APPENDIX
CONCERNING THE
CALCULUS
OF VARIATIONS.

CHAPTER I

CONCERNING THE CALCULUS OF VARIATIONS IN GENERAL

DEFINITION 1

1. *A relation between the variables is said to be varied, if the value, by which the one hence is determined by the other, is considered to be increased by an infinitely small increment, which we will call the variation increment of this quantity, to which it is added.*

EXPLANATION

2. Therefore at first a relation is considered here between the two variables x and y expressed by some equation between the same, from which for the individual attributed values of x agreeing values of y may be determined ; then truly the individual values of y may be conceived to increase by some infinitely small parts , thus these values are to vary from the true values, which are chosen from the proposed relation, so that they may disagree by an infinitely small amount, and in this way the relation between x and y is said to be *varied* and likewise these infinitely small parts to be added are called the *variations* from the true values of y. But at first it is required to be observed here that these variations, by which the individual values of y taken to be augmented, neither are to be equal to each other nor in any way depend on each other in turn, but thus to be allowed arbitrarily by us, so that all beyond one, or any with certain values corresponding to values of y clearly are not allowed to be considered. Evidently these variations are not to be considered to be restricted by some rule nor any given relation between x and y is to be agreed to infer the determination of these variations, which are required to be seen as purely arbitrary.

COROLLARY 1

3. Hence it is apparent that there is a whole world of a difference between variations and differentials, even if both are infinitely small and thus plainly vanish ; indeed a variation is agreed to affect both the value of y and the corresponding value of x, while the differential dy of this at once considers the following value $x+dx$.

COROLLARY 2

4. For if with a proposed relation between x and y, for x itself there is an agreeing y, for $x+dx$ itself truly the agreeing value of y is put as y', then there is $dy = y'-y$; but the variation of y by no means depends on the following value y', so that rather both y and y' are allowed to be given their separate variations as it pleases.

SCHOLIUM

5. This idea of variation, which by itself can be seen to be both exceedingly vague as well as unprofitable, will be illuminated especially, if we can explain more carefully its origin and what agreement concerning that has been reached. Moreover the investigation has led on that account chiefly to the finding of curves, which are predisposed with a certain property of maximum or minimum, from which, lest the thing in general being examined is very obscure, we consider the problem, in which the line of a curve is sought, upon which a weight falling from a given point descends the most quickly to another given point. And here indeed from the nature of the maximums and minimums at once a curve in agreement thus must be prepared, so that, if in place of this some other curve differing from that by an infinitely small amount is substituted, the time of descent upon that shall become utterly the same. Therefore it is required to put a solution in place, so that, while the curve sought is considered as given, also the calculation according to another curve disagreeing with that by an infinitely small amount may be accommodated and thence a redundant expression of the time may be computed; then indeed this distinction on being put equal to zero itself will indicate the nature of the curve sought. But these curves thus most conveniently are to be considered differing by an infinitely small amount from the sought, so that the applied lines corresponding to the individual abscissas may be either increased or diminished by infinitely small amounts, that is, so that *variations* are considered to be undertaken. Indeed generally it is sufficient for a single applied line of this kind to be set up; but nothing stands in the way, why not several and accordingly all applied lines be assigned such variations, since always it is necessary that the same solution be produced. But in this manner not only the strength of the method is illustrated much more clearly, but also thence the solutions of the questions of this kind may be obtained more fully, from which also it is allowed to explain in detail questions regarding other conditions. Since on account of this cause generally it will be seen of necessity, in order that the calculus of variations may be explored to the fullest extent to which indeed it is capable.

DEFINITION 2

6. *For a given relation between the two variable quantities each of these is said to vary, if each separately is considered to be increased by an infinitely small increment ; from which it is apparent, how this may be understood, if for each variable there is attributed its own variation.*

EXPLANATION

7. If the proposed equation shall be some equation between the two variables x and y, by which the mutual relation of these is expressed, this relation by definition can be varied in a twofold manner, with the one, for which with the individual values of x remaining, there is given a variation by the individual y values, truly with the other, in which with the values of y remaining the individual x values are considered to vary. Therefore nothing forbids, that each of the variations likewise is understood to take its own variation, which thus it is allowed to take, so that plainly with no connection tying them together; therefore a twofold variation can be considered, since in the first definition only a single one shall be allowed. But thus here we may consider the situation more generally, so that neither variation shall be restricted by any law nor also do the variations of y depend on the variations of x in any way.

COROLLARY 1

8. Therefore from the case, in which a twofold variation is put in place, the first case only arises as the case if the variations of the other variable clearly are rejected; from which it is evident that the case of the second definition includes the first case.

COROLLARY 2

9. Hence it may be shown further, just as a given relation between two variables can be varied in an infinite number of ways, likewise it is understood, because we assume that these variations are not restricted by any rule, generally all the variations possible of this relation are to be indicated by this reason.

SCHOLIUM 1

10. Indeed now the variations induced by only one variable deal with all the variations possible, which can arise in the proposed relation between the two variables, so that it may seem to be a superfluous calculation for a twofold variation to be applied; now if we may consider more carefully the nature of the problem and the use to which it is to be put, the consideration of a twofold variation may by no means be taken to be unnecessary, as that will be illustrated most clearly geometrically in the following manner.

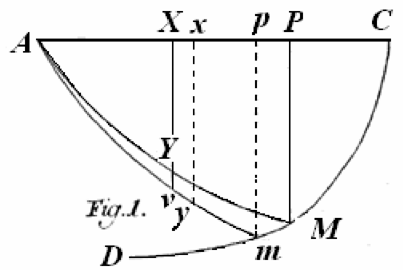


Fig.1.

