OBSERVATIONS
ON THE INTEGRAL FORMULAS $\int x^{p-1}dx(1-x^n)^{\frac{q}{n}-1}$,
ON SETTING $x=1$ AFTER THE INTEGRATION.

Commentary E 321 .

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1. Here we are going to consider the integral formula

$$\int x^{p-1}dx(1-x^n)^{\frac{q}{n}-1},$$

or expressed in this manner,

$$\int \frac{x^{p-1}}{\sqrt[n]{(1-x^n)^{n-q}}}dx,$$

and I assume the exponents *n, p* and *q* to be positive integers, since, if they were not of this form, they could be reduced readily to that. Then the integral of this formula is not to be considered generally here, but only the value of this which it takes, if there may be put $x=1$ after the integration, as the integral vanishes on putting $x=0$. Indeed in the first place there is no reason, why in that case $x=1$, the integral may not be expressed more simply; and besides often in Analysis formulas of this kind may be come upon, and generally it is usual not for the indefinite integral for any value of *x* to be defined, but rather for a particular value such as $x=1$ to be desired.

[It is observed that these integrals are related to the Beta and Gamma functions. Thus here, Euler's $\int x^{p-1}dx(1-x^n)^{\frac{q}{n}-1}$ represents an early form of the modern beta function $\mathrm{B}(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}\,dt$, where $x>0, y>0$. We may note in passing that $\mathrm{B}(x,y)=\frac{(y-1)!}{x(x+1)(x+1)\cdots(x+y-1)}$, where $y>0$ and x is unrestricted. See, e.g. C.S. Ogilvy, *The Beta Function*; A.M.M, Vol. 58, no.7. (1951). pp. 475-479.]

2. Moreover it may be agreed in this case, where after the integration there may be put $x=1$, the integral $\int \frac{x^{p-1}}{\sqrt[n]{(1-x^n)^{n-q}}}dx$ to be expressed in this manner by an infinite product, so that there shall be

$$\frac{(p+q)}{pq}\cdot\frac{n(p+q+n)}{(p+n)(q+n)}\cdot\frac{2n(p+q+2n)}{(p+2n)(q+2n)}\cdot\frac{3n(p+q+3n)}{(p+3n)(q+3n)}\cdot\text{etc.},$$

[See (**) below]

the first factor of which $\frac{(p+q)}{pq}$ indeed is not bound to follow the following law. Yet it is evident this does not prevent the exponents p and q to be interchangeable between each other, thus so that there shall become :

$$\int \frac{x^{p-1}}{\sqrt[n]{(1-x^n)^{n-q}}}dx = \int \frac{x^{q-1}}{\sqrt[n]{(1-x^n)^{n-p}}}dx$$

which equality itself is also easily shown. Truly the same infinite product will lead us to a great many others, by which these integrals will be illustrated further.

3. But in order that I shall have regard to brevity in writing, nor shall I need to be repeatedly writing down this formula $\int \frac{x^{p-1}}{\sqrt[n]{(1-x^n)^{n-q}}}dx$, for any exponent n in place of this I may write $\left(\frac{p}{q}\right)$, thus so that $\left(\frac{p}{q}\right)$ may denote the value of this integral formula $\int \frac{x^{p-1}}{\sqrt[n]{(1-x^n)^{n-q}}}dx$ in the case, where there is put $x=1$ after the integration. And since we have seen [in previous communications] in this case to be

$$\int \frac{x^{p-1}}{\sqrt[n]{(1-x^n)^{n-q}}}dx = \int \frac{x^{q-1}}{\sqrt[n]{(1-x^n)^{n-p}}}dx$$

it is evident there shall be

$$\left(\frac{p}{q}\right) = \left(\frac{q}{p}\right),$$

[To show this symmetry of the Euler's modified Beta Function, consider $\int \frac{x^{p-1}}{\sqrt[n]{(1-x^n)^{n-q}}}dx$,

then on putting $1-x^n = y^n; x^n = 1-y^n$ and $x^p = \left(1-y^n\right)^{\frac{p}{n}}$. Hence,

$px^{p-1}dx = \frac{p}{n}\left(1-y^n\right)^{\frac{p}{n}-1}\cdot -ny^{n-1}dy$; or $x^{p-1}dx = -\left(1-y^n\right)^{\frac{p}{n}-1}\cdot y^{n-1}dy$. Consequently,

$$\int \frac{x^{p-1}}{\sqrt[n]{(1-x^n)^{n-q}}}dx = \int x^{p-1}(1-x^n)^{\frac{q-n}{n}}dx = -\int \left(1-y^n\right)^{\frac{p}{n}-1}\cdot y^{n-1}y^{q-n}dy$$

$$= \int y^{q-1}\left(1-y^n\right)^{\frac{p}{n}-1}dy = \int \frac{y^{q-1}}{\sqrt[n]{(1-y^n)^{n-p}}}dy,$$ on reversing the limits of the integrand.]

thus so that for some value of the exponent n, these expressions $\left(\frac{p}{q}\right)$ and $\left(\frac{q}{p}\right)$ signify the same quantity. Thus, if for the sake of an example, there were $n=4$, there will become

$$\left(\frac{3}{2}\right) = \left(\frac{2}{3}\right) = \int \frac{x^2dx}{\sqrt[4]{(1-x^4)^2}} = \int \frac{xdx}{\sqrt[4]{(1-x^4)}}.$$

Moreover, for the infinite product, there will become :

$$\left(\tfrac{3}{2}\right)=\left(\tfrac{2}{3}\right)=\tfrac{5}{2\cdot 3}\cdot\tfrac{4.9}{6.7}\cdot\tfrac{8.13}{10.11}\cdot\tfrac{12.17}{14.15}\cdot\text{etc.}$$

4. Now I observe in the first place, if the exponents *p* and *q* were greater than the exponent *n,* the integral formula can always be reduced to another form, in which these exponents may be made less than *n*. For since there shall become

$$\int x^{p-1}dx.(1-x^{n})^{\frac{q-n}{n}}=\frac{p-n}{p+q-n}\int x^{p-n-1}dx.(1-x^{n})^{\frac{q-n}{n}},$$

this will be maintained by writing in the usual manner

$$\left(\frac{p}{q}\right)=\frac{p-n}{p+q-n}\left(\frac{p-n}{q}\right),(*)$$

where, if there were $p>n$, the formula shall be returned to another, in which the exponent *p* shall be less than *n,* which also is possible on account of the commutability of the other exponent *q*. On account of which it will suffice for these formulas to be examined to accept any exponent *n* with the exponents *p* and *q* themselves less than *n*, since from all these cases investigated, for which larger values may be had, can there be reduced.

[(*) This can be indicated as follows, from the infinite products : letting

$$I_1=\int x^{p-1}dx.(1-x^{n})^{\frac{q-n}{n}}=\frac{(p+q)}{pq}\cdot\frac{n(p+q+n)}{(p+n)(q+n)}\cdot\frac{2n(p+q+2n)}{(p+2n)(q+2n)}\cdot\frac{3n(p+q+3n)}{(p+3n)(q+3n)}\cdot\text{etc.},$$

$$I_2=\int x^{p-n-1}dx.(1-x^{n})^{\frac{q-n}{n}}=\frac{(p-n+q)}{(p-n)q}\cdot\frac{n(p-n+q+n)}{(p-n+n)(q+n)}\cdot\frac{2n(p-n+q+2n)}{(p-n+2n)(q+2n)}\cdot\frac{3n(p-n+q+3n)}{(p-n+3n)(q+3n)}\cdot\text{etc.}$$

$$=\frac{(p-n+q)}{(p-n)q}\cdot\frac{n(p+q)}{(p)(q+n)}\cdot\frac{2n(p+q+n)}{(p+n)(q+2n)}\cdot\frac{3n(p+q+2n)}{(p+2n)(q+3n)}\cdot\text{etc., then}$$

$$\frac{I_1}{I_2}=\frac{\frac{(p+q)}{pq}\cdot\frac{n(p+q+n)}{(p+n)(q+n)}\cdot\frac{2n(p+q+2n)}{(p+2n)(q+2n)}\cdot\frac{3n(p+q+3n)}{(p+3n)(q+3n)}\cdot\text{etc.}}{\frac{(p-n+q)}{(p-n)q}\cdot\frac{n(p+q)}{(p)(q+n)}\cdot\frac{2n(p+q+n)}{(p+n)(q+2n)}\cdot\frac{3n(p+q+2n)}{(p+2n)(q+3n)}\cdot\text{etc}}$$

$$=\frac{\cancel{(p+q)}}{\cancel{p}\,\cancel{q}}\cdot\frac{\cancel{n}\,\cancel{(p+q+n)}}{\cancel{(p+n)}\,\cancel{(q+n)}}\cdot\frac{\cancel{2n}\,\cancel{(p+q+2n)}}{\cancel{(p+2n)}\,\cancel{(q+2n)}}\cdot\frac{\cancel{3n}\,\cancel{(p+q+3n)}}{\cancel{(p+3n)}\,\cancel{(q+3n)}}\cdots\times\frac{(p-n)\cancel{q}}{(p-n+q)}\cdot\frac{\cancel{(p)}\,\cancel{(q+n)}}{\cancel{n}\,\cancel{(p+q)}}\cdot\frac{\cancel{(p+n)}\,\cancel{(q+2n)}}{\cancel{2n}\,\cancel{(p+q+n)}}\cdot\frac{\cancel{(p+2n)}\,\cancel{(q+3n)}}{\cancel{3n(p+q+2n)}}\cdots$$

$$=\frac{(p-n)}{(p+q-n)}.]$$

5. But the case is apparent at once, for which there is either $p=n$ or $q=n$, to be absolutely or algebraically integrable. For if there were $q=n$, on account of

$$\left(\frac{p}{q}\right)=\int x^{p-1}dx=\frac{x^{p}}{p},$$

on putting $x=1$ there will become $\left(\frac{p}{n}\right)=\frac{1}{p}$, and in a like manner, $\left(\frac{n}{q}\right)=\frac{1}{q}$. And these are the only cases, in which the integrals of our formulas can be shown absolutely, if indeed the exponents p and q do not exceed the exponent n. The integration for all the remaining cases will implicate either the quadrature of the circle or thus of higher quadratures, which here we are going to consider more carefully. Therefore after these formulas $\left(\frac{p}{n}\right)$ or $\left(\frac{n}{q}\right)$, of which the absolute value is $=\frac{1}{p}$, these arise, the value of which is expressed by the quadrature of the circle only; then truly these will follow, which demand a certain higher quadrature, and I will attempt to reduce these higher quadratures both to the simplest form as well as to the smallest number.

6. Since the numbers p and q can be put smaller than the exponent n , these formulas $\left(\frac{p}{q}\right)$ arise through single integrable quadratures of the circle, in which there is $p+q=n$. Indeed let $q=n-p$ and our formula

$$\left(\frac{p}{n-p}\right)=\left(\frac{n-p}{p}\right)=\int\frac{x^{p-1}dx}{\sqrt[n]{(1-x^n)^p}}=\int\frac{x^{q-1}dx}{\sqrt[n]{(1-x^n)^q}}$$

will be expressed by this infinite product

$$\frac{n}{p(n-p)}\cdot\frac{n\cdot 2n}{(n+p)(2n-p)}\cdot\frac{2n\cdot 3n}{(2n+p)(3n-p)}\cdot\frac{3n\cdot 4n}{(3n+p)(4n-p)}\cdot\text{etc.},$$

which, represented in this manner

$$\frac{1}{p}\cdot\frac{nn}{nn-pp}\cdot\frac{4nn}{4nn-pp}\cdot\frac{9nn}{9nn-pp}\cdot\text{etc.}$$

agrees with that product, by which the sines of angles are expressed. Whereby if π may be taken for the semi-circumference of the circle, of which the radius shall be $=1$, and likewise may show the measure of two right angles, there will be

$$\left(\frac{p}{n-p}\right)=\left(\frac{n-p}{p}\right)=\frac{\pi}{n\sin.\frac{p\pi}{n}}=\frac{\pi}{n\sin.\frac{q\pi}{n}}.$$

[We may substitute

$x^n = \sin^2\theta$ into $\int x^{p-1}dx(1-x^n)^{\frac{q}{n}-1}$, where $x^p = \sin x^{\frac{2p}{n}}\theta$ giving

$px^{p-1}dx = \frac{2p}{n}\sin^{\frac{2p}{n}-1}\theta\cos\theta d\theta$, and $(1-x^n)^{\frac{q}{n}-1}$ becomes $\left(1-\sin^2\theta\right)^{\frac{q}{n}-1} = \cos^{\frac{2q}{n}-2}\theta.$

Hence, $\int x^{p-1}dx(1-x^n)^{\frac{q}{n}} = \frac{2}{n}\int \sin^{\frac{2p}{n}-1}\theta d\theta \times \cos^{\frac{2q}{n}-1}\theta$;

In modern terms, on setting the Beta Function to be

(for different n and m)

$B(n,m) = 2\int \sin^{2n-1}\theta \times \cos^{2m-1}d\theta = \Gamma(m-1)\cdot\Gamma\left(n-1\right)/\Gamma(m+n-1),$

where $\Gamma(n)$, etc.

is the well-known gamma function; we have here

$B\left(\frac{p}{n},\frac{q}{n}\right) = 2\int \sin^{\frac{2p}{n}-1}\theta d\theta \times \cos^{\frac{2q}{n}-1}\theta = \frac{\Gamma\left(\frac{p}{n}\right)\Gamma\left(\frac{q}{n}\right)}{\Gamma\left(\frac{q}{n}+\frac{p}{n}\right)}.$

Now, the gamma function can be

expanded in an infinite product: The function $\Gamma(x)$ is equal

to the limit as m goes to infinity of $\Gamma(x) = \frac{m^x m!}{x(x+1)\cdots(x+m)}$. Hence,

$$\frac{\Gamma\left(\frac{p}{n}\right)\Gamma\left(\frac{q}{n}\right)}{\Gamma\left(\frac{q}{n}+\frac{p}{n}\right)} = \lim \frac{\dfrac{m^{\frac{p}{n}}\, m!}{\frac{p}{n}\left(\frac{p}{n}+1\right)\left(\frac{p}{n}+2\right)\cdots\left(\frac{p}{n}+n\right)} \times \dfrac{m^{\frac{q}{n}}\, m!}{\frac{q}{n}\left(\frac{q}{n}+1\right)\left(\frac{q}{n}+2\right)\cdots\left(\frac{q}{n}+m\right)}}{\dfrac{m^{\frac{p+q}{n}}\, m!}{\left(\frac{p+q}{n}\right)\left(\frac{p+q}{n}+1\right)\left(\frac{p+q}{n}+2\right)\cdots\left(\frac{p+q}{n}+m\right)}} \text{ as } m\to\infty$$

$$= \lim \frac{pq}{p+q}\cdot\frac{\left(\frac{p+q+n}{n}\right)\left(\frac{p+q+2n}{n}\right)\cdots\left(\frac{p+q+mn}{n}\right)}{\left(\frac{p+n}{n}\right)\left(\frac{p+2n}{n}\right)\cdots\left(\frac{p+,n}{n}\right)\times\left(\frac{q+n}{n}\right)\left(\frac{q+2n}{n}\right)\cdots\left(\frac{q+mn}{n}\right)}$$

$$= \lim \frac{pq}{p+q}\cdot\frac{n(p+q+n)\cdot 1}{(p+n)(q+n)}\cdot\frac{n(p+q+2n)\cdot 2}{(p+2n)(q+2n)}\cdots\frac{n(p+q+mn)\cdot m}{(p+mn)(q+mn)}\cdots$$, as required.]

[In order to show the connection between these infinite products and the sine function of an angle, we may consider Euler's product for the sine of x:

$$\frac{\sin x}{x}=\left(1-\tfrac{x}{\pi}\right)\left(1+\tfrac{x}{\pi}\right)\left(1-\tfrac{x}{2\pi}\right)\left(1+\tfrac{x}{2\pi}\right)\left(1-\tfrac{x}{3\pi}\right)\left(1+\tfrac{x}{3\pi}\right)\ldots\ldots\ldots$$

$$=\left(1-\tfrac{x^2}{\pi^2}\right)\left(1-\tfrac{x^2}{4\pi^2}\right)\left(1-\tfrac{x^2}{9\pi^2}\right)\ldots\ldots\ldots\ldots;$$ Hence, if $x=\frac{p\pi}{n}$, we have

$$\frac{\sin\frac{p\pi}{n}}{\frac{p\pi}{n}}=\left(1-\frac{p^2}{n^2}\right)\left(1-\frac{p^2}{4n^2}\right)\left(1-\frac{p^2}{9n^2}\right)\ldots\ldots\ldots\ldots=\left(\frac{n^2-p^2}{n^2}\right)\left(\frac{4n^2-p^2}{4n^2}\right)\left(\frac{9n^2-p^2}{9n^2}\right)\ldots.,$$

and hence :]

$$\frac{\frac{p\pi}{n}}{\sin\frac{p\pi}{n}}=\left(\frac{n^2}{n^2-p^2}\right)\left(\frac{4n^2}{4n^2-p^2}\right)\left(\frac{9n^2}{9n^2-p^2}\right)\ldots.$$

We note that if $\left(\frac{p}{n-p}\right)=\left(\frac{n-p}{p}\right)=\frac{1}{p}\cdot\frac{nn}{nn-pp}\cdot\frac{4nn}{4nn-pp}\cdot\frac{9nn}{9nn-pp}\cdot$ etc.,

then this corresponds to $\frac{\pi}{n\sin.\frac{p\pi}{n}}=\frac{\pi}{n\sin.\frac{q\pi}{n}}$.]

