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CONCERNING THE SUMMATION OF SERIES SATISFIED BY THIS FORM

$$\frac{a}{1} + \frac{a^2}{4} + \frac{a^3}{9} + \frac{a^4}{16} + \frac{a^5}{25} + \frac{a^6}{36} + \text{etc.}$$

1. From those, which I first reported some time ago in the literature concerning the summation of the reciprocals of powers [See also E20, E41, E61, E130, & E597], only two cases are able to be derived, in which it is allowed to assign the sum of the series proposed here: the one case evidently, in which $a = 1$, in which I have shown the sum of this series

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \frac{1}{36} + \text{etc.}$$

to be $= \frac{\pi\pi}{6}$, with π denoting the periphery of the circle, the diameter of which is equal to $= 1$; truly the other case in which there is $a = -1$; for then with the sign of this series changed

$$1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \frac{1}{25} - \frac{1}{36} + \text{etc.} ,$$

the sum is $= \frac{\pi\pi}{12}$. Now in addition by a straight-forwards method I have found the sum of this series for the individual case $a = \frac{1}{2}$,

$$\frac{1}{1.2} + \frac{1}{4 \cdot 2^2} + \frac{1}{9.2^3} + \frac{1}{16.2^4} + \text{etc.} ,$$

to be $= \frac{\pi\pi}{12} - \frac{1}{2}\left(l2\right)^2$ with $l2$ denoting the hyperbolic logarithm of two, which is 0.693147180. Nor truly besides these cases can any other case be agreed upon till then, by which one could assign a sum.

2. But the method, by which I arrived at this latter case, can be extended further, thus so that thence several remarkable relations between two or more series of this kind are able to be found. But that method depends on this lemma :

LEMMA

If there is put

$$p = \int \frac{\partial x}{x} ly \quad and \quad q = \int \frac{\partial y}{y} lx,$$

there will be the sum

$$p + q = lx.ly + C ,$$

if indeed the constant may be defined thus, so that it may be satisfied by a single case.

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Hence therefore we may run across the following problems evidently with a changing relation between x and y.

PROBLEM 1

If there should be $x + y = 1$, these two formulas

$$p = \int \frac{\partial x}{x} ly \quad and \quad q = \int \frac{\partial y}{y} lx$$

may be resolved in series, thus so that hence there may be produced

$$p + q = lx.ly + C .$$

SOLUTION

3. Therefore since there shall be $y = 1 - x$, there shall be

$$ly = -x - \frac{xx}{2} - \frac{x^3}{3} - \text{ etc.}$$

and hence

$$p = \int \frac{\partial x}{x} ly = -\frac{x}{1} - \frac{xx}{4} - \frac{x^3}{9} - \frac{x^4}{16} - \text{etc.}$$

[At this stage, the proviso that $|x| < 1$ and $|y| < 1$ needs to be inserted; certain expansions in the following sections hence are invalid, and involve diverging series. This point has to be kept in mind, as it is not referred to again in this translation.]
and in a similar manner on account of

$$x = 1 - y \quad and \quad lx = -y - \frac{yy}{2} - \frac{y^3}{3} - \frac{y^4}{4} - \text{etc.}$$

there will be

$$q = \int \frac{\partial y}{y} lx = -\frac{y}{1} - \frac{y^2}{4} - \frac{y^3}{9} - \frac{y^4}{16} - \text{etc.,}$$

on account of which the sum of these series will be $lx{\cdot}ly + C$. For with the constant C requiring to be defined we may consider the case, in which $x = 0$ and $y = 1$ and thus $lx{\cdot}ly = 0$; then there will be therefore

$$p + q = -1 - \frac{1}{4} - \frac{1}{9} - \frac{1}{16} - \text{etc.} = -\frac{\pi\pi}{6},$$

from which there is elicited $C = -\frac{\pi\pi}{6}$.

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4. Hence as often as there were $x + y = 1$, the sum of these two series taken together

$$\frac{x}{1} + \frac{x^2}{4} + \frac{x^3}{9} + \frac{x^4}{16} + \cdots + \frac{y}{1} + \frac{y^2}{4} + \frac{y^3}{9} + \frac{y^4}{16} + \text{etc.}$$

will be $= \frac{\pi\pi}{6} - lx \cdot ly$; and hence at once there follows the three cases mentioned above. For on taking $x = \frac{1}{2}$ there will be also $y = \frac{1}{2}$ and thus both these series become equal to each other, from which there follows to be :

$$\frac{1}{1.2} + \frac{1}{4.2^2} + \frac{1}{9.2^3} + \frac{1}{16.2^4} + \text{etc.} = \frac{\pi\pi}{12} - \frac{1}{2}\left(l\frac{1}{2}\right)^2 = \frac{\pi\pi}{12} - \frac{1}{2}(l2)^2 .$$

Now in addition, as often as there should be $a + b = 1$ and there may be put in place

$$A = \frac{a}{1} + \frac{a^2}{4} + \frac{a^3}{9} + \text{etc.} \quad \text{and} \quad B = \frac{b}{1} + \frac{b^2}{4} + \frac{b^3}{9} + \text{etc.},$$

there will always be $A + B = \frac{\pi\pi}{6} - la \cdot lb$. Hence therefore, if the sum of one of these series should be known from elsewhere, also the other sum might become known. And this is the that problem itself, that I have discussed formerly.

PROBLEM 2

If there should be $x - y = 1$, these two formulas

$$p = \int \frac{\partial x}{x} ly \quad and \quad q = \int \frac{\partial y}{y} lx$$

may be resolved in series, thus so that hence there may be produced

$$p + q = lx \cdot ly + C .$$

SOLUTION

5. Since here there shall be $y = x - 1$, there will be

$$ly = l(x-1) = lx + l\left(1 - \frac{1}{x}\right) = lx - \frac{1}{x} - \frac{1}{2xx} - \frac{1}{3x^3} - \frac{1}{4x^4} - \text{etc.}$$

and hence

$$p = \int \frac{\partial x}{x} ly = \frac{1}{2}(lx)^2 + \frac{1}{x} + \frac{1}{4x^2} + \frac{1}{9x^3} + \frac{1}{16x^4} + \text{etc.}$$

Then on account of $x = 1 + y$ there will be

$$lx = \frac{y}{1} - \frac{yy}{2} - \frac{y^3}{3} - \frac{y^4}{4} + \text{etc.}$$

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and thus

$$q = \int \frac{\partial y}{y} lx = \frac{y}{1} - \frac{y^2}{4} + \frac{y^3}{9} - \frac{y^4}{16} + \text{etc.}$$

on account of which we will have

$$p + q = lx.ly + C .$$

For the constant requiring to be determined we may consider the case $y = 0$, from which there becomes $x = 1$ and $lx \cdot ly = 0$; then therefore there will be

$$p = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \text{etc} = \frac{\pi\pi}{6} \text{ and } q = 0,$$

from which the constant is defined $C = \frac{\pi\pi}{6}$.

6. Therefore here again we may consider the two series, the sum of which we may be able to assign jointly:

$$\left. \begin{array}{l} \frac{1}{x} + \frac{1}{4x^2} + \frac{1}{9x^3} + \frac{1}{16x^4} + \text{etc.} \\ + \frac{y}{1} - \frac{yy}{4} + \frac{y^3}{9} - \frac{y^4}{16} + \text{etc.} \end{array} \right\} = \frac{\pi\pi}{6} - \frac{1}{2}(lx)^2 + lx \cdot ly = \frac{\pi\pi}{6} + lx \cdot l\frac{y}{\sqrt{x}}.$$

7. But if hence we should have these two series :

$$A = \frac{a}{1} + \frac{a^2}{4} + \frac{a^3}{9} + \frac{a^4}{16} + \text{etc.}$$

and

$$B = \frac{b}{1} - \frac{b^2}{4} + \frac{b^3}{9} - \frac{b^4}{16} + \text{etc.},$$

thus so that there shall be $a = \frac{1}{x}$ and $b = y$, and this relation is given between a and b

$$ab + a = 1,$$

there will be

$$A + B = \frac{\pi\pi}{6} - la \cdot lb\sqrt{a} .$$

We may consider the case, in which

$$b = a \left(= \frac{-1+\sqrt{5}}{2} \text{ on account of } ab + a = 1 \right),$$

and there will be

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$$A + B = 2\left(\frac{a}{1} + \frac{a^3}{9} + \frac{a^5}{25} + \frac{a^7}{49} + \text{etc.}\right);$$

concerning which, with $a = \frac{\sqrt{5}-1}{2}$ present, the sum of this series

$$\frac{a}{1} + \frac{a^3}{9} + \frac{a^5}{25} + \text{etc.}$$

will be

$$\frac{\pi\pi}{12} - \frac{1}{2}la \cdot la\sqrt{a}.$$

8. Then also this case is worthy of note, in which $b = -a$ and $A + B = 0$; for in this case there will be

$$\frac{\pi\pi}{6} = la \cdot lb\sqrt{a}.$$

But because $b = -a$, there will be

$$-aa + a = 1$$

and hence

$$a = \frac{1+\sqrt{-3}}{2} \quad \text{and} \quad b = \frac{1-\sqrt{-3}}{2}.$$

Now since there shall be

$$lb\sqrt{a} = \frac{1}{2}labb,$$

on account of

$$bb = \frac{-1+\sqrt{-3}}{2}$$

there will be $abb = -1$, from which there follows to be

$$\frac{\pi\pi}{6} = l\frac{1+\sqrt{-3}}{2} \cdot \sqrt{-1},$$

that which agrees uncommonly well with the known expression of the periphery of the circle by imaginary logarithms.

9. If we might put here $a = \frac{1}{2}$, there becomes $b = 1$ and thus

$$B = 1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \text{etc.}$$

and hence

$$A + B = \frac{1}{1.2} + \frac{1}{4.2^2} + \frac{1}{9.2^3} + \cdots + \frac{\pi\pi}{12} = \frac{\pi\pi}{6} - \frac{1}{2}(l2)^2,$$

from which there may appear the case of the third mentioned initially.

