CHAPTER IV

THE INTEGRATION OF LOGARITHMIC AND EXPONENTIAL FORMULAS

PROBLEM 18

189. *If X designates an algebraic function of x, to find the integral of the formula Xdxlx.*

SOLUTION

The integral $\int Xdx$ is sought, which if it is put equal to *Z*, and since the differential of the quantity *Zlx* shall be $dZlx+\frac{Zdx}{x}$, there becomes $Zlx=\int dZlx+\int\frac{Zdx}{x}$
and thus

$$\int dZlx=\int Xdxlx=Zlx-\int\frac{Zdx}{x}.$$

And thus the integration of the proposed formula has been reduced to the integration of this formula $\frac{Zdx}{x}$, which, if *Z* should be an algebraic function of *x*, involves the logarithm no further and thus can be treated by the preceding rules. But if $\int Xdx$ cannot be shown to be an algebraic function, hence nothing arises of assistance and it is expedient to rest with an indication of the integral, and to investigate the value of $\int Xdxlx$ by an approximation.

Unless perhaps letting $X=\frac{1}{x}$, in which case evidently it gives

$$\int\frac{dx}{x}lx=\frac{1}{2}(lx)^2+C.$$

COROLLARY 1

190. In the same manner, if with *V* denoting some function of x, the formula $XdxlV$ is proposed with $\int Xdx=Z$ present, the integral of this is equal to $ZlV-\int Z\frac{dV}{V}$, and thus it is reduced towards an algebraic formula, provided that *Z* should be given algebraically.

COROLLARY 2

191. It is agreed to be noted for the particular case $\frac{dx}{x}lx$, if there is put $lx=u$, and *U* can be some algebraic function of *u*, the integration of this formula $\frac{Udx}{x}$ is not difficult, since on account of $\frac{dx}{x}=du$ it changes into *Udu*, and the integration of this is referred to the preceding chapter.

SCHOLIUM

192. This reduction depends on this basis, since when it shall be

$$d.xy = ydx + xdy,$$

hence in turn becomes $xy = \int ydx + \int xdy$ and thus

$$\int ydx = xy - \int xdy,$$

thus in order that in this manner, in general the integration of formulas *ydx* is reduced to *xdy*. But if hence with some proposed formula *Vdx*, *V* is able to be resolved into two factors such as $V = PQ$, thus in order that the integral $\int Pdx = S$ can be assigned, on account of $Pdx = dS$ then $Vdx = PQdx = QdS$ and hence

$$\int Vdx = QS - \int SdQ.$$

A reduction of this kind brings to light a significant use, since the formula $\int SdQ$ should be simpler than the proposed $\int Vdx$ and that in addition can be reduced to a simpler formula in a similar manner. Meanwhile also it comes about conveniently here, that the proposed method produces only a similar formula, in which case the integration can be equally well obtained. Just as if in the reduction to be furthered we should come upon $\int SdQ = T + n\int Vdx$, it might become $\int Vdx = QS - T - n\int Vdx$ and hence

$$\int Vdx = \tfrac{QS-T}{n+1}.$$

Therefore then, there is a conspicuous use for such a reduction, since it leads to either a simpler formula or to the same formula again. And from this principle, we will set out a particular case, in which the formula *Xdxlx* can either be integrated or a series can be established conveniently.

EXAMPLE 1

193. *To find the integral of the differential formula $x^n dxlx$ with n denoting some number.*

Since there is the integral $\int x^n dx = \tfrac{1}{n+1}x^{n+1}$, then

$$\int x^n dxlx = \tfrac{1}{n+1}x^{n+1}lx - \int \tfrac{1}{n+1}x^{n+1}d.lx$$
$$= \tfrac{1}{n+1}x^{n+1}lx - \tfrac{1}{n+1}\int x^n dx = \tfrac{1}{n+1}x^{n+1}lx - \tfrac{1}{(n+1)^2}x^{n+1}$$

and thus

$$\int x^n dxlx = \tfrac{1}{n+1}x^{n+1}\left(lx - \tfrac{1}{n+1}\right).$$

And thus this formula is integrated completely.

COROLLARY 1

194. It may be of help to bear in mind simpler cases, in which *n* is either a positive or negative whole number :

$$\int dxlx = xlx - x, \qquad \int \tfrac{dx}{xx}lx = -\tfrac{1}{x}lx - \tfrac{1}{x},$$
$$\int xdxlx = \tfrac{1}{2}xxlx - \tfrac{1}{4}xx, \qquad \int \tfrac{dx}{x^3}lx = -\tfrac{1}{2xx}lx - \tfrac{1}{4xx},$$
$$\int x^2dxlx = \tfrac{1}{3}x^3lx - \tfrac{1}{9}x^3, \qquad \int \tfrac{dx}{x^4}lx = -\tfrac{1}{3x^3}lx - \tfrac{1}{9x^3},$$
$$\int x^3dxlx = \tfrac{1}{4}x^4lx - \tfrac{1}{16}x^4, \qquad \int \tfrac{dx}{x^5}lx = -\tfrac{1}{4x^4}lx - \tfrac{1}{16x^4}.$$

COROLLARY 2

195. The case $\int \tfrac{dx}{x}lx = \tfrac{1}{2}(lx)^2$, which is entirely on its own, we have noted now above, and indeed it follows also from the reduction to the same formula. For by the above reduction we have :

$$\int \tfrac{dx}{x}lx = lx\cdot lx - \int lx.d.lx = (lx)^2 - \int \tfrac{dx}{x}lx$$

and hence $2\int \tfrac{dx}{x}lx = (lx)^2$, consequently

$$2\int \tfrac{dx}{x}lx = (lx)^2 .$$

EXAMPLE 2

196. *To express the integral of the formula $\tfrac{dx}{1-x}lx$ by means of a series.*

We profit a little from the reduction used before, for it produces :

$$\int \tfrac{dx}{1-x}lx = l\tfrac{1}{1-x}\cdot lx - \int \tfrac{dx}{x}l\tfrac{1}{1-x} .$$

But since there arises :

$$l\frac{1}{1-x} = x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{1}{4}x^4 + \text{etc.},$$

then there becomes

$$\int \frac{dx}{x} l\frac{1}{1-x} = x + \frac{1}{4}x^2 + \frac{1}{9}x^3 + \frac{1}{16}x^4 + \frac{1}{25}x^5 + \text{etc.},$$

and thus

$$\int \frac{dx}{x} lx = l\frac{1}{1-x}\cdot lx - x - \frac{1}{4}x^2 - \frac{1}{9}x^3 - \frac{1}{16}x^4 - \frac{1}{25}x^5 - \text{etc.},$$

since the integral vanishes in the case $x = 0$; for if lx then becomes infinite, yet $l\frac{1}{1-x} = x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{1}{4}x^4 + \text{etc.}$ thus vanishes, in order that, if also it is multiplied by lx, it becomes zero; for in general there arises $x^n lx = 0$ on putting $x = 0$, while n is a positive number.

COROLLARY 1

197. If we put $1 - x = u$, there becomes

$$\frac{dx}{1-x} lx = -\frac{du}{u} l(l-u) = \frac{du}{u} l\frac{1}{1-u}$$

and thus

$$\int \frac{dx}{1-x} lx = C + u + \frac{1}{2}u^2 + \frac{1}{9}u^3 + \frac{1}{16}u^4 + \frac{1}{25}u^5 + \text{etc.};$$

which in order that it vanishes in the case $x = 0$ or $u = 1$, the constant must be taken

$$C = -1 - \frac{1}{2} - \frac{1}{9} - \frac{1}{16} - \frac{1}{25} - \text{etc.} = -\frac{1}{6}\pi\pi.$$

COROLLARY 2

198. Hence on taking $1 - x = u$ or $x + u = 1$ these expressions are equal to each other :

$$-lx\cdot lu - x - \frac{1}{4}x^2 - \frac{1}{9}x^3 - \frac{1}{16}x^4 - \text{etc.}$$

$$= -\frac{1}{6}\pi^2 + u + \frac{1}{4}u^2 + \frac{1}{9}u^3 + \frac{1}{16}u^4 + \text{etc}$$

or there becomes :

$$\frac{1}{6}\pi^2 - lx\cdot lu = x + u + \frac{1}{4}\left(x^2 + u^2\right) + \frac{1}{9}\left(x^3 + u^3\right) + \frac{1}{16}\left(x^4 + u^4\right) + \text{etc.}$$

COROLLARY 3

199. This series certainly converges on putting $x = u = \frac{1}{2}$; hence in this case we have

$$\frac{1}{6}\pi^2 - (l2)^2 = 1 + \frac{1}{2\cdot 4} + \frac{1}{4\cdot 9} + \frac{1}{8\cdot 16} + \frac{1}{16\cdot 25} + \frac{1}{32\cdot 36} + \text{etc.}$$

Hence the sum of this series

$$x+\tfrac{1}{4}x^2+\tfrac{1}{9}x^3+\tfrac{1}{16}x^4+\tfrac{1}{25}x^5+\text{etc.}$$

can be had on only in the case $x = 1$, which is equal to $\frac{\pi\pi}{6}$, but also in the case $x=\frac{1}{2}$, in which case it is equal to $\frac{1}{12}\pi^2-\frac{1}{2}(l2)^2$.

COROLLARY 4

200. If we put $x=\frac{1}{3}$ and $u=\frac{2}{3}$, then the sum of this series

$$1+\frac{5}{3^2\cdot 4}+\frac{9}{3^3\cdot 9}+\frac{17}{3^4\cdot 16}+\frac{33}{3^5\cdot 25}+\frac{65}{3^6\cdot 36}+\text{etc.}$$

of which the general term is $\frac{1+2^n}{3^n nn}$, is equal to $\frac{1}{6}\pi^2-l3\cdot l\frac{3}{2}$, nor hence for the series

$$x+\tfrac{1}{4}x^2+\tfrac{1}{9}x^3+\tfrac{1}{16}x^4+\text{etc.}$$

is it possible for the two cases $x=\frac{1}{3}$ and $x=\frac{2}{3}$ to be summed separately.