7. For the remaining cases also, for which neither $p=n$ nor $q=n$ nor $p+q=n$, the integral cannot be shown, either absolutely or by the quadrature of the circle, but it may be reached by another certain higher quadrature. Truly not only do several different special cases of this kind of quadrature arise, but also several reductions are given, from which it is possible to compare different formulas with each other. Moreover these reductions may be derived from the infinite product I have shown above ; since there shall become

$$\left(\frac{p}{q}\right)=\frac{p+q}{pq}\cdot\frac{n(p+q+n)}{(p+n)(q+n)}\cdot\frac{2n(p+q+2n)}{(p+2n)(q+2n)}\cdot\text{etc.},$$

in a similar manner there shall be

$$\left(\frac{p+q}{r}\right)=\frac{p+q+r}{(p+q)r}\cdot\frac{n(p+q+r+n)}{(p+q+n)(r+n)}\cdot\frac{2n(p+q+r+2n)}{(p+q+2n)(r+2n)}\cdot\text{etc.},$$

from which by multiplying these together, in turn there will be obtained :

$$\left(\frac{p}{q}\right)\left(\frac{p+q}{r}\right)=\frac{p+q+r}{pqr}\cdot\frac{nn(p+q+r+n)}{(p+n)(q+n)(r+n)}\cdot\frac{4nn(p+q+r+2n)}{(p+2n)(q+2n)(r+2n)}\cdot\text{etc.},$$

where the three quantities *p, q, r* can be permuted among themselves.

8. Hence therefore with these quantities *p, q, r* interchanged, we obtain the following reductions:

$$\left(\frac{p}{q}\right)\left(\frac{p+q}{r}\right)=\left(\frac{p}{r}\right)\left(\frac{p+r}{q}\right)=\left(\frac{q}{r}\right)\left(\frac{q+r}{p}\right),$$

from which from the given formulas some more others are able to be determined. Just as if there shall be $q+r=n$ or $r=n-q$, on account of

$$\left(\frac{q+r}{p}\right)=\frac{1}{p} \text{ and } \left(\frac{q}{r}\right)=\frac{\pi}{n\sin.\frac{q\pi}{n}},$$

there will be

$$\left(\frac{p}{q}\right)\left(\frac{p+q}{n-q}\right)=\frac{\pi}{np\sin.\frac{q\pi}{n}}$$

as well as

$$\left(\frac{p}{n-q}\right)\left(\frac{n+p-q}{q}\right)=\frac{\pi}{np\sin.\frac{q\pi}{n}}.$$

Then if there shall be $p+q+r=n$ or $r=n-p-q$, there will become

$$\frac{\pi}{n\sin.\frac{r\pi}{n}}\left(\frac{p}{q}\right)=\frac{\pi}{n\sin.\frac{q\pi}{n}}\left(\frac{p}{r}\right)=\frac{\pi}{n\sin.\frac{p\pi}{n}}\left(\frac{q}{r}\right),$$

from which notable reductions of others to yet others arise, from which multitudes of the necessary quadratures are strongly diminished according to our aim.

9. Truly, in addition, we arrive at the following equations from two formulas assumed for the determined numbers *p, q, r* :

$$\left(\frac{1}{1}\right)\left(\frac{2}{2}\right)=\left(\frac{2}{1}\right)\left(\frac{3}{1}\right),$$
$$\left(\frac{1}{1}\right)\left(\frac{3}{2}\right)=\left(\frac{3}{1}\right)\left(\frac{4}{1}\right),$$
$$\left(\frac{2}{1}\right)\left(\frac{3}{3}\right)=\left(\frac{3}{1}\right)\left(\frac{4}{2}\right)=\left(\frac{3}{2}\right)\left(\frac{5}{1}\right),$$
$$\left(\frac{2}{2}\right)\left(\frac{4}{3}\right)=\left(\frac{3}{2}\right)\left(\frac{5}{2}\right),$$
$$\left(\frac{3}{1}\right)\left(\frac{4}{3}\right)=\left(\frac{3}{3}\right)\left(\frac{6}{1}\right),$$
$$\left(\frac{3}{2}\right)\left(\frac{5}{3}\right)=\left(\frac{3}{3}\right)\left(\frac{6}{2}\right),$$
$$\left(\frac{2}{2}\right)\left(\frac{4}{4}\right)=\left(\frac{4}{2}\right)\left(\frac{6}{2}\right),$$
$$\left(\frac{3}{1}\right)\left(\frac{4}{4}\right)=\left(\frac{4}{1}\right)\left(\frac{5}{3}\right)=\left(\frac{4}{3}\right)\left(\frac{7}{1}\right),$$
$$\left(\frac{2}{1}\right)\left(\frac{5}{3}\right)=\left(\frac{5}{1}\right)\left(\frac{6}{2}\right)=\left(\frac{5}{2}\right)\left(\frac{7}{1}\right),$$
$$\left(\frac{1}{1}\right)\left(\frac{6}{2}\right)=\left(\frac{6}{1}\right)\left(\frac{7}{1}\right),$$

etc.

where indeed more occur, which now are contained in the remaining.

10. I shall assume from these apparent principles established, in which general formula $\int \frac{x^{p-1}dx}{\sqrt[n]{(1-x^n)^{n-q}}}$ presented the numbers *p* and *q* do not exceed the exponent *n,* I will attempt to separate into classes according to the exponent *n*, thus so that the values $n=1,\ n=2,\ n=3,\ n=4$ etc. will be providing the first, second, third, etc, classes.

And indeed the first class, where $n=1$, includes the single formula $\left(\frac{1}{1}\right)$, the value of which is $=1$.

The second class, where $n=2$, contains these formulas $\left(\frac{1}{1}\right)$, $\left(\frac{2}{1}\right)$ and $\left(\frac{2}{2}\right)$, of which the ordering is clear.

The third class, where $n=3$, has these : $\left(\frac{1}{1}\right),\ \left(\frac{2}{1}\right),\ \left(\frac{3}{1}\right),\ \left(\frac{2}{2}\right),\ \left(\frac{3}{2}\right),\ \left(\frac{3}{3}\right)$.

Truly the fourth class, where $n=4$, these

$$\left(\tfrac{1}{1}\right),\ \left(\tfrac{2}{1}\right),\ \left(\tfrac{3}{1}\right),\ \left(\tfrac{4}{1}\right),\ \left(\tfrac{2}{2}\right),\ \left(\tfrac{3}{2}\right),\ \left(\tfrac{4}{2}\right),\ \left(\tfrac{3}{3}\right),\ \left(\tfrac{4}{3}\right),\ \left(\tfrac{4}{4}\right);$$

and thus the number increases in the following classes of the formulas according to triangular numbers. Therefore we will run through these classes in order.

Forms of the 2nd class $\int \frac{x^{p-1}dx}{\sqrt[2]{(1-x^2)^{2-q}}}=\left(\frac{p}{q}\right)$.

Indeed it is seen here that these formulas are to be expressed either absolutely or in terms of the quadrature of the circle; for these $\left(\frac{2}{1}\right)$ and $\left(\frac{2}{2}\right)$ are given absolutely, and the remaining $\left(\frac{1}{1}\right)$ on account of $1+1=2$ is $=\frac{\pi}{2\sin.\frac{\pi}{2}}=\frac{\pi}{2}$; if therefore for brevity we may put $\frac{\pi}{2}=\alpha$, as clearly we will make use of in the following classes, all the formulas of this class are defined thus:

$$\left(\tfrac{2}{1}\right)=1,\ \left(\tfrac{2}{2}\right)=\tfrac{1}{2};$$
$$\left(\tfrac{1}{1}\right)=\alpha.$$

Forms of the 3rd Class $\int \frac{x^{p-1}dx}{\sqrt[3]{(1-x^3)^{3-q}}}=\left(\frac{p}{q}\right)$.

Since here there shall be $n=3$, the formula involving the quadrature of the circle is

$$\left(\tfrac{2}{1}\right)=\frac{\pi}{3\sin.\frac{\pi}{3}};$$

therefore we may put $\left(\frac{2}{1}\right)=\alpha;$ moreover the remaining formulas, which are not given absolutely, involve higher quadratures and indeed the single $\left(\frac{1}{1}\right)$, which we will indicate by the letter A ; with which agreed on we will be able to assign the values of all the formulas of this class:

$$\left(\tfrac{3}{1}\right)=1,\quad \left(\tfrac{3}{2}\right)=\tfrac{1}{2},\quad \left(\tfrac{3}{3}\right)=\tfrac{1}{3};$$
$$\left(\tfrac{2}{1}\right)=\alpha,\ \left(\tfrac{2}{1}\right)=\tfrac{\alpha}{A};$$
$$\left(\tfrac{1}{1}\right)=A.$$

Forms of the 4th class $\int\frac{x^{p-1}dx}{\sqrt[4]{(1-x^4)^{4-q}}}=\left(\frac{p}{q}\right)$

Since here there shall be $n=4$, we have two formulas depending on the quadrature of the circle, the values of which, because they are known, we may indicate thus :

$$\left(\tfrac{3}{1}\right)=\frac{\pi}{4\sin.\frac{\pi}{4}}=\alpha \text{ and } \left(\tfrac{2}{2}\right)=\frac{\pi}{4\sin.\frac{2\pi}{4}}=\beta.$$

Besides truly, there is a need for a formula involving a higher order of quadrature, since it may be agreed that we know all the rest. Indeed we may put $\left(\frac{2}{1}\right)=A$ and all the formulas of this class will be determined thus:

$$\left(\tfrac{4}{1}\right)=1,\quad \left(\tfrac{4}{2}\right)=\tfrac{1}{2},\quad \left(\tfrac{4}{3}\right)=\tfrac{1}{3},\quad \left(\tfrac{4}{4}\right)=\tfrac{1}{4};$$
$$\left(\tfrac{3}{1}\right)=\alpha,\ \left(\tfrac{3}{2}\right)=\tfrac{\beta}{A},\quad \left(\tfrac{3}{3}\right)=\tfrac{\alpha}{2A};$$
$$\left(\tfrac{2}{1}\right)=A,\ \left(\tfrac{2}{1}\right)=\beta;$$
$$\left(\tfrac{1}{1}\right)=\tfrac{\alpha A}{\beta}.$$

Forms of the 5th class $\int\frac{x^{p-1}dx}{\sqrt[5]{(1-x^5)^{5-q}}}=\left(\frac{p}{q}\right)$

Since here there shall be $n = 5$, we will note at once the formulas depending on the quadrature of the circle

$$\left(\tfrac{4}{1}\right)=\frac{\pi}{5\sin.\frac{\pi}{5}}=\alpha \text{ and } \left(\tfrac{3}{2}\right)=\frac{\pi}{5\sin.\frac{2\pi}{5}}=\beta.$$

But in addition there is a need for two new quadratures peculiar to this class, which we will designate thus

$$\left(\tfrac{3}{1}\right)=A \text{ and } \left(\tfrac{2}{2}\right)=B,$$

and from which all the remaining will be defined thus :

$$\left(\tfrac{5}{1}\right)=1,\quad \left(\tfrac{5}{2}\right)=\tfrac{1}{2},\quad \left(\tfrac{5}{3}\right)=\tfrac{1}{3},\quad \left(\tfrac{5}{4}\right)=\tfrac{1}{4},\quad \left(\tfrac{5}{5}\right)=\tfrac{1}{5};$$
$$\left(\tfrac{4}{1}\right)=\alpha,\quad \left(\tfrac{4}{2}\right)=\tfrac{\beta}{A},\quad \left(\tfrac{4}{3}\right)=\tfrac{\beta}{2B},\quad \left(\tfrac{4}{4}\right)=\tfrac{\alpha}{3A};$$
$$\left(\tfrac{3}{1}\right)=A,\quad \left(\tfrac{3}{2}\right)=\beta,\quad \left(\tfrac{3}{3}\right)=\tfrac{\beta\beta}{\alpha B};$$
$$\left(\tfrac{2}{1}\right)=\tfrac{\alpha B}{\beta},\quad \left(\tfrac{2}{2}\right)=B;$$
$$\left(\tfrac{1}{1}\right)=\tfrac{\alpha A}{\beta}.$$

Forms of the 6th order $\int\frac{x^{p-1}dx}{\sqrt[6]{(1-x^6)^{6-q}}}=\left(\frac{p}{q}\right)$.

Here there is $n=6$ and the formulas involving the squares of circles are :

$$\left(\tfrac{5}{1}\right)=\frac{\pi}{6\sin.\frac{\pi}{6}}=\alpha,\quad \left(\tfrac{4}{2}\right)=\frac{\pi}{6\sin.\frac{2\pi}{6}}=\beta,\quad \left(\tfrac{3}{3}\right)=\frac{\pi}{6\sin.\frac{2\pi}{6}}=\gamma.$$

Truly the values of all the others depend in addition on these two quadratures

$$\left(\tfrac{4}{1}\right)=A \text{ and } \left(\tfrac{3}{2}\right)=B,$$

and thus may be taken to be :

$$\left(\tfrac{6}{1}\right)=1,\quad \left(\tfrac{6}{2}\right)=\tfrac{1}{2},\quad \left(\tfrac{6}{3}\right)=\tfrac{1}{3},\quad \left(\tfrac{6}{4}\right)=\tfrac{1}{4},\quad \left(\tfrac{6}{5}\right)=\tfrac{1}{5},\quad \left(\tfrac{6}{6}\right)=\tfrac{1}{6};$$
$$\left(\tfrac{5}{1}\right)=\alpha,\quad \left(\tfrac{5}{2}\right)=\tfrac{\beta}{A},\quad \left(\tfrac{5}{3}\right)=\tfrac{\gamma}{2B},\quad \left(\tfrac{5}{4}\right)=\tfrac{\beta}{3B},\quad \left(\tfrac{5}{5}\right)=\tfrac{\alpha}{4A};$$
$$\left(\tfrac{4}{1}\right)=A,\quad \left(\tfrac{4}{2}\right)=\beta,\quad \left(\tfrac{4}{3}\right)=\tfrac{\beta\gamma}{\alpha B},\quad \left(\tfrac{4}{4}\right)=\tfrac{\beta\gamma A}{2\alpha BB};$$
$$\left(\tfrac{3}{1}\right)=\tfrac{\alpha B}{\beta},\quad \left(\tfrac{3}{2}\right)=B,\quad \left(\tfrac{3}{3}\right)=\gamma;$$
$$\left(\tfrac{2}{1}\right)=\tfrac{\alpha B}{\gamma},\quad \left(\tfrac{2}{2}\right)=\tfrac{\alpha BB}{\gamma A};$$
$$\left(\tfrac{1}{1}\right)=\tfrac{\alpha A}{\beta}.$$

Forms of the 7th class $\int\frac{x^{p-1}dx}{\sqrt[7]{(1-x^7)^{7-q}}}=\left(\frac{p}{q}\right)$.

