PROBLEMA ALGEBRAICVM OB AFFECTIONES PRORSVS SINGVLARES MEMORABILE

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Problema, cuius affectiones hic contemplandas, ita se habet:

Inueniere nouem numeros

A, B, C,
D, E, F,
G, H, I

ita in quadratum disponendos, vt satisfiat duodecim sequentibus conditionibus:

1°. AA+DD+GG = 1
2°. BB+EE+HH = 1
3°. CC+FF+II = 1
4°. AB+DE+GH = 0
5°. AC+DF+GI = 0
6°. BC+EF+HI = 0
7°. AA+BB+CC = 1
8°. DD+EE+FF = 1
9°. GG+HH+II = 1
10°. AD+BE+CF = 0
11°. AG+BH+CI = 0
12°. DG+EH+FI = 0

Circa hoc problema sequentia obseruo.

I. Cum numerus conditionum implendarum superet numerum quantitatum determinandarum, problema hoc plusquam determinatum videtur. Vtcunque enim conditiones praescriptae perpendantur, nulla alio relatio, qua aliquae in reliquis iam contineantur, in iis deprehenditur, nisi quod summa conditionum 7°, 8°, 9° conueniat cum summa conditionum 1°, 2°, 3°; vnde vnica harum duodecim conditionum in reliquis iam contineri videtur; qua remota tamen adhuc vndecim conditiones reliquuntur, quae binario numerum quantitatum incognitarum excedunt. Hic equidem tantum de eiusmodi relatione loquor, quae has conditiones consideranti occurrit, reuera enim aliquot necessariae relationes inter eas intercedunt, quae autem vix ante animaduertuntur, quam problema perfecte fuerit solutum.

II. Deinde obseruo hoc problema non solum non esse plusquam determinatum, sed adeo esse indeterminatum, ita vt nouem numerorum quesitorum tres pro lubitu accipere liceat, nihiloque minus omnibus conditionibus praescriptis satisfieri queat. Dummodo enim sex prioribus conditionibus fuerit satisfactum, relique sex sponte implentur atque omnino fieri non potest, vt sex prioribus satisfiat quin simul omnibus satisfiat. Quocirca problema propositum eiusdem prorsus indolis maneret etiamsi sex posteriores conditiones plane omitterentur; ac tum ei insigne Theorema istud adiungi posset.

Quodsi nouem numeri A, B, C, D, E, F, G, H, I *ita fuerint comparati, vt 6 prioribus conditionibus satisfaciant tum etiam necessario sex posterioribus satisfaciant.*

Quod Theorema pro difficillimo demonstratum venditare non dubito, neque video quomodo demonstratio adornari queat, nisi solutio problematis fuerit explorata.

III. Neque vero hoc problema pro otiosa speculatione seu mero lusu ingenii est habendum, sed potius in doctrina de superficierum natura est maximi momenti. Cum enim natura superficiei per aequationem inter ternas coordinatas tribus axibus inter se normalibus parallelas exprimi soleat, talis aequatio mutandis axibus in infinitum variari potest, etiamsi axium communis intersectio in eodem puncto statuatur. Quoniam igitur eadem superficies infinitis aequationibus diuersis inter ternas coordinatas definiri potest, plurimum interest earum characterem communem nosse, qui in eo consistit, vt si coordinatae ternis quibusdam axibus datis parallelae sint x, y, z; quae autem aliis quibuscunque axibus constituuntur parallelae, fuerint X, Y, Z eorum relatio mutua semper huiusmodi formulis contineatur:

$$\mathrm{X} = \mathrm{A}x + \mathrm{B}y + \mathrm{C}z; \qquad \mathrm{Y} = \mathrm{D}x + \mathrm{E}y + \mathrm{F}z; \qquad \mathrm{Z} = \mathrm{G}x + \mathrm{H}y + \mathrm{I}z,$$

qui nouem coëfficientes ita comparati sint necesse est, vt inde fiat: $\mathrm{XX} + \mathrm{YY} + \mathrm{ZZ} = xx + yy + zz$, quandoquidem his formulis quadratum interualli quo superficiei punctum ab initio coordinatarum distat, exprimitur. Quod fieri nequit, nisi hae sex aequationes habeant locum:

$$\begin{aligned} &\mathrm{AA} + \mathrm{DD} + \mathrm{GG} = 1, \\ &\mathrm{BB} + \mathrm{EE} + \mathrm{HH} = 1, \\ &\mathrm{CC} + \mathrm{FF} + \mathrm{II} = 1, \\ &\mathrm{AB} + \mathrm{DE} + \mathrm{GH} = 0, \\ &\mathrm{AC} + \mathrm{DF} + \mathrm{GI} = 0, \\ &\mathrm{BC} + \mathrm{EF} + \mathrm{HI} = 0 \end{aligned}$$

quae sunt ipsae sex priores conditiones nostri problematis.

IV. Quocunque autem modo hoc problema secundum Algebrae praecepta tentetur, ob tantum incognitarum numerum semper ad calculos vehementer intricatos peruenitur, ex quibus neutiquam solutionem commodam expectare liceat. Theoriam quidem angulorum in subsidium vocando, haud difficulter solutio satis concinna obtinetur, verum haec methodus vix ad alias huius generis quaestiones magis complicatas traduci poterit: veluti si circa 16, 25, 36 etc. numeros, pariter in quadratum disponendos similis quaestio instituatur, vt summa quadratorum per singulas columnas tam verticales quam horizontales sumtorum vnitati aequetur, simul vero summae productorum secundum binas columnas itidem tam verticales quam horizontales ad nihilum redigantur. Methodum ergo etiam ad has quaestiones patentem, quae vtique in Analysi maximi momenti est putanda deinceps sum expositurus, postquam demonstrationem Theorematis II. memorati, atque solutionem problematis initio propositi ope sinuum et cosinuum tradidero.

Demonstratio Theorematis §.II. propositi.

V. Assumo ergo nouem numeros nostros A, B, C, D, E, F, G, H, I ita esse comparatos vt sit

$$1^\circ.\quad AA+DD+GG = 1$$
$$2^\circ.\quad BB+EE+HH = 1$$
$$3^\circ.\quad CC+FF+II = 1$$
$$4^\circ.\quad AB+DE+GH = 0$$
$$5^\circ.\quad AC+DF+GI = 0$$
$$6^\circ.\quad BC+EF+HI = 0$$

quarum tres posteriores ita repraesento:

$$4^\circ.\quad AB = -DE - GH$$
$$5^\circ.\quad AC = -DF - GI$$
$$6^\circ.\quad BC = -EF - HI$$

vnde concludo fore:

$$\frac{4^\circ \cdot 5^\circ}{6^\circ} \cdots = \frac{AA\ BC}{BC} = AA = -\frac{(DE+GH)(DF+GI)}{EF+HI}$$

qui valor ipsius AA in prima aequatione positus dat:

$$-(DE + GH)\ (DF + GI) + (EF + HI)\ (DD + GG) = EF + HI$$

factaque euolutione:

$$-\ DEGI\ - DFGH + DDHI + EFGG = EF + HI$$

cuius aequationis primum membrum manifesto in hos factores resoluitur:

$$(DH - EG)\ (DI - FG) = EF + HI.$$

VI. Cum igitur sit EF + HI = −BC, erit

$$BC = (EG - DH)\,(DI - FG)$$

similique modo colligetur fore

$$AC = (FH - EI)\,(EG - DH) \qquad \text{et} \qquad AB = (DI - FG)\,(FH - EI),$$

quarum duarum posteriorum productum per primam diuisum praebet

$$AA = (FH - EI)^2 \qquad \text{hincque} \qquad A = \pm(FH - EI)$$

quia autem singulos numeros tam negatiue quam positiue capere licet, ambiguitas signi nullam variationem inferre est censenda, vnde sumto superiori habebimus:

$$A = FH - EI;\ B = DI - FG;\ C = EG - DH.$$

Cum autem ex rei natura columnas verticales inter se permutare liceat, hinc per analogiam concludimus fore

$$D = BI - CH;\ E = CG - AI;\ F = AH - BG;$$
$$G = CE - BF;\ H = AF - CD;\ I = BD - AE.$$

VII. En ergo nouem nouas determinationes, quae in sex conditionibus praescriptis necessario inuoluuntur, et quas insuper ad 12 conditiones initio propositas adiicere potuissemus. Verum hae ipsae nouem determinationes, quas sequenti modo indicabo:

13°. A=FH − EI
14°. B=DI − FG
15°. C=EG − DH
16°. D=BI − CH
17°. E=CG − AI
18°. F=AH − BG
19°. G=CE − BF
20°. H=AF − CD
21°. I=BD − AE

facile ad conditiones sex posteriores initio propositas deducunt. Nam formulae 13° per D, 14° per E et 15° per F multiplicatae et in vnam summam collectae dant:

$$AD + BE + CF = + DFH + DEI + EFG - DEI - EFG - DFH = 0,$$

quae est ipsa conditio 10° initio proposita, similique modo 13°. G + 14°. H + 15°. I dabit conditionem 11° et 16°. G + 17°. H + 18°. I conditionem 12° ita vt sit:

10°. AD + BE + CF = 0
11°. AG + BH + CI = 0
12°. DG + EH + FI = 0.

VIII. Denique si in formula no. 13° valores literarum E et F ex formulis 17° et 18° substituantur, emergit haec aequatio:

$$\mathrm{A} = \mathrm{AHH} - \mathrm{BGH} - \mathrm{CGI} + \mathrm{AII} = \mathrm{A}\,(\mathrm{HH}+\mathrm{II}) - \mathrm{G}\,(\mathrm{BH}+\mathrm{CI})$$

at ex aequatione 11° est BH + CI = − AG, vnde colligitur: A = A (GG + HH + II), ideoque vel A = 0 vel GG + HH + II = 1. Cum autem simili modo ex formulis 14°, 15°, 16°, 17°, et 18° eliciantur aequationes:

$$\mathrm{B} = \mathrm{B}\,(\mathrm{GG}+\mathrm{HH}+\mathrm{II});\ \mathrm{C} = \mathrm{C}\,(\mathrm{GG}+\mathrm{HH}+\mathrm{II});\ \mathrm{D} = \mathrm{D}\,(\mathrm{GG}+\mathrm{HH}+\mathrm{II});$$

E = E (GG + HH + II) et F = F (GG + HH + II) neque litterae A, B, C, D, E, F omnes simul euanescant, necesse est sit GG + HH + II = 1 quae est conditio 9° hocque modo ostenditur esse:

$$\begin{aligned} 7^\circ.&\quad \mathrm{AA}+\mathrm{BB}+\mathrm{CC}=1;\\ 8^\circ.&\quad \mathrm{DD}+\mathrm{EE}+\mathrm{FF}=1;\\ 9^\circ.&\quad \mathrm{GG}+\mathrm{HH}+\mathrm{II}=1 \end{aligned}$$

quae est demonstratio completa theorematis propositi.

Solutio Problematis initio propositi.

IX. Statuamus $\mathrm{A} = \cos\zeta$, et cum conditiones 1° et 7° praebeant:

$$\mathrm{DD}+\mathrm{GG} = \sin^2\zeta, \qquad \text{et} \qquad \mathrm{BB}+\mathrm{CC} = \sin^2\zeta$$

his ingenere satisfaciemus ponendo: $\mathrm{B} = \sin\zeta\cos\eta$; $\mathrm{C} = \sin\zeta\sin\eta$; $\mathrm{D} = \sin\zeta\cos\theta$; $\mathrm{G} = \sin\zeta\sin\theta$. Considerentur iam conditiones 17° et 21° quae factis his substitutionibus induent has formas:

$$\begin{aligned} 17^\circ.&\quad \mathrm{E} = \sin^2\zeta\sin\eta\sin\theta - \mathrm{I}\cos\zeta \qquad \text{seu} \qquad \mathrm{E}+\mathrm{I}\cos\zeta = \sin^2\zeta\sin\eta\sin\theta\\ 21^\circ.&\quad \mathrm{I} = \sin^2\zeta\cos\eta\cos\theta - \mathrm{E}\cos\zeta \qquad \text{seu} \qquad \mathrm{I}+\mathrm{E}\cos\zeta = \sin^2\zeta\cos\eta\cos\theta \end{aligned}$$

Hinc (17° − 21° . $\cos\zeta$) et (21° − 17°. $\cos\zeta$) dant:

$$\mathrm{E}(1-\cos^2\zeta) = \sin^2\zeta(\sin\eta\sin\theta - \cos\zeta\cos\eta\cos\theta)$$

$$\mathrm{I}(1-\cos^2\zeta) = \sin^2\zeta(\cos\eta\cos\theta - \cos\zeta\sin\eta\sin\theta)$$

vnde colligitur: $\mathrm{E} = \sin\eta\sin\theta - \cos\zeta\cos\eta\cos\theta$ and $\mathrm{I} = \cos\eta\cos\theta - \cos\zeta\sin\eta\sin\theta$.

X. Simili modo conditiones 18° et 20° modo ante demonstratae, dactis substitionibus suppeditant has aequationes:

$$\begin{aligned} 18^\circ.&\quad \mathrm{F}=\mathrm{H}\cos\zeta - \sin^2\zeta\cos\eta\sin\theta \qquad \text{seu} \qquad \mathrm{F}-\mathrm{H}\cos\zeta = -\sin^2\zeta\cos\eta\sin\theta\\ 20^\circ.&\quad \mathrm{H}=\mathrm{F}\cos\zeta - \sin^2\zeta\sin\eta\cos\theta \qquad \text{seu} \qquad \mathrm{H}-\mathrm{F}\cos\zeta = \sin^2\zeta\sin\eta\cos\theta \end{aligned}$$

vnde formae $(18^\circ + 20^\circ\ .\ \cos\zeta)$ et $(20^\circ + 18^\circ.\ \cos\zeta)$ producunt

$$\mathrm{F}(1-\cos^2\zeta) = -\sin^2\zeta(\cos\eta\sin\theta + \cos\zeta\sin\eta\cos\theta)$$

$$\mathrm{H}(1-\cos^2\zeta) = -\sin^2 z(\sin\eta\cos\theta + \cos\zeta\cos\eta\sin\theta)$$

vnde ob $1-\cos^2\zeta = \sin^2\zeta$ elicitur

$$\mathrm{F} = -\cos\eta\sin\theta - \cos\zeta\sin\eta\cos\theta \qquad \text{et} \qquad \mathrm{H} = -\sin\eta\cos\theta - \cos\zeta\cos\eta\sin\theta$$

sicque nouem numeri conditionibus praescriptis satisfacientes ita sunt definiti, vt tres anguli z, h, q arbitrio nostro relinquantur, in quo criterium solutionis cernitur.

XI. Solutio ergo completa nostri problematis ita se habet, vt nouem numeri quaesiti sequentes sortiantur valores:

$$\begin{array}{lll} \mathrm{A} = \cos\zeta & \mathrm{B} = \sin\zeta\cos\eta & \mathrm{C} = \sin\zeta\sin\eta \\ \mathrm{D} = \sin\zeta\cos\theta & \mathrm{E} = \sin\eta\sin\theta - \cos\zeta\cos\eta\cos\theta & \mathrm{F} = -\cos\eta\sin\theta - \cos\zeta\sin\eta\cos\theta \\ \mathrm{G} = \sin\zeta\sin\theta & \mathrm{H} = -\sin\eta\cos\theta - \cos\zeta\cos\eta\sin\theta & \mathrm{I} = \cos\eta\cos\theta - \cos\zeta\sin\eta\sin\theta \end{array}$$

quibus valoribus non solum sex conditiones priores, quibus problema determinatur, sed etiam sex posteriores, atque adeo etiam nouem nouae §.VII exhibitae, adimplentur. Haecque solutio istum praestat vsum, vt inde facili negotio solutiones in numeris rationalibus, quotcunque libuerit, reperire liceat, tres scilicet angulos ζ, η, θ ita capi opus est, vt eorum tam sinus quam cosinus rationaliter exprimantur. Hinc solutio satis simplex prodibit sumendo $\cos\zeta = 3/5$; $\sin\zeta = 4/5$; $\cos\eta = 3/5$; $\sin\eta = 4/5$; $\cos\theta = 5/13$; $\sin\theta = 12/13$.

Methodus Generalis huiusmodi problemata resoluendi.

XII. Methodus generalis, quam hic sum traditurus, ex principio supra §.III memorato est petita, vbi ostendi problema propositum eo redire, vt ex ternis variabilibus x, y, z aliae tres X, Y, Z per huiusmodi formulas $\alpha x+\beta y+\gamma z$ ita determinentur, vt fiat $\mathrm{X}^2 + \mathrm{Y}^2 + \mathrm{Z}^2 = x^2+y^2+z^2$, haecque determinatio maxime sit generalis; tum enim coëfficientes trium harum formularum $\alpha x + \beta y + \gamma z$ pro nouis variabilibus X, Y, Z resultantium, erunt ipsi illi nouem numeri, qui in problemate desiderantur. Hic igitur duae conditiones probe sunt perpendendae, quarum altera est, vt valores ipsarum X, Y, Z simpliciter per huiusmodi formulas $\alpha x + \beta y + \gamma z$ exprimantur, altera vero vt tum fiat $\mathrm{X}^2 + \mathrm{Y}^2 + \mathrm{Z}^2 = x^2 + y^2 + z^2$. Nisi enim illa conditio adesset, quaestio foret per methodum Diophanteam solutu facilis, dum tantum trium quadratorum summa in tria alia quadrata resolui deberet, id quod nihil habet difficultatis.