EXAMPLE 3

201. *To find the integral of the formula* $\frac{dx}{(1-x)^2}lx$ *and to turn the same into a series.*

Since there shall be $\int\frac{dx}{(1-x)^2}=\frac{1}{1-x}$, then there becomes

$$\int\frac{dx}{(1-x)^2}lx=\frac{1}{1-x}lx-\int\frac{dx}{x(1-x)},$$

but on account of

$\frac{1}{x(1-x)}=\frac{1}{x}+\frac{1}{1-x}$ there becomes

$$\int\frac{dx}{x(1-x)}=lx+l\frac{1}{1-x}$$

from which we deduce the integral

$$\int\frac{dx}{(1-x)^2}lx=\frac{lx}{1-x}-lx-l\frac{1}{1-x}=\frac{xlx}{1-x}-l\frac{1}{1-x}$$

thus taken, in order that it vanishes on putting $x = 0$. Now for the series to be found most conveniently there is put in place $1-x=u$ and our formula becomes

$$=\frac{-du}{uu}l(1-u)=\frac{du}{uu}l\frac{1}{(1-u)}=\frac{du}{uu}\left(u+\tfrac{1}{2}u^2+\tfrac{1}{3}u^3+\tfrac{1}{4}u^4+\tfrac{1}{5}u^5+\text{etc.}\right).$$

On which account for the integral we obtain

$$\int \frac{dx}{(1-x)^2} lx = C + lu + \frac{u}{1\cdot 2} + \frac{uu}{2\cdot 3} + \frac{u^3}{3\cdot 4} + \frac{u^4}{4\cdot 5} + \text{etc.};$$

which expression so that it vanishes on putting $x = 0$ or $u = 1$, the constant must become

$$C = -\frac{1}{1\cdot 2} - \frac{1}{2\cdot 3} - \frac{1}{3\cdot 4} - \frac{1}{4\cdot 5} - \text{etc.} = -1.$$

Whereby on account of $x = 1 - u$ we obtain

$$\frac{u}{1\cdot 2} + \frac{uu}{2\cdot 3} + \frac{u^3}{3\cdot 4} + \frac{u^4}{4\cdot 5} + \text{etc.} = 1 - lu + \frac{(1-u)l(1-u)}{u} + lu$$
$$= 1 + \frac{(1-u)l(1-u)}{u} .$$

COROLLARY 1

202. In a like manner if $dy = \frac{du}{u\sqrt{u}} l \frac{1}{(1-u)}$, then

$$y = -\frac{2}{\sqrt{u}} l \frac{1}{1-u} + \int \frac{2du}{(1-u)\sqrt{u}},$$

but on putting $u = xx$ there is made

$$\int \frac{2du}{(1-u)\sqrt{u}} = 4\int \frac{dx}{1-xx} = 2l \frac{1+x}{1-x}.$$

Hence

$$y = 2l \frac{1+\sqrt{u}}{1-\sqrt{u}} - \frac{2}{\sqrt{u}} l \frac{1}{1-u}.$$

But since through the series

$$dy = \frac{du}{u\sqrt{u}}\left(u + \tfrac{1}{2}uu + \tfrac{1}{3}u^3 + \tfrac{1}{4}u^4 + \text{etc.}\right),$$

there is also

$$y = 2\sqrt{u} + \frac{2}{2\cdot 3}u\sqrt{u} + \frac{2}{3\cdot 5}u^2\sqrt{u} + \frac{2}{4\cdot 7}u^3\sqrt{u} + \text{etc.}$$

COROLLARY 2

203. If hence we multiply by $\frac{\sqrt{u}}{2}$, we obtain

$$u + \frac{uu}{2\cdot 3} + \frac{u^3}{3\cdot 5} + \frac{u^4}{4\cdot 7} + \frac{u^5}{5\cdot 9} + \text{etc.} = \sqrt{u} \cdot l \frac{1+\sqrt{u}}{1-\sqrt{u}} + l(1-u),$$

which sum is also equal to

$$\left(1+\sqrt{u}\right)l\left(1+\sqrt{u}\right) + \left(1-\sqrt{u}\right)l\left(1-\sqrt{u}\right).$$

Whereby on taking $u = 1$ on account of $\left(1-\sqrt{u}\right)l\left(1-\sqrt{u}\right) = 0$ then there becomes

$$1+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 5}+\frac{1}{4\cdot 7}+\frac{1}{5\cdot 9}+\frac{1}{6\cdot 11}+\text{etc.}=2l2.$$

PROBLEM 19

204. *If P denotes a function of x, to find the integral of this formula* $dy=dP(lx)^n$.

SOLUTION

Through the above reduction it can be shown,

$$y=P(lx)^n-\int Pd.(lx)^n=P(lx)^n-n\int\frac{Pdx}{x}(lx)^{n-1}.$$

Hence, if there is put $\int\frac{Pdx}{x}=Q$, in the same manner there is :

$$\int\frac{Pdx}{x}(lx)^{n-1}=Q(lx)^{n-1}-(n-1)\int\frac{Qdx}{x}(lx)^{n-2}.$$

If we wish to progress further in this manner these integrals can be taken :

$$\int\frac{Pdx}{x}=Q,\quad \int\frac{Qdx}{x}=R,\quad \int\frac{Rdx}{x}=S,\quad \int\frac{Sdx}{x}=T\ \text{etc.},$$

and we obtain the integral sought :

$$\int dP(lx)^n=P(lx)^n-nQ(lx)^{n-1}+n(n-1)R(lx)^{n-2}-\ n(n-1)(n-2)S(lx)^{n-3}+\text{etc.},$$

and if the exponent *n* should be a positive whole number, the integral can be expressed in a finite form.

EXAMPLE 1

205. *To assign the integral to the formula* $x^m dx(lx)^2$.

Here there is $n=2$ and $P=\frac{x^{m+1}}{m+1}$ hence

$$Q=\frac{x^{m+1}}{(m+1)^2}\quad\text{and}\quad R=\frac{x^{m+1}}{(m+1)^3}$$

from which we deduce :

$$\int x^m dx(lx)^2=x^{m+1}\left(\frac{(lx)^2}{m+1}-\frac{2lx}{(m+1)^2}+\frac{2\cdot 1}{(m+1)^3}\right),$$

which integral vanishes on putting $x=0$, provided $m+1>0$.

COROLLARY 1

206. Hence on putting $x=1$ there becomes $\int x^m dx(lx)^2 = \frac{2\cdot 1}{(m+1)^3}$. But from the preceding it is apparent, if the formula $\int x^m dxlx$ is to be integrated thus, so that it vanishes on putting $x=0$, then on putting $x=1$ it becomes $\int x^m dxlx = \frac{-1}{(m+1)^2}$.

COROLLARY 2

207. But if there is put $m=-1$, so that there is had $\frac{dx}{x}(lx)^2$, the integral of this will be : e

$$\int \frac{dx}{x}(lx)^2 = \frac{1}{3}(lx)^3,$$

which is the only case to be excepted from the general formula.

EXAMPLE 2

208. *To assign the integral to the formula* $x^{m-1}dx(lx)^3$.

Here there is now $n=3$ and $P=\frac{x^m}{m}$ hence

$$Q=\frac{x^m}{m^2}, \quad R=\frac{x^m}{m^3}, \quad \text{and} \quad S=\frac{x^m}{m^4},$$

from which the integral sought becomes :

$$\int x^{m-1}dx(lx)^3 = x^m\left(\frac{(lx)^3}{m}-\frac{3(lx)^2}{m^2}+\frac{3\cdot 2lx}{m^3}-\frac{3\cdot 2\cdot 1}{m^4}\right),$$

which integral vanishes on putting $x=0$, provided there shall be $m>0$.

COROLLARY 1

209. But if with the integral thus taken, so that it vanishes on putting $x=0$, then there is put in place $x=1$, there becomes

$$\int x^{m-1}dx = \frac{1}{m}, \quad \int x^{m-1}lxdx = -\frac{1}{m^2}, \quad \int x^{m-1}dx(lx)^2 = +\frac{1\cdot 2}{m^3}$$

and

$$\int x^{m-1}dx(lx)^3 = -\frac{1\cdot 2\cdot 3}{m^4}.$$

COROLLARIUM 2

210. But in the case $m=0$ the integral becomes $\int \frac{dx}{x}(lx)^3 = \frac{1}{4}(lx)^4$, which thus cannot be determined in order that it vanishes on putting $x=0$; for it is required to add an infinite constant. But this integral vanishes on putting $x=1$.

EXEMPLE 3

211. *To assign the integral to the formula* $x^{m-1}dx(lx)^n$.

Since here there is $P=\frac{x^m}{m}$, then there arises

$$Q=\frac{x^m}{m^2}, \quad R=\frac{x^m}{m^3}, \quad \text{and} \quad S=\frac{x^m}{m^4},$$

Hence the integral sought emerges:

$$\int x^{m-1}dx(lx)^n = x^m\left(\frac{(lx)^n}{m} - \frac{n(lx)^{n-1}}{m^2} + \frac{n(n-1)(lx)^{n-2}}{m^3} - \frac{n(n-1)(n-2)(lx)^{n-3}}{m^4} + \text{etc.}\right).$$

But in the case $m=0$ it becomes :

$$\int \frac{dx}{x}(lx)^n = \frac{1}{n+1}(lx)^{n+1}.$$

COROLLARY 1

212. If $m>0$, the assigned integral vanishes on putting $x=0$; hence in turn if there is taken $x=1$, the integral will be

$$\int x^{m-1}dx(lx)^n = \pm\frac{1\cdot 2\cdot 3\cdots n}{m^{n+1}},$$

where the + sign prevails, if *n* is an even number, and the lower sign, if *n* is odd.

COROLLARY 2

213. Hence this ambiguity is removed, if in place of *lx* there is written $-l\frac{1}{x}$; for then with the integration put in place in the same way and on putting $x=1$ there arises

$$\int x^{m-1}dx\left(l\frac{1}{x}\right)^n = +\frac{1\cdot 2\cdot 3\cdots n}{m^{n+1}}.$$

SCHOLIUM

214. If the exponent *n* is a fraction, the integral found can be expressed by an infinite series ; just as if there is put $n=-\frac{1}{2}$, then there is found

$$\int\frac{x^{m-1}dx}{\sqrt{lx}}=x^m\left(\frac{1}{m\sqrt{lx}}+\frac{1}{2m^2(lx)^{\frac{3}{2}}}+\frac{1\cdot3}{4m^3(lx)^{\frac{5}{2}}}+\frac{1\cdot3\cdot5}{8m^4(lx)^{\frac{7}{2}}}+\text{etc.}\right),$$

which integral also, in as much as *x* is taken to increase from 0 to 1 initially, can be represented in this manner :

$$\int\frac{x^{m-1}dx}{\sqrt{l\frac{1}{x}}}=\frac{x^m}{\sqrt{l\frac{1}{x}}}\left(\frac{1}{m}+\frac{1}{2m^2lx}+\frac{1\cdot3}{4m^3(lx)^2}+\frac{1\cdot3\cdot5}{8m^4(lx)^3}+\text{etc.}\right),$$

If the exponent *n* is negative, and a whole number, then the integral found is still progressing to infinity ; now in this case it is allowed for the integration to be put in place by other method, by which the integral is reduced to a form of the this kind $\int\frac{Tdx}{lx}$, the integral of this cannot be rendered into anything simpler in any way. Hence we show this reduction by the following problem.

PROBLEM 20

215. *The integration of this formula* $dy=\frac{Xdx}{(lx)^n}$ *is reduced continually to simpler formulas.*

SOLUTION

The proposed formula can be represented thus :

$$dy=Xx\cdot\frac{dx}{x(lx)^n}$$

and since the integral $\int\frac{dx}{x(lx)^n}=\frac{-1}{(n-1)(lx)^{n-1}}$ then

$$y=\frac{-Xx}{(n-1)(lx)^{n-1}}+\frac{1}{n-1}\int\frac{1}{(lx)^{n-1}}\cdot d\left(Xx\right).$$

Whereby if we put continually,

$$d\left(Xx\right)=Pdx,\quad d\left(Px\right)=Qdx,\quad d\left(Qx\right)=Rdx\quad\text{etc.},$$

then this reduction can be continued :

$$y=\frac{-Xx}{(n-1)(lx)^{n-1}}-\frac{Px}{(n-1)(n-2)(lx)^{n-2}}-\frac{Qx}{(n-1)(n-2)(n-3)(lx)^{n-3}}-\text{ etc.},$$

while finally this integral is arrived at :

$$+\frac{1}{(n-1)(n-2)\cdots 1}\int\frac{Vdx}{lx},$$

thus so that, as often as *n* is a positive whole number, the integration is lead to a formula of this kind.

EXAMPLE 1

216. *To investigate the integral of the differential formula* $dy=\frac{x^{m-1}dx}{(lx)^2}$.

