPdx + mQdx)(P+Q)^{k-1} + \frac{uda}{a} - \frac{da}{a}(nP + mQ)(P+Q)^{k-1}x +$
$\frac{da}{a}\int(kn^2P^2dx + (2kmn + n^2 - 2mn + m^2)PQdx + km^2Qdx)\int(P+Q)^{k-2}.$

Et posito

$t = \frac{adu - uda - (nP+mQ)(P+Q)^{k-1}(adx - xda)}{da}$

$t = \int(kn^2P^2dx + (2kmn + n^2 - 2mn + m^2)PQdx + km^2Qdx)(P+Q)^{k-2}$

§31. Cum igitur habeantur tria integralia videndum est, num ea a se invicem pendeant, hoc enim si fuerit, habebitur aequatio algebraica inter *t, u* et *z*, quae dabit loco *t* et *u* substitutis assumtis valoribus aequationem modularem differentialem secundi gradus. Quo autem facilius in casibus particularibus perspici possit, an pendeant a se invicem, ad alias formas eas reduci convenit. Cum igitur sit $z = \int(P+Q)^k dx,$ erit

$u = mz + (n-m)\int(P+Q)^{k-1}Pdx,$ et

$t = (2km + n - m)u - (km^2 - m^2 + mn)z + (n-m)^2(k-1)\int(P+Q)^{k-2}P^2dx.$

Quaerendum itaque est an $\int(P+Q)^{k-2}P^2dx$ reduci possit ad haec $\int(P+Q)^{k-1}Pdx$ et $\int(P+Q)^k dx$. Vel an sit

$\int(P+Q)^{k-2}P^2dx = \alpha\int(P+Q)^{k-1}Pdx + \beta\int(P+Q)^k Pdx + V$ designante *V* quantatem algebraicam quamcumque per *a* et *x* datam, et *α* ac *β* sunt coefficientes ex constantissimis et *a* compositae.

§32. Fiat igitur $V = T(P+Q)^{k-1}$ huis differentiale posito *a* constante sit

$V = dT(P+Q)^{k-1} + (k-1)(TdP + TdQ)(P+Q)^{k-1}.$ Prodibit ergo sequens aequatio

$P^2dx = \alpha P^2dx + \alpha PQdx + \beta P^2dx + 2\beta PQdx + \beta Q^2dx + PdT + QdT + (k-1)TdP + (k-1)TdQ,$

quae per dx dividi poterit. At T ita debet accipi, ut termini respondentes sese destruant, sumtis ad hoc idoneis pro *α* et *β* valoribus.

§33. At si per $\int Pdx$ non absolute determinetur *z* sed quantitas $\int Qdz$, data *Q* utcunque per *a* et *z*, atque *P* per *a* et *x*; habebitur ista aequatio $Qdz = Pdx$ in qua indeterminatae *x* et *z* sunt a se invicem separatae. Modularis vero aequatio hoc modo invenietur : Quia est $Qdx = Pdx$ differentietur utrumque membrum ponendo etiam *a* variabili ope $dP = Adx + Bda$ et $dQ = Cdz + Dda.$ Erit ergo

$Qdz + da\int Ddz = Pdx + da\int Bdx$ seu $Qdz = Pdx + da(\int Bdx - \int Ddz).$ Quae aequatio, si $\int Bdx$ et $\int Ddz$ poterunt eliminari, dabit modularem quaesitam.

§34. Sit P functio $m-1$ dimensionum ipsarum a et x, et Q functio $n-1$ dimensionem ipsarum a et z. His positis erit

$\text{Diff.}\int Pdx = \frac{mda\int Pdx + P(adx - xda)}{a}$, and $\text{Diff.}\int Qdz = \frac{nda\int Qdz + Q(adz - zda)}{a}$. Ex quo eruitur ista aequatio $(m\text{-}n)\int Pdx = \frac{Q(adz-zda)}{da} - \frac{P(adz-zda)}{da}$ ob $\int Pdx = \int Qdz$. Quare si fuerit $m = n$, erit $Qadz - Qzda = Padx - Pxda$, quae est aequatio modularis, seu $\frac{da}{a} = \frac{Qdz - Pdx}{Qz - Px}$.