Since any relation between the two variables may be represented most distinctly by a curved line described in the plane, let *AYM* (Fig. 1) be a curved line defined from the equation between the coordinates $AX = x$ and $XY = y$, which hence shows that given relation ; therefore now any other curved line *Aym* disagreeing with that by an infinitely small amount will represent that varied relation, which, however it is to be considered, always can be considered thus, so that for the same abscissa $AX = x$ it shall agree with the applied line variation X*v* present, with the variation *Yv* a small part of this; which consideration also suffices for most questions brought forward about maxima and minima, where indeed a curve *AM* is accustomed to be considered only from the variation by some elements. But if the question shall be prepared thus, so that between all the curves, which from the given point *A* to some curve *CD* as far as it is allowed to consider, that curve *AYM* may be defined, for which a certain property of maximum or minimum may be agreed upon, then the same property in some other curve *Aym* nearby, also ending in another point *m* of the line *CD* must be agreed upon, and thus with the final curve sought for the point *M*, both variations from the abscissa *AP* as well as the applied line *PM* is to be agreed upon and indeed of this kind, which shall be in agreement with the nature of the line *CD*. Therefore by which a calculation, according to such a variation, leading to the final element being put in place, it is entirely necessary, that most generally the intermediate individual points *Y* of the curve *AM* that some variations are to be attributed both to the abscissas $AX = x$ as well as to the applied lines $XY = y$ and the variation of the one is put in place by a small *Xx*, of the other truly it is $= xy - XY$, from which likewise the nature of the use of twofold variations of this kind are seen most clearly.

SCHOLIUM 2

11. Just as a consideration of the furthest point of a curve requiring to be investigated by us has sufficed for this clear explanation, thus also it is required to gave the variation immediately after the first point. For if between all the lines it is allowed to consider, which lead from a certain given curve *AB* (Fig. 2) to another likewise certain given curve *GD*, that line shall be sought, which shall be provided with some property of the maximum or minimum, then much greater will be the need

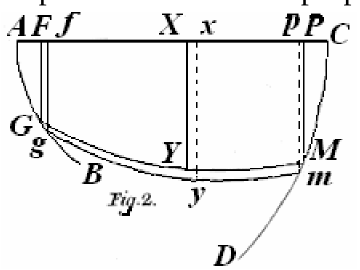


Fig.2.

that the individual variation of both the abscissas *AX* as well as of the applied lines X*Y* be unrestricted by any rule assigned to the calculation, so that hence both *G* sought from the start of the curve, as well as *M* from the end of this are able to carry a variation. But although this illustration from geometry has been chosen, yet it is easily understood that the idea of variation hence needs to be extended wider and not to be without use in the highest absolute analysis . Moreover the most celebrated De La Grange, the shrewdest geometer of Turin * to whom we must refer the first accepted explorations concerning the calculus of variations, indeed transferred the method most ingeniously to non–continuous lines, so as requiring to be referred to a kind of polygons, in which undertaking these twofold variations themselves performed the greatest usefulness.

* [*Miscellanea Taurinensia* : vol. II, 1762, p.173; t. I, *Oeuvres de Lagrange*, p. 335 : (title tr.) *A new method of determining the maxima and minima of indefinite integral formulas* : (in French.) Euler also of course wrote a number of papers on this topic: see the Euler Archives for a listing.],

DEFINITION 3

12. *A relation between three variables determined by two equations is said to be varied, if either one, two or all three are augmented by an infinitely small amount, which are called the variations of these.*

EXPLANATION

13. Since three variable quantities are proposed, as for instance x, y and z, between which two equations are considered to be given, from each one of these one can determine the remaining two, so that it may possible to regard y as well as z as a function of x. Moreover it is customary to define a curved line not described in the same plane in this manner, as long as the individual points of this may be assigned by these three coordinates x, y and z in the accustomed manner. But if now such a curve may be accompanied by some other curve close to itself, so that the difference shall be infinitely small, this new proposed curve will be varied and that relation between these three variables x, y, z can be considered to express the nature of this curve. From which, just as two points nearby, one on the proposed curve and the other taken on the accompanying variation, may be compared with each other, it can happen for the variation that either all three coordinates will produce different variations, or only two, or perhaps a single one, and the differences of these from the principal points of the curve will represent the variations of these ; but which thus here is agreed to consider conveniently, so that the method may be extended entirely to all nearby curves, either these shall be different for the whole treatment of the proposed curve or only differing in some parts from that, thus so that also the lines are discontinuous, provided the principal points shall lie on the main curve, and hence are not excluded. For nor are these varied curves when uninterrupted to be restricted by a rule, so that plainly all possible curves differing by infinitely small amounts from the main curve are themselves to be included.

COROLLARY 1

14. Therefore with some point of a varied curve separated by an infinitely small amount compared with that point of the proposed or principal curve, and hence the variations of the coordinates are understood to be defined.

COROLLARY 2

15. Because again from a single assumed variable x the two remaining y and z and thus a point of the proposed curve is determined, also one is allowed to regard the variations of the individual coordinates as functions of x, provided the magnitudes of these may be regarded as infinitely small.

COROLLARY 3

16. Therefore one is allowed to consider any three functions of x differing from each other in any manner, which suitably multiplied by infinitely small factors are required to represent the three variations of the coordinates. Since likewise it is required to be understood by any three variables, even if they do not refer to geometry.

COROLLARY 4

17. In a similar manner too, if a relation is proposed between two variables only, the variations of these can be considered as functions of another variable, only they shall be infinitely small or, which amounts to the same, multiplied by an infinitely small magnitude.

SCHOLIUM 1

18. But a geometrical consideration is most suitable for illustrating these speculations, which in general are to be considered exceedingly abstract and also may appear aimless [at present]. Therefore the case of three variables, of which a relation is assumed to be defined by two equations, may be set forth in all its splendor by a curve not described in the same plane, provided that the three coordinates are designated by these variables. But if indeed a question concerning these curves of this kind is put in place, so that a property may be defined between these, which shall be some given property of maximum or minimum, it is necessary that the same property may be prepared equally for all other curves differing from that curve by an infinitely small amount, which is decided to be appropriate from the variations introduced into the calculations. But for which variations the greatest use generally will be established here later, thence one can understand, if in place of the two curves *AB* and *CD* (Fig.2 above) there may be given any two surfaces, from one surface of which it is required to draw a curved line of this kind to the other, which may delight in producing the property of some maximum or minimum. Then indeed it is required to consider the variations of the three coordinates, so that for a point of the curve sought at the beginning on the surface *AB*, the variations there will be able to be applied by translation to the same surface, and that in a like manner it is able to be applied at the end to the surface *CD*.

From which it is evident in general that three variables must be introduced into the calculation, so that one is allowed to transfer to the surfaces both these points for the start of the curve as well as for the end of the curve required to be found, clearly the nature of which at each end will determine the mutual relation between the variations.

