SUPPLEMENT,

CONTAINING

AN EXPLANATION CONCERNING THE INTEGRATION OF PARTICULAR CASES OF DIFFERENTIAL EQUATIONS.

THE COMPLETE DEVELOPMENT CONCERNING THE INTEGRATION OF CERTAIN INDIVIDUAL CASES OF DIFFERENTIAL EQUATIONS

1. Since at this stage several methods have been reported in the literature on the integration of differential equations and disagreeing greatly amongst themselves, a question of some reasonable importance emerges, or may there not be given a single equitable method, with the aid of which all those diverse differential equations, which even now can be resolved, may be integrated. For there is no doubt why the finding of such a method should not be reported on at once, to the great advancement of all analysis.

Indeed for most Geometers a separation of the two variables has been considered to provide a sufficient method of this kind, since all the integrations of differential equations either may be integrated by this method, or from that the integrals may be easily recalled. Except moreover because this method is resolved by substitutions, which generally do not require less shrewdness than that by which the solution itself is sought, and sometimes is the only case that ought to be considered, as also this method by no means can be extended to differential equations of the second or of higher order; and such [higher order] equations which have been discussed until now, have been compelled for a long time to call on the aid of other artifices. On account of which the separation of the variables by no means can be considered as the uniform method of the widest extent, in which all integrations which have succeed at this time are themselves included.

2. But indeed I myself may be seen to have shown such a general method some time ago
[See E44, *De infinitis curvis eiusdem generis. Seu methodus inveniendi aequationes pro infinitis curvis eiusdem generis* : *Considering endless*[i.e. families of] *curves of the same kind. Or a method of finding the equations for endless curves of the same kind* . Not found in translation yet. Also attended to in E269, E429, E650, E720, and in §443-§530 vol. I, and §865-§927 vol. II of this treatise in translation.],
while I have shown some proposed differential equation either of the first or higher order always to have given a quantity of this kind, by which if the equation be multiplied, then it becomes integrable, thus so that in this way clearly there shall be no need for some substitution to be sought anxiously elsewhere. From which without doubt differential equations are to be returned integrated with the help of multipliers, as this is a method of the widest extend and to be proclaimed the greatest convenience of this kind, since no integration at this point may be put in place, which cannot be easily resolved in this manner. Since evidently the equation of all differentials of the first order may be contained in this form $Pdx+Qdy=0$, with the letters P and Q denoting some functions of the two variables x and y, there is given a multiplier M of this kind always, likewise a certain function of the two x and y, so that with the multiplication made this form $MPdx+MQdy$ becomes integrable; therefore the integral of which, for an arbitrary constant quantity, will show a comparable integral of the proposed differential equation $Pdx+Qdy=0$, which by the same method also finds a place in differential equations of higher order. Now it is not in mind to extend this argument further here, but rather the excellence of this method to be

established over the separation of variables also in cases of this kind, in which that may be least considered, and likewise here to declare the great usefulness.

3. Evidently as often as the variables x and y now are separable in the differential equation, the whole work now may generally be accustomed to be regarded as complete, since the integral of that equation $Xdx+Ydy=0$, where X may denote a function of x only and Y of y, the integral is apparent :

$$\int Xdx+\int Ydy=\text{Const}.$$

Yet meanwhile on many occasions it can come about by this use, that by no means this agreed most simple form of the integral may be obtained or that at last must be derived thus by several devious routes. Just as from this equation

$$\tfrac{dx}{x}+\tfrac{dy}{y}=0$$

in the first place the logarithmic integral is elicited

$$lx+ly=la,$$

from which indeed there is produced at once algebraically $xy = a$. Truly from this form

$$\tfrac{dx}{aa+xx}+\tfrac{dy}{aa+yy}=0$$

the customary integral is given [a typographical correction has been made here.]

$$\text{Ang. tang.}\tfrac{x}{a}+\text{Ang. tang.}\tfrac{y}{a}=\text{Const}.$$

from which the algebraic form of the integral $\frac{x+y}{aa-xy}=C$ is not deduced so easily. And with this proposed form

$$\tfrac{dx}{\sqrt{(\alpha+\beta x+\gamma xx)}}+\tfrac{dy}{\sqrt{(\alpha+\beta y+\gamma yy)}}=0$$

in general it may not be apparent indeed, whether each part of the integral may be expressed by a arc of a circle or by a logarithm. Yet meanwhile the integral of this can be shown algebraically thus

$$CC(x-y)^2+2\gamma Cxy+\beta C(x+y)+2\alpha C+\tfrac{1}{4}\beta\beta-\alpha\gamma=0,$$

which simplest form surely can only be derived from the transcending integral by several roundabout ways.

4. Indeed from these cases it is seen, how a reduction to the algebraic form is required to be put in place; but some years ago I advanced integrations of this kind, in which here no other method can be outstanding. Just as if this equation shall be proposed

$$\frac{dx}{\sqrt{\left(1+x^4\right)}}+\frac{dy}{\sqrt{\left(1+y^4\right)}}=0$$

and the integration cannot be performed either by logarithms or circular arcs, so that hence the equation may be deduced algebraically by a similar method; yet meanwhile I have shown the integral of this and that thus complete to be expressed algebraically in this manner

$$0=2C+(CC-1)(xx+yy)-2(1+CC)xy+2Cxxyy\,,$$

where C denotes the constant entered by integration. So that also the complete integral of this equation extended much more widely

$$\frac{dx}{\sqrt{\left(\alpha+2\beta x+\gamma xx+2\delta x^3+\varepsilon x^4\right)}}+\frac{dy}{\sqrt{\left(\alpha+2\beta y+\gamma yy+2\delta y^3+\varepsilon y^4\right)}}=0$$

is

$$\begin{aligned}0&=2\alpha C+\beta\beta-\alpha\gamma+2(\beta C-\alpha\delta)(x+y)+(CC-\alpha\varepsilon)(xx+yy)\\&+2(\gamma C-CC-\alpha\varepsilon-\beta\delta)xy+2(\delta C-\beta\varepsilon)xy(x+y)+(2\varepsilon C+\delta\delta-\gamma\varepsilon)xxyy\end{aligned}$$

with C denoting the same arbitrary constant quantity found by the integration. Therefore it is evident from these that the separation of the variables, which the differential equations have provided, clearly provide no help towards the integration of these with the contained algebraic form elicited, by which merit a method of this kind is desired, the benefit of which being that these integrations can be investigated at once from the differential equations, in which affair surely all talented men will not regret attempting.

5. Therefore I have observed that this goal is possible to be obtained with the aid of suitable multipliers, from which the multiplied differential equations thus emerge integrable, so that at once integrals expressed algebraically may be produced. So that which may be examined more clearly, I will begin from the first equation proposed $\frac{dx}{x}+\frac{dy}{y}=0$, which multiplied by xy gives at once $ydx+xdy=0$, the integral of which is $xy=C$. Therefore with the separation removed in this manner the equation is transformed into another, which admits integration, from which it is understood that the method is better with the aid of multipliers of the integrand, because it cannot be expected immediately from the separation of the variables. Likewise it comes about in the equation $\frac{mdx}{x}+\frac{ndy}{y}=0$, which multiplied by $x^m y^n$ gives the integral $x^m y^n=C$, while from that equation proposed it might have turned at once to logarithms. In a similar manner if this separated equation

$$\frac{dx}{1+xx}+\frac{dy}{1+yy}=0$$

be multiplied by $\frac{(1+xx)(1+yy)}{(x+y)^2}$ the equation results :

$$\frac{dx(1+yy)+dy(1+xx)}{(x+y)^2}=0\,,$$

now the integration naturally allows integration and the integration gives

$$\frac{-1+xy}{x+y}=\text{Const.}\quad\text{or}\quad\frac{x+y}{1-xy}=a.$$

Hence truly the equation

$$\frac{2dx}{1+xx}+\frac{dy}{1+yy}=0$$

agrees to be multiplied by $\frac{(xx+1)^2(1+yy)}{(2xy+xx-1)^2}$, so that there is produced

$$\frac{2dx(1+xx)(1+yy)+dy(xx+1)^2}{(2xy+xx-1)^2}=0,$$

the integral of which is found :

$$\frac{xxy-2x-y}{2xy+xx}=\text{Const.}\ \text{or}\ \frac{2x+y-xxy}{2xy+xx-1}=a.$$

6. Weighed against these examples, by which algebraic integrals have been elicited without the aid of separation, this business of preparing multipliers may be objected to as being restricted to these transcending integrals from which the separation of the variables is produced at once, and thus from these the outstanding nature of the method by the preceding multipliers shall be by no means conclusive. To which objection indeed I respond in the first place that the first examples may be set out at once in a similar manner, before the integration by logarithms had been investigated, which therefore is to be considered no help with that being reported. Then indeed, although I may concede in the latter examples the integration by circular arcs may be conveniently sufficient for these suitable multipliers, yet that in working it is discerned less and the same integration without double could be found, before the integral of the formula $\frac{dx}{1+xxx}$ may be agreed upon to be the arc of the tangential circle corresponding to *x*. The true equation reported above [§ 4]

$$\frac{dx}{\sqrt{(1+x^4)}}+\frac{dy}{\sqrt{(1+y^4)}}=0\,,$$

the complete integral of which one can show algebraically, leaves room for no more doubt; since indeed the integral of neither parts may be shown either by logarithms or circular arcs and the form of this shall be referred to a kind of transcending quantity even now unknown, this surely can be considered to bring no help to finding the integral algebraically. And this is to required to be understood with that extending much wider than the equations proposed in § 4, clearly the individual integration in general of which has been elicited by me from the most diverse principles at length.

7. But the method which I used at that time has been lost sight of so much, that scarcely any way may be considered to be permitted to lead to same integrals, and since the separation of the variables clearly brings together nothing there, also scarcely any other method can be hoped for apart from the method considered restricted to multipliers, therefore because then at that stage I was of the opinion that nothing could surpass multipliers, unless in so far as the separation of variables might lead to the same place, since only a differential of the first order is involved. But then with the matter examined more carefully, I have observed that as often as the complete integral of any differential equation is permitted to be shown, from that in turn always a multiplier of this kind is possible to be elicited, if by which the differential equation may be multiplied, not only is it made integrable, but also that itself to be integrated ought to be reproduced from the integral, since now it was known; but towards this generally it is necessary, that the complete integral shall be investigated, while from the particular integrals clearly nothing is possible to be concluded towards this goal.

If indeed the differential equation shall be proposed

$$Pdx + Qdy = 0,$$

of which the complete integral shall be known in every respect, that will agree with the equation, which besides both the two variables x , y and constant quantities contained in the differential equation above, a new quantity depending on our choosing will be included. Which since it may be indicated by the letter C, the value of this may be elicited from the equation of the integral and it may be found that $C = V$ and V surely will be a certain function of x and y; but then from this equation differentiated $0 = dV$ with the differential dV by necessity must be contained in the differential formula $Pdx + Qdy$, so that there shall be

$$dV = M\left(Pdx + Qdy\right),$$

from which form the multiplier M leading to this integral $C = V$ freely presents itself.

8. So that this operation may be illustrated by some examples, first there is taken this equation

$$\frac{mdx}{x}+\frac{ndy}{y}=0$$

of which the complete integral since it shall be $x^m y^n = C$, with differentiation put in place, gives

$$0 = mx^{m-1}y^n dx + nx^m y^{n-1} \quad \text{or} \quad 0 = x^m y^n\left(\frac{mdx}{x}+\frac{ndy}{y}\right),$$

from which it is apparent that the multiplier leading to this integral is $x^m y^n$.

Successively, since the complete integral of this equation

$$\frac{dx}{1+xx}+\frac{dy}{1+yy}=0$$

shall be $1-xy=C(x+y)$, the value of the arbitrary constant hence shall be made $C=\frac{1-xy}{x+y}$, the differentiation of which gives

$$0=\frac{-dx(1+yy)-dy(1+xx)}{(x+y)^2} \quad \text{or} \quad 0=\frac{(1+xx)(1+yy)}{(x+y)^2}\left(\frac{dx}{1+xx}+\frac{dy}{1+yy}\right),$$

from which the multiplier sought is $=\frac{(1+xx)(1+yy)}{(x+y)^2}$.

Again this equation may be proposed

$$\frac{dx}{\sqrt{(\alpha+2\beta x+\gamma xx)}}+\frac{dy}{\sqrt{(\alpha+2\beta y+\gamma yy)}}=0,$$

of which the complete integral

$$CC(x-y)^2-2C(\alpha+\beta x+\beta y+\gamma xy)+\beta\beta-\alpha\gamma=0$$

gives in the first place

$$C=\frac{\alpha+\beta(x+y)+\gamma xy+\sqrt{(\alpha\alpha+2\alpha\beta(x+y)+\alpha\gamma(xx+yy)+4\beta\beta xy+2\beta\gamma xy(x+y)+\gamma\gamma xxyy)}}{(x-y)^2}$$

or

$$C=\frac{\alpha+\beta(x+y)+\gamma xy+\sqrt{(\alpha+2\beta x+\gamma xx)(\alpha+2\beta y+\gamma yy)}}{(x-y)^2},$$

or more neatly [with the negative square root]

$$\frac{\beta\beta-\alpha\gamma}{C}=\alpha+\beta(x+y)+\gamma xy+\sqrt{(\alpha+2\beta x+\gamma xx)(\alpha+2\beta y+\gamma yy)},$$

from which on differentiating there comes about

$$\begin{aligned}0=&\,dx(\beta+\gamma y)+dy(\beta+\gamma x)\\&+\frac{dx(\beta+\gamma x)\sqrt{(\alpha+2\beta y+\gamma yy)}}{\sqrt{(\alpha+2\beta x+\gamma xx)}}+\frac{dy(\beta+\gamma y)\sqrt{(\alpha+2\beta x+\gamma xx)}}{\sqrt{(\alpha+2\beta y+\gamma yy)}}\end{aligned}$$

and hence the multiplier sought is deduced

$$M=(\beta+\gamma x)\sqrt{(\alpha+2\beta y+\gamma yy)}+(\beta+\gamma y)\sqrt{(\alpha+2\beta x+\gamma xx)}.$$

9. In the same manner for a more intricate equation

$$\frac{dx}{\sqrt{(\alpha+2\beta x+\gamma xx+2\delta x^3+\varepsilon x^4)}}+\frac{dy}{\sqrt{(\alpha+2\beta y+\gamma yy+2\delta y^3+\varepsilon y^4)}}=0,$$

from the complete integral shown above [§ 4], a suitable multiplier of this M can be investigated, from which, if it should be known immediately, this same integral would be elicited immediately. Truly here I undertake a much greater work, because moreover on attempting in the first place by no means will it be permitted to lead towards an end; from which indeed I consider it will be better for me, if perhaps I sketch out as it were new outlines and greatly to be desired methods, with the aid of which with the proposed differential equation a suitable multiplier returning that integral may be able to be found.

And indeed at first in this matter it will be pleasing to observe most frequently, if a single multiplier of this kind should be known, from that easily infinitely many others performing the same service can be elicited. Because if indeed the multiplier M returns the differential $Pdx+Qdy=0$ integrable, thus so that there shall be $\int M(Pdx+Qdy)=V$ and thus the equation of the integral $V=C$, because the formula $dV=M(Pdx+Qdy)$ multiplied by any function of the quantity V thus remains integrable, it is evident that this form $Mf:V$, whatever function of V may be taken for $f:V$, always gives a suitable multiplier, since there shall be

$$(Pdx+Qdy)Mf:V=dVf:V$$

and thus integrable. Therefore among these infinitely many suitable multipliers in any case, it will be convenient to select that which performs the work most easily and the integral, if it were algebraic, may show the simplest form. Indeed even if the integral returned actually shall be algebraic, in general it can happen, as that indeed may not be distrusted, unless a suitable multiplier may be in use, just as the above examples indicate abundantly.

10. Therefore let the proposed differential equation be of this form

$$\frac{dx}{X}+\frac{dy}{Y}=0,$$

in which X shall be a function of x only and Y of y only, and it is required to investigate a multiplier of this kind M, by which that equation of the integral is returned algebraically, if indeed it can happen; since it seldom happens, in turn from the form of the multiplier M assumed it will be pleasing to investigate the functions X and Y.

In the first place let the multiplier be

$$M=\frac{XY}{(\alpha+\beta x+\gamma y)^2},$$

so that the integrable quantity ought to be this form

$$\frac{Ydx+Xdy}{(\alpha+\beta x+\gamma y)^2}$$

Hence on taking y constant, the integral is deduced

$$\frac{-Y}{\beta(\alpha+\beta x+\gamma y)}+\Gamma\colon y,$$

moreover with x assumed constant, there will be produced

$$\frac{-X}{\gamma(\alpha+\beta x+\gamma y)}+\Delta\colon x,$$

both forms it is required to be equal to each other; from which there comes about :

$$-\gamma Y+\beta\gamma(\alpha+\beta x+\gamma y)\Gamma{:}y=-\beta X+\beta\gamma(\alpha+\beta x+\gamma y)\Delta{:}x$$

or

$$\beta X-\gamma Y=\beta\gamma(\alpha+\beta x+\gamma y)(\Delta{:}x-\Gamma\colon y)$$

and thus it is apparent that the functions $\Delta\colon x$ and $\Gamma\colon y$ ought to be prepared thus, so that on extrication, the terms from the last member which likewise may contain x and y, mutually cancel each other. From which there is understood to be [*i. e.* no cross terms in xy]

$$\Delta\colon x=m\beta x+\text{ Const. and }\Gamma\colon y=m\gamma y+\text{Const}.$$

Therefore we may put in place

$$\varDelta : x - \varGamma : y = m\beta x - m\gamma y + n$$

and there is formed

$$\beta X - \gamma Y = \beta\gamma \left\{ \begin{matrix} m\beta\beta xx - m\gamma\gamma yy + n\beta x & + n\gamma y & + n\alpha \\ & + m\alpha\beta x - m\alpha\gamma y + f & \\ & & - f \end{matrix} \right\},$$

from which we deduce

$$X = \gamma\left(m\beta\beta xx + \beta\left(m\alpha + n\right)x + f + \tfrac{1}{2}n\alpha\right),$$
$$Y = \gamma\left(m\gamma\gamma yy + \gamma\left(m\alpha - n\right)y + f - \tfrac{1}{2}n\alpha\right),$$

and the algebraic equation of the integral will be

$$m\gamma y - \frac{m\gamma\gamma yy + \gamma(m\alpha - n)y + f - \frac{1}{2}n\alpha}{\alpha + \beta x + \gamma y} = \text{Const.}$$

or

$$m\beta\gamma xy + n\gamma y - f + \tfrac{1}{2}n\alpha = C\left(\alpha + \beta x + \gamma y\right)$$

or in place of C by writing $C + \frac{1}{2}n$ there will be more neatly

$$m\beta\gamma xy - \tfrac{1}{2}n\beta x + \tfrac{1}{2}n\gamma y - f = C\left(\alpha + \beta x + \gamma y\right).$$

11. We may now consider, under what conditions this form of the general equation may emerge from that method of integration

$$\frac{hdx}{Axx + Bx + C} + \frac{kdy}{Dyy + Ey + F} = 0.$$

Therefore by comparison with the values found in place there is deduced

$$A = hm\beta\beta\gamma, \qquad D = km\beta\gamma\gamma,$$
$$B = h\beta\gamma\left(m\alpha + n\right), \quad E = k\beta\gamma\left(m\alpha - n\right),$$
$$C = h\gamma\left(f + \tfrac{1}{2}n\alpha\right), \quad F = k\beta\left(f - \tfrac{1}{2}na\right).$$

Because here the whole undertaking is reduced to accounts of the letters, on taking for the first equations

$$\beta = Ak \text{ and } \gamma = Dh$$

the remainder may be concluded

$$m = \frac{1}{ADhhkk}, \ \ \alpha = \frac{Bk+Eh}{2}, \ \ n = \frac{Bk-Eh}{2ADhhkk} \ \text{ and } \ f = \frac{ACkk+DFhh}{2ADhhkk};$$

besides in addition truly this condition is required, so that there shall be

$$\frac{4AC-BB}{hh} = \frac{4DF-EE}{kk};$$

which if it were considered to be in place, a suitable multiplier will be

$$M = \frac{(Axx+Bx+C)(Dyy+Ey+F)}{hk\left(\frac{1}{2}(Bk+Eh)+Akx+Dhy\right)^2}$$

and the equation of the integral thus resulting on multiplying by *hk* will be

$$xy - \frac{(Bk-Eh)x}{4Dh} + \frac{(Bk-Eh)y}{4Ak} - \frac{ACkk+DFhh}{2ADhk} = G\left(\tfrac{1}{2}(Bk+Eh)+Akx+Dhy\right),$$

which with the unchanging arbitrary constant *G* may be returned to this form

$$\left(x+\frac{B}{2\text{A}}-GDh\right)\left(y+\frac{E}{2\text{D}}-GAk\right) = GGADhk + \frac{(4AC-BB)kk+(4DF-EE)hh}{8ADhk}$$

or

$$\left(\frac{2AX+B}{h}+G\right)\left(\frac{2Dy+E}{k}+G\right) = GG + \frac{4AC-BB}{2hh} + \frac{4DF-EE}{2kk}.$$

12. Behold therefore a theorem not at all to be scorned, even if the truth of this may be able to be disclosed well enough from other principles.

