CHAPTER VI

CONCERNING THE VARIATION OF INTEGRAL FORMULAS INVOLVING THREE VARIABLES THE RELATION OF WHICH IS CONTAINED BY A SINGLE EQUATION

PROBLEM 15

141. *With an equation proposed between the three variables x, y and z, for which some variations* $\delta x,\ \delta y,\ \delta z$ *are given, to define the variations of the differential formula of a first order.*

SOLUTION

Since a single equation is to be given to put in place between the three variables, any of these can be regarded as a function of the two remaining. Therefore *z* will be a function of *x* and *y* themselves and it may be required here to bear in mind the expression $\left(\frac{dz}{dx}\right)=p$ to denote the ratio of the differentials of *z* and *x* themselves, if in that given equation these only are to be treated as variables with the third *y* considered as constant, which likewise is required to be understood concerning the other formula $p'=\left(\frac{dz}{dy}\right)$. In a similar manner also the variations themselves $\delta x,\ \delta y,\ \delta z$ are to be regarded as infinitely small functions of the two variables *x* and *y*, because, if they should depend on the third *z*, this itself is a function of *x* and *y* themselves; from which in a like manner it is understood, which these formulas,

$$\left(\frac{d\delta z}{dx}\right),\quad \left(\frac{d\delta z}{dy}\right),\quad \left(\frac{d\delta x}{dx}\right),\quad \left(\frac{d\delta x}{dy}\right),\quad \text{and}\quad \left(\frac{d\delta y}{dx}\right),\quad \left(\frac{d\delta y}{dy}\right)$$

likewise signify. Therefore since the value of the variation of the formula $\left(\frac{dz}{dx}\right)=p$ shall be

$$p+\delta p=\left(\frac{d(z+\delta z)}{d(x+\delta x)}\right),$$

evidently if here the variable *y* is taken constant, there will be with this condition observed

$$p+\delta p=\left(\frac{d(z+\delta z)}{d(x+\delta x)}\right)=\left(\frac{dz}{dx}+\frac{d\delta z}{dx}-\frac{dzd\delta x}{dx^2}\right),$$

[since

$$p=\frac{dz}{dx} \text{ and } \delta p=(p+\delta p)-p=\left(\frac{d(z+\delta z)}{d(x+\delta x)}-\frac{dz}{dx}\right)$$
$$=\left(\frac{dxd(z+\delta z)-dzd(x+\delta x)}{dxd(x+\delta x)}\right)=\left(\frac{dxdz+dxd\delta z-dzdx-dzd\delta x}{dxdx+d\delta x}\right) \quad \text{etc.]}$$

because therefore the variations δdx and δdz vanish before dx and dz [here we have adopted the correction made by the editor in the O. O. edition]. Therefore hence on account of $\left(\frac{dz}{dx}\right)=p$, the variation sought of the indicated formulas will be had

$$\delta p=\left(\frac{d\delta z}{dx}\right)-\left(\frac{dz}{dx}\cdot\frac{d\delta x}{dx}\right)=\left(\frac{d\delta z}{dx}\right)-p\left(\frac{d\delta x}{dx}\right),$$

since both δz as well as δx shall be functions of x and y and here y may be considered as constant. Moreover in a similar manner there may be found to become

$$\delta p'=\left(\frac{d\delta z}{dy}\right)-p'\left(\frac{d\delta y}{dy}\right),$$

where now the variable x may be considered constant.

COROLLARY 1

142. Here everything has adopted the two variables x and y and as not only the third z, but also all the three variations δx, δy, δz may be considered as functions of these ; moreover it is evident that these three variables can be interchanged among themselves as it pleases.

COROLLARY 2

143. But it is sufficient with these two formulas for differentials of the first order to be used, since one can reduce the others to this, if indeed there shall be

$$\left(\frac{dx}{dz}\right)=\frac{1}{p},\quad \left(\frac{dy}{dz}\right)=\frac{1}{p'},\quad \left(\frac{dy}{dx}\right)=\frac{-p}{p'} \quad \text{and} \quad \left(\frac{dx}{dy}\right)=\frac{-p'}{p},$$

where p and p' are functions of the two x and y.

COROLLARIUM 3

144. Therefore with the variations of these two formulas found

$$p=\left(\frac{dz}{dx}\right) \quad \text{and} \quad p'=\left(\frac{dz}{dy}\right)$$

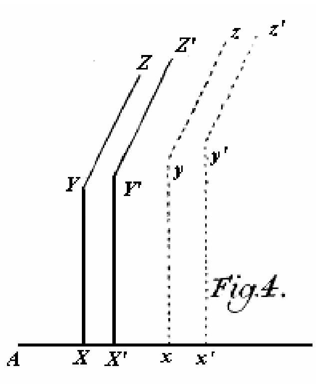
in the same manner the variations of the remaining formulas mentioned hence can easily be found in the same manner.
For there will be

$$\delta\left(\frac{dx}{dz}\right) = -\frac{\delta p}{pp} = -\frac{1}{pp}\left(\frac{d\delta z}{dx}\right) + \frac{1}{p}\left(\frac{d\delta x}{dx}\right),$$
$$\delta\left(\frac{dy}{dz}\right) = -\frac{\delta p'}{p'p'} = -\frac{1}{p'p'}\left(\frac{d\delta z}{dy}\right) + \frac{1}{p'}\left(\frac{d\delta y}{dy}\right),$$
$$\delta\left(\frac{dy}{dx}\right) = -\frac{\delta p}{p'} + \frac{p\delta p'}{p'p'} = -\frac{1}{p'}\left(\frac{d\delta z}{dx}\right) + \frac{p}{p'}\left(\frac{d\delta x}{dx}\right) + \frac{p}{p'p'}\left(\frac{d\delta z}{dy}\right) - \frac{p}{p'}\left(\frac{d\delta y}{dy}\right).$$

SCHOLIUM 1

145. Here before all I note that the differential formulas are not able to have a certain value, unless the two differentials may thus be prepared between themselves, so that the third variable, if three should be considered, or all the rest, if more were present, are taken as constants. Thus in this case, in which a single equation is given between the three variables x, y and z or at least be considered to be given, the formula $\frac{dz}{dx}$ clearly is of no significance, unless the third variable y is taken constant, as these derivatives are usually indicated by including this formula in brackets, even if that can be omitted without risk, because indeed in that case no other variable indicated will be present. So that which may be returned more clearly, some equation may be proposed between the three variables x, y, z, from that the value of z itself may be considered to be elicited, so that z may be equal to a certain function of x and y and with the differential of this taken there may be produced $dz = pdx + p'dy$, where in turn p and p' will be certain functions of x and y themselves and that such that there shall be $\left(\frac{dp}{dy}\right) = \left(\frac{dp'}{dx}\right)$. Now on taking y constant there becomes $dz = pdx$ or $p = \left(\frac{dz}{dx}\right)$, but on taking x constant there will arise $p' = \left(\frac{dz}{dy}\right)$. Then truly also it is clear with z taken constant to become $\frac{dy}{dx} = \frac{-p}{p'}$; but it may be agreed to exclude formulas of this kind, when we may represent both z as well as the variations δx, δy, and δz as functions of x and y themselves.