But truly we may put here $b = \frac{1}{2}$ and there shall be $a = \frac{2}{3}$ and

$$lb\sqrt{a} = \tfrac{1}{2}lbba = \tfrac{1}{2}l\tfrac{1}{6} = -\tfrac{1}{2}l6 \quad \text{et} \quad la = -l\tfrac{3}{2},$$

from which we shall have

$$\left.\begin{array}{l} A = \frac{2}{1\cdot 3} \ + \frac{2^2}{4\cdot 3^2} \ + \frac{2^3}{9\cdot 3^3} + \text{etc.} \\[2mm] +B = \frac{1}{1\cdot 2} - \frac{1}{4\cdot 2^2} \ + \frac{1}{9\cdot 2^3} - \text{etc.} \end{array}\right\} = \frac{\pi\pi}{6} - \tfrac{1}{2}l\tfrac{3}{2}\cdot l6.$$

Hence we may subtract this equation from the first equation :

$$\left.\begin{array}{l} \frac{1}{1\cdot 3} \ + \frac{1}{4\cdot 3^2} \ + \frac{1}{9\cdot 3^3} + \text{etc.} \\[2mm] + \frac{2}{1\cdot 3} + \frac{2^2}{4\cdot 3^2} + \frac{2^3}{9\cdot 3^3} + \text{etc.} \end{array}\right\} = \frac{\pi\pi}{6} - l3\cdot l\tfrac{3}{2}$$

and there will remain

$$\left.\begin{array}{l} \frac{1}{1\cdot 2} \ - \frac{1}{4\cdot 2^2} \ + \frac{1}{9\cdot 2^3} - \frac{1}{16\cdot 2^4} + \text{etc.} \\[2mm] - \frac{1}{1\cdot 3} - \frac{1}{4\cdot 3^2} - \frac{1}{9\cdot 3^3} - \frac{1}{16\cdot 3^4} - \text{etc.} \end{array}\right\} = l3\cdot l\tfrac{3}{2} - \tfrac{1}{2}l\tfrac{3}{2}\cdot l6 = \tfrac{1}{2}\left(l\tfrac{3}{2}\right)^2.$$

And thus we have obtained this noteworthy equation :

$$\frac{1}{1\cdot 2} - \frac{1}{4\cdot 2^2} + \frac{1}{9\cdot 2^3} - \frac{1}{16\cdot 2^4} + \text{etc.} = \tfrac{1}{2}\left(l\tfrac{3}{2}\right)^2 + \frac{1}{1\cdot 3} + \frac{1}{4\cdot 3^2} + \frac{1}{9\cdot 3^3} + \text{etc.} ,$$

where the ratio of the periphery π has left from the inside of the calculation. Truly the same relation can be elicited easier in the following manner.

ANOTHER SOLUTION OF THE SAME PROBLEM

10. With the working of the first part p remaining, the other part q on account of

$$lx = l(1+y) = ly + l\left(1+\tfrac{1}{y}\right)$$

hence

$$lx = ly + \tfrac{1}{y} - \frac{1}{2y^2} + \frac{1}{3y^3} - \text{etc.}$$

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will become

$$q = \int \frac{\partial y}{y}(lx) = \tfrac{1}{2}(ly)^2 - \tfrac{1}{y} + \tfrac{1}{4y^2} - \tfrac{1}{9y^3} + \tfrac{1}{16y^4} - \text{etc.}$$

Therefore now there shall be

$$p + q = lx \cdot ly + C \, ;$$

where the constant C thence can be defined, because on putting $y = 1$ there becomes $x = 2$ and hence

$$p = \tfrac{1}{2}(l2)^2 + \tfrac{\pi\pi}{12} - \tfrac{1}{2}(l2)^2 = \tfrac{\pi\pi}{12}$$

and

$$q = -1 + \tfrac{1}{4} - \tfrac{1}{9} + \tfrac{1}{16} - \text{etc.} = -\tfrac{\pi\pi}{12},$$

with which values substituted for this case there will emerge $p + q = 0 = 0 + C$, and consequently $C = 0$.

11. Truly this constant also can be defined in another way. For the sake of brevity we may put

$$X = \tfrac{1}{x} + \tfrac{1}{4xx} + \tfrac{1}{9x^3} + \tfrac{1}{16x^4} + \text{etc.}$$

and

$$Y = \tfrac{1}{y} - \tfrac{1}{4yy} + \tfrac{1}{9y^3} - \tfrac{1}{16y^4} + \text{etc.,}$$

so that we may have

$$p = \tfrac{1}{2}(lx)^2 + X \quad \text{et} \quad q = \tfrac{1}{2}(ly)^2 - Y,$$

and hence there is made

$$p + q = \tfrac{1}{2}(lx)^2 + \tfrac{1}{2}(ly)^2 + X - Y = lx \cdot ly + C \, ;$$

from which we deduce

$$Y - X = \tfrac{1}{2}(lx)^2 + \tfrac{1}{2}(ly)^2 - lx \cdot ly - C = \tfrac{1}{2}\left(l\tfrac{x}{y}\right)^2 - C,$$

where it is to be noted $y = x - 1$. Now according to the constant C to be defined we may consider the case $x = \infty$, from which there becomes $X = 0$ and $Y = 0$, now in addition $l\tfrac{x}{y} = 0$, with which noted there will be $0 = -C$ and thus $C = 0$.

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12. Hence therefore we have obtained the sum of the two series X and Y of which the difference is expressed by logarithms alone, since there shall be

$$Y - X = \tfrac{1}{2}\left(l\tfrac{x}{y}\right)^2 = \tfrac{1}{2}\left(l\tfrac{y+1}{y}\right)^2$$

on account of $x = y+1$.

From this form, on taking $y = 2$ at once the relation found before emanates forth

$$\tfrac{1}{1\cdot 2} - \tfrac{1}{4\cdot 2^2} + \tfrac{1}{9\cdot 2^3} - \tfrac{1}{16\cdot 2^4} + \text{etc.} = \tfrac{1}{2}\left(l\tfrac{3}{2}\right)^2 + \tfrac{1}{1\cdot 3} + \tfrac{1}{4\cdot 3^2} + \tfrac{1}{9\cdot 3^3} + \text{etc.}$$

Moreover in the same manner now we will consider more generally

$$\tfrac{1}{1\cdot y} - \tfrac{1}{4\cdot y^2} + \tfrac{1}{9\cdot y^3} - \tfrac{1}{16\cdot y^4} + \text{etc.}$$

$$= \tfrac{1}{2}\left(l\tfrac{y+1}{y}\right)^2 + \tfrac{1}{1(y+1)} + \tfrac{1}{4(y+1)^2} + \tfrac{1}{9(y+1)^3} + \text{etc.},$$

where in place of y it is allowed to take whatever it pleases to accept.

PROBLEM 3
If between x and y this relation may be given : $xy + x + y = c$, the two formulas

$$p = \int \tfrac{\partial x}{x} ly \quad and \quad q = \int \tfrac{\partial y}{y} lx,$$

may be resolved series, thus so that hence there may be produced

$$p + q = lx.ly + C,$$

SOLUTION
13. Hence therefore in the first place there will be

$$y = \tfrac{c-x}{1+x},$$

the logarithm of this is expressed by the two following series :

$$ly = \begin{cases} lc - \tfrac{x}{c} - \tfrac{x^2}{2c^2} - \tfrac{x^3}{3c^3} - \tfrac{x^4}{4c^4} - \text{etc.} \\ -x + \tfrac{x^2}{2} - \tfrac{x^3}{3} + \tfrac{x^4}{4} - \tfrac{x^5}{5} + \text{etc.,} \end{cases}$$

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from which

$$p = \int \frac{\partial x}{x} ly = \begin{cases} lc \cdot lx - \frac{x}{c} - \frac{x^2}{4c^2} - \frac{x^3}{9c^3} - \frac{x^4}{16c^4} - \text{etc.} \\ \quad - x + \frac{x^2}{4} - \frac{x^3}{9} + \frac{x^4}{16} - \text{etc.} \end{cases}$$

In a similar manner, since here shall be $x = \frac{c-y}{1+y}$, there will be

$$q = \int \frac{\partial y}{y} lx = \begin{cases} lc \cdot ly - \frac{y}{c} - \frac{y^2}{4c^2} - \frac{y^3}{9c^3} - \frac{y^4}{16c^4} - \text{etc.} \\ \quad - \frac{y}{1} + \frac{y^2}{4} - \frac{y^3}{9} + \frac{y^4}{16} - \text{etc.} \end{cases}$$

And hence there will be $p + q = lx.ly + C$.

14. For the constant requiring to be defined we may consider the case, in which $x = 0$ and thus $p = lc \cdot lx$ and

$$q = \left(lc \right)^2 - 1 - \left. \begin{matrix} \frac{1}{4} - \frac{1}{9} - \frac{1}{16} - \text{etc.} \\ -\frac{c}{1} + \frac{c^2}{4} - \frac{c^3}{9} + \frac{c^4}{16} - \text{etc.} \end{matrix} \right\}$$

or

$$q = \left(lc \right)^2 - \frac{\pi\pi}{6} - \frac{c}{1} + \frac{c^2}{4} - \frac{c^3}{9} + \frac{c^4}{16} - \text{etc.},$$

from which our equation emerges

$$p + q = lc \cdot lx + \left(lc \right)^2 - \frac{\pi\pi}{6} - \frac{c}{1} + \frac{c^2}{4} - \frac{c^3}{9} + \text{etc.} = lc \cdot lx + C,$$

where therefore the terms $lc \cdot lx$ cancel each other out, thus so that there becomes

$$C = \left(lc \right)^2 - \frac{\pi\pi}{6} - \frac{c}{1} + \frac{c^2}{4} - \frac{c^3}{9} + \text{etc.}$$

15. Therefore here there occur five infinite series, which we may indicate in the following manner :

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$$\frac{c}{1} - \frac{c^2}{4} + \frac{c^3}{9} - \frac{c^4}{16} + \text{etc.} = O,$$

$$\frac{x}{c} + \frac{x^2}{4 \cdot c^2} + \frac{x^3}{9 \cdot c^3} + \frac{x^4}{16 \cdot c^4} + \text{etc.} = P,$$

$$\frac{x}{1} - \frac{x^2}{4} + \frac{x^3}{9} - \frac{x^4}{16} + \text{etc.} = Q, ,$$

$$\frac{y}{c} + \frac{y^2}{4 \cdot c^2} + \frac{y^3}{9 \cdot c^3} + \frac{y^4}{16 \cdot c^4} + \text{etc.} = R,$$

$$\frac{y}{1} - \frac{y^2}{4} + \frac{y^3}{9} - \frac{y^4}{16} + \text{etc.} = S;$$

with which letters introduced our equation will be

$$lc \cdot lx - P - Q + lc \cdot ly - R - S = lx \cdot ly + \left(lc\right)^2 - \tfrac{\pi\pi}{6} - O,$$

from which there follows to become

$$O - P - Q - R - S = lx.ly + \left(lc\right)^2 - lc \cdot lx - lc \cdot ly - \tfrac{\pi\pi}{6},$$

which expression is contracted into the following :

$$O - P - Q - R - S = l\tfrac{x}{c}.l\tfrac{y}{c} - \tfrac{\pi\pi}{6}$$

or with the signs changed

$$P + Q + R + S - O = \tfrac{\pi\pi}{6} - l\tfrac{x}{c}.l\tfrac{y}{c}.$$

16. Here a memorable enough case is come across, when $c = 1$, because then there becomes

$$P + Q = \tfrac{2x}{1} + \tfrac{2x^3}{9} + \tfrac{2x^5}{25} + \text{etc.}$$

and

$$R + S = \tfrac{2y}{1} + \tfrac{2y^3}{9} + \tfrac{2y^5}{25} + \text{etc.,}$$

then truly

$$O = \tfrac{\pi\pi}{12},$$

and thus we have gone after a simple enough relation between the two series, which is

$$\left. \begin{array}{l} \frac{x}{1} + \frac{x^3}{9} + \frac{x^5}{25} + \frac{x^7}{49} + \text{etc.} \\[2mm] \frac{y}{1} + \frac{y^3}{9} + \frac{y^5}{25} + \frac{y^7}{49} + \text{etc.} \end{array} \right\} = \tfrac{\pi\pi}{8} - \tfrac{1}{2}lx \cdot ly,$$

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where it is to be noted that there is

$xy + x + y = 1$, and hence either $y = \frac{1-x}{1+x}$ or $x = \frac{1-y}{1+y}$, some examples of which it will be helpful to set out.