SCHOLIUM 2

19. Just as here we have considered three variables, the relation of which may be determined from two equations, thus also it is possible to extend the calculation of the variables to four or more, if indeed the relation may be expressed by so many equations, so that all the remaining may be obtained by their own determination, even if an illustration of this case no longer can be reached from geometry with only three dimensions included, unless perhaps we may wish to call in time as an aid, [as in the case of] a river continually flowing from the surface *AB* to the surface *CD*, but in the lapse of time being continually unchanging, thus so that then also a moment of time is to be assigned, from which some of the flow from a water course stretched out from the surface *AB* to the surface *CD* shall be provided with a certain property of maximum or minimum. To which variables above if we add the changeability of the speed, these serve to increase the number of variables. But in the first place it is understood hence, even if all the variables are assumed to be determined from a single variable, yet an account of the investigation from these, where only two variables are admitted, is to disagree especially, because therefore their single variations must be assigned independently from the rest; nor indeed thence, because between these variables a certain relation is to be recognised, thus also the variations of these are agreed not to be bound by any restrictions; just as is evident from the previous case reported, where a curve stretched out between the two surfaces *AB* and *CD* and with a certain maximum or minimum property thus given certainly has been itself determined, so that with one of the coordinates assumed the two may be determined ; truly from nothing less all the variation curves, which in all regions can be deflected from that, for the individual coordinates receive the variations by no means depending in turn on these alone, with the beginning and the end excepted, where it is required for these to be applied to the given surfaces.

DEFINITION 4

20. *A relation between three variables defined from a single equation, so that one of these is equal to a function of the two remaining, is said to be varied, if either one or two or all three of these variables may be increased by infinitely small amounts, which are called the variations of these.*

EXPLANATION

21. Because here the relation between the three variables put in place is to be defined by a single equation, with two variables assumed arbitrarily the third may be determined at last, so that thus it may be considered as a function of the two variables. Hence that relation is indicated not by a certain curved line, if we wish to transfer the argument to diagrams, but by a certain whole surface, the nature of which is expressed by an equation between the three coordinates; from which it is understood that with the same surface varied, another surface may be represented at an infinitely small separation from that, which variation thus must be the most widely allowed, so that either it is to be restricted only to a certain part of the surface or it may be possible to be extended to the whole surface. As since any point of the given surface therefore is to be compared to that certain other point of the varied surface nearby, it can happen, that not only one of the three coordinates, but also perhaps two or all three may be varied ; from which, where the treatment is put in place in each extent, it may be agreed at once that the variations be attributed to their individual coordinates, which therefore are allowed to be prepared thus so that they may be considered as functions of two variables, since at last from the two determined, a point on the surface may be determined.

COROLLARY 1

22. If therefore the three variables or coordinates shall be *x, y* and *z,* just as it may be permitted to give values as it pleases to the two *x* and *y*, from which the value *z* to be determined may be obtained, likewise the variation of *z* may be agreed to depend on each of these *x* and *y*, because if either one or both may be changed, another variation of *z* must result.

COROLLARY 2

23. Because here it has been observed from the variation of the one variable *z*, likewise for the two remaining it is to be understood, so that thus the variations of each shall be considered as functions of two variables ; because truly an equation is given between *x*, *y*, and *z* , likewise it is the case, for of which of the two functions being considered, because a function of *y* and *z* can be recalled by an equation as a function of *x* and *y*, evidently if in place of *z* its value may be substituted expressed by *x* and *y*.

SCHOLIUM 1

24. This will be required to be used of the variations in place, if a surface needs to be investigated provided with some property of the maximum or minimum, since then the calculation thus is required to be put in place, so that the same properties in those nearby surfaces and varied equally may be agreed upon. Then since an account may usually be prescribed of both the terms in curves provided with the maximum or minimum property, so that they may be terminated either in given points or at given curved lines or thus, here a similar condition is allowed, so that the surface sought shall be defined on all sides or is circumscribed by a certain surface ; so that an account of this last condition may be had, it is entirely necessary, that the most general variations being attributed to all three coordinates by no means depend on each other, from which hence these extrema arise able to be applied according to the nature of the terms of the surface. Here indeed it must be admitted that the method of maximas and minimas scarcely at this point has advanced to investigations of this kind and great difficulties occur here, overcoming which may be seen to require a much greater advance in analysis. Truly on account of that cause itself accordingly more will have to be forthcoming, so that the principles of this method, which the principles of this method may comprise, certainly may be made firm and likewise put forwards clearly and distinctly.

SCHOLIUM 2

25. Scarcely is there a need here, I consider, to turn one's mind to extending this calculus in a similar manner to more than three variables, even if geometrical questions may not be able to provide further explanation; for the analysis itself is not limited to a number of dimensions as geometry certainly is agreed to be. But when more variables may be considered, everything before is agreed to be considered, each relation of these may be expressed only by a single mutual equation or by several ; which as many as can be done, so that many with one variable only are lacking in a number of variables, in which case one may be allowed consider all as functions of one variable. But if a relation is constructed from fewer equations, the individual variables will be functions of two or more variables, and in whatever single case also the variations must be treated functions of just as many variables, if indeed we wish to explain this calculus most generally.

DEFINITION 5

26. *The calculus of variations is the method of finding the variation, as an expression recovered from some number of variables put together, while the variations are assigned either by all or by some variables.*

EXPLANATION

27. In this definition there shall be no mention of a relation, as we assume to have at this point between the variables ; since indeed this calculus chiefly shall be occupied with investigating this relation itself, which evidently shall be provided with the property of maximum or minimum, as long as that at this stage is unknown, an account of this by no means is allowed to be considered in the calculation, but rather it is agreed to treat this thus, as if the variables plainly were not to be connected between each other by a relation. Therefore a calculation thus has been agreed to be put in place, so that, if for the individual variables, which are entering into the calculation, some variations are to be assigned, with all kinds of expressions, which in some manner were constructed from that, the variations thence to be arising may be shown to be investigated; from which found in general then finally questions of this kind occur to be disclosed, such a relation between the variables may be required to be put in place, so that the variation found in that shall be either zero, as usually comes about in an investigation of maximas or minimas, or certainly some other way shall be provided, as the nature of the question might examine. In this manner if the precepts of this calculus are treated, nothing hinders why questions of this kind should not be treated, in which immediately some relation produced between the variables is assumed as given and may be desired of certain resolved expressions from these formed from the variation of the variables. From which it is understood this calculation can be adapted to most questions of the most diverse kinds.