§35. Sin vero m et n non sint aequales, aequatio modularis erit differentiales secundi gradus. Nam cum sit $(m\text{-}n)\int Pdx = \frac{Q(adx-zda)}{da} - \frac{P(adz-zda)}{da}$ erit

$$\text{Diff.}\frac{Q(adz-zda)}{da} - \frac{P(adx-xda)}{da} = \frac{m(m\text{-}n)da\int Pdx}{a} + \frac{(m\text{-}n)P(adx-xda)}{a}$$
$$= \frac{mQ(adz-zda) - nP(adx-xda)}{a}.$$

Quae aequatio est modularis quaesita.

§36. Si in aequatione proposita $dz + Pdx = 0$ indeterminatae non fuerint a se invicem separatae, ita ut P sit functio involvens x et z et a; debebit per quantitatem quandam R multiplicari, quo formula $Rdz + PRdx$ ut differentiale integralis cuiusdam S possit considerari. Erit itaque $dS = Rdz + PRdx = 0,$ ideoque S = Const. Sed ad quantitatem R inveniendam, sit $dP = Adx + Bdz$ et $dR = Ddx + Edz,$ ubi a tantisper pro constant habemus. His positis erit $\text{d.}PR = (DP + AR)dx + (EP + BR)dz,$ quocirca debet esse $D = EP + BR.$ At ob $D = \frac{dR - Edz}{dx}$ fiet $Edz + EPdx + BRdx = dR.$ Cum vero sit $dz + Pdx = 0,$ habebitur $dR = BRdx,$ et $lR = \int Bdx.$ Cognita vero est B ex dato P, et quia B et z et x involuit, Bdx integrari debet ope aequationis $dz + Pdx = 0$, si quidem fieri potest. Sit itaque $\int Bdx = K,$ erit $R = e^K$ posito $le = 1$.

§37. Cum igitur sit $dS = e^K dz + e^K Pdx = 0,$ ad aequationem modularem inveniendam sit $dK = Fdx + Gdz + Hda,$ eritque $de^K = e^K(Fdx + Gdz + Hda).$ Sumatur deinde integrale ipsius $e^K Hdz$ posito tantum z variabili, x vero et a constantibus, quo facto erit aequatio modularis $e^K dz + e^K Pdx + da\int e^K Hdz = 0,$ seu diviso per e^K haec $dz + Pdx + e^{-K}da\int e^K Hdz = 0.$ Alia aequatio modularis invenitur, posito $dP = Adx + Bdz + Cda,$ erit enim ipsius $e^K P$ differentiale posito x et z constante hoc $e^K(Cda + PHda)$. Integretur $e^K dx(C + PH)$ posito tantum x variabili, quo facto erit aequatio modularis $dz + Pdx + e^{-K}da\int e^K dx(C + PH) = 0.$ Sed huiusmodi aequationes nisi R possit sine aequatione proposita $dz + Pdx = 0$ determinari, nullius fere sunt usus.

§38. Consideremus igitur casus particulares, sitque in aequatione $dz + Pdx = 0, P$ functio nullius dimensionis ipsarum x et z, non computatis constantibus et modulo a. Formula vero $dz + Pdx$ integrabilis semper redditur si dividatur per $z + Px$,

quamobrem erit $S = \int \frac{dz+Pdx}{z+Px} = \text{Const.}$ Fit autem $\int \frac{dz+Pdx}{z+Px} = l(z+Px) - \int \frac{xdP}{z+Px}$.
Deinde posito $z = tx$, fiet P functio ipsius t tantum quae sit T. Quare erit
$S = l(z+Px) - \int \frac{dT}{t+T}$ quod per quadraturus potest exhiberi.