Because $n=7$, the formulas depending on the quadrature of the circle will be designated thus

$$\left(\tfrac{6}{1}\right)=\frac{\pi}{7\sin.\frac{\pi}{7}}=\alpha,\ \left(\tfrac{5}{2}\right)=\frac{\pi}{7\sin.\frac{2\pi}{7}}=\beta,\ \left(\tfrac{4}{3}\right)=\frac{\pi}{7\sin.\frac{2\pi}{7}}=\gamma,$$

moreover truly these quadratures are introduced

$$\left(\tfrac{6}{1}\right)=A,\ \left(\tfrac{4}{2}\right)=B,\ \left(\tfrac{3}{3}\right)=C,$$

from which the formulas from all given will be determined thus :

$$\left(\tfrac{7}{1}\right)=1,\quad \left(\tfrac{7}{2}\right)=\tfrac{1}{2},\quad \left(\tfrac{7}{3}\right)=\tfrac{1}{3},\quad \left(\tfrac{7}{4}\right)=\tfrac{1}{4},\quad \left(\tfrac{7}{5}\right)=\tfrac{1}{5},\quad \left(\tfrac{7}{6}\right)=\tfrac{1}{6},\quad \left(\tfrac{7}{7}\right)=\tfrac{1}{7};$$
$$\left(\tfrac{6}{1}\right)=\alpha,\quad \left(\tfrac{6}{2}\right)=\tfrac{\beta}{A},\quad \left(\tfrac{6}{3}\right)=\tfrac{\gamma}{2B},\quad \left(\tfrac{6}{4}\right)=\tfrac{\beta}{3C},\quad \left(\tfrac{6}{5}\right)=\tfrac{\beta}{4B},\quad \left(\tfrac{6}{6}\right)=\tfrac{\alpha}{5A};$$
$$\left(\tfrac{5}{1}\right)=A,\quad \left(\tfrac{5}{2}\right)=\beta,\quad \left(\tfrac{5}{3}\right)=\tfrac{\beta\gamma}{AB},\quad \left(\tfrac{5}{4}\right)=\tfrac{\gamma\gamma A}{2\alpha BC},\quad \left(\tfrac{5}{5}\right)=\tfrac{\beta\gamma A}{2\alpha BC};$$
$$\left(\tfrac{4}{1}\right)=\tfrac{\alpha B}{\beta},\quad \left(\tfrac{4}{2}\right)=B,\quad \left(\tfrac{4}{3}\right)=\gamma,\quad \left(\tfrac{4}{4}\right)=\tfrac{\gamma\gamma}{\alpha C};$$
$$\left(\tfrac{3}{1}\right)=\tfrac{\alpha C}{\gamma},\quad \left(\tfrac{3}{2}\right)=\tfrac{\alpha BC}{\gamma A},\quad \left(\tfrac{3}{3}\right)=C;$$
$$\left(\tfrac{2}{1}\right)=\tfrac{\alpha B}{\gamma},\ \left(\tfrac{2}{2}\right)=\tfrac{\alpha\beta BB}{\gamma\gamma A};$$
$$\left(\tfrac{1}{1}\right)=\tfrac{\alpha A}{\beta}.$$

Forms of the 8th class $\int\frac{x^{p-1}dx}{\sqrt[8]{(1-x^8)^{8-q}}}=\left(\frac{p}{q}\right).$

Because here there is $n=8$, the formulas implicating the quadratures of the circle will be:

$$\left(\tfrac{7}{1}\right)=\frac{\pi}{8\sin.\frac{\pi}{8}}=\alpha,\ \left(\tfrac{6}{2}\right)=\frac{\pi}{8\sin.\frac{2\pi}{8}}=\beta,$$
$$\left(\tfrac{5}{3}\right)=\frac{\pi}{8\sin.\frac{3\pi}{8}}=\gamma,\ \left(\tfrac{4}{4}\right)=\frac{\pi}{8\sin.\frac{4\pi}{8}}=\delta.$$

Now truly the three often repeated formula will be considered as known

$$\left(\tfrac{6}{1}\right)=A,\ \left(\tfrac{5}{2}\right)=B,\ \left(\tfrac{4}{3}\right)=C,$$

and from these all the formulas of this class will be determined thus :

$$\left(\tfrac{8}{1}\right)=1,\quad \left(\tfrac{8}{2}\right)=\tfrac{1}{2},\quad \left(\tfrac{8}{3}\right)=\tfrac{1}{3},\quad \left(\tfrac{8}{4}\right)=\tfrac{1}{4},\quad \left(\tfrac{8}{5}\right)=\tfrac{1}{5},\quad \left(\tfrac{8}{6}\right)=\tfrac{1}{6},\quad \left(\tfrac{8}{7}\right)=\tfrac{1}{7},\quad \left(\tfrac{8}{8}\right)=\tfrac{1}{8};$$

$$\left(\tfrac{7}{1}\right)=\alpha,\quad \left(\tfrac{7}{2}\right)=\tfrac{\beta}{A},\quad \left(\tfrac{7}{3}\right)=\tfrac{\gamma}{2B},\quad \left(\tfrac{7}{4}\right)=\tfrac{\delta}{3C},\quad \left(\tfrac{7}{5}\right)=\tfrac{\gamma}{4C},\quad \left(\tfrac{7}{6}\right)=\tfrac{\beta}{5B},\quad \left(\tfrac{7}{7}\right)=\tfrac{\alpha}{6A};$$

$$\left(\tfrac{6}{1}\right)=A,\quad \left(\tfrac{6}{2}\right)=\beta,\quad \left(\tfrac{6}{3}\right)=\tfrac{\beta\gamma}{\alpha B},\quad \left(\tfrac{6}{4}\right)=\tfrac{\gamma\delta A}{2\alpha BC},\quad \left(\tfrac{6}{5}\right)=\tfrac{\gamma\delta A}{3\alpha CC},\quad \left(\tfrac{6}{6}\right)=\tfrac{\beta\gamma A}{4\alpha BC};$$

$$\left(\tfrac{5}{1}\right)=\tfrac{aB}{\beta},\quad \left(\tfrac{5}{2}\right)=B,\quad \left(\tfrac{5}{3}\right)=\gamma,\quad \left(\tfrac{5}{4}\right)=\tfrac{\gamma\delta}{\alpha C},\quad \left(\tfrac{5}{5}\right)=\tfrac{\gamma\gamma\delta A}{2\alpha\beta CC};$$

$$\left(\tfrac{4}{1}\right)=\tfrac{\alpha C}{\gamma},\quad \left(\tfrac{4}{2}\right)=\tfrac{\alpha BC}{\gamma A},\quad \left(\tfrac{4}{3}\right)=C,\quad \left(\tfrac{4}{4}\right)=\delta;$$

$$\left(\tfrac{3}{1}\right)=\tfrac{\alpha C}{\delta},\quad \left(\tfrac{3}{2}\right)=\tfrac{\alpha\beta CC}{\gamma\delta A},\quad \left(\tfrac{3}{3}\right)=\tfrac{\alpha CC}{\delta A};$$

$$\left(\tfrac{2}{1}\right)=\tfrac{\alpha B}{\gamma},\quad \left(\tfrac{2}{2}\right)=\tfrac{\alpha\beta BC}{\gamma\delta A};$$

$$\left(\tfrac{1}{1}\right)=\tfrac{\alpha A}{\beta}.$$

Hence it is allowed to continue these reductions to the following classes, as far as it may be desired. Therefore just as hence we may explain in general how the integrations of the individual formulas themselves may be found :

Establishing the general form $\int \frac{x^{p-1}dx}{\sqrt[n]{(1-x^n)^{n-q}}}=\left(\frac{p}{q}\right)$.

Therefore in the first place these formulas shall be absolute integrals

$$\left(\tfrac{n}{1}\right)=1,\quad \left(\tfrac{n}{2}\right)=\tfrac{1}{2},\quad \left(\tfrac{n}{3}\right)=\tfrac{1}{3},\quad \left(\tfrac{n}{4}\right)=\tfrac{1}{4},\text{etc.},$$

then the formulas depending on the quadrature of the circle are

$$\left(\tfrac{n-1}{1}\right)=\alpha,\quad \left(\tfrac{n-2}{2}\right)=\beta,\quad \left(\tfrac{n-3}{3}\right)=\gamma,\quad \left(\tfrac{n-4}{4}\right)=\delta,\text{etc.},$$

the progression of which quantities finally is reverted into itself, since there shall also

$$\left(\tfrac{4}{n-4}\right)=\delta,\quad \left(\tfrac{3}{n-3}\right)=\gamma,\quad \left(\tfrac{2}{n-2}\right)=\beta,\quad \left(\tfrac{1}{n-1}\right)=\alpha.$$

In addition truly higher quadratures must be called into help, which are represented thus :

$$\left(\tfrac{n-2}{1}\right)=A,\quad \left(\tfrac{n-3}{2}\right)=B,\quad \left(\tfrac{n-4}{3}\right)=C,\quad \left(\tfrac{n-5}{4}\right)=D,\text{etc.},$$

the number of which in any case is determined at once, because these formulas finally are returned into themselves.

But from these formulas allowed, all belonging to the same class will be able to be defined generally. Moreover, we will have from the formula $\left(\frac{n-1}{1}\right)=\alpha$, as we have ordered the formulas above, by descending downwards:

$$\left(\tfrac{n-1}{1}\right)=\alpha,\ \left(\tfrac{n-2}{1}\right)=A,\ \left(\tfrac{n-3}{1}\right)=\tfrac{\alpha B}{\beta},\ \left(\tfrac{n-4}{1}\right)=\tfrac{\alpha C}{\gamma},$$
$$\left(\tfrac{n-5}{1}\right)=\tfrac{\alpha D}{\delta},\ \left(\tfrac{n-6}{1}\right)=\tfrac{\alpha E}{\varepsilon},\ \text{etc.},$$

which values taken backwards thus may are obtained :

$$\left(\tfrac{1}{1}\right)=\tfrac{\alpha A}{\beta},\ \left(\tfrac{2}{1}\right)=\tfrac{\alpha B}{\gamma},\ \left(\tfrac{3}{1}\right)=\tfrac{\alpha C}{\delta},\ \text{etc.}$$

Then truly from that same $\left(\frac{n-1}{1}\right)=\alpha$ by progressing horizontally these same formulas are defined

$$\left(\tfrac{n-1}{1}\right)=\alpha,\ \left(\tfrac{n-1}{2}\right)=\tfrac{\beta}{A},\ \left(\tfrac{n-1}{3}\right)=\tfrac{\gamma}{2B},\ \left(\tfrac{n-1}{4}\right)=\tfrac{\delta}{3C},\ \text{etc.},$$

of which the last will become:

$$\left(\tfrac{n-1}{n-1}\right)=\frac{\alpha}{(n-2)A},$$

the second from last:

$$\left(\tfrac{n-1}{n-2}\right)=\frac{\beta}{(n-3)B},$$

the third from last:

$$\left(\tfrac{n-1}{n-3}\right)=\frac{\gamma}{(n-4)C},\ \text{etc.}$$

In a similar manner from the formula $\left(\frac{n-2}{2}\right)=\beta$, by descending as well as by progressing horizontally, we will obtain the values of the others; as indeed by descending

$$\left(\tfrac{n-2}{2}\right)=\beta,\ \left(\tfrac{n-3}{2}\right)=B,\ \left(\tfrac{n-4}{2}\right)=\tfrac{\alpha BC}{\gamma A},\ \left(\tfrac{n-5}{2}\right)=\tfrac{\alpha\beta CD}{\gamma\delta A},$$
$$\left(\tfrac{n-6}{2}\right)=\tfrac{\alpha\beta DE}{\delta\varepsilon A},\ \left(\tfrac{n-7}{2}\right)=\tfrac{\alpha\beta EF}{\varepsilon\xi A}\ \text{etc.},$$

where the last becomes :

$$\left(\tfrac{2}{2}\right)=\frac{\alpha\beta BC}{\gamma\delta A},$$

the second last :

$$\left(\tfrac{3}{2}\right)=\frac{\alpha\beta CD}{\delta\varepsilon A}$$

etc.;

but by progressing horizontally :

$$\left(\frac{n-2}{2}\right)=\beta,\ \left(\frac{n-2}{3}\right)=\frac{\beta\gamma}{\alpha B},\ \left(\frac{n-2}{4}\right)=\frac{\gamma\delta A}{2\alpha BC},\ \left(\frac{n-2}{5}\right)=\frac{\delta\varepsilon A}{3\alpha CD},$$
$$\left(\frac{n-2}{6}\right)=\frac{\varepsilon\xi A}{4\alpha DE},\ \left(\frac{n-2}{7}\right)=\frac{\xi\eta A}{5\alpha EF}\ \text{ etc.,}$$

of which the last will become :

$$\left(\frac{n-2}{n-2}\right)=\frac{\beta\gamma A}{(n-4)\alpha BC}$$

the second last :

$$\left(\frac{n-2}{n-3}\right)=\frac{\gamma\delta A}{(n-5)\alpha CD}$$

etc.

Again, from the formula $\left(\frac{n-3}{3}\right)=\gamma$ by descending we arrive at these formulas

$$\left(\frac{n-3}{3}\right)=\gamma,\ \left(\frac{n-4}{3}\right)=C,\ \left(\frac{n-5}{3}\right)=\frac{\alpha CD}{\delta A},\ \left(\frac{n-6}{3}\right)=\frac{\alpha\beta CDE}{\delta\varepsilon AB},$$
$$\left(\frac{n-7}{3}\right)=\frac{\alpha\beta\gamma DEF}{\delta\varepsilon\xi AB},\ \left(\frac{n-8}{3}\right)=\frac{\alpha\beta\gamma EFG}{\varepsilon\xi\eta AB}\ \text{ etc.}$$

and by progressing horizontally :

$$\left(\frac{n-3}{3}\right)=\gamma,\ \left(\frac{n-3}{4}\right)=\frac{\gamma\delta}{\alpha C},\ \left(\frac{n-3}{5}\right)=\frac{\gamma\delta\varepsilon A}{2\alpha\beta CD},\ \left(\frac{n-3}{6}\right)=\frac{\delta\varepsilon\xi AB}{3\alpha\beta CDE},$$
$$\left(\frac{n-3}{7}\right)=\frac{\varepsilon\xi\eta AB}{4\alpha\beta DEF},\ \left(\frac{n-3}{8}\right)=\frac{\xi\eta\theta AB}{5\alpha\beta EFG}\ \text{ etc.}$$

In a similar manner from the formula $\left(\frac{n-4}{4}\right)=\delta$, by descending we obtain

$$\left(\frac{n-4}{4}\right)=\delta,\ \left(\frac{n-5}{4}\right)=D,\ \left(\frac{n-6}{4}\right)=\frac{\alpha DE}{\varepsilon A},\ \left(\frac{n-7}{4}\right)=\frac{\alpha\beta DEF}{\varepsilon\xi AB},$$
$$\left(\frac{n-8}{4}\right)=\frac{\alpha\beta\gamma DEFG}{\varepsilon\xi\eta ABC},\ \left(\frac{n-9}{4}\right)=\frac{\alpha\beta\gamma\delta EFGH}{\varepsilon\xi\eta\theta ABC}\ \text{ etc.}$$

and by progressing horizontally:

$$\left(\frac{n-4}{4}\right)=\delta,\ \left(\frac{n-4}{5}\right)=\frac{\delta\varepsilon}{\alpha D},\ \left(\frac{n-4}{6}\right)=\frac{\delta\varepsilon\xi A}{2\alpha\beta DE},\ \left(\frac{n-4}{7}\right)=\frac{\delta\varepsilon\xi\eta AB}{3\alpha\beta\gamma DEF},$$
$$\left(\frac{n-4}{8}\right)=\frac{\varepsilon\xi\eta\theta ABC}{4\alpha\beta\gamma DEFG},\ \left(\frac{n-4}{9}\right)=\frac{\xi\eta\theta\iota ABC}{5\alpha\beta\gamma EFGH}\ \text{ etc.}$$

And by this account all the values of the formulas are found at last.

We may adapt these general reductions to

$$\textit{The forms of the } 9^{th} \textit{ class } \int \frac{x^{p-1}dx}{\sqrt[9]{(1-x^9)^{9-q}}} = \left(\frac{p}{q}\right).$$

Whereas $n=9$, the formulas involving the quadratures of the circle will become :

$$\left(\tfrac{8}{1}\right)=\alpha,\ \left(\tfrac{7}{2}\right)=\beta,\ \left(\tfrac{6}{3}\right)=\gamma,\ \left(\tfrac{5}{4}\right)=\delta,\text{etc.},$$

hence $\varepsilon=\delta,\ \xi=\gamma,\ \eta=\beta,\ \theta=\alpha.$

Then the new quadratures required here are put in place :

$$\left(\tfrac{7}{1}\right)=A,\ \left(\tfrac{6}{2}\right)=B,\ \left(\tfrac{5}{3}\right)=C,\ \left(\tfrac{4}{4}\right)=D,$$

and thus there will become:

$$E=C,\ F=B \text{ et } G=A;$$

and with these four values granted the values of all the ninth class will be able to be assigned, which we will represent in a similar order, as we have done thus far:

$$\left(\tfrac{9}{1}\right)=1,\quad \left(\tfrac{9}{2}\right)=\tfrac{1}{2},\quad \left(\tfrac{9}{3}\right)=\tfrac{1}{3},\quad \left(\tfrac{9}{4}\right)=\tfrac{1}{4},\quad \left(\tfrac{9}{5}\right)=\tfrac{1}{5},$$

$$\left(\tfrac{9}{6}\right)=\tfrac{1}{6},\quad \left(\tfrac{9}{7}\right)=\tfrac{1}{7},\quad \left(\tfrac{9}{8}\right)=\tfrac{1}{8}\quad \left(\tfrac{9}{9}\right)=\tfrac{1}{9};$$

$$\left(\tfrac{8}{1}\right)=\alpha,\quad \left(\tfrac{8}{2}\right)=\tfrac{\beta}{A},\quad \left(\tfrac{8}{3}\right)=\tfrac{\gamma}{2B},\quad \left(\tfrac{8}{4}\right)=\tfrac{\delta}{3C},\quad \left(\tfrac{8}{5}\right)=\tfrac{\delta}{4D},$$

$$\left(\tfrac{8}{6}\right)=\tfrac{\gamma}{5C},\quad \left(\tfrac{8}{7}\right)=\tfrac{\beta}{6B},\quad \left(\tfrac{8}{8}\right)=\tfrac{\alpha}{7A}\ ;$$

$$\left(\tfrac{7}{1}\right)=A,\quad \left(\tfrac{7}{2}\right)=\beta,\quad \left(\tfrac{7}{3}\right)=\tfrac{\beta\gamma}{\alpha B},\quad \left(\tfrac{7}{4}\right)=\tfrac{\gamma\delta A}{2\alpha BC},\quad \left(\tfrac{7}{5}\right)=\tfrac{\delta\delta A}{3\alpha CD},$$

$$\left(\tfrac{7}{6}\right)=\tfrac{\gamma\delta A}{4\alpha CD},\quad \left(\tfrac{7}{7}\right)=\tfrac{\beta\gamma A}{5\alpha BC};$$

$$\left(\tfrac{6}{1}\right)=\tfrac{aB}{\beta},\quad \left(\tfrac{6}{2}\right)=B,\quad \left(\tfrac{6}{3}\right)=\gamma,\quad \left(\tfrac{6}{4}\right)=\tfrac{\gamma\delta}{\alpha C},\quad \left(\tfrac{6}{5}\right)=\tfrac{\gamma\delta\delta A}{2\alpha\beta CD},\quad \left(\tfrac{6}{6}\right)=\tfrac{\gamma\delta\delta AB}{3\alpha\beta CCD}\ ;$$