COROLLARY 1

28. Hence questions requiring to be treated in this calculus are returned here , so that for some proposed expression from however many variables assembled in some manner an increment of this may be defined, if the individual variables are increased by their own variations.

COROLLARY 2

29. Therefore the calculus of variations is entirely similar to the differential calculus, provided infinitely small increments are attributed from the variables in both places. But in as much as now we have observed [§3 & 4], the variations disagree with differentials and thus likewise since they are able to be consistent with these, so far as a great distinction must be recognised between each.

SCHOLIUM.

30. From the observations reported above, this distinction shall be especially evident ; for when the calculus is referring to a curved line, as when it is required to be compared with another close to itself, by differentiation we are progressing from some point on the curve to another point on the same curve ; but when we jump from this curve to another close to itself, the passage provided it is very small, shall be by variations. Likewise it comes about on surfaces referred to others close to themselves, where the differentials are considered to be on the same surface, truly with the variations there is a jump from one to another. The account is entirely the same, if the matter is considered analytically without any regard to geometric figures, where it is required always to distinguish carefully the variations of variable quantities from their own differentials, which variations finally it is agreed to indicate by a different sign.

HYPOTHESIS .

31. *We will designate in what follows, the variation of any kind of varied quantity by the letter δ prefixed to the same quantity, thus so that $\delta x, \delta y, \delta z$ may designate the variations of the quantities x, y, z, and if V should be some expression constructed from these, the variation of this will be indicate by use in this manner δV .*

COROLLARY 1

32. Hence δx signifies that infinitely small increment, by which the quantity *x* is considered to be increased, so that a varied value of this may be produced, from which in turn the value of the variation of *x* is understood to be $x + \delta x$.

COROLLARY 2

33. Therefore in as much as the expression *V* is composed from the variables *x*, *y* and *z*, if in place of these the values of the variations are written $x + \delta x,\ y + \delta y$ and $z + \delta z$ and from a value for *V* resulting in this manner, *V* itself is subtracted, there will be the variation dV .

COROLLARY 3

34. Therefore at this stage everything can itself be considered likewise as in the differential calculus, if *V* were some function of *x, y* and *z,* with the differential of this assumed in the customary manner, as everywhere there is written δ in place of *d* , there will be had the variation of this δV .

SCHOLIUM 1

35. Therefore as often as *V* is some function of the variable quantities *x, y, z,* the variation of this from these same rules thence is elicited and from the differential of this, from which the calculus of variations can be seen to agree with the differential calculus in a straight forwards manner, since only the difference of sign shall be cause for a moment's thought. The truth to be considered carefully here is that not all the quantities, of which the variations are required, in general is it possible to take as functions; on account of which also in the definition [§ 26] I have used with the name *expression*, to which I attribute far more significance. Indeed in as much as one is not allowed to consider the mutual relation of the variations, which is unknown, so far expressions of this kind, or formulas in which the differentials and also integrals of variables are present, no longer are to be considered as pure functions of the variables, and the variation demands special precepts both of the differential as well as the integral formulas ; and thus the whole business is reduced to this, that just as often as it is agreed to investigate the variations of each kind of formulas, so we may show from which our treatment will arise in two parts.

SCHOLIUM 2

36. But in this treatment the greatest distinction emerges from the number of variables, which if it should exceed two, at this point scarcely is it evident how the calculation shall be performed. Since indeed with several variables introduced also an otherwise tedious examination of the differentiation has to be considered, and while generally only the differentials of two are usually compared between each other, as if the remaining variables remain constant, a similar account also will be had with the variations, in which even now so great difficulties occur, to that it may scarcely be apparent, how one may overcome these ; before everything surely it will be necessary to explain the first principles of this calculation most carefully, and from the innermost parts of the thing the precepts of the calculation may be satisfied, in which generally the greatest difficulties are accustomed to be stumbled on. Therefore this first calculation shall be applied to two variables only, just as that indeed has been accustomed to be treated at this point, I will attempt to explain the variations both of the differential formulas as well as of the integrations we are about to investigate; then truly, if here any of the light might flow forth from the treatment itself, I may progress to contemplating three or more variables.

APPENDIX
DE
CALCULO
VARIATIONUM.

CAPUT I

DE CALCULO VARIATIONUM IN GENERE

DEFINITIO 1

1. *Relatio inter binas variabiles variari dicitur, si valor, quo altera inde per alteram determinatur, incremento infinite parvo augeri concipiatur, quod incrementum variationem eius quantitatis, cui adiicitur, vocabimus.*

EXPLICATIO

2. Primum ergo hic consideratur relatio inter binas variabiles x et y quaecunque aequatione quacunque inter easdem expressa, qua pro singulis valoribus ipsi x tributis valores ipsius y convenientes determinentur; tum vero singuli valores ipsius y particulis infinite parvis utcunque augeri concipiantur, ita ut hi valores variati a veris, quos ex relatione proposita sortiuntur, infinite parum discrepent, atque hoc modo relatio illa inter x et y *variari* dicitur simulque particulae illae infinite parvae valoribus veris ipsius y adiunctae *variationes* appellantur. Imprimis autem hic notandum est has variationes, quibus singuli valores ipsius y augeri concipiuntur, neque inter se statui aequales neque ullo modo a se invicem pendentes, sed ita arbitrio nostro permitti, ut omnes praeter unam vel aliquas certis valoribus ipsius y respondentes plane ut nullas spectare liceat. Nulli scilicet legi hae variationes adstrictae sunt concipiendae neque relatio inter x et y data ullam determinationem in istas variationes inferre est censenda, quas ut prorsus arbitrarias spectare oportet.

COROLLARIUM 1

3. Hinc patet variationes toto coelo differre a differentialibus, etiamsi utraque sint infinite parva ideoque plane evanescant; variatio enim afficit eundem valorem ipsius y eidem valori ipsius x convenientem, dum eius differentiale dy simul sequentem valorem $x+dx$ respicit.