XIII. Quoniam vero rem eo deducere animus est, vt methodus ad quaestiones continuo magis complicatas extendi queat, a casu simplicissimo exordiar, quo propositis tantum duabus variabilibus x et y, ex iis aliae duae X et Y per huiusmodi formulas $\alpha x + \beta y$ definiri debeant vt fiat $\mathrm{X}^2 + \mathrm{Y}^2 = x^2 + y^2$. Hunc in finem posito

$$\mathrm{X} = \alpha x + \beta y \qquad \text{et} \qquad \mathrm{Y} = \gamma x + \delta y$$

necesse est, fiat:

$$\alpha\alpha + \gamma\gamma = 1; \quad \beta\beta + \delta\delta = 1; \quad \alpha\beta + \gamma\delta = 0.$$

Statuamus ergo $\alpha = \cos\zeta$ et $\beta = \cos\eta$, vt habeatur $\gamma = \sin\zeta$ et $\delta = \sin\eta$, sicque duabus prioribus conditionibus satisfiat: tum vero tertia dabit $\cos\zeta\cos\eta + \sin\zeta\sin\eta = \cos(\zeta-\eta) = 0$, ex quo erit $\zeta - \eta = 90°$, ideoque $\eta = \zeta - 90°$, ac propterea $\cos\eta = \sin\zeta$ et $\sin\eta = -\cos\zeta$. Vnde patet si capiatur:

$$\mathrm{X} = x\cos\zeta + y\sin\zeta \qquad \text{et} \qquad \mathrm{Y} = x\sin\zeta - y\cos\zeta$$

fore $\mathrm{X}^2 + \mathrm{Y}^2 = x^2 + y^2$.

XIV. Hoc lemmate praemisso ex propositis tribus variabilibus x, y, z primo alias tres x', y', z' ita definio vt sit

$$x' = x\cos\zeta + y\sin\zeta; \qquad y' = x\sin\zeta - y\cos z; \qquad z' = z$$

hoc enim modo certo erit

$$x'x' + y'y' + z'z' = xx + yy + zz.$$

Deinde ex his simili modo alias x'', y'', z'' deduco, vt sit

$$x'' = x'; \qquad y'' = y'\cos\eta + z'\sin\eta; \qquad z'' = y'\sin\eta - z'\cos\eta$$

atque hinc tandem quaesitas X, Y, Z ita definio:

$$\mathrm{X} = z''\cos\theta + x''\sin\theta; \qquad \mathrm{Y} = y''; \qquad Z = z''\sin\theta - x''\cos\theta$$

sic enim vtique fiet:

$$\mathrm{X}^2 + \mathrm{Y}^2 + \mathrm{Z}^2 = x''x'' + y''y'' + z''z'' = x'x' + y'y' + z'z' = xx + yy + zz.$$

XV. Ex hac autem triplici positione sequitur fore:

$$x'' = x\cos\zeta + y\sin\zeta; \qquad y'' = x\sin\zeta\cos\eta - y\cos\zeta\cos\eta + z\sin\eta;$$
$$z'' = x\sin\zeta\sin\eta - y\cos\zeta\sin\eta - z\cos\eta$$

tum vero

$$\begin{aligned}
\mathrm{X} &= x(\sin\zeta\sin\eta\cos\theta + \cos\zeta\sin\theta) - y(\cos\zeta\sin\eta\cos\theta - \sin\zeta\sin\theta) - z\cos\eta\cos\theta \\
\mathrm{Y} &= x\sin\zeta\cos\eta - y\cos\zeta\cos\eta + z\sin\eta \\
\mathrm{Z} &= x(\sin\zeta\sin\eta\sin\theta - \cos\zeta\cos\theta) - y(\cos\zeta\sin\eta\sin\theta + \sin\zeta\cos\theta) - z\cos\eta\sin\theta
\end{aligned}$$

quae formulae cum ante inuentis conueniunt.

XVI. Hanc solutionem esse generalem vel inde patet, quod ea complectatur tres angulos arbitrarios ζ, η, θ, qui per tres transformationes quas instituimus, sunt introducti. Vis enim huius methodi in hoc consistit, vt quauis transformatione duae tantum quantitates varientur, dum scilicet in earum locum duae aliae vna cum angulo arbitrario introducuntur, tertia manente immutata. Hinc duae operationes iam quidem solutionem problematis suppeditant, sed nondum completam, ob defectum vnius quantitatis arbitrariae. Quamobrem tot transformationes institui oportet donec tot huiusmodi quantitates arbitrariae fuerint ingressae quot ad maximam solutionis extensionem requiruntur. Supra autem iam obseruaui, cum quaestio circa nouem numeros versetur ac tantum sex conditiones praescribantur, tres eorum manere indeterminatos, quemadmodum etiam in solutione hic data ob angulos ζ, η, θ arbitrio nostro relictos, tres numeri A, B, D pro lubitu accipi possunt.

XVII. Hinc autem dubium nasci posset, quod cum qualibet transformatione nouus angulus introducatur, aucto transformationum numero nostri problematis solutio multo adhuc generalior obtineri posset. Verum tamen qui huius rei periculum facere voluerit, mox animaduertet, nouum angulum introductum cum aliquo praecedentium in vnum coalescere ita vt quotcunquae transformationes suscipiantur, numerus angulorum vere arbitrariorum non vltra ternarium augeri queat. Adiiciamus enim insuper hanc transformationem ponendo:

$$\mathrm{X}' = \mathrm{X}; \quad \mathrm{Y}' = \mathrm{Y}\cos\lambda - \mathrm{Z}\sin\lambda; \quad \mathrm{Z}' = \mathrm{Y}\sin\lambda + \mathrm{Z}\cos\lambda$$

fietque

$$\begin{aligned}
\mathrm{X}' &= x(\sin\zeta\sin\eta\cos\theta + \cos\zeta\sin\theta) + y(\sin\zeta\sin\theta - \cos\zeta\cos\eta\cos\theta) - z\cos\eta\cos\theta \\
\mathrm{Y}' &= x(\sin\zeta\cos\eta\cos\lambda - \sin\zeta\sin\eta\sin\theta\sin\lambda + \cos\zeta\cos\theta\sin\lambda \\
&- y(\cos\zeta\cos\eta\cos\lambda - \cos\zeta\sin\eta\sin\theta\sin\lambda - \sin\zeta\cos\theta\sin\lambda \\
&+ z(\sin\eta\cos\lambda + \cos\eta\sin\theta\sin\lambda) \\
\mathrm{Z}' &= x(\sin\zeta\cos\eta\sin\lambda + \sin\zeta\sin\eta\sin\theta\cos\lambda - \cos\zeta\cos\theta\cos\lambda \\
&- y(\cos\zeta\cos\eta\sin\lambda + \cos\zeta\sin\eta\sin\theta\sin\lambda - \sin\zeta\cos\theta\cos\lambda \\
&+ z(\sin\eta\sin\lambda - \cos\eta\sin\theta\cos\lambda)
\end{aligned}$$

vbi etsi quatuor anguli adsunt ζ, η, θ et λ, tamen inde non plures tribus coëfficientes pro lubitu assignare licet: quod quidem non facile perspicitur, et nonnisi per plures ambages ostendi posse videtur: cum tamen ex rei natura res sit prorsus manifesta.

XVIII. Etiam maxime arduum videatur has quatuor quantitates indeterminatas a tres reuocare haecque inuestigatio omnino singulares calculi euolutiones postulet, tamen ratio in eo sita haud difficulter deprehenditur, quod bis inter easdem quantitates cognomines y et z transformatio sit instituta. Scilicet in secunda quantitates y', z' in y'', z'' ope anguli η et in quarta quantitates cognomines Y et Z ope anguli λ in Y' et Z' sunt transformatae. Quae duae transformationes si immediate se exciperent ponendo exempli gratia

$$\text{primum} \quad y' = y\cos\zeta + z\sin\zeta; \quad z' = y\sin\zeta - z\cos\zeta$$

tum vero

$$y'' = y'\cos\eta + z'\sin\eta; \quad z'' = y'\sin\eta - z'\cos\eta$$

coniunctim prodiret:

$$y'' = y\cos(\zeta - \eta) + z\sin(\zeta - \eta) \quad \text{et} \quad z'' = -y\sin(\zeta - \eta) + z\cos(\zeta - \eta)$$

sicque duplex illa transformatio manifesto vnicae ope anguli $\zeta - \eta$ factae aequiualeret. Quod etiam euenire est intelligendum, etiamsi huiusmodi binae transformationes inter quantitates cognomines non immediate se excipiant.

XIX. Hinc cum quaelibet transformatio inter duas tantum quantitates variabiles instituatur, hanc regulam stabiliri conuenit, vt hae transformationes semper inter binas variabiles diuersi nominis suscipiantur; quo pacto numerus transformationum ita determinatur, vt plures forent inutiles. Ita cum in nostro problemate tres habeantur quantitates variabiles litteris x, y, z indicatae, plures quam tres transformationes locum habere nequeunt, dum vna inter x et y, alia inter x et z, et tertia inter y et z instituitur hoc modo

$$\begin{array}{lll} x' = x\cos\zeta + y\sin\zeta & x'' = x'\cos\eta + z'\sin\eta & x''' = x'' \\ y' = x\sin\zeta - y\cos\zeta & y'' = y' & y''' = y''\cos\theta + z''\sin\theta \\ z' = z & z'' = x'\sin\eta - z'\cos\eta & z''' = y''\sin\theta - z''\cos\theta \end{array}$$

vbi in prima quantitas nominis z, in secunda nominis y, in tertia vero nominis x inuariata relinquitur.

XX. Hanc regulam obseruantes methodum hanc per istiusmodi transformationes procedentem facile ad eiusmodi problemata accomodare poterimus, quibus plures quam tres quantitates variabiles proponuntur, quas simili modo in alias totidem transformari oporteat, vt quadratorum summa maneat eadem. Pluribus scilicet transformationibus inter binas tantum instituendis opus erit, vbi tantum erit cauendum, ne inter binas cognomines bis transformatio instituatur. Quo obseruato, solutio non ante erit completa, quam inter omnes binas diuersi nominis tales transformationes fuerint absolutae cuiusmodi diuersae combinationes habebuntur sex, si quatuor propositae sint quantitates, decem vero si quinque et ita porro. Cuiusmodi problemata aliquot cum solutionibus hic subiungam.

Problema.

Quatuor quantitates v, x, y, z ita in alias per huiusmodi formulas $\alpha v + \beta x + \gamma y + \delta z$ transformare, vt summa quadratorum maneat eadem vel ponendo

$$\begin{array}{ll} \mathrm{V} = \mathrm{A}v + \mathrm{B}x + \mathrm{C}y + \mathrm{D}z; & \mathrm{Y} = \mathrm{I}v + \mathrm{K}x + \mathrm{L}y + \mathrm{M}z \\ \mathrm{X} = \mathrm{E}v + \mathrm{F}x + \mathrm{G}y + \mathrm{H}z; & \mathrm{Z} = \mathrm{N}v + \mathrm{O}x + \mathrm{P}y + \mathrm{Q}z \end{array}$$

hos sedecim coëfficientes ita definire vt fiat

$$\mathrm{VV} + \mathrm{XX} + \mathrm{YY} + \mathrm{ZZ} = vv + xx + yy + zz$$

quem in finem sequentibus 10 conditionibus satisfieri oportet:

$$
\begin{array}{ll}
1^\circ. & \mathrm{AA + EE + II + NN} = 1 \\
2^\circ. & \mathrm{BB + FF + KK + OO} = 1 \\
3^\circ. & \mathrm{CC + GG + LL + PP} = 1 \\
4^\circ. & \mathrm{DD + HH + MM + QQ} = 0 \\
5^\circ. & \mathrm{AB + EF + IK + NO} = 0 \\
6^\circ. & \mathrm{AC + EG + IL + NP} = 0 \\
7^\circ. & \mathrm{AD + EH + IM + NQ} = 1 \\
8^\circ. & \mathrm{BC + FG + KL + OP} = 1 \\
9^\circ. & \mathrm{BD + FH + KM + OQ} = 1 \\
10^\circ. & \mathrm{CD + GH + LM + PQ} = 0
\end{array}
$$

XXI. Cum hic sedecim numeri ex 10 conditionibus inueniendi proponantur, euidens est eorum sex arbitrio nostro relinqui, seu quod eodem redit solutionem completam sex quantitates arbitrarias complecti debere.

Methodum autem ante expositam sequentes reuera solutionem sex transformationibus absolui deprehendimus, quae ita repraesentari possunt:

$$
\begin{array}{ccc}
\text{I.} & \text{II.} & \text{III.} \\
x^{\mathrm{I}} = x\cos\alpha + y\sin\alpha & x^{\mathrm{II}} = x^{\mathrm{I}}\cos\beta + z^{\mathrm{I}}\sin\beta & x^{\mathrm{III}} = x^{\mathrm{II}}\cos\gamma + v^{\mathrm{II}}\sin\gamma \\
y^{\mathrm{I}} = x\sin\alpha - y\cos\alpha & y^{\mathrm{II}} = y^{\mathrm{I}} & y^{\mathrm{III}} = y^{\mathrm{II}} \\
z^{\mathrm{I}} = z & z^{\mathrm{II}} = x^{\mathrm{I}}\sin\beta - z^{\mathrm{I}}\cos\beta & z^{\mathrm{III}} = z^{\mathrm{II}} \\
v^{\mathrm{I}} = v & v^{\mathrm{II}} = v^{\mathrm{I}} & v^{\mathrm{III}} = x^{\mathrm{II}}\sin\gamma - v^{\mathrm{II}}\cos\gamma
\end{array}
$$

$$
\begin{array}{ccc}
\text{IV.} & \text{V.} & \text{VI.} \\
x^{\mathrm{IV}} = x^{\mathrm{III}} & x^{\mathrm{V}} = x^{\mathrm{IV}} & x^{\mathrm{VI}} = x^{\mathrm{V}} = \mathrm{X} \\
y^{\mathrm{IV}} = y^{\mathrm{III}}\cos\delta + z^{\mathrm{III}}\sin\delta & y^{\mathrm{V}} = y^{\mathrm{IV}}\cos\epsilon + v^{\mathrm{IV}}\sin\epsilon & y^{\mathrm{VI}} = y^{\mathrm{V}} = \mathrm{Y} \\
z^{\mathrm{IV}} = y^{\mathrm{III}}\sin\delta - z^{\mathrm{III}}\cos\delta & z^{\mathrm{V}} = z^{\mathrm{IV}} & z^{\mathrm{VI}} = z^{\mathrm{V}}\cos\zeta + v^{\mathrm{V}}\sin\zeta = \mathrm{Z} \\
v^{\mathrm{VI}} = v^{\mathrm{III}} & v^{\mathrm{V}} = y^{\mathrm{IV}}\sin\epsilon - v^{\mathrm{VI}}\cos\epsilon & v^{\mathrm{VI}} = z^{\mathrm{V}}\sin\zeta - v^{\mathrm{V}}\cos\zeta = \mathrm{V}
\end{array}
$$

in quas formulas reuera sex anguli arbitrarii ingrediuntur vt solutionis completae indoles postulat.

XXII. Iam perspicuum est ope harum reductionum nouas quantitates X, Y, Z, V ita per primum assumtas x, y, z, v expressum iri, vt fiat $\mathrm{X} = \mathrm{A}x + \mathrm{B}y + \mathrm{C}z + \mathrm{D}v$, silimiterque etiam reliquae vnde facta euolutione coëfficientes ipsarum x, y, z, v in quatuor formis pro

X, Y, Z, V oriundis ipsos eos sedecim numeros praebebunt, qui requiruntur, pro solutione problematis propositi. Quae cum per se sint manifesta, non opus esse arbitror singulos valores harum sedecim litterarum euoluere. Ceterum cum in harum sex transformationum prima binae x et y, in secunda x et z, in tertia x et v, in quarta y et z, in quinta y et v et in sexta z et v sint transformatae, quae sunt omnes combinationes possibiles; in hoc ipso etiam continetur criterium solutionis completae.