If this differential equation

$$\frac{hdx}{Axx+Bx+C} + \frac{kdy}{Dyy+Ey+F} = 0$$

were prepared thus, so that there should be

$$\frac{4AC-BB}{hh} = \frac{4DF-EE}{kk}$$

then the complete integral of this will be algebraic and expressed by this equation

$$\frac{2Ax+B}{h} \cdot \frac{2Dy+E}{k} + G\left(\frac{2Ax+B}{h} + \frac{2Dy+E}{k}\right) = \frac{4AC-BB}{2hh} + \frac{4DF-EE}{2kk},$$

where G may denote the arbitrary constant entering through the integration. Now this integral is found, if the proposed equation is taken by this multiplier

$$\frac{(Axx+Bx+C)(Dyy+Ey+F)}{\left(\frac{2Ax+B}{h}+\frac{2Dy+E}{k}\right)^2}.$$

13. Just as we have given the form of the multiplier M

$$\frac{XY}{(\alpha+\beta x+\gamma y)^2}$$

thus also it will be allowed to be used with more complicated forms, because in general indeed it is unable to be bettered. Moreover we may consider the multiplier

$$M=\frac{YX}{(\alpha+\beta x+\gamma y\delta xy)^2}$$

so that this integrable equation shall be effected

$$\frac{Ydx+Xdy}{(\alpha+\beta x+\gamma y+\delta xy)^2}=0.$$

the integration of which leads to this equation

$$\frac{-Y}{(\beta+\delta y)(\alpha+\beta x+\gamma y+\delta xy)}+\Gamma{:}y=\frac{-X}{(\gamma+\delta x)(\alpha+\beta x+\gamma y+\delta xy)}+\Delta{:}x,$$

which may be transformed into this

$$\frac{X}{\gamma+\delta x}-\frac{Y}{\beta+\delta y}=(\alpha+\beta x+\gamma y+\delta xy)(\Delta{:}\,x-\Gamma{:}\,y),$$

where it is evident there must be put in place

$$\Delta{:}\,x=\frac{\zeta x+\eta}{\gamma+\delta x} \quad\text{and}\quad \Gamma{:}\,y=\frac{\zeta y+\theta}{\beta+\delta y}$$

in order that no terms arise, which may include each variable at the same time.
Hence therefore there comes about :

$$\frac{X}{\gamma+\delta x}-\frac{Y}{\beta+\delta y}=\underset{+f}{\eta y}+\frac{(\alpha+\beta x)(\zeta x+\eta)}{\gamma+\delta x}\underset{-\ f}{-\theta x}-\frac{(\alpha+\gamma y)(\zeta y+\theta)}{\beta+\delta y},$$

from which it is concluded

$$X = (\alpha + \beta x)(\zeta x + \eta) - (\gamma + \delta x)(\theta x + f),$$
$$Y = (\alpha + \gamma y)(\zeta y + \theta) - (\beta + \delta y)(\eta y + f)$$

or if, on expanding

$$X = (\beta\zeta - \delta\theta)xx + (\alpha\zeta + \beta\eta - \gamma\theta - \delta f)x + \alpha\eta - \gamma f,$$
$$Y = (\gamma\zeta - \delta\eta)yy + (\alpha\zeta + \gamma\theta - \beta\eta - \delta f)y + \alpha\theta - \beta f,$$

and the equation of the integral will be

$$\frac{\zeta x + \eta}{\gamma + \delta x} - \frac{X}{(\gamma + \delta x)(\alpha + \beta x + \gamma y + \delta xy)} = \text{Const.},$$

which with the value found substituted in place of X, this will change into this form

$$\frac{\zeta xy + \eta y + \theta x + f}{\alpha + \beta x + \gamma y + \delta xy} = \text{Const.}$$

14. We may transform this integral again to the form

$$\frac{hdx}{Axx + Bx + C} + \frac{kdy}{Dyy + Ey + F} = 0$$

and it is required to make:

$$A = h(\beta\zeta - \delta\theta), \qquad D = k(\gamma\zeta - \delta\eta),$$
$$B = h(\alpha\zeta + \beta\eta - \gamma\theta - \delta f), \; E = k(\alpha\zeta + \gamma\theta - \beta\eta - \delta f),$$
$$C = h(\alpha\eta - \gamma f), \qquad F = k(\alpha\theta - \beta f).$$

The first equations give :

$$\theta = \frac{\beta\zeta}{\delta} - \frac{A}{\delta h}, \quad \eta = \frac{\gamma\zeta}{\delta} - \frac{D}{\delta k},$$

indeed the second give :

$$f = \frac{\alpha\zeta}{\delta} - \frac{Bk + Eh}{2\delta hk} \quad \text{and} \quad \delta = \frac{2A\gamma k - 2D\beta h}{Bk - Eh},$$

from the third it is deduced :

$$\frac{2Ck(A\gamma k-D\beta h)}{Bk-Eh}=\frac{\gamma}{2}(Bk+Eh)-D\alpha h,$$
$$\frac{2Fh(A\gamma k-D\beta h)}{Bk-Eh}=\frac{\beta}{2}(Bk+Eh)-A\alpha k.$$

Hence with α removed there arises

$$\frac{2(ACkk-DFhh)(Ak\gamma-Dh\beta)}{Bk-Eh}=\frac{1}{2}(Ak\gamma-Dh\beta)(Bk+Eh),$$

from which, since it cannot occur that

$$Ak\gamma-Dh\beta=0,$$

because otherwise there would be made $\delta=0$ and the quantities θ,η,f become infinite, then truly, which is to be noted in particular, the equation of the integral might appear to be Const. = Const., in which case therefore nothing would be indicated, it is necessary that there shall be

$$4(ACkk-DFhh)=BBkk-EEhh$$

or

$$\frac{4AC-BB}{hh}=\frac{4DF-EE}{kk}$$

as before.

Because here especially it deserves to be brought to mind, which is , although the three letters β, γ and δ remain indefinite, yet the equation of the integral agrees with the preceding constant quantity; indeed there is produced

$$\frac{2\zeta hk}{Bk-Eh}-\frac{k(2Ax+B)+h(2Dy+E)}{2(Aky-Dh\beta)xy+(Bk-Eh)(\beta x+\gamma y)+2(Ck\beta-Fhy)}=\text{Const.}$$

or

$$\frac{yky(2Ax+B)+\beta k(Bx+2C)-\beta hx(2Dy+E)-yh(Ey+2F)}{k(2Ax+B)+h(2Dy+E)}=\text{Const.},$$

which form, however the letters β and γ may be taken, always shows the true value of the integral. Which since it may be less evident, it may be sufficient to make clear that both the parts involving β and γ separately assume the same relation defined between x and y . For with these two equations taken together

$$\frac{2Akxy+Bky-Ehy-2Fh}{2Akx+2Dhy+Bk+Eh}=\text{Const.},\quad \frac{-2Dhxy-Ehx+Bkx+2Ck}{2Akx+2Dhy+Bk+Eh}=\text{Const.}$$

the former may be multiplied by *Dh*, the latter by *Ak* and the sum becomes

$$\frac{Ak(Bk-Eh)x+Dh(Bk-Eh)y+2ACkk-2DFhh}{2Akx+2Dhy+Bk+Eh},$$

the value of which everywhere is a constant $=\frac{Bk-Eh}{2}$, therefore because

$$\frac{2ACkk-2DFhh}{Bk+Eh}=\frac{Bk-Eh}{2},$$

from which the proposition may be apparent.

15. Now I progress to a harder form of the equation, which shall be

$$\frac{dx}{\sqrt{X}}+\frac{dy}{\sqrt{Y}}=0,$$

and let the multiplier rendering that integrable be

$$M=P\sqrt{X}+Q\sqrt{Y},$$

thus so that the equation admitting integration shall be

$$Pdx+Qdy+\frac{Qdx\sqrt{Y}}{\sqrt{X}}+\frac{Pdy\sqrt{X}}{\sqrt{Y}}=0,$$

each member of which shall be required to be integrable separately.

For the prior therefore there will be $\left(\frac{dP}{dy}\right)=\left(\frac{dQ}{dx}\right)$, truly the integral of the posterior may be put in place $2V\sqrt{XY}$, from which it is deduced

$$Q=2X\left(\frac{dV}{dx}\right)+V\frac{dX}{dx} \quad \text{and} \quad P=2Y\left(\frac{dV}{dy}\right)+V\frac{dY}{dy}$$

and on account of the first condition

$$2Y\left(\frac{ddV}{dy^2}\right)+\frac{3dY}{dy}\left(\frac{dV}{dx}\right)+V\frac{ddY}{dy^2}=2X\left(\frac{ddV}{dx^2}\right)+\frac{3dX}{dx}\left(\frac{dV}{dx}\right)+V\frac{ddX}{dx^2},$$

from which equation, if in place of *V* we may have selected a certain function of *x* and *y*, it can be discerned, how suitable values may be obtained for the functions *X* and *Y*.

16. At first we may give a constant value of V, suppose $V = 1$, and we arrive at this condition

$$\frac{ddY}{dy^2} = \frac{ddX}{dx^2}$$

which equality is unable to remain, unless each member separately is equal to a constant quantity, which shall be $= 2a$, from which we deduce

$$X = axx + bx + c \text{ and } Y = ayy + dy + e$$

and hence again

$$P = \frac{dY}{dx} = 2ay + d \text{ and } Q = \frac{dX}{dx} = 2ax + b,$$

from which the equation of the complete integral is deduced

$$2axy + dx + by + 2\sqrt{XY} = \text{Const.}$$

On account of which that differential equation

$$\frac{dx}{\sqrt{(axx+bx+c)}} + \frac{dy}{\sqrt{(ayy+dy+e)}} = 0$$

is returned integrable with the aid of the multiplier

$$M = (2ay + d)\sqrt{(axx + bx + c)} + (2ax + b)\sqrt{(ayy + dy + e)}$$

and then the complete integral may be found

$$2axy + dx + by + 2\sqrt{(axx + bx + c)(ayy + dy + e)} = C$$

or with the irrationality removed

$$CC - 2C(2axy + dx + by) = (4ae - dd)xx + (4ac - bb)yy + 2bdxy + 4bex + 4cdy + 4ce.$$

Moreover this differential may be extended much wider from that which I reported initially in §3.

17. Now we may attribute this value of V

$$V = \frac{1}{(\alpha + \beta x + \gamma y)^2};$$

if indeed in place of the exponent 2 I may assume it to be indefinite, soon it would become apparent that this power ought to be accepted. Therefore there will be

$$\left(\frac{dV}{dx}\right)=\frac{-2\beta}{(\alpha+\beta x+\gamma y)^3}, \qquad \left(\frac{dV}{dy}\right)=\frac{-2\gamma}{(\alpha+\beta x+\gamma y)^3},$$
$$\left(\frac{ddV}{dx^2}\right)=\frac{6\beta\beta}{(\alpha+\beta x+\gamma y)^4} \text{ and } \left(\frac{ddV}{dy^2}\right)=\frac{6\gamma\gamma}{(\alpha+\beta x+\gamma y)^4}.$$

But with these values substituted the following two forms arise

$$12\beta\beta X-\frac{6\beta dX}{dx}(\alpha+\beta x+\gamma y)+\frac{ddX}{dx^2}(\alpha+\beta x+\gamma y)^2$$
$$=12\gamma\gamma Y-\frac{6\gamma dY}{dy}(\alpha+\beta x+\gamma y)\ +\frac{ddY}{dy^2}(\alpha+\beta x+\gamma y)^2;$$

therefore because in the former *y*, in the other *x* no dimensions beyond two arise, it is evident in the formulas

$$\frac{ddX}{dx^2} \text{ and } \frac{ddY}{dy^2}$$

the variables *x* and *y* must have just as many dimensions , because otherwise the terms from *x* and *y* may not be able to be mixed equal on both sides. Therefore since these functions *X* and *Y* shall have ascended to the fourth order, we may put

$$X=Ax^4+2Bx^3+Cx^2+2Dx+E \text{ and } Y=\mathfrak{A}y^4+2\mathfrak{B}y^3+\mathfrak{C}y^2+2\mathfrak{D}y+\mathfrak{E}.$$

Thus from the substitution for the first part there is produced

$$\begin{array}{lllll}
+12\beta\beta Ax^4 & +24\beta\beta Bx^3 & +12\beta\beta Cxx & +24\beta\beta Dx & +12\beta\beta E\\
-24\beta\beta A & -36\beta\beta B & -12\beta\beta C & -12\beta\beta D & -12\alpha\beta D\\
+12\beta\beta A & -24\alpha\beta A & -36\alpha\beta B & -12\alpha\beta C & +2\alpha\alpha C\\
 & +12\beta\beta B & +2\beta\beta C & +4\alpha\beta C & \\
 & +24\alpha\beta A & +24\alpha\beta B & +12\alpha\alpha B & \\
 & & +12\alpha\alpha A & & \\
-24\beta\gamma Ax^3y & -36\beta\gamma Bx^2y & -12\beta\gamma Cxy & -12\beta\gamma Dy & \\
+24\beta\gamma A & +24\beta\gamma B & +4\beta\gamma C & +4\alpha\gamma C & \\
 & +24\alpha\gamma A & +24\alpha\gamma B & & \\
+12\gamma\gamma Axxyy & +12\gamma\gamma Bxyy & +2\gamma\gamma Cyy, & &
\end{array}$$

which terms may be set out in order

$$\begin{aligned}&12\gamma\gamma Axxyy+12\gamma\gamma Bxyy+12\gamma\left(2\alpha A-\beta B\right)xxy\\&+\ 2\gamma\gamma Cyy+8\gamma\left(3\alpha B-\beta C\right)xy+2\left(6\alpha\alpha A-6\alpha\beta B+\beta\beta C\right)xx\\&+\ 4\gamma\left(\alpha C-3\beta D\right)y+4\left(3\alpha\alpha B-2a\beta C+3\beta\beta D\right)x+2\left(\alpha\alpha C-6\alpha\beta D+6\beta\beta E\right).\end{aligned}$$

In a similar manner the other part will be

$$\begin{aligned}&12\beta\beta\mathfrak{A}xxyy+12\beta\beta\mathfrak{B}xxy+12\beta\left(2\alpha\mathfrak{A}-\gamma\mathfrak{B}\right)xyy\\&+\ 2\beta\beta\mathfrak{C}xx+8\beta\left(3\alpha\mathfrak{B}-\gamma\mathfrak{C}\right)xy+2\left(6\alpha\alpha\mathfrak{A}-6\alpha\gamma\mathfrak{B}+\gamma\gamma\mathfrak{C}\right)yy\\&+\ 4\beta\left(\alpha\mathfrak{C}-3\gamma\mathfrak{D}\right)x+4\left(3\alpha\alpha\mathfrak{B}-2a\gamma\mathfrak{C}+3\gamma\gamma\mathfrak{D}\right)y+2\left(\alpha\alpha\mathfrak{C}-6\alpha\gamma\mathfrak{D}+6\gamma\gamma\mathfrak{E}\right).\end{aligned}$$

18. Now the homologous terms of each form may be equated to each other and will be required to be satisfied by the following equations

$$\begin{array}{c|l}xxyy & \gamma\gamma A=\beta\beta\mathfrak{A},\\ xxy & 2\alpha\gamma A-\beta\gamma B=\beta\beta\mathfrak{B},\\ xyy & \gamma\gamma B=2\alpha\beta\mathfrak{A}-\beta\beta\mathfrak{B},\\ xx & 6\alpha\alpha A-6\alpha\beta B+\beta\beta C=\beta\beta\mathfrak{C},\\ yy & \beta\beta C=6\alpha\alpha\mathfrak{A}-6\alpha\gamma\mathfrak{B}+\gamma\gamma\mathfrak{C},\\ xy & 3\alpha\gamma B-\beta\gamma C=3\alpha\beta\mathfrak{B}-\beta\gamma\mathfrak{C},\\ x & 3\alpha\alpha B-2\alpha\beta C+3\beta\beta D=\alpha\beta\mathfrak{C}-3\beta\gamma\mathfrak{D},\\ y & \alpha\gamma C-3\beta\gamma D=3\alpha\alpha\mathfrak{B}-2\alpha\gamma\mathfrak{C}+3\gamma\gamma\mathfrak{D},\\ 1 & \alpha\alpha C-6\alpha\beta D+6\beta\beta E=\alpha\alpha\mathfrak{C}-6\alpha\gamma\mathfrak{D}+6\gamma\gamma\mathfrak{E}.\end{array}$$

Moreover the first three equations determined only give

$$\beta=\frac{2\alpha A\sqrt{\mathfrak{A}}}{B\sqrt{\mathfrak{A}}+\mathfrak{B}\sqrt{A}}\quad\text{and}\quad\gamma=\frac{2\alpha\mathfrak{A}\sqrt{A}}{B\sqrt{\mathfrak{A}}+\mathfrak{B}\sqrt{A}}.$$

likewise the fourth and fifth support the single determination

$$C-\mathfrak{C}=\frac{3\left(\mathfrak{A}BB-A\mathfrak{B}\mathfrak{B}\right)}{2A\mathfrak{A}}=\frac{3}{2}\left(\frac{BB}{A}-\frac{\mathfrak{B}\mathfrak{B}}{\mathfrak{A}}\right),$$

and the same also from the sixth follows; therefore there is put in place

$$C=\frac{3BB}{2A}+n\ \text{ and }\ \mathfrak{C}=\frac{3\mathfrak{B}\mathfrak{B}}{2\mathfrak{A}}+n.$$

The seventh and eighth also involve a single determination

$$\frac{D\sqrt{A}+\mathfrak{D}\sqrt{\mathfrak{A}}}{B\sqrt{\mathfrak{A}}+\mathfrak{B}\sqrt{A}}=\frac{A\mathfrak{B}\mathfrak{B}+\mathfrak{A}BB-B\mathfrak{B}\sqrt{A\mathfrak{A}}+2nA\mathfrak{A}}{4A\mathfrak{A}\sqrt{A\mathfrak{A}}}$$

or

$$D\sqrt{A}+\mathfrak{D}\sqrt{\mathfrak{A}}=\frac{B^3}{4A\sqrt{A}}+\frac{\mathfrak{B}^3}{4\mathfrak{A}\sqrt{\mathfrak{A}}}+\frac{nB}{2\sqrt{A}}+\frac{n\mathfrak{B}}{2\sqrt{\mathfrak{A}}};$$

therefore there is put in place

$$D=\frac{B^3}{4AA}+\frac{nB}{2A}+\frac{m}{2\sqrt{A}} \text{ and } \mathfrak{D}=\frac{\mathfrak{B}^3}{4\mathfrak{A}\mathfrak{A}}+\frac{n\mathfrak{B}}{2\mathfrak{A}}-\frac{m}{2\sqrt{\mathfrak{A}}},$$

which values substituted into the final equation give

$$24\left(AE-\mathfrak{A}\mathfrak{E}\right)=\frac{3B^4}{2AA}+\frac{6nBB}{A}+\frac{12mB}{\sqrt{A}}-\frac{3\mathfrak{B}^4}{2\mathfrak{A}\mathfrak{A}}-\frac{6n\mathfrak{B}\mathfrak{B}}{\mathfrak{A}}+\frac{12m\mathfrak{B}}{\sqrt{\mathfrak{A}}},$$

whereby conveniently it will be allowed to put in place

$$E=\frac{B^4}{16A^3}+\frac{nBB}{4AA}+\frac{mB}{2A\sqrt{A}}+\frac{l}{A},$$
$$\mathfrak{E}=\frac{\mathfrak{B}^4}{16\mathfrak{A}^3}+\frac{n\mathfrak{B}\mathfrak{B}}{4\mathfrak{A}\mathfrak{A}}-\frac{m\mathfrak{B}}{2\mathfrak{A}\sqrt{\mathfrak{A}}}+\frac{l}{\mathfrak{A}}.$$