COROLLARIUM 2

4. Si enim ex relatione inter x et y proposita ipsi x conveniat y, ipsi $x+dx$ vero valor ipsius y conveniens ponatur y', tum est $dy = y' - y$; at variatio ipsius y neutiquam pendet a valore sequente y', quin potius utrique y et y' pro lubitu suam variationem seorsim tribuere licet.

SCHOLION

5. Haec variationum idea, quae per se tam nimis vaga quam sterilis videri queat, maxime illustrabitur, si eius originem et quo pacto ad eam est perventum accuratius exposuerimus. Perduxit autem eo potissimum quaestio de curvis inveniendis, quae certa quadam maximi minimive proprietate sint praeditae, unde, ne rem in genere considerando obscuritas offundatur, problema contemplemur, quo linea curva quaeritur, super qua grave delabens e dato puncto citissime ad aliud punctum datum descendat. Atque hic quidem ex natura maximorum et minimorum statim

constat curvam ita debere esse comparatam, ut, si eius loco alia curva quaecunque infinite parum ab illa discrepans substituatur, tempus descensus super ea idem prorsus sit futurum.
Solutionem ergo ita institui oportet, ut, dum curva quaesita tanquam data spectatur, calculus quoque ad aliam curvam infinite parum ab ea discrepantem accommodetur indeque discrimen, quod in temporis expressionem redundat, supputetur; tum enim hoc ipsum discrimen nihilo aequale positum naturam curvae quaesitae declarabit. Curvae autem istae infinite parum a quaesita discrepantes commodissime ita considerantur, ut applicatae singulis abscissis respondentes particulis infinite parvis vel augeantur vel minuantur, hoc est, ut *variationes* recipere concipiantur. Vulgo quidem sufficit huiusmodi variationem in unica applicata constituisse; nihil autem impedit, quominus pluribus atque adeo omnibus applicatis tales variationes assignentur, cum semper ad eandem solutionem perduci sit necesse. Hoc autem modo non solum vis methodi multo luculentius illustratur, sed etiam inde solutiones quaestionum huius generis pleniores obtinentur, unde etiam quaestiones ad alias conditiones spectantes enucleare licet. Quam ob causam omnino necessarium videtur, ut calculus variationum in amplissima extensione, cuius quidem est capax, pertractetur.

DEFINITIO 2

6. *Pro data relatione inter binas variabiles quantitates utraque earum variari dicitur, si utraque seorsim incremento infinite parvo augeri concipiatur; unde patet, quomodo intelligendum sit, si utrique variabili sua tribuatur variatio.*

EXPLICATIO

7. Si proposita sit aequatio quaecunque inter binas variabiles x et y, qua earum relatio mutua exprimitur, haec relatio per definitionem duplici modo variari potest, altero, quo manentibus valoribus x singulis y variatio tribuitur, altero vero, quo manentibus valoribus y singuli x variari concipiuntur. Nihil igitur prohibet, quominus utraque variabilis simul suas variationes recipere intelligatur, quas adeo ita capere licet, ut nullo plane nexu inter se cohaereant; duplex ergo hic variatio consideratur, cum in definitione prima unica tantum sit admissa. Rem autem hic ita generaliter contemplamur, ut neutra variatio ulli legi sit adstricta neque etiam variationes ipsius y ullo modo a variationibus ipsius x pendeant.

COROLLARIUM 1

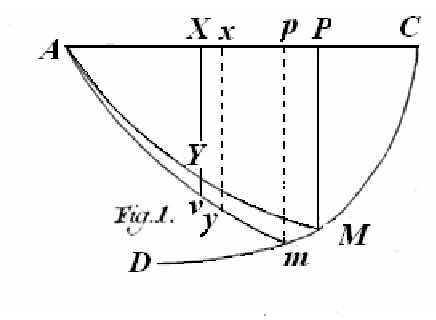
8. Ex casu ergo, quo duplex variatio statuitur, casus prior tanquam species nascitur, si variationes alterius variabilis plane reiiciantur; unde manifestum est casum definitionis secundae in se complecti casum primae.

COROLLARIUM 2

9. Hinc magis elucet, quemadmodum data relatio inter binas variabiles infinitis modis variari possit, simulque intelligitur, quoniam has variationes nulli legi adstrictas assumimus, omnes omnino illius relationis variationes possibiles hac ratione indicari.

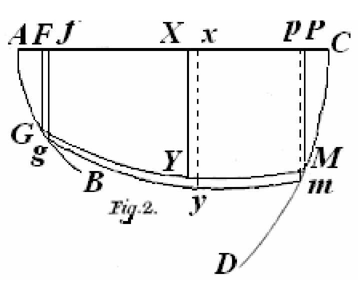
SCHOLION 1

10. Variationes quidem alterutri tantum variabili inductae iam omnes variationes possibiles, quae in propositam relationem inter binas variabiles cadere possunt, comprehendunt, ut superfluum videri possit calculum ad duplicem variationem accommodari; verum si indolem rei usumque, cui destinatur, attentius contemplemur, duplicis variationis consideratio neutiquam supervacanea deprehendetur, id quod per Geometriam evidentissime sequentem in modum illustrabitur.



Cum relatio quaecunque inter binas variabiles distinctissime per lineam curvam in plano descriptam repraesentetur, sit *AYM* (Fig. 1) linea curva aequatione inter coordinatas $AX= x$ et $XY = y$ definita, quae ergo datam illam relationem exhibeat; iam igitur quaelibet linea curva alia *Aym* ab illa infinite parum discrepans relationem illam variatam repraesentabit, quae, quomodocunque se habeat, semper ita considerari potest, ut eidem abscissae $AX = x$ conveniat applicata variata X*v* existente particula *Yv* eius variatione; quae consideratio quoque pro plerisque circa maxima et minima prolatis quaestionibus sufficit, ubi adeo curva *AM* in nonnullis tantum elementis variari solet concipi. At si quaestio ita sit comparata, ut inter omnes curvas, quas a dato puncto *A* ad datam quampiam curvam *CD* usque ducere licet, ea definiatur *AYM*, cui maximi minimive proprietas quaedam conveniat, tum eadem proprietas in aliam quamcunque curvam proximam *Aym* etiam in alio lineae *CD* puncto *m* terminatam aeque competere debet sicque pro ultimo curvae quaesitae puncto *M* tam abscissa *AP* quam applicata *PM* variationem recipere est censenda et huiusmodi quidem, quae naturae lineae *CD* sit consentanea. Quo igitur calculus ad talem variationem ultimo elemento inductam accommodari queat, omnino necesse est, ut pro singulis curvae *AM* punctis intermediis *Y* generalissime tam abscissae $AX = x$ quam applicatae $XY = y$ variationes tribuantur quaecunque illiusque variatio statuatur particula *Xx,* huius vero $= xy - XY$, ex quo indoles simulque usus huiusmodi duplicis variationis clarissime perspicitur.