XXIII. Quoniam autem hic occurunt quatuor quantitates x, y, z, v in singulis operationibus duae transformationes binarum institui possunt, quo pacto euolutio valorem quaesitorum non mediocriter subleuantur, vti iterum cauendum ne inter easdem binas litteras plus vna transformatione suscipiatur. Sic autem totum negotium tribus operationibus absolui poterit hoc modo:

I.	II.	III.
$x' = x\cos\alpha + y\sin\alpha$	$x'' = x'\cos\gamma + z'\sin\gamma$	$x''' = x''\cos\epsilon + v''\sin\epsilon = \mathrm{X}$
$y' = x\sin\alpha - y\cos\alpha$	$y'' = y'\cos\delta + v'\sin\delta$	$y''' = y''\cos\zeta + z''\sin\zeta = \mathrm{Y}$
$z' = z\cos\beta + v\sin\beta$	$z'' = x'\sin\gamma - z'\cos\gamma$	$z''' = y''\sin\zeta + z''\cos\zeta = \mathrm{Z}$
$v' = z\sin\beta - v\cos\beta$	$v'' = y''\sin\delta + v'\cos\delta$	$v''' = x''\sin\epsilon - v''\cos\epsilon = \mathrm{V}$

Harum formularum euolutio pro sedecim numeris quaesitis sequentes praebet valores:

$$
\begin{aligned}
\mathrm{A} &= +\cos\alpha\cos\gamma\cos\epsilon + \sin\alpha\sin\delta\sin\epsilon;\\
\mathrm{B} &= +\sin\alpha\cos\gamma\cos\epsilon - \cos\alpha\sin\delta\sin\epsilon;\\
\mathrm{C} &= +\cos\beta\sin\gamma\cos\epsilon - \sin\beta\cos\delta\sin\epsilon;\\
\mathrm{D} &= +\sin\beta\sin\gamma\cos\epsilon + \cos\beta\cos\delta\sin\epsilon;\\
\mathrm{E} &= +\sin\alpha\cos\delta\cos\zeta + \cos\alpha\sin\gamma\sin\zeta;\\
\mathrm{F} &= -\cos\alpha\cos\delta\cos\zeta + \sin\alpha\sin\gamma\sin\zeta;\\
\mathrm{G} &= +\sin\beta\sin\delta\cos\zeta - \cos\beta\cos\gamma\sin\zeta;\\
\mathrm{H} &= -\cos\beta\sin\delta\cos\zeta - \sin\beta\cos\gamma\sin\zeta;\\
\mathrm{I} &= +\sin\alpha\cos\delta\sin\zeta - \cos\alpha\sin\gamma\cos\zeta;\\
\mathrm{K} &= -\cos\alpha\cos\delta\sin\zeta - \sin\alpha\sin\gamma\cos\zeta;\\
\mathrm{L} &= +\sin\beta\sin\delta\sin\zeta + \cos\beta\cos\gamma\cos\zeta;\\
\mathrm{M} &= -\cos\beta\sin\delta\sin\zeta + \sin\beta\cos\gamma\cos\zeta;\\
\mathrm{N} &= +\cos\alpha\cos\gamma\sin\epsilon - \sin\alpha\sin\delta\cos\epsilon;\\
\mathrm{O} &= +\sin\alpha\cos\gamma\sin\epsilon + \cos\alpha\sin\delta\cos\epsilon;\\
\mathrm{P} &= +\cos\beta\sin\gamma\sin\epsilon + \sin\beta\cos\delta\cos\epsilon;\\
\mathrm{Q} &= +\sin\beta\sin\gamma\sin\epsilon - \cos\beta\cos\delta\cos\epsilon.
\end{aligned}
$$

XXIV. Circa hos autem sedecim valores, quibus decem conditiones in problemate allatae implentur, hanc insignem proprietatem locum habere obseruo, vt iisdem quoque sequentibus decem conditionibus satisfiat:

$$\begin{array}{ll}
11^\circ. & \mathrm{AA + BB + CC + DD} = 1 \\
12^\circ. & \mathrm{EE + FF + GG + HH} = 1 \\
13^\circ. & \mathrm{II + KK + LL + MM} = 1 \\
14^\circ. & \mathrm{NN + OO + PP + QQ} = 1 \\
15^\circ. & \mathrm{AE + BF + CG + DH} = 0 \\
16^\circ. & \mathrm{AI + BK + CL + DM} = 0 \\
17^\circ. & \mathrm{AN + BO + CP + DQ} = 0 \\
18^\circ. & \mathrm{EI + FK + GL + HM} = 0 \\
19^\circ. & \mathrm{EN + FO + GP + HQ} = 0 \\
20^\circ. & \mathrm{IN + KO + LP + MQ} = 0
\end{array}$$

Quod est Theorema prorsus memorabile ac simile ei, quod initio circa nouem tantum numeros demonstraui. Eo autem modo, quo ibi demonstrationem adornaui, hic quidem ob litterarum multitudinem vti non licebit; sed quoniam ad hos valores generales successiue peruenire docui, demonstratio ita conuenientissime conficietur, vt si haec proprietas in valoribus quibusque antecedentibus, locum habuerit, eadem quoque in sequentibus per transformationem inde deriuatis locum habere ostendatur.

XXV. Consideremus igitur valores quoscunque intermedios qui per quatuor primitiuas quantitates x, y, z, v ita definiantur, vt sit

$$x^{(n)} = \mathfrak{A}x + \mathfrak{B}y + \mathfrak{C}z + \mathfrak{D}v; \qquad y^{(n)} = \mathfrak{E}x + \mathfrak{F}y + \mathfrak{G}z + \mathfrak{H}v$$

$$z^{(n)} = \mathfrak{I}x + \mathfrak{K}y + \mathfrak{L}z + \mathfrak{M}v; \qquad v^{(n)} = \mathfrak{N}x + \mathfrak{O}y + \mathfrak{P}z + \mathfrak{Q}v$$

vbi coëfficientes ita sint comparati, vt supra memoratis conditionibus satisfaciant; scilicet vt sit:

$$\begin{array}{rcl}
\mathfrak{AA} + \mathfrak{BB} + \mathfrak{CC} + \mathfrak{DD} &=& 1; \\
\mathfrak{EE} + \mathfrak{FF} + \mathfrak{GG} + \mathfrak{HH} &=& 1; \\
\mathfrak{II} + \mathfrak{KK} + \mathfrak{LL} + \mathfrak{MM} &=& 1; \\
\mathfrak{NN} + \mathfrak{OO} + \mathfrak{PP} + \mathfrak{QQ} &=& 1; \\
\mathfrak{AE} + \mathfrak{BF} + \mathfrak{CG} + \mathfrak{DH} &=& 0; \\
\mathfrak{AI} + \mathfrak{BK} + \mathfrak{CL} + \mathfrak{DM} &=& 0; \\
\mathfrak{AN} + \mathfrak{BO} + \mathfrak{CP} + \mathfrak{DQ} &=& 0; \\
\mathfrak{EI} + \mathfrak{FK} + \mathfrak{GL} + \mathfrak{HM} &=& 0; \\
\mathfrak{EN} + \mathfrak{FO} + \mathfrak{GP} + \mathfrak{HQ} &=& 0; \\
\mathfrak{IN} + \mathfrak{KO} + \mathfrak{LP} + \mathfrak{MQ} &=& 0,
\end{array}$$

quae conditionibus vtique in prima positione locum habent, vbi est $x^{(n)} = x$, $y^{(n)} = y$, $z^{(n)} = z$,

$v^{(n)} = v$; siquidem tum habetur:

$$\begin{array}{llll} \mathfrak{A}=1; & \mathfrak{E}=0; & \mathfrak{I}=0; & \mathfrak{N}=0; \\ \mathfrak{B}=0; & \mathfrak{F}=1; & \mathfrak{K}=0; & \mathfrak{O}=0; \\ \mathfrak{C}=0; & \mathfrak{G}=0; & \mathfrak{L}=1; & \mathfrak{P}=0; \\ \mathfrak{D}=0; & \mathfrak{H}=0; & \mathfrak{M}=0; & \mathfrak{Q}=1. \end{array}$$

XXVI. Ponamus ex illis valoribus per transformationes sequentes ita deriuari

vt posito	prodeant hi valores deriuati
$x^{(n+1)} = x^{(n)}\cos\theta + y^{(n)}\sin\theta$	$x^{(n+1)} = \mathfrak{A}'x + \mathfrak{B}'y + \mathfrak{C}'z + \mathfrak{D}'v$
$y^{(n+1)} = x^{(n)}\sin\theta - y^{(n)}\cos\theta$	$y^{(n+1)} = \mathfrak{E}'x + \mathfrak{F}'y + \mathfrak{G}'z + \mathfrak{H}'v$
$z^{(n+1)} = z^{(n)}$	$z^{(n+1)} = \mathfrak{I}'x + \mathfrak{K}'y + \mathfrak{L}'z + \mathfrak{M}'v$
$v^{(n+1)} = v^{(n)}$	$v^{(n+1)} = \mathfrak{N}'x + \mathfrak{O}'y + \mathfrak{P}'z + \mathfrak{Q}'v$

eritque:

$$\begin{array}{rcllll} \mathfrak{A}' &=& \mathfrak{A}\cos\theta + \mathfrak{E}\sin\theta; & \mathfrak{E}' = \mathfrak{A}\sin\theta - \mathfrak{E}\cos\theta; & \mathfrak{I}' = \mathfrak{I}; & \mathfrak{N}' = \mathfrak{N} \\ \mathfrak{B}' &=& \mathfrak{B}\cos\theta + \mathfrak{F}\sin\theta; & \mathfrak{F}' = \mathfrak{B}\sin\theta - \mathfrak{F}\cos\theta; & \mathfrak{K}' = \mathfrak{K}; & \mathfrak{O}' = \mathfrak{O} \\ \mathfrak{C}' &=& \mathfrak{C}\cos\theta + \mathfrak{G}\sin\theta; & \mathfrak{G}' = \mathfrak{C}\sin\theta - \mathfrak{G}\cos\theta; & \mathfrak{L}' = \mathfrak{L}; & \mathfrak{P}' = \mathfrak{P} \\ \mathfrak{D}' &=& \mathfrak{D}\cos\theta + \mathfrak{H}\sin\theta; & \mathfrak{H}' = \mathfrak{D}\sin\theta - \mathfrak{H}\cos\theta; & \mathfrak{M}' = \mathfrak{M}; & \mathfrak{Q}' = \mathfrak{Q} \end{array}$$

Vnde quidem hae conditiones iam sponte implentur:

$$\begin{array}{rcl} \mathfrak{I}'\mathfrak{I}' + \mathfrak{K}'\mathfrak{K}' + \mathfrak{L}'\mathfrak{L}' + \mathfrak{M}'\mathfrak{M}' &=& 1 \\ \mathfrak{I}'\mathfrak{N}' + \mathfrak{K}'\mathfrak{O}' + \mathfrak{L}'\mathfrak{P}' + \mathfrak{M}'\mathfrak{Q}' &=& 0 \\ \mathfrak{N}'\mathfrak{N}' + \mathfrak{O}'\mathfrak{O}' + \mathfrak{P}'\mathfrak{P}' + \mathfrak{Q}'\mathfrak{Q}' &=& 1. \end{array}$$

XXVII. Reliquis vero etiam conditionibus satisfieri facile ostenditur; erit enim:

$$\begin{array}{rcl} \mathfrak{A}'\mathfrak{A}' + \mathfrak{B}'\mathfrak{B}' + \mathfrak{C}'\mathfrak{C}' + \mathfrak{D}'\mathfrak{D}' &=& +(\mathfrak{A}\mathfrak{A} + \mathfrak{B}\mathfrak{B} + \mathfrak{C}\mathfrak{C} + \mathfrak{D}\mathfrak{D})\cos^2\theta \\ && +(\mathfrak{E}\mathfrak{E} + \mathfrak{F}\mathfrak{F} + \mathfrak{G}\mathfrak{G} + \mathfrak{H}\mathfrak{H})\sin^2\theta \\ && +2(\mathfrak{A}\mathfrak{E} + \mathfrak{B}\mathfrak{F} + \mathfrak{C}\mathfrak{G} + \mathfrak{D}\mathfrak{H})\sin\theta\cos\theta \\ &=& 1\cdot\cos^2\theta + 1\cdot\sin^2\theta + 0\cdot\sin\theta\cos\theta = 1 \end{array}$$

quod simili modo de summa quadratorum secundae columnae $\mathfrak{E}'\mathfrak{E}' + \mathfrak{F}'\mathfrak{F}' + \mathfrak{G}'\mathfrak{G}' + \mathfrak{H}'\mathfrak{H}'$ ostenditur. Deinde etiam res manifesta est circa summam productorum:

$$\begin{array}{rl} & \mathfrak{A}'\mathfrak{I}' + \mathfrak{B}'\mathfrak{K}' + \mathfrak{C}'\mathfrak{L}' + \mathfrak{D}'\mathfrak{M}' \\ = & -(\mathfrak{A}\mathfrak{I} + \mathfrak{B}\mathfrak{K} + \mathfrak{C}\mathfrak{L} + \mathfrak{D}\mathfrak{M})\cos\theta + (\mathfrak{E}\mathfrak{I} + \mathfrak{F}\mathfrak{K} + \mathfrak{G}\mathfrak{L} + \mathfrak{H}\mathfrak{M})\sin\theta = 0 \end{array}$$

pariterque etiam circa has summas:

$$\begin{aligned}\mathfrak{A}'\mathfrak{N}' + \mathfrak{B}'\mathfrak{O}' + \mathfrak{C}'\mathfrak{P}' + \mathfrak{D}'\mathfrak{Q}' &= 0;\\ \mathfrak{E}'\mathfrak{J}' + \mathfrak{F}'\mathfrak{K}' + \mathfrak{G}'\mathfrak{L}' + \mathfrak{H}'\mathfrak{M}' &= 0 \quad \text{et} \quad \mathfrak{E}'\mathfrak{N}' + \mathfrak{F}'\mathfrak{O}' + \mathfrak{G}'\mathfrak{P}' + \mathfrak{H}'\mathfrak{Q}' = 0\end{aligned}$$

vnde tantum relinquitur haec:

$$\begin{aligned}\mathfrak{A}'\mathfrak{E}' + \mathfrak{B}'\mathfrak{F}' + \mathfrak{C}'\mathfrak{G}' + \mathfrak{D}'\mathfrak{H}' &= +(\mathfrak{A}\mathfrak{A} + \mathfrak{B}\mathfrak{B} + \mathfrak{C}\mathfrak{C} + \mathfrak{D}\mathfrak{D}) \sin\theta\cos\theta\\ &\quad -(\mathfrak{E}\mathfrak{E} + \mathfrak{F}\mathfrak{F} + \mathfrak{G}\mathfrak{G} + \mathfrak{H}\mathfrak{H}) \sin\theta\cos\theta\\ &\quad +(\mathfrak{A}\mathfrak{E} + \mathfrak{B}\mathfrak{F} + \mathfrak{C}\mathfrak{G} + \mathfrak{D}\mathfrak{H}) \sin^2\theta\\ &\quad -(\mathfrak{A}\mathfrak{E} + \mathfrak{B}\mathfrak{F} + \mathfrak{C}\mathfrak{G} + \mathfrak{D}\mathfrak{H}) \cos^2\theta\\ &= +\sin\theta\cos\theta - \sin\theta\cos\theta + 0\sin^2\theta - 0\cos^2\theta = 0.\end{aligned}$$

XXVIII. Cum igitur harum decem conditionum veritas in positione prima, vti iam ostendi, sit manifesta, etiam in positione secunda per transformationem binarum inde deducta quoque subsistet, hincque etiam in omnibus sequentibus positionibus simili modo ex praecedentibus deductis. Quocirca etiam solutio generalis sex transformationibus vti in §.21. absoluta ita erit comparata, vt non solum decem conditionibus in problemate praescriptis, sed etiam alteris illis decem §.24 commemoratis satisfaciat: hocque ita vt decem prioribus conditionibus satisfieri nequeat, quin simul decem posterioribus satisfiat. Atque hinc iam facile colligitur eandem proprietatem etiam in problematibus, vbi similis quaestio circa 25, 36 pluresque numeros instituitur, semper locum habere debere. Progredior igitur ad sequens.

Problema.

Inuenire 25 numeros A, B, C, D etc, ita in formam quadrati disponendos:

A, B, C, D, E
F, G, H, I, K
L, M, N, O, P
Q, R, S, T, U
V, W, X, Y, Z

vt summae quadratorum ex singulis columnis tam verticalibus quam horizontalibus desumtorum vnitate aequantur, summae productorum autem ex binis columnis siue verticalibus siue horizontalibus formatorum euanescant.