17. 1ˢᵗ. If $x = \frac{1}{2}$, there will be $y = \frac{1}{3}$, from which the equation follows

$$\left. \begin{array}{l} \frac{1}{1\cdot2} + \frac{1}{9\cdot2^3} + \frac{1}{25\cdot2^5} + \frac{1}{49\cdot2^7} + \text{etc.} \\ \frac{1}{1} + \frac{1}{9\cdot3^3} + \frac{1}{25\cdot3^5} + \frac{1}{49\cdot3^7} + \text{etc.} \end{array} \right\} = \frac{\pi\pi}{8} - \frac{1}{2}l2\cdot l3.$$

2ⁿᵈ. If $x = \frac{1}{4}$, there will be $y = \frac{3}{5}$ and thus

$$\left. \begin{array}{l} \frac{1}{1\cdot4} + \frac{1}{9\cdot4^3} + \frac{1}{25\cdot4^5} + \frac{1}{49\cdot4^7} + \text{etc.} \\ \frac{3}{1\cdot5} + \frac{3^3}{9\cdot5^3} + \frac{3^5}{25\cdot5^5} + \frac{3^7}{49\cdot5^7} + \text{etc.} \end{array} \right\} = \frac{\pi\pi}{8} - \frac{1}{2}l4\cdot l\frac{5}{3}.$$

3ʳᵈ. So that also the case is given, in which $x = y$, which comes about on putting

$$x = y = -1 + \sqrt{2} = a \ ;$$

therefore then there is made

$$\frac{a}{1} + \frac{a^3}{9} + \frac{a^5}{25} + \frac{a^7}{49} + \text{etc.} = \frac{\pi\pi}{16} - \frac{1}{4}(la)^2 .$$

18. Therefore in general also, whatever c should be, it will be worth the effort to consider the case in which there becomes $x = y$, which arises if

$$x = y = -1 + \sqrt{1+c} = a \ ;$$

therefore then there will be

$$P = R = \frac{a}{c} + \frac{a^2}{4\cdot c^2} + \frac{a^3}{9\cdot c^3} + \frac{a^4}{16\cdot c^4} + \text{ etc.,}$$

$$Q = S = \frac{a}{1} - \frac{a^2}{4} + \frac{a^3}{9} + \frac{a^4}{16} + \text{ etc.,}$$

from which this equation is deduced :

$$\left. \begin{array}{l} \frac{a}{1\cdot c} + \frac{a^2}{4\cdot c^2} + \frac{a^3}{9\cdot c^3} + \frac{a^4}{16\cdot c^4} + \text{ etc.} \\ + \frac{a}{1} - \frac{a^2}{4} + \frac{a^3}{9} + \frac{a^4}{16} + \text{ etc.} \end{array} \right\} = \frac{\pi\pi}{12} - \frac{1}{2}\left(l\frac{a}{c}\right)^2 + \frac{1}{2}\left(\frac{c}{1} - \frac{c^2}{4} + \frac{c^3}{9} - \text{etc.}\right).$$

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Hence it is possible to derive several outstanding relations between the three series of this kind, which hence appear rational, as often as $1 + c$ should be square.

19. Several other relations between the two numbers *x* and *y* are permitted of course satisfied by these general forms :

$$xy \pm \alpha x \pm \beta y = \gamma ,$$

but which on putting $x = \beta t$ and $y = \alpha u$ is changed into this simpler form :

$$tu \pm t \pm u = \tfrac{\gamma}{\alpha\beta},$$

where only a variation of the signs enters into the calculation. Truly because hence generally three or more series are found, I will not linger setting out more than one here, but chiefly I will get involved with these cases, in which a relation is defined between only two series of these kind, which therefore I have included in the following theorems.

THEOREM I

20. *If these two series should be considered:*

$$X = \tfrac{x}{1} + \tfrac{x^2}{4} + \tfrac{x^3}{9} + \tfrac{x^4}{16} + etc.$$

and

$$Y = \tfrac{y}{1} + \tfrac{y^2}{4} + \tfrac{y^3}{9} + \tfrac{y^4}{16} + etc.$$

and there should be $x + y = 1$, *then there will be always*

$$X + Y = \tfrac{\pi\pi}{6} - lx \cdot ly,$$

the demonstration of this theorem thus has been treated in § 4.

COROLLARY I

21. Here before all it is evident that the sums of these series cannot be real, as soon as either *x* or *y* should exceed one. Indeed the sums in these cases is considered to increase to infinity; truly so that may become imaginary, since, on account of negative *y*, the logarithm of an imaginary *y* comes about.

COROLLARY II

22. The use of this theorem chiefly is discerned in these cases, in which x falls a little short of one and thus the first series X converges little ; then indeed the other Y therefore will converge more. Just as if there should be $x = \frac{9}{10}$ there will be

$$X = \frac{9}{10} + \frac{9^2}{4 \cdot 10^2} + \frac{9^3}{9 \cdot 10^3} + \frac{9^4}{16 \cdot 10^4} + \text{etc.,}$$

a series hardly converging, yet the sum of this by our theorem can easily be assigned approximately. For indeed there shall be

$$Y = \frac{1}{10} + \frac{1}{4 \cdot 10^2} + \frac{1}{9 \cdot 10^3} + \frac{1}{16 \cdot 10^4} + \text{etc.,}$$

which series is especially converging, and as there will be

$$X = \frac{\pi\pi}{6} - l10 \cdot l\frac{10}{9} - Y$$

COROLLARY III

23. Thus in general, if we should put in place $x = \frac{m}{m+n}$ and $y = \frac{n}{m+n}$, there will be

$$X = \frac{m}{1(m+n)} + \frac{m^2}{4(m+n)^2} + \frac{m^3}{9(m+n)^3} + \text{etc.}$$

and

$$Y = \frac{n}{1(m+n)} + \frac{n^2}{4(m+n)^2} + \frac{n^3}{9(m+n)^3} + \text{etc.}$$

then therefore there will be

$$X + Y = \frac{\pi\pi}{6} - l\frac{m+n}{m} \cdot l\frac{m+n}{n}.$$

THEOREM II

24. *If these two series should be considered :*

$$X = \frac{1}{x} - \frac{1}{4xx} + \frac{1}{9x^3} - \frac{1}{16x^4} + etc.,$$

$$Y = \frac{1}{y} + \frac{1}{4yy} + \frac{1}{9y^3} + \frac{1}{16y^4} + etc.$$

with the equation arising

$$y = x + 1,$$

there will always be

$$X - Y = \frac{1}{2}\left(l\frac{y}{x}\right)^2 = \frac{1}{2}\left(l\frac{x+1}{x}\right)^2,$$

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the demonstration of which is deduced from § 12, provided the letters *x, y* and *X, Y* are permuted.

COROLLARY I

25. Because here there shall be $y = x+1$, the latter series, *Y,* will converge more than the former *X.* So that also, if the former series, *X*, were thus diverging, which happens when *x* is a fraction less than one, the latter nevertheless remains converging. Just as if there should be $x = \frac{1}{2}$, then there will be $y = \frac{3}{2}$; truly the series themselves will be

$$X = \frac{2}{1} - \frac{2^2}{4} + \frac{2^3}{9} - \frac{2^4}{16} + \frac{2^5}{25} - \text{etc.}$$

and

$$Y = \frac{2}{3} + \frac{2^2}{4\cdot 3^2} + \frac{2^3}{9\cdot 3^3} + \frac{2^4}{16\cdot 3^4} + \text{etc.;}$$

consequently there will be

$$X - Y = \frac{1}{2}\left(l3\right)^2.$$

Truly because the latter series, *Y,* converges a little, we may reduce that by the first theorem in this manner :

$$\frac{2}{1\cdot 3} + \frac{2^2}{4\cdot 3^2} + \frac{2^3}{9\cdot 3^3} + \text{etc.} = \frac{\pi\pi}{6} - l3\cdot l\frac{3}{2} - \frac{1}{1\cdot 3} - \frac{1}{4\cdot 3^2} - \frac{1}{9\cdot 3^3} - \text{etc.}$$

and hence we will have this summation :

$$\frac{2}{1} - \frac{2^2}{4} + \frac{2^3}{9} - \frac{2^4}{16} + \text{etc.} = \frac{1}{2}\left(l3\right)^2 + \frac{\pi\pi}{6} - l3\cdot l\frac{3}{2} - \left(\frac{1}{1\cdot 3} + \frac{1}{4\cdot 3^2} + \frac{1}{9\cdot 3^3} + \text{etc}\right).$$

COROLLARY II

26. Now in general we may assume $x = \frac{1}{n}$, so that there shall be the series to be summed

$$X = \frac{n}{1} - \frac{n^2}{4} + \frac{n^3}{9} - \frac{n^4}{16} + \text{etc.,}$$

then truly on account of $y = \frac{1+n}{n}$, the other series will be

$$Y = \frac{n}{n+1} + \frac{nn}{4\left(n+1\right)^2} + \frac{n^3}{9\left(n+1\right)^3} + \text{etc.}$$

and hence

$$X = \frac{1}{2}\left(l\left(n+1\right)\right)^2 + Y.$$

But now by theorem I there is

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$$Y = \tfrac{\pi\pi}{6} - l(n+1)\cdot l\,\tfrac{n+1}{n} - \tfrac{1}{n+1} - \tfrac{1}{4(n+1)^2} - \tfrac{1}{9(n+1)^3} - \text{etc.},$$

with which value substituted there will be

$$X = \tfrac{1}{2}\left(l(n+1)\right)^2 + \tfrac{\pi\pi}{6} - l(n+1)\cdot l\,\tfrac{n+1}{n} - \left(\tfrac{1}{n+1} + \tfrac{1}{4(n+1)^2} + \tfrac{1}{9(n+1)^3} + \text{etc.}\right),$$

which expression is contracted into this :

$$\tfrac{n}{1} - \tfrac{n^2}{4} + \tfrac{n^3}{9} - \tfrac{n^4}{16} + \text{etc.}$$

$$= \tfrac{1}{2}l(n+1)\cdot l\,\tfrac{nn}{n+1} + \tfrac{\pi\pi}{6} - \left(\tfrac{1}{n+1} + \tfrac{1}{4(n+1)^2} + \tfrac{1}{9(n+1)^3} + \text{etc.}\right).$$

THEOREM III

27. *If these two series may be considered:*

$$X = \tfrac{x}{1} - \tfrac{x^2}{4} + \tfrac{x^3}{9} - \tfrac{x^4}{16} + etc.$$

and

$$Y = \tfrac{1}{x} - \tfrac{1}{4x^2} + \tfrac{1}{9x^3} - \tfrac{1}{16x^4} + etc.,$$

there will be

$$X + Y = \tfrac{\pi\pi}{6} + \tfrac{1}{2}\left(lx\right)^2.$$

The demonstration is not contained in the preceding, truly it may be set off in this simple manner :

Since by the integral formula there shall be

$$X = \int \tfrac{\partial x}{x} l(1+x),$$

in place of x on writing $\tfrac{1}{x}$ there will be

$$Y = \int \tfrac{\partial x}{x} l\,\tfrac{x}{1+x}$$

or

$$Y = -\int \tfrac{\partial x}{x} l(1+x) + \int \tfrac{\partial x}{x} lx$$

and hence on adding

$$X + Y = +\int \tfrac{\partial x}{x} lx = \tfrac{1}{2}\left(lx\right)^2 + C,$$

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where the constant is most easily defined from the case $x = 1$. Because indeed in this case there shall be both X and $Y = \frac{\pi\pi}{12}$ the constant will be $C = \frac{\pi\pi}{6}$ and thus

$$X + Y = \frac{\pi\pi}{6} + \frac{1}{2}(lx)^2 .$$

COROLLARY I

28. But if therefore for x a number however great may be taken, with the aid of this theorem the sum of the series *X*, which is especially diverging, may be assigned easily, since it may be reduced to the series *Y,* which from that is converging more, by which the former diverges more.