SCHOLION 2



11. Quemadmodum consideratio ultimi puncti curvae investigandae nobis hanc insignem dilucidationem suppeditavit, ita etiam subinde primo puncto variationem tribui oportet. Veluti si inter omnes lineas, quas a data quadam curva *AB* (Fig. 2) ad aliam quandam itidem datam *GD* ductas concipere licet, ea sit quaerenda, quae maximi minimive cuiuspiam proprietate sit praedita, tum multo magis erit necessarium tam singulis abscissis *AX* quam applicatis X *Y* variationes quascunque nulla lege adstrictas in calculo assignari, ut deinceps tam ad initii *G* curvae quaesitae quam eius finis *M* variationem transferri possint. Quanquam autem haec illustratio ex Geometria est desumta, tamen facile intelligitur ideam variationum inde petitam multo latius patere atque in Analysi absoluta summo usu non esse carituram. Celeberrimus autem DE LA GRANGE, acutissimus Geometra Taurinensis, cui primas speculationes de calculo variationum acceptas referre debemus, hanc methodum adeo ingeniosissime transtulit ad lineas non continuas veluti ad polygonorum genus referendas, in quo negotio hae duplices variationes ipsi summam praestiterunt utilitatem.

DEFINITIO 3

12. *Relatio inter tres variabiles duabus aequationibus determinata variari dicitur, si earum vel una vel duae vel omnes tres particulis infinite parvisaugeantur, quae earum variationes appellantur.*

EXPLICATIO

13. Cum tres proponantur variabiles quantitates veluti x, y et z, inter quas duae aequationes dari concipiuntur, ex unaquaque earum binas reliquas determinare licet, ita ut tam y quam z tanquam functio ipsius x spectari possit. Hoc autem modo definiri solet linea curva non in eodem plano descripta, dum singula eius puncta per has ternas coordinatas x, y et z more solito assignantur. Quodsi iam talis curva alia quacunque sibi proxima comitetur, ut differentia sit infinite parva, haec nova curva propositae erit variata ac relatio illa inter ternas variabiles x, y, z variata eius naturam exprimere est concipienda. Ex quo, prout bina puncta proxima, alterum in ipsa curva proposita, alterum in variata comitante assumtum, inter se comparantur, fieri potest, ut pro variata vel omnes tres coordinatae prodeant diversae vel duae tantum vel saltern unica, harumque differentiae a coordinatis principalis curvae earum variationes repraesentabunt; quas autem hic ita generalissime contemplari convenit, ut ad omnes omnino curvas proximas extendantur, sive eae per totum tractum a curva proposita fuerint diversae sive tantum in quibusdam portionibus ab ea aberrent, ita ut etiam lineae non continuae, dummodo principali sint proximae, hinc non excludantur. Neque enim hae curvae variatae ubi continuitatis legi sunt adstringendae, ut omnes plane curvas possibiles infinite parum a principali aberrantes in se complectantur.

COROLLARIUM 1

14. Cum puncto ergo quocunque curvae propositae seu principalis comparatur punctum quodpiam curvae variatae infinite parum ab illo dissitum hincque coordinatarum variationes definiri intelliguntur.

COROLLARIUM 2

15. Quia porro ex assumta variabili una x binae reliquae y et z ideoque punctum curvae propositae determinatur, etiam variationes singularum coordinatarum tanquam functiones ipsius x spectare licet, dummodo earum quantitas ut infinite parva spectetur.

COROLLARIUM 3

16. Tres ergo quascunque functiones ipsius x utcunque inter se diversas concipere licet, quae per factores infinite parvos multiplicatae idoneae erunt ad ternas variationes coordinatarum repraesentandas. Quod idem de ternis quibuscunque variabilibus est tenendum, etiamsi non ad Geometriam referantur.

COROLLARIUM 4

17. Simili quoque modo, si relatio tantum inter duas variabiles proponatur, earum variationes tanquam functiones alterius variabilis spectari possunt, modo sint infinite parvae seu, quod eodem redit, per quantitatem infinite parvam multiplicatae.

SCHOLION 1

18. Consideratio autem geometrica maxime est idonea ad has speculationes illustrandas, quae in genere consideratae nimis abstractae atque etiam vagae videri queant. Casus igitur trium variabilium, quarum relatio duabus aequationibus definiri assumitur, luculentissime per curvam non in eodem plano descriptam explicatur, dum illis variabilibus ternae coordinatae designantur. Quodsi enim de huiusmodi curvis quaestio instituatur, ut inter eas definiatur ea, quae maximi minimive proprietate quapiam sit praedita, necesse est, ut eadem proprietas in omnes alias curvas ab ea infinite parum aberrantes aeque competat, id quod ex variationibus debite in calculum introductis est diiudicandum. Cuinam autem usui summa generalitas in variationibus hic stabilita sit futura, inde intelligere licet, si loco duarum curvarum *AB* et *CD* (Fig. 2) datae sint duae quaecunque superficies, a quarum illa ad hanc eiusmodi lineam curvam duci oporteat, quae maximi minimive quapiam gaudeat proprietate. Tum enim ternarum coordinatarum variationes ita generales considerari oportet, ut curvae quaesitae puncto ad initium in superficiem *AB* translato variationes ibi ad eandem superficiem accommodari possint idque simili modo in fine ad superficiem *CD* fieri queat. Ex quo perspicuum est in genere tres variationes in calculum introduci debere, ut eas tam pro initio quam pro fine curvae investigandae ad superficies terminatrices transferre liceat, quippe quarum indoles in utroque termino relationem mutuam inter variationes determinabit.