XXIX. Ex praecedentibus intelligitur hoc problema eo reduci, vt sumtis istis 25 numeris pro coëfficientibus, quinque variabiles u, v, x, y, z per huiusmodi formulas in alias transformentur:

$$\begin{aligned}\mathrm{U} &= \mathrm{A}u + \mathrm{B}v + \mathrm{C}x + \mathrm{D}y + \mathrm{E}z\\ \mathrm{V} &= \mathrm{F}u + \mathrm{G}v + \mathrm{H}x + \mathrm{I}y + \mathrm{K}z\\ \mathrm{X} &= \mathrm{L}u + \mathrm{M}v + \mathrm{N}x + \mathrm{O}y + \mathrm{P}z\\ \mathrm{Y} &= \mathrm{Q}u + \mathrm{R}v + \mathrm{S}x + \mathrm{T}y + \mathrm{U}z\\ \mathrm{Z} &= \mathrm{V}u + \mathrm{W}v + \mathrm{X}x + \mathrm{Y}y + \mathrm{Z}z\end{aligned}$$

vt fiat UU + VV + XX + YY + ZZ $= uu + vv + xx + yy + zz$. Quod ergo problema, cum quinque quantitates 10 combinationes diuersas binarum admittant, per decem transformationes successiue in binis instituendas, resoluetur, hoc modo:

I.

$$\begin{aligned} u^{\mathrm{I}} &= u\cos\alpha + v\sin\alpha \\ v^{\mathrm{I}} &= u\sin\alpha - v\cos\alpha \\ x^{\mathrm{I}} &= x \\ y^{\mathrm{I}} &= y \\ z^{\mathrm{I}} &= z \end{aligned}$$

II.

$$\begin{aligned} u^{\mathrm{II}} &= u^{\mathrm{I}}\cos\beta + x^{\mathrm{I}}\sin\beta \\ v^{\mathrm{II}} &= v^{\mathrm{I}} \\ x^{\mathrm{II}} &= u^{\mathrm{I}}\sin\beta - x^{\mathrm{I}}\cos\beta \\ y^{\mathrm{II}} &= y^{\mathrm{I}} \\ z^{\mathrm{II}} &= z^{\mathrm{I}} \end{aligned}$$

III.

$$\begin{aligned} u^{\mathrm{III}} &= u^{\mathrm{II}}\cos\gamma + y^{\mathrm{II}}\sin\gamma \\ v^{\mathrm{III}} &= v^{\mathrm{II}} \\ x^{\mathrm{III}} &= x^{\mathrm{II}} \\ y^{\mathrm{III}} &= u^{\mathrm{II}}\sin\gamma - y^{\mathrm{II}}\cos\gamma \\ z^{\mathrm{III}} &= z^{\mathrm{II}} \end{aligned}$$

IV.

$$\begin{aligned} u^{\mathrm{IV}} &= u^{\mathrm{III}}\cos\delta + z^{\mathrm{III}}\sin\delta \\ v^{\mathrm{IV}} &= v^{\mathrm{III}} \\ x^{\mathrm{IV}} &= x^{\mathrm{III}} \\ y^{\mathrm{VI}} &= y^{\mathrm{III}} \\ z^{\mathrm{IV}} &= u^{\mathrm{III}}\sin\delta - z^{\mathrm{III}}\cos\delta \end{aligned}$$

V.

$$\begin{aligned} u^{\mathrm{V}} &= u^{\mathrm{IV}} \\ v^{\mathrm{V}} &= v^{\mathrm{IV}}\cos\epsilon + x^{\mathrm{IV}}\sin\epsilon \\ x^{\mathrm{V}} &= v^{\mathrm{IV}}\sin\epsilon - x^{\mathrm{IV}}\cos\epsilon \\ y^{\mathrm{V}} &= y^{\mathrm{IV}} \\ z^{\mathrm{V}} &= z^{\mathrm{IV}} \end{aligned}$$

VI.

$$\begin{aligned} u^{\mathrm{VI}} &= u^{\mathrm{V}} \\ v^{\mathrm{VI}} &= v^{\mathrm{V}}\cos\zeta + y^{\mathrm{IV}}\sin\zeta \\ x^{\mathrm{VI}} &= x^{\mathrm{V}} \\ y^{\mathrm{VI}} &= v^{\mathrm{V}}\sin\zeta - y^{\mathrm{V}}\cos\zeta \\ z^{\mathrm{VI}} &= z^{\mathrm{V}} \end{aligned}$$

VII.

$$\begin{aligned} u^{\mathrm{VII}} &= u^{\mathrm{VI}} \\ v^{\mathrm{VII}} &= v^{\mathrm{VI}}\cos\eta + z^{\mathrm{VI}}\sin\eta \\ x^{\mathrm{VII}} &= x^{\mathrm{VI}} \\ y^{\mathrm{VII}} &= y^{\mathrm{VI}} \\ z^{\mathrm{VII}} &= v^{\mathrm{VI}}\sin\eta - z^{\mathrm{VI}}\cos\eta \end{aligned}$$

VIII.

$$\begin{aligned} u^{\mathrm{VIII}} &= u^{\mathrm{VII}} \\ v^{\mathrm{VIII}} &= v^{\mathrm{VII}} \\ x^{\mathrm{VIII}} &= x^{\mathrm{VII}}\cos\theta + y^{\mathrm{VII}}\sin\theta \\ y^{\mathrm{VIII}} &= y^{\mathrm{VII}}\sin\theta - y^{\mathrm{VII}}\cos\theta \\ z^{\mathrm{VIII}} &= z^{\mathrm{VII}} \end{aligned}$$

IX.

$$\begin{aligned} u^{\mathrm{IX}} &= u^{\mathrm{VIII}} \\ v^{\mathrm{IX}} &= v^{\mathrm{VIII}} \\ x^{\mathrm{IX}} &= x^{\mathrm{VIII}}\cos\kappa + z^{\mathrm{VIII}}\sin\kappa \\ y^{\mathrm{IX}} &= y^{\mathrm{VIII}} \\ z^{\mathrm{IX}} &= x^{\mathrm{VIII}}\sin\kappa - z^{\mathrm{VIII}}\cos\kappa \end{aligned}$$

X.

$$\begin{aligned} u^{\mathrm{X}} &= u^{\mathrm{IX}} &&= \mathrm{U} \\ v^{\mathrm{X}} &= v^{\mathrm{IX}} &&= \mathrm{V} \\ x^{\mathrm{X}} &= x^{\mathrm{IX}} &&= \mathrm{X} \\ y^{\mathrm{X}} &= y^{\mathrm{IX}}\cos\lambda + z^{\mathrm{IX}}\sin\lambda &&= \mathrm{Y} \\ z^{\mathrm{X}} &= y^{\mathrm{IX}}\sin\lambda - z^{\mathrm{IX}}\cos\lambda &&= \mathrm{Z} \end{aligned}$$

XXX. His ergo operationibus decem anguli arbitrarii introducuntur, in quo character solutionis completae seu generalis consistit. Cum enim conditiones ex columnis verticalibus petitae problemati soluendo sufficiat, indeque alterae conditiones ad columnas horizontales spectantes sponte impleantur; quadratorum summae praebent 5, producta vero ex binis 10 aequationes; ita vt omnino 15 conditionibus sit satisfaciendum; quare cum 25 numeri inuestigandi proponantur, ex iis decem adhuc manebunt indeterminati, in quo etiam solutio hic data egregie consentit, dum plures quam 10 transformationes, quae quidem circa binas quantitates diuersas instituantur, locum habere nequeunt.

XXXI. Quo illarum formularum euolutio facilior reddatur, qualibet operatione duae transformationes coniungi possunt, prorsus vt in solutione praecedentis problematis est factum. Has autem coniunctiones ita capi conuenit, vt quantitas solitaria nullam mutationem patiens in omnibus sit diuersa: id quod euenit si binae praecedentium transformationum hoc modo coniugantur:

(I, VIII), (II, VII), (III, IX), (IV, VI), (V, X)

vnde sequentes quinque transformationes oriuntur:

I.

$$\begin{aligned} u^{\mathrm{I}} &= u\cos\alpha + v\sin\alpha \\ v^{\mathrm{I}} &= u\sin\alpha - v\cos\alpha \\ x^{\mathrm{I}} &= x\cos\beta + y\sin\beta \\ y^{\mathrm{I}} &= x\sin\beta - y\cos\beta \\ z^{\mathrm{I}} &= z \end{aligned}$$

II.

$$\begin{aligned} u^{\mathrm{II}} &= u^{\mathrm{I}}\cos\gamma + x^{\mathrm{I}}\sin\gamma \\ v^{\mathrm{II}} &= v^{\mathrm{I}}\cos\delta + z^{\mathrm{I}}\sin\delta \\ x^{\mathrm{II}} &= u^{\mathrm{I}}\sin\gamma - x^{\mathrm{I}}\cos\gamma \\ y^{\mathrm{II}} &= y^{\mathrm{I}} \\ z^{\mathrm{II}} &= v^{\mathrm{I}}\sin\delta - z^{\mathrm{I}}\cos\delta \end{aligned}$$

III.

$$\begin{aligned} u^{\mathrm{III}} &= u^{\mathrm{II}}\cos\epsilon + y^{\mathrm{II}}\sin\epsilon \\ v^{\mathrm{III}} &= v^{\mathrm{II}} \\ x^{\mathrm{III}} &= x^{\mathrm{II}}\cos\zeta + z^{\mathrm{II}}\sin\zeta \\ y^{\mathrm{III}} &= u^{\mathrm{II}}\sin\epsilon - y^{\mathrm{II}}\cos\epsilon \\ z^{\mathrm{III}} &= z^{\mathrm{II}}\sin\zeta - z^{\mathrm{II}}\cos\zeta \end{aligned}$$

IV.

$$\begin{aligned} u^{\mathrm{IV}} &= u^{\mathrm{III}}\cos\eta + z^{\mathrm{III}}\sin\eta \\ v^{\mathrm{IV}} &= v^{\mathrm{III}}\cos\theta + y^{\mathrm{III}}\sin\theta \\ x^{\mathrm{IV}} &= x^{\mathrm{III}} \\ y^{\mathrm{IV}} &= v^{\mathrm{III}}\sin\theta - y^{\mathrm{III}}\cos\theta \\ z^{\mathrm{IV}} &= u^{\mathrm{III}}\sin\eta - z^{\mathrm{III}}\cos\eta \end{aligned}$$

V.

$$\begin{aligned} u^{\mathrm{V}} &= u^{\mathrm{IV}} \\ v^{\mathrm{V}} &= v^{\mathrm{IV}}\cos\kappa + x^{\mathrm{IV}}\sin\kappa \\ x^{\mathrm{V}} &= v^{\mathrm{IV}}\sin\kappa - x^{\mathrm{IV}}\cos\kappa \\ y^{\mathrm{V}} &= y^{\mathrm{IV}}\cos\lambda + z^{\mathrm{IV}}\sin\lambda \\ z^{\mathrm{V}} &= y^{\mathrm{IV}}\sin\lambda - z^{\mathrm{IV}}\cos\lambda \end{aligned}$$

XXXII. Simili modo problemata huius generis circa 36 pluresque numeros, quorum quidem multitudo est numerus quadratus resolui possunt; vbi pro calculo contrahendo non solum duas, sed etiam tres ac deinceps plures transformationes in vna operatione complecti licebit; atque hic perpetuo pulcherrimus consensus inter solutionem generalem ex omnibus combinationibus eliciendam ac rei naturam deprehendetur. Posito enim in genere quantitatum quaesitarum numero $= nn$, quadratorum summae vnitati aequandae praebent n conditiones, productorum autem ex binis nihilo aequandae $(nn - n)/2$, sicque coniunctim $(nn + n)/2$ conditiones, quo numero a numero quaesitorum nn ablato, restat $(nn - n)/2$, ac propterea totidem ex quaesitis manebunt indeterminati, seu solutio generalis totidem quantitates arbitrarias complecti debet, secundum regulam autem supra expositam in hunc finem $(nn - n)/2$ transformationibus est vtendum, quibus ergo praecise tot anguli arbitrarii in calculum introducuntur.

Problematis initio propositi solutio generalis in numeris rationalibus.

XXXIII. Coronidis loco solutionem problematis nostri e methodo Diophantea petitam, subiungam, quae sequenti modo satis concinne exhiberi potest.

Sumantur pro lubitu quatuor numeri p, q, r, s ac posita quadratorum eorum summa $pp+qq+rr+ss=u$ nouem numeri quaesiti ita determinati reperiuntur:

$$\begin{array}{lcl}
\mathrm{A} &=& \dfrac{pp+qq-rr-ss}{u}; \quad \mathrm{B}=\dfrac{2qr+2ps}{u}; \quad \mathrm{C}=\dfrac{2qs-2pr}{u}; \\
\mathrm{D} &=& \dfrac{2qr-2ps}{u}; \mathrm{E}=\dfrac{pp-qq+rr-ss}{u}; \quad \mathrm{F}=\dfrac{2pq+2rs}{u}; \\
\mathrm{G} &=& \dfrac{2qs+2pr}{u}; \quad \mathrm{H}=\dfrac{2rs-2pq}{u}; \quad \mathrm{I}=\dfrac{pp-qq-rr+ss}{u}.
\end{array}$$

Hinc simplicissimi numeri, qui quidem inter se omnes sint inaequales, colliguntur sequentes in quadratum dispositi:

$+\frac{47}{57}$	$+\frac{28}{57}$	$-\frac{16}{57}$
$+\frac{4}{57}$	$+\frac{23}{57}$	$+\frac{52}{57}$
$+\frac{32}{57}$	$-\frac{44}{57}$	$+\frac{17}{57}$

hic est $p=6$, $q=4$, $r=2$, $s=1$.

$+\frac{53}{63}$	$+\frac{26}{63}$	$-\frac{22}{63}$
$-\frac{2}{63}$	$+\frac{43}{63}$	$+\frac{46}{63}$
$+\frac{34}{57}$	$-\frac{38}{63}$	$+\frac{37}{63}$

vbi est $p=7$, $q=3$, $r=2$, $s=1$.

En adhuc alia fere aeque simplicia exempla

$+\frac{51}{71}$	$-\frac{42}{71}$	$-\frac{26}{71}$
$-\frac{18}{71}$	$+\frac{19}{71}$	$-\frac{66}{71}$
$+\frac{46}{71}$	$+\frac{54}{71}$	$+\frac{3}{71}$

$+\frac{86}{99}$	$+\frac{38}{99}$	$-\frac{31}{99}$
$-\frac{14}{99}$	$+\frac{79}{99}$	$+\frac{58}{99}$
$+\frac{47}{99}$	$-\frac{46}{99}$	$+\frac{74}{99}$

Pro casu sedecim numerorum.

XXXIV. Si pro casu sedecim numerorum simili modo in quadratum disponendorum solutio in rationalibus desideretur, vnde facile numeros non nimis magnos reperire liceat; methodus supra data ad hunc finem difficulter accomodatur. Alio autem modo prorsus singulari sequentem solutionem latissime patentem sum nactus, vbi sumtis pro lubitu octo numeris a, b, c, d, p, q, r, s, sedecim numeri in quadratum dispositi ita se habent

$+ap+bq+cr+ds$	$+aq-bp+cs-dr$	$+ar-bs-cp+dq$	$+as+br-cq-dp$
$+aq-bp-cs+dr$	$-ap-bq+cr+ds$	$-as-br-cq-dp$	$+ar-bs+cp-dq$
$+ar+bs-cp-dq$	$+as-br-cq+dp$	$-ap+bq-cr+ds$	$-aq-bp-cs-dr$
$+as-br+cq-dp$	$-ar-bs-cp-dq$	$+aq+bp-cs-dr$	$-ap+bq+cr-ds$

vbi summa quadratorum in singulis columnis siue horizontalibus siue verticalibus prodit vbique eadem

$$= (aa+bb+cc+dd)(pp+qq+rr+ss).$$

Quare vt hae summae vnitati aequentur, hanc expressionem quadratum reddi, per eiusque radicem singulos numeros diuidi oportet. Tum vero hi sedecim numeri etiam hac gaudent proprietate, vt summa productorum ex binis columnis siue horizontalibus siue verticalibus sumtorum vbique euanescat.

XXXV. Hinc ergo facile plurima exempla in numeris satis exiguis deduci possunt, inter quae sequens ideo notatu dignum videtur, quod omnes numeri sint se inaequales

$+37$	$+4$	$+1$	$+12$
-6	$+33$	-18	$+9$
$+11$	$+8$	-7	-36
-2	$+19$	$+34$	-3

quadrata

1369	16	1	144	1530
36	1089	324	81	1530
121	64	49	1296	1530
4	361	1156	9	1530
1530	1530	1530	1530	summae

ac de productis binorum res est manifesta: cum sit

$$-6\cdot 37+4\cdot 33-1\cdot 18+9\cdot 12=0$$

$$+4\cdot 37-6\cdot 33+8\cdot 11-2\cdot 19=0.\text{etc.}$$

Generales autem formas inspicienti facile patebit, per eas omnes illas 20 conditiones §§.20 et 24 allatas perfecte impleri, siquidem summae quaternorum quadratorum ad vnitatem reuocentur.

XXXVI. Solutio haec eo maiorem attentionem meretur quod ad eam nulla certa methodo, sed potius quasi diuinando sum perductus: et quoniam ea adeo octo numeros arbitrarios implicat, qui quidem facta reductione ad vnitatem, ad septem rediguntur, vix dubitare licet, quin ista solutio sit vniuersalis et omnes prorsus solutiones possibiles in se complectatur. Si quis ergo viam directam ad hanc solutionem manuducentem inuestigauerit, insignia certe subsidia Analysi attulisse erit censendus. Vtrum autem similes solutiones pro amplioribus quadratis, quae numeris 25, 36 et maioribus constant, expectare liceat, vix affirmare ausim. Non solum autem hinc Algebra communis sed etiam Methodus Diophantea maxima incrementa videtur.

Problema curiosum.

Inuenire sedecim numeros

A	B	C	D
E	F	G	H
I	K	L	M
N	O	P	Q

ita in quadratum disponendos, vt non solum summae quadratorum per columnas tam horizontales quam verticales sumtorum sed etiam eae quae per diagonales sumuntur, scilicet $A^2 + F^2 + L^2 + Q^2$ et $D^2 + G^2 + K^2 + N^2$ sint omnes inter se aequales, ac praeterea producta binorum ita sumtorum, vt supra est praeceptum, euanescant, scilicet

$$\begin{aligned} AE + BF + CG + DH &= 0 \qquad AB + EF + IK + NO = 0 \\ AI + BK + CL + DM &= 0 \qquad AC + EG + IL + NP = 0 \\ AN + BO + CP + DQ &= 0 \qquad AD + EH + IM + NQ = 0 \\ EI + FK + GL + HM &= 0 \qquad BC + FG + KL + OP = 0 \\ EH + FO + GP + HQ &= 0 \qquad BD + FH + KM + OQ = 0 \\ IH + KO + LP + MQ &= 0 \qquad CD + GH + LM + PQ = 0. \end{aligned}$$

Solutio:

Hic ergo proponuntur 22 conditiones, quibus satisfieri oportet; omissis autem duabus ad diagonales spectantibus, sequens forma generalis reliquas omnes adimplet,

$+ap+bq+cr+ds$	$+ar-bs-cp+dq$	$-as-br+cq+dp$	$+aq-bp+cs-dr$
$-aq+bp+cs-dr$	$+as+br+cq+dp$	$+ar-bs+cp-dq$	$+ap+bq-cr-ds$
$+ar+bs-cp-dq$	$-ap+bq-cr+ds$	$+aq+bp+cs+dr$	$+as-br-cq+dp$
$-as+br-cq+dp$	$-aq-bp+cs+dr$	$-ap+bq+cr-ds$	$+ar+bs+cp+dq$

vbi summa quaternorum quadratorum ex columnis tam horizontalibus quam verticalibus sumtorum est

$$(aa+bb+cc+dd)(pp+qq+rr+ss)$$

cui vt etiam summae quadratorum per diagonales sumtorum aequentur, sequentes binas aequationes confici oportet:

$$+abpq+abrs+acpr+acqs+adps+adqr+bcqr+bcps+bdqs+bdpr+cdrs+cdpq=0$$

$$-abpq-abrs+acpr+acps-adps-adqr-bcqr-bcps+bdqs+bdpr-cdrs-cdpq=0$$

ex quibus deducuntur hae duae:

$$\begin{aligned} (ac+bd)(pr+qs) &= 0, \\ (ab+cd)(pq+rs)+(ad+bc)(ps+qr) &= 0. \end{aligned}$$

Vnde hae duae determinationes eliciuntur:

$$\text{I.}\quad pr+qs=0\quad\text{et}\quad\text{II.}\quad \frac{a}{c}=\frac{-d(pq+rs)-b(ps+qr)}{b(pq+rs)+d(ps+qr)}$$

ita vt adhuc sex litterae arbitrio nostro relinquantur.