19. But since we will have selected $V=\frac{1}{(\alpha+\beta x+\gamma y)^2}$, there will be

$$Q=\frac{-4\beta\left(Ax^4+2Bx^3+Cxx+2Dx+E\right)}{(\alpha+\beta x+\gamma y)^3}+\frac{2\left(2Ax^3+3Bxx+Cx+D\right)}{(\alpha+\beta x+\gamma y)^2},$$

$$P=\frac{-4\gamma\left(\mathfrak{A}x^4+2\mathfrak{B}x^3+\mathfrak{C}xx+2\mathfrak{D}x+\mathfrak{E}\right)}{(\alpha+\beta x+\gamma y)^3}+\frac{2\left(2\mathfrak{A}x^3+3\mathfrak{B}xx+\mathfrak{C}x+\mathfrak{D}\right)}{(\alpha+\beta x+\gamma y)^2},$$

or

$$Q=\frac{2\gamma y\left(2Ax^3+3Bxx+Cx+D\right)+2(2\alpha A-\beta B)x^3+2(3\alpha B-\beta C)xx+2(\alpha C-3\beta D)x+2(\alpha D-2\beta E)}{(\alpha+\beta x+\gamma y)^3},$$

$$P=\frac{2\beta x\left(2\mathfrak{A}y^{3}+3\mathfrak{B}yy+\mathfrak{C}y+\mathfrak{D}\right)+2(2\alpha\mathfrak{A}-\gamma\mathfrak{B})y^{3}+2(3\alpha\mathfrak{B}-\gamma\mathfrak{C})yy+2(\alpha\mathfrak{C}-3\gamma\mathfrak{D})y+2(\alpha\mathfrak{D}-2\gamma\mathfrak{E})}{(\alpha+\beta x+\gamma y)^{3}},$$

from which it is correct to investigate the integral of the formula $Pdx+Qdy$; to which if successively there is added to that $\frac{2\sqrt{XY}}{(\alpha+\beta x+\gamma y)^{2}}$, the sum equal to a constant quantity will show the complete integral of the equation

$$\frac{dx}{\sqrt{X}}+\frac{dy}{\sqrt{Y}}=0.$$

But for that integral required to be found from the prior values for P and Q shown, it may be noted separately to become

$$\int Qdy=\frac{2\beta\left(Ax^{4}+2Bx^{3}+Cxx+2Dx+E\right)}{\gamma(\alpha+\beta x+\gamma y)^{2}}-\frac{2\left(2Ax^{3}+3Bxx+Cx+D\right)}{(\alpha+\beta x+\gamma y)}+\Gamma\colon x,$$

$$\int Pdx=\frac{2\gamma\left(\mathfrak{A}x^{4}+2\mathfrak{B}x^{3}+\mathfrak{C}xx+2\mathfrak{D}x+\mathfrak{E}\right)}{\beta(\alpha+\beta x+\gamma y)^{2}}-\frac{2\left(2\mathfrak{A}x^{3}+3\mathfrak{B}xx+\mathfrak{C}x+\mathfrak{D}\right)}{\beta(\alpha+\beta x+\gamma y)}+\Delta\colon y,$$

which two expressions ought to be equal; which

$$\Gamma\colon x=\frac{2(Axx+Bx+N)}{\beta\gamma}\quad\text{and}\quad\Delta\colon x=\frac{2(\mathfrak{A}xx+\mathfrak{B}x+\mathfrak{N})}{\beta\gamma}$$

may be put in place at the end and there becomes

$\tfrac{1}{2}\beta\gamma(\alpha+\beta x+\gamma y)^{2}\int Qdy$	$\tfrac{1}{2}\beta\gamma(\alpha+\beta x+\gamma y)^{2}\int Pdx$
$+A\gamma\gamma xxyy$	$+\mathfrak{A}\beta\beta xxyy$
$+B\gamma\gamma xyy$	$+\beta(2\mathfrak{A}\alpha-\mathfrak{B}\gamma)xyy$
$+\gamma(2A\alpha-B\beta)xxy$	$+\mathfrak{B}\beta\beta xxy$
$+N\gamma\gamma yy$	$+(\mathfrak{A}\alpha\alpha-\mathfrak{B}\alpha\gamma+\mathfrak{N}\gamma\gamma)yy$
$+(A\alpha\alpha-B\alpha\beta+N\beta\beta)xx$	$+\mathfrak{N}\beta\beta xx$
$+\gamma(2B\alpha-C\beta+2N\beta\beta)xy$	$+\beta(2\mathfrak{B}\alpha-\mathfrak{C}\gamma+2\mathfrak{N}\gamma)xy$
$+\gamma(2N\alpha-D\beta)y$	$+(\mathfrak{B}\alpha\alpha-\mathfrak{C}\alpha\gamma+\mathfrak{D}\gamma\gamma+2\mathfrak{N}\alpha\gamma)y$
$+(B\alpha\alpha-C\alpha\beta+D\beta\beta+2N\alpha\beta)x$	$+\beta(2\mathfrak{N}\alpha-\mathfrak{D}\gamma)x$
$+E\beta\beta-D\alpha\beta+N\alpha\alpha$	$+\mathfrak{E}\gamma\gamma-\mathfrak{D}\alpha\gamma+\mathfrak{N}\alpha\alpha.$

20. These conditions are in perfect agreement with the preceding (§18), only if there is selected

$$N=\tfrac{1}{6}C \text{ and } \mathfrak{N}=\tfrac{1}{6}\mathfrak{C}$$

We may divide the individual terms by $\beta\gamma$, so that the value of the formula may be produced

$$\tfrac{1}{2}\beta\gamma\left(\alpha+\beta x+\gamma y\right)^2\int Qdy\ ,$$

whereby with the values found before substituted it will be found

$$\begin{gathered}xxyy\sqrt{A\mathfrak{A}}+Bxyy\sqrt{\tfrac{\mathfrak{A}}{A}}+\mathfrak{B}xxy\sqrt{\tfrac{A}{\mathfrak{A}}}\\+\tfrac{1}{6}Cyy\sqrt{\tfrac{\mathfrak{A}}{A}}+\tfrac{1}{6}\mathfrak{C}xx\sqrt{\tfrac{A}{\mathfrak{A}}}+\left(\tfrac{B\mathfrak{B}}{\sqrt{A\mathfrak{A}}}-\tfrac{2}{3}n\right)xy\\+\left(\tfrac{BB\mathfrak{B}}{4A\sqrt{A\mathfrak{A}}}-\tfrac{nB}{3A}+\tfrac{n\mathfrak{B}}{6\sqrt{A\mathfrak{A}}}-\tfrac{m}{2\sqrt{A}}\right)y+\left(\tfrac{B\mathfrak{B}\mathfrak{B}}{4\mathfrak{A}\sqrt{A\mathfrak{A}}}-\tfrac{n\mathfrak{B}}{3\mathfrak{A}}+\tfrac{nB}{6\sqrt{A\mathfrak{A}}}+\tfrac{m}{2\sqrt{\mathfrak{A}}}\right)x\\+\tfrac{BB\mathfrak{B}\mathfrak{B}}{16A\mathfrak{A}\sqrt{A\mathfrak{A}}}+\tfrac{n\left(B\sqrt{\mathfrak{A}}+\mathfrak{B}\sqrt{A}\right)^2}{16A\mathfrak{A}\sqrt{A\mathfrak{A}}}-\tfrac{nB\mathfrak{B}}{4A\mathfrak{A}}+\tfrac{m\left(B\sqrt{\mathfrak{A}}-\mathfrak{B}\sqrt{A}\right)}{4A\mathfrak{A}}+\tfrac{l}{\sqrt{A\mathfrak{A}}}.\end{gathered}$$

Let this form for the sake of brevity $=S$ and the complete integral will be

$$\frac{S+\sqrt{XY}}{(\alpha+\beta x+\gamma y)^2}=\text{Const.}$$

or

$$S+\sqrt{XY}=\text{Const.}\left(B\sqrt{\mathfrak{A}}+\mathfrak{B}\sqrt{A}+2Ax\sqrt{\mathfrak{A}}+2\mathfrak{A}y\sqrt{A}\right)^2,$$

because also it is possible with this form to be shown more neatly

$$S+\sqrt{XY}=\text{Const.}\left(\tfrac{B}{\sqrt{A}}+\tfrac{\mathfrak{B}}{\sqrt{\mathfrak{A}}}+2x\sqrt{A}+2y\sqrt{\mathfrak{A}}\right)^2.$$

Whereby, while the functions X and Y with the conditions defined before shall be given, in this manner the complete integral of the differential equation will be had

$$\frac{dx}{\sqrt{X}}+\frac{dy}{\sqrt{Y}}=0.$$

21. This investigation can be put in place considerably more generally on attributing such a value to V

$$\frac{1}{(\alpha+\beta x+\gamma y+\delta xy)^2};$$

from which moreover we may be able to overcome the calculation difficulties, I observe, provided the variables x and y be increased or diminished by a constant quantity, since it can be reduced to this form $\frac{1}{(a+xy)^2}$; but with the calculation performed a restoration is easily put in place.

Therefore I will consider this form of differential equation

$$\frac{dx}{\sqrt{X}}+\frac{dy}{\sqrt{Y}}=0,$$

which integral I assume to be returned with the aid of the multiplier $P\sqrt{X}+Q\sqrt{Y}$ *:* as this formula must be integrated

$$Pdx+Qdy+\frac{Qdx\sqrt{Y}}{\sqrt{X}}+\frac{Pdy\sqrt{X}}{\sqrt{Y}}=0.$$

The integral for the latter parts is put in place $=2V\sqrt{XY}$ and there becomes, as we have seen,

$$Q=2X\left(\frac{dV}{dx}\right)+V\frac{dX}{dx} \text{ and } P=2Y\left(\frac{dV}{dy}\right)+V\frac{dY}{dy}.$$

Therefore if $V=\frac{1}{(a+xy)^2}$ and thus

$$\left(\frac{dV}{dx}\right)=\frac{-2y}{(a+xy)^3} \quad \text{and} \quad \left(\frac{dV}{dy}\right)=\frac{-2x}{(a+xy)^3},$$

thus so that we may have

$$Q=\frac{-4Xy}{(a+xy)^3}+\frac{dX}{dx}\frac{1}{(a+xy)^2} \text{ and } P=\frac{-4Yx}{(a+xy)^3}+\frac{dY}{dy}\frac{1}{(a+xy)^2}.$$

But now it must be effected, that the formula $Pdx+Qdy$ admits to integration; the integral of this may be taken in two ways in the end, while either y or x is taken constant, and thus we will obtain

$$\int Pdx=\frac{4Y}{yy(a+xy)}-\frac{2aY}{yy(a+xy)^2}-\frac{dY}{ydy}\cdot\frac{1}{a+xy}+\frac{\Gamma:y}{yy},$$

$$\int Qdx=\frac{4X}{xx(a+xy)}-\frac{2aX}{xx(a+xy)^2}-\frac{dX}{xdx}\cdot\frac{1}{a+xy}+\frac{\Delta:x}{xx},$$

which two forms it is required to be returned equal to each other. Therefore on multiplying by

$xxyy(a+xy)^2$ we will have

$$4xxY(a+xy)-2axxY-\tfrac{xxydY}{dy}(a+xy)+xx\Gamma\colon y\cdot(a+xy)^2$$
$$=4yyX(a+xy)-\ 2ayyX-\tfrac{xyydX}{dx}(a+xy)+yy\Delta\colon x\cdot(a+xy)^2,$$

from which we produce

$$X=Ax^4+2Bx^3+Cxx+2Dx+E,\ \Delta\colon x=Lxx+Mx+N,$$

$$Y=\mathfrak{A}y^4+2\mathfrak{B}y^3+\mathfrak{C}yy+2\mathfrak{D}y+\mathfrak{E},\ \Gamma\colon y=\mathfrak{L}yy+\mathfrak{M}y+\mathfrak{N},$$

$$\tfrac{dX}{dx}=4Ax^3+6Bxx+2Cx+2D \text{ and } \tfrac{dY}{dy}=4\mathfrak{A}y^3+6\mathfrak{B}yy+2\mathfrak{C}y+2\mathfrak{D}.$$

Hence our expressions adopt these forms

$$xxyy(a+xy)^2\int Qdy \quad\Big|\quad xxyy(a+xy)^2\int Pdx$$

$+Lx^4y^4$	$+\mathfrak{L}x^4y^4$
$+Mx^3y^4$	$+2\mathfrak{B}x^3y^4$
$+2Bx^4y^3$	$+\mathfrak{M}x^4y^3$
$+Nxxy^4$	$-2a\mathfrak{A}xxy^4$
$+2(C+aL)x^3x^3$	$+2(\mathfrak{C}+a\mathfrak{L})x^3y^3$
$-2aAx^4yy$	$+\mathfrak{N}x^4yy$
$+2(3D+aM)xxy^3$	$-2a\mathfrak{B}xxy^3$
$-2aBx^3yy$	$+2(3\mathfrak{D}+a\mathfrak{M})x^3yy$
$+aaLxxyy$	$+aa\mathfrak{L}xxyy$
$+2(2E+aN)xy^3$	$+0xy^3$
$+0x^3y$	$+2(2\mathfrak{E}+a\mathfrak{N})x^3y$
$+(2aD+aaM)xyy$	$+0xyy$
$+0xxy$	$+(2a\mathfrak{D}+aa\mathfrak{M})xxy$
$+(2aE+aaN)yy$	$+0yy$
$+0xx$	$+(2a\mathfrak{E}+aa\mathfrak{N})xx.$

22. The equating together of these formulas provides the following determinations

$$\mathfrak{L}=L,\ M=2\mathfrak{B},\ \mathfrak{M}=2B,\ N=-2a\mathfrak{A},\ \mathfrak{N}=-2aA,$$
$$\mathfrak{C}=C,\ D=-a\mathfrak{B},\ \mathfrak{D}=-aB,\ E=aa\mathfrak{A},\ \mathfrak{E}=aaA,$$

so that this differential equation may be considered

$$\frac{dx}{\sqrt{(Ax^4+2Bx^3+Cxx+2Dx+E)}}+\frac{dy}{\sqrt{\left(\frac{E}{aa}y^4-\frac{2D}{a}y^3+Cyy-2aBy+aaA\right)}}=0$$

the complete integral of this is

$$\frac{2Bxxy-\frac{2D}{a}xyy-2aAxx-\frac{2E}{a}yy+2Cxy-2aBx+2Dy+2\sqrt{XY}}{(a+xy)^2}=\text{Const.}$$

Here I note, if we put $y=\frac{-a}{z}$, to produce the initial equation reported

$$\frac{dx}{\sqrt{(Ax^4+2Bx^3+Cxx+2Dx+E)}}+\frac{dz}{\sqrt{(Az^4+2Bz^3+Czz+2Dz+E)}}=0,$$

therefore the integral of this also by the most natural principles of integration can now be assigned, certainly since I may be able to deduce indirectly from that by the previous method. Evidently the integral is

$$Axxzz+Bxz(x+z)+Cxz+D(x+z)+E+G(x-z)^2$$
$$=\sqrt{\left(Ax^4+2Bx^3+Cxx+2Dx+E\right)\left(Az^4+2Bz^3+Czz+2Dz+E\right)},$$

which adopts this form freed from irrationality

$$GG(x-z)+\ 2G\left(Axxzz+Bxz(x+z)+Cxz+D(x+z)+E\right)$$
$$+(BB-AC)xxzz-2ADxz(x+z)-AE(x+z)^2-2BDxz$$
$$-2BE(x+z)+DD-CE=0,$$

which equation in this form agrees with the above [§ 4]

$$(2AG+BB-AC)xxzz+2(BG-AD)xz(x+z)+(GG-AE)(x+z)^2$$
$$-\ 2(2GG+BD-\ CG)xz+2(DG-BE)(x+z)+2EG+DD-CE=0.$$

23. Now if we may wish to examine carefully, under what conditions this differential equation may admit to integration

$$\frac{dx}{\sqrt{\left(Ax^4+2Bx^3+Cxx+2Dx+E\right)}}+\frac{dy}{\sqrt{\left(\mathfrak{A}y^4+2\mathfrak{B}y^3+\mathfrak{C}yy+2\mathfrak{D}y+\mathfrak{E}\right)}}=0,$$

we may consider this to arise from that by putting in place $z=\frac{fy+g}{hy+k}$, thus so that the integral equation will become

$$(2AG+BB-AC)xx(fy+g)^2+2(BG-AD)x(fy+g)(hxy+kx+fy+g)$$
$$+(GG-AE)(hxy+kx+fy+g)^2-2(2GG-CG+BD)x(fy+g)(hy+k)$$
$$+2(DG-BE)(hy+k)(hxy+kx+fy+g)+(2EG+DD-CE)(hy+k)^2=0.$$

But indeed the coefficients $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$, $\mathfrak{E}$ thus must be determined from these quantities *f, g, h, k*, so that there shall be

$$\mathfrak{A}(fk-gh)^2 = Af^4 + 2Bf^3h + Cffhh + 2Dfh^3 + Eh^4,$$
$$\mathfrak{B}(fk-gh)^2 = 2Af^3g + Bff(3gh+fk) + Cfh(fk+gh) + Dhh(3fk+gh) + 2Eh^3k,$$
$$\mathfrak{C}(fk-gh)^2 = 6Affgg + 6Bfg(fk+gh) + C(fk+gh)^2 + 6Dhk(fk+gh) + 6Ehhkk + 2Cfghk,$$
$$\mathfrak{D}(fk-gh)^2 = 2Afg^3 + Bgg(gh+3fk) + Cgk(fk+gh) + Dkk(fk+3gh) + 2Ehk^3,$$
$$\mathfrak{E}(fk-gh)^2 = Ag^4 + 2Bg^3k + Cggkk + 2Dgk^3 + Ek^4.$$

24. But we may consider, how far in general we are able to undertake setting out the calculation of the problem.

Therefore let the proposed equation be $\frac{dx}{\sqrt{X}} + \frac{dy}{\sqrt{Y}} = 0$, which multiplied by $P\sqrt{X} + Q\sqrt{Y}$ is made integrable, and there shall be the integral

$$\int (Pdx + Qdy) + \frac{2\sqrt{XY}}{(\alpha+\beta x+\gamma y+\delta xy)^2} = \text{Const.}$$

and there will be, as we have seen,

$$Q = \frac{-4X(\beta+\delta y)}{(\alpha+\beta x+\gamma y+\delta xy)^3} + \frac{dX}{dx(\alpha+\beta x+\gamma y+\delta xy)^2},$$
$$P = \frac{-4Y(\gamma+\delta x)}{(\alpha+\beta x+\gamma y+\delta xy)^3} + \frac{dY}{dy(\alpha+\beta x+\gamma y+\delta xy)^2},$$

from which we deduce

$$(\gamma+\delta x)^2(\alpha+\beta x+\gamma y+\delta xy)^2 \int Qdy = 2(\beta\gamma-\alpha\delta)X$$
$$+\left(4\delta X - (\gamma+\delta x)\tfrac{dX}{dx}\right)(\alpha+\beta x+\gamma y+\delta xy) + (\alpha+\beta x+\gamma y+\delta xy)^2 \varDelta : x$$

and in a similar manner

$$(\beta+\delta y)^2(\alpha+\beta x+\gamma y+\delta xy)^2 \int Pdx = 2(\beta\gamma-\alpha\delta)Y$$
$$+\left(4\delta Y - (\beta+\delta y)\tfrac{dY}{dy}\right)(\alpha+\beta x+\gamma y+\delta xy) + (\alpha+\beta x+\gamma y+\delta xy)^2 \varGamma : y,$$

which two forms must be agreed upon to be produced, thus so that the first admits to division by $(\gamma+\delta x)^2$, the other indeed by $(\beta+\delta y)^2$ may show the same function. On account of which it is necessary, as the first may admit division by $(\gamma+\delta x)^2$, the latter by $(\beta+\delta y)^2$, therefore everything is required to be satisfied to that requirement.