SCHOLION 2

19. Quemadmodum hic tres variabiles sumus contemplati, quarum relatio duabus aequationibus determinatur, ita etiam calculus variabilium ad quatuor pluresve extendi potest, siquidem relatio per tot aequationes exprimatur, ut per unicam variabilem reliquae omnes determinationem suam nanciscantur, etiamsi huius casus illustratio non amplius ex Geometria tribus tantum dimensionibus inclusa peti queat, nisi forte tempus in subsidium vocare velimus, fluvium continuum a superficie *AB* ad superficiem *CD* profluentem, sed temporis lapsu iugiter immutatum considerantes, ita ut tum etiam temporis momentum sit assignandum, quo quaepiam fluvii vena a superficie *AB* ad superficiem *CD* porrecta maximi vel minimi proprietate quadam sit praedita. Ad quas variabiles si insuper celeritatis mutabilitatem adiiciamus, haec maiori variationum numero illustrando inservire poterunt. Imprimis autem hinc intelligitur, etiamsi omnes variabiles per unicam determinari assumantur, rationem investigationis tamen ab ea, ubi duae tantum variabiles admittuntur, maxime discrepare, propterea quod singulis suae variationes a reliquis non pendentes tribui debent; neque enim inde, quod inter variabiles ipsas certa quaedam relatio agnoscitur, ideo quoque earum variationes ulli relationi adstrictae sunt censendae; veluti ex casu ante allato manifestum est, ubi curva inter binas superficies *AB* et *CD* porrecta et certa maximi minimive proprietate praedita utique ita est in se determinata, ut sumta coordinatarum una binae reliquae determinentur; nihilo vero minus curvae variatae omnes, quae in omnes plagas ab illa deflectere possunt, pro singulis coordinatis recipiunt variationes neutiquam a se invicem pendentes solo initio ac fine excepto, ubi eas ad datas superficies accommodari oportet.

DEFINITIO 4

20. *Relatio inter ternas variabiles unica aequatione definita, ut una earum aequetur functioni binarum reliquarum, variari dicitur,* si *vel una [vel duae] vel omnes tres illae variabiles particulis infinite parvis augeantur, quae earum variationes vocantur*.

EXPLICATIO

21. Quoniam hic relatio inter ternas variabiles unica aequatione definiri ponitur, duabus pro arbitrio sumtis tertia demum determinatur, ita ut pro functione duarum variabilium sit habenda. Ea ergo relatione non quaedam linea curva, si rem ad figuras transferre velimus, indicatur, sed tota quaedam superficies, cuius natura aequatione inter ternas coordinatas exprimitur; ex quo intelligitur eadem relatione variata aliam superficiem ab illa infinite parum dissidentem repraesentari, quae variatio ita latissime patere debet, ut variatio vel tantum ad quampiam superficiei portionem restringi vel per totam extendi possit. Prout igitur cum quovis superficiei datae puncto aliud punctum superficiei variatae illi quidem proximum comparatur, fieri potest, ut non solum trium coordinatarum una, sed etiam duae vel adeo omnes tres varientur; unde, quo tractatio in omni amplitudine instituatur, conveniet statim singulis coordinatis suas tribui variationes, quas propterea ita comparatas esse oportet, ut tanquam functiones binarum variabilium spectari possint, cum binis demum determinatis superficiei punctum determinetur.

COROLLARIUM 1

22. Si igitur tres variabiles seu coordinatae sint x, y et z, quemadmodum ex relatione binis x et y pro lubitu valores tribuere licet, unde z valorem determinatum obtineat, itidem variatio ipsius z ab utraque illarum x et y pendere censenda est, quandoquidem, sive alterutra sive ambae mutentur, alia variatio ipsius z resultare debet.

COROLLARIUM 2

23. Quod hic de variatione unius z observatum est, perinde de binis reliquis est intelligendum, ita ut singularum variationes sint tanquam functiones binarum variabilium considerandae; quoniam vera inter x et y et z aequatio datur, perinde est, quarumnam binarum functiones concipiantur, quia functio ipsarum y et z per aequationem ad functionem ipsarum x et y revocari potest, si scilicet loco z suus valor per x et y expressus substituatur.

SCHOLION 1

24. Hac variationum institutione erit utendum, si superficies fuerit investiganda, quae maximi minimive quapiam proprietate sit praedita, quandoquidem calculum tum ita instrui oportet, ut eadem proprietas in superficies illi proximas ac variatas aeque competat. Deinde cum in curvis maximi minimive proprietate praeditis amborum terminorum ratio praescribi soleat, ut vel in datis punctis vel ad datas lineas curvas vel adeo superficies terminentur, similis conditio hic est admittenda, ut superficies quaerenda circumquaque definiatur vel data quadam superficie circumscribatur; cuius posterioris conditionis ut ratio haberi possit, omnino necesse est, ut omnibus tribus coordinatis variationes generalissimae a se invicem neutiquam pendentes tribuantur, quo eae deinceps in extrema ora ad naturam superficiei terminantis accommodari queant. Hic quidem fatendum est methodum maximorum et minimorum vix adhuc ad huiusmodi investigationes esse promotam tantasque difficultates hic occurrere, ad quas superandas multo maiora Analyseos incrementa requiri videntur. Verum ob hanc ipsam causam eo magis erit enitendum, ut principia huius methodi, quae calculo variationum continentur, solide stabiliantur simulque clare ac distincte proponantur.

SCHOLION 2

25. Vix opus esse arbitror hic animadvertere istum calculum simili modo ad plures tribus variabiles amplificari posse, etiamsi quaestiones geometricae non amplius dilucidationem suppeditent; ipsa enim Analysis non uti Geometria certo dimensionum numero limitari est censenda. Quando autem plures variabiles considerantur, ante omnia perpendi convenit, utrum earum relatio mutua unica tantum aequatione exprimatur an pluribus; quae tot esse possunt, ut multitudo unitate tantum a numero variabilium deficiat, quo casu omnes tanquam functiones unius spectare licet. Sin autem paucioribus aequationibus constet relatio, singulae variabiles erunt functiones duarum pluriumve variabilium et quolibet quoque casu variationes singulis tributae tanquam functiones totidem variabilium tractari debent, siquidem hunc caIculum generalissime expedire velimus.