Euoluamus exemplum sumendo $p=6$, $q=3$, $r=1$, $s=-2$ vnde cum fiat $\frac{a}{c}=\frac{-16d+9b}{16b-9d}$, sit $d=0$, $b=1$, $a=9$, $c=16$ et quadratum omnibus conditionibus satisfaciens erit

+73	−85	+65	+11
−53	+31	+107	+41
−89	−67	+1	−67
−29	−65	−35	+103

vbi summae quaternorum quadratorum secundum columnas tam horizontales quam verticales, itidemque secumdum diagonales sumtorum, prodeunt = 16900 ex quo si hi numeri diuiderentur per 130, hae summae omnes ad vnitatem redigerentur.

Si quem hic offendant numeri 65 et 67 bis occurrentes, adiungam aliud huiusmodi quadratum minoribus adeo numeris expressum.

+68	−29	+41	−37
−17	+31	+79	+32
+59	+28	−23	+61
−11	−65	+8	+49

vbi quaternorum quadratorum summa est 8515.

Notetur denique in his quadratis etiam quadrata tam numerorum angularium, quam mediorum eandem summam producere.

Algebraic problem remarkable on account of its truly extraordinary properties

by Leonhard Euler

in the New Commentaries of the Imperial Academy of Sciences in Saint Petersburg of 1770

Compiled and translated by Johan Sten, including corrections suggested by Johan E. Mebius.

Eneström number 407

The problem whose properties I undertake to consider here is thus formulated:

Find the nine numbers

$$\begin{array}{c} \text{A, B, C,} \\ \text{D, E, F,} \\ \text{G, H, I} \end{array}$$

arranged in this way in a square so that the twelve following conditions are satisfied:

$$\begin{array}{rl}
1^\circ. & \text{AA+DD+GG} = 1 \\
2^\circ. & \text{BB+EE+HH} = 1 \\
3^\circ. & \text{CC+FF+II} = 1 \\
4^\circ. & \text{AB+DE+GH} = 0 \\
5^\circ. & \text{AC+DF+GI} = 0 \\
6^\circ. & \text{BC+EF+HI} = 0 \\
7^\circ. & \text{AA+BB+CC} = 1 \\
8^\circ. & \text{DD+EE+FF} = 1 \\
9^\circ. & \text{GG+HH+II} = 1 \\
10^\circ. & \text{AD+BE+CF} = 0 \\
11^\circ. & \text{AG+BH+CI} = 0 \\
12^\circ. & \text{DG+EH+FI} = 0
\end{array}$$

Concerning this problem I observe the following.

I. As the number of conditions required to be satisfied surpasses the number of quantities to be determined, the problem is in all cases seen to be over-determined. For whatever be the assessed prescribed conditions, no other relation may be taken from these, which now is satisfied by the remainder, unless the sum of the condition 7°, 8°, 9° should agree with 1°, 2°, 3°; from which a single one of the 12 conditions is seen now to be contained by the rest. With which [relation] removed at this stage, eleven conditions are still left, which exceed the unknown quantities by the number two. Here indeed I speak only about a relation of this kind, which these considered conditions meet; for in truth a number of necessary conditions exists between these, but which scarcely were noticed before now, as then the problem would have been perfectly solved.

II. Further I observe that the problem is not only over-determined, but even indeterminate, so that of the nine numbers searched for three can be chosen at will, and nothing less can satisfy all the prescribed conditions. Namely on condition that the six first conditions should be satisfied, the remaining six are satisfied at once, and on the whole it cannot be that the six former are satisfied without satisfying all the rest at the same time. For this reason the same proposed problem remains of this quite straight forwards kind even if the latter six conditions clearly should be omitted; and then to these notes the following conspicous Theorem can be added:

If the nine numbers A, B, C, D, E, F, G, H, I *are chosen so that they satisfy the six former conditions, then they will necessarily also satisfy the six latter.*

I do not doubt that the Theorem would most difficult to show; neither can I see how to approach its demonstration, unless the solution of the problem should be investigated.

III. Nor is this problem suited for leisure speculation or for mere amusement of the mind, but rather it is of the greatest importance in the science of the nature of surfaces. Namely as the nature of surfaces is usually expressed by an equation between three coordinate axes parallel with the three axes normal to each other, such an equation can be transformed without limit by modifying the axes, even if the intersection of the common axes is put in place at the same point. Because then the same surface can be defined by an infinite number of diverse equations between three coordinates, it is of the greatest concern to know the common characteristic of these, which depends on this, that if the coordinates of any three given parallel axes shall be x, y, z, then on the other hand for some other parallel axes put in place, their mutual relations with X, Y, Z, are always contained in these kind of formulas;

$$\mathrm{X} = \mathrm{A}x + \mathrm{B}y + \mathrm{C}z; \qquad \mathrm{Y} = \mathrm{D}x + \mathrm{E}y + \mathrm{F}z; \qquad \mathrm{Z} = \mathrm{G}x + \mathrm{H}y + \mathrm{I}z,$$

which nine coefficients are to be chosen so that necessarily $\mathrm{XX} + \mathrm{YY} + \mathrm{ZZ} = xx + yy + zz$, as these formulas express the square of the distance of the surface from the origin. This is not possible unless these six equations will be fulfilled:

$$\begin{aligned}
&\mathrm{AA} + \mathrm{DD} + \mathrm{GG} = 1,\\
&\mathrm{BB} + \mathrm{EE} + \mathrm{HH} = 1,\\
&\mathrm{CC} + \mathrm{FF} + \mathrm{II} = 1,\\
&\mathrm{AB} + \mathrm{DE} + \mathrm{GH} = 0,\\
&\mathrm{AC} + \mathrm{DF} + \mathrm{GI} = 0,\\
&\mathrm{BC} + \mathrm{EF} + \mathrm{HI} = 0
\end{aligned}$$

which are the same six former conditions of our problem.

IV. But in whatever way by the rules of Algebra this problem is attacked, because of the many unknown numbers it always becomes more and more complicated by calculation, from which by no means can a solution be expected. Indeed, by calling on the aid of the theory of angles, a suitable solution is obtained without difficulty, but truly this method can hardly be applied to other more complicated questions of this kind: just as if around numbers 16, 25, 36 etc., equally arranged in a square, a similar inquiry would be set up, that the sum through each column of the squares taken both vertically and horizontally be equal to unity, simultaneously as the sum of the products of two columns in the same way both vertically and horizontally render zero. I will next, after the demonstration of the Theorem put on record in §.II, be explaining the method which thus extends also to these inquiries and is certainly thought to be the most important in Analysis, and I will present the solution for the problem proposed initially by means of sines and cosines.

Proof of Theorem proposed in §.II.

V. Take, then, our nine numbers A, B, C, D, E, F, G, H, I chosen so that

$$
\begin{aligned}
1^\circ.&\quad \mathrm{AA+DD+GG} = 1\\
2^\circ.&\quad \mathrm{BB+EE+HH} = 1\\
3^\circ.&\quad \mathrm{CC+FF+II} = 1\\
4^\circ.&\quad \mathrm{AB+DE+GH} = 0\\
5^\circ.&\quad \mathrm{AC+DF+GI} = 0\\
6^\circ.&\quad \mathrm{BC+EF+HI} = 0
\end{aligned}
$$

which three latter are expressed thus:

$$
\begin{aligned}
4^\circ.&\quad \mathrm{AB} = -\mathrm{DE} - \mathrm{GH}\\
5^\circ.&\quad \mathrm{AC} = -\mathrm{DF} - \mathrm{GI}\\
6^\circ.&\quad \mathrm{BC} = -\mathrm{EF} - \mathrm{HI}
\end{aligned}
$$

whence I conclude that:

$$\frac{4^\circ \cdot 5^\circ}{6^\circ} \cdots = \frac{\mathrm{AA\ BC}}{\mathrm{BC}} = \mathrm{AA} = -\frac{\mathrm{(DE+GH)(DF+GI)}}{\mathrm{EF+HI}}$$

which value put in AA of the first equation gives:

$$-(\mathrm{DE} + \mathrm{GH})\ (\mathrm{DF} + \mathrm{GI}) + (\mathrm{EF} + \mathrm{HI})\ (\mathrm{DD} + \mathrm{GG}) = \mathrm{EF} + \mathrm{HI}$$

and after expanding:

$$-\ \mathrm{DEGI}\ - \mathrm{DFGH} + \mathrm{DDHI} + \mathrm{EFGG} = \mathrm{EF} + \mathrm{HI}$$

the first member of which clearly resolves in these factors:

$$(\mathrm{DH} - \mathrm{EG})\ (\mathrm{DI} - \mathrm{FG}) = \mathrm{EF} + \mathrm{HI}.$$

VI. As EF + HI = −BC, then

$$\mathrm{BC} = (\mathrm{EG} - \mathrm{DH})\ (\mathrm{DI} - \mathrm{FG})$$

and in the same way it is obtained that

$$\mathrm{AC} = (\mathrm{FH} - \mathrm{EI})\ (\mathrm{EG} - \mathrm{DH}) \qquad \text{and} \qquad \mathrm{AB} = (\mathrm{DI} - \mathrm{FG})\ (\mathrm{FH} - \mathrm{EI}),$$

the product of the two latter ones divided by the first gives

$$AA = (FH - EI)^2 \qquad \text{and hence} \qquad A = \pm(FH - EI)$$

because while separate numbers can be taken, be they negative or positive, thinking that the ambiguous sign can make no difference, whence taking the upper sign we have:

$$A = FH - EI;\ B = DI - FG;\ C = EG - DH.$$

But since from the very nature of the vertical columns, permutation is allowed between them, we hence conclude through analogy

$$D = BI - CH;\ E = CG - AI;\ F = AH - BG;$$

$$G = CE - BF;\ H = AF - CD;\ I = BD - AE.$$

VII. Here are therefore nine new boundary conditions, which are necessarily involved in the six given before, and which we could add to the 12 conditions initially proposed. Here are truly the nine boundary conditions themselves, which I indicate in the following way:

13°. A=FH – EI
14°. B=DI – FG
15°. C=EG – DH
16°. D=BI – CH
17°. E=CG – AI
18°. F=AH – BG
19°. G=CE – BF
20°. H=AF – CD
21°. I=BD – AE

which easily lead to the six conditions proposed initially. For the formula 13° multiplied by D, 14° by E and 15° by F and collected in one sum gives:

$$AD + BE + CF = + DFH + DEI + EFG - DEI - EFG - DFH = 0,$$

which is the same condition as 10° initially proposed, and in the same way 13°. G + 14°. H + 15°. I gives condition 11° and 16°. G + 17°. H + 18°. I condition 12°, so that:

10°. AD + BE + CF = 0
11°. AG + BH + CI = 0
12°. DG + EH + FI = 0.

VIII. Further if in the formula 13° the values of the symbols E and F are substituted from formulas 17° and 18°, this equation emerges:

$$A = AHH - BGH - CGI + AII = A\,(HH + II) - G\,(BH + CI)$$

but from equation 11° we have that BH + CI = − AG, from which it is gathered: A = A (GG + HH + II), so that either A = 0 or GG + HH + II = 1. But as in the same way from the formulas 14°, 15°, 16°, 17°, and 18°, the equations:

$$\mathrm{B = B\ (GG + HH + II);\ C = C\ (GG + HH + II);\ D = D\ (GG + HH + II);}$$

E = E (GG + HH + II) and F = F (GG + HH + II) are drawn, and neither are all the letters A, B, C, D, E, F simultaneously zero, so it is necessary that GG + HH + II = 1, which is condition 9° represented in the following way:

$$\begin{aligned} &7^\circ.\quad \mathrm{AA + BB + CC = 1;}\\ &8^\circ.\quad \mathrm{DD + EE + FF = 1;}\\ &9^\circ.\quad \mathrm{GG + HH + II = 1,} \end{aligned}$$

which is the completing demonstration of our theorem.

Solution of the initially proposed Problem.

IX. Let A = $\cos\zeta$, and with the conditions 1° and 7° they give:

$$\mathrm{DD + GG} = \sin^2\zeta, \qquad \text{and} \qquad \mathrm{BB + CC} = \sin^2\zeta$$

which we will satisfy by setting: B = $\sin\zeta\cos\eta$; C = $\sin\zeta\sin\eta$; D = $\sin\zeta\cos\theta$; G = $\sin\zeta\sin\theta$. Considering now conditions 17° and 21° which after these substitutions take these forms:

$$\begin{aligned} &17^\circ.\quad \mathrm{E} = \sin^2\zeta\sin\eta\sin\theta - \mathrm{I}\cos\zeta \qquad \text{or} \qquad \mathrm{E}+\mathrm{I}\cos\zeta = \sin^2\zeta\sin\eta\sin\theta\\ &21^\circ.\quad \mathrm{I} = \sin^2\zeta\cos\eta\cos\theta - \mathrm{E}\cos\zeta \qquad \text{or} \qquad \mathrm{I}+\mathrm{E}\cos\zeta = \sin^2\zeta\cos\eta\cos\theta \end{aligned}$$

Hence (17° − 21° . $\cos\zeta$) and (21° − 17°. $\cos\zeta$) give:

$$\mathrm{E}(1-\cos^2\zeta) = \sin^2\zeta(\sin\eta\sin\theta - \cos\zeta\cos\eta\cos\theta)$$

$$\mathrm{I}(1-\cos^2\zeta) = \sin^2\zeta(\cos\eta\cos\theta - \cos\zeta\sin\eta\sin\theta)$$

whence it is concluded that: E = $\sin\eta\sin\theta - \cos\zeta\cos\eta\cos\theta$ and I = $\cos\eta\cos\theta - \cos\zeta\sin\eta\sin\theta$.

X. Similarly the conditions 18° and 20° in the same way as earlier, with the substitutions duly made, supply us with these equations:

$$\begin{aligned} &18^\circ.\quad \mathrm{F}=\mathrm{H}\cos\zeta - \sin^2\zeta\cos\eta\sin\theta \qquad \text{or} \qquad \mathrm{F} - \mathrm{H}\cos\zeta = -\sin^2\zeta\cos\eta\sin\theta\\ &20^\circ.\quad \mathrm{H}=\mathrm{F}\cos\zeta - \sin^2\zeta\sin\eta\cos\theta \qquad \text{or} \qquad \mathrm{H} - \mathrm{F}\cos\zeta = \sin^2\zeta\sin\eta\cos\theta \end{aligned}$$

from which (18° + 20° . $\cos\zeta$) and (20° + 18°. $\cos\zeta$) produce

$$\mathrm{F}(1-\cos^2\zeta) = -\sin^2\zeta(\cos\eta\sin\theta + \cos\zeta\sin\eta\cos\theta)$$

$$\mathrm{H}(1-\cos^2\zeta) = -\sin^2 z(\sin\eta\cos\theta + \cos\zeta\cos\eta\sin\theta)$$

which because of $1 - \cos^2\zeta = \sin^2\zeta$ render

$$\mathrm{F} = -\cos\eta\sin\theta - \cos\zeta\sin\eta\cos\theta \qquad \text{and} \qquad \mathrm{H} = -\sin\eta\cos\theta - \cos\zeta\cos\eta\sin\theta$$

and so the nine numbers satisfying the prescribed conditions are so defined, that three angles ζ, η, θ are left for our choice, in which the criteria of our problem are completely included.

XI. The complete solution to our problem, then, is that the nine numbers searched for will be chosen the following values:

$$\begin{array}{lll} \mathrm{A} = \cos\zeta & \mathrm{B} = \sin\zeta\cos\eta & \mathrm{C} = \sin\zeta\sin\eta \\ \mathrm{D} = \sin\zeta\cos\theta & \mathrm{E} = \sin\eta\sin\theta - \cos\zeta\cos\eta\cos\theta & \mathrm{F} = -\cos\eta\sin\theta - \cos\zeta\sin\eta\cos\theta \\ \mathrm{G} = \sin\zeta\sin\theta & \mathrm{H} = -\sin\eta\cos\theta - \cos\zeta\cos\eta\sin\theta & \mathrm{I} = \cos\eta\cos\theta - \cos\zeta\sin\eta\sin\theta \end{array}$$

which values not only fulfil the six former conditions, defining the problem, but also the six latter, as well as even the nine new ones expressed in §.VII. And for this solution to be most suitable for use, which easily allows to discover solutions in rational numbers, no matter how many, there is a need to take three angles ζ, η, θ expressed rationally by sines and cosines. Hence taking a sufficiently simple solution $\cos\zeta = 3/5$; $\sin\zeta = 4/5$; $\cos\eta = 3/5$; $\sin\eta = 4/5$; $\cos\theta = 5/13$; $\sin\theta = 12/13$.