Here there is $n=2$ and $X=x^{m-1}$ from which there arises $P=mx^{m-1}$ and hence the integral

$$y=\int\frac{x^{m-1}dx}{(lx)^2}=-\frac{x^m}{lx}+\frac{m}{1}\int\frac{x^{m-1}dx}{lx}$$

But the integral of the formula $\frac{x^{m-1}dx}{lx}$ cannot be shown, except in the case $m=0$, in which it becomes $\int\frac{dx}{xlx}=llx$. Now if $m=0$, the integration of the proposed formula is independent; for the integral is given completely by

$$\int\frac{dx}{x(lx)^2}=-\frac{1}{lx}+C.$$

EXAMPLE 2

217. *To investigate the integral of the cases of the differential formula* $dy=\frac{x^{m-1}dx}{(lx)^n}$, *in which n is a positive integer.*

Since there shall be $X=x^{m-1}$, then there becomes

$$P=\frac{d.Xx}{dx}=mx^{m-1},$$

then indeed

$$Q=\frac{d.Px}{dx}=m^2x^{m-1},\quad R=m^3x^{m-1},\quad S=m^4x^{m-1}\quad \text{etc.}$$

Whereby the integral hence is formed thus, in order that it shall become :

$$y=\int\frac{x^{m-1}dx}{(lx)^n}=-\frac{x^m}{(n-1)(lx)^{n-1}}-\frac{mx^m}{(n-1)(n-2)(lx)^{n-2}}-\frac{m^2x^m}{(n-1)(n-2)(n-3)(lx)^{n-3}}$$
$$-\cdots+\frac{m^{n-1}}{(n-1)(n-2)\cdots 1}\int\frac{x^{m-1}dx}{lx}.$$

COROLLARY

218. Hence for the *n* successive numbers 1, 2, 3, 4 etc. being substituted we have these reductions :

$$\int\frac{x^{m-1}dx}{(lx)^2}=-\frac{x^m}{lx}+\frac{m}{1}\int\frac{x^{m-1}dx}{lx},$$
$$\int\frac{x^{m-1}dx}{(lx)^3}=-\frac{x^m}{2(lx)^2}-\frac{mx^m}{2\cdot 1lx}+\frac{m^2}{2\cdot 1}\int\frac{x^{m-1}dx}{lx},$$
$$\int\frac{x^{m-1}dx}{(lx)^4}=-\frac{x^m}{3(lx)^3}-\frac{mx^m}{3\cdot 2(lx)^2}-\frac{m^2x^m}{3\cdot 2\cdot 1lx}+\frac{m^3}{3\cdot 2\cdot 1}\int\frac{x^{m-1}dx}{lx}.$$

SCHOLIUM

219. Hence these integrations depend on the formula $\int\frac{x^{m-1}dx}{lx}$, which on putting $x^m=z$ on account of $x^{m-1}dx=\frac{1}{m}dz$ and $lx=\frac{1}{m}lz$ is reduced to that most simple form $\int\frac{dz}{lz}$; the integral of this, if it were possible to be assigned, would be rendering a considerable service by its use in analysis, now at this point by no artifax is it possible to be shown, either through logarithms or angles ; but we have shown below (§ 228) how it may be expressed by a series. Hence this formula $\int\frac{dz}{lz}$ suffices to be considered as a special kind of transcendental function, which certainly deserves to be investigated more precisely. But the same transcending quantity arises frequently in the integrations of exponential formulas, which we have established in this chapter, on account of which they make a brief contact with logarithms, as the one kind of integral can easily be converted into the other ; just as with the formula $\frac{dz}{lz}$ considered in the following manner, on putting $lz=x$, in order that $z=e^x$ and $dz=e^xdx$, it is transformed into this exponential $e^x\frac{dx}{x}$, hence the integration of this formula is equally obscure. Therefore we set out manageable formulas and indeed of this kind, which cannot be reduced to an algebraic form by any easy substitution. Just as if V should be some function of v and letting $v=a^x$, then the formula Vdx on account of $x=\frac{lv}{lx}$ and $dx=\frac{dv}{vla}$ is changed into $\frac{Vdv}{vla}$, in which account the variable v is algebraic. Hence therefore we exclude formulas of this kind, $\frac{a^xdx}{\sqrt{(1+a^{nx})}}$, clearly which on putting $a^x=v$ present nothing of difficulty.

PROBLEM 21

220. *To investigate the integration of the differential formula* $a^x Xdx$ *, for some function* X *of* x.

SOLUTION 1

Since there shall be $d.a^x = a^x dxla$, then in turn $\int a^x dx = \frac{1}{la}\cdot a^x$; whereby if the formula proposed is resolved into these factors $X \cdot a^x dx$, by reduction there is had

$$\int a^x Xdx = \tfrac{1}{la}a^x X - \tfrac{1}{la}\int a^x dX \ .$$

But if further we put $d\mathrm{X} = Pdx$, in order that there arises

$$\int a^x Pdx = \tfrac{1}{la}a^x P - \tfrac{1}{la}\int a^x dP$$

this reduction appears :

$$\int a^x Xdx = \tfrac{1}{la}a^x X - \frac{1}{(la)^2}a^x P + \frac{1}{(la)^2}\int a^x dP.$$

If again we put $dP = Qdx$, this reduction is had :

$$\int a^x Xdx = \tfrac{1}{la}a^x X - \frac{1}{(la)^2}a^x P + \frac{1}{(la)^3}a^x Q - \frac{1}{(la)^3}\int a^x dQ.$$

and thus it is allowed to progress further on putting $dQ = Rdx,\ dR = Sdx$ etc., until the calculation arrives at a formula which is either integrable, or which has been put into its most simple general form.

SOLUTION 2

A resolution of the formula into factors can be put in place in another way ; putting

$$\int Xdx = P \quad \text{or} \quad Xdx = dP$$

and thus the related formula $a^x dP$ can be obtained :

$$\int a^x Xdx = a^x P - la\int a^x Pdx$$

in a like manner if we put $\int Pdx = Q$, then we obtain :

$$\int a^{x} X dx = a^{x} P - la \cdot a^{x} Q + (la)^{2} \int a^{x} Q dx .$$

Again we put $\int Q dx = R$ and we follow in this manner :

$$\int a^{x} X dx = a^{x} P - la \cdot a^{x} Q + (la)^{2} \cdot a^{x} R - (la)^{3} \int a^{x} R dx ,$$

and the process is allowed to progress as far as it pleases, until we arrive at a formula which is either integrable or expressed in its simplest general form.

COROLLARY 1

221. With the first solution it is always permitted to be used, since the functions *P, Q, R* etc. are elicited by the differentiation of the function *X*, as long as

$$P = \frac{dX}{dx}, \quad Q = \frac{dP}{dx}, \quad R = \frac{dQ}{dx} \quad \text{etc.}$$

Whereby if *X* were a whole function of rationals [*i.e.* a polynomial], finally the reduction arrives at the formula $\int a^{x} dx = \frac{1}{la} \cdot a^{x}$ and thus in these cases the completed integral can be shown.

COROLLARIUM 2

222. There is not a place for the other solution unless the integral *P* of the formula *Xdx* can be assigned ; neither also is it allowed to be continued, except to the extent that the following integrals

$$\int P dx = Q, \quad \int Q dx = R \quad \text{etc.}$$

are in succession.

EXAMPLE 1

223. *To define the integral of the formula* $a^{x} x^{n} dx$, *with n denoting a positive number.*

Since there shall be $X = x^{n}$, using the first solution we have :

$$\int a^{x} x^{n} dx = \frac{1}{la} \cdot a^{x} x^{n} - \frac{n}{la} \int a^{x} x^{n-1} dx ;$$

hence on putting for *n* successively the numbers 0, 1, 2, 3 etc., since in the first case the integral is in agreement, we can elicit the following integrals :

$$\int a^x dx = \tfrac{1}{la} a^x,$$

$$\int a^x x dx = \tfrac{1}{la} a^x x - \frac{1}{(la)^2} a^x,$$

$$\int a^x x^2 dx = \tfrac{1}{la} a^x x^2 - \frac{2}{(la)^2} a^x x + \frac{2\cdot 1}{(la)^2} a^x,$$

$$\int a^x x^3 dx = \tfrac{1}{la} a^x x^3 - \frac{3}{(la)^2} a^x x^2 + \frac{3\cdot 2}{(la)^3} a^x x - \frac{3\cdot 2\cdot 1}{(la)^4} a^x$$

etc.,

from which in general for whatever exponent n we conclude :

$$\int a^x dx = a^x \left(\frac{x^n}{la} - \frac{nx^{n-1}}{(la)^2} + \frac{n(n-1)x^{n-2}}{(la)^3} - \frac{n(n-1)(n-2)x^{n-3}}{(la)^4} + \text{ etc.} \right),$$

to which above expression it is required to add an arbitrary constant, in order that the complete integral can be obtained.

COROLLARY

224. If the integral must thus be determined so that it vanishes on putting $x = 0$, then there becomes :

$$\int a^x dx = \tfrac{1}{la} a^x - \tfrac{1}{la},$$

$$\int a^x x dx = a^x \left(\frac{x}{la} - \frac{1}{(la)^2} \right) + \frac{1}{(la)^2},$$

$$\int a^x x^2 dx = a^x \left(\frac{x^2}{la} - \frac{2x}{(la)^2} + \frac{2\cdot 1}{(la)^3} \right) - \frac{2\cdot 1}{(la)^3},$$

$$\int a^x x^3 dx = a^x \left(\frac{x^3}{la} - \frac{3x^2}{(la)^2} + \frac{3\cdot 2x}{(la)^3} - \frac{3\cdot 2\cdot 1}{(la)^4} \right) + \frac{3\cdot 2\cdot 1}{(la)^4}$$

etc.,

EXAMPLE 2

225. *To find the integral of the formula* $\frac{a^x dx}{x^n}$, *if n denotes a certain positive number.*

Here we can conveniently use the other formula, since there shall be $X = \frac{1}{x^n}$, then

$$P = \frac{-1}{(n-1)x^{n-1}}$$

and hence this reduction results :

$$\int \frac{a^x dx}{x^n} = \frac{-a^x}{(n-1)x^{n-1}} + \frac{la}{(n-1)} \int \frac{a^x dx}{x^{n-1}}.$$

Therefore it is evident on putting $n = 1$ that nothing can be concluded ; which is that case mentioned above, embracing a particular kind of transcendental function $\int \frac{a^x dx}{x}$, with which granted we are able to show the integrals of the following cases :

$$\int \frac{a^x dx}{x^2} = C - \frac{a^x}{1x} + \frac{la}{1} \int \frac{a^x dx}{x},$$

$$\int \frac{a^x dx}{x^3} = C - \frac{a^x}{2x^2} - \frac{a^x la}{2 \cdot 1x} + \frac{(la)^2}{2 \cdot 1} \int \frac{a^x dx}{x},$$

$$\int \frac{a^x dx}{x^4} = C - \frac{a^x}{3x^3} - \frac{a^x la}{3 \cdot 2x} + \frac{a^x (la)^2}{3 \cdot 2 \cdot 1x} + \frac{(la)^3}{3 \cdot 2 \cdot 1} \int \frac{a^x dx}{x},$$

from which in general we conclude

$$\int \frac{a^x dx}{x^n} = C - \frac{a^x}{(n-1)x^{n-1}} - \frac{a^x la}{(n-1)(n-2)x^{n-2}} - \frac{a^x (la)^2}{(n-1)(n-2)(n-3)x^{n-3}}$$
$$- \cdots - \frac{a^x (la)^{n-2}}{(n-1)(n-2) \cdots 1x} + \frac{a^x (la)^{n-1}}{(n-1)(n-2) \cdots 1} \int \frac{a^x dx}{x}.$$

COROLLARY 1

226. Hence with this transcendent quantity granted, $\int \frac{a^x dx}{x}$, we are able to integrate the formula $a^x x^m dx$, whether the exponent *m* should be a positive or negative number. Indeed from these cases the integration does not depend on that new transcendent quantity.