SCHOLIUM 2



146. One may illustrate this argument much more clearly from geometry. For our three variables x, y, z may specify the three coordinates AX, XY, YZ (Fig. 4), between which the proposed equation will be assigned some certain surface, on which the ordinate $YZ = z$ will be defined, which certainly can be regarded as a certain function of the two remaining variables $AX = x$ and $XY = y$, thus so that with these two taken x and y as it pleases, the third $YZ = z$ may be determined from the proposed equation. Because if now some other surface may be considered differing from that by an infinitesimal amount and this thus compared with that, so that any point z of this may be brought together with the proposed point Z, yet thus, so that the interval Zz shall always be infinitely small, the variations may be represented thus, so that there shall be

$$\delta x = Ax - AX = Xx,\ \delta y = xy - XY \text{ et } \delta z = yz - YZ;$$

and since these variations are allowed wholly by our choice they may not depend on each other in a single way, these also can be regarded as functions of the two *x* and *y* and that thus, so that it may not depend on the rest, but each can be produced arbitrarily. So that hence it is understood also, because the nearby surface must be different from the proposed surface, by no means does there become

$$\delta z = p\delta x + p'\delta y,$$

if indeed for the proposed surface there should be

$$dz = pdx + p'dy;$$

otherwise the point *z* would be on the same surface, from which in general thus it is required to prepare the three functions of x and *y* themselves with the variations δx, δy, et δz, so that there shall not be

$$\delta z = p\delta x + p'\delta y,$$

but rather it shall disagree with this value in some way ; where indeed from the first place it is to be noted thus that these functions be prepared more widely, so that discontinuities may not be excluded and thus for argument's sake the variations only at a single point or at any rate are able to be set up at a little distance. But lest there should be a place for any doubt remaining here, it is to be noted well from this, because we have put *z* a function of this kind of *x* and *y*, so that there shall be

$$dz = pdx + p'dy,$$

it does not follow at all also that

$$\delta z = p\delta x + p'\delta y,$$

just as we have assumed above, because here therefore we have attributed the variation of *z* itself not to be depending in any way on the variations of *x* and *y*.

PROBLEM 16

147. *With an equation proposed between the three variables x, y, z, to which some variations* δx, δy, δz *are attributed, to investigate the variations of the differential formulas of the second orders*

$$q=\left(\frac{ddz}{dx^2}\right),\quad q'=\left(\frac{ddz}{dxdy}\right),\quad \textit{and}\quad q''=\left(\frac{ddz}{dy^2}\right)$$

SOLUTION

Here again *z* may be considered as a function of *x* and *y*, of which also the three variation functions δx, δy, δz in no manner are depending on each other in turn. Because in the preceding problem we have put

$$p=\left(\frac{dz}{dx}\right)\quad \text{and}\quad p'=\left(\frac{dz}{dy}\right)$$

we may consider the aid by calling these formulas

$$q=\left(\frac{dp}{dx}\right),\quad p'=\left(\frac{dp}{dy}\right)=\left(\frac{dp'}{dx}\right)\quad \text{and}\quad q''=\left(\frac{dp'}{dy}\right)$$

and here an account of the variations δp and $\delta p'$ is required to be considered, which we have found

$$\delta p=\left(\frac{d\delta z}{dx}\right)-p\left(\frac{d\delta x}{dx}\right)\quad \text{and}\quad \delta p'=\left(\frac{d\delta z}{dy}\right)-p'\left(\frac{d\delta y}{dy}\right).$$

Therefore in a similar manner we may find on performing the calculation in the first place

$$\delta q=\left(\frac{d\delta p}{dx}\right)-q\left(\frac{d\delta x}{dx}\right)$$

where $\left(\frac{d\delta p}{dx}\right)$ is found, if the value δp may be differentiated on putting y constant and with the differential divided by *dx*, from which there arises

$$\left(\frac{d\delta p}{dx}\right)=\left(\frac{dd\delta z}{dx^2}\right)-q\left(\frac{d\delta x}{dx}\right)-p\left(\frac{dd\delta x}{dx^2}\right)$$

on account of $q=\left(\frac{dp}{dx}\right)$, from which we infer

$$\delta q=\left(\frac{dd\delta z}{dx^2}\right)-2q\left(\frac{d\delta x}{dx}\right)-p\left(\frac{dd\delta x}{dx^2}\right)$$

In the same manner on account of $q'=\left(\frac{dp}{dy}\right)$ there will be

$$\delta q' = \left(\frac{d\delta p}{dy}\right) - q'\left(\frac{d\delta y}{dy}\right),$$

but

$$\left(\frac{d\delta p}{dy}\right) = \left(\frac{dd\delta z}{dxdy}\right) - q'\left(\frac{d\delta x}{dx}\right) - p\left(\frac{dd\delta x}{dxdy}\right)$$

and thus

$$\delta q' = \left(\frac{dd\delta z}{dxdy}\right) - q'\left(\frac{d\delta x}{dx}\right) - q'\left(\frac{d\delta y}{dy}\right) - p\left(\frac{dd\delta x}{dxdy}\right).$$

Moreover the other value $q' = \left(\frac{dp'}{dx}\right)$ treated in a similar manner gives

$$\delta q' = \left(\frac{dd\delta z}{dxdy}\right) - q'\left(\frac{d\delta x}{dx}\right) - q'\left(\frac{d\delta y}{dy}\right) - p'\left(\frac{dd\delta y}{dxdy}\right).$$

of which value from that inconvenient discrepancy it involves, is required soon to be examined more precisely. But from the third formula $q'' = \left(\frac{dp'}{dy}\right)$ there is elicited

$$\delta q'' = \left(\frac{dd\delta z}{dy^2}\right) - 2q''\left(\frac{d\delta y}{dy}\right) - p'\left(\frac{dd\delta y}{dy^2}\right).$$