$$\left(\tfrac{5}{1}\right)=\tfrac{\alpha C}{\gamma},\quad \left(\tfrac{5}{2}\right)=\tfrac{\alpha BC}{\gamma A},\quad \left(\tfrac{5}{3}\right)=C,\quad \left(\tfrac{5}{4}\right)=\delta,\quad \left(\tfrac{5}{5}\right)=\tfrac{\delta\delta}{\alpha D};$$

$$\left(\tfrac{4}{1}\right)=\tfrac{\alpha D}{\delta},\quad \left(\tfrac{4}{2}\right)=\tfrac{\alpha\beta CD}{\gamma\delta A},\quad \left(\tfrac{4}{3}\right)=\tfrac{\alpha CD}{\delta A},\quad \left(\tfrac{4}{4}\right)=D;$$

$$\left(\tfrac{3}{1}\right)=\tfrac{\alpha C}{\delta},\quad \left(\tfrac{3}{2}\right)=\tfrac{\alpha\beta CD}{\delta\delta A},\quad \left(\tfrac{3}{3}\right)=\tfrac{\alpha\beta CCD}{\delta\delta AB};$$

$$\left(\tfrac{2}{1}\right)=\tfrac{\alpha B}{\gamma},\quad \left(\tfrac{2}{2}\right)=\tfrac{\alpha\beta BC}{\gamma\delta A};$$

$$\left(\tfrac{1}{1}\right)=\tfrac{\alpha A}{\beta}.$$

The order of these formulas also in general deserve to be noted by proceeding diagonally from the left to the right, where indeed two kinds of progressions occur, just as if we begin vertically from the first series or from the top horizontally . In this manner by beginning from a vertical series :

$$\left(\tfrac{n-1}{1}\right)=\alpha,\quad \left(\tfrac{n-2}{2}\right)=\tfrac{\beta}{\alpha}\times\left(\tfrac{n-1}{1}\right),\quad \left(\tfrac{n-3}{3}\right)=\tfrac{\gamma}{\beta}\times\left(\tfrac{n-2}{2}\right),\quad \left(\tfrac{n-4}{4}\right)=\tfrac{\delta}{\gamma}\times\left(\tfrac{n-3}{3}\right)$$

$$\left(\tfrac{n-2}{1}\right)=A,\quad \left(\tfrac{n-3}{2}\right)=\tfrac{B}{A}\times\left(\tfrac{n-2}{1}\right),\quad \left(\tfrac{n-4}{3}\right)=\tfrac{C}{B}\times\left(\tfrac{n-3}{2}\right),\quad \left(\tfrac{n-5}{4}\right)=\tfrac{D}{C}\times\left(\tfrac{n-4}{3}\right)$$

$$\left(\tfrac{n-3}{1}\right)=\tfrac{\alpha B}{\beta},\quad \left(\tfrac{n-4}{2}\right)=\tfrac{BC}{\gamma A}\times\left(\tfrac{n-3}{1}\right),\quad \left(\tfrac{n-5}{3}\right)=\tfrac{\gamma D}{\delta B}\times\left(\tfrac{n-4}{2}\right),\quad \left(\tfrac{n-6}{4}\right)=\tfrac{\delta E}{\varepsilon C}\times\left(\tfrac{n-5}{3}\right)$$

$$\left(\tfrac{n-4}{1}\right)=\tfrac{\alpha C}{\gamma},\quad \left(\tfrac{n-5}{2}\right)=\tfrac{\beta D}{\delta A}\times\left(\tfrac{n-4}{1}\right),\quad \left(\tfrac{n-6}{3}\right)=\tfrac{\gamma E}{\varepsilon B}\times\left(\tfrac{n-5}{2}\right),\quad \left(\tfrac{n-7}{4}\right)=\tfrac{\delta F}{\xi C}\times\left(\tfrac{n-6}{3}\right)$$

$$\left(\tfrac{n-5}{1}\right)=\tfrac{\alpha D}{\delta},\quad \left(\tfrac{n-6}{2}\right)=\tfrac{\beta E}{\varepsilon A}\times\left(\tfrac{n-5}{1}\right),\quad \left(\tfrac{n-7}{3}\right)=\tfrac{\gamma F}{\xi B}\times\left(\tfrac{n-6}{2}\right),\quad \left(\tfrac{n-8}{4}\right)=\tfrac{\delta G}{\eta C}\times\left(\tfrac{n-7}{3}\right)$$

$$\left(\tfrac{n-6}{1}\right)=\tfrac{\alpha E}{\varepsilon},\quad \left(\tfrac{n-7}{2}\right)=\tfrac{\beta F}{\xi A}\times\left(\tfrac{n-6}{1}\right),\quad \left(\tfrac{n-8}{3}\right)=\tfrac{\gamma G}{\eta B}\times\left(\tfrac{n-7}{2}\right),\quad \left(\tfrac{n-9}{4}\right)=\tfrac{\delta H}{\theta C}\times\left(\tfrac{n-8}{3}\right)$$

etc.,

then from the above, by beginning horizontally:

$$\left(\tfrac{n}{1}\right)=1,\quad \left(\tfrac{n-1}{2}\right)=\tfrac{\beta}{\alpha}\times\left(\tfrac{n}{1}\right),\quad \left(\tfrac{n-2}{3}\right)=\tfrac{\gamma A}{\alpha B}\times\left(\tfrac{n-1}{2}\right),\quad \left(\tfrac{n-3}{4}\right)=\tfrac{\delta B}{\beta C}\times\left(\tfrac{n-2}{3}\right)$$

$$\left(\tfrac{n}{2}\right)=\tfrac{1}{2},\quad \left(\tfrac{n-1}{3}\right)=\tfrac{\gamma}{B}\times\left(\tfrac{n}{2}\right),\quad \left(\tfrac{n-2}{4}\right)=\tfrac{\delta A}{\alpha C}\times\left(\tfrac{n-1}{3}\right),\quad \left(\tfrac{n-3}{5}\right)=\tfrac{\varepsilon B}{\beta D}\times\left(\tfrac{n-2}{4}\right)$$

$$\left(\tfrac{n}{3}\right)=\tfrac{1}{3},\quad \left(\tfrac{n-1}{4}\right)=\tfrac{\delta}{C}\times\left(\tfrac{n}{3}\right),\quad \left(\tfrac{n-2}{5}\right)=\tfrac{\varepsilon A}{\alpha D}\times\left(\tfrac{n-1}{4}\right),\quad \left(\tfrac{n-3}{6}\right)=\tfrac{\xi B}{\beta E}\times\left(\tfrac{n-2}{5}\right)$$

$$\left(\tfrac{n}{4}\right)=\tfrac{1}{4},\quad \left(\tfrac{n-1}{5}\right)=\tfrac{\varepsilon}{D}\times\left(\tfrac{n}{4}\right),\quad \left(\tfrac{n-2}{6}\right)=\tfrac{\xi A}{\alpha E}\times\left(\tfrac{n-1}{5}\right),\quad \left(\tfrac{n-3}{7}\right)=\tfrac{\eta B}{\beta F}\times\left(\tfrac{n-2}{6}\right)$$

$$\left(\tfrac{n}{5}\right)=\tfrac{1}{5},\quad \left(\tfrac{n-1}{6}\right)=\tfrac{\xi}{E}\times\left(\tfrac{n}{5}\right),\quad \left(\tfrac{n-2}{7}\right)=\tfrac{\eta A}{\alpha F}\times\left(\tfrac{n-1}{6}\right),\quad \left(\tfrac{n-3}{8}\right)=\tfrac{\theta B}{\beta G}\times\left(\tfrac{n-2}{7}\right)$$

etc.

Where the law, on which these formulas in turn depend, is clear enough, but only if we may note in each series of letters $\alpha,\ \beta, \gamma,\ \delta$ etc. and *A, B, C, D* etc. the following terms shall be equal to the preceding first terms themselves.

CONCLUSION.

Therefore since we may prevail to show that the formulas of the second class to be granted only from the quadrature of the circle, the formulas of the third class require in addition a known quadrature, either, for example, this formula

$$\int\frac{dx}{\sqrt[3]{(1-x^3)^2}}=A,\ \text{or this,}\ \int\frac{xdx}{\sqrt[3]{(1-x^3)^2}}=\frac{\alpha}{A},$$

since, with one given, likewise the other is given. But if we may express the same formulas per an infinite product, the value of these will be found

$$\int\frac{dx}{\sqrt[3]{(1-x^3)^2}}=\frac{2}{1}\cdot\frac{3}{4}\cdot\frac{5}{4}\cdot\frac{6\cdot 8}{7\cdot 7}\cdot\frac{9\cdot 11}{10\cdot 10}\cdot\frac{12\cdot 14}{13\cdot 13}\cdot\text{etc.},$$

from which its magnitude can be readily deduced approximately ; in a similar manner there becomes

$$\int\frac{xdx}{\sqrt[3]{(1-x^3)}}=1\cdot\frac{3\cdot 7}{5\cdot 5}\cdot\frac{6\cdot 10}{8\cdot 8}\cdot\frac{9\cdot 13}{11\cdot 11}\cdot\frac{12\cdot 16}{14\cdot 14}\cdot\text{etc.}$$

From which we will be able to integrate all the formulas of the fourth class, but only if besides one of the quadratures of the circle were known from these four formulas $\left(\frac{2}{1}\right)$, $\left(\frac{1}{1}\right)$, $\left(\frac{3}{2}\right)$, $\left(\frac{3}{3}\right)$, which provide these forms

$$[c.f.:\int x^{p-1}dx(1-x^n)^{\frac{q}{n}-1}=\left(\frac{p}{q}\right)=\frac{p+q}{pq}\cdot\frac{n(p+q+n)}{(p+n)(q+n)}\cdot\frac{2n(p+q+2n)}{(p+2n)(q+2n)}\cdot\text{etc.,}]:$$

$$\int\frac{xdx}{\sqrt[4]{(1-x^4)^3}}=\frac{1}{2}\int\frac{dx}{\sqrt[4]{(1-xx)^3}}=\int\frac{dx}{\sqrt{(1-x^4)}}=A,$$
$$\int\frac{dx}{\sqrt[4]{(1-x^4)^3}}=\frac{\alpha A}{\beta},\quad\int\frac{xxdx}{\sqrt{(1-x^4)}}=\frac{\alpha}{2A},$$
$$\int\frac{xxdx}{\sqrt[4]{(1-x^4)}}=\int\frac{xdx}{\sqrt[4]{(1-x^4)}}=\frac{1}{2}\int\frac{dx}{\sqrt[4]{(1-xx)}}=\frac{\beta}{A};$$

moreover, by being produced indefinitely, there will become

$$A=\frac{3}{1\cdot 2}\cdot\frac{4\cdot 7}{5\cdot 6}\cdot\frac{8\cdot 11}{9\cdot 10}\cdot\frac{12\cdot 15}{13\cdot 14}\cdot\frac{16\cdot 19}{17\cdot 18}\cdot\text{etc.}$$

The fifth class demands two higher quadratures $\left(\frac{3}{1}\right)=A$ and $\left(\frac{2}{2}\right)=B$, in place of which the other two are presumed to depend, which indeed may be seen more easily, even if indeed some may be able to be computed more easily than others, on account of the prime number 5.

Two quadratures also are required for the sixth class: $\left(\frac{4}{1}\right)=A$ and $\left(\frac{3}{2}\right)=B$.
Truly that in place of the other, for which there was a need in the third class, can be assumed here, so that only a single new form shall be required to be used. Since indeed there shall be

$$\left(\frac{2}{2}\right)=\int\frac{xdx}{\sqrt[6]{(1-x^6)^4}}=\frac{1}{2}\int\frac{dx}{\sqrt[3]{(1-x^3)^2}}=\frac{\alpha BB}{\gamma A},$$

there will become

$$\frac{2\alpha BB}{\gamma A}=\int\frac{dx}{\sqrt[3]{(1-x^3)^2}},$$

which is a formula required for the third class. Therefore with this given, if in addition the formula may be known:

$$\left(\tfrac{3}{2}\right)=\int\frac{xdx}{\sqrt{(1-x^6)}}=\frac{1}{2}\int\frac{dx}{\sqrt{(1-x^3)}}=B,$$

or also this :

$$\left(\tfrac{4}{3}\right)=\int\frac{xxdx}{\sqrt[3]{(1-x^6)}}=\frac{1}{3}\int\frac{dx}{\sqrt[3]{(1-xx)}}=\frac{\beta\gamma}{\alpha B},$$

which are the most simple in this kind, all the rest will be able to be defined by these. But with these combined there is apparent to become :

$$\int\frac{dx}{\sqrt{(1-x^6)}}\cdot\int\frac{dx}{\sqrt[3]{(1-xx)}}=\frac{6\beta\gamma}{\alpha}=\frac{\pi}{\sqrt{3}}.$$

In a similar manner it may be deduced from the formulas of the fourth order, that :

$$\int\frac{dx}{\sqrt{(1-x^4)}}\cdot\int\frac{dx}{\sqrt[4]{(1-x^2)}}=\frac{\pi}{2},$$

hence a large number of theorems of this kind can be deduced, among which this one especially is notable :

$$\int\frac{dx}{\sqrt[m]{(1-x^n)}}\cdot\int\frac{dx}{\sqrt[n]{(1-x^m)}}=\frac{\pi\sin.\frac{(m-n)\pi}{mn}}{(m-n)\sin.\frac{\pi}{m}\cdot\sin.\frac{\pi}{n}},$$

which, if m and n shall be fractions, may be transformed into this :

$$\int\frac{x^{q-1}dx}{\sqrt[r]{(1-x^p)^s}}\cdot\int\frac{x^{s-1}dx}{\sqrt[p]{(1-x^r)^q}}=\frac{\pi\sin.\left(\frac{s}{r}-\frac{q}{p}\right)\pi}{(ps-qr)\sin.\frac{q\pi}{p}\cdot\sin.\frac{s\pi}{r}}.$$

For truly in general, there is:

$$\left(\frac{n-p}{q}\right)\left(\frac{n-q}{p}\right)=\frac{\left(\frac{n-p}{p}\right)\left(\frac{n-q}{q}\right)}{(q-p)\left(\frac{n-q+p}{q-p}\right)},$$

which gives this form :

$$\int\frac{x^{p-1}dx}{\sqrt[n]{(1-x^n)^q}}\cdot\int\frac{x^{q-1}dx}{\sqrt[n]{(1-x^n)^p}}=\frac{\pi\sin.\frac{(q-p)\pi}{n}}{n(q-p)\sin.\frac{p\pi}{n}\cdot\sin.\frac{q\pi}{n}},$$

from which not only the preceding theorems may be able to be derived easily but also many others.

For on putting $n=\frac{pq}{m}$ we will have :

$$\int\frac{x^{m-1}dx}{\sqrt[p]{(1-x^q)^m}}\cdot\int\frac{x^{m-1}dx}{\sqrt[q]{(1-x^p)^m}}=\frac{\pi\sin.\left(\frac{m}{p}-\frac{m}{q}\right)\pi}{m(q-p)\sin.\frac{m\pi}{q}\cdot\sin.\frac{m\pi}{p}},$$

which thus is permitted to be extended further

$$\int\frac{x^{p-1}dx}{\sqrt[n]{(1-x^m)^q}}\cdot\int\frac{x^{q-1}dx}{\sqrt[m]{(1-x^n)^p}}=\frac{\pi\sin.\left(\frac{q}{n}-\frac{p}{m}\right)\pi}{(mq-np)\sin.\frac{p\pi}{m}\cdot\sin.\frac{q\pi}{n}};$$

In which if there may be put $n=2q$, there will become

$$\int\frac{x^{p-1}dx}{\sqrt{(1-x^m)}}\cdot\int\frac{x^{q-1}dx}{\sqrt[m]{(1-x^{2q})^p}}=\frac{\pi\sin.\frac{p\pi}{m}}{q(m-2p)\sin.\frac{p\pi}{m}}.$$

But if $x^{2q}=1-y^m$ may be put into the latest formula of the integral, there will become :

$$\int\frac{x^{q-1}dx}{\sqrt[m]{(1-x^{2q})^p}}=\frac{m}{2q}\int\frac{y^{n-p-1}dy}{\sqrt{(1-y^m)}},$$

from which on writing x for y :

$$\int\frac{x^{p-1}dx}{\sqrt{(1-x^m)}}\cdot\int\frac{x^{m-p-1}dx}{\sqrt{(1-x^m)}}=\frac{2\pi\sin.\frac{p\pi}{m}}{m(m-2p)\sin.\frac{p\pi}{m}}.$$

In a like manner if there may be put in general $1-x^n=y^m$ for the other integral formula, there becomes :

$$\int\frac{x^{q-1}dx}{\sqrt[m]{(1-x^n)^p}}=\frac{m}{n}\int\frac{y^{m-p-1}dy}{\sqrt[n]{(1-y^m)^{n-q}}},$$

from which on again writing x for y there will be obtained :

$$\int\frac{x^{p-1}dx}{\sqrt[n]{(1-x^m)}^q}\cdot\int\frac{x^{m-p-1}dx}{\sqrt[n]{(1-x^m)^{n-q}}}=\frac{n\pi\sin.\left(\frac{q}{n}-\frac{p}{m}\right)\pi}{m(mq-np)\sin.\frac{p\pi}{m}\cdot\sin.\frac{q\pi}{n}},$$

which value is reduced to $\frac{n\pi}{m(mq-np)}\left(\cot.\frac{p\pi}{m}-\cot.\frac{q\pi}{n}\right)$. And hence this neater form results:

$$\int \frac{x^{\frac{m-r}{2}-1}dx}{\sqrt[n]{(1-x^m)^{\frac{n-s}{2}}}}\cdot\int \frac{x^{\frac{m+r}{2}-1}dx}{\sqrt[n]{(1-x^m)^{\frac{n+s}{2}}}} = \frac{2n\pi\left(\text{tang.}\frac{r\pi}{2m}-\text{tang.}\frac{s\pi}{2n}\right)}{m(nr-ms)}.$$