COROLLARY II

29. Now truly with the aid of the theorem the following series

$$Y = \frac{1}{x} - \frac{1}{4x^2} + \frac{1}{9x^3} - \text{etc.}$$

is reduced to this form:

$$Y = \frac{1}{2}\left(l\,\frac{x+1}{x}\right)^2 + \frac{1}{x+1} + \frac{1}{4(x+1)^2} + \frac{1}{9(x+1)^3} + \text{etc.},$$

with which value substituted the following equation will be produced :

$$\frac{x}{1} - \frac{x^2}{4} + \frac{x^3}{9} - \frac{x^4}{16} + \text{etc.} = \frac{\pi\pi}{6} + \frac{1}{2}(lx)^2 - \frac{1}{2}\left(l\,\frac{x+1}{x}\right)^2 - \left(\frac{1}{x+1} + \frac{1}{4(x+1)^2} + \frac{1}{9(x+1)^3} + \text{etc.}\right),$$

which expression agrees exceedingly well with the above § 26, because there is

$$\frac{1}{2}l(x+1) \cdot l\,\frac{xx}{x+1} = \frac{1}{2}(lx)^2 - \frac{1}{2}\left(l\,\frac{x+1}{x}\right)^2 ,$$

as readily will become apparent on expanding.

THEOREM IV

30. *If these series should be considered :*

$$X = \frac{x}{1} + \frac{x^3}{9} + \frac{x^5}{25} + \text{etc.} \quad and \quad Y = \frac{y}{1} + \frac{y^3}{9} + \frac{y^5}{25} + \text{etc.}$$

with the equation arising
$$xy + x + y = 1$$
or if
$$x = \frac{1-y}{1+y} \quad or \quad y = \frac{1-x}{1+x},$$
there will be
$$X + Y = \frac{\pi\pi}{8} - \frac{1}{2}lx \cdot ly .$$

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The demonstration is evident from §16.

COROLLARY I

31. Here again, as above, the sums of these series is observed to become imaginary as soon as the letters x and y become greater than one. But if there should be $x < 1$, then always another series of the same form can be shown, the sum of which depends on that. Thus if there should be $x = \frac{1}{2}$, there will be $y = \frac{1}{3}$. But if x approaches close to one, such as $x = \frac{9}{10}$, the other series, *Y,* converges greatly.

COROLLARY II

32. In these four theorems all the cases may be seen to be included, from which two series of this kind are allowed to be compared with each other. To the showing of which we may attach the following special theorem, which I have arrived at finally by long winding calculations, but which now can be deduced conveniently enough from the preceding theorems.

A SPECIAL THEOREM

33. *If these series may be considered related to each other:*

$$A = \tfrac{1}{1 \cdot 3} + \tfrac{1}{9 \cdot 3^3} + \tfrac{1}{25 \cdot 3^5} + etc.$$

and

$$B = \tfrac{1}{1 \cdot 3} + \tfrac{1}{4 \cdot 3^2} + \tfrac{1}{9 \cdot 3^3} + etc.$$

then there will be

$$2A + B = \tfrac{\pi\pi}{6} - \tfrac{1}{2}\left(l3\right)^2.$$

DEMONSTRATION

Since from the first theorem, on taking $x = y = \frac{1}{2}$, there will be

$$\tfrac{1}{1 \cdot 2} + \tfrac{1}{4 \cdot 2^2} + \tfrac{1}{9 \cdot 2^3} + \text{etc.} = \tfrac{\pi\pi}{12} - \tfrac{1}{2}\left(l2\right)^2,$$

this resolved series can be represented in the following manner:

$$2\left(\tfrac{1}{1 \cdot 2} + \tfrac{1}{9 \cdot 2^3} + \tfrac{1}{25 \cdot 2^5} + \text{etc.}\right) - 1\left(\tfrac{1}{1 \cdot 2} - \tfrac{1}{4 \cdot 2^2} + \tfrac{1}{9 \cdot 2^3} - \text{etc.}\right) = \tfrac{\pi\pi}{12} - \tfrac{1}{2}\left(l2\right)^2.$$

Now truly by theorem IV, on taking $x = \frac{1}{2}$ and $y = \frac{1}{3}$, we will have this equation:

$$\tfrac{1}{1 \cdot 2} + \tfrac{1}{9 \cdot 2^3} + \tfrac{1}{25 \cdot 2^5} + \text{etc.} = \tfrac{\pi\pi}{8} - \tfrac{1}{2} l2 \cdot l3 - \tfrac{1}{1 \cdot 3} - \tfrac{1}{9 \cdot 3^3} - \tfrac{1}{25 \cdot 3^5} - \text{etc.}$$

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Then truly from the second theorem, on taking $x = 2$ and $y = 3$, there will be

$$\tfrac{1}{1\cdot2} - \tfrac{1}{4\cdot2^2} + \tfrac{1}{9\cdot2^3} - \tfrac{1}{16\cdot2^4} + \text{etc.} = \tfrac{1}{2}\left(l\tfrac{3}{2}\right)^2 + \tfrac{1}{1\cdot3} + \tfrac{1}{4\cdot3^2} + \tfrac{1}{9\cdot3^3} + \text{etc.}$$

Now these values may be substituted in place of that series, and from the left hand part there will be produced

$$\left.\begin{array}{l} \tfrac{\pi\pi}{4} - l2\cdot l3 - 2\left(\tfrac{1}{1\cdot3} + \tfrac{1}{9\cdot2^3} + \tfrac{1}{25\cdot2^5} + \text{etc.}\right) \\[2mm] -\tfrac{1}{2}\left(l\tfrac{3}{2}\right)^2 - \left(\tfrac{1}{1\cdot3} + \tfrac{1}{4\cdot3^2} + \tfrac{1}{9\cdot3^3} + \text{etc.}\right) \end{array}\right\} = \tfrac{\pi\pi}{12} - \tfrac{1}{2}\left(l2\right)^2.$$

From which we conclude to become

$$\left.\begin{array}{l} 2\left(\tfrac{1}{1\cdot3} + \tfrac{1}{9\cdot3^3} + \tfrac{1}{25\cdot3^5} + \text{etc.}\right) \\[2mm] +1\left(\tfrac{1}{1\cdot3} + \tfrac{1}{4\cdot3^2} + \tfrac{1}{9\cdot3^3} + \text{etc.}\right) \end{array}\right\} = \tfrac{\pi\pi}{6} - l2\cdot l3 - \tfrac{1}{2}\left(l\tfrac{3}{2}\right)^2 + \tfrac{1}{2}\left(l2\right)^2.$$

$$= \tfrac{\pi\pi}{6} - \tfrac{1}{2}\left(l3\right)^2 \qquad \left(\text{on account of } \left(l\tfrac{3}{2}\right)^2 = \left(l3\right)^2 - 2l2\cdot l3 + \left(l2\right)^2\right).$$

34. But however the theorems given here may be combined with each other, scarcely another relation between the two series can be elicited, but much less thence it allows a simple series of this kind to be elicited, the sum of which can be shown completely, besides the case now indicated, which therefore here we may put to be shown jointly.

$$1 + \tfrac{1}{4} + \tfrac{1}{9} + \tfrac{1}{16} + \tfrac{1}{25} + \text{etc.} = \tfrac{\pi\pi}{6},$$
$$1 - \tfrac{1}{4} + \tfrac{1}{9} - \tfrac{1}{16} + \tfrac{1}{25} - \text{etc.} = \tfrac{\pi\pi}{12},$$
$$\tfrac{1}{1\cdot2} + \tfrac{1}{4\cdot2^2} + \tfrac{1}{9\cdot2^3} + \tfrac{1}{16\cdot2^4} + \text{etc.} = \tfrac{\pi\pi}{12} - \tfrac{1}{2}\left(l2\right)^2,$$
$$1 + \tfrac{1}{9} + \tfrac{1}{25} + \tfrac{1}{49} + \text{etc.} = \tfrac{\pi\pi}{8}.$$

Now besides it is possible for this series to be adjoined :

$$\tfrac{a}{1} + \tfrac{a^3}{9} + \tfrac{a^5}{25} + \tfrac{a^7}{49} + \text{etc.} = \tfrac{\pi\pi}{16} - \tfrac{1}{4}\left(la\right)^2$$

with $a = \sqrt{2} - 1$ arising.

But although in this series the value of a shall be irrational and thus any powers seen to be arising must be considered separately, yet the numerators also constitute a recurring

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series, in which any term can be defined by the two preceding terms with the aid of this formula:

$$a^{n+4} = 6a^{n+2} - a^{n},$$

the truth of which it may be helpful to show, because it shall be, on dividing by a^{n}, $a^{4} = 6aa - 1$. Because for $a = \sqrt{2} - 1$, there will be $a^{2} = 3 - 2\sqrt{2}$ and $a^{4} = 17 - 12\sqrt{2}$, from which the truth shall be evident.