25. We may set out the first value with the parts depending on y distinguished

$$\text{I. } 2(\beta\gamma-\alpha\delta)X+4\delta(\alpha+\beta x)X-(\alpha+\beta x)(\gamma+\delta x)\tfrac{dX}{dx}+(\alpha+\beta x)^2\varDelta\!:x,$$
$$\text{II. } +y(\gamma+\delta x)\left(4\delta X-(\gamma+\delta x)\tfrac{dX}{dx}+2(\alpha+\beta x)\varDelta\!:x\right),$$
$$\text{III. } +yy(y+\delta x)^2\varDelta\!:x,$$

which expression must be divisible by $(\gamma+\delta x)^2$; therefore since the third part shall be divisible at once, we may put for the second

$$(\alpha+\beta x)\varDelta\!:x+2\delta X=(\gamma+\delta x)R$$

and the first part will be

$$2(\beta\gamma-\alpha\delta)X+2\delta(a+\beta x)X+(a+\beta x)(\gamma+\delta x)R-(a+\beta x)(\gamma+\delta x)\tfrac{dX}{dx},$$

which returns to this form

$$(\gamma+\delta x)\left(2\beta X+(a+\beta x)R-(a+\beta x)\tfrac{dX}{dx}\right),$$

thus so that

$$2\beta X+(a+\beta x)\left(R-\tfrac{dX}{dx}\right)$$

also ought to allow division by $\gamma+\delta x$. For which condition on taking

$$R=\tfrac{\beta}{\delta}\varDelta:x-\tfrac{\alpha+\beta x}{\delta}\varDelta':x+(\gamma+\delta x)S,$$

is satisfied, from which there comes about

$$X=\tfrac{\beta\gamma-\alpha\delta}{2\delta\delta}\varDelta:x-\tfrac{(\alpha+\beta x)(\gamma+\delta x)}{2\delta\delta}\varDelta':x+\tfrac{(\gamma+\delta x)^2}{2\delta}S,$$

and thus the first part will be

$$(\gamma+\delta x)^2\left(\tfrac{\beta}{\delta}R-\tfrac{(\alpha+\beta x)dR}{2\delta dx}\right)+\tfrac{1}{2}(\alpha+\beta x)(\gamma+\delta x)^2S$$

or

$$(\gamma+\delta x)^2\left\{\begin{aligned}&\tfrac{\beta\beta}{\delta\delta}\varDelta\colon x-\tfrac{\beta(\alpha+\beta x)}{\delta\delta}\varDelta'\colon x+\tfrac{(\alpha+\beta x)^2}{2\delta\delta}\varDelta''\colon x\\&+\tfrac{\beta(\alpha+\beta x)}{\delta}S-\tfrac{(\alpha+\beta x)(\gamma+\delta x)}{2\delta}\cdot\tfrac{dS}{dx}\end{aligned}\right\};$$

then with the second

$$y(\gamma+\delta x)^2\left\{\begin{aligned}&\tfrac{2\beta}{\delta}\varDelta\colon x-\tfrac{\alpha+\beta x}{\delta}\varDelta'\colon x+\tfrac{(\alpha+\beta x)(\gamma+\delta x)}{2\delta\delta}\varDelta''\colon x\\&+(\gamma+\delta x)S-\tfrac{(\gamma+\delta x)^2}{2\delta}\cdot\tfrac{dS}{dx}\end{aligned}\right\};$$

and with the third

$$yy(\gamma+\delta x)^2\varDelta\colon x.$$

On account of which the value of the formula

$$(\alpha+\beta x+\gamma y+\delta xy)^2\int Qdy$$

will be

$$\begin{aligned}&\tfrac{\beta\beta}{\delta\delta}\varDelta\colon x+\tfrac{2\beta}{\delta}y\varDelta\colon x+yy\varDelta\colon x-\tfrac{\beta(\alpha+\beta x)}{\delta\delta}\varDelta'\colon x-\tfrac{\alpha+\beta x}{\delta}y\varDelta'\colon x\\&+\tfrac{(\alpha+\beta x)^2}{2\delta\delta}\varDelta''\colon x+\tfrac{(\alpha+\beta x)(\gamma+\delta x)}{2\delta\delta}y\varDelta''\colon x\\&+\tfrac{\beta}{\delta}(\gamma+\delta x)S+(\gamma+\delta x)yS-\tfrac{(\alpha+\beta x)(\gamma+\delta x)}{2\delta}\cdot\tfrac{dS}{dx}-\tfrac{(\gamma+\delta x)^2}{2\delta}y\tfrac{dS}{dx}\end{aligned}$$

or thus expressed more concisely

$$\begin{aligned}&\tfrac{(\beta+\delta y)^2}{\delta\delta}\varDelta\colon x-\tfrac{(\alpha+\beta x)(\beta+\delta y)}{\delta\delta}\varDelta'\colon x\\&+\tfrac{(\alpha+\beta x)(\alpha+\beta x+\gamma y+\delta xy)}{2\delta\delta}\varDelta''\colon x+\tfrac{(\gamma+\delta x)(\beta+\delta y)}{\delta}S-\tfrac{(\gamma+\delta x)(\alpha+\beta x+\gamma y+\delta xy)}{2\delta}\cdot\tfrac{dS}{dx},\end{aligned}$$

to which the other must become equal, which is

$$\begin{aligned}&\tfrac{(\gamma+\delta x)^2}{\delta\delta}\varGamma\colon y-\tfrac{(\alpha+\gamma y)(\gamma+\delta x)}{\delta\delta}\varGamma'\colon y\\&+\tfrac{(\alpha+\gamma y)(\alpha+\beta x+\gamma y+\delta xy)}{2\delta\delta}\varGamma''\colon y+\tfrac{(\beta+\delta y)(\gamma+\delta x)}{\delta}\mathfrak{S}-\tfrac{(\beta+\delta y)(\alpha+\beta x+\gamma y+\delta xy)}{2\delta}\cdot\tfrac{d\mathfrak{S}}{dx}.\end{aligned}$$

26. But if now we may put

$$\varDelta : x = \delta\delta\left(Axx + 2Bx + C\right) \text{ and } S = \delta\left(Dxx + 2Ex + F\right),$$

likewise

$$\varGamma : y = \delta\delta\left(\mathfrak{A}xx + 2\mathfrak{B}x + \mathfrak{C}\right) \text{ and } \mathfrak{S} = \delta\left(\mathfrak{D}xx + 2\mathfrak{E}x + \mathfrak{F}\right),$$

our expression may be found thus expanded out

$\left(\alpha+\beta x+\gamma y+\delta xy\right)^2\int Qdy$	$\left(\alpha+\beta x+\gamma y+\delta xy\right)^2\int Pdx$
$+\delta\delta Axxyy$	$+\delta\delta\mathfrak{A}xxyy$
$+2\delta\delta Bxyy$	$+\delta\left(\gamma\mathfrak{A}-\beta\mathfrak{D}+\delta\mathfrak{E}\right)xyy$
$+\delta\left(\beta A-\gamma D+\delta E\right)xxy$	$+2\delta\delta\mathfrak{B}xxy$
$+\delta\delta Cyy$	$+\delta\left(\gamma\mathfrak{E}-\alpha\mathfrak{D}\right)yy$
$+\delta\left(\beta E-\alpha D\right)xx$	$+\delta\delta\mathfrak{C}xx$
$+\left(2\beta\delta B+\left(\beta\gamma-\alpha\delta\right)A-\gamma\gamma D+\delta\delta F\right)xy$	$+\left(2\gamma\delta\mathfrak{B}+\left(\beta\gamma-\alpha\delta\right)\mathfrak{A}-\beta\beta\mathfrak{D}+\delta\delta\mathfrak{F}\right)xy$
$+\left(\alpha\gamma A-2\alpha\delta B+2\beta\delta C-\gamma\gamma E+\gamma\delta F\right)y$	$+\left(\gamma\delta\mathfrak{F}+\left(\beta\gamma-\alpha\delta\right)\mathfrak{E}-\alpha\beta\mathfrak{D}\right)y$
$+\left(\beta\delta F+\left(\beta\gamma-\alpha\delta\right)E-\alpha\gamma D\right)x$	$+\left(\alpha\beta\mathfrak{A}-2\alpha\delta\mathfrak{B}+2\gamma\delta\mathfrak{C}-\beta\beta\mathfrak{E}+\beta\delta\mathfrak{F}\right)x$
$+\alpha\alpha A-2\alpha\beta B+\beta\beta C-\alpha\gamma E+\beta\gamma F$	$+\alpha\alpha\mathfrak{A}-2\alpha\beta\mathfrak{B}+\beta\beta\mathfrak{C}-\alpha\gamma\mathfrak{E}+\beta\gamma\mathfrak{F},$

from which not unless the following six determinations may be deduced,

$$\mathfrak{A}=A,$$
$$\mathfrak{B}=\tfrac{\beta A-\gamma D}{2\delta}+\tfrac{1}{2}E,$$
$$\mathfrak{C}=\tfrac{\beta E-\alpha D}{\delta},$$
$$\mathfrak{D}=\tfrac{2\gamma\delta B-\gamma\gamma A-\delta\delta C}{\alpha\delta-\beta\gamma},$$
$$\mathfrak{E}=\tfrac{2\alpha\delta B-\alpha\gamma A-\beta\delta C}{\alpha\delta-\beta\gamma},$$
$$\mathfrak{F}=F-\tfrac{\gamma E}{\delta}-\tfrac{\alpha\beta\gamma A-2\alpha\beta\delta B+\beta\beta\delta C}{\delta\left(\alpha\delta-\beta\gamma\right)};$$

for it is satisfied by all these conditions. Therefore since all the letters *A*, *B*, *C*, *D*, *E*, *F* together with α, β, γ, δ remain left to our choice, from which then the function is deduced

$$\begin{aligned}2X &= \delta\delta D x^4 + 2\delta(\delta E + \gamma D - \beta A)x^3 \\ &+ (\delta\delta F + 4\gamma\delta E + \gamma\gamma D - 2\beta\delta B - (\beta\gamma + 3\alpha\delta)A)xx \\ &+ 2(\gamma\delta F + \gamma\gamma E - \alpha\gamma A - 2\alpha\delta B)x + \gamma\gamma F - 2a\gamma B + (\beta\gamma - \alpha\delta)C.\end{aligned}$$

27. But I shall not pursue this calculation further, since now indeed it may stand up to be a direct method and to have uncovered the similar natures of the matter, which led to the same integrations generally, which at one time at length I elicited by other principles. Therefore in the increase of this skill it will make a difference that the new method be examined thoroughly with great interest by all.

This in the end I observe also that another form of multipliers can be used, with the aid of which such a form

$$\frac{dx}{\sqrt{X}} + \frac{dy}{\sqrt{Y}} = 0$$

may be returned integrable. Evidently the multiplier $M = P + Q\sqrt{XY}$ may be put in place, so that this must be the form of the integral

$$\frac{Pdx}{\sqrt{X}} + Qdy\sqrt{X} + \frac{Pdy}{\sqrt{Y}} + Qdx\sqrt{Y} = 0.$$

The integral of the first part may be devised $= 2R\sqrt{X}$, indeed of the latter part $= 2S\sqrt{Y}$ so that the complete integral shall be

$$R\sqrt{X} + S\sqrt{Y} = \text{Const.},$$

and from the expansion done there is found

$$P = RdX + 2X\left(\frac{dR}{dx}\right), \quad P = \frac{SdY}{dy} + 2Y\left(\frac{dS}{dy}\right),$$
$$Q = 2\left(\frac{dR}{dy}\right), \qquad Q = 2\left(\frac{dS}{dx}\right).$$

Therefore since there must be $\left(\frac{dR}{dy}\right) = \left(\frac{dS}{dx}\right)$, it is evident the formula $Rdx + Sdy$ must be integrable. Moreover it is not worth the effort, that this integral be considered algebraic, but it is sufficient that the character of the integration shall be provided.

28. For there is assumed

$$R=\frac{y}{\alpha+\beta xy+\gamma xxyy} \text{ and } S=\frac{x}{\alpha+\beta xy+\gamma xxyy}$$

and there will be

$$Q=\frac{2\alpha-2\gamma xxyy}{(\alpha+\beta xy+\gamma xxyy)^2}$$

and

$$P=\frac{ydX}{dx(\alpha+\beta xy+\gamma xxyy)}-\frac{2Xyy(\beta+2\gamma xy)}{(\alpha+\beta xy+\gamma xxyy)^2}$$

likewise

$$P=\frac{xdY}{dy(\alpha+\beta xy+\gamma xxyy)}-\frac{2Yxx(\beta+2\gamma xy)}{(\alpha+\beta xy+\gamma xxyy)^2},$$

thus so that there may be considered

$$\begin{aligned}(\alpha+\beta xy+\gamma xxyy)^2 P &= \tfrac{ydX}{dx}(\alpha+\beta xy+\gamma xxyy)-2yyX(\beta+2\gamma xy)\\ &= \tfrac{xdY}{dy}(\alpha+\beta xy+\gamma xxyy)-2xxY(\beta+2\gamma xy).\end{aligned}$$

There may be put in place

$$\begin{aligned}X &= Ax^4+2Bx^3+Cxx+2Dx+E\\ Y &= \mathfrak{A}y^4+2\mathfrak{B}y^3+\mathfrak{C}yy+2\mathfrak{D}y+\mathfrak{E}\end{aligned}$$

and the values from these two may be understood to be made equal to each other, so that there shall be

$$\beta=0,\quad B=0,\quad \mathfrak{B}=0,\quad D=0,\quad \text{and}\quad \mathfrak{D}=0;$$

then indeed these become

$$\begin{aligned}&\text{I.} \ = -2\gamma Cx^3y^3+4\alpha Ax^3y-4\gamma Exy^3+2\alpha Cxy,\\ &\text{II.} \ = -2\gamma\mathfrak{C}x^3y^3+4\alpha\mathfrak{A}x^3y-4\gamma\mathfrak{E}xy^3+2\alpha\mathfrak{C}xy,\end{aligned}$$

from which it is deduced

$$\mathfrak{C}=C,\quad \frac{\alpha}{\gamma}=-\frac{\mathfrak{E}}{A}=\frac{-E}{\mathfrak{A}} \quad \text{or} \quad \mathfrak{A}\mathfrak{E}=AE.$$

Hence there will be

$$X = Ax^4 + Cxx - \tfrac{\alpha}{\gamma}\mathfrak{A}, \quad Y = \mathfrak{A}y^4 + \mathfrak{C}yy - \tfrac{\alpha}{\gamma}A$$

and the complete integral of the equation

$$\frac{dx}{\sqrt{\left(Ax^4 + Cxx - \frac{\alpha}{\gamma}\mathfrak{A}\right)}} + \frac{dy}{\sqrt{\left(\mathfrak{A}y^4 + \mathfrak{C}yy - \frac{\alpha}{\gamma}A\right)}} = 0$$

will be

$$y\sqrt{\left(Ax^4 + Cxx - \tfrac{\alpha}{\gamma}\mathfrak{A}\right)} + x\sqrt{\left(\mathfrak{A}y^4 + \mathfrak{C}yy - \tfrac{\alpha}{\gamma}A\right)} = \text{Const.}\left(\alpha + \gamma xxyy\right).$$

29. From these examples it is easily understood that almost a new kind of analysis is to be desired at this stage, where operations of this kind are able to be put in place in a certain order and to be extended further, from which goal indeed until now we have been far removed. Yet meanwhile that, which so far I have explained, may be considered to be of the greatest interest according to the general principles of integration mentioned initially requiring to be established, since from the benefit of which so far these especially arduous integrations are able to be extricated by suitable multipliers and from known principles, which have been considered transcending. Indeed for me, since first I might have fallen upon these, by no other way was it being considered to deduce from that [precept] besides that, for which then I had a need; for I was not yet able to judge always, how often the complete integral of a differential equation of some kind should be constructed from that multiplier, from which it can be concluded that an integral be returned, which conclusion, if the integral were only a particular one, would by no means be of worth. On account of which, of these particular integrations, which at one time likewise I had followed from the same principle with others, by far otherwise has an account been prepared, nor at this point is it possible to understand, how by a certain direct and natural method one may be able to come upon the same.

30. Therefore from that it will be more worthwhile to examine more carefully the nature of these particular integrals, which indeed will be made by the consideration of the simplest case. I had found therefore that the particular integral of this differential equation

$$dx\sqrt{(1+xx)} + dy\sqrt{(1+yy)} + nydx + nxdy = 0$$

to be

$$xx + yy + 2xy\sqrt{(1+nn)} = nn$$

and also I have discovered innumerable integrals in a similar manner for differential equations of this kind, which neither depend on logarithms nor on the quadrature of the circle; whereby this equation is being examined thus, as if it may not be integrated by logarithms. Here therefore in the first place it is sought, in what direct way this particular integral may be concluded from the form of the differential, then how the differential equation must be prepared, so that it may be able to be shown with such a particular integral.

Therefore according to these questions I note at first that an algebraic equation is to be the complete integral of this differential equation

$$\frac{dx}{\sqrt{(1+xx)}}+\frac{dy}{\sqrt{(1+yy)}}=0,$$

then indeed to follow from that

$$x+y\sqrt{(1+nn)}=n\sqrt{(1+yy)} \text{ and } y+x\sqrt{(1+nn)}=n\sqrt{(1+xx)},$$

thus so that both $\sqrt{(1+xx)}$ as well as $\sqrt{(1+yy)}$ may be able to be expressed rationally by x and y. Hence since therefore by differentiation there shall be

$$\frac{xdx}{\sqrt{(1+xx)}}=\frac{dy+dx\sqrt{(1+nn)}}{n} \text{ and } \frac{ydy}{\sqrt{(1+yy)}}=\frac{dx+dy\sqrt{(1+nn)}}{n},$$

if some multiple of the form

$$\frac{dx}{\sqrt{(1+xx)}}+\frac{dy}{\sqrt{(1+yy)}}=0,$$

may be added to these, always a differential equation is to be produced, which at any rate may be satisfied by a particular algebraic equation. In general therefore a particular integral of this differential equation

$$\frac{dx+Pxdx}{\sqrt{(1+xx)}}+\frac{dy+Qydy}{\sqrt{(1+yy)}}=\frac{Pdy+Qdx+(Pdx+Qdy)\sqrt{(1+nn)}}{n}$$

will be

$$xx+yy+2xy\sqrt{(1+nn)}=nn.$$

Now let there be $P=x$ and $Q=y$ and for this equation

$$dx\sqrt{(1+xx)}+dy\sqrt{(1+yy)}=\frac{xdy+ydx+(xdx+ydy)\sqrt{(1+nn)}}{n},$$

indeed will be satisfied with the integral becoming

$$xdx+ydy=-(xdy+ydx)\sqrt{(1+nn)},$$

thus so that this differential equation may be considered

$$dx\sqrt{(1+xx)}+dy\sqrt{(1+yy)}+nxdy+nydx=0,$$

therefore for which the integral given above agrees particularly.

31. Now we may transfer this to more widely extended cases, and after the complete integral had been found of this equation

$$\frac{dx}{\sqrt{X}}+\frac{dy}{\sqrt{Y}}=0,$$

which shall be $W=\text{Const.}$, it may be noted hence always each root value $\sqrt{X}$ and $\sqrt{Y}$ is to be defined by rational functions of x and y. Therefore thus let there be

$$\sqrt{X}=R \text{ and } \sqrt{Y}=S$$

and thus

$$\frac{dX}{\sqrt{X}}=2dR \text{ and } \frac{dY}{\sqrt{Y}}=2dS.$$

Now let P be some function of x and Q of y and hence this equation may be put together

$$\frac{dx+PdX}{\sqrt{X}}+\frac{dy+QdY}{\sqrt{Y}}-2PdR-2QdS=0,$$

for which the algebraic equation $W=\text{Const.}$ surely satisfies particularly. Hence if P and Q thus may be taken, so that the formula $PdR+QdS$ admits integration, the integral of which shall be $=V$ the transcending equation will arise

$$\int\frac{dx+PdX}{\sqrt{X}}+\int\frac{dy+QdY}{\sqrt{Y}}-2V=\text{Const.},$$

to which equation $W=\text{Const.}$ or, it is satisfied particularly by the values thence deduced, $\sqrt{X}=R$ or $\sqrt{Y}=S$. Therefore it may be seen how such a way of reasoning brings to light particular integrals of this kind otherwise being found most difficult.

SUPPLEMENTUM,

CONTINENS

EVOLUTIONEM CASUUM SINGULARIUM CIRCA INTEGRATIONEM

AEQUATIONUM DIFFERENTIALIUM.

EVOLUTIO CASUUM PRORSUS SINGULARIUM CIRCA INTEGRATIONEM AEQUATIONUM DIFFERENTIALIUM

1. Cum adhuc plurimae atque inter se maxime discrepantes methodi sint in medium allatae aequationes differentiales integrandi, quaestio exoritur summi sane momenti, an non unica detur eaque aequabilis methodus, cuius ope omnes illas diversas aequationes differentiales, quas etiamnum resolvere licuit, integrari queant. Nullum enim est dubium, quin inventio talis methodi maxima incrementa in universam Analysin esset allatura.

Pluribus Geometris quidem separatio binarum variabilium huiusmodi methodum suppeditare est visa, cum omnes aequationum differentialium integrationes vel hac ratione sint integratae vel eo facile possint revocari. Praeterquam autem quod haec methodus substitutionibus absolvitur, quae plerumque non minorem sagacitatem postulant quam id ipsum, quod quaeritur, ac nonnunquam soli casui deberi videntur, haec methodus etiam neutiquam extenditur ad aequationes differentiales secundi altiorumque graduum; et qui tales aequationes adhuc tractaverunt, longe alia artificia in subsidium vocare sunt coacti. Quamobrem separationem variabilium nequaquam tanquam methodum uniformem ac latissime patentem spectare licet, quae omnes integrationes, quae adhuc successerunt, in se complectatur.