SCHOLIUM 1

148. In the origin of the discrepancy of the variation $\delta q'$ sought from the twin value

$$q' = \left(\frac{dp}{dy}\right) = \left(\frac{dp'}{dx}\right)$$

I note that the variation arises from these formulas expressing either the quantity *x* or the quantity *y* to be considered as constant, as the denominator of any member indicates. Truly if we assume the quantity *x* to remain constant, whatever the change arising meanwhile of the other *y*, the nature of the argument demands, that also the variations of *x* itself undergoes no change, because moreover otherwise it comes about, if the variation δx also depends on the quantity *y*, as likewise with the other variable *y* while it is made constant is understood. From which it is evident, if the variations δx and δy likewise are taken to depend on both the variables *x* and *y*, that by itself is opposed to the hypothesis, from which either is always put to remain constant. On account of which this inconvenience cannot otherwise be avoided, unless we put in place the variation of *x* to be completely independent from the other variability *y*, nor the variation of this δy to depend on the other *x*. Moreover, if δx is determined by *x* alone and δy by *y* only, so that there shall be both

$$\left(\frac{d\delta x}{dy}\right) = 0 \quad \text{and} \quad \left(\frac{d\delta y}{dx}\right) = 0,$$

also there will be

$$\left(\frac{dd\delta x}{dxdy}\right)=0 \quad \text{and} \quad \left(\frac{dd\delta y}{dxdy}\right)=0$$

and thus both those disagreeing values found for $\delta q'$ are produced in agreement.

SCHOLIUM 2

149. But with all the doubts in this investigation it may occur to us most happily, if we may attribute variations to the quantity *z* only with the two *x* and *y* remaining clearly invariant, thus so that both $\delta x=0$ as well as $\delta y=0$, with which agreed upon not only is the calculation come to a conclusion, but also the use of this calculus of variations is scarcely restricted. But if indeed we may compare some surface with another close to itself, nothing hinders, why we should not refer the individual proposed points of the surface to those nearly points, to which the same two coordinates *x* and *y* correspond, and only the third variation *z* is permitted. So that also this supposition, when we will progress to integral formulas, there with a greater necessity, always whenever the whole matter leads to integral formulas of this kind, which require a double integration, in the one of which only *x*, in the other truly only *y* may be treated as a variable; therefore unless no variations of these put in place, great disadvantages may thence be brought into the calculation, which since by themselves generally shall be the most difficult, may be considered the least deliberated on, as from this part the difficulties may be multiplied. On account of which thus I am about to bring out this treatment, so that in the future clearly always I will give no variations for the two variables *x* and *y* and I will be assuming only the third variable *z* to be augmented with some variation δz, where indeed I regard δz as some function of *x* and *y* themselves either continuous or discontinuous.

PROBLEM 17

150. *If z were some function of x and y themselves and to that there is attributed the variation δz depending equally in some manner on x and y, to investigate the variations of all the formulas of the differential of any order.*

SOLUTION

For the differentials of the first order these two formulas may be considered

$$p=\left(\frac{dz}{dx}\right) \text{ and } p'=\left(\frac{dz}{dy}\right),$$

the variations of which, since *x* and *y* are permitted no variation, thus they will be had from what was found above

$$\delta p=\left(\frac{d\delta z}{dx}\right) \text{ and } \delta p'=\left(\frac{d\delta z}{dy}\right).$$

For the differentials of the second order these three formulas are considered

$$q=\left(\frac{ddz}{dx^2}\right),\ q'=\left(\frac{ddz}{dxdy}\right),\ \text{and}\quad q''=\left(\frac{ddz}{dy^2}\right),$$

thus so that there shall be

$$q=\left(\frac{dp}{dx}\right),\ q'=\left(\frac{dp}{dy}\right)=\left(\frac{dp'}{dx}\right)\ \text{ and }\quad q''=\left(\frac{dp'}{dy}\right),$$

the variations of which from the preceding problem on account of $\delta x=0$ and $\delta y=0$ are

$$\delta q=\left(\frac{dd\delta z}{dx^2}\right),\ \delta q'=\left(\frac{dd\delta z}{dxdy}\right),\quad \delta q''=\left(\frac{dd\delta z}{dy^2}\right).$$

In a similar manner if we should rise to differentials of the third order, these four formulas occur

$$r=\left(\frac{d^3z}{dx^3}\right),\quad r'=\left(\frac{d^3z}{dx^2dy}\right),\quad r''=\left(\frac{d^3z}{dxdy^2}\right),\quad r'''=\left(\frac{d^3z}{dy^3}\right),$$

of which the variations thus expressed it is evident go to

$$\delta r=\left(\frac{d^3\delta z}{dx^3}\right),\quad \delta r'=\left(\frac{d^3\delta z}{dx^2dy}\right),\quad \delta r''=\left(\frac{d^3\delta z}{dxdy^2}\right),\quad \delta r'''=\left(\frac{d^3\delta z}{dy^3}\right),$$

from which it may be apparent by itself, how the variations of the differential formulas of higher orders may be required to be expressed.

COROLLARY 1

151. Hence now it is to be evident in general for the differential formula of any order $\left(\frac{d^{\mu+\nu}z}{dx^\mu dy^\nu}\right)$ the variation of this $=\left(\frac{d^{\mu+\nu}\delta z}{dx^\mu dy^\nu}\right)$, in which all the above forms may be contained.

COROLLARY 2

152. From this in turn also it is evident in place of the differentials of the first order with the letters introduced p, p', of the second order with the letters q, q', q'', of the third order with the letters r, r', r'', r''', of the fourth order with the letters $s, s', s'', s''', s^{\text{IV}}$ etc. the kind of the differentials to be taken, just as also with the letters of this kind above we have supported a kind of differentials.

SCHOLIUM

153. Because the two variables x and y in turn are completely independent from each other, thus so that the one may retain the same value, while the other is varied by all the values possible, it being evident that the differential form of this kind $\frac{dy}{dx}$, certainly which clearly is to have no special significance, at no time can find a place in the calculation. Now otherwise , since the quantity z shall be a function of x and y, these formulas $\frac{dz}{dx}$, $\frac{dz}{dy}$ and all the rest, which I have considered above, have a definite significance nor any others can enter into the calculation. Then because here one is allowed always to reduce pertaining questions from that, so that z can be considered as a function of the two x and y, formulas of this kind $\frac{dy}{dx}$, where the quantity z may be had for a constant, hence are completely excluded nor any other besides the above mentioned may be agreed to be admitted into the calculation; and thus all the expressions besides the variables x, y, z free from the integral formulas do not implicate other differential formulas except these, the variations of which have been indicated here.