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DE SUMMATIONE SERlERUM
IN HAC FORMA CONTENTARUM

$$\frac{a}{1} + \frac{a^2}{4} + \frac{a^3}{9} + \frac{a^4}{16} + \frac{a^5}{25} + \frac{a^6}{36} + \text{etc.}$$

1. Ex iis, quae olim primus de summatione potestatum reciprocarum in medium attuli, duo tantum casus derivari possunt, quibus summam seriei hic propositae assignare licet: alter scilicet, quo $a = 1$, ubi ostendi huius seriei

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \frac{1}{36} + \text{etc.}$$

summam esse $= \frac{\pi\pi}{6}$, denotante π peripheriam circuli, cuius diameter $= 1$; alter vero casus est, quo $a = -1$; tum enim mutatis signis huius seriei

$$1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \frac{1}{25} - \frac{1}{36} + \text{etc.}$$

summa est $= \frac{\pi\pi}{12}$. Praeterea vero methodo prorsus singulari inveni casu $a = \frac{1}{2}$ huius seriei

$$\frac{1}{1.2} + \frac{1}{4 \cdot 2^2} + \frac{1}{9.2^3} + \frac{1}{16.2^4} + \text{etc.}$$

summam esse $= \frac{\pi\pi}{12} - \frac{1}{2}(l2)^2$ denotante *l2* logarithmum hyperbolicum binarii, qui est 0.693147180. Neque vero praeter hos casus ullus alius adhuc constat, quo summam assignare liceat.

2. Methodus autem, qua hunc postremum casum sum adeptus, ulterius extendi potest, ita ut inde plurimae insignes relationes inter binas pluresve series huius formae reperiri queant. Innititur autem ista methodus hoc lemmate:

LEMMA

Si ponatur

$$p = \int \frac{\partial x}{x} ly \quad et \quad q = \int \frac{\partial y}{y} lx,$$

erit summa

$$p + q = lx.ly + C,$$

siquidem constans ita definiatur, ut unico casui satisfaciat.

Hinc igitur sequentia problemata percurramus pro varia scilicet relatione inter *x* et *y*.

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PROBLEMA 1

Si fuerit $x + y = 1$, *binas illas formulas*

$$p = \int \frac{\partial x}{x} ly \quad et \quad q = \int \frac{\partial y}{y} lx$$

in series resolvere, ita ut hinc prodeat

$$p + q = lx.ly + C.$$

SOLUTIO

3. Cum igitur sit $y = 1 - x$, erit

$$ly = -x - \frac{xx}{2} - \frac{x^3}{3} - \text{ etc.}$$

hincque

$$p = \int \frac{\partial x}{x} ly = -\frac{x}{1} - \frac{xx}{4} - \frac{x^3}{9} - \frac{x^4}{16} - \text{etc.}$$

similique modo ob

$$x = 1 - y \quad et \quad lx = -y - \frac{yy}{2} - \frac{y^3}{3} - \frac{y^4}{4} - \text{etc.}$$

erit

$$q = \int \frac{\partial y}{y} lx = -\frac{y}{1} - \frac{y^2}{4} - \frac{y^3}{9} - \frac{y^4}{16} - \text{etc.},$$

quamobrem harum duarum serierum summa erit $lx \cdot ly + C$. Pro constante C definienda consideremus casum, quo $x = 0$ et $y = 1$ ideoque $lx \cdot ly = 0$; tum igitur erit

$$p + q = -1 - \frac{1}{4} - \frac{1}{9} - \frac{1}{16} - \text{etc.} = -\frac{\pi\pi}{6},$$

unde elicitur $C = -\frac{\pi\pi}{6}$.

4. Quoties ergo fuerit $x + y = 1$, summa harum duarum serierum iunctim sumtarum

$$\frac{x}{1} + \frac{x^2}{4} + \frac{x^3}{9} + \frac{x^4}{16} + \cdots + \frac{y}{1} + \frac{y^2}{4} + \frac{y^3}{9} + \frac{y^4}{16} + \text{etc.}$$

erit $= \frac{\pi\pi}{6} - lx \cdot ly$; hincque statim sequitur tertius casus supra memoratus. Sumto enim $x = \frac{1}{2}$ erit quoque $y = \frac{1}{2}$ ideoque ambae hae series inter se aequales [evadunt], unde sequitur fore

$$\frac{1}{1.2} + \frac{1}{4.2^2} + \frac{1}{9.2^3} + \frac{1}{16.2^4} + \text{etc.} = \frac{\pi\pi}{12} - \frac{1}{2}\left(l\frac{1}{2}\right)^2 = \frac{\pi\pi}{12} - \frac{1}{2}(l2)^2.$$

Praeterea vero, quoties fuerit $a + b = 1$ ponaturque

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$$A = \tfrac{a}{1} + \tfrac{a^2}{4} + \tfrac{a^3}{9} + \text{etc.} \quad \text{et} \quad B = \tfrac{b}{1} + \tfrac{b^2}{4} + \tfrac{b^3}{9} + \text{etc.},$$

semper erit $A + B = \frac{\pi\pi}{6} - la \cdot lb$. Hinc ergo, si alterius harum serierum summa aliunde esset cognita, etiam alterius summa innotesceret. Hocque est illud ipsum problema, quod iam olim tractavi.

PROBLEMA 2

Si fuerit $x - y = 1$, binas illas formulas

$$p = \int \tfrac{\partial x}{x} ly \quad et \quad q = \int \tfrac{\partial y}{y} lx$$

in series resolvere, ita ut hinc prodeat

$$p + q = lx \cdot ly + C.$$

SOLUTIO

5. Cum hic sit $y = x - 1$, erit

$$ly = l(x-1) = lx + l\left(1 - \tfrac{1}{x}\right) = lx - \tfrac{1}{x} - \tfrac{1}{2xx} - \tfrac{1}{3x^3} - \tfrac{1}{4x^4} - \text{etc.}$$

hincque

$$p = \int \tfrac{\partial x}{x} ly = \tfrac{1}{2}(lx)^2 + \tfrac{1}{x} + \tfrac{1}{4x^2} + \tfrac{1}{9x^3} + \tfrac{1}{16x^4} + \text{etc.}$$

Deinde ob $x = 1 + y$ erit

$$lx = \tfrac{y}{1} - \tfrac{yy}{2} - \tfrac{y^3}{3} - \tfrac{y^4}{4} + \text{etc.}$$

ideoque

$$q = \int \tfrac{\partial y}{y} lx = \tfrac{y}{1} - \tfrac{y^2}{4} + \tfrac{y^3}{9} - \tfrac{y^4}{16} + \text{etc.}$$

quamobrem hababimus

$$p + q = lx.ly + C.$$

Pro constante determinanda consideremus casum $y = 0$, quo fit $x = 1$ et $lx \cdot ly = 0$; tum igitur erit

$$p = 1 + \tfrac{1}{4} + \tfrac{1}{9} + \tfrac{1}{16} + \text{etc} = \tfrac{\pi\pi}{6} \quad \text{et} \quad q = 0,$$

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unde definitur constans $C = \frac{\pi\pi}{6}$.

6. Hic igitur iterum duas habemus series, quarum coniunctim summam assignare valemus:

$$\left.\begin{array}{l} \frac{1}{x} + \frac{1}{4x^2} + \frac{1}{9x^3} + \frac{1}{16x^4} + \text{etc.} \\ + \frac{y}{1} - \frac{yy}{4} + \frac{y^3}{9} - \frac{y^4}{16} + \text{etc.} \end{array}\right\} = \frac{\pi\pi}{6} - \frac{1}{2}\left(lx\right)^2 + lx \cdot ly = \frac{\pi\pi}{6} + lx \cdot l\frac{y}{\sqrt{x}}.$$

7. Quodsi ergo habeantur hae duae series:

$$A = \frac{a}{1} + \frac{a^2}{4} + \frac{a^3}{9} + \frac{a^4}{16} + \text{etc.}$$

et

$$B = \frac{b}{1} - \frac{b^2}{4} + \frac{b^3}{9} - \frac{b^4}{16} + \text{etc.},$$

ita ut sit $a = \frac{1}{x}$ et $b = y$, atque inter a et b haec datur relatio

$$ab + a = 1,$$

erit

$$A + B = \frac{\pi\pi}{6} - la \cdot lb\sqrt{a}.$$

Consideremus casum, quo

$$b = a\left(= \frac{-1+\sqrt{5}}{2} \quad \text{ob } ab + a = 1\right),$$

eritque

$$A + B = 2\left(\frac{a}{1} + \frac{a^3}{9} + \frac{a^5}{25} + \frac{a^7}{49} + \text{etc.}\right);$$

quocirca, existente $a = \frac{\sqrt{5}-1}{2}$, huius seriei

$$\frac{a}{1} + \frac{a^3}{9} + \frac{a^5}{25} + \text{etc.}$$

summa erit

$$\frac{\pi\pi}{12} - \frac{1}{2}la \cdot la\sqrt{a}.$$

8. Deinde etiam hic notatu dignus est casus, quo $b = -a$ atque $A + B = 0$; hoc enim casu erit

$$\frac{\pi\pi}{6} = la \cdot lb\sqrt{a}.$$

At quia $b = -a$, erit

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$$-aa + a = 1$$

hincque

$$a = \frac{1+\sqrt{-3}}{2} \quad et \quad b = \frac{1-\sqrt{-3}}{2}.$$

Iam cum sit

$$lb\sqrt{a} = \tfrac{1}{2}labb,$$

ob

$$bb = \frac{-1+\sqrt{-3}}{2}$$

erit $abb = -1$, unde sequitur fore

$$\tfrac{\pi\pi}{6} = l\frac{1+\sqrt{-3}}{2} \cdot \sqrt{-1},$$

id quod egregie convenit cum expressione cognita peripheriae circuli per logarithmos imaginarios.

9. Si poneremus hic $a = \tfrac{1}{2}$, foret $b = 1$ ideoque

$$B = 1 - \tfrac{1}{4} + \tfrac{1}{9} - \tfrac{1}{16} + \text{etc.}$$

hincque

$$A + B = \tfrac{1}{1.2} + \tfrac{1}{4 \cdot 2^2} + \tfrac{1}{9.2^3} + \cdots + \tfrac{\pi\pi}{12} = \tfrac{\pi\pi}{6} - \tfrac{1}{2}\left(l2\right)^2,$$

unde prodiret tertius casus initio memoratus.