2. Talem autem methodum universalem iam pridem mihi equidem indicasse videor, dum ostendi proposita quacunque aequatione differentiali sive primi sive altioris gradus semper dari eiusmodi quantitatem, per quam si aequatio multiplicetur, evadat integrabilis, ita ut hoc modo nulla plane substitutione alibi anxie quaerenda sit opus. Ex quo non dubito hanc methodum aequationes differentiales ope multiplicationum ad integrabilitatem revocandi tanquam latissime patentem atque naturae maxime convenientenl pronunciare, cum nulla integratio adhuc sit expedita, quae hoc modo non facile absolvi possit. Cum scilicet omnis aequatio differentialis primi gradus in hac forma $Pdx+Qdy=0$ contineatur denotantibus litteris P et Q functiones quascunque binarum variabilium x et y, semper datur eiusmodi multiplicator M, itidem functio quaedam alubarum variabilium x et y, ut facta multiplicatione haec forma $MPdx+MQdy$ fiat integrabilis; cuius propterea integrale quantitati constanti arbitrariae aequatum exhibebit aequationem integralem aequationis differentialis propositae $Pdx+Qdy=0$, quae eadem ratio quoque in aequationibus differentialibus altiorum graduum locum habet. Verum hoc argumentum hic fusius exponere non est animus, sed potius praestantiam huius methodi prae separatione variabilium etiam eiusmodi casibus, quibus id minime videatur, simulque summam eius utilitatem hic declarare constitui.

3. Quoties scilicet in aequatione differentiali variabiles x et y iam sunt separatae, totum negotium vulgo ut iam confectum spectari solet, quandoquidem huius aequationis $Xdx+Ydy=0$, ubi X denotat functionem solius x et Y solius y, integrale in promtu est

$$\int Xdx + \int Ydy = \text{Const}.$$

Interim tamen saepenumero usu venire potest, ut hoc pacta neutiquam forma integralis simplicissima obtineatur vel ea demum per plures ambages inde derivari debeat. Veluti ex hac aequatione

$$\frac{dx}{x} + \frac{dy}{y} = 0$$

primo elicitur integrale logarithmicum

$$lx + ly = la,$$

unde quidem statim se prodit algebraicum $xy = a$. Verum ex hac forma

$$\frac{dx}{aa+xx} + \frac{dy}{aa+yy} = 0$$

integratio solita praebet

$$\text{Ang. tang.}\frac{x}{a} + \text{Ang. tang.}\frac{y}{a} = \text{Const.}$$

unde non tam facile forma integralis algebraica $\frac{x+y}{aa-xy} = C$ deducitur. Ac proposita hac forma

$$\frac{dx}{\sqrt{(\alpha+\beta x+\gamma xx)}} + \frac{dy}{\sqrt{(\alpha+\beta y+\gamma yy)}} = 0$$

in genere ne patet quidem, utrum utraque pars integralis arcu circulari an logarithmo exprimatur. Interim tamen eius integrale ita algebraice exhiberi potest

$$CC(x-y)^2 + 2\gamma Cxy + \beta C(x+y) + 2\alpha C + \tfrac{1}{4}\beta\beta - \alpha\gamma = 0,$$

quae certe forma simplicissima nonnisi per plures ambages ex integrali transcendente derivatur.

4. His quidem casibus perspicitur, quomodo reductionem ad formam algebraicam institui oporteat; sed ante aliquot annos eiusmodi integrationes protuli, in quibus ne hoc quidem ullo modo praestari potest. Veluti si proposita sit haec aequatio

$$\frac{dx}{\sqrt{(1+x^4)}} + \frac{dy}{\sqrt{(1+y^4)}} = 0$$

integrationem neque per logarithmos neque arcus circulares expedire licet, ut inde deinceps simili ratione aequatio algebraica colligi posset; interim tamen ostendi huius integrale idque adeo completum hoc modo algebraice exprimi

$$0 = 2C + (CC-1)(xx+yy) - 2(1+CC)xy + 2Cxxyy,$$

ubi C denotat constantem per integrationem ingressam. Quin etiam huius aequationis multo latius patentis

$$\frac{dx}{\sqrt{\left(\alpha+2\beta x+\gamma xx+2\delta x^3+\varepsilon x^4\right)}}+\frac{dy}{\sqrt{\left(\alpha+2\beta y+\gamma yy+2\delta y^3+\varepsilon y^4\right)}}=0$$

integrale completum est

$$\begin{aligned}0&=2\alpha C+\beta\beta-\alpha\gamma+2\left(\beta C-\alpha\delta\right)\left(x+y\right)+\left(CC-\alpha\varepsilon\right)\left(xx+yy\right)\\&+2\left(\gamma C-CC-\alpha\varepsilon-\beta\delta\right)xy+2\left(\delta C-\beta\varepsilon\right)xy\left(x+y\right)+\left(2\varepsilon C+\delta\delta-\gamma\varepsilon\right)xxyy\end{aligned}$$

denotante C item constantem quantitatem arbitrariam per integrationem inventam. His igitur casibus perspicuum est separationem variabilium, qua aequationes differentiales sunt praeditae, nihil plane iuvare ad integralia earum forma algebraica contenta eruenda, ex quo merito eiusmodi methodus desideratur, cuius beneficio haec integralia statim ex aequationibus differentialibus investigari potuissent, in quo negotio certe omnes ingenii vires tentasse non pigebit.

5. Observavi igitur hunc scopum ope multiplicatorum idoneorum obtineri posse, quibus aequationes differentiales multiplicatae ita integrabiles evadant, ut integraIia statim algebraice expressa prodeant. Quod quo clarius perspiciatur, ab aequationem primum proposita $\frac{dx}{x}+\frac{dy}{y}=0$ exordiar, quae per xy multiplicata statim praebet $ydx+xdy=0$, cuius integrale est $xy=C$. Hoc ergo modo sublata separatione aequatio in aliam transformatur, quae integrationem admittit, ex quo intelligitur methodum ope multiplicatorum integrandi id praestare, quod a separatione variabilium immediate expectari nequeat. Idem evenit in aequatione $\frac{mdx}{x}+\frac{ndy}{y}=0$, quae per $x^m y^n$ multiplicata integrale praebet $x^m y^n=C$, dum ex ipsa aequatione. proposita statim ad logarithmos fuisset perventum. Simili modo si haec aequatio separata

$$\frac{dx}{1+xx}+\frac{dy}{1+yy}=0$$

multiplicetur in $\frac{(1+xx)(1+yy)}{(x+y)^2}$ aequatio resultans

$$\frac{dx(1+yy)+dy(1+xx)}{(x+y)^2}=0,$$

integrationem iam sponte admittit praebetque integrata

$$\frac{-1+xy}{x+y}=\text{Const.}\quad\text{seu}\quad\frac{x+y}{1-xy}=a.$$

Hanc vero aequationem

$$\frac{2dx}{1+xx}+\frac{dy}{1+yy}=0$$

multiplicari convenit in $\frac{(xx+1)^2(1+yy)}{(2xy+xx-1)^2}$, ut prodeat

$$\frac{2dx(1+xx)(1+yy)+dy(xx+1)^2}{(2xy+xx-1)^2}=0,$$

cuius integrale reperitur

$$\frac{xxy-2x-y}{2xy+xx}=\text{Const.} \text{ seu } \frac{2x+y-xxy}{2xy+xx-1}=a.$$

6. Contra haec exempla, quibus integralia algebraica sine subsidio separationis sunt eruta, obiicietur multiplicatores negotium hoc conficientes ex ipsis integralibus illis transcendentibus, ad quae separatio variabilium immediate perducit, esse conclusos iisque adeo praestantiam methodi per multiplicatores procedentis neutiquam probali. Cui quidem obiectioni primum respondeo priora exempla statim ab inventis integrationis principiis simili modo fuisse expedita, antequam integratio per logarithmos erat explorata, quae ergo nullum subsidium eo attulisse est censenda. Tum vero, quamvis concedam in posterioribus exemplis integrationem per arcus circulares multiplicatores illos idoneos commode suppeditasse, id tamen in ipsa evolutione minus cernitur eademque integratio sine dubio inveniri potuisset, antequam constaret formulae $\frac{dx}{1+xxx}$ integrale esse arcum circuli tangenti x respondentem. Verum aequatio supra [§ 4] allata

$$\frac{dx}{\sqrt{(1+x^4)}}+\frac{dy}{\sqrt{(1+y^4)}}=0,$$

cuius integrale completum algebraice exhibere licet, nulli amplius dubio locum relinquit; cum enim neutrius partis integrale ne concessis quidem logarithmis vel arcubus circularibus exhiberi possit eiusque forma ad genus quantitatum transcendentium etiamnum incognitum sit referenda, haec certe nullum auxilium ad integrale algebraicum inveniendum attulisse censeri potest. Atque hoc multo magis de aequatione illa latius patente in § 4 proposita est tenendum, quippe cuius integratio omnino singularis ex principiis longe diversissimis a me est eruta.

7. Methodus autem, qua tum sum usus, tantopere est abscondita, ut vix ulla via ad eadem integralia perducens patere videatur, et cum separatio variabilium nihil plane eo contulisset, vix etiam quicquam ab altera methodo ad multiplicatores adstricta sperari posse videbatur, propterea quod tum ipse adhuc in ea opinione versabar per multiplicatores nihil praestari posse, nisi quatenus separatio variabilium eodem manuducat, quandoquidem quaestio differentialia tantum primi gradus implicaret. Deinceps autem re diligentius considerata perspexi, quoties aequationis cuiusque differentialis integrale completum exhibere licet, ex eo vicissim semper eiusmodi multiplicatorem elici posse, per quem si aequatio differentialis multiplicetur, non solum fiat integrabilis, sed etiam integrata id ipsum integrale, quod iam erat cognitum, reproducere debeat; ad hoc autem omnino necesse est, ut integrale completum sit exploratum, dum ex integralibus particularibus nihil plane pro hoc scopo concludere licet.

Si enim proposita sit aequatio differentialis

$$Pdx + Qdy = 0,$$

cuius integrale completum undecunque sit cognitum, constabit id aequatione, quae praeter binas variabiles x et y et quantitates constantes in ipsa aequatione differentiali contentas insuper quantitatem constantem novam prorsus ab arbitrio nostro pendentem complectetur. Quae si littera C indicetur, eruatur eius valor ex aequatione integrali ac reperiatur $C = V$ eritque V certa quaedam functio ipsarum x et y; tum autem hac aequatione differentiata $0 = dV$ differentiale dV necessario ita formulam differentialem $Pdx + Qdy$ continere debet, ut sit

$$dV = M\left(Pdx + Qdy\right),$$

ex qua forma multiplicator M ad hoc integrale $C = V$ perducens sponte se offert.

8. Quo haec operatio aliquot exemplis illustretur, sumatur primo haec aequatio

$$\frac{mdx}{x} + \frac{ndy}{y} = 0$$

cuius integrale completum cum sit $x^m y^n = C$, instituta differentiatione prodit

$$0 = mx^{m-1}y^n dx + nx^m y^{n-1} \quad \text{seu} \quad 0 = x^m y^n \left(\frac{mdx}{x} + \frac{ndy}{y}\right),$$

unde patet multiplicatorem ad hoc integrale ducentem esse $x^m y^n$.

Deinde cum huius aequationis

$$\frac{dx}{1+xx} + \frac{dy}{1+yy} = 0$$

integrale completum sit $1 - xy = C\left(x + y\right)$, valor constantis arbitrariae hinc fit $C = \frac{1-xy}{x+y}$, cuius differentiatio praebet

$$0 = \frac{-dx(1+yy)-dy(1+xx)}{(x+y)^2} \quad \text{seu} \quad 0 = \frac{(1+xx)(1+yy)}{(x+y)^2}\left(\frac{dx}{1+xx} + \frac{dy}{1+yy}\right),$$

unde multiplicator quaesitus est $= \frac{(1+xx)(1+yy)}{(x+y)^2}$.

Proposita porro sit haec aequatio

$$\frac{dx}{\sqrt{(\alpha+2\beta x+\gamma xx)}} + \frac{dy}{\sqrt{(\alpha+2\beta y+\gamma yy)}} = 0,$$

cuius integrale completum

$$CC\left(x - y\right)^2 - 2C\left(\alpha + \beta x + \beta y + \gamma xy\right) + \beta\beta - \alpha\gamma = 0$$

dat primo

$$C = \frac{\alpha+\beta(x+y)+\gamma xy+\sqrt{\left(\alpha\alpha+2\alpha\beta(x+y)+\alpha\gamma(xx+yy)+4\beta\beta xy+2\beta\gamma xy(x+y)+\gamma\gamma xxyy\right)}}{(x-y)^2}$$

seu

$$C=\frac{\alpha+\beta(x+y)+\gamma xy+\sqrt{(\alpha+2\beta x+\gamma xx)(\alpha+2\beta y+\gamma yy)}}{(x-y)^2},$$

vel concinnius

$$\frac{\beta\beta-\alpha\gamma}{C}=\alpha+\beta(x+y)+\gamma xy+\sqrt{(\alpha+2\beta x+\gamma xx)(\alpha+2\beta y+\gamma yy)},$$

unde differentiando fit

$$0=dx(\beta+\gamma y)+dy(\beta+\gamma x)$$
$$+\frac{dx(\beta+\gamma x)\sqrt{(\alpha+2\beta y+\gamma yy)}}{\sqrt{(\alpha+2\beta x+\gamma xx)}}+\frac{dy(\beta+\gamma y)\sqrt{(\alpha+2\beta x+\gamma xx)}}{\sqrt{(\alpha+2\beta y+\gamma yy)}}$$

hincque colligitur multiplicator quaesitus

$$M=(\beta+\gamma x)\sqrt{(\alpha+2\beta y+\gamma yy)}+(\beta+\gamma y)\sqrt{(\alpha+2\beta x+\gamma xx)}.$$

9. Simili modo pro aequatione magis complexa

$$\frac{dx}{\sqrt{(\alpha+2\beta x+\gamma xx+2\delta x^3+\varepsilon x^4)}}+\frac{dy}{\sqrt{(\alpha+2\beta y+\gamma yy+2\delta y^3+\varepsilon y^4)}}=0$$

ex eius integrali completo supra [§ 4] exhibito multiplicator idoneus M investigari poterit, ex quo, si statim fuisset cognitus, idem hoc integrale immediate elici potuisset. Verum hic opus multo maius molior, quod autem primo conatu neutiquam ad finem perducere licebit; ex quo satis mihi equidem praestitisse videbor, si saltem prima quasi lineamenta novae atque maxime desiderandae methodi adumbra vero, cuius ope proposita huiusmodi aequatione differentiali rnultiplicator idoneus eam reddens integrabilem inveniri queat.

Ac primo quidem in hoc negotio plurimum observasse iuvabit, si unicus huiusmodi multiplicator innotuerit, ex eo facile infinitos alios idem officium praestantes erui posse. Quodsi enim multiplicator M aequationem differentialem $Pdx+Qdy=0$ integrabilem reddat, ita ut sit $\int M(Pdx+Qdy)=V$ ideoque aequatio integralis $V=C$, quoniam formula $dV=M(Pdx+Qdy)$ per functionem quamcunque quantitatis V multiplicata perinde manet integrabilis, perspicuum est hanc formam $Mf:V$ quaecunque functio ipsius V pro $f:V$ accipiatur, semper multiplicatorem idoneum praebere, cum sit

$$(Pdx+Qdy)Mf:V=dVf:V$$

ideoque integrabile. Inter infinitos igitur hos multiplicatores idoneos quovis casu eum eligi conveniet, qui negotium facillime couficiat et integrale, si fuerit algebraicum, forma simplicissima exhibeat. Etiamsi enim integrale revera sit algebraicum, omnino fieri potest, ut id ne suspicari q.uidem liceat, nisi multiplicator idoneus in usum vocetur, quemadmodum superiora exempla abunde declarant.

10. Sit ergo aequatio differentialis proposita huius formae

$$\frac{dx}{X}+\frac{dy}{Y}=0,$$

in qua X sit functio solius x et Y solius y, atque investigari oporteat eiusmodi multiplicatorem M, quo illa aequatio algebraice integrabilis reddatur, siquidem fieri potest; quod cum raro eveniat, vicissim assumta multiplicatoris forma M indagasse iuvabit functiones X et Y.

Sit primo multiplicator

$$M=\frac{XY}{(\alpha+\beta x+\gamma y)^2},$$

ut integrabilis esse debeat haec forma

$$\frac{Ydx+Xdy}{(\alpha+\beta x+\gamma y)^2}$$

Hinc sumta y constante colligitur integrale

$$\frac{-Y}{\beta(\alpha+\beta x+\gamma y)}+\Gamma:y,$$

sumta autem x constante prodit

$$\frac{-X}{\gamma(\alpha+\beta x+\gamma y)}+\Delta:x,$$

quas ambas formas inter se aequales esse oportet; unde fit

$$-\gamma Y+\beta\gamma(\alpha+\beta x+\gamma y)\Gamma:y=-\beta X+\beta\gamma(\alpha+\beta x+\gamma y)\Delta:x$$

seu

$$\beta X-\gamma Y=\beta\gamma(\alpha+\beta x+\gamma y)(\Delta:x-\Gamma:y)$$

sicque patet functiones $\Delta:x$ et $\Gamma:y$ ita comparatas esse debere, ut evoluto posteriori membro termini, qui simul x et y continerent, se mutuo tollant.

Ex quo intelligitur fore

$$\Delta:x=m\beta x+\text{ Const. et }\Gamma:y=m\gamma y+\text{Const}.$$

Statuamus ergo

$$\Delta:x-\Gamma:y=m\beta x-m\gamma y+n$$

fietque

$$\beta X-\gamma Y=\beta\gamma\left\{\begin{array}{r} m\beta\beta xx-m\gamma\gamma yy+n\beta x \quad +n\gamma y \quad +n\alpha \\ +m\alpha\beta x-m\alpha\gamma y+f \\ -f \end{array}\right\},$$

unde colligimus

$$X=\gamma\left(m\beta\beta xx+\beta(m\alpha+n)x+f+\tfrac{1}{2}n\alpha\right),$$
$$Y=\gamma\left(m\gamma\gamma yy+\gamma(m\alpha-n)y+f-\tfrac{1}{2}n\alpha\right),$$

et integralis aequatio algebraica erit

$$m\gamma y-\frac{m\gamma\gamma yy+\gamma(m\alpha-n)y+f-\frac{1}{2}n\alpha}{\alpha+\beta x+\gamma y}=\text{Const.}$$

seu

$$m\beta\gamma xy+n\gamma y-f+\tfrac{1}{2}n\alpha=C(\alpha+\beta x+\gamma y)$$

vel loco C scribendo $C+\frac{1}{2}n$ erit concinnius

$$m\beta\gamma xy-\tfrac{1}{2}n\beta x+\tfrac{1}{2}n\gamma y-f=C(\alpha+\beta x+\gamma y).$$

11. Videamus iam, sub quibus conditionibus haec forma aequationis generalis ista ratione integrabilis evadat

$$\frac{hdx}{Axx+Bx+C}+\frac{kdy}{Dyy+Ey+F}=0.$$

Comparatione ergo cum valoribus inventis instituta colligitur

$$A=hm\beta\beta\gamma, \qquad D=km\beta\gamma\gamma,$$
$$B=h\beta\gamma(m\alpha+n), \quad E=k\beta\gamma(m\alpha-n),$$
$$C=h\gamma\left(f+\tfrac{1}{2}n\alpha\right), \quad F=k\beta\left(f-\tfrac{1}{2}na\right).$$

Quoniam hic totum negotium ad rationes litterarum reducitur, sumtis pro primis aequalitatibus

$$\beta=Ak \text{ et } \gamma=Dh$$

concluduntur reliquae

$$m=\frac{1}{ADhhkk}, \quad \alpha=\frac{Bk+Eh}{2}, \quad n=\frac{Bk-Eh}{2ADhhkk} \quad \text{et} \quad f=\frac{ACkk+DFhh}{2ADhhkk};$$

praeterea vero haec conditio requiritur, ut sit

$$\frac{4AC-BB}{hh}=\frac{4DF-EE}{kk};$$

quae si habuerit locum, multiplicator idoneus erit

$$M=\frac{(Axx+Bx+C)(Dyy+Ey+F)}{hk\left(\frac{1}{2}(Bk+Eh)+Akx+Dhy\right)^2}$$

et aequatio integralis inde resultans erit per *hk* multiplicando

$$xy-\frac{(Bk-Eh)x}{4Dh}+\frac{(Bk-Eh)y}{4Ak}-\frac{ACkk+DFhh}{2ADhk}=G\left(\tfrac{1}{2}(Bk+Eh)+Akx+Dhy\right),$$

quae immutata constante arbitraria *G* ad hanc formam revocatur

$$\left(x+\tfrac{B}{2A}-GDh\right)\left(y+\tfrac{E}{2D}-GAk\right)=GGADhk+\frac{(4AC-BB)kk+(4DF-EE)hh}{8ADhk}$$

seu

$$\left(\tfrac{2AX+B}{h}+G\right)\left(\tfrac{2Dy+E}{k}+G\right)=GG+\frac{4AC-BB}{2hh}+\frac{4DF-EE}{2kk}.$$