PROBLEM 18

154. *If z shall be a function of x and y and to that there may be attributed some variation δz depending on x and y, then indeed were V some quantity composed from the three variables x, y, z and of their differentials of any order, to investigate the variation δV of this.*

SOLUTION

So that in the expression V the kinds of differentials are raised, we may put, as we have used up to this stage,

$$p=\left(\frac{dz}{dx}\right),\quad p'=\left(\frac{dz}{dy}\right),$$

$$q=\left(\frac{ddz}{dx^2}\right),\quad q'=\left(\frac{ddz}{dxdy}\right),\quad q''=\left(\frac{ddz}{dy^2}\right),$$

$$r=\left(\frac{d^3z}{dx^3}\right),\quad r'=\left(\frac{d^3z}{dx^2dy}\right),\quad r''=\left(\frac{d^3z}{dxdy^2}\right),\quad r'''=\left(\frac{d^3z}{dy^3}\right)$$

etc.,

the variations of which formulas arising from the variation of z we may define thus, as on putting that variation on account of distinctness $\delta z=\omega$, so that as it is required to consider any function of the two variables x and y, there shall be

$$\delta p = \left(\tfrac{d\omega}{dx}\right), \quad \delta p' = \left(\tfrac{d\omega}{dy}\right),$$
$$\delta q = \left(\tfrac{dd\omega}{dx^2}\right), \ \delta q' = \left(\tfrac{dd\omega}{dxdy}\right), \quad \delta q'' = \left(\tfrac{dd\omega}{dy^2}\right),$$
$$\delta r = \left(\tfrac{d^3\omega}{dx^3}\right), \quad \delta r' = \left(\tfrac{d^3\omega}{dx^2dy}\right), \quad \delta r'' = \left(\tfrac{d^3\omega}{dxdy^2}\right), \quad \delta r''' = \left(\tfrac{d^3\omega}{dy^3}\right)$$
etc.,

But with these substitutions made the proposed expression V becomes a function of these quantities $x, y, z, p, p', q, q', q'', r, r', r'', r'''$ etc. Therefore the differential of this adopts such a form

$$\begin{aligned} dV = Ldx + Mdy + Ndz + Pdp &+ Qdq &+ Rdr \\ + P'dp' &+ Q'dq' &+ R'dr' \\ &+ Q''dq'' &+ R''dr'' \\ && + R'''dr''' \end{aligned}$$
etc.

Now because V so far only takes a variation, as far as it may be varied from the quantities from which it is composed, but the two x and y are put in place unchanged, the variation of this, as we request, will be

$$\begin{aligned} \delta V = N\delta z + P\delta p &+ Q\delta q &+ R\delta \\ + P'\delta p' &+ Q'\delta q' &+ R'\delta r' \\ &+ Q''dq'' &+ R''\delta r'' \\ && + R'''\delta r''' \end{aligned}$$
etc.,

and if in place of the variation δz we may write ω, we will have the variations found on substituting

$$\begin{aligned} \delta V = N\omega + P\left(\tfrac{d\omega}{dx}\right) &+ Q\left(\tfrac{dd\omega}{dx^2}\right) &+ R\left(\tfrac{d^3\omega}{dx^3}\right) \\ + P'\left(\tfrac{d\omega}{dy}\right) &+ Q'\left(\tfrac{dd\omega}{dxdy}\right) &+ R'\left(\tfrac{d^3\omega}{dx^2dy}\right) \\ &+ Q''\left(\tfrac{dd\omega}{dy^2}\right) &+ R''\left(\tfrac{d^3\omega}{dxdy^2}\right) \\ && + R'''\left(\tfrac{d^3\omega}{dy^3}\right) \end{aligned}$$
etc.,

the formation of which, if perhaps also the differentials of higher orders are present, by themselves becomes evident.

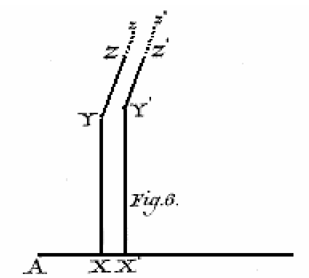
COROLLARY 1

155. Since ω may be seen as a function of the two variables *x* and *y*, of the individual parts which constitute the variation δV , the significance has been determined and this variation may be agreed to be defined perfectly.

COROLLARY 2

156. But in whatever manner the expression *V* shall be provided with differentials, whenever a certain value to be indicated has been agreed upon, with the substitutions used it must be free from a kind of differentials.

COROLLARY 3



157. If our three variables refer to a surface, so that the coordinates of this shall be $AX = x$, $XY = y$, $YZ = z$ (Fig. 6), only the ordinate $YZ = z$ is understood to take an infinitely small increment everywhere $Zz = \delta z = \omega$, thus so that the points *z* may fall on another surface disagreeing with that by an infinitely small amount.

SCHOLIUM

158. Here this must occur with doubt arising, because we have said the quantity *z* be regarded as a function of the two *x* and *y*; for since we have attributed no variations to *x* and *y* themselves, if in the expression *V* in place of *z* its value may be substituted in terms of *x* and *y*, that itself will change into a separate function of *x* and *y*, nor therefore is it to undertake any variation. Indeed it is to be noted, even if *z* may be considered as a function of *x* and *y*, yet that generally to be unknown, when evidently the nature of this finally may be elicited from the condition of the variation; but if now from the beginning is may be given, yet, as long as the variation is sought, this function *z* is agreed to be considered as if unknown and one is not allowed at all to substitute a value in place of this expressed by *x* and *y*, before the variation, to be sure which depends on *z* only, should have been explored thoroughly.