At vero faciamus hic $b = \tfrac{1}{2}$ eritque $a = \tfrac{2}{3}$ et

$$lb\sqrt{a} = \tfrac{1}{2}lbba = \tfrac{1}{2}l\tfrac{1}{6} = -\tfrac{1}{2}l6 \quad et \quad la = -l\tfrac{3}{2},$$

unde habebimus

$$\left.\begin{array}{l} A = \tfrac{2}{1\cdot3} + \tfrac{2^2}{4\cdot3^2} + \tfrac{2^3}{9\cdot3^3} + \text{etc.} \\[2mm] +B = \tfrac{1}{1\cdot2} - \tfrac{1}{4\cdot2^2} + \tfrac{1}{9\cdot2^3} - \text{etc.} \end{array}\right\} = \tfrac{\pi\pi}{6} - \tfrac{1}{2}l\tfrac{3}{2} \cdot l6.$$

Subtrahamus hinc ex problemate primo hanc aequationem:

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$$\left. \begin{array}{l} \frac{1}{1 \cdot 3} + \frac{1}{4 \cdot 3^2} + \frac{1}{9 \cdot 3^3} + \text{etc.} \\[2mm] + \frac{2}{1 \cdot 3} + \frac{2^2}{4 \cdot 3^2} + \frac{2^3}{9 \cdot 3^3} + \text{etc.} \end{array} \right\} = \frac{\pi\pi}{6} - l3 \cdot l\tfrac{3}{2}$$

et remanebit

$$\left. \begin{array}{l} \frac{1}{1 \cdot 2} - \frac{1}{4 \cdot 2^2} + \frac{1}{9 \cdot 2^3} - \frac{1}{16 \cdot 2^4} + \text{etc.} \\[2mm] - \frac{1}{1 \cdot 3} - \frac{1}{4 \cdot 3^2} - \frac{1}{9 \cdot 3^3} - \frac{1}{16 \cdot 3^4} - \text{etc.} \end{array} \right\} = l3 \cdot l\tfrac{3}{2} - \tfrac{1}{2} l\tfrac{3}{2} \cdot l6 = \tfrac{1}{2}\left(l\tfrac{3}{2} \right)^2 .$$

Sicque nacti sumus hanc aequationem notatu dignam:

$$\frac{1}{1 \cdot 2} - \frac{1}{4 \cdot 2^2} + \frac{1}{9 \cdot 2^3} - \frac{1}{16 \cdot 2^4} + \text{etc.} = \tfrac{1}{2}\left(l\tfrac{3}{2} \right)^2 + \frac{1}{1 \cdot 3} + \frac{1}{4 \cdot 3^2} + \frac{1}{9 \cdot 3^3} + \text{etc.} ,$$

ubi ratio peripheriae π penitus e calculo excessit. Verum eadem relatio sequenti modo facilius eruitur.

ALIA SOLUTIO EIUSDEM PROBLEMATIS

10. Manente evolutione prioris partis *p,* altera pars *q* ob

$$lx = l(1 + y) = ly + l\left(1 + \tfrac{1}{y} \right)$$

hinc

$$lx = ly + \frac{1}{y} - \frac{1}{2y^2} + \frac{1}{3y^3} - \text{etc.}$$

erit

$$q = \int \frac{\partial y}{y}(lx) = \tfrac{1}{2}(ly)^2 - \frac{1}{y} + \frac{1}{4y^2} - \frac{1}{9y^3} + \frac{1}{16y^4} - \text{etc.}$$

Nunc igitur erit

$$p + q = lx \cdot ly + C ;$$

ubi constans *C* inde definiri potest, quod posito $y = 1$ fit $x = 2$ hincque

$$p = \tfrac{1}{2}(l2)^2 + \frac{\pi\pi}{12} - \tfrac{1}{2}(l2)^2 = \frac{\pi\pi}{12}$$

et

$$q = -1 + \tfrac{1}{4} - \tfrac{1}{9} + \tfrac{1}{16} - \text{etc.} = -\frac{\pi\pi}{12},$$

quibus valoribus substitutis pro hoc casu prodit $p + q = 0 = 0 + C$, consequenter $C = 0$.

11. Verum haec constans etiam alio modo definiri potest. Ponamus brevitatis gratia

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$$X = \tfrac{1}{x} + \tfrac{1}{4xx} + \tfrac{1}{9x^3} + \tfrac{1}{16x^4} + \text{etc.}$$

et

$$Y = \tfrac{1}{y} - \tfrac{1}{4yy} + \tfrac{1}{9y^3} - \tfrac{1}{16y^4} + \text{etc.},$$

ut habeamus

$$p = \tfrac{1}{2}(lx)^2 + X \quad \text{et} \quad q = \tfrac{1}{2}(ly)^2 - Y,$$

hincque fiet

$$p + q = \tfrac{1}{2}(lx)^2 + \tfrac{1}{2}(ly)^2 + X - Y = lx \cdot ly + C\,;$$

unde deducimus

$$Y - X = \tfrac{1}{2}(lx)^2 + \tfrac{1}{2}(ly)^2 - lx \cdot ly - C = \tfrac{1}{2}\left(l\tfrac{x}{y}\right)^2 - C,$$

ubi notandum est esse $y = x - 1$. Iam ad constantem C definiendam consideretur casus $x = \infty$, quo fit $X = 0$ et $Y = 0$, praeterea vero $l\tfrac{x}{y} = 0$, quibus notatis erit $0 = -C$ ideoque $C = 0$.

12. Hinc igitur nacti sumus duas series X et Y quarum differentia per solos logarithmos exprimitur, cum sit

$$Y - X = \tfrac{1}{2}\left(l\tfrac{x}{y}\right)^2 = \tfrac{1}{2}\left(l\tfrac{y+1}{y}\right)^2$$

ob $x = y + 1$.

Ex hac forma sumto $y = 2$ statim fluit relatio ante inventa

$$\tfrac{1}{1\cdot2} - \tfrac{1}{4\cdot2^2} + \tfrac{1}{9\cdot2^3} - \tfrac{1}{16\cdot2^4} + \text{etc.} = \tfrac{1}{2}\left(l\tfrac{3}{2}\right)^2 + \tfrac{1}{1\cdot3} + \tfrac{1}{4\cdot3^2} + \tfrac{1}{9\cdot3^3} + \text{etc.}$$

Simili autem modo nunc multo generalius habebimus

$$\tfrac{1}{1\cdot y} - \tfrac{1}{4\cdot y^2} \;+\; \tfrac{1}{9\cdot y^3} - \tfrac{1}{16\cdot y^4} \;+ \text{etc.}$$

$$= \tfrac{1}{2}\left(l\tfrac{y+1}{y}\right)^2 + \tfrac{1}{1(y+1)} + \tfrac{1}{4(y+1)^2} + \tfrac{1}{9(y+1)^3} + \text{etc.},$$

ubi loco y quicquid lubuerit accipere licet.

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PROBLEMA 3

Si inter x et y haec detur relatio: $xy + x + y = c$, binas formulas

$$p = \int \frac{\partial x}{x} ly \quad et \quad q = \int \frac{\partial y}{y} lx,$$

in series resolvere, ita ut hinc prodeat

$$p + q = lx.ly + C,$$

SOLUTIO

13. Hinc igitur primo erit

$$y = \frac{c-x}{1+x},$$

eius logarithmus per duas series sequentes exprimitur:

$$ly = \begin{cases} lc - \frac{x}{c} - \frac{x^2}{2c^2} - \frac{x^3}{3c^3} - \frac{x^4}{4c^4} - \text{etc.} \\ -x + \frac{x^2}{2} - \frac{x^3}{3} + \frac{x^4}{4} - \frac{x^5}{5} + \text{etc.}, \end{cases}$$

unde

$$p = \int \frac{\partial x}{x} ly = \begin{cases} lc \cdot lx - \frac{x}{c} - \frac{x^2}{4c^2} - \frac{x^3}{9c^3} - \frac{x^4}{16c^4} - \text{etc.} \\ \quad -x + \frac{x^2}{4} - \frac{x^3}{9} + \frac{x^4}{16} - \text{etc.} \end{cases}$$

Simili modo, cum sit $x = \frac{c-y}{1+y}$, erit

$$q = \int \frac{\partial y}{y} lx = \begin{cases} lc \cdot ly - \frac{y}{c} - \frac{y^2}{4c^2} - \frac{y^3}{9c^3} - \frac{y^4}{16c^4} - \text{etc.} \\ \quad -\frac{y}{1} + \frac{y^2}{4} - \frac{y^3}{9} + \frac{y^4}{16} - \text{etc.} \end{cases}$$

Atque hinc erit $p + q = lx.ly + C$.

14. Pro constante definienda consideremus casum, quo $x = 0$ ideoque
$p = lc \cdot lx$ et

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$$q = \left(lc\right)^2 - 1 - \left.\begin{array}{l} \frac{1}{4} - \frac{1}{9} - \frac{1}{16} - \text{etc.} \\ -\frac{c}{1} + \frac{c^2}{4} - \frac{c^3}{9} + \frac{c^4}{16} - \text{etc.} \end{array}\right\}$$

sive

$$q = \left(lc\right)^2 - \frac{\pi\pi}{6} - \frac{c}{1} + \frac{c^2}{4} - \frac{c^3}{9} + \frac{c^4}{16} - \text{etc.},$$

unde aequatio nostra evadit

$$p + q = lc \cdot lx + \left(lc\right)^2 - \frac{\pi\pi}{6} - \frac{c}{1} + \frac{c^2}{4} - \frac{c^3}{9} + \text{etc.} = lc \cdot lx + C,$$

ubi ergo termini $lc \cdot lx$ se mutuo destruunt, ita ut sit

$$C = \left(lc\right)^2 - \frac{\pi\pi}{6} - \frac{c}{1} + \frac{c^2}{4} - \frac{c^3}{9} + \text{etc.}$$

15. Hic ergo quinque occurrunt series infinitae, quas sequenti modo indicemus:

$$\frac{c}{1} - \frac{c^2}{4} + \frac{c^3}{9} - \frac{c^4}{16} + \text{etc.} = O,$$

$$\frac{x}{c} + \frac{x^2}{4 \cdot c^2} + \frac{x^3}{9 \cdot c^3} + \frac{x^4}{16 \cdot c^4} + \text{etc.} = P,$$

$$\frac{x}{1} - \frac{x^2}{4} + \frac{x^3}{9} - \frac{x^4}{16} + \text{etc.} = Q,,$$

$$\frac{y}{c} + \frac{y^2}{4 \cdot c^2} + \frac{y^3}{9 \cdot c^3} + \frac{y^4}{16 \cdot c^4} + \text{etc.} = R,$$

$$\frac{y}{1} - \frac{y^2}{4} + \frac{y^3}{9} - \frac{y^4}{16} + \text{etc.} = S;$$

quibus litteris introductis nostra aequatio erit

$$lc \cdot lx - P - Q + lc \cdot ly - R - S = lx \cdot ly + \left(lc\right)^2 - \frac{\pi\pi}{6} - O,$$

unde sequitur fore

$$O - P - Q - R - S = lx.ly + \left(lc\right)^2 - lc \cdot lx - lc \cdot ly - \frac{\pi\pi}{6},$$

quae expressio contrahitur in sequentem:

$$O - P - Q - R - S = l\frac{x}{c}.l\frac{y}{c} - \frac{\pi\pi}{6}$$

sive mutatis signis

$$P + Q + R + S - O = \frac{\pi\pi}{6} - l\frac{x}{c}.l\frac{y}{c}.$$

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16. Hic casus satis memorabilis occurrit, quando $c = 1$, quia tum fit

$$P + Q = \frac{2x}{1} + \frac{2x^3}{9} + \frac{2x^5}{25} + \text{etc.}$$

et

$$R + S = \frac{2y}{1} + \frac{2y^3}{9} + \frac{2y^5}{25} + \text{etc.,}$$

tum vero

$$O = \frac{\pi\pi}{12},$$

sicque inter binas series satis simplicem relationem sumus assecuti, quae est

$$\left.\begin{array}{l} \frac{x}{1} + \frac{x^3}{9} + \frac{x^5}{25} + \frac{x^7}{49} + \text{etc.} \\[2mm] \frac{y}{1} + \frac{y^3}{9} + \frac{y^5}{25} + \frac{y^7}{49} + \text{etc.} \end{array}\right\} = \frac{\pi\pi}{8} - \frac{1}{2} lx \cdot ly,$$

ubi notandum est fore

$xy + x + y = 1$, hinc sive $y = \frac{1-x}{1+x}$ sive $x = \frac{1-y}{1+y}$, cuius aliquot exempla evolvisse iuvabit.