COROLLARY 2

227. But if *m* were a fractional number, neither solution can complete the work, but each shows an infinite series for the integral. Just as if there is put $m = -\frac{1}{2}$, then we have from the first solution :

$$\int \frac{a^x dx}{\sqrt{x}} = a^x\left(\frac{1}{la} + \frac{1}{2x(la)^2} + \frac{1\cdot3}{4x^2(la)^3} + \frac{1\cdot3\cdot5}{8x^3(la)^4}\right):\sqrt{x} + C,$$

but from the latter,

$$\int \frac{a^x dx}{\sqrt{x}} = C + \frac{a^x}{\sqrt{x}}\left(\frac{2x}{1} - \frac{4x^2 la}{1\cdot3} + \frac{8x^3(la)^2}{1\cdot3\cdot5} - \frac{16x^4(la)^3}{1\cdot3\cdot5\cdot7} + \text{etc.}\right).$$

SCHOLIUM 1

228. Hence the transcendent quantity $\int \frac{a^x dx}{x}$ can be expressed following the progressing powers of *x*. For since there becomes

$$a^x = 1 + xla + \frac{x^2(la)^2}{1\cdot2} + \frac{x^3(la)^3}{1\cdot2\cdot3} + \text{etc.},$$

then there is

$$\int \frac{a^x dx}{x} = C + lx + \frac{xla}{1} + \frac{x^2(la)^2}{1\cdot2\cdot2} + \frac{x^3(la)^3}{1\cdot2\cdot3\cdot3} + \frac{x^4(la)^4}{1\cdot2\cdot3\cdot4\cdot4} + \text{etc.},$$

and if we take a number for *a*, the hyperbolic logarithm of which is unity, that we shall indicate by the number *e*, then we have

$$\int \frac{e^x dx}{x} = C + lx + \frac{x}{1} + \frac{1}{2}\cdot\frac{x^2}{1\cdot2} + \frac{1}{3}\cdot\frac{x^3}{1\cdot2\cdot3} + \frac{1}{4}\cdot\frac{x^4}{1\cdot2\cdot3\cdot4} + \text{etc.}$$

And hence on putting also $e^x = z$, in order that $x = lz$, we are able to integrate the above mentioned $\frac{dz}{lz}$ by a series :

$$\int \frac{dz}{lz} = C + llz + \frac{lz}{1} + \frac{1}{2}\cdot\frac{(lz)^2}{1\cdot2} + \frac{1}{3}\cdot\frac{(lz)^3}{1\cdot2\cdot3} + \frac{1}{4}\cdot\frac{(lz)^4}{1\cdot2\cdot3\cdot4} + \text{etc.};$$

because the integral if it must vanish on putting $z = 0$, then the constant *C* becomes infinite, from which nothing can be concluded for the remaining cases. Likewise there is an inconvenience, if we introduce the vanishing case to $z = 1$, since $llz = l0$ becomes infinite. It is apparent besides, if the integral shall be real for values of *z* smaller than unity, where *lz* is negative, then in turn for values greater than unity it becomes imaginary. Hence therefore the nature of this transcendental function becomes known a little.

SCHOLIUM 2

229. When either the integration does not follow or the series found before are seen to be less suitable, then the quantity a^x can be resolved into a series at once and the integral can be shown by a series without other help from the formula $a^x Xdx$; for it becomes :

$$\int a^x Xdx = \int Xdx + \frac{la}{1}\int Xxdx + \frac{(la)^2}{1\cdot 2}\int Xx^2dx + \frac{(la)^3}{1\cdot 2\cdot 3}\int Xx^3dx + \text{etc.}$$

Thus, if there is put $X = x^n$, there is had

$$\int a^x x^n dx = C + \frac{x^{n+1}}{n+1} + \frac{x^{n+2}la}{1(n+2)} + \frac{x^{n+3}(la)^2}{1\cdot 2(n+3)} + \frac{x^{n+4}(la)^3}{1\cdot 2\cdot 3(n+4)} + \text{etc.},$$

where it is to be noted, if *n* is a whole negative number, for example, $n = -i$, in place of $\frac{x^{n+i}}{n+i}$ there must be written *lx*.

EXAMPLE 3

230. *To express the integral of the formula* $\frac{a^x dx}{1-x}$ *by an infinites series.*

By the first solution we obtain on account of $X = \frac{1}{1-x}$

$$P = \frac{dX}{dx} = \frac{1}{(1-x)^2}, \quad Q = \frac{dP}{dx} = \frac{1\cdot 2}{(1-x)^3}, \quad R = \frac{dQ}{dx} = \frac{1\cdot 2\cdot 3}{(1-x)^4} \quad \text{etc.}$$

and hence the following series :

$$\int \frac{a^x dx}{1-x} = a^x\left(\frac{1}{(1-x)la} - \frac{1}{(1-x)^2(la)^2} + \frac{1\cdot 2}{(1-x)^3(la)^3} - \frac{1\cdot 3\cdot 5}{(1-x)^4(la)^4} + \text{etc.}\right).$$

Other series can be found, if either a^x or the fraction $\frac{1}{1-x}$ is expanded out in a series. But it is considered most convenient, because the series by being formed is elicited ; for the sake of brevity for *a* we take the number *e*, so that *le* = 1, and there is put in place $dy = \frac{e^x dx}{1-x}$ or

$$\frac{dy}{dx}(1-x) - 1 - x - \frac{x^2}{1\cdot 2} - \frac{x^3}{1\cdot 2\cdot 3} - \frac{x^4}{1\cdot 2\cdot 3\cdot 4} - \text{etc.} = 0;$$

now for *y* this series is formed :

$$y = \int \frac{e^x dx}{1-x} = A + Bx + Cx^2 + Dx^3 + Ex^4 + Fx^5 + \text{etc.}$$

and it becomes on making the substitution [i.e. as usual, the coefficients are equated to zero in the above equation on substitution for y]:

$$\left.\begin{array}{llllll} B & +2Cx & +3Dx^2 & +4Ex^3 & +5Fx^4 & +\text{etc} \\ & -B & -2C & -3D & -4E & \\ -1 & -1 & -\frac{1}{2} & -\frac{1}{6} & -\frac{1}{24} & \end{array}\right\} = 0,$$

from which these determined quantities are elicited :

$$B = 1, \quad C = \tfrac{1}{2}(1+1), \quad D = \tfrac{1}{3}\left(1+1+\tfrac{1}{2}\right),$$
$$E = \tfrac{1}{4}\left(1+1+\tfrac{1}{2}+\tfrac{1}{6}\right), \quad F = \tfrac{1}{5}\left(1+1+\tfrac{1}{2}+\tfrac{1}{6}+\tfrac{1}{24}\right) \quad \text{etc.}$$

PROBLEM 22

231. *To investigate the integral of the differential formula* $dy = x^{nx}dx$ *, and to express it by an infinite series.*

SOLUTION

This cannot be performed more conveniently, than by converting the formula for the exponential x^{nx} into an infinite series, which is

$$x^{nx} = 1 + nxlx + \frac{n^2x^2(lx)^2}{1\cdot 2} + \frac{n^3x^3(la)^3}{1\cdot 2\cdot 3} + \frac{n^4x^4(la)^4}{1\cdot 2\cdot 3\cdot 4} + \text{etc.},$$

with which on being multiplied by dx and with the individual terms integrated becomes :

$$\int dx = x,$$
$$\int xdxlx = x^2\left(\frac{lx}{2} - \frac{1}{2^2}\right),$$
$$\int x^2dx(lx)^2 = x^3\left(\frac{(lx)^2}{3} - \frac{2lx}{3^2} + \frac{2\cdot 1}{3^3}\right),$$
$$\int x^3dx(lx)^3 = x^4\left(\frac{(lx)^3}{4} - \frac{3(lx)^2}{4^2} + \frac{3\cdot 2lx}{4^3} - \frac{3\cdot 2\cdot 1}{4^4}\right),$$
$$\int x^4dx(lx)^4 = x^5\left(\frac{(lx)^4}{5} - \frac{4(lx)^3}{5^2} + \frac{4\cdot 3(lx)^2}{5^3} - \frac{4\cdot 3\cdot 2lx}{5^4} + \frac{4\cdot 3\cdot 2\cdot 1}{5^5}\right)$$
etc.

Whereby if these series are substituted and the following powers of lx are set out, the integral sought can be expressed by these innumerable infinite series

$$y=\int x^{nx}dx = x\left(1-\frac{nx}{2^2}\quad+\frac{n^2x^2}{3^3}-\frac{n^3x^3}{4^4}+\frac{n^4x^4}{5^5}-\text{etc.}\right)$$
$$+\frac{nx^2lx}{1}\left(\frac{1}{2^1}-\frac{nx}{3^2}+\frac{n^2x^2}{4^3}-\frac{n^3x^3}{5^4}+\frac{n^4x^4}{6^5}-\text{etc.}\right)$$
$$+\frac{n^2x^3(lx)^2}{1\cdot 2}\left(\frac{1}{3^1}-\frac{nx}{4^2}+\frac{n^2x^2}{5^3}-\frac{n^3x^3}{6^4}+\frac{n^4x^4}{7^5}-\text{etc.}\right)$$
$$+\frac{n^3x^4(lx)^2}{1\cdot 2\cdot 3}\left(\frac{1}{4^1}-\frac{nx}{5^2}+\frac{n^2x^2}{6^3}-\frac{n^3x^3}{7^4}+\frac{n^4x^4}{8^5}-\text{etc.}\right)$$
etc.,

which integral has thus been taken, to that it vanishes on putting $x=0$.

COROLLARY

232. Therefore with this integration law established : if $x=1$ is put in place [*i.e.* as the upper limit of the sum], the value of the integral $\int x^{nx}dx$ is equal to this series

$$1-\frac{n}{2^2}+\frac{n^2}{3^3}-\frac{n^3}{4^4}+\frac{n^4}{5^5}-\text{etc.},$$

which is worthy of note on account of the elegance of all the terms.

SCHOLIUM

233. In the same way the integral of this formula is found :

$$y=\int x^{nx}x^m dx=\int x^m dx\left(1+nxlx+\frac{n^2x^2(lx)^2}{1\cdot 2}+\frac{n^3x^3(lx)^3}{1\cdot 2\cdot 3}+\text{etc.}\right);$$

it becomes with the individual terms integrated :

$$\int x^m dx=\frac{x^{m+1}}{m+1},$$
$$\int x^{m+1}dxlx = x^{m+2}\left(\frac{lx}{m+2}-\frac{1}{(m+2)^2}\right),$$
$$\int x^{m+2}dx(lx)^2 = x^{m+3}\left(\frac{(lx)^2}{m+3}-\frac{2lx}{(m+3)^2}+\frac{2\cdot 1}{(m+3)^3}\right),$$
$$\int x^{m+3}dx(lx)^3 = x^{m+4}\left(\frac{(lx)^3}{m+4}-\frac{3(lx)^2}{(m+4)^2}+\frac{3\cdot 2lx}{(m+4)^3}-\frac{3\cdot 2\cdot 1}{(m+4)^4}\right)$$
etc.