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CAPUT VI

DE VARIATIONE FORMULARUM DIFFERENTIALIUM TRES VARIABILES INVOLVENTIUM QUARUM RELATIO UNICA AEQUATIONE CONTINETUR

PROBLEMA 15

141. *Proposita aequatione inter tres variabiles x, y et z, quibus variationes quaecunque* $\delta x,\ \delta y,\ \delta z$ *tribuuntur, definire variationes formularum differentialium primi gradus.*

SOLUTIO

Cum unica aequatio inter tres variabiles dari ponitur, quaelibet earum tanquam functio binarum reliquarum spectari potest. Erit ergo *z* functio ipsarum *x* et *y* et meminisse hic oportet expressionem $\left(\frac{dz}{dx}\right)=p$ denotare rationem differentialium ipsarum *z* et *x*, si in aequatione illa data hae solae ut variabiles tractentur tertia *y* pro constante habita, quod idem de altera formula $p'=\left(\frac{dz}{dy}\right)$ est tenendum. Simili modo ipsae quoque variationes $\delta x,\ \delta y,\ \delta z$ ut functiones infinite parvae binarum variabilium *x* et *y* spectari possunt, quoniam, si etiam a tertia *z* penderent, haec ipsa est functio ipsarum *x* et *y;* unde simul intelligitur, quid istae formulae

$$\left(\frac{d\delta z}{dx}\right),\quad \left(\frac{d\delta z}{dy}\right),\quad \text{item}\quad \left(\frac{d\delta x}{dx}\right),\quad \left(\frac{d\delta x}{dy}\right),\quad \text{et}\quad \left(\frac{d\delta y}{dx}\right),\quad \left(\frac{d\delta y}{dy}\right)$$

significant. Cum igitur valor variatus formulae $\left(\frac{dz}{dx}\right)=p$ sit

$$p+\delta p=\left(\frac{d(z+\delta z)}{d(x+\delta x)}\right),$$

si scilicet hic variabilis *y* constans sumatur, erit hac conditione observata

$$p+\delta p=\left(\frac{d(z+\delta z)}{d(x+\delta x)}\right)=\left(\frac{dz}{dx}+\frac{d\delta z}{dx}-\frac{dzd\delta x}{dx^2}\right),$$

propterea quod variationes δdx et δdz prae *dx* et *dz* evanescunt. Hinc ergo ob $\left(\frac{dz}{dx}\right)=p$ habebitur variatio quaesita

$$\delta p=\left(\frac{d\delta z}{dx}\right)-\left(\frac{dz}{dx}\cdot\frac{d\delta x}{dx}\right)=\left(\frac{d\delta z}{dx}\right)-p\left(\frac{d\delta x}{dx}\right),$$

quarum formularum significatus, cum tam δz quam δx sint functiones ipsarum *x* et *y* hicque *y* constans habeatur, per se est manifestus. Simili autem modo reperietur fore

$$\delta p' = \left(\frac{d\delta z}{dy}\right) - p'\left(\frac{d\delta y}{dy}\right),$$

ubi iam variabilis *x* pro constante habetur.

COROLLARIUM 1

142. Hic omnia ad binas variabiles *x* et *y* sunt perducta atque ut earum functiones spectantur non solum tertia *z*, sed etiam omnes tres variationes $\delta x,\ \delta y,\ \delta z$; manifestum autem est has tres variabiles pro lubitu inter se permutari posse.

COROLLARIUM 2

143. Sufficit autem his binis formulis pro differentialibus primi gradus uti, quoniam reliquas ad has reducere licet, siquidem sit

$$\left(\frac{dx}{dz}\right) = \frac{1}{p}, \quad \left(\frac{dy}{dz}\right) = \frac{1}{p'} \quad \text{et} \quad \left(\frac{dy}{dx}\right) = \frac{-p}{p'} \quad \text{et} \quad \left(\frac{dx}{dy}\right) = \frac{-p'}{p},$$

ubi *p* et *p'* sunt functiones binarum *x* et *y*.

COROLLARIUM 3

144. Inventis ergo variationibus harum duarum formularum

$$p = \left(\frac{dz}{dx}\right) \quad \text{et} \quad p' = \left(\frac{dz}{dy}\right)$$

reliquarum formularum modo memoratarum variationes hinc facile reperientur.
Erit enim

$$\delta\left(\frac{dx}{dz}\right) = -\frac{\delta p}{pp} = -\frac{1}{pp}\left(\frac{d\delta z}{dx}\right) + \frac{1}{p}\left(\frac{d\delta x}{dx}\right),$$
$$\delta\left(\frac{dy}{dz}\right) = -\frac{\delta p'}{p'p'} = -\frac{1}{p'p'}\left(\frac{d\delta z}{dy}\right) + \frac{1}{p'}\left(\frac{d\delta y}{dy}\right),$$
$$\delta\left(\frac{dy}{dx}\right) = -\frac{\delta p}{p'} + \frac{p\delta p'}{p'p'} = -\frac{1}{p'}\left(\frac{d\delta z}{dx}\right) + \frac{p}{p'}\left(\frac{d\delta x}{dx}\right) + \frac{p}{p'p'}\left(\frac{d\delta z}{dy}\right) - \frac{p}{p'}\left(\frac{d\delta y}{dy}\right).$$

SCHOLION 1

145. Hic ante omnia observo formulas differentiales certum valorem habere non posse, nisi duo differentialia ita inter se comparentur, ut tertia variabilis, si tres habeantur, seu reliquae omnes, si plures adsint, constantes accipiantur. Ita hoc casu, quo inter tres variabiles *x*, *y* et *z* unica aequatio datur vel saltem dari concipitur, formula $\frac{dz}{dx}$ nullum plane habet significatum, nisi tertia variabilis *y* constans sumatur, quam conditionem vinculis includendo hanc formulam innuere consueverunt, etiamsi ea tuto omitti possent, quoniam alioquin ne ullus quidem significatus adesset. Quod quo magis perspicuum reddatur, quaecunque aequatio inter ternas variabiles x, y, z proponatur, ex ea valor ipsius z elici concipiatur, ut z aequetur certae functioni ipsarum x et y eiusque sumto differentiali prodeat $dz = pdx + p'dy$, ubi iterum p et p' certae erunt functiones ipsarum x et y idque tales, ut sit $\left(\frac{dp}{dy}\right) = \left(\frac{dp'}{dx}\right)$. Sumta nunc y constante fit $dz = pdx$ seu $p = \left(\frac{dz}{dx}\right)$, sumta autem x constante prodit $p' = \left(\frac{dz}{dy}\right)$. Tum vero etiam manifestum est sumta z constante fore $\frac{dy}{dx} = \frac{-p}{p'}$; huiusmodi autem formulas excludi conveniet, quando tam z quam variationes δx, δy, et δz ut functiones ipsarum x et y repraesentamus.

SCHOLION 2

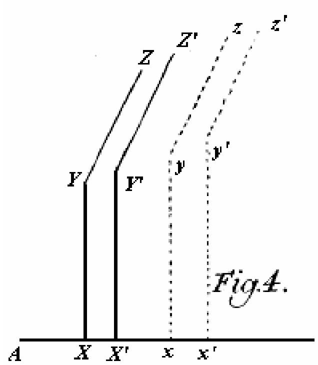


Fig.4.