12. En ergo theorema minime spernendum, etiamsi eius veritas ex aliis principiis satis manifesta esse queat.

Si haec aequatio differentialis

$$\frac{hdx}{Axx+Bx+C}+\frac{kdy}{Dyy+Ey+F}=0$$

ita fuerit comparata, ut sit

$$\frac{4AC-BB}{hh}=\frac{4DF-EE}{kk}$$

tum eius integrale completum erit algebraicum atque hac aequatione expressum

$$\frac{2Ax+B}{h}\cdot\frac{2Dy+E}{k}+G\left(\frac{2Ax+B}{h}+\frac{2Dy+E}{k}\right)=\frac{4AC-BB}{2hh}+\frac{4DF-EE}{2kk},$$

ubi G denotat constantem arbitrariam per integrationem invectam. Hoc vero integrale invenitur, si aequatio proposita ducatur in hunc multiplicatorem

$$\frac{(Axx+Bx+C)(Dyy+Ey+F)}{\left(\frac{2Ax+B}{h}+\frac{2Dy+E}{k}\right)^2}.$$

13. Quemadmodum multiplicatori *M* tribuimus formam

$$\frac{XY}{(\alpha+\beta x+\gamma y)^2}$$

ita etiam formis magis complicatis uti licebit, quod quidem in genere praestari nequit. Evolvamus autem multiplicatorem

$$M = \frac{YX}{(\alpha+\beta x+\gamma y\delta xy)^2}$$

ut haec aequatio integrabilis sit efficienda

$$\frac{Ydx+Xdy}{(\alpha+\beta x+\gamma y+\delta xy)^2} = 0.$$

cuius integratio ad hanc perducit aequationem

$$\frac{-Y}{(\beta+\delta y)(\alpha+\beta x+\gamma y+\delta xy)} + \Gamma{:}y = \frac{-X}{(\gamma+\delta x)(\alpha+\beta x+\gamma y+\delta xy)} + \varDelta{:}x,$$

quae transformatur in hanc

$$\frac{X}{\gamma+\delta x} - \frac{Y}{\beta+\delta y} = (\alpha+\beta x+\gamma y+\delta xy)(\varDelta{:}x - \Gamma{:}y),$$

ubi evidens est statui debere

$$\varDelta{:}x = \frac{\zeta x+\eta}{\gamma+\delta x} \quad \text{et} \quad \Gamma{:}y = \frac{\zeta y+\theta}{\beta+\delta y}$$

ut nulli termini occurrant, qui utramque variabilem simul complectantur.
Hinc ergo fiet

$$\frac{X}{\gamma+\delta x} - \frac{Y}{\beta+\delta y} = \eta y + \frac{(\alpha+\beta x)(\zeta x+\eta)}{\gamma+\delta x} - \theta x - \frac{(\alpha+\gamma y)(\zeta y+\theta)}{\beta+\delta y},$$
$$+f \qquad\qquad\qquad\qquad -f$$

unde concludimus

$$X = (\alpha+\beta x)(\zeta x+\eta) - (\gamma+\delta x)(\theta x+f),$$
$$Y = (\alpha+\gamma y)(\zeta y+\theta) - (\beta+\delta y)(\eta y+f)$$

sive evolvendo

$$X = (\beta\zeta - \delta\theta)xx + (\alpha\zeta + \beta\eta - \gamma\theta - \delta f)x + \alpha\eta - \gamma f,$$
$$Y = (\gamma\zeta - \delta\eta)yy + (\alpha\zeta + \gamma\theta - \beta\eta - \delta f)y + \alpha\theta - \beta f,$$

et aequatio integralis erit

$$\frac{\zeta x + \eta}{\gamma + \delta x} - \frac{X}{(\gamma + \delta x)(\alpha + \beta x + \gamma y + \delta xy)} = \text{Const.},$$

quae loco X substituto valore invento abit in hanc formam

$$\frac{\zeta xy + \eta y + \theta x + f}{\alpha + \beta x + \gamma y + \delta xy} = \text{Const.}$$

14. Transferamus haec iterum ad formam

$$\frac{hdx}{Axx + Bx + C} + \frac{kdy}{Dyy + Ey + F} = 0$$

fieri oportet

$$A = h(\beta\zeta - \delta\theta), \qquad D = k(\gamma\zeta - \delta\eta),$$
$$B = h(\alpha\zeta + \beta\eta - \gamma\theta - \delta f), \quad E = k(\alpha\zeta + \gamma\theta - \beta\eta - \delta f),$$
$$C = h(\alpha\eta - \gamma f), \qquad F = k(\alpha\theta - \beta f).$$

Primae aequationes praebent

$$\theta = \frac{\beta\zeta}{\delta} - \frac{A}{\delta h}, \quad \eta = \frac{\gamma\zeta}{\delta} - \frac{D}{\delta k},$$

secundae vero

$$f = \frac{\alpha\zeta}{\delta} - \frac{Bk + Eh}{2\delta hk} \quad \text{et} \quad \delta = \frac{2A\gamma k - 2D\beta h}{Bk - Eh},$$

unde ex tertiis colligitur

$$\frac{2Ck(A\gamma k - D\beta h)}{Bk - Eh} = \frac{\gamma}{2}(Bk + Eh) - D\alpha h,$$
$$\frac{2Fh(A\gamma k - D\beta h)}{Bk - Eh} = \frac{\beta}{2}(Bk + Eh) - A\alpha k.$$

Hinc α elidendo fit

$$\frac{2(ACkk - DFhh)(Ak\gamma - Dh\beta)}{Bk - Eh} = \frac{1}{2}(Ak\gamma - Dh\beta)(Bk + Eh),$$

unde, cum esse nequeat

$$Ak\gamma - Dh\beta = 0,$$

quia alioquin fieret $\delta = 0$ et quantitates θ, η, f infinitae, tum vero, quod praecipue est notandum, aequatio integralis prodiret Const. = Const., quo ergo casu nihil indicaretur, necesse est, ut sit

$$4(ACkk - DFhh) = BBkk - EEhh$$

seu

$$\frac{4AC-BB}{hh} = \frac{4DF-EE}{kk}$$

ut ante.

Quod autem hic maxime animadverti meretur, est, quod, etsi tres litterae β, γ et δ manent indefinitae, aequatio tamen integralis a praecedente nonnisi quantitate constante discrepat; prodit enim

$$\frac{2\zeta hk}{Bk-Eh} - \frac{k(2Ax+B)+h(2Dy+E)}{2(Aky-Dh\beta)xy+(Bk-Eh)(\beta x+\gamma y)+2(Ck\beta-Fhy)} = \text{Const.}$$

seu

$$\frac{yky(2Ax+B)+\beta k(Bx+2C)-\beta hx(2Dy+E)-yh(Ey+2F)}{k(2Ax+B)+h(2Dy+E)} = \text{Const.},$$

quae forma, quomodocunque accipiantur litterae β et γ, semper veram aequationem integralem exhibet. Quod cum minus sit perspicuum, ostendisse sufficiet ambas partes β et γ involventes seorsim sumtas eandem relationem inter x et y definire. Constitutis enim his duabus aequationibus

$$\frac{2Akxy+Bky-Ehy-2Fh}{2Akx+2Dhy+Bk+Eh} = \text{Const.}, \quad \frac{-2Dhxy-Ehx+Bkx+2Ck}{2Akx+2Dhy+Bk+Eh} = \text{Const.}$$

multiplicetur prior per Dh, posterior per Ak fietque summa

$$\frac{Ak(Bk-Eh)x+Dh(Bk-Eh)y+2ACkk-2DFhh}{2Akx+2Dhy+Bk+Eh},$$

cuius valor utique est constans $= \frac{Bk-Eh}{2}$, propterea quod

$$\frac{2ACkk-2DFhh}{Bk+Eh} = \frac{Bk-Eh}{2},$$

unde patet propositum.

15. Progredior nunc ad formam aequationum magis arduam, quae sit

$$\frac{dx}{\sqrt{X}} + \frac{dy}{\sqrt{Y}} = 0,$$

sitque multiplicator eam reddens integrabilem

$$M = P\sqrt{X} + Q\sqrt{Y} \ ,$$

ita ut aequatio integrationem admittens sit

$$Pdx + Qdy + \frac{Qdx\sqrt{Y}}{\sqrt{X}} + \frac{Pdy\sqrt{X}}{\sqrt{Y}} = 0,$$

cuius utrumque membrum seorsim integrabile sit oportet.

Pro priore ergo erit $\left(\frac{dP}{dy}\right) = \left(\frac{dQ}{dx}\right)$, posterioris vero integrale statuatur $2V\sqrt{XY}$, unde colligitur

$$Q = 2X\left(\frac{dV}{dx}\right) + V\frac{dX}{dx} \quad \text{et} \quad P = 2Y\left(\frac{dV}{dy}\right) + V\frac{dY}{dy}$$

et ob priorem conditionem

$$2Y\left(\frac{ddV}{dy^2}\right) + \frac{3dY}{dy}\left(\frac{dV}{dx}\right) + V\frac{ddY}{dy^2} = 2X\left(\frac{ddV}{dx^2}\right) + \frac{3dX}{dx}\left(\frac{dV}{dx}\right) + V\frac{ddX}{dx^2},$$

ex qua aequatione, si loco *V* sumserimus certam functionem ipsarum *x* et *y*, dispiciendum est, quomodo idonei valores pro functionibus *X* et *Y* obtineantur.

16. Demus primo ipsi *V* valorem constantem, puta *V* = 1, ae pervenimus ad hanc conditionem

$$\frac{ddY}{dy^2} = \frac{ddX}{dx^2}$$

quae aequalitas subsistere nequit, nisi utrumque membrum seorsim aequetur quantitati constanti, quae sit = *2a*, unde colligemus

$$X = axx + bx + c \ \text{ et } \ Y = ayy + dy + e$$

hincque porro

$$P = \frac{dY}{dx} = 2ay + d \ \text{ et } \ Q = \frac{dX}{dx} = 2ax + b,$$

unde aequatio integralis completa colligitur

$$2axy + dx + by + 2\sqrt{XY} = \text{Const.}$$

Quocirca ista aequatio differentialis

$$\frac{dx}{\sqrt{(axx+bx+c)}} + \frac{dy}{\sqrt{(ayy+dy+e)}} = 0$$

integrabilis redditur ope multiplicatoris

$$M = (2ay+d)\sqrt{(axx+bx+c)} + (2ax+b)\sqrt{(ayy+dy+e)}$$

ac tum integrale completum reperietur

$$2axy + dx + by + 2\sqrt{(axx+bx+c)(ayy+dy+e)} = C$$

seu sublata irrationalitate

$$CC - 2C(2axy+dx+by) = (4ae-dd)xx + (4ac-bb)yy + 2bdxy + 4bex + 4cdy + 4ce.$$

Haec autem aequatio differentialis multo latius patet illa, quam initio § 3 attuleram.

17. Tribuamus nunc ipsi V hunc valorem

$$V = \frac{1}{(\alpha+\beta x+\gamma y)^2};$$

si enim loco exponentis 2 indefinitum sumsissem, mox patuisset hanc potestatem accipi debuisse. Erit ergo

$$\left(\frac{dV}{dx}\right) = \frac{-2\beta}{(\alpha+\beta x+\gamma y)^3}, \qquad \left(\frac{dV}{dy}\right) = \frac{-2\gamma}{(\alpha+\beta x+\gamma y)^3},$$
$$\left(\frac{ddV}{dx^2}\right) = \frac{6\beta\beta}{(\alpha+\beta x+\gamma y)^4} \text{ et } \left(\frac{ddV}{dy^2}\right) = \frac{6\gamma\gamma}{(\alpha+\beta x+\gamma y)^4}.$$

His autem valoribus substitutis seqnentes oriuntur binae formae

$$12\beta\beta X - \frac{6\beta dX}{dx}(\alpha+\beta x+\gamma y) + \frac{ddX}{dx^2}(\alpha+\beta x+\gamma y)^2$$
$$= 12\beta\beta Y - \frac{6\gamma dY}{dy}(\alpha+\beta x+\gamma y) + \frac{ddY}{dy^2}(\alpha+\beta x+\gamma y)^2;$$

quia igitur in priore y, in altera x non ultra duas dimensiones assurgit, evidens est in formulis

$$\frac{ddX}{dx^2} \text{ et } \frac{ddY}{dy^2}$$

variabiles *x* et *y* totidem dimensiones habere debere , quia alioquin termini ex *x* et *y* mixti utrinque aequales fieri non possent. Cum ergo ipsae functiones *X* et *Y* ad quartum gradum sint ascensurae, ponamus

$$X = Ax^4 + 2Bx^3 + Cx^2 + 2Dx + E \text{ et } Y = \mathfrak{A}y^4 + 2\mathfrak{B}y^3 + \mathfrak{C}y^2 + 2\mathfrak{D}y + \mathfrak{E}.$$

Facta iam substitutione pro priori parte prodit

$$\begin{array}{lllll}
+12\beta\beta Ax^4 & +24\beta\beta Bx^3 & +12\beta\beta Cxx & +24\beta\beta Dx & +12\beta\beta E \\
-24\beta\beta A & -36\beta\beta B & -12\beta\beta C & -12\beta\beta D & -12\alpha\beta D \\
+12\beta\beta A & -24\alpha\beta A & -36\alpha\beta B & -12\alpha\beta C & +2\alpha\alpha C \\
 & +12\beta\beta B & +2\beta\beta C & +4\alpha\beta C & \\
 & +24\alpha\beta A & +24\alpha\beta B & +12\alpha\alpha B & \\
 & & +12\alpha\alpha A & & \\
-24\beta\gamma Ax^3y & -36\beta\gamma Bx^2y & -12\beta\gamma Cxy & -12\beta\gamma Dy & \\
+24\beta\gamma A & +24\beta\gamma B & +4\beta\gamma C & +4\alpha\gamma C & \\
 & +24\alpha\gamma A & +24\alpha\gamma B & & \\
+12\gamma\gamma Axxyy & +12\gamma\gamma Bxyy & +2\gamma\gamma Cyy, & &
\end{array}$$

qui termini in ordinem disponantur

$$\begin{aligned}
&12\gamma\gamma Axxyy + 12\gamma\gamma Bxyy + 12\gamma\left(2\alpha A - \beta B\right)xxy \\
&+ 2\gamma\gamma Cyy + 8\gamma\left(3\alpha B - \beta C\right)xy + 2\left(6\alpha\alpha A - 6\alpha\beta B + \beta\beta C\right)xx \\
&+ 4\gamma\left(\alpha C - 3\beta D\right)y + 4\left(3\alpha\alpha B - 2a\beta C + 3\beta\beta D\right)x + 2\left(\alpha\alpha C - 6\alpha\beta D + 6\beta\beta E\right).
\end{aligned}$$

Simili vero modo altera pars erit

$$\begin{aligned}
&12\beta\beta\mathfrak{A}xxyy + 12\beta\beta\mathfrak{B}xxy + 12\beta\left(2\alpha\mathfrak{A} - \gamma\mathfrak{B}\right)xyy \\
&+ 2\beta\beta\mathfrak{C}xx + 8\beta\left(3\alpha\mathfrak{B} - \gamma\mathfrak{C}\right)xy + 2\left(6\alpha\alpha\mathfrak{A} - 6\alpha\gamma\mathfrak{B} + \gamma\gamma\mathfrak{C}\right)yy \\
&+ 4\beta\left(\alpha\mathfrak{C} - 3\gamma\mathfrak{D}\right)x + 4\left(3\alpha\alpha\mathfrak{B} - 2a\gamma\mathfrak{C} + 3\gamma\gamma\mathfrak{D}\right)y + 2\left(\alpha\alpha\mathfrak{C} - 6\alpha\gamma\mathfrak{D} + 6\gamma\gamma\mathfrak{E}\right).
\end{aligned}$$

18. Coaequentur nunc inter se termini homologi utriusque formae et sequentibus aequationibus erit satisfaciendum

$$\begin{array}{r|l} xxyy & \gamma\gamma A=\beta\beta\mathfrak{A}, \\ xxy & 2\alpha\gamma A-\beta\gamma B=\beta\beta\mathfrak{B}, \\ xyy & \gamma\gamma B=2\alpha\beta\mathfrak{A}-\beta\beta\mathfrak{B}, \\ xx & 6\alpha\alpha A-6\alpha\beta B+\beta\beta C=\beta\beta\mathfrak{C}, \\ yy & \beta\beta C=6\alpha\alpha\mathfrak{A}-6\alpha\gamma\mathfrak{B}+\gamma\gamma\mathfrak{C}, \\ xy & 3\alpha\gamma B-\beta\gamma C=3\alpha\beta\mathfrak{B}-\beta\gamma\mathfrak{C}, \\ x & 3\alpha\alpha B-2\alpha\beta C+3\beta\beta D=\alpha\beta\mathfrak{C}-3\beta\gamma\mathfrak{D}, \\ y & \alpha\gamma C-3\beta\gamma D=3\alpha\alpha\mathfrak{B}-2\alpha\gamma\mathfrak{C}+3\gamma\gamma\mathfrak{D}, \\ 1 & \alpha\alpha C-6\alpha\beta D+6\beta\beta E=\alpha\alpha\mathfrak{C}-6\alpha\gamma\mathfrak{D}+6\gamma\gamma\mathfrak{E}. \end{array}$$

Tres autem primae aequationes tantum duas dant determinationes

$$\beta=\frac{2\alpha A\sqrt{\mathfrak{A}}}{B\sqrt{\mathfrak{A}}+\mathfrak{B}\sqrt{A}} \quad \text{et} \quad \gamma=\frac{2\alpha\mathfrak{A}\sqrt{A}}{B\sqrt{\mathfrak{A}}+\mathfrak{B}\sqrt{A}}.$$

quarta et quinta itidem unicam determinationem suppeditant

$$C-\mathfrak{C}=\frac{3(\mathfrak{A}BB-A\mathfrak{B}\mathfrak{B})}{2A\mathfrak{A}}=\frac{3}{2}\left(\frac{BB}{A}-\frac{\mathfrak{B}\mathfrak{B}}{\mathfrak{A}}\right),$$

quae eadem quoque ex sexta sequitur; statuatur ergo

$$C=\frac{3BB}{2A}+n \quad \text{et} \quad \mathfrak{C}=\frac{3\mathfrak{B}\mathfrak{B}}{2\mathfrak{A}}+n.$$

Septima et octava etiam unicam determinationem involvunt

$$\frac{D\sqrt{A}+\mathfrak{D}\sqrt{\mathfrak{A}}}{B\sqrt{\mathfrak{A}}+\mathfrak{B}\sqrt{A}}=\frac{A\mathfrak{B}\mathfrak{B}+\mathfrak{A}BB-B\mathfrak{B}\sqrt{A\mathfrak{A}}+2nA\mathfrak{A}}{4A\mathfrak{A}\sqrt{A\mathfrak{A}}}$$

vel

$$D\sqrt{A}+\mathfrak{D}\sqrt{\mathfrak{A}}=\frac{B^3}{4A\sqrt{A}}+\frac{\mathfrak{B}^3}{4\mathfrak{A}\sqrt{\mathfrak{A}}}+\frac{nB}{2\sqrt{A}}+\frac{n\mathfrak{B}}{2\sqrt{\mathfrak{A}}};$$

statuatur ergo

$$D=\frac{B^3}{4AA}+\frac{nB}{2A}+\frac{m}{2\sqrt{A}} \quad \text{et} \quad \mathfrak{D}=\frac{\mathfrak{B}^3}{4\mathfrak{A}\mathfrak{A}}+\frac{n\mathfrak{B}}{2\mathfrak{A}}-\frac{m}{2\sqrt{\mathfrak{A}}},$$

qui valores in ultima aequatione substituti praebent

$$24(AE-\mathfrak{A}\mathfrak{E})=\frac{3B^4}{2AA}+\frac{6nBB}{A}+\frac{12mB}{\sqrt{A}}-\frac{3\mathfrak{B}^4}{2\mathfrak{A}\mathfrak{A}}-\frac{6n\mathfrak{B}\mathfrak{B}}{\mathfrak{A}}+\frac{12m\mathfrak{B}}{\sqrt{\mathfrak{A}}},$$

quare commode statui licebit

$$E=\frac{B^4}{16A^3}+\frac{nBB}{4AA}+\frac{mB}{2A\sqrt{A}}+\frac{l}{A},$$

$$\mathfrak{E}=\frac{\mathfrak{B}^4}{16\mathfrak{A}^3}+\frac{n\mathfrak{B}\mathfrak{B}}{4\mathfrak{A}\mathfrak{A}}-\frac{m\mathfrak{B}}{2\mathfrak{A}\sqrt{\mathfrak{A}}}+\frac{l}{\mathfrak{A}}.$$