But if hence the integral is to be determined thus, so that it vanishes on putting $x = 0$, then indeed there may be put in place $x = 1$, for in this case the value of the integral of the formula $\int x^{nx} x^m dx$ may be expressed by this memorable enough series

$$\frac{1}{m+1} - \frac{n}{(m+2)^2} + \frac{nn}{(m+3)^3} - \frac{n^3}{(m+4)^4} + \frac{n^4}{(m+5)^5} - \text{ etc.},$$

which, as has been shown, cannot have a place whenever *m* is a negative integer. I will not add other examples of the exponential formulas, because generally integrals are expressed in an exceedingly awkward manner, but the method by which these are to be treated has been set out here sufficiently. Yet meanwhile special attention is deserved for formulas admitting complete integration, which are contained in this form $e^x(dP + Pdx)$, of which the integral is clearly $e^x P$. But in cases of this kind it is with difficulty that the integral can be found and the rules treated, and generally much must be attributed to guesswork. Just as if this formula should be proposed :

$$\frac{e^x xdx}{(1+x)^2}$$

which is supposed to have an easy integral, if that is given, it will be of the form :

$$\frac{e^x z}{1+x}$$

Hence with the differential of this

$$\frac{e^x\left(dz(1+x)+xzdx\right)}{(1+x)^2}$$

since compared with that gives

$$dz(1+x) + xzdx = xdx,$$

where it is apparent at once that $z = 1$, since, unless this should be apparent, it is recognized from the rules with difficulty. Where I cross to the other kind of transcendental formulas now received in analysis, which either angles or the sines or tangents of angles are embraced.

CAPUT IV

DE INTEGRATIONE FORMULARUM LOGARITHMICARUM ET EXPONENTIALIUM

PROBLEMA 18

189. *Si X designet functionem algebraicam ipsius x, invenire integrale formulae Xdxlx.*

SOLUTIO

Quaeratur integrale $\int Xdx$, quod sit = *Z*, et cum quantitatis *Zlx* differentiale sit $dZlx + \frac{Zdx}{x}$, erit

$$Zlx = \int dZlx + \int \frac{Zdx}{x}$$

ideoque

$$\int dZlx = \int Xdxlx = Zlx - \int \frac{Zdx}{x}.$$

Sicque integratio formulae propositae reducta est ad integrationem huius $\frac{Zdx}{x}$, quae, si *Z* fuerit functio algebraica ipsius *x*, non amplius logarithmum involvit ideoque per praecedentes regulas tractari poterit. Sin autem $\int Xdx$ algebraice exhiberi nequeat, hinc nihil subsidii nascitur expedietque indicatione integralis $\int Xdxlx$ acquiescere eiusque valorem per approximationem investigare.

Nisi forte sit $X = \frac{1}{x}$, quo casu manifesto dat

$$\int \frac{dx}{x} lx = \frac{1}{2}(lx)^2 + C.$$

COROLLARIUM 1

190. Eodem modo, si denotante *V* functionem quamcunque ipsius *x* proposita sit formula *XdxlV* ; erit existente $\int Xdx = Z$ eius integrale = $ZlV - \int Z\frac{dV}{V}$ sicque ad formulam algebraicam reducitur, si modo *Z* algebraice detur.

COROLLARIUM 2

191. Pro casu singulari $\frac{dx}{x}lx$ notare licet, si posito *lx* = *u* fuerit *U* functio quaecunque algebraica ipsius *u*, integrationem huius formulae $\frac{Udx}{x}$ non fore difficilem, quia ob $\frac{dx}{x} = du$ abit in *Udu*, cuius integratio ad praecedentia capita refertur.

SCHOLION

192. Haec reductio innititur isti fundamento, quod, cum sit

$$d.xy = ydx + xdy,$$

hinc vicissim fiat $xy = \int ydx + \int xdy$ ideoque

$$\int ydx = xy - \int xdy,$$

ita ut hoc modo in genere integratio formulae ydx ad integrationem formulae xdy reducatur. Quodsi ergo proposita quacunque formula Vdx functio Vin duos factores, puta $V = PQ$, resolvi queat, ita ut integrale $\int Pdx = S$ assignari queat, ob $Pdx = dS$ erit $Vdx = P\,Qdx = QdS$ hincque

$$\int Vdx = QS - \int SdQ.$$

Huiusmodi reductio insignem usum affert, cum formula $\int SdQ$ simplicior fuerit quam proposita $\int Vdx$ eaque insuper simili modo ad simpliciorem reduci queat. Interdum etiam commode evenit, ut hac methodo tandem ad formulam propositae similem perveniatur, quo casu integratio pariter obtinetur. Veluti si ulteriori reductione inveniremus $\int SdQ = T + n\int Vdx$, foret utique $\int Vdx = QS - T - n\int Vdx$ hincque

$$\int Vdx = \tfrac{QS-T}{n+1}.$$

Tum igitur talis reductio insignem praestat usum, cum vel ad formulam simpliciorem vel ad eandem perducit. Atque ex hoc principio praecipuos casus, quibus formula $Xdxlx$ vel integrationem admittit vel per seriem commode exhiberi potest, evolvamus.

EXEMPLUM 1

193. *Formulae differentialis $x^n dxlx$ integrale invenire denotante n numerum quemcunque.*

Cum sit $\int x^n dx = \tfrac{1}{n+1}x^{n+1}$, erit

$$\int x^n dxlx = \tfrac{1}{n+1}x^{n+1}lx - \int \tfrac{1}{n+1}x^{n+1}d.lx$$
$$= \tfrac{1}{n+1}x^{n+1}lx - \tfrac{1}{n+1}\int x^n dx = \tfrac{1}{n+1}x^{n+1}lx - \tfrac{1}{(n+1)^2}x^{n+1}$$

ideoque

$$\int x^n dxlx = \tfrac{1}{n+1}x^{n+1}\left(lx - \tfrac{1}{n+1}\right).$$

Sicque haec formula absolute est integrabilis.

COROLLARIUM 1

194. Casus simpliciores, quibus n est numerus integer sive positivus sive negativus, tenuisse iuvabit

$$\int dxlx = xlx - x, \qquad \int \frac{dx}{xx} lx = -\frac{1}{x} lx - \frac{1}{x},$$
$$\int xdxlx = \frac{1}{2} xxlx - \frac{1}{4} xx, \qquad \int \frac{dx}{x^3} lx = -\frac{1}{2xx} lx - \frac{1}{4xx},$$
$$\int x^2 dxlx = \frac{1}{3} x^3 lx - \frac{1}{9} x^3, \qquad \int \frac{dx}{x^4} lx = -\frac{1}{3x^3} lx - \frac{1}{9x^3},$$
$$\int x^3 dxlx = \frac{1}{4} x^4 lx - \frac{1}{16} x^4, \qquad \int \frac{dx}{x^5} lx = -\frac{1}{4x^4} lx - \frac{1}{16x^4}.$$

COROLLARIUM 2

195. Casum $\int \frac{dx}{x} lx = \frac{1}{2}(lx)^2$, qui est omnino singularis, iam supra annotavimus, sequitur vero etiam ex reductione ad eandem formulam. Namque per superiorem reductionem habemus

$$\int \frac{dx}{x} lx = lx \cdot lx - \int lx.d.lx = (lx)^2 - \int \frac{dx}{x} lx$$

hincque $2\int \frac{dx}{x} lx = (lx)^2$, consequenter

$$2\int \frac{dx}{x} lx = (lx)^2 .$$

EXEMPLUM 2

196. *Formulae* $\frac{dx}{1-x} lx$ *integrale per seriem exprimere.*

Reductione ante adhibita parum lucramur; prodit enim

$$\int \frac{dx}{1-x} lx = l\frac{1}{1-x} \cdot lx - \int \frac{dx}{x} l\frac{1}{1-x} .$$

Cum autem sit

$$l\frac{1}{1-x} = x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{1}{4}x^4 + \text{etc.},$$

erit

$$\int \frac{dx}{x} l\frac{1}{1-x} = x + \frac{1}{4}x^2 + \frac{1}{9}x^3 + \frac{1}{16}x^4 + \frac{1}{25}x^5 + \text{etc.},$$

ideoque

$$\int \frac{dx}{x} lx = l\frac{1}{1-x} \cdot lx - x - \frac{1}{4}x^2 - \frac{1}{9}x^3 - \frac{1}{16}x^4 - \frac{1}{25}x^5 - \text{etc.},$$

quod integrale evanescit casu $x = 0$; etsi enim lx tum in infinitum abit, tamen $l\frac{1}{1-x} = x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{1}{4}x^4 + \text{etc.}$ ita evanescit, ut, etiam si per lx multiplicetur, in nihilum abeat; est enim in genere $x^n lx = 0$ posito $x = 0$, dum n numerus positivus.

COROLLARIUM 1

197. Si ponamus $1 - x = u$, fit

$$\frac{dx}{1-x}lx = -\frac{du}{u}l(l-u) = \frac{du}{u}l\frac{1}{1-u}$$

ideoque

$$\int\frac{dx}{1-x}lx = C + u + \frac{1}{2}u^2 + \frac{1}{9}u^3 + \frac{1}{16}u^4 + \frac{1}{25}u^5 + \text{etc.};$$

quae ut etiam casu $x = 0$ seu $u = 1$ evanescat, capi debet

$$C = -1 - \frac{1}{2} - \frac{1}{9} - \frac{1}{16} - \frac{1}{25} - \text{etc.} = -\frac{1}{6}\pi\pi.$$

COROLLARIUM 2

198. Sumto ergo $1 - x = u$ seu $x + u = 1$ aequales erunt inter se hae expressiones

$$-lx\cdot lu - x - \frac{1}{4}x^2 - \frac{1}{9}x^3 - \frac{1}{16}x^4 - \text{etc.}$$
$$= -\frac{1}{6}\pi^2 + u + \frac{1}{4}u^2 + \frac{1}{9}u^3 + \frac{1}{16}u^4 + \text{etc}$$

seu erit

$$\frac{1}{6}\pi^2 - lx\cdot lu = x + u + \frac{1}{4}\left(x^2 + u^2\right) + \frac{1}{9}\left(x^3 + u^3\right) + \frac{1}{16}\left(x^4 + u^4\right) + \text{etc.}$$

COROLLARIUM 3

199. Haec series maxime convergit ponendo $x = u = \frac{1}{2}$; hoc ergo casu habebimus

$$\frac{1}{6}\pi^2 - (l2)^2 = 1 + \frac{1}{2\cdot 4} + \frac{1}{4\cdot 9} + \frac{1}{8\cdot 16} + \frac{1}{16\cdot 25} + \frac{1}{32\cdot 36} + \text{etc.}$$

Huius ergo seriei

$$x + \frac{1}{4}x^2 + \frac{1}{9}x^3 + \frac{1}{16}x^4 + \frac{1}{25}x^5 + \text{etc.}$$

summa habetur non solum casu $x = 1$, quo est $= \frac{\pi\pi}{6}$,sed etiam casu $x = \frac{1}{2}$,
quo est $= \frac{1}{12}\pi^2 - \frac{1}{2}(l2)^2$.