146. Ex Geometria hoc argumentum multo clarius illustrare licet. Denotent enim tres nostrae variabiles x, y, z ternas coordinatas AX, XY, YZ (Fig. 4), inter quas aequatio proposita certam quandam superficiem assignabit, in qua ordinata $YZ = z$ terminabitur, quae utique tanquam certa functio binarum reliquarum $AX = x$ et $XY = y$ spectari potest, ita ut sumtis pro lubitu his binis x et y tertia $YZ = z$ ex aequatione proposita determinetur. Quodsi iam alia superficies quaecunque concipiatur ab ista infinite parum discrepans eaque ita cum hac comparetur, ut eius punctum quodvis z cum propositae puncto Z conferatur, ita tamen, ut intervallum Zz sit semper infinite parvum, variationes ita repraesentabuntur, ut sit

$$\delta x = Ax - AX = Xx,\ \delta y = xy - XY \text{ et } \delta z = yz - YZ;$$

et cum hae variationes prorsus arbitrio nostro permittantur neque uno modo a se invicem pendeant, eae etiam tanquam functiones binarum x et y spectari possunt idque ita, ut nulla a reliquis pendeat, sed unaquaeque pro arbitrio fingi queat. Quin etiam hinc intelligitur, quoniam superficies proxima a proposita diversa esse debet, neutiquam fore

$$\delta z = p\delta x + p'\delta y,$$

siquidem pro superficie proposita fuerit

$$dz = pdx + p'dy;$$

alioquin punctum z foret in eadem superficie, ex quo omnino ternas functiones ipsarum x et y pro variationibus δx, δy, et δz ita comparatas esse oportet, ut non sit

$$\delta z = p\delta x + p'\delta y,$$

sed potius ab hoc valore quomodocumque discrepet; ubi quidem imprimis notandum est has functiones ita late patere, ut discontinuae non excludantur atque adeo pro lubitu variationes tantum in unico puncto vel saltem exiguo spatia constitui queant. Ne autem hic ulli dubio locus relinquatur, probe notandum est ex eo, quod ponimus *z* eiusmodi functionem ipsarum *x* et *y*, ut sit

$$dz = pdx + p'dy,$$

minime sequi fore quoque

$$\delta z = p\delta x + p'\delta y,$$

quemadmodum supra assumsimus, propterea quod hic ipsi *z* propriam tribuimus variationem neutiquam pendentem a variationibus ipsarum *x* et *y*.

PROBLEMA 16

147. *Proposita aequatione inter tres variabiles x, y, z, quibus variationes quaecunque* δx, δy, δz *tribuuntur, investigare variationes formularum differentialium secundi gradus*

$$q = \left(\frac{ddz}{dx^2}\right), \quad q' = \left(\frac{ddz}{dxdy}\right), \quad et \quad q'' = \left(\frac{ddz}{dy^2}\right)$$

SOLUTIO

Hic iterum *z* spectatur ut functio ipsarum *x* et *y*, quarum etiam sunt functiones ternae variationes δx, δy, δz nullo modo a se invicem pendentes. Quoniam in praecedente problemate posuimus

$$p = \left(\frac{dz}{dx}\right) \quad \text{et} \quad p' = \left(\frac{dz}{dy}\right)$$

his formulis in subsidium vocatis habebimus

$$q = \left(\frac{dp}{dx}\right), \quad p' = \left(\frac{dp}{dy}\right) = \left(\frac{dp'}{dx}\right) \quad \text{et} \quad q'' = \left(\frac{dp'}{dy}\right)$$

hicque ratio variationum δp et $\delta p'$ est habenda, quas invenimus

$$\delta p = \left(\frac{d\delta z}{dx}\right) - p\left(\frac{d\delta x}{dx}\right) \quad \text{et} \quad \delta p' = \left(\frac{d\delta z}{dy}\right) - p'\left(\frac{d\delta y}{dy}\right).$$

Simili ergo modo calculum subducendo reperiemus primo

$$\delta q = \left(\frac{d\delta p}{dx}\right) - q\left(\frac{d\delta x}{dx}\right)$$

ubi $\left(\frac{d\delta p}{dx}\right)$ invenitur, si valor δp differentietur posita *y* constante ac differentiale per *dx* dividatur, unde oritur

$$\left(\frac{d\delta p}{dx}\right)=\left(\frac{dd\delta z}{dx^2}\right)-q\left(\frac{d\delta x}{dx}\right)-p\left(\frac{dd\delta x}{dx^2}\right)$$

ob $q=\left(\frac{dp}{dx}\right)$, unde concludimus

$$\delta q=\left(\frac{dd\delta z}{dx^2}\right)-2q\left(\frac{d\delta x}{dx}\right)-p\left(\frac{dd\delta x}{dx^2}\right)$$

Eodem modo ob $q'=\left(\frac{dp}{dy}\right)$ erit

$$\delta q'=\left(\frac{d\delta p}{dy}\right)-q'\left(\frac{d\delta y}{dy}\right),$$

at

$$\left(\frac{d\delta p}{dy}\right)=\left(\frac{dd\delta z}{dxdy}\right)-q'\left(\frac{d\delta x}{dx}\right)-p\left(\frac{dd\delta x}{dxdy}\right)$$

ideoque

$$\delta q'=\left(\frac{dd\delta z}{dxdy}\right)-q'\left(\frac{d\delta x}{dx}\right)-q'\left(\frac{d\delta y}{dy}\right)-p\left(\frac{dd\delta x}{dxdy}\right).$$

Alter autem valor $q'=\left(\frac{dp'}{dx}\right)$ simili modo tractatus praebet

$$\delta q'=\left(\frac{dd\delta z}{dxdy}\right)-q'\left(\frac{d\delta x}{dx}\right)-q'\left(\frac{d\delta y}{dy}\right)-p'\left(\frac{dd\delta y}{dxdy}\right).$$

cuius valoris ab illo discrepantia incommodum involvit mox accuratius examinandum.
Ex tertia autem formula $q''=\left(\frac{dp'}{dy}\right)$ elicitur

$$\delta q''=\left(\frac{dd\delta z}{dy^2}\right)-2q''\left(\frac{d\delta y}{dy}\right)-p'\left(\frac{dd\delta y}{dy^2}\right).$$