General method to solve this kind of problems

XII. The general method which I will be delivering here is beyond the principles given in §.III aimed at showing whenever the proposed problem reverts to this: from the three variables x, y, z determine other three X, Y, Z with this kind of formulas $\alpha x + \beta y + \gamma z$, such that $\mathrm{X}^2 + \mathrm{Y}^2 + \mathrm{Z}^2 = x^2 + y^2 + z^2$, so that the determination is the most general; then the coefficients of three such formulas $\alpha x + \beta y + \gamma z$ for the resulting new variables X, Y, Z are in fact themselves the nine numbers which are desired in the problem. The two conditions are therefore chosen carefully, one being that the values of X, Y, Z themselves are most easily expressed through these kind of formulas $\alpha x + \beta y + \gamma z$, and the other truly that $\mathrm{X}^2 + \mathrm{Y}^2 + \mathrm{Z}^2 = x^2 + y^2 + z^2$. If not this condition was present, the inquiry would be easily resolved by the Diophantean method, as long as only the sum of three squares would resolve in other three squares, in which there are no difficulties.

XIII. Because the mind is set to deduce this to the fact, that the method can be extended to even more complicated inquiries, the simplest case is first considered, that of proposing no more than two variables x and y, and to define from them two other X and Y by such kind of formulas $\alpha x + \beta y$ so that $\mathrm{X}^2 + \mathrm{Y}^2 = x^2 + y^2$. Putting to this end

$$\mathrm{X} = \alpha x + \beta y \qquad \text{and} \qquad \mathrm{Y} = \gamma x + \delta y$$

it is necessary to set:

$$\alpha\alpha + \gamma\gamma = 1; \quad \beta\beta + \delta\delta = 1; \quad \alpha\beta + \gamma\delta = 0.$$

Therefore, stating $\alpha = \cos\zeta$ and $\beta = \cos\eta$, then $\gamma = \sin\zeta$ and $\delta = \sin\eta$, and so they satisfy the two prior conditions: then truly the third gives $\cos\zeta\cos\eta + \sin\zeta\sin\eta = \cos(\zeta - \eta) = 0$, from which $\zeta - \eta = 90°$, and therefore $\eta = \zeta - 90°$, and because $\cos\eta = \sin\zeta$ and $\sin\eta = -\cos\zeta$. From which it is clear, if we take:

$$\mathrm{X} = x\cos\zeta + y\sin\zeta \qquad \text{and} \qquad \mathrm{Y} = x\sin\zeta - y\cos\zeta$$

that $\mathrm{X}^2 + \mathrm{Y}^2 = x^2 + y^2$.

XIV. Extending this lemma by proposing three variables x, y, z, I first define three others x', y', z' so that

$$x' = x\cos\zeta + y\sin\zeta; \qquad y' = x\sin\zeta - y\cos z; \qquad z' = z$$

in which way it is indeed certain that

$$x'x' + y'y' + z'z' = xx + yy + zz.$$

Next, from these I deduce other x'', y'', z'' in a similar way so that

$$x'' = x'; \qquad y'' = y'\cos\eta + z'\sin\eta; \qquad z'' = y'\sin\eta - z'\cos\eta$$

and hence finally the X, Y, Z searched for defined such that:

$$\mathrm{X} = z''\cos\theta + x''\sin\theta; \qquad \mathrm{Y} = y''; \qquad Z = z''\sin\theta - x''\cos\theta$$

indeed so that it will be certain that:

$$\mathrm{X}^2 + \mathrm{Y}^2 + \mathrm{Z}^2 = x''x'' + y''y'' + z''z'' = x'x' + y'y' + z'z' = xx + yy + zz.$$

XV. But from this there follows three positions:

$$x'' = x\cos\zeta + y\sin\zeta; \qquad y'' = x\sin\zeta\cos\eta - y\cos\zeta\cos\eta + z\sin\eta;$$

$$z'' = x\sin\zeta\sin\eta - y\cos\zeta\sin\eta - z\cos\eta$$

then truly

$$\begin{aligned}
\mathrm{X} &= x(\sin\zeta\sin\eta\cos\theta + \cos\zeta\sin\theta) - y(\cos\zeta\sin\eta\cos\theta - \sin\zeta\sin\theta) - z\cos\eta\cos\theta \\
\mathrm{Y} &= x\sin\zeta\cos\eta - y\cos\zeta\cos\eta + z\sin\eta \\
\mathrm{Z} &= x(\sin\zeta\sin\eta\sin\theta - \cos\zeta\cos\theta) - y(\cos\zeta\sin\eta\sin\theta + \sin\zeta\cos\theta) - z\cos\eta\sin\theta
\end{aligned}$$

which agree with the formulas found earlier.

XVI. The generality of this solution is clear from the fact that it embraces three arbitrary angles ζ, η, θ, which are introduced through the three transformations we have established. The strength of this method indeed consists in this, that just as in a transformation of no more than two changing quantities, provided of course in their place two others, one for an arbitrary angle are introduced, the third remains unchanged. Hence the two operations certainly supply the solution for the problem, but yet not completely, because of the one lacking arbitrary quantity. As many transformations must therefore be established, while such a number of arbitrary quantities will enter in this way, as the largest extension of the solution requires. Above I already observed, while turning about the problem of nine numbers and only six conditions were prescribed, that three of them remain undetermined, just as also in the solution given here by angles ζ, η, θ is left for our choice, the three numbers A, B, D, can be picked as it pleases us.

XVII. From this there can rise a doubt, however, that since with a transformation new angles are anyway introduced, by increasing the number of our transformations, the solution of even more general problems could be obtained. Yet, however true, whoever wants to make this risky thing will soon notice that the new angle introduced will coalesce with some earlier one so that no matter how many transformations will be taken for consideration, the number of truly arbitrary angles cannot rise above three. Let us add, for example, this transformation, putting:

$$\mathrm{X}' = \mathrm{X}; \quad \mathrm{Y}' = \mathrm{Y}\cos\lambda - \mathrm{Z}\sin\lambda; \quad \mathrm{Z}' = \mathrm{Y}\sin\lambda + \mathrm{Z}\cos\lambda$$

resulting in

$$\begin{aligned}
\mathrm{X}' &= x(\sin\zeta\sin\eta\cos\theta + \cos\zeta\sin\theta) + y(\sin\zeta\sin\theta - \cos\zeta\cos\eta\cos\theta) - z\cos\eta\cos\theta \\
\mathrm{Y}' &= x(\sin\zeta\cos\eta\cos\lambda - \sin\zeta\sin\eta\sin\theta\sin\lambda + \cos\zeta\cos\theta\sin\lambda \\
&- y(\cos\zeta\cos\eta\cos\lambda - \cos\zeta\sin\eta\sin\theta\sin\lambda - \sin\zeta\cos\theta\sin\lambda \\
&+ z(\sin\eta\cos\lambda + \cos\eta\sin\theta\sin\lambda) \\
\mathrm{Z}' &= x(\sin\zeta\cos\eta\sin\lambda + \sin\zeta\sin\eta\sin\theta\cos\lambda - \cos\zeta\cos\theta\cos\lambda \\
&- y(\cos\zeta\cos\eta\sin\lambda + \cos\zeta\sin\eta\sin\theta\sin\lambda - \sin\zeta\cos\theta\cos\lambda \\
&+ z(\sin\eta\sin\lambda - \cos\eta\sin\theta\cos\lambda)
\end{aligned}$$

where, even if four angles are present, ζ, η, θ and λ, nevertheless for no more than three of them can coefficients be chosen freely: but it is certainly not easily seen, and not without many turns can be seen to be demonstrated, whereas the fact would nevertheless be very clear in itself.

XVIII. Also it is the most difficult to see these four undetermined quantities go back to three and this investigation would demand wholly singular developments, nevertheless a rule in its place with no difficulty is discovered, because among these same-named quantities y and z a transformation is made twice. Namely in the second the quantities y', z' are transformed in y'', z'' by means of the angle η and in the fourth the same-named quantities Y and Z are transformed by means of the angle λ into Y$'$ and Z$'$. To see that the two transformations follow each other so immediately, take for example:

$$\text{first} \quad y' = y\cos\zeta + z\sin\zeta; \quad z' = y\sin\zeta - z\cos\zeta$$

then truly

$$y'' = y'\cos\eta + z'\sin\eta; \quad z'' = y'\sin\eta - z'\cos\eta$$

taken together yield:

$$y'' = y\cos(\zeta - \eta) + z\sin(\zeta - \eta) \quad \text{et} \quad z'' = -y\sin(\zeta - \eta) + z\cos(\zeta - \eta)$$

and so the transformation doubled is clearly equivalent to one made with the angle $\zeta - \eta$. This also comes for understanding, although these kinds of two transformations between same-named quantities do not immediately take out each other.

XIX. Hence when making any transformation between only two variables, this rule can be established, that these transformations would always be set up among two variables with different names; when agreeing to this the number of transformations are so determined that many of them are useless. So when in our problem there are three variable quantities indicated x, y, z, more than three transformations cannot take place, provided that one among x and y, the other among x and z, and the third among y and z are established in this way

$$\begin{array}{lll} x' = x\cos\zeta + y\sin\zeta & x'' = x'\cos\eta + z'\sin\eta & x''' = x'' \\ y' = x\sin\zeta - y\cos\zeta & y'' = y' & y''' = y''\cos\theta + z''\sin\theta \\ z' = z & z'' = x'\sin\eta - z'\cos\eta & z''' = y''\sin\theta - z''\cos\theta \end{array}$$

where in the first the quantity named z, in the second the one named y, and in the third truly the one named x remain invariant.

XX. Observing this rule we can easily adapt the method via such kind of previous transformations to problems, in which more than three variables are displayed, which in the same way should be transformed into as many other problems, so that the sum of the squares remain the same. Evidently there is a need for several transformations established between the two variables only, where it must be avoided that not two transformations are performed among two with the same names. Observing this, the solution is not complete before two transformations among all with diverse names are complete, of which kind of diverse combinations there are six, if the quantities proposed are four, and ten if the quantities are five and so on. I will join herein some such problems with their solutions.

Problem.

To transform four quantities v, x, y, z in four others using this kind of formulas $\alpha v+\beta x+\gamma y+\delta z$, that the sum of the squares remains the same or putting

$$\begin{array}{ll} \mathrm{V} = \mathrm{A}v + \mathrm{B}x + \mathrm{C}y + \mathrm{D}z; & \mathrm{Y} = \mathrm{I}v + \mathrm{K}x + \mathrm{L}y + \mathrm{M}z \\ \mathrm{X} = \mathrm{E}v + \mathrm{F}x + \mathrm{G}y + \mathrm{H}z; & \mathrm{Z} = \mathrm{N}v + \mathrm{O}x + \mathrm{P}y + \mathrm{Q}z \end{array}$$

these 16 coefficients are so defined that

$$\mathrm{VV} + \mathrm{XX} + \mathrm{YY} + \mathrm{ZZ} = vv + xx + yy + zz$$

and in the end they should satisfy the following 10 conditions:

$$
\begin{array}{ll}
1^\circ. & \mathrm{AA} + \mathrm{EE} + \mathrm{II} + \mathrm{NN} = 1 \\
2^\circ. & \mathrm{BB} + \mathrm{FF} + \mathrm{KK} + \mathrm{OO} = 1 \\
3^\circ. & \mathrm{CC} + \mathrm{GG} + \mathrm{LL} + \mathrm{PP} = 1 \\
4^\circ. & \mathrm{DD} + \mathrm{HH} + \mathrm{MM} + \mathrm{QQ} = 0 \\
5^\circ. & \mathrm{AB} + \mathrm{EF} + \mathrm{IK} + \mathrm{NO} = 0 \\
6^\circ. & \mathrm{AC} + \mathrm{EG} + \mathrm{IL} + \mathrm{NP} = 0 \\
7^\circ. & \mathrm{AD} + \mathrm{EH} + \mathrm{IM} + \mathrm{NQ} = 1 \\
8^\circ. & \mathrm{BC} + \mathrm{FG} + \mathrm{KL} + \mathrm{OP} = 1 \\
9^\circ. & \mathrm{BD} + \mathrm{FH} + \mathrm{KM} + \mathrm{OQ} = 1 \\
10^\circ. & \mathrm{CD} + \mathrm{GH} + \mathrm{LM} + \mathrm{PQ} = 0
\end{array}
$$

XXI. While proposing here to find sixteen numbers from ten conditions, it is evident that six of them will be left for our choice, or in order to give the same a complete solution, six arbitrary quantities must be included.

But before presenting the method, we see the following solution passes through six transformations, which can be represented as follows:

I. II. III.

$$
\begin{array}{lll}
x^{\mathrm{I}} = x\cos\alpha + y\sin\alpha & x^{\mathrm{II}} = x^{\mathrm{I}}\cos\beta + z^{\mathrm{I}}\sin\beta & x^{\mathrm{III}} = x^{\mathrm{II}}\cos\gamma + v^{\mathrm{II}}\sin\gamma \\
y^{\mathrm{I}} = x\sin\alpha - y\cos\alpha & y^{\mathrm{II}} = y^{\mathrm{I}} & y^{\mathrm{III}} = y^{\mathrm{II}} \\
z^{\mathrm{I}} = z & z^{\mathrm{II}} = x^{\mathrm{I}}\sin\beta - z^{\mathrm{I}}\cos\beta & z^{\mathrm{III}} = z^{\mathrm{II}} \\
v^{\mathrm{I}} = v & v^{\mathrm{II}} = v^{\mathrm{I}} & v^{\mathrm{III}} = x^{\mathrm{II}}\sin\gamma - v^{\mathrm{II}}\cos\gamma
\end{array}
$$

IV. V. VI.

$$
\begin{array}{lll}
x^{\mathrm{IV}} = x^{\mathrm{III}} & x^{\mathrm{V}} = x^{\mathrm{IV}} & x^{\mathrm{VI}} = x^{\mathrm{V}} = \mathrm{X} \\
y^{\mathrm{IV}} = y^{\mathrm{III}}\cos\delta + z^{\mathrm{III}}\sin\delta & y^{\mathrm{V}} = y^{\mathrm{IV}}\cos\epsilon + v^{\mathrm{IV}}\sin\epsilon & y^{\mathrm{VI}} = y^{\mathrm{V}} = \mathrm{Y} \\
z^{\mathrm{IV}} = y^{\mathrm{III}}\sin\delta - z^{\mathrm{III}}\cos\delta & z^{\mathrm{V}} = z^{\mathrm{IV}} & z^{\mathrm{VI}} = z^{\mathrm{V}}\cos\zeta + v^{\mathrm{V}}\sin\zeta = \mathrm{Z} \\
v^{\mathrm{VI}} = v^{\mathrm{III}} & v^{\mathrm{V}} = y^{\mathrm{IV}}\sin\epsilon - v^{\mathrm{VI}}\cos\epsilon & v^{\mathrm{VI}} = z^{\mathrm{V}}\sin\zeta - v^{\mathrm{V}}\cos\zeta = \mathrm{V}
\end{array}
$$

in which formulas actually six arbitrary angles are involved so it inherently aspires to be the complete solution.

XXII. As is already clear, by means of these reductions the new quantities X, Y, Z, V are expressed by means of the first quantities assumed x, y, z, v, making $\mathrm{X} = \mathrm{A}x + \mathrm{B}y + \mathrm{C}z + \mathrm{D}v$, and similarly also the remaining, whence having performed the developments, the coefficients of x, y, z, v themselves provide through the four forms for the created X, Y, Z, V the 16 numbers required for solving the proposed problem. As they are clear by themselves, there is no need to pursue the singular values for these 16 letters. And besides, as in these six transformations the first two x and y, in the second x and z, in the third x and v, in the fourth y and z, in the fifth y and v and in the sixth z and v are transformed, which are all the possible combinations; the complete solution is contained even in itself.