COROLLARIUM 4

200. Si ponamus $x=\frac{1}{3}$ et $u=\frac{2}{3}$, erit huius seriei

$$1+\frac{5}{3^2\cdot 4}+\frac{9}{3^3\cdot 9}+\frac{17}{3^4\cdot 16}+\frac{33}{3^5\cdot 25}+\frac{65}{3^6\cdot 36}+\text{etc.}$$

cuius terminus generalis $\frac{1+2^n}{3^n nn}$, summa $=\frac{1}{6}\pi^2-l3\cdot l\frac{3}{2}$, neque vero hinc seriei

$$x+\frac{1}{4}x^2+\frac{1}{9}x^3+\frac{1}{16}x^4+\text{etc.}$$

binos casus $x=\frac{1}{3}$ et $x=\frac{2}{3}$ seorsim summare licet.

EXEMPLUM 3

201. *Formulae* $\frac{dx}{(1-x)^2}lx$ *integrale invenire idemque in seriem convertere.*

Cum sit $\int\frac{dx}{(1-x)^2}=\frac{1}{1-x}$, erit

$$\int\frac{dx}{(1-x)^2}lx=\frac{1}{1-x}lx-\int\frac{dx}{x(1-x)},$$

at ob
$\frac{1}{x(1-x)}=\frac{1}{x}+\frac{1}{1-x}$ fit

$$\int\frac{dx}{x(1-x)}=lx+l\frac{1}{1-x}$$

unde colligimus integrale

$$\int\frac{dx}{(1-x)^2}lx=\frac{lx}{1-x}-lx-l\frac{1}{1-x}=\frac{xlx}{1-x}-l\frac{1}{1-x}$$

ita sumtum, ut evanescat posito $x = 0$. Iam pro serie commodissime invenienda statuatur $1-x=u$ et nostra formula fit

$$=\frac{-du}{uu}l(1-u)=\frac{du}{uu}l\frac{1}{(1-u)}=\frac{du}{uu}\left(u+\frac{1}{2}u^2+\frac{1}{3}u^3+\frac{1}{4}u^4+\frac{1}{5}u^5+\text{etc.}\right).$$

Quocirca integrando nanciscimur

$$\int\frac{dx}{(1-x)^2}lx=C+lu+\frac{u}{1\cdot 2}+\frac{uu}{2\cdot 3}+\frac{u^3}{3\cdot 4}+\frac{u^4}{4\cdot 5}+\text{etc.};$$

quae expressio ut etiam evanescat facto $x = 0$ seu $u = 1$, oportet sit

$$C=-\frac{1}{1\cdot 2}-\frac{1}{2\cdot 3}-\frac{1}{3\cdot 4}-\frac{1}{4\cdot 5}-\text{etc.}=-1.$$

Quare ob $x=1-u$ obtinebimus

$$\frac{u}{1\cdot 2}+\frac{uu}{2\cdot 3}+\frac{u^3}{3\cdot 4}+\frac{u^4}{4\cdot 5}+\text{etc.}=1-lu+\frac{(1-u)l(1-u)}{u}+lu$$
$$=1+\frac{(1-u)l(1-u)}{u}\ .$$

COROLLARIUM 1

202. Simili modo si $dy=\frac{du}{u\sqrt{u}}l\frac{1}{(1-u)}$, erit

$$y=-\frac{2}{\sqrt{u}}l\frac{1}{1-u}+\int\frac{2du}{(1-u)\sqrt{u}},$$

at posito $u = xx$ fit

$$\int\frac{2du}{(1-u)\sqrt{u}}=4\int\frac{dx}{1-xx}=2l\frac{1+x}{1-x}.$$

Ergo

$$y=2l\frac{1+\sqrt{u}}{1-\sqrt{u}}-\frac{2}{\sqrt{u}}l\frac{1}{1-u}.$$

At quia per seriem

$$dy=\frac{du}{u\sqrt{u}}\left(u+\tfrac{1}{2}uu+\tfrac{1}{3}u^3+\tfrac{1}{4}u^4+\text{etc.}\right),$$

erit etiam

$$y=2\sqrt{u}+\frac{2}{2\cdot 3}u\sqrt{u}+\frac{2}{3\cdot 5}u^2\sqrt{u}+\frac{2}{4\cdot 7}u^3\sqrt{u}+\text{etc.}$$

COROLLARIUM 2

203. Si ergo multiplicemus per $\frac{\sqrt{u}}{2}$, adipiscimur

$$u+\frac{uu}{2\cdot 3}+\frac{u^3}{3\cdot 5}+\frac{u^4}{4\cdot 7}+\frac{u^5}{5\cdot 9}+\text{etc.}=\sqrt{u}\cdot l\frac{1+\sqrt{u}}{1-\sqrt{u}}+l(1-u),$$

quae summa est etiam

$$=\left(1+\sqrt{u}\right)l\left(1+\sqrt{u}\right)+\left(1-\sqrt{u}\right)l\left(1-\sqrt{u}\right).$$

Quare sumto $u=1$ ob $\left(1-\sqrt{u}\right)l\left(1-\sqrt{u}\right)=0$ erit

$$1+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 5}+\frac{1}{4\cdot 7}+\frac{1}{5\cdot 9}+\frac{1}{6\cdot 11}+\text{ etc.}=2l2.$$

PROBLEMA 19

204. *Si P denotet functionem ipsius x, invenire integrale huius formulae* $dy=dP(lx)^n$.

SOLUTIO

Per reductionem supra monstratam fit

$$y = P(lx)^n - \int Pd.(lx)^n = P(lx)^n - n\int \frac{Pdx}{x}(lx)^{n-1}.$$

Hinc, si sit $\int \frac{Pdx}{x} = Q$, erit simili modo

$$\int \frac{Pdx}{x}(lx)^{n-1} = Q(lx)^{n-1} - (n-1)\int \frac{Qdx}{x}(lx)^{n-2}.$$

Quo modo si ulterius progredimur haecque integralia capere liceat

$$\int \frac{Pdx}{x} = Q, \quad \int \frac{Qdx}{x} = R, \quad \int \frac{Rdx}{x} = S, \quad \int \frac{Sdx}{x} = T \text{ etc.},$$

obtinebimus integrale quaesitum

$$\int dP(lx)^n = P(lx)^n - nQ(lx)^{n-1} + n(n-1)R(lx)^{n-2} - n(n-1)(n-2)S(lx)^{n-3} + \text{etc.},$$

ac si exponens *n* fuerit numerus integer positivus, integrale forma finita exprimetur.

EXEMPLUM 1

205. *Formulae* $x^m dx(lx)^2$ *integrale assignare.*

Hic est *n* = 2 et $P = \frac{x^{m+1}}{m+1}$ hinc

$$Q = \frac{x^{m+1}}{(m+1)^2} \quad \text{et} \quad R = \frac{x^{m+1}}{(m+1)^3}$$

unde colligimus

$$\int x^m dx(lx)^2 = x^{m+1}\left(\frac{(lx)^2}{m+1} - \frac{2lx}{(m+1)^2} + \frac{2\cdot 1}{(m+1)^3}\right),$$

quod integrale evanescit posito $x = 0$, dum sit $m+1 > 0$.

COROLLARIUM 1

206. Hinc posito $x = 1$ fit $\int x^m dx(lx)^2 = \frac{2\cdot 1}{(m+1)^3}$. Ex praecedentibns autem patet, si formula $\int x^m dxlx$ ita integretur, ut evaneseat posito $x = 0$, tum facto $x = 1$ fieri $\int x^m dxlx = \frac{-1}{(m+1)^2}$.

COROLLARIUM 2

207. At si sit $m=-1$, ut habeatur $\frac{dx}{x}(lx)^2$, erit eius integrale

$$\int \frac{dx}{x}(lx)^2 = \frac{1}{3}(lx)^3,$$

qui solus casus ex formula generali est excipiendus.

EXEMPLUM 2

208. *Formulae* $x^{m-1}dx(lx)^3$ *integrale assignare.*

Hic est $n=3$ et $P=\frac{x^m}{m}$ hinc

$$Q=\frac{x^m}{m^2},\quad R=\frac{x^m}{m^3},\quad \text{et}\quad S=\frac{x^m}{m^4},$$

unde integrale quaesitum fit

$$\int x^{m-1}dx(lx)^3 = x^m\left(\frac{(lx)^3}{m}-\frac{3(lx)^2}{m^2}+\frac{3\cdot 2lx}{m^3}-\frac{3\cdot 2\cdot 1}{m^4}\right),$$

quod integrale evanescit posito $x=0$, dum sit $m>0$.

COROLLARIUM 1

209. Quodsi integrali ita sumto, ut evaneseat posito $x=0$, tum ponatur $x=1$, erit

$$\int x^{m-1}dx=\frac{1}{m},\quad \int x^{m-1}lxdx=-\frac{1}{m^2},\quad \int x^{m-1}dx(lx)^2=+\frac{1\cdot 2}{m^3}$$

et

$$\int x^{m-1}dx(lx)^3=-\frac{1\cdot 2\cdot 3}{m^4}.$$

COROLLARIUM 2

210. Casu autem $m=0$ erit integrale $\int \frac{dx}{x}(lx)^3=\frac{1}{4}(lx)^4$, quod ita determinari nequit, ut evanescat posito $x=0$; oporteret enim constantem infinitam adiici. Hoc autem integrale evanescit posito $x=1$.

EXEMPLUM 3

211. *Formulae* $x^{m-1}dx(lx)^n$ *integrale assignare.*

Cum hic sit $P=\frac{x^m}{m}$, erit

$$Q=\frac{x^m}{m^2},\quad R=\frac{x^m}{m^3},\quad \text{et}\quad S=\frac{x^m}{m^4},$$

Hinc integrale quaesitum prodit

$$\int x^{m-1}dx(lx)^n = x^m\left(\frac{(lx)^n}{m} - \frac{n(lx)^{n-1}}{m^2} + \frac{n(n-1)(lx)^{n-2}}{m^3} - \frac{n(n-1)(n-2)(lx)^{n-3}}{m^4} + \text{etc.}\right).$$

Casu autem $m = 0$ est

$$\int \tfrac{dx}{x}(lx)^n = \tfrac{1}{n+1}(lx)^{n+1}.$$

COROLLARIUM 1

212. Si $m > 0$, integrale assignatum evanescit posito $x = 0$; deinceps ergo si sumatur $x = 1$, erit integrale

$$\int x^{m-1}dx(lx)^n = \pm\tfrac{1\cdot 2\cdot 3\cdots n}{m^{n+1}},$$

ubi signum + valet, si *n* sit numerus par, inferius vero, si *n* impar.

COROLLARIUM 2

213. Haec ergo ambiguitas tollitur, si loco *lx* scribatur $-l\tfrac{1}{x}$; tum enim integratione eodem modo instituta positoque $x = 1$ fiet

$$\int x^{m-1}dx\left(l\tfrac{1}{x}\right)^n = +\tfrac{1\cdot 2\cdot 3\cdots n}{m^{n+1}}.$$

SCHOLION

214. Si exponens *n* sit numerus fractus, integrale inventum per seriem infinitam exprimitur; veluti si sit $n = -\tfrac{1}{2}$, reperitur

$$\int \tfrac{x^{m-1}dx}{\sqrt{lx}} = x^m\left(\frac{1}{m\sqrt{lx}} + \frac{1}{2m^2(lx)^{\frac{3}{2}}} + \frac{1\cdot 3}{4m^3(lx)^{\frac{5}{2}}} + \frac{1\cdot 3\cdot 5}{8m^4(lx)^{\frac{7}{2}}} + \text{etc.}\right),$$

quae etiam, quatenus initio *x* ab 0 ad 1 crescere sumitur, hoc modo repraesentari potest

$$\int \tfrac{x^{m-1}dx}{\sqrt{l\frac{1}{x}}} = \tfrac{x^m}{\sqrt{l\frac{1}{x}}}\left(\frac{1}{m} + \frac{1}{2m^2lx} + \frac{1\cdot 3}{4m^3(lx)^2} + \frac{1\cdot 3\cdot 5}{8m^4(lx)^3} + \text{etc.}\right),$$

Si exponens *n* sit negativus, etsi integer, tamen integrale inventum in infinitum progredietur; verum hoc casu alia ratione integrationem instituere licet, qua tandem reducitur ad huiusmodi formulam $\int \tfrac{Tdx}{lx}$, cuius integratio nullo modo simplicior reddi potest. Hanc ergo reductionem sequenti problemate doceamus.