19. Cum autem sumserimus $V=\frac{1}{(\alpha+\beta x+\gamma y)^2}$, erit

$$Q=\frac{-4\beta\left(Ax^4+2Bx^3+Cxx+2Dx+E\right)}{(\alpha+\beta x+\gamma y)^3}+\frac{2\left(2Ax^3+3Bxx+Cx+D\right)}{(\alpha+\beta x+\gamma y)^2},$$

$$P=\frac{-4\gamma\left(\mathfrak{A}x^4+2\mathfrak{B}x^3+\mathfrak{C}xx+2\mathfrak{D}x+\mathfrak{E}\right)}{(\alpha+\beta x+\gamma y)^3}+\frac{2\left(2\mathfrak{A}x^3+3\mathfrak{B}xx+\mathfrak{C}x+\mathfrak{D}\right)}{(\alpha+\beta x+\gamma y)^2},$$

sive

$$Q=\frac{2\gamma y\left(2Ax^3+3Bxx+Cx+D\right)+2(2\alpha A-\beta B)x^3+2(3\alpha B-\beta C)xx+2(\alpha C-3\beta D)x+2(\alpha D-2\beta E)}{(\alpha+\beta x+\gamma y)^3},$$

$$P=\frac{2\beta x\left(2\mathfrak{A}y^3+3\mathfrak{B}yy+\mathfrak{C}y+\mathfrak{D}\right)+2(2\alpha\mathfrak{A}-\gamma\mathfrak{B})y^3+2(3\alpha\mathfrak{B}-\gamma\mathfrak{C})yy+2(\alpha\mathfrak{C}-3\gamma\mathfrak{D})y+2(\alpha\mathfrak{D}-2\gamma\mathfrak{E})}{(\alpha+\beta x+\gamma y)^3},$$

unde investigari oportet integrale formulae $Pdx+Qdy$; ad quod si deinceps addatur ea $\frac{2\sqrt{XY}}{(\alpha+\beta x+\gamma y)^2}$, aggregatum quantitati constanti aequatum exhibebit integrale completum aequationis

$$\frac{dx}{\sqrt{X}}+\frac{dy}{\sqrt{Y}}=0.$$

Pro illo autem integrali inveniendo ex prioribus valoribus pro P et Q exhibitis notetur fore separatim

$$\int Qdy=\frac{2\beta\left(Ax^4+2Bx^3+Cxx+2Dx+E\right)}{\gamma(\alpha+\beta x+\gamma y)^2}-\frac{2\left(2Ax^3+3Bxx+Cx+D\right)}{(\alpha+\beta x+\gamma y)}+\Gamma : x,$$

$$\int Pdx=\frac{2\gamma\left(\mathfrak{A}x^4+2\mathfrak{B}x^3+\mathfrak{C}xx+2\mathfrak{D}x+\mathfrak{E}\right)}{\beta(\alpha+\beta x+\gamma y)^2}-\frac{2\left(2\mathfrak{A}x^3+3\mathfrak{B}xx+\mathfrak{C}x+\mathfrak{D}\right)}{\beta(\alpha+\beta x+\gamma y)}+\Delta : y,$$

quae duae expressiones aequales esse debent; quem in finem ponatur

$$\Gamma{:}x=\frac{2(Axx+Bx+N)}{\beta\gamma}\quad\text{et}\quad\Delta{:}x=\frac{2(\mathfrak{A}xx+\mathfrak{B}x+\mathfrak{N})}{\beta\gamma}$$

fietque

$$\left.\begin{aligned}
&\tfrac{1}{2}\beta\gamma(\alpha+\beta x+\gamma y)^2\int Qdy\\
&+A\gamma\gamma xxyy\\
&+B\gamma\gamma xyy\\
&+\gamma(2A\alpha-B\beta)xxy\\
&+N\gamma\gamma yy\\
&+(A\alpha\alpha-B\alpha\beta+N\beta\beta)xx\\
&+\gamma(2B\alpha-C\beta+2N\beta\beta)xy\\
&+\gamma(2N\alpha-D\beta)y\\
&+(B\alpha\alpha-C\alpha\beta+D\beta\beta+2N\alpha\beta)x\\
&+E\beta\beta-D\alpha\beta+N\alpha\alpha
\end{aligned}\right|
\begin{aligned}
&\tfrac{1}{2}\beta\gamma(\alpha+\beta x+\gamma y)^2\int Pdx\\
&+\mathfrak{A}\beta\beta xxyy\\
&+\beta(2\mathfrak{A}\alpha-\mathfrak{B}\gamma)xyy\\
&+\mathfrak{B}\beta\beta xxy\\
&+(\mathfrak{A}\alpha\alpha-\mathfrak{B}\alpha\gamma+\mathfrak{N}\gamma\gamma)yy\\
&+\mathfrak{N}\beta\beta xx\\
&+\beta(2\mathfrak{B}\alpha-\mathfrak{C}\gamma+2\mathfrak{N}\gamma)xy\\
&+(\mathfrak{B}\alpha\alpha-\mathfrak{C}\alpha\gamma+\mathfrak{D}\gamma\gamma+2\mathfrak{N}\alpha\gamma)y\\
&+\beta(2\mathfrak{N}\alpha-\mathfrak{D}\gamma)x\\
&+\mathfrak{E}\gamma\gamma-\mathfrak{D}\alpha\gamma+\mathfrak{N}\alpha\alpha.
\end{aligned}$$

20. Hae conditiones cum praecedentibus (§18) perfecte conveniunt, si modo sumatur

$$N=\tfrac{1}{6}C\quad\text{et}\quad\mathfrak{N}=\tfrac{1}{6}\mathfrak{C}$$

Dividamus singulos terminos per $\beta\gamma$, ut prodeat valor formulae

$$\tfrac{1}{2}\beta\gamma(\alpha+\beta x+\gamma y)^2\int Qdy\,,$$

qui substitutis valoribus ante inventis reperietur

$$\begin{aligned}
&xxyy\sqrt{A\mathfrak{A}}+Bxyy\sqrt{\tfrac{\mathfrak{A}}{A}}+\mathfrak{B}xxy\sqrt{\tfrac{A}{\mathfrak{A}}}\\
&+\tfrac{1}{6}Cyy\sqrt{\tfrac{\mathfrak{A}}{A}}+\tfrac{1}{6}\mathfrak{C}xx\sqrt{\tfrac{A}{\mathfrak{A}}}+\left(\tfrac{B\mathfrak{B}}{\sqrt{A\mathfrak{A}}}-\tfrac{2}{3}n\right)xy\\
&+\left(\tfrac{BB\mathfrak{B}}{4A\sqrt{A\mathfrak{A}}}-\tfrac{nB}{3A}+\tfrac{n\mathfrak{B}}{6\sqrt{A\mathfrak{A}}}-\tfrac{m}{2\sqrt{A}}\right)y+\left(\tfrac{B\mathfrak{B}\mathfrak{B}}{4\mathfrak{A}\sqrt{A\mathfrak{A}}}-\tfrac{n\mathfrak{B}}{3\mathfrak{A}}+\tfrac{nB}{6\sqrt{A\mathfrak{A}}}+\tfrac{m}{2\sqrt{\mathfrak{A}}}\right)x\\
&+\tfrac{BB\mathfrak{B}\mathfrak{B}}{16A\mathfrak{A}\sqrt{A\mathfrak{A}}}+\tfrac{n\left(B\sqrt{\mathfrak{A}}+\mathfrak{B}\sqrt{A}\right)^2}{16A\mathfrak{A}\sqrt{A\mathfrak{A}}}-\tfrac{nB\mathfrak{B}}{4A\mathfrak{A}}+\tfrac{m\left(B\sqrt{\mathfrak{A}}-\mathfrak{B}\sqrt{A}\right)}{4A\mathfrak{A}}+\tfrac{l}{\sqrt{A\mathfrak{A}}}.
\end{aligned}$$

Sit haec forma brevitatis gratia $=S$ eritque integrale completum

$$\frac{S+\sqrt{XY}}{(\alpha+\beta x+\gamma y)^2}=\text{Const.}$$

seu

$$S+\sqrt{XY}=\text{Const.}\left(B\sqrt{\mathfrak{A}}+\mathfrak{B}\sqrt{A}+2Ax\sqrt{\mathfrak{A}}+2\mathfrak{A}y\sqrt{A}\right)^2,$$

quod etiam hac forma concinniori exhiberi potest

$$S+\sqrt{XY}=\text{Const.}\left(\frac{B}{\sqrt{A}}+\frac{\mathfrak{B}}{\sqrt{\mathfrak{A}}}+2x\sqrt{A}+2y\sqrt{\mathfrak{A}}\right)^2.$$

Quare, dum functiones *X* et *Y* conditionibus ante definitis sint praeditae, hoc modo habebitur integrale completum aequationis differentialis

$$\frac{dx}{\sqrt{X}}+\frac{dy}{\sqrt{Y}}=0.$$

21. Haec investigatio aliquanto generalius institui potest tribuendo ipsi *V* talem valorem

$$\frac{1}{(\alpha+\beta x+\gamma y+\delta xy)^2};$$

quo facilius autem calculi molestias superare queamus, observo, dummodo variabiles *x* et *y* quantitate constante augeantur vel minuantur, cum ad hanc formam $\frac{1}{(a+xy)^2}$ reduci posse; expedito autem calculo restitutio facile instituetur.

Considerabo ergo hanc aequationis differentialis formam

$$\frac{dx}{\sqrt{X}}+\frac{dy}{\sqrt{Y}}=0,$$

quam integrabilem reddi assumo ope multiplicatoris $P\sqrt{X}+Q\sqrt{Y}$: ut integrari debeat haec formula

$$Pdx+Qdy+\frac{Qdx\sqrt{Y}}{\sqrt{X}}+\frac{Pdy\sqrt{X}}{\sqrt{Y}}=0.$$

Statuatur partis posterioris integrale $=2V\sqrt{XY}$ fietque, ut vidimus,

$$Q=2X\left(\frac{dV}{dx}\right)+V\frac{dX}{dx}\quad\text{et}\quad P=2Y\left(\frac{dV}{dy}\right)+V\frac{dY}{dy}.$$

Sit igitur $V=\frac{1}{(a+xy)^2}$ ideoque

$$\left(\frac{dV}{dx}\right)=\frac{-2y}{(a+xy)^3} \quad \text{et} \quad \left(\frac{dV}{dy}\right)=\frac{-2x}{(a+xy)^3},$$

ita ut habeamus

$$Q=\frac{-4Xy}{(a+xy)^3}+\frac{dX}{dx}\frac{1}{(a+xy)^2} \quad \text{et} \quad P=\frac{-4Yx}{(a+xy)^3}+\frac{dY}{dy}\frac{1}{(a+xy)^2}.$$

Nunc autem effici debet, ut formula $Pdx+Qdy$ integrationem admittat; hunc in finem duplici modo eius integrale capiatur, dum vel y vel x constans accipitur, sicque obtinebimus

$$\int Pdx=\frac{4Y}{yy(a+xy)}-\frac{2aY}{yy(a+xy)^2}-\frac{dY}{ydy}\cdot\frac{1}{a+xy}+\frac{\Gamma{:}y}{yy},$$

$$\int Qdx=\frac{4X}{xx(a+xy)}-\frac{2aX}{xx(a+xy)^2}-\frac{dX}{xdx}\cdot\frac{1}{a+xy}+\frac{\Delta{:}x}{xx},$$

quas duas formas inter se aequales reddi oportet. Multiplicando ergo per $xxyy(a+xy)^2$ habebimus

$$4xxY(a+xy)-2axxY-\frac{xxydY}{dy}(a+xy)+xx\Gamma{:}y\cdot(a+xy)^2$$
$$=4yyX(a+xy)-2ayyX-\frac{xyydX}{dx}(a+xy)+yy\Delta{:}x\cdot(a+xy)^2,$$

unde fingamus

$$X=Ax^4+2Bx^3+Cxx+2Dx+E,\ \Delta{:}x=Lxx+Mx+N,$$

$$Y=\mathfrak{A}y^4+2\mathfrak{B}y^3+\mathfrak{C}yy+2\mathfrak{D}y+\mathfrak{E},\ \Gamma{:}y=\mathfrak{L}yy+\mathfrak{M}y+\mathfrak{N},$$

$$\frac{dX}{dx}=4Ax^3+6Bxx+2Cx+2D \quad \text{et} \quad \frac{dY}{dy}=4\mathfrak{A}y^3+6\mathfrak{B}yy+2\mathfrak{C}y+2\mathfrak{D}.$$

Hinc nostrae expressiones induent has formas

$xxyy(a+xy)^2\int Qdy$	$xxyy(a+xy)^2\int Pdx$
$+Lx^4y^4$	$+\mathfrak{L}x^4y^4$
$+Mx^3y^4$	$+2\mathfrak{B}x^3y^4$
$+2Bx^4y^3$	$+\mathfrak{M}x^4y^3$
$+Nxxy^4$	$-2a\mathfrak{A}xxy^4$
$+2(C+aL)x^3x^3$	$+2(\mathfrak{C}+a\mathfrak{L})x^3y^3$
$-2aAx^4yy$	$+\mathfrak{N}x^4yy$
$+2(3D+aM)xxy^3$	$-2a\mathfrak{B}xxy^3$
$-2aBx^3yy$	$+2(3\mathfrak{D}+a\mathfrak{M})x^3yy$
$+aaLxxyy$	$+aa\mathfrak{L}xxyy$
$+2(2E+aN)xy^3$	$+0xy^3$
$+0x^3y$	$+2(2\mathfrak{E}+a\mathfrak{N})x^3y$
$+(2aD+aaM)xyy$	$+0xyy$
$+0xxy$	$+(2a\mathfrak{D}+aa\mathfrak{M})xxy$
$+(2aE+aaN)yy$	$+0yy$
$+0xx$	$+(2a\mathfrak{E}+aa\mathfrak{N})xx$

22. Harum formarum coaequatio suppeditat sequentes determinationes

$$\mathfrak{L}=L,\ M=2\mathfrak{B},\ \mathfrak{M}=2B,\ N=-2a\mathfrak{A},\ \mathfrak{N}=-2aA,$$
$$\mathfrak{C}=C,\ D=-a\mathfrak{B},\ \mathfrak{D}=-aB,\ E=aa\mathfrak{A},\ \mathfrak{E}=aaA,$$

ita ut habeatur haec aequatio differentialis

$$\frac{dx}{\sqrt{\left(Ax^4+2Bx^3+Cxx+2Dx+E\right)}}+\frac{dy}{\sqrt{\left(\frac{E}{aa}y^4-\frac{2D}{a}y^3+Cyy-2aBy+aaA\right)}}=0$$

cuius integrale completum est

$$\frac{2Bxxy-\frac{2D}{a}xyy-2aAxx-\frac{2E}{a}yy+2Cxy-2aBx+2Dy+2\sqrt{XY}}{(a+xy)^2}=\text{Const.}$$

Hic observo, si ponamus $y=\frac{-a}{z}$, prodire aequationem initio allatam

$$\frac{dx}{\sqrt{\left(Ax^4+2Bx^3+Cxx+2Dx+E\right)}}+\frac{dz}{\sqrt{\left(Az^4+2Bz^3+Czz+2Dz+E\right)}}=0,$$

cuius propterea integrale nunc etiam per principia integrationis maxime naturalia assignari potest, cum antea methodo admodum indirecta eo fuissem deductus. Integrale quippe est

$$\begin{aligned}&Axxzz+Bxz(x+z)+Cxz+D(x+z)+E+G(x-z)^2\\&=\sqrt{\left(Ax^4+2Bx^3+Cxx+2Dx+E\right)\left(Az^4+2Bz^3+Czz+2Dz+E\right)},\end{aligned}$$

quae ab irrationalitate liberata induit hanc formam

$$\begin{aligned}&GG(x-z)+\ 2G\left(Axxzz+Bxz(x+z)+Cxz+D(x+z)+E\right)\\&+(BB-AC)xxzz-2ADxz(x+z)-AE(x+z)^2-2BDxz\\&-2BE(x+z)+DD-CE=0,\end{aligned}$$

quae aequatio in hanc formam reducta cum superiori [§ 4] convenit

$$\begin{aligned}&(2AG+BB-AC)xxzz+2(BG-AD)xz(x+z)+(GG-AE)(x+z)^2\\&-\ 2(2GG+BD-\ CG)xz+2(DG-BE)(x+z)+2EG+DD-CE=0.\end{aligned}$$

23. Si nunc scrutari velimus, sub quibus conditionibus haec aequatio differentialis integrationem admittat

$$\frac{dx}{\sqrt{\left(Ax^4+2Bx^3+Cxx+2Dx+E\right)}}+\frac{dy}{\sqrt{\left(\mathfrak{A}y^4+2\mathfrak{B}y^3+\mathfrak{C}yy+2\mathfrak{D}y+\mathfrak{E}\right)}}=0,$$

concipiamus hanc nasci ex illa ponendo $z=\frac{fy+g}{hy+k}$, ita ut aequatio integralis futura sit

$$\begin{aligned}&(2AG+BB-AC)xx(fy+g)^2+2(BG-AD)x(fy+g)(hxy+kx+fy+g)\\&+(GG-AE)(hxy+kx+fy+g)^2-2(2GG-CG+BD)x(fy+g)(hy+k)\\&+2(DG-BE)(hy+k)(hxy+kx+fy+g)+(2EG+DD-CE)(hy+k)^2=0.\end{aligned}$$

At vero coefficientes $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$, $\mathfrak{E}$ ex his quantitatibus *f, g, h, k* ita definiuntur, ut sit

$$\mathfrak{A}(fk-gh)^2 = Af^4 + 2Bf^3h + Cffhh + 2Dfh^3 + Eh^4$$
$$\mathfrak{B}(fk-gh)^2 = 2Af^3g + Bff(3gh+fk) + Cfh(fk+gh) + Dhh(3fk+gh) + 2Eh^3k,$$
$$\mathfrak{C}(fk-gh)^2 = 6Affgg + 6Bfg(fk+gh) + C(fk+gh)^2 + 6Dhk(fk+gh) + 6Ehhkk + 2Cfghk,$$
$$\mathfrak{D}(fk-gh)^2 = 2Afg^3 + Bgg(gh+3fk) + Cgk(fk+gh) + Dkk(fk+3gh) + 2Ehk^3,$$
$$\mathfrak{E}(fk-gh)^2 = Ag^4 + 2Bg^3k + Cggkk + 2Dgk^3 + Ek^4.$$

24. Videamus autem, quousque problema in genere aggressi calculum expedire queamus.

Sit igitur proposita aequatio $\frac{dx}{\sqrt{X}} + \frac{dy}{\sqrt{Y}} = 0$, quae per $P\sqrt{X} + Q\sqrt{Y}$ multiplicata fiat integrabilis, sitque integrale

$$\int(Pdx + Qdy) + \frac{2\sqrt{XY}}{(\alpha+\beta x+\gamma y+\delta xy)^2} = \text{Const.}$$

eritque, ut vidimus,

$$Q = \frac{-4X(\beta+\delta y)}{(\alpha+\beta x+\gamma y+\delta xy)^3} + \frac{dX}{dx(\alpha+\beta x+\gamma y+\delta xy)^2},$$
$$P = \frac{-4Y(\gamma+\delta x)}{(\alpha+\beta x+\gamma y+\delta xy)^3} + \frac{dY}{dy(\alpha+\beta x+\gamma y+\delta xy)^2},$$

unde colligimus

$$(\gamma+\delta x)^2(\alpha+\beta x+\gamma y+\delta xy)^2\int Qdy = 2(\beta\gamma-\alpha\delta)X$$
$$+\left(4\delta X - (\gamma+\delta x)\tfrac{dX}{dx}\right)(\alpha+\beta x+\gamma y+\delta xy) + (\alpha+\beta x+\gamma y+\delta xy)^2 \varDelta{:}x$$

similique modo

$$(\beta+\delta y)^2(\alpha+\beta x+\gamma y+\delta xy)^2\int Pdx = 2(\beta\gamma-\alpha\delta)Y$$
$$+\left(4\delta Y - (\beta+\delta y)\tfrac{dY}{dy}\right)(\alpha+\beta x+\gamma y+\delta xy) + (\alpha+\beta x+\gamma y+\delta xy)^2 \varGamma{:}y,$$

quae duae formae ad consensum perduci debent, ita ut prima per $(\gamma+\delta x)^2$, altera vero per $(\beta+\delta y)^2$ divisa eandem functionem exhibeant. Quamobrem necesse est, ut prior per $(\gamma+\delta x)^2$, posterior per $(\beta+\delta y)^2$ divisionem admittat, cui ergo requisito ante omnia est satisfaciendum.