Since the foundation of these reductions shall be placed in this equality :

$$\left(\frac{n-p}{q}\right)\left(\frac{n-q}{p}\right)=\frac{\left(\frac{n}{q-p}\right)\left(\frac{n-p}{p}\right)\left(\frac{n-q}{p}\right)}{\left(\frac{n-q+p}{q-p}\right)},$$

which is reduced to this form:

$$\left(\frac{n-p}{q}\right)\left(\frac{n-q}{p}\right)\left(\frac{n-q+p}{q-p}\right)=\left(\frac{n}{q-p}\right)\left(\frac{n-p}{p}\right)\left(\frac{n-q}{q}\right),$$

the truth of this can be shown directly in this manner.
In the reduction treated in § 8, with $n-q, q-p, q$ taken for these three numbers *p, q, r*

$$[i.e. \left(\frac{p}{q}\right)\left(\frac{p+q}{r}\right)=\left(\frac{p}{r}\right)\left(\frac{p+r}{q}\right)=\left(\frac{q}{r}\right)\left(\frac{q+r}{p}\right),]$$

we will have :

$$\left(\frac{n-q}{q-p}\right)\left(\frac{n-p}{q}\right)=\left(\frac{n-q}{q}\right)\left(\frac{n}{q-p}\right);$$

then truly with $n-q,\ q-p,\ p$ taken in place of these :

$$[\ i.e.\ n-q,\ q-p,\ p \text{ in place of } p,\ q,\ r \text{ in } \left(\frac{p}{q}\right)\left(\frac{p+q}{r}\right)=\left(\frac{p}{r}\right)\left(\frac{p+r}{q}\right)=\left(\frac{q}{r}\right)\left(\frac{q+r}{p}\right);]$$

$$\left(\frac{n-q}{q-p}\right)\left(\frac{n-p}{p}\right)=\left(\frac{n-q}{p}\right)\left(\frac{n-q+p}{q-p}\right)$$

with which equations taken in place of these multiplied together in turn and with the formula $\left(\frac{n-q}{q-p}\right)$ removed from each side by division, there will become :

$$[i.e.\ \frac{\left(\frac{n-q}{q-p}\right)\left(\frac{n-p}{q}\right)}{\sqrt{\left(\frac{n-q}{q-p}\right)}\left(\frac{n-p}{p}\right)}=\frac{\left(\frac{n-q}{q}\right)\left(\frac{n}{q-p}\right)}{\left(\frac{n-q}{p}\right)\left(\frac{n-q+p}{q-p}\right)};\left(\frac{n-p}{q}\right)\left(\frac{n-q}{p}\right)\left(\frac{n-q+p}{q-p}\right)=\left(\frac{n-q}{q}\right)\left(\frac{n-p}{p}\right)\left(\frac{n}{q-p}\right).]$$

$$\left(\frac{n-p}{q}\right)\left(\frac{n-q}{p}\right)\left(\frac{n-q+p}{q-p}\right)=\left(\frac{n}{q-p}\right)\left(\frac{n-p}{p}\right)\left(\frac{n-q}{p}\right).$$

We can show also an equality of this third kind not depending on the exponent *n*, evidently

$$\left(\frac{s}{p}\right)\left(\frac{r+s}{q}\right)\left(\frac{p+s}{r}\right)=\left(\frac{r+s}{p}\right)\left(\frac{s}{q}\right)\left(\frac{q+s}{r}\right)=\left(\frac{r}{p}\right)\left(\frac{r+s}{q}\right)\left(\frac{p+r}{s}\right)=\left(\frac{r+s}{p}\right)\left(\frac{r}{q}\right)\left(\frac{q+r}{s}\right).$$

which thus involves four letters not depending on n and is similar to the equality of the binary formulas produced between

$$\left(\frac{r}{p}\right)\left(\frac{p+r}{q}\right)=\left(\frac{q+r}{p}\right)\left(\frac{r}{q}\right)=\left(\frac{q}{p}\right)\left(\frac{p+q}{r}\right).$$

Moreover the equality between the products of the third formulas also may be had thus:

$$\left(\frac{p}{q}\right)\left(\frac{r}{s}\right)\left(\frac{p+q}{r+s}\right)=\left(\frac{q}{r}\right)\left(\frac{s}{p}\right)\left(\frac{q+r}{p+s}\right)=\left(\frac{p}{r}\right)\left(\frac{q}{s}\right)\left(\frac{p+r}{q+s}\right)$$
$$=\left(\frac{p}{q}\right)\left(\frac{p+q}{r}\right)\left(\frac{p+q+r}{s}\right)=\left(\frac{p}{q}\right)\left(\frac{p+q}{s}\right)\left(\frac{p+q+s}{r}\right)\text{etc.}$$

For in these the letters *p, q, r, s* can be interchanged between each other in some manner.

OBSERVATIONES
CIRCA INTEGRALIA FORMULARUM $\int x^{p-1}dx(1-x^n)^{\frac{q}{n}-1}$
POSITO POST INTEGRATIONEM $x=1$

Commentatio 321 indicis ENESTROEMIANI
Melanges de philosophie et de mathematique de la societe royale de Turin 32 (1762/5), 1766, p. 156-177.

1. Formulam integralem

$$\int x^{p-1}dx(1-x^n)^{\frac{q}{n}-1}$$

seu hoc modo expressam

$$\int \frac{x^{p-1}}{\sqrt[n]{(1-x^n)^{n-q}}}dx$$

hic consideraturus assumo exponentes *n, p* et *q* esse numeros integros positivos, quandoquidem, si tales non essent, facile ad hanc formam perduci possent. Deinde huius formulae integrale non in genere hic perpendere constitui, sed eius tantum valorem, quem recipit, si post integrationem statuatur $x=1$, postquam scilicet integratio ita fuerit instituta, ut integrale evanescat posito $x=0$. Primum enim nullum est dubium, quin hoc casu $x=1$ integrale multo simplicius exprimatur; ac praeterea quoties in Analysi ad huiusmodi formulas pervenitur, plerumque non tam integrale indefinitum pro quocunque valore ipsius *x* quam definitum valori $x=1$ utpote praecipuo desiderari solet.

2. Constat autem casu, quo post integrationem ponitur $x=1$, integrale $\int \frac{x^{p-1}}{\sqrt[n]{(1-x^n)^{n-q}}}dx$ hoc modo per productum infinitorum factorum exprimi, ut sit

$$\frac{(p+q)}{pq}\cdot\frac{n(p+q+n)}{(p+n)(q+n)}\cdot\frac{2n(p+q+2n)}{(p+2n)(q+2n)}\cdot\frac{3n(p+q+3n)}{(p+3n)(q+3n)}\cdot\text{etc.},$$

cuius quidem primus factor $\frac{(p+q)}{pq}$ non legi sequentium adstringitur. Hoc tamen non obstante perspicuum est exponentes *p* et *q* inter se esse commutabiles, ita ut sit

$$\int \frac{x^{p-1}}{\sqrt[n]{(1-x^n)^{n-q}}}dx = \int \frac{x^{q-1}}{\sqrt[n]{(1-x^n)^{n-p}}}dx$$

quae quidem aequalitas etiam facile per se ostenditur. Verum productum istud infinitum nos ad alia multo maiora perducet, quibus haec integralia magis illustrabuntur.

3. Ut autem brevitati in scribendo consulam neque opus habeam scripturam huius formuale $\int \frac{x^{p-1}}{\sqrt[n]{(1-x^n)^{n-q}}}dx$ toties repetere, pro quovis exponente n eius loco scribam $\left(\frac{p}{q}\right)$, ita ut $\left(\frac{p}{q}\right)$ denotet valorem formulae integralis $\int \frac{x^{p-1}}{\sqrt[n]{(1-x^n)^{n-q}}}dx$ casu, quo post integrationem ponitur $x=1$. Et quoniam vidimus esse hoc casu

$$\int \frac{x^{p-1}}{\sqrt[n]{(1-x^n)^{n-q}}}dx = \int \frac{x^{q-1}}{\sqrt[n]{(1-x^n)^{n-p}}}dx$$

manifestum est fore

$$\left(\frac{p}{q}\right) = \left(\frac{q}{p}\right),$$

ita ut pro quovis valore exponentis n hae expressiones $\left(\frac{p}{q}\right)$ et $\left(\frac{q}{p}\right)$ eandem significent quantitatem. Ita si fuerit exempli gratia $n=4$, erit

$$\left(\frac{3}{2}\right) = \left(\frac{2}{3}\right) = \int \frac{x^2}{\sqrt[4]{(1-x^4)^2}}dx = \int \frac{x}{\sqrt[4]{(1-x^4)}}dx.$$

Per productum autem infinitum habebitur

$$\left(\frac{3}{2}\right) = \left(\frac{2}{3}\right) = \frac{5}{2\cdot 3}\cdot\frac{4.9}{6.7}\cdot\frac{8.13}{10.11}\cdot\frac{12.17}{14.15}\cdot \text{etc.}$$

4. Iam primum observo, si exponentes p et q fuerint maiores exponente n, formulam integralem semper ad aliam reduci posse, in qua hi exponentes infra n deprimantur. Cum enim sit

$$\int \frac{x^{p-1}dx}{\sqrt[n]{(1-x^n)^{n-q}}} = \frac{p-n}{p+q-n}\int \frac{x^{p-n-1}dx}{\sqrt[n]{(1-x^n)^{n-q}}}$$

erit recepto hic scribendi more

$$\left(\frac{p}{q}\right) = \frac{p-n}{p+q-n}\left(\frac{p-n}{q}\right),$$

quo, si fuerit $p > n$, formula ad aliam, in qua exponens p minor sit quam n, revocatur, quod etiam ob commutabilitatem de altero exponente q est tenendum. Quamobrem nobis has formulas examinaturis sufficiet pro quovis exponente n exponentes p et q ipso n minores accipere, quoniam his expeditis omnes casus, quibus maiores habituri essent valores, eo reduci possunt.

5. Statim autem patet casus, quibus est vel $p=n$ vel $q=n$, absolute seu algebraice esse integrabiles. Si enim fuerit $q=n$, ob

$$\left(\frac{p}{q}\right) = \int x^{p-1}dx = \frac{x^p}{p},$$

posito $x=1$ erit $\left(\frac{p}{n}\right) = \frac{1}{p}$ similique modo $\left(\frac{n}{q}\right) = \frac{1}{q}$. Atque hi soli sunt casus, quibus integrale nostrae formulae absolute exhiberi potest, si quidem p et q exponentem n non excedant. Reliquis casibus omnibus integratio vel quadraturam circuli vel adeo altiores quadraturas implicabit, quas hic accuratius perpendere animus est. Post eas igitur formulas $\left(\frac{p}{n}\right)$ seu $\left(\frac{n}{q}\right)$, quarum valor absolute est $=\frac{1}{p}$, veniunt eae, quarum valor per solam circuli quadraturam exprimitur; tum vero sequentur eae, quae altiorem quandam quadraturam postulant, atque has altiores quadraturas tam ad simplicissimam formam quam ad minimum numerum revocare conabor.

6. Cum numeri p et q exponente n minores ponantur, eae formulae $\left(\frac{p}{q}\right)$ per solam circuli quadraturam integrabiles existunt, in quibus est $p+q=n$.
Sit enim $q=n-p$ et formula nostra

$$\left(\frac{p}{n-p}\right) = \left(\frac{n-p}{p}\right) = \int \frac{x^{p-1}dx}{\sqrt[n]{(1-x^n)^p}} = \int \frac{x^{q-1}dx}{\sqrt[n]{(1-x^n)^q}}$$

hoc producto infinito exprimetur

$$\frac{n}{p(n-p)} \cdot \frac{n \cdot 2n}{(n+p)(2n-p)} \cdot \frac{2n \cdot 3n}{(2n+p)(3n-p)} \cdot \frac{3n \cdot 4n}{(3n+p)(4n-p)} \cdot \text{etc.},$$

quod hoc modo repraesentatum

$$\frac{1}{p} \cdot \frac{nn}{nn-pp} \cdot \frac{4nn}{4nn-pp} \cdot \frac{9nn}{9nn-pp} \cdot \text{etc.}$$

congruit cum eo producto, quo sinus angulorum expressi. Quare si n sumatur ad semicircumferentiam circuli, cuius radius sit $=1$, simulque mensuram duorum angulorum rectorum exhibeat, erit

$$\left(\frac{p}{n-p}\right) = \left(\frac{n-p}{p}\right) = \frac{\pi}{n\sin.\frac{p\pi}{n}} = \frac{\pi}{n\sin.\frac{q\pi}{n}}.$$

7. Ceteris casibus, quibus neque $p=n$ neque $q=n$ neque $p+q=n$, integrale etiam neque absolute neque per quadraturam circuli exhiberi potest, sed aliam quandam altiorem quadraturam complectitur. Neque vero singuli casus diversi peculiarem huius modi quadraturam exigunt, sed plures dantur reductiones, quibus diversas formulas inter se comparare licet. Hae autem reductiones derivantur ex producto infinito supra exhibito; cum enim sit

$$\left(\frac{p}{q}\right)=\frac{p+q}{pq}\cdot\frac{n(p+q+n)}{(p+n)(q+n)}\cdot\frac{2n(p+q+2n)}{(p+2n)(q+2n)}\cdot\text{etc.},$$

erit simili modo

$$\left(\frac{p+q}{r}\right)=\frac{p+q+r}{(p+q)r}\cdot\frac{n(p+q+r+n)}{(p+q+n)(r+n)}\cdot\frac{2n(p+q+r+2n)}{(p+q+2n)(r+2n)}\cdot\text{etc.},$$

quibus in se invicem ductis obtinetur

$$\left(\frac{p}{q}\right)\left(\frac{p+q}{r}\right)=\frac{p+q+r}{pqr}\cdot\frac{nn(p+q+r+n)}{(p+n)(q+n)(r+n)}\cdot\frac{4nn(p+q+r+2n)}{(p+2n)(q+2n)(r+2n)}\cdot\text{etc.},$$

ubi ternae quantitates *p, q, r* sunt inter se permutabiles.

8. Hinc ergo permutandis his quantitatibus *p, q, r* consequimur sequentes reductiones

$$\left(\frac{p}{q}\right)\left(\frac{p+q}{r}\right)=\left(\frac{p}{r}\right)\left(\frac{p+r}{q}\right)=\left(\frac{q}{r}\right)\left(\frac{q+r}{p}\right),$$

unde ex datis aliquot formulis plures aliae determinari possunt. Veluti si sit $q+r=n$ seu $r=n-q$, ob

$$\left(\frac{q+r}{p}\right)=\frac{1}{p}\ \text{ et }\ \left(\frac{q}{r}\right)=\frac{\pi}{n\sin.\frac{q\pi}{n}},$$

erit

$$\left(\frac{p}{q}\right)\left(\frac{p+q}{n-q}\right)=\frac{\pi}{np\sin.\frac{q\pi}{n}}$$

nec non

$$\left(\frac{p}{n-q}\right)\left(\frac{n+p-q}{q}\right)=\frac{\pi}{np\sin.\frac{q\pi}{n}}.$$

Deinde si sit $p+q+r=n$ seu $r=n-p-q$, erit

$$\frac{\pi}{n\sin.\frac{r\pi}{n}}\left(\frac{p}{q}\right)=\frac{\pi}{n\sin.\frac{q\pi}{n}}\left(\frac{p}{r}\right)=\frac{\pi}{n\sin.\frac{p\pi}{n}}\left(\frac{q}{r}\right),$$

unde insignes reductiones aliarum ad alias oriuntur, quibus multitudo quadraturarum ad nostrum scopum necessariarum vehementer diminuitur.