PROBLEMA 20

215. *Integrationem huius formulae* $dy = \frac{Xdx}{(lx)^n}$ *continuo ad formulas simpliciores reducere.*

SOLUTIO

Formula proposita ita repraesentetur

$$dy = Xx \cdot \frac{dx}{x(lx)^n}$$

et cum sit $\int \frac{dx}{x(lx)^n} = \frac{-1}{(n-1)(lx)^{n-1}}$ erit

$$y = \frac{-Xx}{(n-1)(lx)^{n-1}} + \frac{1}{n-1}\int \frac{1}{(lx)^{n-1}} \cdot d(Xx).$$

Quare si ponamus continuo

$$d(Xx) = Pdx, \quad d(Px) = Qdx, \quad d(Qx) = Rdx \quad \text{etc.},$$

erit hanc reductionem continuando

$$y = \frac{-Xx}{(n-1)(lx)^{n-1}} - \frac{Px}{(n-1)(n-2)(lx)^{n-2}} - \frac{Qx}{(n-1)(n-2)(n-3)(lx)^{n-3}} - \text{ etc.},$$

donec tandem perveniatur ad hanc integralem

$$+\frac{1}{(n-1)(n-2)\cdots 1}\int \frac{Vdx}{lx},$$

ita ut, quoties n fuerit numerus integer positivus, integratio tandem ad huiusmodi formulam perducatur.

EXEMPLUM 1

216. *Formulae differentialis* $dy = \frac{x^{m-1}dx}{(lx)^2}$ *integrale investigare.*

Hic est $n = 2$ et $X = x^{m-1}$ unde fit $P = mx^{m-1}$ hincque integrale

$$y = \int \frac{x^{m-1}dx}{(lx)^2} = -\frac{x^m}{lx} + \frac{m}{1}\int \frac{x^{m-1}dx}{lx}$$

At formulae $\frac{x^{m-1}dx}{lx}$ integrale exhiberi nequit, nisi casu $m = 0$, quo fit $\int \frac{dx}{xlx} = llx$. Verum si $m = 0$, formulae propositae integratio ne hinc quidem pendet; fit enim absolute

$$\int \frac{dx}{x(lx)^2} = -\frac{1}{lx} + C.$$

EXEMPLUM 2

217. *Formulae differentialis* $dy = \frac{x^{m-1}dx}{(lx)^n}$ *integrale investigare casibus, quibus n est numerus integer positivus.*

Cum sit $X = x^{m-1}$, erit

$$P = \frac{d.Xx}{dx} = mx^{m-1},$$

tum vero

$$Q = \frac{d.Px}{dx} = m^2x^{m-1}, \quad R = m^3x^{m-1}, \quad S = m^4x^{m-1} \quad \text{etc.}$$

Quare integrale hinc ita formabitur, ut sit

$$y = \int \frac{x^{m-1}dx}{(lx)^n} = -\frac{x^m}{(n-1)(lx)^{n-1}} - \frac{mx^m}{(n-1)(n-2)(lx)^{n-2}} - \frac{m^2x^m}{(n-1)(n-2)(n-3)(lx)^{n-3}} - \cdots + \frac{m^{n-1}}{(n-1)(n-2)\cdots 1}\int \frac{x^{m-1}dx}{lx}.$$

COROLLARIUM

218. Pro *n* ergo successive numeros 1, 2, 3, 4 etc. substituendo habebimus istas reductiones

$$\int \frac{x^{m-1}dx}{(lx)^2} = -\frac{x^m}{lx} + \frac{m}{1}\int \frac{x^{m-1}dx}{lx},$$

$$\int \frac{x^{m-1}dx}{(lx)^3} = -\frac{x^m}{2(lx)^2} - \frac{mx^m}{2\cdot 1 lx} + \frac{m^2}{2\cdot 1}\int \frac{x^{m-1}dx}{lx},$$

$$\int \frac{x^{m-1}dx}{(lx)^4} = -\frac{x^m}{3(lx)^3} - \frac{mx^m}{3\cdot 2(lx)^2} - \frac{m^2x^m}{3\cdot 2\cdot 1 lx} + \frac{m^3}{3\cdot 2\cdot 1}\int \frac{x^{m-1}dx}{lx}.$$

SCHOLION

219. Hae ergo integrationes pendent a formula $\int \frac{x^{m-1}dx}{lx}$, quae posito $x^m = z$ ob $x^{m-1}dx = \frac{1}{m}dz$ et $lx = \frac{1}{m}lz$ reducitur ad hanc simplicissimam formam $\int \frac{dz}{lz}$; cuius integrale si assignari posset, amplissimum usum in Analysi esset allaturum, verum nullis adhuc artificiis neque per logarithmos neque angulos exhiberi potuit; quomodo autem per seriem exprimi possit, infra ostendemus (§ 228). Videtur ergo haec formula $\int\frac{dz}{lz}$ singularem speciem functionum transcendentium suppeditare, quae utique accuratiorem evolutionem meretur. Eadem autem quantitas transcendens in integrationibus formularum exponentialium frequenter occurrit, quas in hoc capite tractare instituimus, propterea quod cum logarithmicis tam arcte cohaerent, ut alterum genus facile in alterum converti possit; veluti ipsa formula modo; considerata $\frac{dz}{lz}$ posito $lz = x$, ut sit $z = e^x$ et $dz = e^x dx$, transformatur in hanc exponentialem $e^x\frac{dx}{x}$, cuius ergo integratio aeque est abscondita. Formulas igitur tractabiles evolvamus et huiusmodi quidem, quae non obvia substitutione ad formam algebraicam reduci possunt. Veluti si V fuerit functio quaecunque ipsius v sitque $v = a^x$, formula Vdx ob $x = \frac{lv}{lx}$ et $dx = \frac{dv}{vla}$ abit in $\frac{Vdv}{vla}$, quae ratione variabilis v est algebraica. Huiusmodi ergo formulas $\frac{a^x dx}{\sqrt{(1+a^{nx})}}$, quippe quae posito $a^x = v$ nihil habent difficultatis, hinc excludimus.

PROBLEMA 21

220. *Formulae differentialis $a^x Xdx$ denotante X functionem quamcunque ipsius x integrale investigare.*

SOLUTIO 1

Cum sit $d.a^x = a^x dxla$, erit vicissim $\int a^x dx = \frac{1}{la}\cdot a^x$; quare si formula proposita in hos factores resolvatur $X\cdot a^x dx$, habebitur per reductionem

$$\int a^x Xdx = \frac{1}{la}a^x X - \frac{1}{la}\int a^x dX .$$

Quodsi ulterius ponamus $d\mathrm{X} = Pdx$, ut sit

$$\int a^x Pdx = \frac{1}{la}a^x P - \frac{1}{la}\int a^x dP$$

prodibit haec reductio

$$\int a^x Xdx = \frac{1}{la}a^x X - \frac{1}{(la)^2}a^x P + \frac{1}{(la)^2}\int a^x dP.$$

Si porro ponamus $dP = Qdx$, habebitur haec reductio

$$\int a^x Xdx = \frac{1}{la}a^x X - \frac{1}{(la)^2}a^x P + \frac{1}{(la)^3}a^x Q - \frac{1}{(la)^3}\int a^x dQ.$$

sicque ulterius ponendo $dQ = Rdx$, $dR = Sdx$ etc. progredi licet, donec ad formulam vel integrabilem vel in suo genere simplicissimam perveniatur.

SOLUTIO 2

Alio modo resolutio formulae in factores institui potest; ponatur

$$\int Xdx = P \quad \text{seu} \quad Xdx = dP$$

et formula ita relata $a^x dP$ habebitur

$$\int a^x Xdx = a^x P - la\int a^x Pdx$$

simili modo si ponamus $\int Pdx = Q$, obtinebimus

$$\int a^x Xdx = a^x P - la \cdot a^x Q + (la)^2 \int a^x Qdx .$$

Ponamus porro $\int Qdx = R$ et consequimur

$$\int a^x Xdx = a^x P - la \cdot a^x Q + (la)^2 \cdot a^x R - (la)^3 \int a^x Rdx$$

hocque modo, quousque lubuerit, progredi licet, donec ad formulam vel integrabilem vel in suo genere simplicissimam perveniamus.

COROLLARIUM 1

221. Priori solutione semper uti licet, quia functiones *P, Q, R* etc. per differentiationem functionis *X* eliciuntur, dum est

$$P = \tfrac{dX}{dx}, \quad Q = \tfrac{dP}{dx}, \quad R = \tfrac{dQ}{dx} \quad \text{etc.}$$

Quare si *X* fuerit functio rationalis integra, tandem ad formulam pervenietur $\int a^x dx = \frac{1}{la} \cdot a^x$ ideoque his casibus integrale absolute exhiberi potest.

COROLLARIUM 2

222. Altera solutio locum non invenit, nisi formulae *Xdx* integrale *P*assignari queat; neque etiam eam continuare licet, nisi quatenus sequentes integrationes

$$\int Pdx = Q, \quad \int Qdx = R \quad \text{etc.}$$

succedunt.

EXEMPLUM 1

223. *Formulae* $a^x x^n dx$ *integrale definire denotante n numerum integrum positivum.*

Cum sit $X = x^n$, solutione prima utentes habebimus

$$\int a^x x^n dx = \frac{1}{la}\cdot a^x x^n - \frac{n}{la}\int a^x x^{n-1} dx\,;$$

hinc ponendo pro *n* successive numeros 0, 1, 2, 3 etc., quia primo casu integratio constat, sequentia integralia eruemus

$$\int a^x dx = \frac{1}{la}a^x,$$

$$\int a^x x dx = \frac{1}{la}a^x x - \frac{1}{(la)^2}a^x,$$

$$\int a^x x^2 dx = \frac{1}{la}a^x x^2 - \frac{2}{(la)^2}a^x x + \frac{2\cdot 1}{(la)^2}a^x,$$

$$\int a^x x^3 dx = \frac{1}{la}a^x x^3 - \frac{3}{(la)^2}a^x x^2 + \frac{3\cdot 2}{(la)^3}a^x x - \frac{3\cdot 2\cdot 1}{(la)^4}a^x$$

etc.,

unde in genere pro quovis exponente *n* concludimus

$$\int a^x dx = a^x\left(\frac{x^n}{la} - \frac{nx^{n-1}}{(la)^2} + \frac{n(n-1)x^{n-2}}{(la)^3} - \frac{n(n-1)(n-2)x^{n-3}}{(la)^4} + \text{ etc.}\right),$$

ad quam expressionem insuper constantem arbitrariam adiici oportet, ut integrale completum obtineatur.