SCHOLION 1

148. In originem discrepantiae variationis $\delta q'$ ex gemino valore

$$q'=\left(\frac{dp}{dy}\right)=\left(\frac{dp'}{dx}\right)$$

natae inquisiturus observo in his formulis variationem exprimentibus vel quantitatem *x* vel quantitatem *y* pro constanti haberi, prout denominator cuiuscunque membri declarat. Verum si quantitatem *x* constantem manere sumimus, utcunque interea altera *y* mutabilis existit, natura rei postulat, ut etiam variationes ipsius *x* nullam mutationem subeant, quod autem secus evenit, si variatio δx quoque a quantitate *y* pendeat, quod idem de altera variabili *y*, dum constans ponitur, est tenendum. Ex quo manifestum est, si variationes δx et δy simul ab ambabus variabilibus *x* et *y* pendere sumantur, id ipsi hypothesi, qua alterutra perpetuo constans ponitur, adversari. Quamobrem hoc incommodum aliter vitari nequit, nisi statuamus variationem ipsius *x* prorsus non ab altera variabili *y* neque huius variationem δy ab altera *x* pendere. Sin autem δx per solam *x* et δy per solam *y* determinatur, ut sit

$$\text{et}\quad \left(\frac{d\delta x}{dy}\right)=0 \quad\text{et}\quad \left(\frac{d\delta y}{dx}\right)=0,$$

erit etiam

$$\left(\frac{dd\delta x}{dxdy}\right)=0 \quad\text{et}\quad \left(\frac{dd\delta y}{dxdy}\right)=0$$

sicque ambo illi valores discrepantes pro $\delta q'$ inventi ad consensum perducuntur.

SCHOLION 2

149. Omnibus autem dubiis in hac investigatione felicissime occurremus, si soli quantitati *z* variationes tribuamus binis reliquis *x* et *y* plane invariatis relictis, ita ut sit tam $\delta x=0$ quam $\delta y=0$, quo pacta non solum calculo consulitur, sed etiam usus huius calculi variationum vix restringitur. Quodsi enim superficiem quamcunque cum alia sibi proxima comparamus, nihil impedit, quominus singula proposita superficiei puncta ad ea proximae puncta referamus, quibus eaedem binae coordinatae *x* et *y* respondeant, solaque tertia *z* variationem patiatur. Quin etiam haec suppositio, cum ad formulas integrales progrediemur, eo magis est necessaria, quandoquidem semper totum negotium ad eiusmodi formulas integrales perducitur, quae duplicem integrationem requirunt, in quarum altera sola *x*, in altera vero sola *y* ut variabilis tractatur; nisi ergo harum variationes nullae statuantur, maxima incommoda inde in calculum inveherentur; qui cum per se plerumque sit difficillimus, minime consultum videtur, ut ex hac parte difficultates multiplicentur. Quamobrem hanc tractationem ita sum expediturus, ut in posterum perpetuo binis variabilibus *x* et *y* nullas plane variationes tribuam solamque tertiam *z* variatione quacunque δz augeri assumam, ubi quidem δz ut functionem quamcunque ipsarum *x* et *y* sive continuam sive discontinuam sum spectaturus.

PROBLEMA 17

150. *Si z fuerit functio quaecunque ipsarum x et y eique tribuatur variatio δz pariter utcunque ab x et y pendens, investigare variationes formularum omnium differentialium cuiuscunque ordinis.*

SOLUTIO

Pro differentialibus primi gradus habentur hae duae formulae

$$p=\left(\frac{dz}{dx}\right) \quad\text{et}\quad p'=\left(\frac{dz}{dy}\right),$$

quarum variationes, cum *x* et *y* nullam variationem pati concipiantur, ex supra inventis ita se habebunt

$$\delta p=\left(\frac{d\delta z}{dx}\right) \quad\text{et}\quad \delta p'=\left(\frac{d\delta z}{dy}\right).$$

Pro differentialibus secundi ordinis hae tres formulae habentur

$$q=\left(\frac{ddz}{dx^2}\right),\ q'=\left(\frac{ddz}{dxdy}\right),\ \text{et}\quad q''=\left(\frac{ddz}{dy^2}\right),$$

ita ut sit

$$q=\left(\frac{dp}{dx}\right),\ q'=\left(\frac{dp}{dy}\right)=\left(\frac{dp'}{dx}\right)\ \text{et}\quad q''=\left(\frac{dp'}{dy}\right),$$

quarum variationes ex praecedente problemate ob $\delta x=0$ et $\delta y=0$ sunt

$$\delta q=\left(\frac{dd\delta z}{dx^2}\right),\ \delta q'=\left(\frac{dd\delta z}{dxdy}\right),\quad \delta q''=\left(\frac{dd\delta z}{dy^2}\right).$$

Simili modo si ad differentialia tertii ordinis ascendamus, hae quatuor formulae occurrunt

$$r=\left(\frac{d^3z}{dx^3}\right),\quad r'=\left(\frac{d^3z}{dx^2dy}\right),\quad r''=\left(\frac{d^3z}{dxdy^2}\right),\quad r'''=\left(\frac{d^3z}{dy^3}\right),$$

quarum variationes ita expressum iri manifestum est

$$\delta r=\left(\frac{d^3\delta z}{dx^3}\right),\quad \delta r'=\left(\frac{d^3\delta z}{dx^2dy}\right),\quad \delta r''=\left(\frac{d^3\delta z}{dxdy^2}\right),\quad \delta r'''=\left(\frac{d^3\delta z}{dy^3}\right),$$

unde per se patet, quomodo variationes formularum differentialium superiorum ordinum sint exprimendae.

COROLLARIUM 1

151. Hinc iam manifestum est fore in genere pro formula differentiali cuiuscunque ordinis $\left(\frac{d^{\mu+\nu}z}{dx^\mu dy^\nu}\right)$ eius variationem $=\left(\frac{d^{\mu+\nu}\delta z}{dx^\mu dy^\nu}\right)$, in qua forma superiores omnes continentur.

COROLLARIUM 2

152. Deinde etiam perspicuum est introducendis loco differentialium primi ordinis litteris p, p', secundi ordinis litteris q, q', q'', tertii ordinis litteris r, r', r'', r''', quarti ordinis litteris $s, s', s'', s''', s^{\text{IV}}$ etc. speciem differentialium tolli, quemadmodum etiam supra huiusmodi litteris speciem differentialium sustulimus.