DEFINITIO 5

26. *Calculus variationum est methodus inveniendi variationem, quam recipit expressio ex quotcunque variabilibus utcunque conflata, dum variabilibus vel omnibus vel aliquibus variationes tribuuntur.*

EXPLICATIO

27. In hac definitione nulla fit mentio relationis, quam hactenus inter variabiles dari assumsimus; cumenim hic calculus potissimum in hac ipsa relatione investiganda sit occupatus, quae scilicet maximi minimive proprietate sit praedita, quamdiu ea adhuc est incognita, eius rationem in calculo neutiquam habere licet, sed potius eum ita tractari convenit, quasi variabiles nulla plane relatione inter se essent connexae. Calculum igitur ita instrui convenit, ut, si singulis variabilibus, quae in calculum ingrediuntur, variationes tribuantur quaecunque, omnis generis expressionum, quae utcunque ex iis fuerint conflatae, variationes inde oriundae investigari doceantur; quibus in genere inventis tum demum eiusmodi quaestiones evolvendae occurrunt, qualem relationem inter variabiles statui oporteat, ut variatio illa inventa sit vel nulla, uti in investigatione maximorum seu minimorum usu venit, vel alio certo quodam modo sit comparata, prout natura quaestionum exegerit. Hoc modo si istius calculi praecepta tradantur, nihil impedit, quominus etiam eiusmodi quaestiones tractentur, in quibus statim relatio quaedam inter variabiles tanquam data assumitur ac certae cuiusdam expressionis ex iis formatae variatio ex variabilium variationibus nata desideratur. Ex quo intelligitur hunc calculum ad quaestiones plurimas diversissimi generis accommodari posse.

COROLLARIUM 1

28. Quaestiones ergo in hoc calculo tractandae huc redeunt, ut proposita expressione quacunque ex quotcunque variabilibus utcunque conflata eius incrementum definiatur, si singulae variabiles suis variationibus augeantur.

COROLLARIUM 2

29. Similis igitur omnino est calculus variationum calculo differentiali, dum in utroque variabilibus incrementa infinite parva tribuuntur. Quatenus autem, uti iam observavimus [§ 3, 4], variationes a differentialibus discrepant adeoque simul cum iis consistere possunt, eatenus summum discrimen inter utrumque calculum est agnoscendum.

SCHOLION.

30. Ex observationibus supra allatis discrimen hoc maxime fit manifestum; ubi enim calculus refertur ad lineam curvam, quam cum alia sibi proxima comparari oportet, per differentialia a puncto quovis curvae ad alia puncta eiusdem curvae progredimur; quando autem ab hac curva ad alteram; sibi proximam transilimus, transitus, quatenus est infinite parvus, fit per variationes. Idem evenit in superficiebus ad alias sibi proximas relatis, ubi differentialia in eadem superficie concipiuntur, variationibus vero ab una in alteram transilitur. Eadem omnino est ratio, si res analytice consideretur sine ullo respectu ad figuras geometricas, ubi semper variationes quantitatum variabilium a suis differentialibus sollicite distingui oportet, quem in finem variationes signo diverso indicari conveniet.

HYPOTHESIS .

31. Variationem cuiusque quantitatis variabilis littera δ *eidem quantitati praefixa in posterum designabimus, ita ut* $\delta x, \delta y, \delta z$ *designent variationes quantitatum x, y, z, ac si V fuerit expressio quaecunque ex iis conflata, eius variatio hoc modo* δV *nobis indicabitur.*

COROLLARIUM 1

32. Significat ergo δx incrementum illud infinite parvum, quo quantitas *x* augeri concipitur, ut eiusdem valor variatus prodeat, ex quo vicissim intelligitur valorem variatum ipsius *x* fore $x+\delta x$.

COROLLARIUM 2

33. Quatenus ergo expressio *V* ex variabilibus *x*, *y* et *z* conflatur, si earum loco scribantur valores variati $x+\delta x$, $y+\delta y$ et $z+\delta z$ atque a valore hoc modo pro *V* resultante subtrahatur ipsa *V* residuum erit variatio dV .

COROLLARIUM 3

34. Hactenus ergo omnia perinde se habent atque in calculo differentiali, ac si *V* fuerit functio quaecunque ipsarum *x*, *y* et *z*, sumto eius differentiali more solito tantum ubique loco *d* scribatur δ et habebitur eius variatio δV .

SCHOLION 1

35. Quoties ergo V est functio quaecunque quantitatum variabilium *x, y, z,* eius variatio iisdem regulis inde elicitur ac differentiale eius, ex quo calculus variationum prorsus cum calculo differentiali congruere videri posset, cum sola signi diversitas levis sit momenti. Verum probe perpendendum est hic non omnes quantitates, quarum variationes requiruntur, in genere functionum comprehendi posse; quamobrem etiam in definitione [§ 26] vocabulo *expressionis* sum usus, cui longe ampliorem significatum attribuo. Quatenus enim ad relationem mutuam variabilium respicere non licet, quia est incognita, eatenus eiusmodi expressiones seu formulae, in quas variabilium differentialia atque etiam integralia ingrediuntur, non amplius tanquam merae functiones variabilium spectari possunt ac formularum tam differentialium quam integralium variatio peculiaria praecepta postulat; sicque totum negotium huc redit, ut, quemadmodum formularum utriusque generis variationes investigari conveniat, doceamus, ex quo tractatio nostra evadit bipartita.

SCHOLION 2

36. In ipsa autem tractatione maximum exoritur discrimen ex numero variabilium, qui si binarium superet, vix adhuc perspicitur, quomodo calculus sit expediendus. Cum enim pluribus introductis variabilibus etiam differentialium consideratio longe aliter expendatur, dum plerumque binarum tantum differentialia ita inter se comparari solent, quasi reliquae variabiles manerent constantes, similis quoque ratio in variationibus erit habenda, in quo etiamnunc tantae difficultates occurrunt, ut vix pateat, quomodo eas superare liceat; ante omnia certe prima huius calculi principia accuratissime evolvi erit necesse, ut ex intima rei natura calculi praecepta repetantur, in quo plerumque summae difficultates offendi solent. Primum igitur hunc calculum ad duas tantum variabiles acconlmodatum, quemadmodum is quidem adhuc tractari est solitus, explicare conabor variationes tam formularum differentialium quam integralium investigaturus; tum vero, si quid lucis ex ipsa hac tractatione affulserit, quoque ad tres pluresve variabiles contemplandas progrediar.