9. Praeterea vero pro *p, q, r* numeris determinatis assumendis sequentes adipiscimur productorum ex binis formulis aequalitates

$$\left(\tfrac{1}{1}\right)\left(\tfrac{2}{2}\right)=\left(\tfrac{2}{1}\right)\left(\tfrac{3}{1}\right),$$
$$\left(\tfrac{1}{1}\right)\left(\tfrac{3}{2}\right)=\left(\tfrac{3}{1}\right)\left(\tfrac{4}{1}\right),$$
$$\left(\tfrac{2}{1}\right)\left(\tfrac{3}{3}\right)=\left(\tfrac{3}{1}\right)\left(\tfrac{4}{2}\right)=\left(\tfrac{3}{2}\right)\left(\tfrac{5}{1}\right),$$
$$\left(\tfrac{2}{2}\right)\left(\tfrac{4}{3}\right)=\left(\tfrac{3}{2}\right)\left(\tfrac{5}{2}\right),$$
$$\left(\tfrac{3}{1}\right)\left(\tfrac{4}{3}\right)=\left(\tfrac{3}{3}\right)\left(\tfrac{6}{1}\right),$$
$$\left(\tfrac{3}{2}\right)\left(\tfrac{5}{3}\right)=\left(\tfrac{3}{3}\right)\left(\tfrac{6}{2}\right),$$
$$\left(\tfrac{2}{2}\right)\left(\tfrac{4}{4}\right)=\left(\tfrac{4}{2}\right)\left(\tfrac{6}{2}\right),$$
$$\left(\tfrac{3}{1}\right)\left(\tfrac{4}{4}\right)=\left(\tfrac{4}{1}\right)\left(\tfrac{5}{3}\right)=\left(\tfrac{4}{3}\right)\left(\tfrac{7}{1}\right),$$
$$\left(\tfrac{2}{1}\right)\left(\tfrac{5}{3}\right)=\left(\tfrac{5}{1}\right)\left(\tfrac{6}{2}\right)=\left(\tfrac{5}{2}\right)\left(\tfrac{7}{1}\right),$$
$$\left(\tfrac{1}{1}\right)\left(\tfrac{6}{2}\right)=\left(\tfrac{6}{1}\right)\left(\tfrac{7}{1}\right),$$
etc.

ubi quidem plures occurrunt, quae iam in reliquis continentur.

10. His quasi principiis praemissis formulam generalem $\int\frac{x^{p-1}dx}{\sqrt[n]{(1-x^n)^{n-q}}}$, in qua numeros p et q exponentem n non superare assumo, in classes ex exponente n petitas distinguam, ita ut valores $n=1,\ n=2,\ n=3,\ n=4$ etc. classes primam, secundam, tertiam etc. sint praebituri.

Ac prima quidem classis, qua $n=1$, unicam formulam complectitur $\left(\tfrac{1}{1}\right)$, cuius valor est $=1$. Secunda classis, qua $n=2$, has formulas $\left(\tfrac{1}{1}\right),\left(\tfrac{2}{1}\right)$ et $\left(\tfrac{2}{2}\right)$ continet, quarum evolutio per se est manifesta. Tertia classis, qua $n=3$, has habet
$\left(\tfrac{1}{1}\right),\left(\tfrac{2}{1}\right),\left(\tfrac{3}{1}\right),\left(\tfrac{2}{2}\right),\left(\tfrac{3}{2}\right),\left(\tfrac{3}{3}\right).$
Quarta vero classis, qua $n=4$, istas

$$\left(\tfrac{1}{1}\right),\left(\tfrac{2}{1}\right),\left(\tfrac{3}{1}\right),\left(\tfrac{4}{1}\right),\left(\tfrac{2}{2}\right),\left(\tfrac{3}{2}\right),\left(\tfrac{4}{2}\right),\left(\tfrac{3}{3}\right),\left(\tfrac{4}{3}\right),\left(\tfrac{4}{4}\right);$$

sicque in sequentibus classibus formularum numerus secundum numeros trigonales crescit. Has igitur classes ordine percurramus.

Classis 2^dae^ formae $\int\frac{x^{p-1}dx}{\sqrt[2]{(1-x^2)^{2-q}}}=\left(\tfrac{p}{q}\right).$

Perspicuum hic quidem est istas formulas vel absolute vel per quadraturam circuli exprimi; nam hae $\left(\frac{2}{1}\right)$ et $\left(\frac{2}{2}\right)$ absolute dantur et reliqua $\left(\frac{1}{1}\right)$ ob $1+1=2$ est $=\frac{\pi}{2\sin.\frac{\pi}{2}}=\frac{\pi}{2}$; si ergo causa brevitatis ponamus $\frac{\pi}{2}=\alpha$, uti scilicet in sequentibus classibus faciemus, omnes formulae huius classis ita definiuntur:

$$\left(\frac{2}{1}\right)=1,\ \left(\frac{2}{2}\right)=\frac{1}{2};$$
$$\left(\frac{1}{1}\right)=\alpha.$$

$$\textit{Classis } 3^{ae} \textit{ formae } \int\frac{x^{p-1}dx}{\sqrt[3]{(1-x^3)^{3-q}}}=\left(\frac{p}{q}\right).$$

Cum hic sit $n=3$, formula quadraturam circuli involvens est

$$\left(\frac{2}{1}\right)=\frac{\pi}{3\sin.\frac{\pi}{3}};$$

ponamus ergo $\left(\frac{2}{1}\right)=\alpha$; reliquae autem formulae, quae non absolute dantur, altiorem quadraturam involvunt et quidem unicam $\left(\frac{1}{1}\right)$, quam littera A indicemus; qua concessa valores omnium formularum huius classis assignare poterimus:

$$\left(\frac{3}{1}\right)=1,\quad \left(\frac{3}{2}\right)=\frac{1}{2},\quad \left(\frac{3}{3}\right)=\frac{1}{3};$$
$$\left(\frac{2}{1}\right)=\alpha,\ \left(\frac{2}{1}\right)=\frac{\alpha}{A};$$
$$\left(\frac{1}{1}\right)=A.$$

$$\textit{Classis } 4^{tae} \textit{ formae } \int\frac{x^{p-1}dx}{\sqrt[4]{(1-x^4)^{4-q}}}=\left(\frac{p}{q}\right)$$

Cum hic sit $n=4$, duas habemus formulas a quadratura circuli pendentes, quarum valores, quia sunt cogniti, ita indicemus:

$$\left(\frac{3}{1}\right)=\frac{\pi}{4\sin.\frac{\pi}{4}}=\alpha \text{ et } \left(\frac{2}{2}\right)=\frac{\pi}{4\sin.\frac{2\pi}{4}}=\beta.$$

Praeterea vero unica opus est formula altiorem quadraturam involvente, qua concessa reliquas omnes cognoscemus. Ponamus enim $\left(\frac{2}{1}\right)=A$ et omnes formulae huius classis ita determinabuntur:

$$\left(\tfrac{4}{1}\right)=1,\quad \left(\tfrac{4}{2}\right)=\tfrac{1}{2},\quad \left(\tfrac{4}{3}\right)=\tfrac{1}{3},\quad \left(\tfrac{4}{4}\right)=\tfrac{1}{4};$$
$$\left(\tfrac{3}{1}\right)=\alpha,\ \left(\tfrac{3}{2}\right)=\tfrac{\beta}{A},\quad \left(\tfrac{3}{3}\right)=\tfrac{\alpha}{2A};$$
$$\left(\tfrac{2}{1}\right)=A,\ \left(\tfrac{2}{1}\right)=\beta;$$
$$\left(\tfrac{1}{1}\right)=\tfrac{\alpha A}{\beta}.$$

Classis 5^tae formae $\int\frac{x^{p-1}dx}{\sqrt[5]{(1-x^5)^{5-q}}}=\left(\frac{p}{q}\right)$

Cum hic sit $n = 5$, notemus statim formulas a quadratura circuli pendentes

$$\left(\tfrac{4}{1}\right)=\frac{\pi}{5\sin.\frac{\pi}{5}}=\alpha \text{ et } \left(\tfrac{3}{2}\right)=\frac{\pi}{5\sin.\frac{2\pi}{5}}=\beta.$$

Duabus autem insuper novis quadraturis opus est huic classi peculiaribus, quas ita designemus

$$\left(\tfrac{3}{1}\right)=A \text{ et } \left(\tfrac{2}{2}\right)=B,$$

ex quibus reliquae omnes ita definientur:

$$\left(\tfrac{5}{1}\right)=1,\quad \left(\tfrac{5}{2}\right)=\tfrac{1}{2},\quad \left(\tfrac{5}{3}\right)=\tfrac{1}{3},\quad \left(\tfrac{5}{4}\right)=\tfrac{1}{4},\quad \left(\tfrac{5}{5}\right)=\tfrac{1}{5};$$
$$\left(\tfrac{4}{1}\right)=\alpha,\quad \left(\tfrac{4}{2}\right)=\tfrac{\beta}{A},\quad \left(\tfrac{4}{3}\right)=\tfrac{\beta}{2B},\quad \left(\tfrac{4}{4}\right)=\tfrac{\alpha}{3A};$$
$$\left(\tfrac{3}{1}\right)=A,\quad \left(\tfrac{3}{2}\right)=\beta,\quad \left(\tfrac{3}{3}\right)=\tfrac{\beta\beta}{\alpha B};$$
$$\left(\tfrac{2}{1}\right)=\tfrac{\alpha B}{\beta},\ \left(\tfrac{2}{2}\right)=B;$$
$$\left(\tfrac{1}{1}\right)=\tfrac{\alpha A}{\beta}.$$

Classis 6^tae formae $\int\frac{x^{p-1}dx}{\sqrt[6]{(1-x^6)^{6-q}}}=\left(\frac{p}{q}\right).$

Hic est $n=6$ et formulae quadraturam circuli involventes sunt

$$\left(\tfrac{5}{1}\right)=\frac{\pi}{6\sin.\frac{\pi}{6}}=\alpha,\ \left(\tfrac{4}{2}\right)=\frac{\pi}{6\sin.\frac{2\pi}{6}}=\beta,\ \left(\tfrac{3}{3}\right)=\frac{\pi}{6\sin.\frac{2\pi}{6}}=\gamma.$$

Reliquarum vero omnium valores insuper a binis hisce quadraturis pendent

$$\left(\tfrac{4}{1}\right)=A \text{ et } \left(\tfrac{3}{2}\right)=B,$$

atque ita se habere deprehenduntur:

$$\left(\tfrac{6}{1}\right)=1,\quad \left(\tfrac{6}{2}\right)=\tfrac{1}{2},\quad \left(\tfrac{6}{3}\right)=\tfrac{1}{3},\quad \left(\tfrac{6}{4}\right)=\tfrac{1}{4},\quad \left(\tfrac{6}{5}\right)=\tfrac{1}{5},\quad \left(\tfrac{6}{6}\right)=\tfrac{1}{6};$$
$$\left(\tfrac{5}{1}\right)=\alpha,\quad \left(\tfrac{5}{2}\right)=\tfrac{\beta}{A},\quad \left(\tfrac{5}{3}\right)=\tfrac{\gamma}{2B},\quad \left(\tfrac{5}{4}\right)=\tfrac{\beta}{3B},\quad \left(\tfrac{5}{5}\right)=\tfrac{\alpha}{4A};$$
$$\left(\tfrac{4}{1}\right)=A,\quad \left(\tfrac{4}{2}\right)=\beta,\quad \left(\tfrac{4}{3}\right)=\tfrac{\beta\gamma}{\alpha B},\quad \left(\tfrac{4}{4}\right)=\tfrac{\beta\gamma A}{2\alpha BB};$$
$$\left(\tfrac{3}{1}\right)=\tfrac{\alpha B}{\beta},\quad \left(\tfrac{3}{2}\right)=B,\quad \left(\tfrac{3}{3}\right)=\gamma;$$
$$\left(\tfrac{2}{1}\right)=\tfrac{\alpha B}{\gamma},\quad \left(\tfrac{2}{2}\right)=\tfrac{\alpha BB}{\gamma A};$$
$$\left(\tfrac{1}{1}\right)=\tfrac{\alpha A}{\beta}.$$

$$\textit{Classis } 7^{mas} \textit{ formae } \int\frac{x^{p-1}dx}{\sqrt[7]{(1-x^7)^{7-q}}}=\left(\frac{p}{q}\right).$$

Quia $n=7$, formulae a quadratura circuli pendentes ita designentur

$$\left(\tfrac{6}{1}\right)=\frac{\pi}{7\sin.\frac{\pi}{7}}=\alpha,\ \left(\tfrac{5}{2}\right)=\frac{\pi}{7\sin.\frac{2\pi}{7}}=\beta,\ \left(\tfrac{4}{3}\right)=\frac{\pi}{7\sin.\frac{2\pi}{7}}=\gamma,$$

praeterea vero hae quadraturae introducantur

$$\left(\tfrac{6}{1}\right)=A,\ \left(\tfrac{4}{2}\right)=B,\ \left(\tfrac{3}{3}\right)=C,$$

quibus datis omnis formulae ita determinabuntur:

$$\left(\tfrac{7}{1}\right)=1,\quad \left(\tfrac{7}{2}\right)=\tfrac{1}{2},\quad \left(\tfrac{7}{3}\right)=\tfrac{1}{3},\quad \left(\tfrac{7}{4}\right)=\tfrac{1}{4},\quad \left(\tfrac{7}{5}\right)=\tfrac{1}{5},\quad \left(\tfrac{7}{6}\right)=\tfrac{1}{6},\quad \left(\tfrac{7}{7}\right)=\tfrac{1}{7};$$
$$\left(\tfrac{6}{1}\right)=\alpha,\quad \left(\tfrac{6}{2}\right)=\tfrac{\beta}{A},\quad \left(\tfrac{6}{3}\right)=\tfrac{\gamma}{2B},\quad \left(\tfrac{6}{4}\right)=\tfrac{\beta}{3C},\quad \left(\tfrac{6}{5}\right)=\tfrac{\beta}{4B},\quad \left(\tfrac{6}{6}\right)=\tfrac{\alpha}{5A};$$
$$\left(\tfrac{5}{1}\right)=A,\quad \left(\tfrac{5}{2}\right)=\beta,\quad \left(\tfrac{5}{3}\right)=\tfrac{\beta\gamma}{AB},\quad \left(\tfrac{5}{4}\right)=\tfrac{\gamma\gamma A}{2\alpha BC},\quad \left(\tfrac{5}{5}\right)=\tfrac{\beta\gamma A}{2\alpha BC};$$
$$\left(\tfrac{4}{1}\right)=\tfrac{\alpha B}{\beta},\quad \left(\tfrac{4}{2}\right)=B,\quad \left(\tfrac{4}{3}\right)=\gamma,\quad \left(\tfrac{4}{4}\right)=\tfrac{\gamma\gamma}{\alpha C};$$
$$\left(\tfrac{3}{1}\right)=\tfrac{\alpha C}{\gamma},\quad \left(\tfrac{3}{2}\right)=\tfrac{\alpha BC}{\gamma A},\quad \left(\tfrac{3}{3}\right)=C;$$
$$\left(\tfrac{2}{1}\right)=\tfrac{\alpha B}{\gamma},\quad \left(\tfrac{2}{2}\right)=\tfrac{\alpha\beta BB}{\gamma\gamma A};$$
$$\left(\tfrac{1}{1}\right)=\tfrac{\alpha A}{\beta}.$$

$$\textit{Classis } 8^{tas} \textit{ formae } \int\frac{x^{p-1}dx}{\sqrt[8]{(1-x^8)^{8-q}}}=\left(\frac{p}{q}\right).$$

Quia hic est $n=8$, formulae quadraturam circuli implicantes erunt

$$\left(\tfrac{7}{1}\right)=\frac{\pi}{8\sin.\frac{\pi}{8}}=\alpha,\quad \left(\tfrac{6}{2}\right)=\frac{\pi}{8\sin.\frac{2\pi}{8}}=\beta,$$
$$\left(\tfrac{5}{3}\right)=\frac{\pi}{8\sin.\frac{3\pi}{8}}=\gamma,\quad \left(\tfrac{4}{4}\right)=\frac{\pi}{8\sin.\frac{4\pi}{8}}=\delta.$$

Nunc vero tres frequentes formulae tanquam cognitae spectentur

$$\left(\tfrac{6}{1}\right)=A,\ \left(\tfrac{5}{2}\right)=B,\ \left(\tfrac{4}{3}\right)=C,$$

atque ex his omnes formulae huius classis ita determinabuntur:

$$\left(\tfrac{8}{1}\right)=1,\quad \left(\tfrac{8}{2}\right)=\tfrac{1}{2},\quad \left(\tfrac{8}{3}\right)=\tfrac{1}{3},\quad \left(\tfrac{8}{4}\right)=\tfrac{1}{4},\quad \left(\tfrac{8}{5}\right)=\tfrac{1}{5},\quad \left(\tfrac{8}{6}\right)=\tfrac{1}{6},\quad \left(\tfrac{8}{7}\right)=\tfrac{1}{7},\quad \left(\tfrac{8}{8}\right)=\tfrac{1}{8};$$
$$\left(\tfrac{7}{1}\right)=\alpha,\quad \left(\tfrac{7}{2}\right)=\tfrac{\beta}{A},\quad \left(\tfrac{7}{3}\right)=\tfrac{\gamma}{2B},\quad \left(\tfrac{7}{4}\right)=\tfrac{\delta}{3C},\quad \left(\tfrac{7}{5}\right)=\tfrac{\gamma}{4C},\quad \left(\tfrac{7}{6}\right)=\tfrac{\beta}{5B},\quad \left(\tfrac{7}{7}\right)=\tfrac{\alpha}{6A};$$
$$\left(\tfrac{6}{1}\right)=A,\quad \left(\tfrac{6}{2}\right)=\beta,\quad \left(\tfrac{6}{3}\right)=\tfrac{\beta\gamma}{\alpha B},\quad \left(\tfrac{6}{4}\right)=\tfrac{\gamma\delta A}{2\alpha BC},\quad \left(\tfrac{6}{5}\right)=\tfrac{\gamma\delta A}{3\alpha CC},\quad \left(\tfrac{6}{6}\right)=\tfrac{\beta\gamma A}{4\alpha BC};$$
$$\left(\tfrac{5}{1}\right)=\tfrac{\alpha B}{\beta},\quad \left(\tfrac{5}{2}\right)=B,\quad \left(\tfrac{5}{3}\right)=\gamma,\quad \left(\tfrac{5}{4}\right)=\tfrac{\gamma\delta}{\alpha C},\quad \left(\tfrac{5}{5}\right)=\tfrac{\gamma\gamma\delta A}{2\alpha\beta CC};$$
$$\left(\tfrac{4}{1}\right)=\tfrac{\alpha C}{\gamma},\quad \left(\tfrac{4}{2}\right)=\tfrac{\alpha BC}{\gamma A},\quad \left(\tfrac{4}{3}\right)=C,\quad \left(\tfrac{4}{4}\right)=\delta;$$
$$\left(\tfrac{3}{1}\right)=\tfrac{\alpha C}{\delta},\quad \left(\tfrac{3}{2}\right)=\tfrac{\alpha\beta CC}{\gamma\delta A},\quad \left(\tfrac{3}{3}\right)=\tfrac{\alpha CC}{\delta A};$$
$$\left(\tfrac{2}{1}\right)=\tfrac{\alpha B}{\gamma},\quad \left(\tfrac{2}{2}\right)=\tfrac{\alpha\beta BC}{\gamma\delta A};$$
$$\left(\tfrac{1}{1}\right)=\tfrac{\alpha A}{\beta}.$$

Hinc istas reductiones ad sequentes classes, quousque libuerit, continuare licet. Quemadmodum ergo hinc in genere singularum formularum integralia se sint habitura, exponamus.