17. 1^0. Si $x = \frac{1}{2}$, erit $y = \frac{1}{3}$, unde sequitur aequatio

$$\left.\begin{array}{l} \frac{1}{1\cdot2} + \frac{1}{9\cdot2^3} + \frac{1}{25\cdot2^5} + \frac{1}{49\cdot2^7} + \text{etc.} \\[2mm] \frac{1}{1} + \frac{1}{9\cdot3^3} + \frac{1}{25\cdot3^5} + \frac{1}{49\cdot3^7} + \text{etc.} \end{array}\right\} = \frac{\pi\pi}{8} - \frac{1}{2} l2 \cdot l3.$$

2^0. Si $x = \frac{1}{4}$, erit $y = \frac{3}{5}$ ideoque

$$\left.\begin{array}{l} \frac{1}{1\cdot4} + \frac{1}{9\cdot4^3} + \frac{1}{25\cdot4^5} + \frac{1}{49\cdot4^7} + \text{etc.} \\[2mm] \frac{3}{1\cdot5} + \frac{3^3}{9\cdot5^3} + \frac{3^5}{25\cdot5^5} + \frac{3^7}{49\cdot5^7} + \text{etc.} \end{array}\right\} = \frac{\pi\pi}{8} - \frac{1}{2} l4 \cdot l\frac{5}{3}.$$

$3°$. Quin etiam datur casus, quo $x = y$, quod evenit ponendo

$$x = y = -1 + \sqrt{2} = a \;;$$

tum igitur fiet

$$\frac{a}{1} + \frac{a^3}{9} + \frac{a^5}{25} + \frac{a^7}{49} + \text{etc.} = \frac{\pi\pi}{16} - \frac{1}{4}\left(la\right)^2.$$

18. In genere igitur etiam, quicquid fuerit *c,* operae pretium erit casum perpendere, quo fit $x = y$, quod evenit si

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$$x = \ y = -1 + \sqrt{1+c} = a \ ;$$

tum igitur erit

$$P = R = \tfrac{a}{c} + \tfrac{a^2}{4 \cdot c^2} + \tfrac{a^3}{9 \cdot c^3} + \tfrac{a^4}{16 \cdot c^4} + \ \text{etc.},$$

$$Q = S = \tfrac{a}{1} - \tfrac{a^2}{4} + \tfrac{a^3}{9} + \tfrac{a^4}{16} + \text{etc.},$$

unde deducitur ista aequatio:

$$\left.\begin{array}{l} \tfrac{a}{1 \cdot c} + \tfrac{a^2}{4 \cdot c^2} + \tfrac{a^3}{9 \cdot c^3} + \tfrac{a^4}{16 \cdot c^4} + \ \text{etc.} \\[2mm] + \tfrac{a}{1} - \tfrac{a^2}{4} + \tfrac{a^3}{9} + \tfrac{a^4}{16} + \text{etc.} \end{array}\right\} = \tfrac{\pi\pi}{12} - \tfrac{1}{2}\left(l\,\tfrac{a}{c}\right)^2 + \tfrac{1}{2}\left(\tfrac{c}{1} - \tfrac{c^2}{4} + \tfrac{c^3}{9} - \text{etc.}\right).$$

Hinc plurimas egregias relationes inter ternas huiusmodi series derivare licet, quae ergo evadunt rationales, quoties fuerit $1 + c$ quadratum.

19. Plures alias relationes inter binos numeros x et y evolvere liceret in hac scilicet forma generali contentas:

$$xy \pm \alpha x \pm \beta y = \gamma \,,$$

quae autem posito $x = \beta t$ et $y = \alpha u$ in hanc simpliciorem mutatur:

$$tu \pm t \pm u = \tfrac{\gamma}{\alpha\beta}\,,$$

ubi iantum varietas signorum in computum venit. Verum quia hinc plerumque tres pluresve series reperiuntur, alteriori evolutioni hic non immoror, sed potissimum iis casibus inhaerebo, quibus relatio inter duas tantum huiusmodi series definitur, quos igitur in sequentibus theorematibus sum complexurus.

THEOREMA I

20*. Si habeantur hae duae series:*

$$X = \tfrac{x}{1} + \tfrac{x^2}{4} + \tfrac{x^3}{9} + \tfrac{x^4}{16} + etc.$$

et

$$Y = \tfrac{y}{1} + \tfrac{y^2}{4} + \tfrac{y^3}{9} + \tfrac{y^4}{16} + etc.$$

fueritque $x + y = 1$, tum semper erit

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$$X + Y = \tfrac{\pi\pi}{6} - lx \cdot ly,$$

eius theorematis demonstratio in § 4 iam est tradita.

COROLLARIUM I

21. Hic ante omnia manifestum est summas harum serierum reales esse non posse, simulac vel x vel y unitatem superaverit. Summa quidem his casibus videtur in infinitum exerescere; verum ea fit adeo imaginaria, cum, ob y negativum, logarithmus y maginarius evadat.

COROLLARIUM II

22. Usus huius theorematis potissimum iis casibus cernitur, quibus x parum ab unitate deficit ideoque prior series X parum convergit; tum enim altera Y eo magis converget. Veluti si fuerit $x = \tfrac{9}{10}$ erit

$$X = \tfrac{9}{10} + \tfrac{9^2}{4 \cdot 10^2} + \tfrac{9^3}{9 \cdot 10^3} + \tfrac{9^4}{16 \cdot 10^4} + \text{etc.,}$$

series vix convergens, cuius tamen summa per nostrum theorema facile quam proxime assignari poterit. Cum enim sit

$$Y = \tfrac{1}{10} + \tfrac{1}{4 \cdot 10^2} + \tfrac{1}{9 \cdot 10^3} + \tfrac{1}{16 \cdot 10^4} + \text{etc.,}$$

quae series est maxime convergens, erit utique

$$X = \tfrac{\pi\pi}{6} - l10 \cdot l\tfrac{10}{9} - Y$$

COROLLARIUM III

23. Ita in genere, si statuamus $x = \tfrac{m}{m+n}$ et $y = \tfrac{n}{m+n}$, erit

$$X = \tfrac{m}{1(m+n)} + \tfrac{m^2}{4(m+n)^2} + \tfrac{m^3}{9(m+n)^3} + \text{etc.}$$

et

$$Y = \tfrac{n}{1(m+n)} + \tfrac{n^2}{4(m+n)^2} + \tfrac{n^3}{9(m+n)^3} + \text{etc.}$$

tum igitur erit

$$X + Y = \tfrac{\pi\pi}{6} - l\tfrac{m+n}{m} \cdot l\tfrac{m+n}{n}.$$

THEOREMA II

24. *Si habeantur hae duae series:*

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$$X = \tfrac{1}{x} - \tfrac{1}{4xx} + \tfrac{1}{9x^3} - \tfrac{1}{16x^4} + etc.,$$

$$Y = \tfrac{1}{y} + \tfrac{1}{4yy} + \tfrac{1}{9y^3} + \tfrac{1}{16y^4} + etc.$$

existente

$$y = x + 1$$

semper erit

$$X - Y = \tfrac{1}{2}\left(l\tfrac{y}{x}\right)^2 = \tfrac{1}{2}\left(l\tfrac{x+1}{x}\right)^2,$$

cuius demonstratio colligitur ex § 12, dummodo litterae *x, y* et *X, Y* permutantur.

COROLLARIUM I

25. Quia hic est $y = x + 1$, posterior series, *Y,* magis convergit quam prior *X*. Quin etiam, si prior series, *X*, fuerit adeo divergens, quod evenit, quando *x* est fractio unitate minor, posterior nihilominus manet convergens. Veluti si fuerit $x = \tfrac{1}{2}$, erit $y = \tfrac{3}{2}$; ipsae vero series erunt

$$X = \tfrac{2}{1} - \tfrac{2^2}{4} + \tfrac{2^3}{9} - \tfrac{2^4}{16} + \tfrac{2^5}{25} - \text{etc.}$$

et

$$Y = \tfrac{2}{3} + \tfrac{2^2}{4 \cdot 3^2} + \tfrac{2^3}{9 \cdot 3^3} + \tfrac{2^4}{16 \cdot 3^4} + \text{etc.};$$

consequenter erit

$$X - Y = \tfrac{1}{2}(l3)^2.$$

Quia vero posterior series, *Y,* parum convergit, eam per theorema primum hoc modo reducimus:

$$\tfrac{2}{1 \cdot 3} + \tfrac{2^2}{4 \cdot 3^2} + \tfrac{2^3}{9 \cdot 3^3} + \text{etc.} = \tfrac{\pi\pi}{6} - l3 \cdot l\tfrac{3}{2} - \tfrac{1}{1 \cdot 3} - \tfrac{1}{4 \cdot 3^2} - \tfrac{1}{9 \cdot 3^3} - \text{etc.}$$

hincque habebimus hanc summationem:

$$\tfrac{2}{1} - \tfrac{2^2}{4} + \tfrac{2^3}{9} - \tfrac{2^4}{16} + \text{etc.} = \tfrac{1}{2}(l3)^2 + \tfrac{\pi\pi}{6} - l3 \cdot l\tfrac{3}{2} - \left(\tfrac{1}{1 \cdot 3} + \tfrac{1}{4 \cdot 3^2} + \tfrac{1}{9 \cdot 3^3} + \text{etc}\right).$$

COROLLARIUM II

26. Sumamus nunc in genere $x = \tfrac{1}{n}$, ut sit series summanda

$$X = \tfrac{n}{1} - \tfrac{n^2}{4} + \tfrac{n^3}{9} - \tfrac{n^4}{16} + \text{etc.},$$

tum vero ob $y = \frac{1+n}{n}$ altera series erit

$$Y = \frac{n}{n+1} + \frac{nn}{4(n+1)^2} + \frac{n^3}{9(n+1)^3} + \text{etc.}$$

hincque

$$X = \tfrac{1}{2}\left(l(n+1)\right)^2 + Y\,.$$

At vero per theorema I est

$$Y = \frac{\pi\pi}{6} - l(n+1)\cdot l\frac{n+1}{n} - \frac{1}{n+1} - \frac{1}{4(n+1)^2} - \frac{1}{9(n+1)^3} - \text{etc.},$$

quo valore substituto erit

$$X = \tfrac{1}{2}\left(l(n+1)\right)^2 + \frac{\pi\pi}{6} - l(n+1)\cdot l\frac{n+1}{n} - \left(\frac{1}{n+1} + \frac{1}{4(n+1)^2} + \frac{1}{9(n+1)^3} + \text{etc.}\right),$$

quae expressio contrahitur in hanc:

$$\frac{n}{1} - \frac{n^2}{4} + \frac{n^3}{9} - \frac{n^4}{16} + \text{etc.}$$

$$= \tfrac{1}{2}l(n+1)\cdot l\frac{nn}{n+1} + \frac{\pi\pi}{6} - \left(\frac{1}{n+1} + \frac{1}{4(n+1)^2} + \frac{1}{9(n+1)^3} + \text{etc.}\right).$$