§39. Ad aequationem modularem igitur inveniendam nil aliud est agendum, nisi ut $\int \frac{dz+Pdx}{z+Px}$ differentietur posito quoque modulo a variabili. Ponatur igitur $dP = Adx + Bdz + Cda$, ubi erit $Ax + Bz = 0.$ Differentietur nunc coefficiens ipsius dx, nempe $\frac{P}{z+Px}$ posito tantum a variabili, erit eius differentiale $\frac{Czda}{(z+Px)^2}$. Deinde integretur $\frac{Czdz}{(z+Px)^2}$ tantum x pro variabili habia, quo facto erit aequatio modularis quaesita $dz + Pdx + (z+Px)da\int \frac{Czdx}{(z+Px)^2} = 0$. Simili modo ex coefficiente ipsius dz qui est $\frac{1}{z+Px}$ prodit haec aequatio modularis $dz + Pdx - (z+Px)da\int \frac{Cxdz}{(z+Px)^2} = 0$, in qua integratione z tantum pro variabili habetur. Sive etiam haec
$dz + Pdx = (z+Px)da\int \frac{Cdt}{(t+T)^2}$ in qua C et T per solum t et a dantur.

§40. Praemittere hic non possum, quin generalem aequationum homogenearum, uti a *Cel. Ioh. Bernoulli* vocantur, quae omnes hac aequatione $dz + Pdx = 0$ continentur, resolutionem adiiciam. Namque reperitur ex (§. 38)
$l(z+Px) = \int \frac{dT}{t+T} = l(t+T) - \int \frac{dt}{t+T}$ ubi $= \frac{z}{x}$ et $T = P.$ Prodibit igitur $lx + \int \frac{dt}{t+T} = 0$ seu adiecta constante $l\frac{c}{x} = \int \frac{dt}{t+T}$. Ut si proposita sit aequatio
$nxdz + dx\sqrt{(x^2+z^2)} = 0$ fiet $P = \frac{\sqrt{(x^2+z^2)}}{nx}$, positque $z = tx$, erit $T = \frac{(1+tt)}{n}$ ideoque
$l\frac{c}{x} = \int \frac{ndt}{nt+\sqrt{(1+tt)}}$ fiat $\sqrt{(1+tt)} = t+s$ erit $t = \frac{1-ss}{2s}$ et $dt = \frac{-ds(1+ss)}{2ss}$. Quare erit
$l\frac{c}{x} = \int \frac{-nds(1+ss)}{(n+1)s-(n-1)s^3} = \frac{-n}{n+1}ls + \frac{n^2}{n^2-1}l((n-1)s^2-n-1).$

§41. Quo tamen usus calculi §36 in casu speciali appareat, sit aequatio proposita $dz + pzdx - qdx = 0,$ in qua p et q utcunque in a et x dantur. Quae aequatio cum illa generali $dz + Pdx = 0$ collata dat $P = pz - q,$ ex quo fiet $B = p$, et $lR = \int pdx$, seu $R = e^{\int pdx}$. Cum igitur $\int pdx$ per quadraturas possit assignari, cognitus est valor ipsius R, ideoque aequatio proposita per $e^{\int pdx}$ multiplicata sit integrabilis : erit igitur
$e^{\int pdx}dz + e^{\int pdx}pzdx - e^{\int pdx}qdx = 0$ huius integralis $e^{\int pdx}z$
$= \int e^{\int pdx}qdx$ seu $z = e^{-\int pdx}\int e^{\int pdx}qdx$. Differentiari itaque debet
$e^{-\int pdx}\int e^{\int pdx}qdx$ positis et a et x variabilibus, et differentiale ipsi dz aequale poni, quo facto habebitur aequatio modularis. Positis igitur

$dp = fdx + gda$ et $dq = hdx + ida$ prodibit ista aequatio modularis

$$dz = -e^{-\int pdx}(pdx + da\int gdx)\int e^{\int pdx}qdx + qdx + e^{-\int pdx}da\int e^{\int pdx}(idx + qdx\int gdx),$$

seu posito brevitatis gratia $\int e^{\int pdx}qdx = T$ erit

$$dz = -e^{-\int pdx}Tpdx + qdx + e^{-\int pdx}da\int e^{\int pdx}idx - e^{-\int pdx}da\int Tgdx.$$

Ex qua operatione intelligi potest, ad aequationem modularem inveniendam id maxime esse efficiendum, ut in aequatione proposita indeterminatae a se invicem separentur.