XXIII. However, because with the four quantities x, y, z, v that occur here, two transformations of two can be set up in single operations, which agreed to the development of the values searched for are not very ordinarily pulled out, so that again care must be taken not to undertake more than one transformation among two letters. But so the whole work can be acquitted in three operations in this way:

I.	II.	III.
$x' = x\cos\alpha + y\sin\alpha$	$x'' = x'\cos\gamma + z'\sin\gamma$	$x''' = x''\cos\epsilon + v''\sin\epsilon = \mathrm{X}$
$y' = x\sin\alpha - y\cos\alpha$	$y'' = y'\cos\delta + v'\sin\delta$	$y''' = y''\cos\zeta + z''\sin\zeta = \mathrm{Y}$
$z' = z\cos\beta + v\sin\beta$	$z'' = x'\sin\gamma - z'\cos\gamma$	$z''' = y''\sin\zeta + z''\cos\zeta = \mathrm{Z}$
$v' = z\sin\beta - v\cos\beta$	$v'' = y''\sin\delta + v'\cos\delta$	$v''' = x''\sin\epsilon - v''\cos\epsilon = \mathrm{V}$

Development of these formulas gives the following values for the sixteen numbers searched for:

$$
\begin{aligned}
\mathrm{A} &= +\cos\alpha\cos\gamma\cos\epsilon + \sin\alpha\sin\delta\sin\epsilon;\\
\mathrm{B} &= +\sin\alpha\cos\gamma\cos\epsilon - \cos\alpha\sin\delta\sin\epsilon;\\
\mathrm{C} &= +\cos\beta\sin\gamma\cos\epsilon - \sin\beta\cos\delta\sin\epsilon;\\
\mathrm{D} &= +\sin\beta\sin\gamma\cos\epsilon + \cos\beta\cos\delta\sin\epsilon;\\
\mathrm{E} &= +\sin\alpha\cos\delta\cos\zeta + \cos\alpha\sin\gamma\sin\zeta;\\
\mathrm{F} &= -\cos\alpha\cos\delta\cos\zeta + \sin\alpha\sin\gamma\sin\zeta;\\
\mathrm{G} &= +\sin\beta\sin\delta\cos\zeta - \cos\beta\cos\gamma\sin\zeta;\\
\mathrm{H} &= -\cos\beta\sin\delta\cos\zeta - \sin\beta\cos\gamma\sin\zeta;\\
\mathrm{I} &= +\sin\alpha\cos\delta\sin\zeta - \cos\alpha\sin\gamma\cos\zeta;\\
\mathrm{K} &= -\cos\alpha\cos\delta\sin\zeta - \sin\alpha\sin\gamma\cos\zeta;\\
\mathrm{L} &= +\sin\beta\sin\delta\sin\zeta + \cos\beta\cos\gamma\cos\zeta;\\
\mathrm{M} &= -\cos\beta\sin\delta\sin\zeta + \sin\beta\cos\gamma\cos\zeta;\\
\mathrm{N} &= +\cos\alpha\cos\gamma\sin\epsilon - \sin\alpha\sin\delta\cos\epsilon;\\
\mathrm{O} &= +\sin\alpha\cos\gamma\sin\epsilon + \cos\alpha\sin\delta\cos\epsilon;\\
\mathrm{P} &= +\cos\beta\sin\gamma\sin\epsilon + \sin\beta\cos\delta\cos\epsilon;\\
\mathrm{Q} &= +\sin\beta\sin\gamma\sin\epsilon - \cos\beta\cos\delta\cos\epsilon.
\end{aligned}
$$

XXIV. Concerning the sixteen values, which fulfil the ten conditions in the given problem, I observe this conspicous property to take place, that the same also satisfy the following ten conditions:

$$\begin{array}{ll} 11^\circ. & \mathrm{AA + BB + CC + DD} = 1 \\ 12^\circ. & \mathrm{EE + FF + GG + HH} = 1 \\ 13^\circ. & \mathrm{II + KK + LL + MM} = 1 \\ 14^\circ. & \mathrm{NN + OO + PP + QQ} = 1 \\ 15^\circ. & \mathrm{AE + BF + CG + DH} = 0 \\ 16^\circ. & \mathrm{AI + BK + CL + DM} = 0 \\ 17^\circ. & \mathrm{AN + BO + CP + DQ} = 0 \\ 18^\circ. & \mathrm{EI + FK + GL + HM} = 0 \\ 19^\circ. & \mathrm{EN + FO + GP + HQ} = 0 \\ 20^\circ. & \mathrm{IN + KO + LP + MQ} = 0 \end{array}$$

This is a very remarkable Theorem and similar to that which I demonstrated for only nine numbers. For the demonstration which I prepare here, however, the same way does not work because of the multitude of letters; but because I will teach how to get these values successively, the demonstration will be prepared most conveniently, when in case that these properties take place in some previous value, it is shown that the same will also take place in the following ones derived from them by transformation.

XXV. Let us therefore consider some intermediate values, defined through the four primitive quantities x, y, z, v so that

$$x^{(n)} = \mathfrak{A}x + \mathfrak{B}y + \mathfrak{C}z + \mathfrak{D}v; \qquad y^{(n)} = \mathfrak{E}x + \mathfrak{F}y + \mathfrak{G}z + \mathfrak{H}v$$

$$z^{(n)} = \mathfrak{I}x + \mathfrak{K}y + \mathfrak{L}z + \mathfrak{M}v; \qquad v^{(n)} = \mathfrak{N}x + \mathfrak{O}y + \mathfrak{P}z + \mathfrak{Q}v$$

where the coefficients are prepared so that they satisfy the above noted conditions *viz.*

$$\begin{array}{rcl} \mathfrak{AA} + \mathfrak{BB} + \mathfrak{CC} + \mathfrak{DD} &=& 1; \\ \mathfrak{EE} + \mathfrak{FF} + \mathfrak{GG} + \mathfrak{HH} &=& 1; \\ \mathfrak{II} + \mathfrak{KK} + \mathfrak{LL} + \mathfrak{MM} &=& 1; \\ \mathfrak{NN} + \mathfrak{OO} + \mathfrak{PP} + \mathfrak{QQ} &=& 1; \\ \mathfrak{AE} + \mathfrak{BF} + \mathfrak{CG} + \mathfrak{DH} &=& 0; \\ \mathfrak{AI} + \mathfrak{BK} + \mathfrak{CL} + \mathfrak{DM} &=& 0; \\ \mathfrak{AN} + \mathfrak{BO} + \mathfrak{CP} + \mathfrak{DQ} &=& 0; \\ \mathfrak{EI} + \mathfrak{FK} + \mathfrak{GL} + \mathfrak{HM} &=& 0; \\ \mathfrak{EN} + \mathfrak{FO} + \mathfrak{GP} + \mathfrak{HQ} &=& 0; \\ \mathfrak{IN} + \mathfrak{KO} + \mathfrak{LP} + \mathfrak{MQ} &=& 0, \end{array}$$

which conditions certainly take place at the first arrangement, where $x^{(n)} = x$, $y^{(n)} = y$,

$z^{(n)} = z$, $v^{(n)} = v$; in fact, then we have:

$$\begin{array}{llll}
\mathfrak{A} = 1; & \mathfrak{E} = 0; & \mathfrak{J} = 0; & \mathfrak{N} = 0; \\
\mathfrak{B} = 0; & \mathfrak{F} = 1; & \mathfrak{K} = 0; & \mathfrak{O} = 0; \\
\mathfrak{C} = 0; & \mathfrak{G} = 0; & \mathfrak{L} = 1; & \mathfrak{P} = 0; \\
\mathfrak{D} = 0; & \mathfrak{H} = 0; & \mathfrak{M} = 0; & \mathfrak{Q} = 1.
\end{array}$$

XXVI. Let us set the following ones derived from these values so

that by setting	these derived values are obtained
$x^{(n+1)} = x^{(n)}\cos\theta + y^{(n)}\sin\theta$	$x^{(n+1)} = \mathfrak{A}'x + \mathfrak{B}'y + \mathfrak{C}'z + \mathfrak{D}'v$
$y^{(n+1)} = x^{(n)}\sin\theta - y^{(n)}\cos\theta$	$y^{(n+1)} = \mathfrak{E}'x + \mathfrak{F}'y + \mathfrak{G}'z + \mathfrak{H}'v$
$z^{(n+1)} = z^{(n)}$	$z^{(n+1)} = \mathfrak{J}'x + \mathfrak{K}'y + \mathfrak{L}'z + \mathfrak{M}'v$
$v^{(n+1)} = v^{(n)}$	$v^{(n+1)} = \mathfrak{N}'x + \mathfrak{O}'y + \mathfrak{P}'z + \mathfrak{Q}'v$

which become

$$\begin{array}{lll}
\mathfrak{A}' = \mathfrak{A}\cos\theta + \mathfrak{E}\sin\theta; & \mathfrak{E}' = \mathfrak{A}\sin\theta - \mathfrak{E}\cos\theta; & \mathfrak{J}' = \mathfrak{J}; \quad \mathfrak{N}' = \mathfrak{N} \\
\mathfrak{B}' = \mathfrak{B}\cos\theta + \mathfrak{F}\sin\theta; & \mathfrak{F}' = \mathfrak{B}\sin\theta - \mathfrak{F}\cos\theta; & \mathfrak{K}' = \mathfrak{K}; \quad \mathfrak{O}' = \mathfrak{O} \\
\mathfrak{C}' = \mathfrak{C}\cos\theta + \mathfrak{G}\sin\theta; & \mathfrak{G}' = \mathfrak{C}\sin\theta - \mathfrak{G}\cos\theta; & \mathfrak{L}' = \mathfrak{L}; \quad \mathfrak{P}' = \mathfrak{P} \\
\mathfrak{D}' = \mathfrak{D}\cos\theta + \mathfrak{H}\sin\theta; & \mathfrak{H}' = \mathfrak{D}\sin\theta - \mathfrak{H}\cos\theta; & \mathfrak{M}' = \mathfrak{M}; \quad \mathfrak{Q}' = \mathfrak{Q}
\end{array}$$

From this, indeed, the following conditions are directly satisfied:

$$\begin{aligned}
\mathfrak{J}'\mathfrak{J}' + \mathfrak{K}'\mathfrak{K}' + \mathfrak{L}'\mathfrak{L}' + \mathfrak{M}'\mathfrak{M}' &= 1 \\
\mathfrak{J}'\mathfrak{N}' + \mathfrak{K}'\mathfrak{O}' + \mathfrak{L}'\mathfrak{P}' + \mathfrak{M}'\mathfrak{Q}' &= 0 \\
\mathfrak{N}'\mathfrak{N}' + \mathfrak{O}'\mathfrak{O}' + \mathfrak{P}'\mathfrak{P}' + \mathfrak{Q}'\mathfrak{Q}' &= 1.
\end{aligned}$$

XXVII. That the remaining conditions are also satisfied can be easily shown *viz.*

$$\begin{aligned}
\mathfrak{A}'\mathfrak{A}' + \mathfrak{B}'\mathfrak{B}' + \mathfrak{C}'\mathfrak{C}' + \mathfrak{D}'\mathfrak{D}' &= +(\mathfrak{A}\mathfrak{A} + \mathfrak{B}\mathfrak{B} + \mathfrak{C}\mathfrak{C} + \mathfrak{D}\mathfrak{D})\cos^2\theta \\
&\quad +(\mathfrak{E}\mathfrak{E} + \mathfrak{F}\mathfrak{F} + \mathfrak{G}\mathfrak{G} + \mathfrak{H}\mathfrak{H})\sin^2\theta \\
&\quad +2(\mathfrak{A}\mathfrak{E} + \mathfrak{B}\mathfrak{F} + \mathfrak{C}\mathfrak{G} + \mathfrak{D}\mathfrak{H})\sin\theta\cos\theta \\
&= 1\cdot\cos^2\theta + 1\cdot\sin^2\theta + 0\cdot\sin\theta\cos\theta = 1
\end{aligned}$$

in the same way as the sum of the squares of the second column $\mathfrak{E}'\mathfrak{E}' + \mathfrak{F}'\mathfrak{F}' + \mathfrak{G}'\mathfrak{G}' + \mathfrak{H}'\mathfrak{H}'$ is shown. Then the fact is clear about the sum of the products:

$$\begin{aligned}
&\mathfrak{A}'\mathfrak{J}' + \mathfrak{B}'\mathfrak{K}' + \mathfrak{C}'\mathfrak{L}' + \mathfrak{D}'\mathfrak{M}' \\
= \;& -(\mathfrak{A}\mathfrak{J} + \mathfrak{B}\mathfrak{K} + \mathfrak{C}\mathfrak{L} + \mathfrak{D}\mathfrak{M})\cos\theta + (\mathfrak{E}\mathfrak{J} + \mathfrak{F}\mathfrak{K} + \mathfrak{G}\mathfrak{L} + \mathfrak{H}\mathfrak{M})\sin\theta = 0
\end{aligned}$$

and equally also about these sums:

$$\begin{aligned}\mathfrak{A}'\mathfrak{N}' + \mathfrak{B}'\mathfrak{O}' + \mathfrak{C}'\mathfrak{P}' + \mathfrak{D}'\mathfrak{Q}' &= 0;\\ \mathfrak{E}'\mathfrak{I}' + \mathfrak{F}'\mathfrak{K}' + \mathfrak{G}'\mathfrak{L}' + \mathfrak{H}'\mathfrak{M}' &= 0 \quad \text{et} \quad \mathfrak{E}'\mathfrak{N}' + \mathfrak{F}'\mathfrak{O}' + \mathfrak{G}'\mathfrak{P}' + \mathfrak{H}'\mathfrak{Q}' = 0\end{aligned}$$

whence only these are left:

$$\begin{aligned}\mathfrak{A}'\mathfrak{E}' + \mathfrak{B}'\mathfrak{F}' + \mathfrak{C}'\mathfrak{G}' + \mathfrak{D}'\mathfrak{H}' &= +(\mathfrak{A}\mathfrak{A} + \mathfrak{B}\mathfrak{B} + \mathfrak{C}\mathfrak{C} + \mathfrak{D}\mathfrak{D})\sin\theta\cos\theta\\ &\quad -(\mathfrak{E}\mathfrak{E} + \mathfrak{F}\mathfrak{F} + \mathfrak{G}\mathfrak{G} + \mathfrak{H}\mathfrak{H})\sin\theta\cos\theta\\ &\quad +(\mathfrak{A}\mathfrak{E} + \mathfrak{B}\mathfrak{F} + \mathfrak{C}\mathfrak{G} + \mathfrak{D}\mathfrak{H})\sin^2\theta\\ &\quad -(\mathfrak{A}\mathfrak{E} + \mathfrak{B}\mathfrak{F} + \mathfrak{C}\mathfrak{G} + \mathfrak{D}\mathfrak{H})\cos^2\theta\\ &= +\sin\theta\cos\theta - \sin\theta\cos\theta + 0\sin^2\theta - 0\cos^2\theta = 0.\end{aligned}$$

XXVIII. As then the truth of these ten conditions in the first position, as already shown, is manifest, they also subsist in the second position derived by transformation of two, and hence also in all the consequent positions deduced similarly as the preceding ones. For this reason also the general solution of six transformations given as in §.21 will be done so that not only the ten prescribed conditions for the problem but also that the other ten mentioned in §.24 are satisfied: and this such that the ten first conditions are not satisfied, unless the latter ten are simultaneously satisfied. And hence it is now easily gathered that the same properties must also take place in the problems where similar question about 25, 36 and more numbers are set up. Let us proceed to the following.

Problem.

Find 25 numbers A, B, C, D etc., so that when arranged in a square form:

A, B, C, D, E
F, G, H, I, K
L, M, N, O, P
Q, R, S, T, U
V, W, X, Y, Z

the sum of the squares from single columns, vertical or horizontal, taken together equal unity, but the sum formed by the products of two columns, vertical or horizontal, vanish.

XXIX. From the preceding it is understood that the problem reduces to this, that taken 25 of these numbers as coefficients, five variables u, v, x, y, z will transform into other ones by such formulas:

$$\begin{aligned}\mathrm{U} &= \mathrm{A}u + \mathrm{B}v + \mathrm{C}x + \mathrm{D}y + \mathrm{E}z\\ \mathrm{V} &= \mathrm{F}u + \mathrm{G}v + \mathrm{H}x + \mathrm{I}y + \mathrm{K}z\\ \mathrm{X} &= \mathrm{L}u + \mathrm{M}v + \mathrm{N}x + \mathrm{O}y + \mathrm{P}z\\ \mathrm{Y} &= \mathrm{Q}u + \mathrm{R}v + \mathrm{S}x + \mathrm{T}y + \mathrm{U}z\\ \mathrm{Z} &= \mathrm{V}u + \mathrm{W}v + \mathrm{X}x + \mathrm{Y}y + \mathrm{Z}z\end{aligned}$$

which makes UU + VV + XX + YY + ZZ = $uu + vv + xx + yy + zz$. Therefore, as five quantities admit 10 diverse combinations, through ten successive transformations of two, the problem is resolved in the following manner:

I.

$$\begin{aligned} u^{\mathrm{I}} &= u\cos\alpha + v\sin\alpha \\ v^{\mathrm{I}} &= u\sin\alpha - v\cos\alpha \\ x^{\mathrm{I}} &= x \\ y^{\mathrm{I}} &= y \\ z^{\mathrm{I}} &= z \end{aligned}$$

II.

$$\begin{aligned} u^{\mathrm{II}} &= u^{\mathrm{I}}\cos\beta + x^{\mathrm{I}}\sin\beta \\ v^{\mathrm{II}} &= v^{\mathrm{I}} \\ x^{\mathrm{II}} &= u^{\mathrm{I}}\sin\beta - x^{\mathrm{I}}\cos\beta \\ y^{\mathrm{II}} &= y^{\mathrm{I}} \\ z^{\mathrm{II}} &= z^{\mathrm{I}} \end{aligned}$$

III.

$$\begin{aligned} u^{\mathrm{III}} &= u^{\mathrm{II}}\cos\gamma + y^{\mathrm{II}}\sin\gamma \\ v^{\mathrm{III}} &= v^{\mathrm{II}} \\ x^{\mathrm{III}} &= x^{\mathrm{II}} \\ y^{\mathrm{III}} &= u^{\mathrm{II}}\sin\gamma - y^{\mathrm{II}}\cos\gamma \\ z^{\mathrm{III}} &= z^{\mathrm{II}} \end{aligned}$$

IV.