THEOREMA III

27. *Si habeantur hae duae series:*

$$X = \frac{x}{1} - \frac{x^2}{4} + \frac{x^3}{9} - \frac{x^4}{16} + etc.$$

et

$$Y = \frac{1}{x} - \frac{1}{4x^2} + \frac{1}{9x^3} - \frac{1}{16x^4} + etc.,$$

erit

$$X + Y = \frac{\pi\pi}{6} + \tfrac{1}{2}\left(lx\right)^2.$$

Demonstratio in praecedentibus non continetur, verum ea hoc modo facile adornatur:

Cum per formulam integralem sit

$$X = \int \frac{\partial x}{x} l(1+x),$$

loco x scribendo $\frac{1}{x}$ erit

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$$Y = \int \frac{\partial x}{x} l \frac{x}{1+x}$$

sive

$$Y = -\int \frac{\partial x}{x} l(1+x) + \int \frac{\partial x}{x} lx$$

hincque addendo

$$X + Y = +\int \frac{\partial x}{x} lx = \tfrac{1}{2}(lx)^2 + C,$$

ubi constans ex casu $x = 1$ facillime definitur. Quia enim hoc casu fit tam X quam $Y = \frac{\pi\pi}{12}$ erit constans $C = \frac{\pi\pi}{6}$ ideoque

$$X + Y = \frac{\pi\pi}{6} + \tfrac{1}{2}(lx)^2.$$

COROLLARIUM I

28. Quodsi ergo pro x numerus quantumvis magnus accipiatur, ope huius theorematis summa seriei X, quae maxime est divergens, facillime assignatur, cum reducatur ad seriem Y, quae eo magis est convergens, quo magis prior divergit.

COROLLARIUM II

29. Nunc vero ope theorematis secundi series

$$Y = \tfrac{1}{x} - \tfrac{1}{4x^2} + \tfrac{1}{9x^3} - \text{etc.}$$

reducitur ad hanc formam:

$$Y = \tfrac{1}{2}\left(l\tfrac{x+1}{x}\right)^2 + \tfrac{1}{x+1} + \tfrac{1}{4(x+1)^2} + \tfrac{1}{9(x+1)^3} + \text{etc.},$$

quo valore substituto prodibit sequens aequatio:

$$\tfrac{x}{1} - \tfrac{x^2}{4} + \tfrac{x^3}{9} - \tfrac{x^4}{16} + \text{etc.} = \tfrac{\pi\pi}{6} + \tfrac{1}{2}(lx)^2 - \tfrac{1}{2}\left(l\tfrac{x+1}{x}\right)^2 - \left(\tfrac{1}{x+1} + \tfrac{1}{4(x+1)^2} + \tfrac{1}{9(x+1)^3} + \text{etc.}\right),$$

quae expressio cum superiori § 26 egregie convenit, quia est

$$\tfrac{1}{2}l(x+1) \cdot l\tfrac{xx}{x+1} = \tfrac{1}{2}(lx)^2 - \tfrac{1}{2}\left(l\tfrac{x+1}{x}\right)^2,$$

uti evolventi facile patebit.

THEOREMA IV

30. *Si habeantur hae series:*

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$$X = \frac{x}{1} + \frac{x^3}{9} + \frac{x^5}{25} + etc. \quad et \quad Y = \frac{y}{1} + \frac{y^3}{9} + \frac{y^5}{25} + etc.$$

existente

$$xy + x + y = 1$$

sive

$$x = \frac{1-y}{1+y} \quad vel \quad y = \frac{1-x}{1+x},$$

erit

$$X + Y = \frac{\pi\pi}{8} - \frac{1}{2} lx \cdot ly.$$

Demonstratio manifesta est ex §16.

COROLLARIUM I

31. Hic iterum, ut supra, observandum est summas harum serierum fieri imaginarias, simulac litterae x et y unitatem superaverint. At si fuerit $x < 1$, tum semper alia series eiusdem formae exhiberi potest, cuius summa ab illa pendeat. Ita si fuerit $x = \frac{1}{2}$, erit $y = \frac{1}{3}$. At si x prope ad unitatem accedat, veluti $x = \frac{9}{10}$, altera series, *Y,* maxime converget.

COROLLARIUM II

32. In his quatuor theorematibus omnes casus contineri videntur, quibus binas huiusmodi series inter se comparare licet. Ad quod ostendendum sequens theorema speciale subiungamus, quod demum per longas calculi ambages sum adeptus, quod autem nunc satis commode ex praecedentibus theorematibus deduci potest.

THEOREMA SPECIALE

33. *Si habeantur hae series sibi affines:*

$$A = \frac{1}{1\cdot3} + \frac{1}{9\cdot3^3} + \frac{1}{25\cdot3^5} + etc.$$

et

$$B = \frac{1}{1\cdot3} + \frac{1}{4\cdot3^2} + \frac{1}{9\cdot3^3} + etc.$$

tum erit

$$2A + B = \frac{\pi\pi}{6} - \frac{1}{2}\left(l3\right)^2.$$

DEMONSTRATIO

Cum ex theoremate primo, sumto $x = y = \frac{1}{2}$, sit

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$$\frac{1}{1\cdot 2} + \frac{1}{4\cdot 2^2} + \frac{1}{9\cdot 2^3} + \text{etc.} = \frac{\pi\pi}{12} - \frac{1}{2}\left(l2\right)^2,$$

haec series sequenti modo resoluta repraesentari potest:

$$2\left(\frac{1}{1\cdot 2} + \frac{1}{9\cdot 2^3} + \frac{1}{25\cdot 2^5} + \text{etc.}\right) - 1\left(\frac{1}{1\cdot 2} - \frac{1}{4\cdot 2^2} + \frac{1}{9\cdot 2^3} - \text{etc.}\right) = \frac{\pi\pi}{12} - \frac{1}{2}\left(l2\right)^2.$$

Nunc vero per theorema IV, sumto $x = \frac{1}{2}$ et $y = \frac{1}{3}$, habemus hanc aequationem:

$$\frac{1}{1\cdot 2} + \frac{1}{9\cdot 2^3} + \frac{1}{25\cdot 2^5} + \text{etc.} = \frac{\pi\pi}{8} - \frac{1}{2}l2\cdot l3 - \frac{1}{1\cdot 3} - \frac{1}{9\cdot 3^3} - \frac{1}{25\cdot 3^5} - \text{etc.}$$

Deinde vero ex theoremate secundo, sumto $x = 2$ et $y = 3$, erit

$$\frac{1}{1\cdot 2} - \frac{1}{4\cdot 2^2} + \frac{1}{9\cdot 2^3} - \frac{1}{16\cdot 2^4} + \text{etc.} = \frac{1}{2}\left(l\tfrac{3}{2}\right)^2 + \frac{1}{1\cdot 3} + \frac{1}{4\cdot 3^2} + \frac{1}{9\cdot 3^3} + \text{etc.}$$

Substituantur iam hi valores loco illarum serierum, ac pro parte sinistra prodibit

$$\left.\begin{array}{l} \frac{\pi\pi}{4} - l2\cdot l3 - 2\left(\frac{1}{1\cdot 3} + \frac{1}{9\cdot 2^3} + \frac{1}{25\cdot 2^5} + \text{etc.}\right) \\[2mm] \quad - \frac{1}{2}\left(l\tfrac{3}{2}\right)^2 - \left(\frac{1}{1\cdot 3} + \frac{1}{4\cdot 3^2} + \frac{1}{9\cdot 3^3} + \text{etc.}\right) \end{array}\right\} = \frac{\pi\pi}{12} - \frac{1}{2}\left(l2\right)^2.$$

Unde concludimus fore

$$\left.\begin{array}{l} 2\left(\frac{1}{1\cdot 3} + \frac{1}{9\cdot 3^3} + \frac{1}{25\cdot 3^5} + \text{etc.}\right) \\[2mm] +1\left(\frac{1}{1\cdot 3} + \frac{1}{4\cdot 3^2} + \frac{1}{9\cdot 3^3} + \text{etc.}\right) \end{array}\right\} = \frac{\pi\pi}{6} - l2\cdot l3 - \frac{1}{2}\left(l\tfrac{3}{2}\right)^2 + \frac{1}{2}\left(l2\right)^2.$$

$$= \frac{\pi\pi}{6} - \frac{1}{2}\left(l3\right)^2 \qquad \left(\text{ob } \left(l\tfrac{3}{2}\right)^2 = \left(l3\right)^2 - 2l2\cdot l3 + \left(l2\right)^2\right).$$

34. Quomodocunque autem theoremata hic data inter se combinentur, vix alia relatio inter binas huiusmodi series elici potest, multo minus autem inde eiusmodi series simplices eruere licet, quarum summa absolute exhiberi queat, praeter casus iam indicatos, quos igitur hic coniunctim ob oculos ponamus.

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$$1 + \tfrac{1}{4} + \tfrac{1}{9} + \tfrac{1}{16} + \tfrac{1}{25} + \text{etc.} = \tfrac{\pi\pi}{6},$$

$$1 - \tfrac{1}{4} + \tfrac{1}{9} - \tfrac{1}{16} + \tfrac{1}{25} - \text{etc.} = \tfrac{\pi\pi}{12},$$

$$\tfrac{1}{1\cdot 2} + \tfrac{1}{4\cdot 2^2} + \tfrac{1}{9\cdot 2^3} + \tfrac{1}{16\cdot 2^4} + \text{etc.} = \tfrac{\pi\pi}{12} - \tfrac{1}{2}\left(l2\right)^2,$$

$$1 + \tfrac{1}{9} + \tfrac{1}{25} + \tfrac{1}{49} + \text{etc.} = \tfrac{\pi\pi}{8}.$$

Praeterea vero adiungi potest adhuc ista series:

$$\tfrac{a}{1} + \tfrac{a^3}{9} + \tfrac{a^5}{25} + \tfrac{a^7}{49} + \text{etc.} = \tfrac{\pi\pi}{16} - \tfrac{1}{4}\left(la\right)^2$$

existente $a = \sqrt{2} - 1$.

Quanquam autem in hac serie valor ipsius a sit irrationalis ideoque quaevis potestas seorsim evolvi debere videatur, tamen numeratores etiam seriem recurrentem constituunt, in qua quilibet terminus per binos praecedentes definiri potest ope huius formulae:

$$a^{n+4} = 6a^{n+2} - a^n,$$

cuius veritas iude elucet, quod sit, per a^n dividendo, $a^4 = 6aa - 1$. Quia enim $a = \sqrt{2} - 1$, erit $a^2 = 3 - 2\sqrt{2}$ et $a^4 = 17 - 12\sqrt{2}$, unde veritas fit manifesta.