Evolutio formae generalis $\int\frac{x^{p-1}dx}{\sqrt[n]{(1-x^n)^{n-q}}}=\left(\frac{p}{q}\right)$.

Primo ergo absolute integrabiles sunt hae formulae

$$\left(\tfrac{n}{1}\right)=1,\ \left(\tfrac{n}{2}\right)=\tfrac{1}{2},\ \left(\tfrac{n}{3}\right)=\tfrac{1}{3},\ \left(\tfrac{n}{4}\right)=\tfrac{1}{4},\text{etc.},$$

deinde formulae a quadratura circuli pendentes sunt

$$\left(\tfrac{n-1}{1}\right)=\alpha,\ \left(\tfrac{n-2}{2}\right)=\beta,\ \left(\tfrac{n-3}{3}\right)=\gamma,\ \left(\tfrac{n-4}{4}\right)=\delta,\text{etc.},$$

quarum quantitatum progressio tandem in se revertitur, cum sit etiam

$$\left(\tfrac{4}{n-4}\right)=\delta,\ \left(\tfrac{3}{n-3}\right)=\gamma,\ \left(\tfrac{2}{n-2}\right)=\beta,\ \left(\tfrac{1}{n-1}\right)=\alpha.$$

Praeterea vero altiores quadraturae in subsidium vocari debent, quae ita repraesentantur

$$\left(\tfrac{n-2}{1}\right)=A,\ \ \left(\tfrac{n-3}{2}\right)=B,\ \left(\tfrac{n-4}{3}\right)=C,\ \left(\tfrac{n-5}{4}\right)=D,\text{etc.},$$

quarum numerus quovis casu sponte determinatur, quia hae formulae tandem in se revertuntur.

His autem formulis admissis omnes omnino ad eandem classem pertinentes definiri poterunt. Habebimus autem a formula $\left(\tfrac{n-1}{1}\right)=\alpha$ *,* uti supra istas formulas ordinavimus, deorsum descendendo

$$\left(\tfrac{n-1}{1}\right)=\alpha,\ \ \left(\tfrac{n-2}{1}\right)=A,\ \left(\tfrac{n-3}{1}\right)=\tfrac{\alpha B}{\beta},\ \left(\tfrac{n-4}{1}\right)=\tfrac{\alpha C}{\gamma},$$
$$\left(\tfrac{n-5}{1}\right)=\tfrac{\alpha D}{\delta},\ \left(\tfrac{n-6}{1}\right)=\tfrac{\alpha E}{\varepsilon},\ \ \text{etc.},$$

qui valores retro sumti ita se habent

$$\left(\tfrac{1}{1}\right)=\tfrac{\alpha A}{\beta},\ \ \left(\tfrac{2}{1}\right)=\tfrac{\alpha B}{\gamma},\ \left(\tfrac{3}{1}\right)=\tfrac{\alpha C}{\delta},\ \ \text{etc.}$$

Tum vero ab eadem formula $\left(\tfrac{n-1}{1}\right)=\alpha$ horizontaliter progrediendo definiuntur istae formulae

$$\left(\tfrac{n-1}{1}\right)=\alpha,\ \ \left(\tfrac{n-1}{2}\right)=\tfrac{\beta}{A},\ \left(\tfrac{n-1}{3}\right)=\tfrac{\gamma}{2B},\ \left(\tfrac{n-1}{4}\right)=\tfrac{\delta}{3C},\ \text{etc.},$$

quarum ultima erit

$$\left(\tfrac{n-1}{n-1}\right)=\frac{\alpha}{(n-2)A},$$

penultima

$$\left(\tfrac{n-1}{n-2}\right)=\frac{\beta}{(n-3)B},$$

antepenultima

$$\left(\tfrac{n-1}{n-3}\right)=\frac{\gamma}{(n-4)C},\ \text{etc.}$$

Simili modo a formula $\left(\tfrac{n-2}{2}\right)=\beta$ tam descendendo quam progrediendo horizontaliter valores aliarum impetrabimus ac descendendo quidem

$$\left(\tfrac{n-2}{2}\right)=\beta,\quad \left(\tfrac{n-3}{2}\right)=B,\quad \left(\tfrac{n-4}{2}\right)=\frac{\alpha BC}{\gamma A},\quad \left(\tfrac{n-5}{2}\right)=\frac{\alpha\beta CD}{\gamma\delta A},$$
$$\left(\tfrac{n-6}{2}\right)=\frac{\alpha\beta DE}{\delta\varepsilon A},\quad \left(\tfrac{n-7}{2}\right)=\frac{\alpha\beta EF}{\varepsilon\xi A}\quad \text{etc.,}$$

ubi erit ultima

$$\left(\tfrac{2}{2}\right)=\frac{\alpha\beta BC}{\gamma\delta A},$$

penultima

$$\left(\tfrac{3}{2}\right)=\frac{\alpha\beta CD}{\delta\varepsilon A}$$

etc.;

at horizontaliter progrediendo

$$\left(\tfrac{n-2}{2}\right)=\beta,\quad \left(\tfrac{n-2}{3}\right)=\frac{\beta\gamma}{\alpha B},\quad \left(\tfrac{n-2}{4}\right)=\frac{\gamma\delta A}{2\alpha BC},\quad \left(\tfrac{n-2}{5}\right)=\frac{\delta\varepsilon A}{3\alpha CD},$$
$$\left(\tfrac{n-2}{6}\right)=\frac{\varepsilon\xi A}{4\alpha DE},\quad \left(\tfrac{n-2}{7}\right)=\frac{\xi\eta A}{5\alpha EF}\quad \text{etc.,}$$

quarum erit ultima

$$\left(\tfrac{n-2}{n-2}\right)=\frac{\beta\gamma A}{(n-4)\alpha BC}$$

penultima

$$\left(\tfrac{n-2}{n-3}\right)=\frac{\gamma\delta A}{(n-5)\alpha CD}$$

etc.

Porro a formula $\left(\tfrac{n-3}{3}\right)=\gamma$ descendendo pervenimus ad has formulas

$$\left(\tfrac{n-3}{3}\right)=\gamma,\quad \left(\tfrac{n-4}{3}\right)=C,\quad \left(\tfrac{n-5}{3}\right)=\frac{\alpha CD}{\delta A},\quad \left(\tfrac{n-6}{3}\right)=\frac{\alpha\beta CDE}{\delta\varepsilon AB},$$
$$\left(\tfrac{n-7}{3}\right)=\frac{\alpha\beta\gamma DEF}{\delta\varepsilon\xi AB},\quad \left(\tfrac{n-8}{3}\right)=\frac{\alpha\beta\gamma EFG}{\varepsilon\xi\eta AB}\quad \text{etc.}$$

et horizontaliter progrediendo

$$\left(\tfrac{n-3}{3}\right)=\gamma,\quad \left(\tfrac{n-3}{4}\right)=\frac{\gamma\delta}{\alpha C},\quad \left(\tfrac{n-3}{5}\right)=\frac{\gamma\delta\varepsilon A}{2\alpha\beta CD},\quad \left(\tfrac{n-3}{6}\right)=\frac{\delta\varepsilon\xi AB}{3\alpha\beta CDE},$$
$$\left(\tfrac{n-3}{7}\right)=\frac{\varepsilon\xi\eta AB}{4\alpha\beta DEF},\quad \left(\tfrac{n-3}{8}\right)=\frac{\xi\eta\theta AB}{5\alpha\beta EFG}\quad \text{etc.}$$

Pari modo a formula $\left(\tfrac{n-4}{4}\right)=\delta$ descendendo nanciscimur

$$\left(\tfrac{n-4}{4}\right)=\delta,\quad \left(\tfrac{n-5}{4}\right)=D,\quad \left(\tfrac{n-6}{4}\right)=\frac{\alpha DE}{\varepsilon A},\quad \left(\tfrac{n-7}{4}\right)=\frac{\alpha\beta DEF}{\varepsilon\xi AB},$$
$$\left(\tfrac{n-8}{4}\right)=\frac{\alpha\beta\gamma DEFG}{\varepsilon\xi\eta ABC},\quad \left(\tfrac{n-9}{4}\right)=\frac{\alpha\beta\gamma\delta EFGH}{\varepsilon\xi\eta\theta ABC}\quad \text{etc.}$$

et horizontaliter progrediendo

$$\left(\tfrac{n-4}{4}\right)=\delta,\quad \left(\tfrac{n-4}{5}\right)=\frac{\delta\varepsilon}{\alpha D},\quad \left(\tfrac{n-4}{6}\right)=\frac{\delta\varepsilon\xi A}{2\alpha\beta DE},\quad \left(\tfrac{n-4}{7}\right)=\frac{\delta\varepsilon\xi\eta AB}{3\alpha\beta\gamma DEF},$$
$$\left(\tfrac{n-4}{8}\right)=\frac{\varepsilon\xi\eta\theta ABC}{4\alpha\beta\gamma DEFG},\quad \left(\tfrac{n-4}{9}\right)=\frac{\xi\eta\theta\iota ABC}{5\alpha\beta\gamma EFGH}\quad \text{etc.}$$

Atque hac ratione tandem ommum formularum valores reperiuntur.

Accommodemus has generalas reductiones ad

$$\textit{Classis } 9^{nae} \textit{ formae } \int\frac{x^{p-1}dx}{\sqrt[9]{(1-x^{9})^{9-q}}}=\left(\tfrac{p}{q}\right).$$

Ubi ob $n=9$ formulae quadraturam circuli involventes erunt

$$\left(\tfrac{8}{1}\right)=\alpha,\quad \left(\tfrac{7}{2}\right)=\beta,\quad \left(\tfrac{6}{3}\right)=\gamma,\quad \left(\tfrac{5}{4}\right)=\delta, \text{etc.},$$

hinc $\varepsilon=\delta,\ \xi=\gamma,\ \eta=\beta,\ \theta=\alpha.$

Deinde novae quadraturae huc requisitae ponantur

$$\left(\tfrac{7}{1}\right)=A,\quad \left(\tfrac{6}{2}\right)=B,\quad \left(\tfrac{5}{3}\right)=C,\quad \left(\tfrac{4}{4}\right)=D,$$

sicque erit

$$E=C,\ F=B \text{ et } G=A;$$

atque his quatuor valoribus concessis omnium formularum nonae classis valores assignari poterunt, quos simili ordine, ut hactenus fecimus, repraesentemus:

$\left(\frac{9}{1}\right)=1,\quad \left(\frac{9}{2}\right)=\frac{1}{2},\quad \left(\frac{9}{3}\right)=\frac{1}{3},\quad \left(\frac{9}{4}\right)=\frac{1}{4},\quad \left(\frac{9}{5}\right)=\frac{1}{5},$

$\left(\frac{9}{6}\right)=\frac{1}{6},\quad \left(\frac{9}{7}\right)=\frac{1}{7},\quad \left(\frac{9}{8}\right)=\frac{1}{8}\quad \left(\frac{9}{9}\right)=\frac{1}{9};$

$\left(\frac{8}{1}\right)=\alpha,\quad \left(\frac{8}{2}\right)=\frac{\beta}{A},\quad \left(\frac{8}{3}\right)=\frac{\gamma}{2B},\quad \left(\frac{8}{4}\right)=\frac{\delta}{3C},\quad \left(\frac{8}{5}\right)=\frac{\delta}{4D},$

$\left(\frac{8}{6}\right)=\frac{\gamma}{5C},\quad \left(\frac{8}{7}\right)=\frac{\beta}{6B},\quad \left(\frac{8}{8}\right)=\frac{\alpha}{7A}\quad;$

$\left(\frac{7}{1}\right)=A,\quad \left(\frac{7}{2}\right)=\beta,\quad \left(\frac{7}{3}\right)=\frac{\beta\gamma}{\alpha B},\quad \left(\frac{7}{4}\right)=\frac{\gamma\delta A}{2\alpha BC},\quad \left(\frac{7}{5}\right)=\frac{\delta\delta A}{3\alpha CD},$

$\left(\frac{7}{6}\right)=\frac{\gamma\delta A}{4\alpha CD},\quad \left(\frac{7}{7}\right)=\frac{\beta\gamma A}{5\alpha BC};$

$\left(\frac{6}{1}\right)=\frac{aB}{\beta},\quad \left(\frac{6}{2}\right)=B,\quad \left(\frac{6}{3}\right)=\gamma,\quad \left(\frac{6}{4}\right)=\frac{\gamma\delta}{\alpha C},\quad \left(\frac{6}{5}\right)=\frac{\gamma\delta\delta A}{2\alpha\beta CD},\quad \left(\frac{6}{6}\right)=\frac{\gamma\delta\delta AB}{3\alpha\beta CCD}\quad;$

$\left(\frac{5}{1}\right)=\frac{\alpha C}{\gamma},\quad \left(\frac{5}{2}\right)=\frac{\alpha BC}{\gamma A},\quad \left(\frac{5}{3}\right)=C,\quad \left(\frac{5}{4}\right)=\delta,\quad \left(\frac{5}{5}\right)=\frac{\delta\delta}{\alpha D};$

$\left(\frac{4}{1}\right)=\frac{\alpha D}{\delta},\quad \left(\frac{4}{2}\right)=\frac{\alpha\beta CD}{\gamma\delta A},\quad \left(\frac{4}{3}\right)=\frac{\alpha CD}{\delta A},\quad \left(\frac{4}{4}\right)=D;$

$\left(\frac{3}{1}\right)=\frac{\alpha C}{\delta},\quad \left(\frac{3}{2}\right)=\frac{\alpha\beta CD}{\delta\delta A},\quad \left(\frac{3}{3}\right)=\frac{\alpha\beta CCD}{\delta\delta AB};$

$\left(\frac{2}{1}\right)=\frac{\alpha B}{\gamma},\quad \left(\frac{2}{2}\right)=\frac{\alpha\beta BC}{\gamma\delta A};$