25. Evolvamus priorem valorem partibus ab *y* pendentibus distinguendis

$$\text{I. } 2(\beta\gamma-\alpha\delta)X+4\delta(\alpha+\beta x)X-(\alpha+\beta x)(\gamma+\delta x)\tfrac{dX}{dx}+(\alpha+\beta x)^2\varDelta\colon x,$$
$$\text{II. } +y(\gamma+\delta x)\left(4\delta X-(\gamma+\delta x)\tfrac{dX}{dx}+2(\alpha+\beta x)\varDelta\colon x\right),$$
$$\text{III. } +yy(y+\delta x)^2\varDelta\colon x,$$

quae expressio per $(\gamma+\delta x)^2$ divisibilis esse debet; cum ergo tertia pars sponte sit divisibilis, pro secunda ponamus

$$(\alpha+\beta x)\varDelta\colon x+2\delta X=(\gamma+\delta x)R$$

et prima pars erit

$$2(\beta\gamma-\alpha\delta)X+2\delta(a+\beta x)X+(a+\beta x)(\gamma+\delta x)R-(a+\beta x)(\gamma+\delta x)\tfrac{dX}{dx},$$

quae redit ad hanc formam

$$(\gamma+\delta x)\left(2\beta X+(a+\beta x)R-(a+\beta x)\tfrac{dX}{dx}\right),$$

ita ut

$$2\beta X+(a+\beta x)\left(R-\tfrac{dX}{dx}\right)$$

adhuc divisionem per $\gamma+\delta x$ admittere debeat. Cui conditioni satisfit sumendo

$$R=\tfrac{\beta}{\delta}\varDelta:x-\tfrac{\alpha+\beta x}{\delta}\varDelta':x+(\gamma+\delta x)S,$$

unde fit

$$X=\tfrac{\beta\gamma-\alpha\delta}{2\delta\delta}\varDelta:x-\tfrac{(\alpha+\beta x)(\gamma+\delta x)}{2\delta\delta}\varDelta':x+\tfrac{(\gamma+\delta x)^2}{2\delta}S,$$

ideoque prima pars erit

$$(\gamma+\delta x)^2\left(\tfrac{\beta}{\delta}R-\tfrac{(\alpha+\beta x)dR}{2\delta dx}\right)+\tfrac{1}{2}(\alpha+\beta x)(\gamma+\delta x)^2 S$$

sive

$$(\gamma+\delta x)^2\left\{\begin{aligned}&\tfrac{\beta\beta}{\delta\delta}\varDelta\colon x-\tfrac{\beta(\alpha+\beta x)}{\delta\delta}\varDelta':x+\tfrac{(\alpha+\beta x)^2}{2\delta\delta}\varDelta'':x\\&+\tfrac{\beta(\alpha+\beta x)}{\delta}S-\tfrac{(\alpha+\beta x)(\gamma+\delta x)}{2\delta}\cdot\tfrac{dS}{dx}\end{aligned}\right\};$$

deinde secunda

$$y(\gamma+\delta x)^2\left\{\begin{array}{l}\frac{2\beta}{\delta}\varDelta:x-\frac{\alpha+\beta x}{\delta}\varDelta':x+\frac{(\alpha+\beta x)(\gamma+\delta x)}{2\delta\delta}\varDelta'':x\\+(\gamma+\delta x)S-\frac{(\gamma+\delta x)^2}{2\delta}\cdot\frac{dS}{dx}\end{array}\right\};$$

ac tertia

$$yy(\gamma+\delta x)^2\varDelta:x.$$

Quocirca formulae

$$(\alpha+\beta x+\gamma y+\delta xy)^2\int Qdy$$

valor erit

$$\begin{aligned}&\frac{\beta\beta}{\delta\delta}\varDelta:x+\frac{2\beta}{\delta}y\varDelta:x+yy\varDelta:x-\frac{\beta(\alpha+\beta x)}{\delta\delta}\varDelta':x-\frac{\alpha+\beta x}{\delta}y\varDelta':x\\&+\frac{(\alpha+\beta x)^2}{2\delta\delta}\varDelta'':x+\frac{(\alpha+\beta x)(\gamma+\delta x)}{2\delta\delta}y\varDelta'':x\\&+\frac{\beta}{\delta}(\gamma+\delta x)S+(\gamma+\delta x)yS-\frac{(\alpha+\beta x)(\gamma+\delta x)}{2\delta}\cdot\frac{dS}{dx}-\frac{(\gamma+\delta x)^2}{2\delta}y\frac{dS}{dx}\end{aligned}$$

seu ita concinnius expressus

$$\begin{aligned}&\frac{(\beta+\delta y)^2}{\delta\delta}\varDelta:x-\frac{(\alpha+\beta x)(\beta+\delta y)}{\delta\delta}\varDelta':x\\&+\frac{(\alpha+\beta x)(\alpha+\beta x+\gamma y+\delta xy)}{2\delta\delta}\varDelta'':x+\frac{(\gamma+\delta x)(\beta+\delta y)}{\delta}S-\frac{(\gamma+\delta x)(\alpha+\beta x+\gamma y+\delta xy)}{2\delta}\cdot\frac{dS}{dx},\end{aligned}$$

cui alter aequalis fieri, debet, qui est

$$\begin{aligned}&\frac{(\gamma+\delta x)^2}{\delta\delta}\varGamma:y-\frac{(\alpha+\gamma y)(\gamma+\delta x)}{\delta\delta}\varGamma':y\\&+\frac{(\alpha+\gamma y)(\alpha+\beta x+\gamma y+\delta xy)}{2\delta\delta}\varGamma'':y+\frac{(\beta+\delta y)(\gamma+\delta x)}{\delta}\mathfrak{S}-\frac{(\beta+\delta y)(\alpha+\beta x+\gamma y+\delta xy)}{2\delta}\cdot\frac{d\mathfrak{S}}{dx}.\end{aligned}$$

26. Quodsi iam ponamus

$$\varDelta:x=\delta\delta(Axx+2Bx+C)\ \text{ et }\ S=\delta(Dxx+2Ex+F),$$

item

$$\varGamma:y=\delta\delta(\mathfrak{A}xx+2\mathfrak{B}x+\mathfrak{C})\ \text{ et }\ \mathfrak{S}=\delta(\mathfrak{D}xx+2\mathfrak{E}x+\mathfrak{F}),$$

reperientur nostrae expressiones ita evolutae

$$\begin{aligned}&(\alpha+\beta x+\gamma y+\delta xy)^2\int Qdy\\
&+\delta\delta Axxyy\\
&+2\delta\delta Bxyy\\
&+\delta(\beta A-\gamma D+\delta E)xxy\\
&+\delta\delta Cyy\\
&+\delta(\beta E-\alpha D)xx\\
&+(2\beta\delta B+(\beta\gamma-\alpha\delta)A-\gamma\gamma D+\delta\delta F)xy\\
&+(\alpha\gamma A-2\alpha\delta B+2\beta\delta C-\gamma\gamma E+\gamma\delta F)y\\
&+(\beta\delta F+(\beta\gamma-\alpha\delta)E-\alpha\gamma D)x\\
&+\alpha\alpha A-2\alpha\beta B+\beta\beta C-\alpha\gamma E+\beta\gamma F\end{aligned}$$

$$\begin{aligned}&(\alpha+\beta x+\gamma y+\delta xy)^2\int Pdx\\
&+\delta\delta\mathfrak{A}xxyy\\
&+\delta(\gamma\mathfrak{A}-\beta\mathfrak{D}+\delta\mathfrak{E})xyy\\
&+2\delta\delta\mathfrak{B}xxy\\
&+\delta(\gamma\mathfrak{E}-\alpha\mathfrak{D})yy\\
&+\delta\delta\mathfrak{C}xx\\
&+(2\gamma\delta\mathfrak{B}+(\beta\gamma-\alpha\delta)\mathfrak{A}-\beta\beta\mathfrak{D}+\delta\delta\mathfrak{F})xy\\
&+(\gamma\delta\mathfrak{F}+(\beta\gamma-\alpha\delta)\mathfrak{E}-\alpha\beta\mathfrak{D})y\\
&+(\alpha\beta\mathfrak{A}-2\alpha\delta\mathfrak{B}+2\gamma\delta\mathfrak{C}-\beta\beta\mathfrak{E}+\beta\delta\mathfrak{F})x\\
&+\alpha\alpha\mathfrak{A}-2\alpha\beta\mathfrak{B}+\beta\beta\mathfrak{C}-\alpha\gamma\mathfrak{E}+\beta\gamma\mathfrak{F},\end{aligned}$$

unde nonnisi sequentes sex determinationes deducuntur

$$\begin{aligned}&\mathfrak{A}=A,\\
&\mathfrak{B}=\tfrac{\beta A-\gamma D}{2\delta}+\tfrac{1}{2}E,\\
&\mathfrak{C}=\tfrac{\beta E-\alpha D}{\delta},\\
&\mathfrak{D}=\tfrac{2\gamma\delta B-\gamma\gamma A-\delta\delta C}{\alpha\delta-\beta\gamma},\\
&\mathfrak{E}=\tfrac{2\alpha\delta B-\alpha\gamma A-\beta\delta C}{\alpha\delta-\beta\gamma},\\
&\mathfrak{F}=F-\tfrac{\gamma E}{\delta}-\tfrac{\alpha\beta\gamma A-2\alpha\beta\delta B+\beta\beta\delta C}{\delta(\alpha\delta-\beta\gamma)};\end{aligned}$$

his enim omnibus illis conditionibus satisfit. Sic igitur omnes litterae *A*, *B*, *C*, *D*, *E*, *F* una cum α, β, γ, δ arbitrio nostro manent relictae, ex quibus deinde colligitur functio

$$\begin{aligned}2X&=\delta\delta Dx^4+2\delta(\delta E+\gamma D-\beta A)x^3\\
&+(\delta\delta F+4\gamma\delta E+\gamma\gamma D-2\beta\delta B-(\beta\gamma+3\alpha\delta)A)xx\\
&+2(\gamma\delta F+\gamma\gamma E-\alpha\gamma A-2\alpha\delta B)x+\gamma\gamma F-2a\gamma B+(\beta\gamma-\alpha\delta)C.\end{aligned}$$

27. Hunc autem calculum ulterius non prosequor, cum nunc quidem sufficiat methodum directam et rei naturae conformem aperuisse, quae ad easdem integrationes omnino singulares,

quas olim ex longe aliis principiis erueram, perducat. In augmentum igitur huius scientiae plurimum intererit istam novam methodum omni studio penitius scrutari.

Hunc in finem adhuc observo aliam multiplicatoris formam adhiberi posse, cuius ope talis forma

$$\frac{dx}{\sqrt{X}}+\frac{dy}{\sqrt{Y}}=0$$

integrabilis reddi queat. Statuatur scilicet multiplicator $M=P+Q\sqrt{XY}$,
ut integrabilis esse debeat haec forma

$$\frac{Pdx}{\sqrt{X}}+Qdy\sqrt{X}+\frac{Pdy}{\sqrt{Y}}+Qdx\sqrt{Y}=0$$

Fingatur prioris partis integrale $=2R\sqrt{X}$, posterioris vero $=2S\sqrt{Y}$ ut integrale completum sit

$$R\sqrt{X}+S\sqrt{Y}=\text{Const.},$$

et facta evolutione reperitur

$$P=RdX+2X\left(\frac{dR}{dx}\right),\quad P=\frac{SdY}{dy}+2Y\left(\frac{dS}{dy}\right),$$

$$Q=2\left(\frac{dR}{dy}\right),\qquad Q=2\left(\frac{dS}{dx}\right).$$

Cum igitur debeat esse $\left(\frac{dR}{dy}\right)=\left(\frac{dS}{dx}\right)$, manifestum est formulam $Rdx+Sdy$ integrabilem esse debere. Non autem opus est, ut ea algebraicum habeat integrale, sed sufficit, ut integrationis charactere sit praedita.

28. Sumatur enim

$$R=\frac{y}{\alpha+\beta xy+\gamma xxyy}\ \text{ et }\ S=\frac{x}{\alpha+\beta xy+\gamma xxyy}$$

eritque

$$Q=\frac{2\alpha-2\gamma xxyy}{(\alpha+\beta xy+\gamma xxyy)^2}$$

et

$$P=\frac{ydX}{dx(\alpha+\beta xy+\gamma xxyy)}-\frac{2Xyy(\beta+2\gamma xy)}{(\alpha+\beta xy+\gamma xxyy)^2}$$

simulque

$$P=\frac{xdY}{dy(\alpha+\beta xy+\gamma xxyy)}-\frac{2Yxx(\beta+2\gamma xy)}{(\alpha+\beta xy+\gamma xxyy)^2},$$

ita ut habeatur

$$\left(\alpha+\beta xy+\gamma xxyy\right)^2 P=\tfrac{ydX}{dx}\left(\alpha+\beta xy+\gamma xxyy\right)-2yyX\left(\beta+2\gamma xy\right)$$
$$=\tfrac{xdY}{dy}\left(\alpha+\beta xy+\gamma xxyy\right)-2xxY\left(\beta+2\gamma xy\right).$$

Statuatur

$$X=Ax^4+2Bx^3+Cxx+2Dx+E$$
$$Y=\mathfrak{A}y^4+2\mathfrak{B}y^3+\mathfrak{C}yy+2\mathfrak{D}y+\mathfrak{E}$$

ac duo illi valores inter se aequandi postulare deprehenduntur, ut sit

$$\beta=0,\quad B=0,\quad \mathfrak{B}=0,\quad D=0,\quad \text{et}\quad \mathfrak{D}=0;$$

tum vero ii fient

$$\text{I.}\quad =-2\gamma Cx^3y^3+4\alpha Ax^3y-4\gamma Exy^3+2\alpha Cxy,$$
$$\text{II.}\quad =-2\gamma\mathfrak{C}x^3y^3+4\alpha\mathfrak{A}x^3y-4\gamma\mathfrak{E}xy^3+2\alpha\mathfrak{C}xy,$$

unde colligitur

$$\mathfrak{C}=C,\quad \tfrac{\alpha}{\gamma}=-\tfrac{\mathfrak{E}}{A}=\tfrac{-E}{\mathfrak{A}}\quad \text{seu}\quad \mathfrak{A}\mathfrak{E}=AE.$$

Erit ergo

$$X=Ax^4+Cxx-\tfrac{\alpha}{\gamma}\mathfrak{A},\quad Y=\mathfrak{A}y^4+\mathfrak{C}yy-\tfrac{\alpha}{\gamma}A$$

et aequationis

$$\frac{dx}{\sqrt{\left(Ax^4+Cxx-\frac{\alpha}{\gamma}\mathfrak{A}\right)}}+\frac{dy}{\sqrt{\left(\mathfrak{A}y^4+\mathfrak{C}yy-\frac{\alpha}{\gamma}A\right)}}=0$$

integrale completum erit

$$y\sqrt{\left(Ax^4+Cxx-\tfrac{\alpha}{\gamma}\mathfrak{A}\right)}+x\sqrt{\left(\mathfrak{A}y^4+\mathfrak{C}yy-\tfrac{\alpha}{\gamma}A\right)}=\text{Const.}\left(\alpha+\gamma xxyy\right).$$

29. Ex his exemplis facile intelligitur fere novum adhuc Analyseos genus desiderari, quo huiusmodi operationes certo ordine institui atque ulterius extendi queant, a quo quidem scopo adhuc longissime sumus remoti. Interim tamen ea, quae hactenus exposui, maximi momenti esse videntur ad universalitatem principii integrandi initio memorati stabiliendam, cum adeo eius beneficio per multiplicatores idoneos eae integrationes, quae maxime arduae et cognita principia, transcendentes erant visae, expediri queant. Mihi quidem, cum primum in eas

incidissem, nulla alia via eo deducere videbatur praeter eam, qua tum eram usus; nondum enim animadverteram semper, quoties cuiuscunque aequationis differentialis integrale completum constaret, ex eo multiplicatorem, quo illa integrabilis reddatur, concludi posse, quae conclusio, si integrale tantum fuisset particulare, neutiquam valuisset. Quamobrem integrationum illarum particularium, quas olim simul ex eodem principio alieno eram consecutus, longealiter est ratio comparata, neque adhuc perspicere licet, quomodo methodo quadam directa et naturali ad easdem perveniri queat.

30. Eo magis igitur operae erit pretium indolem harum integrationum particularium accuratius examinari, quod quidem contemplatione casus simplicissimi fiet. Huius igitur aequationis differentialis

$$dx\sqrt{(1+xx)}+dy\sqrt{(1+yy)}+nydx+nxdy=0$$

integrale particulare inveneram esse

$$xx+yy+2xy\sqrt{(1+nn)}=nn$$

similiaque integralia innumerabilia etiam inveni pro eiusmodi aequationibus differentialibus, quae neque a logarithmis neque a circuli quadratura pendent; quare haec aequatio ita spectetur, quasi non per logarithmos integrari posset. Hic igitur primo quaeritur, qua via directa hoc integrale particulare ex forma differentiali concludi queat, deinde quomodo aequatio differentialis comparata esse debeat, ut tale integrale particulare exhiberi queat.

Ad has ergo quaestiones primum observo aequationem algebraicam esse integrale completum istius aequationis differentialis

$$\frac{dx}{\sqrt{(1+xx)}}+\frac{dy}{\sqrt{(1+yy)}}=0,$$

tum vero ex illa sequi

$$x+y\sqrt{(1+nn)}=n\sqrt{(1+yy)} \quad \text{et} \quad y+x\sqrt{(1+nn)}=n\sqrt{(1+xx)},$$

ita ut tam $\sqrt{(1+xx)}$ quam $\sqrt{(1+yy)}$ rationaliter per x et y exprimi queat.
Cum igitur hinc sit differentiando

$$\frac{xdx}{\sqrt{(1+xx)}}=\frac{dy+dx\sqrt{(1+nn)}}{n} \quad \text{et} \quad \frac{ydy}{\sqrt{(1+yy)}}=\frac{dx+dy\sqrt{(1+nn)}}{n},$$

si harum formarum multipla quaecunque ad illam

$$\frac{dx}{\sqrt{(1+xx)}}+\frac{dy}{\sqrt{(1+yy)}}=0,$$

addantur, semper prodire aequationem differentialem, cui aequatio algebraica particulariter saltem satisfaciat. In genere ergo huius aequationis differentialis

$$\frac{dx+Pxdx}{\sqrt{(1+xx)}}+\frac{dy+Qydy}{\sqrt{(1+yy)}}=\frac{Pdy+Qdx+(Pdx+Qdy)\sqrt{(1+nn)}}{n}$$

integrale particulare erit

$$xx+yy+2xy\sqrt{(1+nn)}=nn.$$

Sit iam $P=x$ et $Q=y$ ac satisfiet huic aequationi

$$dx\sqrt{(1+xx)}+dy\sqrt{(1+yy)}=\frac{xdy+ydx+(xdx+ydy)\sqrt{(1+nn)}}{n}$$

ex integrali vero fit

$$xdx+ydy=-(xdy+ydx)\sqrt{(1+nn)},$$

ita ut habeatur haec aequatio differentialis

$$dx\sqrt{(1+xx)}+dy\sqrt{(1+yy)}+nxdy+nydx=0,$$

cui ergo integrale supra datum particulariter convenit.

31. Transferamus iam haec ad casus latius patentes, et postquam huius aequationis

$$\frac{dx}{\sqrt{X}}+\frac{dy}{\sqrt{Y}}=0$$

inventum fuerit integrale completum, quod sit $W=\text{Const.}$, notetur hinc semper utrumque valorem radicalem $\sqrt{X}$ et $\sqrt{Y}$ per functiones rationales ipsarum x et y definiri. Sit ergo ideoque

$$\sqrt{X}=R \quad \text{et} \quad \sqrt{Y}=S$$

ideoque

$$\frac{dX}{\sqrt{X}}=2dR \quad \text{et} \quad \frac{dY}{\sqrt{Y}}=2dS.$$

Sit iam P functio ipsius x et Q ipsius y hincque conflatur ista aequatio

$$\frac{dx+PdX}{\sqrt{X}}+\frac{dy+QdY}{\sqrt{Y}}-2PdR-2QdS=0,$$

cui aequatio algebraica $W = \text{Const.}$ certe particulariter satisfacit. Hinc si P et Q ita accipiantur, ut formula $PdR + QdS$ integrationem admittat, cuius integrale sit $= V$ orietur aequatio transcendens

$$\int \frac{dx + PdX}{\sqrt{X}} + \int \frac{dy + QdY}{\sqrt{Y}} - 2V = \text{Const.},$$

cui aequationi $W = \text{Const.}$ seu, valoribus inde deductis, $\sqrt{X} = R$ seu $\sqrt{Y} = S$ particulariter satisfacit. Tale ergo ratiocinium viam ad huiusmodi integrationes particulares alioquin inventu difficillimas patefacere videtur.