$$\begin{aligned} u^{\mathrm{IV}} &= u^{\mathrm{III}}\cos\delta + z^{\mathrm{III}}\sin\delta \\ v^{\mathrm{IV}} &= v^{\mathrm{III}} \\ x^{\mathrm{IV}} &= x^{\mathrm{III}} \\ y^{\mathrm{VI}} &= y^{\mathrm{III}} \\ z^{\mathrm{IV}} &= u^{\mathrm{III}}\sin\delta - z^{\mathrm{III}}\cos\delta \end{aligned}$$

V.

$$\begin{aligned} u^{\mathrm{V}} &= u^{\mathrm{IV}} \\ v^{\mathrm{V}} &= v^{\mathrm{IV}}\cos\epsilon + x^{\mathrm{IV}}\sin\epsilon \\ x^{\mathrm{V}} &= v^{\mathrm{IV}}\sin\epsilon - x^{\mathrm{IV}}\cos\epsilon \\ y^{\mathrm{V}} &= y^{\mathrm{IV}} \\ z^{\mathrm{V}} &= z^{\mathrm{IV}} \end{aligned}$$

VI.

$$\begin{aligned} u^{\mathrm{VI}} &= u^{\mathrm{V}} \\ v^{\mathrm{VI}} &= v^{\mathrm{V}}\cos\zeta + y^{\mathrm{IV}}\sin\zeta \\ x^{\mathrm{VI}} &= x^{\mathrm{V}} \\ y^{\mathrm{VI}} &= v^{\mathrm{V}}\sin\zeta - y^{\mathrm{V}}\cos\zeta \\ z^{\mathrm{VI}} &= z^{\mathrm{V}} \end{aligned}$$

VII.

$$\begin{aligned} u^{\mathrm{VII}} &= u^{\mathrm{VI}} \\ v^{\mathrm{VII}} &= v^{\mathrm{VI}}\cos\eta + z^{\mathrm{VI}}\sin\eta \\ x^{\mathrm{VII}} &= x^{\mathrm{VI}} \\ y^{\mathrm{VII}} &= y^{\mathrm{VI}} \\ z^{\mathrm{VII}} &= v^{\mathrm{VI}}\sin\eta - z^{\mathrm{VI}}\cos\eta \end{aligned}$$

VIII.

$$\begin{aligned} u^{\mathrm{VIII}} &= u^{\mathrm{VII}} \\ v^{\mathrm{VIII}} &= v^{\mathrm{VII}} \\ x^{\mathrm{VIII}} &= x^{\mathrm{VII}}\cos\theta + y^{\mathrm{VII}}\sin\theta \\ y^{\mathrm{VIII}} &= y^{\mathrm{VII}}\sin\theta - y^{\mathrm{VII}}\cos\theta \\ z^{\mathrm{VIII}} &= z^{\mathrm{VII}} \end{aligned}$$

IX.

$$\begin{aligned} u^{\mathrm{IX}} &= u^{\mathrm{VIII}} \\ v^{\mathrm{IX}} &= v^{\mathrm{VIII}} \\ x^{\mathrm{IX}} &= x^{\mathrm{VIII}}\cos\kappa + z^{\mathrm{VIII}}\sin\kappa \\ y^{\mathrm{IX}} &= y^{\mathrm{VIII}} \\ z^{\mathrm{IX}} &= x^{\mathrm{VIII}}\sin\kappa - z^{\mathrm{VIII}}\cos\kappa \end{aligned}$$

X.

$$\begin{aligned} u^{\mathrm{X}} &= u^{\mathrm{IX}} &&= \mathrm{U} \\ v^{\mathrm{X}} &= v^{\mathrm{IX}} &&= \mathrm{V} \\ x^{\mathrm{X}} &= x^{\mathrm{IX}} &&= \mathrm{X} \\ y^{\mathrm{X}} &= y^{\mathrm{IX}}\cos\lambda + z^{\mathrm{IX}}\sin\lambda &&= \mathrm{Y} \\ z^{\mathrm{X}} &= y^{\mathrm{IX}}\sin\lambda - z^{\mathrm{IX}}\cos\lambda &&= \mathrm{Z} \end{aligned}$$

XXX. These operations therefore introduce ten arbitrary angles, wherein the nature of the complete or general solution is included. In fact, while the conditions from the vertical columns suffice to solve the posed problem, the other conditions from the horizontal columns satisfy it automatically; the square of the sums give five and the product between two 10 equations; thus satisfying in all 15 conditions; therefore, since 25 numbers is investigated, among which ten till now are kept indeterminate, which also agrees very well with the solution given here, then there cannot take place more than 10 transformations, instituted between two separate quantities.

XXXI. To make the formulation easier to develop, in whatever way two operations can be joined, it is made in the very same way as the solution of the preceding problem. It is convenient to take these combinations such that the solitary quantity suffering no change is different in every one: that will happen if two preceding transformations are joined in the following way:

(I, VIII), (II, VII), (III, IX), (IV, VI), (V, X)

from which the following five transformations are born:

I.

$$\begin{aligned} u^{\mathrm{I}} &= u\cos\alpha + v\sin\alpha \\ v^{\mathrm{I}} &= u\sin\alpha - v\cos\alpha \\ x^{\mathrm{I}} &= x\cos\beta + y\sin\beta \\ y^{\mathrm{I}} &= x\sin\beta - y\cos\beta \\ z^{\mathrm{I}} &= z \end{aligned}$$

II.

$$\begin{aligned} u^{\mathrm{II}} &= u^{\mathrm{I}}\cos\gamma + x^{\mathrm{I}}\sin\gamma \\ v^{\mathrm{II}} &= v^{\mathrm{I}}\cos\delta + z^{\mathrm{I}}\sin\delta \\ x^{\mathrm{II}} &= u^{\mathrm{I}}\sin\gamma - x^{\mathrm{I}}\cos\gamma \\ y^{\mathrm{II}} &= y^{\mathrm{I}} \\ z^{\mathrm{II}} &= v^{\mathrm{I}}\sin\delta - z^{\mathrm{I}}\cos\delta \end{aligned}$$

III.

$$\begin{aligned} u^{\mathrm{III}} &= u^{\mathrm{II}}\cos\epsilon + y^{\mathrm{II}}\sin\epsilon \\ v^{\mathrm{III}} &= v^{\mathrm{II}} \\ x^{\mathrm{III}} &= x^{\mathrm{II}}\cos\zeta + z^{\mathrm{II}}\sin\zeta \\ y^{\mathrm{III}} &= u^{\mathrm{II}}\sin\epsilon - y^{\mathrm{II}}\cos\epsilon \\ z^{\mathrm{III}} &= z^{\mathrm{II}}\sin\zeta - z^{\mathrm{II}}\cos\zeta \end{aligned}$$

IV.

$$\begin{aligned} u^{\mathrm{IV}} &= u^{\mathrm{III}}\cos\eta + z^{\mathrm{III}}\sin\eta \\ v^{\mathrm{IV}} &= v^{\mathrm{III}}\cos\theta + y^{\mathrm{III}}\sin\theta \\ x^{\mathrm{IV}} &= x^{\mathrm{III}} \\ y^{\mathrm{IV}} &= v^{\mathrm{III}}\sin\theta - y^{\mathrm{III}}\cos\theta \\ z^{\mathrm{IV}} &= u^{\mathrm{III}}\sin\eta - z^{\mathrm{III}}\cos\eta \end{aligned}$$

V.

$$\begin{aligned} u^{\mathrm{V}} &= u^{\mathrm{IV}} \\ v^{\mathrm{V}} &= v^{\mathrm{IV}}\cos\kappa + x^{\mathrm{IV}}\sin\kappa \\ x^{\mathrm{V}} &= v^{\mathrm{IV}}\sin\kappa - x^{\mathrm{IV}}\cos\kappa \\ y^{\mathrm{V}} &= y^{\mathrm{IV}}\cos\lambda + z^{\mathrm{IV}}\sin\lambda \\ z^{\mathrm{V}} &= y^{\mathrm{IV}}\sin\lambda - z^{\mathrm{IV}}\cos\lambda \end{aligned}$$

XXXII. In the same way problems of this sort with 36 and more numbers, of which indeed there is a great number of squares, can be resolved; where for bringing together the calculations not only two, but also three and next even more transformations can be included in one operation; and here continually more beautiful agreement is noticed between the general solution from all the taken combinations and the nature of the facts. Indeed, setting in general the number of quantities searched for $= nn$, the sum of squares equal to one give n conditions, but the products giving zero are $(nn - n)/2$, so that together there are $(nn + n)/2$ conditions, which number snatches away from the number of quantities searched for, leaving $(nn - n)/2$, and therefore as many of the searched quantities remain undetermined, or the general solution must contain as many arbitrary quantities, but according to the rule explained above $(nn-n)/2$ transformations are used to this goal, of which therefore precisely as many arbitrary angles are introduced in the calculation.

General solution of the problem initially proposed in rational numbers

XXXIII. As a final mark I will join an attack on the solution of our problem by the Diophantean method, which can be shown beautifully as follows.

Taking four numbers p, q, r, s at will and putting the squares of their sum $pp+qq+rr+ss=u$, the nine numbers searched for are found to be determined as:

$$\begin{aligned}
\mathrm{A} &= \frac{pp+qq-rr-ss}{u}; \quad \mathrm{B} = \frac{2qr+2ps}{u}; \quad \mathrm{C} = \frac{2qs-2pr}{u};\\
\mathrm{D} &= \frac{2qr-2ps}{u}; \mathrm{E} = \frac{pp-qq+rr-ss}{u}; \quad \mathrm{F} = \frac{2pq+2rs}{u};\\
\mathrm{G} &= \frac{2qs+2pr}{u}; \quad \mathrm{H} = \frac{2rs-2pq}{u}; \quad \mathrm{I} = \frac{pp-qq-rr+ss}{u}.
\end{aligned}$$

Hence the simplest numbers, which are unequal to one another as well, are collected as follows as arranged in a square:

$+\frac{47}{57}$	$+\frac{28}{57}$	$-\frac{16}{57}$
$+\frac{4}{57}$	$+\frac{23}{57}$	$+\frac{52}{57}$
$+\frac{32}{57}$	$-\frac{44}{57}$	$+\frac{17}{57}$

that is for $p=6$, $q=4$, $r=2$, $s=1$.

$+\frac{53}{63}$	$+\frac{26}{63}$	$-\frac{22}{63}$
$-\frac{2}{63}$	$+\frac{43}{63}$	$+\frac{46}{63}$
$+\frac{34}{57}$	$-\frac{38}{63}$	$+\frac{37}{63}$

where $p=7$, $q=3$, $r=2$, $s=1$.

And here is yet another almost as simple example

$+\frac{51}{71}$	$-\frac{42}{71}$	$-\frac{26}{71}$
$-\frac{18}{71}$	$+\frac{19}{71}$	$-\frac{66}{71}$
$+\frac{46}{71}$	$+\frac{54}{71}$	$+\frac{3}{71}$

$+\frac{86}{99}$	$+\frac{38}{99}$	$-\frac{31}{99}$
$-\frac{14}{99}$	$+\frac{79}{99}$	$+\frac{58}{99}$
$+\frac{47}{99}$	$-\frac{46}{99}$	$+\frac{74}{99}$

The case of sixteen numbers.

XXXIV. If a solution in rationals is desired for the case of sixteen numbers similarly arranged in a square, out of which not very large ones can be found easily, the method given above is not well adapted for this purpose. However, I have got another very singular way wide open, where taking eight numbers at will a, b, c, d, p, q, r, s, the sixteen numbers are arranged in a square as follows

$+ap+bq+cr+ds$	$+aq-bp+cs-dr$	$+ar-bs-cp+dq$	$+as+br-cq-dp$
$+aq-bp-cs+dr$	$-ap-bq+cr+ds$	$-as-br-cq-dp$	$+ar-bs+cp-dq$
$+ar+bs-cp-dq$	$+as-br-cq+dp$	$-ap+bq-cr+ds$	$-aq-bp-cs-dr$
$+as-br+cq-dp$	$-ar-bs-cp-dq$	$+aq+bp-cs-dr$	$-ap+bq+cr-ds$

where the sums of the squares in the individual columns either with the horizontal or vertical always produce the same value

$$= (aa+bb+cc+dd)(pp+qq+rr+ss).$$

Therefore in order that these sums equal unity, in the expression to render square, the individual numbers must be divided by the root of each. Then truly these sixteen numbers also enjoy this property, that the sum of the products among two columns taken either horizontally or vertically vanish everywhere.

XXXV. Hence it is therefore easy to deduce several examples in small enough numbers, among which the following for that reason are seen worth noticing, that all numbers are equal

$+37$	$+4$	$+1$	$+12$
-6	$+33$	-18	$+9$
$+11$	$+8$	-7	-36
-2	$+19$	$+34$	-3

squared

1369	16	1	144	1530
36	1089	324	81	1530
121	64	49	1296	1530
4	361	1156	9	1530
1530	1530	1530	1530	summed

and of the products of two it is an evident fact that:

$$-6\cdot 37 + 4\cdot 33 - 1\cdot 18 + 9\cdot 12 = 0$$

$$+4\cdot 37 - 6\cdot 33 + 8\cdot 11 - 2\cdot 19 = 0.\text{etc.}$$

But by inspecting these general forms it is easily shown that they satisfied perfectly all the 20 conditions given in §§.20 and 24, and indeed so that the four squares sum up to unity is recovered.

XXXVI. This solution deserves the more attention since beside it I have not found any fixed method better than as if guessing: and because it involves precisely eight numbers, which indeed by reduction towards unity reduce to seven, it can hardly be doubted that this solution would not be universal and include every possible solution. Therefore who investigates the direct way leading to this solution will certainly be thinking to bring results by the help of Analysis. But whether it can be expected to hold beyond similar solutions for larger squares than 25, 36 and greater, I would not dare to affirm. Hence not only common Algebra but also the Diophantean Method would be seen to grow very much.

A curious problem.

Invent sixteen numbers

A	B	C	D
E	F	G	H
I	K	L	M
N	O	P	Q

arranged in a square such that not only the sum of the squares of the columns taken both horizontally and vertically but also those which are taken along the diagonals, namely $A^2 + F^2 + L^2 + Q^2$ and $D^2 + G^2 + K^2 + N^2$ are all equal, and moreover the product of two of them as above vanish *viz.*

$$\begin{aligned} AE + BF + CG + DH &= 0 \qquad AB + EF + IK + NO = 0 \\ AI + BK + CL + DM &= 0 \qquad AC + EG + IL + NP = 0 \\ AN + BO + CP + DQ &= 0 \qquad AD + EH + IM + NQ = 0 \\ EI + FK + GL + HM &= 0 \qquad BC + FG + KL + OP = 0 \\ EH + FO + GP + HQ &= 0 \qquad BD + FH + KM + OQ = 0 \\ IH + KO + LP + MQ &= 0 \qquad CD + GH + LM + PQ = 0. \end{aligned}$$

Solution:

Therefore the 22 conditions that should be satisfied are these; but laying aside the two considered on the diagonals, all the remaining ones fulfil the following general form

$+ap+bq+cr+ds$	$+ar-bs-cp+dq$	$-as-br+cq+dp$	$+aq-bp+cs-dr$
$-aq+bp+cs-dr$	$+as+br+cq+dp$	$+ar-bs+cp-dq$	$+ap+bq-cr-ds$
$+ar+bs-cp-dq$	$-ap+bq-cr+ds$	$+aq+bp+cs+dr$	$+as-br-cq+dp$
$-as+br-cq+dp$	$-aq-bp+cs+dr$	$-ap+bq+cr-ds$	$+ar+bs+cp+dq$

where the sum of the four squares from the columns either taken horizontally or vertically is

$$(aa+bb+cc+dd)(pp+qq+rr+ss)$$

which, as they also equal the sum through the squares of the diagonals, would prepare for the following two equations:

$$+abpq+abrs+acpr+acqs+adps+adqr+bcqr+bcps+bdqs+bdpr+cdrs+cdpq=0$$

$$-abpq-abrs+acpr+acps-adps-adqr-bcqr-bcps+bdqs+bdpr-cdrs-cdpq=0$$

from which the following two are deduced:

$$\begin{aligned} (ac+bd)(pr+qs) &= 0, \\ (ab+cd)(pq+rs)+(ad+bc)(ps+qr) &= 0. \end{aligned}$$

From here two conditions are pulled out:

$$\text{I.} \quad pr+qs=0 \quad \text{and} \quad \text{II.} \quad \frac{a}{c}=\frac{-d(pq+rs)-b(ps+qr)}{b(pq+rs)+d(ps+qr)}$$

so that thus far we are left with six arbitrary letters.

Let us evaluate an example taking $p=6$, $q=3$, $r=1$, $s=-2$ from which by letting $\frac{a}{c}=\frac{-16d+9b}{16b-9d}$, would give $d=0$, $b=1$, $a=9$, $c=16$ and the square of all the satisfying conditions is

+73	−85	+65	+11
−53	+31	+107	+41
−89	−67	+1	−67
−29	−65	−35	+103

where the sum of the four squares along the columns taken either horizontally or vertically, and similarly along the diagonals, give = 16900 from which, if this number be divided by 130, all these sums reduce to unity.

If it offends someone that the numbers 65 and 67 occur here twice, I will join another such square expressed in a bit smaller numbers.

+68	−29	+41	−37
−17	+31	+79	+32
+59	+28	−23	+61
−11	−65	+8	+49

where the sum of the four squares is 8515.

Finally it is noted in this square that also the squares of the numbers in the corners and in the middle produce the same sum.