SCHOLION

153. Quoniam binae variabiles x et y prorsus a se invicem non pendent, ita ut altera adeo eundem valorem retinere queat, dum altera per omnes valores possibiles variatur, evidens huiusmodi formulam differentialem $\frac{dy}{dx}$, quippe quae nullum plane significatum certum esset habitura, in calculo nunquam locum invenire posse. Contra vero, cum quantitas z sit functio ipsarum x et y, hae formulae $\frac{dz}{dx}$, $\frac{dz}{dy}$ et reliquae omnes, quas supra sum contemplatus, definitos habent significatus neque ullae aliae in calculum ingredi possunt. Deinde quia semper quaestiones huc pertinentes eo reducere licet, ut z tanquam functio binarum x et y spectari possit, eiusmodi formulae $\frac{dy}{dx}$, ubi quantitas z esset pro constanti habita, hinc prorsus excluduntur neque ullae aliae praeter supra memoratas in calculo admitti sunt censendae; sicque omnes expressiones a formulis integralibus liberae praeter ipsas variabiles x, y, z alias formulas differentiales non implicabunt praeter eas, quarum variationes hic sunt indicatae.

PROBLEMA 18

154. *Si z sit functio ipsarum x et y eique tribuatur variatio δz utcunque ab x et y pendens, tum vero fuerit V quantitas quomodocunque ex tribus variabilibus x, y, z earumque differentialibus cuiuscunque ordinis composita, eius variationem δV investigare.*

SOLUTIO

Ut in expressione V species differentialium tollantur, ponamus, ut hactenus fecimus,

$$p=\left(\frac{dz}{dx}\right),\quad p'=\left(\frac{dz}{dy}\right),$$

$$q=\left(\frac{ddz}{dx^2}\right),\quad q'=\left(\frac{ddz}{dxdy}\right),\quad q''=\left(\frac{ddz}{dy^2}\right),$$

$$r=\left(\frac{d^3z}{dx^3}\right),\quad r'=\left(\frac{d^3z}{dx^2dy}\right),\quad r''=\left(\frac{d^3z}{dxdy^2}\right),\quad r'''=\left(\frac{d^3z}{dy^3}\right)$$

etc.,

quarum formularum variationes a variatione ipsius z oriundas ita definimus, ut posita evidentiae gratia ista variatione $\delta z=\omega$, quam ut functionem quamcunque binarum variabilium x et y spectari oportet, sit

$$\delta p=\left(\frac{d\omega}{dx}\right),\quad \delta p'=\left(\frac{d\omega}{dy}\right),$$

$$\delta q=\left(\frac{dd\omega}{dx^2}\right),\quad \delta q'=\left(\frac{dd\omega}{dxdy}\right),\quad \delta q''=\left(\frac{dd\omega}{dy^2}\right),$$

$$\delta r=\left(\frac{d^3\omega}{dx^3}\right),\quad \delta r'=\left(\frac{d^3\omega}{dx^2dy}\right),\quad \delta r''=\left(\frac{d^3\omega}{dxdy^2}\right),\quad \delta r'''=\left(\frac{d^3\omega}{dy^3}\right)$$

etc.,

Illis autem factis substitutionibus expressio proposita V fiet functio harum quantitatum $x, y, z, p, p', q, q', q'', r, r', r'', r'''$ etc. Eius ergo differentiale talem induet formam

$$\begin{aligned} dV = Ldx + Mdy + Ndz + Pdp &+ Qdq + Rdr \\ + P'dp' &+ Q'dq' + R'dr' \\ &+ Q''dq'' + R''dr'' \\ &+ R'''dr''' \end{aligned}$$

etc.

Quoniam nunc formula V eatenus tantum variationem recipit, quatenus quantitates, ex quibus componitur, variantur, binae autem x et y immunes statuuntur, eius variatio, quam quaerimus, erit

$$\begin{aligned} \delta V = N\delta z + P\delta p &+ Q\delta q + R\delta \\ + P'\delta p' &+ Q'\delta q' + R'\delta r' \\ &+ Q''dq'' + R''\delta r'' \\ &+ R'''\delta r''' \end{aligned}$$

etc.,

ac si loco variationis δz scribamus ω, habebimus variationes inventas substituendo

$$\begin{aligned} \delta V = N\omega + P\left(\tfrac{d\omega}{dx}\right) &+ Q\left(\tfrac{dd\omega}{dx^2}\right) + R\left(\tfrac{d^3\omega}{dx^3}\right) \\ + P'\left(\tfrac{d\omega}{dy}\right) &+ Q'\left(\tfrac{dd\omega}{dxdy}\right) + R'\left(\tfrac{d^3\omega}{dx^2dy}\right) \\ &+ Q''\left(\tfrac{dd\omega}{dy^2}\right) + R''\left(\tfrac{d^3\omega}{dxdy^2}\right) \\ &+ R'''\left(\tfrac{d^3\omega}{dy^3}\right) \end{aligned}$$

etc.,

cuius formatio, si forte etiam differentialia altiorum graduum ingrediantur, per se est manifesta.

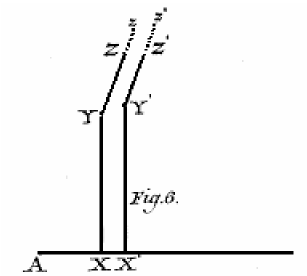
COROLLARIUM 1

155. Cum ω spectetur ut functio binarum variabilium x et y, singularum partium, quae variationem δV constituunt, significatus est determinatus atque haec variatio perfecte definita est censenda.

COROLLARlUM 2

156. Quomodocunque autem expressio V differentialibus sit referta, quandoquidem valorem certum indicare est censenda, substitutionibus adhibitis semper a specie differentialium liberari debet.

COROLLARIUM 3



157. Si nostrae tres variabiles ad superficiem referantur, ut sint eius coordinatae $AX = x$, $XY = y$, $YZ = z$ (Fig. 6), sola ordinata $YZ = z$ ubique incrementum infinite parvum $Zz = \delta z = \omega$ accipere intelligitur, ita ut puncta z cadant in aliam superficiem ab illa infinite parum discrepantem.

SCHOLION

158. Dubio hic occurri debet inde oriundo, quod quantitatem z ut functionem binarum x et y spectandam esse diximus; quoniam enim ipsis x et y nullas variationes tribuimus, si in expressione V loco z eius valor in x et y substitueretur, ea ipsa in meram functionem ipsarum x et y abiret neque propterea ullam variationem esset receptura. Verum notandum est, tametsi z ut functio ipsarum x et y consideratur, eam tamen plerumque esse incognitam, quando scilicet eius naturam demum ex conditione variationis erui oportet; sin autem iam ab initio esset data, tamen, dum variatio quaeritur, functionem hanc z quasi incognitam spectari convenit minimeque eius loco valorem per x et y expressum substitui licet, antequam variatio, quippe quae a sola z pendet, penitus fuerit explorata.