$\left(\frac{1}{1}\right)=\frac{\alpha A}{\beta}.$

Ordo harum formularum etiam in genere diagonaliter a sinistra ad dextram procedendo notari meretur, ubi quidem duo genera progressionum occurrunt, prout vel a prima serie verticali vel a suprema horizontali incipimus. Hoc modo primum a serie verticali incipiendo:

$$\left(\frac{n-1}{1}\right)=\alpha,\quad \left(\frac{n-2}{2}\right)=\frac{\beta}{\alpha}\times\left(\frac{n-1}{1}\right),\quad \left(\frac{n-3}{3}\right)=\frac{\gamma}{\beta}\times\left(\frac{n-2}{2}\right),\quad \left(\frac{n-4}{4}\right)=\frac{\delta}{\gamma}\times\left(\frac{n-3}{3}\right)$$

$$\left(\frac{n-2}{1}\right)=A,\quad \left(\frac{n-3}{2}\right)=\frac{B}{A}\times\left(\frac{n-2}{1}\right),\quad \left(\frac{n-4}{3}\right)=\frac{C}{B}\times\left(\frac{n-3}{2}\right),\quad \left(\frac{n-5}{4}\right)=\frac{D}{C}\times\left(\frac{n-4}{3}\right)$$

$$\left(\frac{n-3}{1}\right)=\frac{\alpha B}{\beta},\quad \left(\frac{n-4}{2}\right)=\frac{BC}{\gamma A}\times\left(\frac{n-3}{1}\right),\quad \left(\frac{n-5}{3}\right)=\frac{\gamma D}{\delta B}\times\left(\frac{n-4}{2}\right),\quad \left(\frac{n-6}{4}\right)=\frac{\delta E}{\varepsilon C}\times\left(\frac{n-5}{3}\right)$$

$$\left(\frac{n-4}{1}\right)=\frac{\alpha C}{\gamma},\quad \left(\frac{n-5}{2}\right)=\frac{\beta D}{\delta A}\times\left(\frac{n-4}{1}\right),\quad \left(\frac{n-6}{3}\right)=\frac{\gamma E}{\varepsilon B}\times\left(\frac{n-5}{2}\right),\quad \left(\frac{n-7}{4}\right)=\frac{\delta F}{\xi C}\times\left(\frac{n-6}{3}\right)$$

$$\left(\frac{n-5}{1}\right)=\frac{\alpha D}{\delta},\quad \left(\frac{n-6}{2}\right)=\frac{\beta E}{\varepsilon A}\times\left(\frac{n-5}{1}\right),\quad \left(\frac{n-7}{3}\right)=\frac{\gamma F}{\xi B}\times\left(\frac{n-6}{2}\right),\quad \left(\frac{n-8}{4}\right)=\frac{\delta G}{\eta C}\times\left(\frac{n-7}{3}\right)$$

$$\left(\frac{n-6}{1}\right)=\frac{\alpha E}{\varepsilon},\quad \left(\frac{n-7}{2}\right)=\frac{\beta F}{\xi A}\times\left(\frac{n-6}{1}\right),\quad \left(\frac{n-8}{3}\right)=\frac{\gamma G}{\eta B}\times\left(\frac{n-7}{2}\right),\quad \left(\frac{n-9}{4}\right)=\frac{\delta H}{\theta C}\times\left(\frac{n-8}{3}\right)$$

etc.,

deinde a suprema horizontali incipiendo:

$$\left(\tfrac{n}{1}\right)=1,\quad \left(\tfrac{n-1}{2}\right)=\tfrac{\beta}{\alpha}\times\left(\tfrac{n}{1}\right),\quad \left(\tfrac{n-2}{3}\right)=\tfrac{\gamma A}{\alpha B}\times\left(\tfrac{n-1}{2}\right),\quad \left(\tfrac{n-3}{4}\right)=\tfrac{\delta B}{\beta C}\times\left(\tfrac{n-2}{3}\right)$$
$$\left(\tfrac{n}{2}\right)=\tfrac{1}{2},\quad \left(\tfrac{n-1}{3}\right)=\tfrac{\gamma}{B}\times\left(\tfrac{n}{2}\right),\quad \left(\tfrac{n-2}{4}\right)=\tfrac{\delta A}{\alpha C}\times\left(\tfrac{n-1}{3}\right),\quad \left(\tfrac{n-3}{5}\right)=\tfrac{\varepsilon B}{\beta D}\times\left(\tfrac{n-2}{4}\right)$$
$$\left(\tfrac{n}{3}\right)=\tfrac{1}{3},\quad \left(\tfrac{n-1}{4}\right)=\tfrac{\delta}{C}\times\left(\tfrac{n}{3}\right),\quad \left(\tfrac{n-2}{5}\right)=\tfrac{\varepsilon A}{\alpha D}\times\left(\tfrac{n-1}{4}\right),\quad \left(\tfrac{n-3}{6}\right)=\tfrac{\xi B}{\beta E}\times\left(\tfrac{n-2}{5}\right)$$
$$\left(\tfrac{n}{4}\right)=\tfrac{1}{4},\quad \left(\tfrac{n-1}{5}\right)=\tfrac{\varepsilon}{D}\times\left(\tfrac{n}{4}\right),\quad \left(\tfrac{n-2}{6}\right)=\tfrac{\xi A}{\alpha E}\times\left(\tfrac{n-1}{5}\right),\quad \left(\tfrac{n-3}{7}\right)=\tfrac{\eta B}{\beta F}\times\left(\tfrac{n-2}{6}\right)$$
$$\left(\tfrac{n}{5}\right)=\tfrac{1}{5},\quad \left(\tfrac{n-1}{6}\right)=\tfrac{\xi}{E}\times\left(\tfrac{n}{5}\right),\quad \left(\tfrac{n-2}{7}\right)=\tfrac{\eta A}{\alpha F}\times\left(\tfrac{n-1}{6}\right),\quad \left(\tfrac{n-3}{8}\right)=\tfrac{\theta B}{\beta G}\times\left(\tfrac{n-2}{7}\right)$$

etc.

Ubi lex, qua hae formulae a se invicem pendent, satis est perspicua, si modo notemus in utraque litterarum serie α, β, γ, δ etc. et *A, B, C, D* etc. terminos primum antecedentes inter se esse aequales.

CONCLUSIO

Cum igitur formulas secundae classis sola concessa circuli quadratura exhibere valeamus, formulae tertiae classis insuper requirunt quadraturam contentam vel hac formula

$$\int\frac{dx}{\sqrt[3]{(1-x^3)^2}}=A \text{ vel hac } \int\frac{xdx}{\sqrt[3]{(1-x^3)^2}}=\frac{\alpha}{A},$$

quandoquidem, data una, simul altera datur. Quodsi istas formulas per productum infinitum exprimamus, earum valor reperitur

$$\int\frac{dx}{\sqrt[3]{(1-x^3)^2}}=\frac{2}{1}\cdot\frac{3}{4}\cdot\frac{5}{4}\cdot\frac{6\cdot8}{7\cdot7}\cdot\frac{9\cdot11}{10\cdot10}\cdot\frac{12\cdot14}{13\cdot13}\cdot\text{etc.},$$

unde eius quantitas vero proxime satis expedite colligi potest; simili modo est

$$\int\frac{xdx}{\sqrt[3]{(1-x^3)}}=1\cdot\frac{3\cdot7}{5\cdot5}\cdot\frac{6\cdot10}{8\cdot8}\cdot\frac{9\cdot13}{11\cdot11}\cdot\frac{12\cdot16}{14\cdot14}\cdot\text{etc.}$$

Deinde omnes formulas quartae classis integrare poterimus, si modo praeter circuli quadraturam una ex his quatuor formulis fuerit cognita $\left(\frac{2}{1}\right)$, $\left(\frac{1}{1}\right)$, $\left(\frac{3}{2}\right)$, $\left(\frac{3}{3}\right)$, quae praebent has formas

$$\int\frac{xdx}{\sqrt[4]{(1-x^4)^3}}=\frac{1}{2}\int\frac{dx}{\sqrt[4]{(1-xx)^3}}=\int\frac{dx}{\sqrt{(1-x^4)}}=A,$$
$$\int\frac{dx}{\sqrt[4]{(1-x^4)^3}}=\frac{\alpha A}{\beta},\quad \int\frac{xxdx}{\sqrt[4]{(1-x^4)}}=\frac{\alpha}{2A},$$
$$\int\frac{xxdx}{\sqrt[4]{(1-x^4)}}=\int\frac{xdx}{\sqrt[4]{(1-x^4)}}=\frac{1}{2}\int\frac{dx}{\sqrt[4]{(1-xx)}}=\frac{\beta}{A};$$

at per productum infinitum erit

$$A = \frac{3}{1\cdot 2}\cdot\frac{4\cdot 7}{5\cdot 6}\cdot\frac{8\cdot 11}{9\cdot 10}\cdot\frac{12\cdot 15}{13\cdot 14}\cdot\frac{16\cdot 19}{17\cdot 18}\cdot\text{etc.}$$

Quinta classis postulate duas quadraturas altiores $\left(\frac{3}{1}\right) = A$ et $\left(\frac{2}{2}\right) = B$, quarum loco aliae binae ab his pendentes assumi possunt, quae quidem faciliores videantur, etsi ob 5 numerum primum aliae aliis vix simpliciores reputari queant.

Pro sexta classe etiam duae quadraturae requiruntur $\left(\frac{4}{1}\right) = A$ et $\left(\frac{3}{2}\right) = B$.

Verum hic loco alterius ea, quae in tertia classe opus erat, assumi potest, ut unica tantum nova sit adhibenda. Cum enim sit

$$\left(\frac{2}{2}\right) = \int\frac{xdx}{\sqrt[6]{(1-x^6)^4}} = \frac{1}{2}\int\frac{dx}{\sqrt[3]{(1-x^3)^2}} = \frac{\alpha BB}{\gamma A},$$

erit

$$\frac{2\alpha BB}{\gamma A} = \int\frac{dx}{\sqrt[3]{(1-x^3)^2}},$$

quae est formula ad classem tertiam requisita. Hac ergo data si insuper innotescat formula

$$\left(\frac{3}{2}\right) = \int\frac{xdx}{\sqrt{(1-x^6)}} = \frac{1}{2}\int\frac{dx}{\sqrt{(1-x^3)}} = B,$$

vel etiam haec

$$\left(\frac{4}{3}\right) = \int\frac{xxdx}{\sqrt[3]{(1-x^6)}} = \frac{1}{3}\int\frac{dx}{\sqrt[3]{(1-xx)}} = \frac{\beta\gamma}{\alpha B},$$

quae sunt simplicissimae in hoc genere, reliquae omnes per has definiri poterunt.
His autem combinatis patet fore

$$\int\frac{dx}{\sqrt{(1-x^6)}}\cdot\int\frac{dx}{\sqrt[3]{(1-xx)}} = \frac{6\beta\gamma}{\alpha} = \frac{\pi}{\sqrt{3}}.$$

Simili modo ex formulis quartae classis colligitur

$$\int\frac{dx}{\sqrt{(1-x^4)}}\cdot\int\frac{dx}{\sqrt[4]{(1-x^2)}} = \frac{\pi}{2},$$

cuiusmodi theorematum ingens multitudo hinc deduci potest, inter quae hoc imprimis est notabile

$$\int \frac{dx}{\sqrt[m]{(1-x^n)}} \cdot \int \frac{dx}{\sqrt[n]{(1-x^m)}} = \frac{\pi \sin.\frac{(m-n)\pi}{mn}}{(m-n)\sin.\frac{\pi}{m}\cdot\sin.\frac{\pi}{n}},$$

quod, si m et n sint numeri fracti, in hanc formam transmutatur

$$\int \frac{x^{q-1}dx}{\sqrt[r]{(1-x^p)^s}} \cdot \int \frac{x^{s-1}dx}{\sqrt[p]{(1-x^r)^q}} = \frac{\pi \sin.\left(\frac{s}{r}-\frac{q}{p}\right)\pi}{(ps-qr)\sin.\frac{q\pi}{p}\cdot\sin.\frac{s\pi}{r}}.$$

In genere vero est

$$\left(\frac{n-p}{q}\right)\left(\frac{n-q}{p}\right) = \frac{\left(\frac{n-p}{p}\right)\left(\frac{n-q}{q}\right)}{(q-p)\left(\frac{n-q+p}{q-p}\right)},$$

quod hanc formam praebet

$$\int \frac{x^{p-1}dx}{\sqrt[n]{(1-x^n)^q}} \cdot \int \frac{x^{q-1}dx}{\sqrt[n]{(1-x^n)^p}} = \frac{\pi \sin.\frac{(q-p)\pi}{n}}{n(q-p)\sin.\frac{p\pi}{n}\cdot\sin.\frac{q\pi}{n}},$$

unde non solum praecedentia theoremata, sed alia plura facile derivantur.
Posito enim $n=\frac{pq}{m}$ habebimus

$$\int \frac{x^{m-1}dx}{\sqrt[p]{(1-x^q)^m}} \cdot \int \frac{x^{m-1}dx}{\sqrt[q]{(1-x^p)^m}} = \frac{\pi \sin.\left(\frac{m}{p}-\frac{m}{q}\right)\pi}{m(q-p)\sin.\frac{m\pi}{q}\cdot\sin.\frac{m\pi}{p}},$$

quam ita latius extendere licet

$$\int \frac{x^{p-1}dx}{\sqrt[n]{(1-x^m)^q}} \cdot \int \frac{x^{q-1}dx}{\sqrt[m]{(1-x^n)^p}} = \frac{\pi \sin.\left(\frac{q}{n}-\frac{p}{m}\right)\pi}{(mq-np)\sin.\frac{p\pi}{m}\cdot\sin.\frac{q\pi}{n}};$$

In qua si ponatur $n=2q$, erit

$$\int \frac{x^{p-1}dx}{\sqrt{(1-x^m)}} \cdot \int \frac{x^{q-1}dx}{\sqrt[m]{(1-x^{2q})^p}} = \frac{\pi \sin.\frac{p\pi}{m}}{q(m-2p)\sin.\frac{p\pi}{m}}.$$

At in posteriori formula integrali si ponatur $x^{2q}=1-y^m$, erit

$$\int \frac{x^{q-1}dx}{\sqrt[m]{(1-x^{2q})^p}} = \frac{m}{2q}\int \frac{y^{n-p-1}dy}{\sqrt{(1-y^m)}},$$

unde scripto x pro y

$$\int \frac{x^{p-1}dx}{\sqrt{(1-x^m)}} \cdot \int \frac{x^{m-p-1}dx}{\sqrt{(1-x^m)}} = \frac{2\pi \sin.\frac{p\pi}{m}}{m(m-2p)\sin.\frac{p\pi}{m}}.$$

Simili modo si in genere ponatur pro altera formula integrali $1-x^n = y^m$,
fiet

$$\int \frac{x^{q-1}dx}{\sqrt[m]{(1-x^n)^p}} = \frac{m}{n}\int \frac{y^{m-p-1}dy}{\sqrt[n]{(1-y^m)^{n-q}}},$$

unde scripto iterum x pro y obtinebitur

$$\int \frac{x^{p-1}dx}{\sqrt[n]{(1-x^m)}^q} \cdot \int \frac{x^{m-p-1}dx}{\sqrt[n]{(1-x^m)^{n-q}}} = \frac{n\pi \sin.\left(\frac{q}{n}-\frac{p}{m}\right)\pi}{m(mq-np)\sin.\frac{p\pi}{m}\cdot\sin.\frac{q\pi}{n}},$$

qui valor reducitur ad $\frac{n\pi}{m(mq-np)}\left(\cot.\frac{p\pi}{m}-\cot.\frac{q\pi}{n}\right)$. Atque hinc ista forma concinnior resultat

$$\int \frac{x^{\frac{m-r}{2}-1}dx}{\sqrt[n]{(1-x^m)^{\frac{n-s}{2}}}} \cdot \int \frac{x^{\frac{m+r}{2}-1}dx}{\sqrt[n]{(1-x^m)^{\frac{n+s}{2}}}} = \frac{2n\pi\left(\text{tang.}\frac{r\pi}{2m}-\text{tang.}\frac{s\pi}{2n}\right)}{m(nr-ms)}.$$

Cum fundamentum harum reductionum situm sit in hac aequalitate

$$\left(\frac{n-p}{q}\right)\left(\frac{n-q}{p}\right) = \frac{\left(\frac{n}{q-p}\right)\left(\frac{n-p}{p}\right)\left(\frac{n-q}{p}\right)}{\left(\frac{n-q+p}{q-p}\right)},$$

quae ad hanc formam reducitur

$$\left(\frac{n-p}{q}\right)\left(\frac{n-q}{p}\right)\left(\frac{n-q+p}{q-p}\right) = \left(\frac{n}{q-p}\right)\left(\frac{n-p}{p}\right)\left(\frac{n-q}{p}\right),$$

eius veritas hoc modo directe ostendi potest.
Sumtis in reductione § 8 tradita pro numeris ternis *p, q, r* his $n-q, q-p,\ q$ habebimus

$$\left(\frac{n-q}{q-p}\right)\left(\frac{n-p}{q}\right) = \left(\frac{n-q}{q}\right)\left(\frac{n}{q-p}\right);$$

tum vero sumtis eorum loco $n-q,\ q-p,\ p$ erit

$$\left(\frac{n-q}{q-p}\right)\left(\frac{n-p}{p}\right) = \left(\frac{n-q}{p}\right)\left(\frac{n-q+p}{q-p}\right),$$

quibus aequationibus in se invicem ductis et formula $\left(\frac{n-q}{q-p}\right)$ utrinque communi per divisionem sublata erit

$$\left(\frac{n-p}{q}\right)\left(\frac{n-q}{p}\right)\left(\frac{n-q+p}{q-p}\right)=\left(\frac{n}{q-p}\right)\left(\frac{n-p}{p}\right)\left(\frac{n-q}{p}\right).$$

Quin etiam huiusmodi ternarum aequalitas ab exponente n non pendens exhiberi potest, scilicet

$$\left(\frac{s}{p}\right)\left(\frac{r+s}{q}\right)\left(\frac{p+s}{r}\right)=\left(\frac{r+s}{p}\right)\left(\frac{s}{q}\right)\left(\frac{q+s}{r}\right)=\left(\frac{r}{p}\right)\left(\frac{r+s}{q}\right)\left(\frac{p+r}{s}\right)=\left(\frac{r+s}{p}\right)\left(\frac{r}{q}\right)\left(\frac{q+r}{s}\right).$$

quae quatuor adeo litteras ab n non pendentes involvit ac similis est aequalitati inter binarum formularum producta

$$\left(\frac{r}{p}\right)\left(\frac{p+r}{q}\right)=\left(\frac{q+r}{p}\right)\left(\frac{r}{q}\right)=\left(\frac{q}{p}\right)\left(\frac{p+q}{r}\right).$$

Aequalitas autem inter ternarum formularum producta habetur etiam ista

$$\left(\frac{p}{q}\right)\left(\frac{r}{s}\right)\left(\frac{p+q}{r+s}\right)=\left(\frac{q}{r}\right)\left(\frac{s}{p}\right)\left(\frac{q+r}{p+s}\right)=\left(\frac{p}{r}\right)\left(\frac{q}{s}\right)\left(\frac{p+r}{q+s}\right)$$
$$=\left(\frac{p}{q}\right)\left(\frac{p+q}{r}\right)\left(\frac{p+q+r}{s}\right)=\left(\frac{p}{q}\right)\left(\frac{p+q}{s}\right)\left(\frac{p+q+s}{r}\right)\text{etc.}$$

In his enim litterae *p, q, r, s* utcumque inter se permutari possunt.