COROLLARIUM

224. Si integrale ita determinari debeat, ut evanescat posito $x = 0$, erit

$$\int a^x dx = \frac{1}{la}a^x - \frac{1}{la},$$

$$\int a^x x dx = a^x\left(\frac{x}{la} - \frac{1}{(la)^2}\right) + \frac{1}{(la)^2},$$

$$\int a^x x^2 dx = a^x\left(\frac{x^2}{la} - \frac{2x}{(la)^2} + \frac{2\cdot 1}{(la)^3}\right) - \frac{2\cdot 1}{(la)^3},$$

$$\int a^x x^3 dx = a^x\left(\frac{x^3}{la} - \frac{3x^2}{(la)^2} + \frac{3\cdot 2x}{(la)^3} - \frac{3\cdot 2\cdot 1}{(la)^4}\right) + \frac{3\cdot 2\cdot 1}{(la)^4}$$

etc.,

EXEMPLUM 2

225. *Formulae* $\frac{a^x dx}{x^n}$ *integrale investigare, si quidem n denotet numerum integrum positivum.*

Hic commode altera solutione utemur, ubi, cum sit $X = \frac{1}{x^n}$, erit

$$P = \frac{-1}{(n-1)x^{n-1}}$$

hincque resultat ista reductio

$$\int \frac{a^x dx}{x^n} = \frac{-a^x}{(n-1)x^{n-1}} + \frac{la}{(n-1)}\int \frac{a^x dx}{x^{n-1}}.$$

Perspicuum igitur est posito $n = 1$ hinc nihil concludi posse; qui est ipse casus supra memoratus $\int \frac{a^x dx}{x}$ singularem speciem transcendentium functionum complectens, qua admissa integralia sequentium casuum exhibere poterimus:

$$\int \frac{a^x dx}{x^2} = C - \frac{a^x}{1x} + \frac{la}{1}\int \frac{a^x dx}{x},$$

$$\int \frac{a^x dx}{x^3} = C - \frac{a^x}{2x^2} - \frac{a^x la}{2\cdot 1x} + \frac{(la)^2}{2\cdot 1}\int \frac{a^x dx}{x},$$

$$\int \frac{a^x dx}{x^4} = C - \frac{a^x}{3x^3} - \frac{a^x la}{3\cdot 2x} + \frac{a^x (la)^2}{3\cdot 2\cdot 1x} + \frac{(la)^3}{3\cdot 2\cdot 1}\int \frac{a^x dx}{x},$$

unde in genere colligimus

$$\int \frac{a^x dx}{x^n} = C - \frac{a^x}{(n-1)x^{n-1}} - \frac{a^x la}{(n-1)(n-2)x^{n-2}} - \frac{a^x (la)^2}{(n-1)(n-2)(n-3)x^{n-3}}$$
$$- \cdots - \frac{a^x (la)^{n-2}}{(n-1)(n-2)\cdots 1x} + \frac{a^x (la)^{n-1}}{(n-1)(n-2)\cdots 1}\int \frac{a^x dx}{x}.$$

COROLLARIUM 1

226. Admissa ergo quantitate transcendente $\int \frac{a^x dx}{x}$ hanc formulam $a^x x^m dx$ integrare poterimus, sive exponens *m* fuerit numerus integer positivus sive negativus. Illis quidem casibus integratio ab ista nova quantitate transcendente non pendet.

COROLLARIUM 2

227. At si *m* fuerit fractus numerus, neutra solutio negotium conficit, sed utraque seriem infinitam pro integrali exhibet. Veluti si sit $m = -\frac{1}{2}$, habebimus ex priore

$$\int \frac{a^x dx}{\sqrt{x}} = a^x \left(\frac{1}{la} + \frac{1}{2x(la)^2} + \frac{1\cdot 3}{4x^2(la)^3} + \frac{1\cdot 3\cdot 5}{8x^3(la)^4} \right) : \sqrt{x} + C,$$

ex posteriore autem

$$\int \frac{a^x dx}{\sqrt{x}} = C + \frac{a^x}{\sqrt{x}}\left(\frac{2x}{1} - \frac{4x^2 la}{1\cdot 3} + \frac{8x^3 (la)^2}{1\cdot 3\cdot 5} - \frac{16x^4 (la)^3}{1\cdot 3\cdot 5\cdot 7} + \text{etc.}\right).$$

SCHOLION 1

228. Hinc quantitas transcendens $\int \frac{a^x dx}{x}$ per seriem exprimi potest secundum potestates ipsius x progredientem. Cum enim sit

$$a^x = 1 + xla + \frac{x^2 (la)^2}{1\cdot 2} + \frac{x^3 (la)^3}{1\cdot 2\cdot 3} + \text{etc.},$$

erit

$$\int \frac{a^x dx}{x} = C + lx + \frac{xla}{1} + \frac{x^2 (la)^2}{1\cdot 2\cdot 2} + \frac{x^3 (la)^3}{1\cdot 2\cdot 3\cdot 3} + \frac{x^4 (la)^4}{1\cdot 2\cdot 3\cdot 4\cdot 4} + \text{etc.},$$

ac si pro a sumamus numerum, cuius logarithmus hyperbolicus est unitas, quem numerum littera e indicemus, habebimus

$$\int \frac{e^x dx}{x} = C + lx + \frac{x}{1} + \frac{1}{2}\cdot\frac{x^2}{1\cdot 2} + \frac{1}{3}\cdot\frac{x^3}{1\cdot 2\cdot 3} + \frac{1}{4}\cdot\frac{x^4}{1\cdot 2\cdot 3\cdot 4} + \text{etc.}$$

Atque hinc etiam ponendo $e^x = z$, ut sit $x = lz$, formulam supra memoratam $\frac{dz}{lz}$ per seriem integrare poterimus:

$$\int \frac{dz}{lz} = C + llz + \frac{lz}{1} + \frac{1}{2}\cdot\frac{(lz)^2}{1\cdot 2} + \frac{1}{3}\cdot\frac{(lz)^3}{1\cdot 2\cdot 3} + \frac{1}{4}\cdot\frac{(lz)^4}{1\cdot 2\cdot 3\cdot 4} + \text{etc.};$$

quod integrale si debeat evanescere sumto $z = 0$, constans C fit infinita, unde pro reliquis casibus nihil concludi potest. Idem incommodum locum habet, si evanescens reddamus casu $z = 1$, quia $llz = l0$ fit infinitum. Caeterum patet, si integrale sit reale pro valoribus ipsius z unitate minoribus, ubi lz est negativus, tum pro valoribus unitate maioribus fieri imaginarium et vicissim. Hinc ergo natura huius functionis transcendentis parum cognoscitur.

SCHOLION 2

229. Quando vel integratio non succedit vel series ante inventae minus idoneae videntur, hinc quantitatem a^x in seriem resolvendo statim sine aliis subsidiis formulae $a^x Xdx$ integrale per seriem exhiberi potest; erit enim

$$\int a^x Xdx = \int Xdx + \frac{la}{1}\int Xxdx + \frac{(la)^2}{1\cdot 2}\int Xx^2 dx + \frac{(la)^3}{1\cdot 2\cdot 3}\int Xx^3 dx + \text{etc.}$$

Ita, si sit $X = x^n$, habebitur

$$\int a^x x^n dx = C + \frac{x^{n+1}}{n+1} + \frac{x^{n+2} la}{1(n+2)} + \frac{x^{n+3}(la)^2}{1\cdot 2(n+3)} + \frac{x^{n+4}(la)^3}{1\cdot 2\cdot 3(n+4)} + \text{etc.},$$

ubi notandum, si *n* fuerit numerus integer negativus, puta $n = -i$, loco $\frac{x^{n+i}}{n+i}$ scribi debere *lx*.

EXEMPLUM 3

230. *Formulae* $\frac{a^x dx}{1-x}$ *integrale per seriem infinitam exprimere.*

Per priorem solutionem obtinemus ob $X = \frac{1}{1-x}$

$$P = \frac{dX}{dx} = \frac{1}{(1-x)^2}, \quad Q = \frac{dP}{dx} = \frac{1\cdot 2}{(1-x)^3}, \quad R = \frac{dQ}{dx} = \frac{1\cdot 2\cdot 3}{(1-x)^4} \quad \text{etc.}$$

hincque sequentem seriem

$$\int \frac{a^x dx}{1-x} = a^x \left(\frac{1}{(1-x)la} - \frac{1}{(1-x)^2 (la)^2} + \frac{1\cdot 2}{(1-x)^3 (la)^3} - \frac{1\cdot 3\cdot 5}{(1-x)^4 (la)^4} + \text{etc.} \right).$$

Aliae series reperiuntur, si vel a^x vel fractio $\frac{1}{1-x}$ in seriem evolvatur. Commodissima autem videtur, quae seriem fingendo eruitur; brevitatis gratia pro *a* sumamus numerum *e*, ut *le* = 1, ac statuatur $dy = \frac{e^x dx}{1-x}$ seu

$$\frac{dy}{dx}(1-x) - 1 - x - \frac{x^2}{1\cdot 2} - \frac{x^3}{1\cdot 2\cdot 3} - \frac{x^4}{1\cdot 2\cdot 3\cdot 4} - \text{etc.} = 0;$$

iam pro *y* fingatur haec series

$$y = \int \frac{e^x dx}{1-x} = A + Bx + Cx^2 + Dx^3 + Ex^4 + Fx^5 + \text{etc.}$$

eritque facta substitutione

$$\left.\begin{array}{llllll} B & +2Cx & +3Dx^2 & +4Ex^3 & +5Fx^4 & +\text{etc} \\ & -B & -2C & -3D & -4E & \\ -1 & -1 & -\frac{1}{2} & -\frac{1}{6} & -\frac{1}{24} & \end{array}\right\} = 0,$$

unde eliciuntur istae determinationes

$$B=1,\quad C=\tfrac{1}{2}(1+1),\quad D=\tfrac{1}{3}\left(1+1+\tfrac{1}{2}\right),$$
$$E=\tfrac{1}{4}\left(1+1+\tfrac{1}{2}+\tfrac{1}{6}\right),\quad F=\tfrac{1}{5}\left(1+1+\tfrac{1}{2}+\tfrac{1}{6}+\tfrac{1}{24}\right)\quad \text{etc.}$$

PROBLEMA 22

231. *Formulae differentialis* $dy=x^{nx}dx$ *integrale investigare ac per seriem infinitam exprimere.*

SOLUTIO

Commodius hoc praestari nequit, quam ut formula exponentialis x^{nx} in seriem infinitam convertatur, quae est

$$x^{nx}=1+nxlx+\frac{n^2x^2(lx)^2}{1\cdot 2}+\frac{n^3x^3(la)^3}{1\cdot 2\cdot 3}+\frac{n^4x^4(la)^4}{1\cdot 2\cdot 3\cdot 4}+\text{etc.},$$

qua per *dx* multiplicata et singulis terminis integratis erit

$$\int dx=x,$$
$$\int xdxlx = x^2\left(\frac{lx}{2}-\frac{1}{2^2}\right),$$
$$\int x^2dx(lx)^2 = x^3\left(\frac{(lx)^2}{3}-\frac{2lx}{3^2}+\frac{2\cdot 1}{3^3}\right),$$
$$\int x^3dx(lx)^3 = x^4\left(\frac{(lx)^3}{4}-\frac{3(lx)^2}{4^2}+\frac{3\cdot 2lx}{4^3}-\frac{3\cdot 2\cdot 1}{4^4}\right),$$
$$\int x^4dx(lx)^4 = x^5\left(\frac{(lx)^4}{5}-\frac{4(lx)^3}{5^2}+\frac{4\cdot 3(lx)^2}{5^3}-\frac{4\cdot 3\cdot 2lx}{5^4}+\frac{4\cdot 3\cdot 2\cdot 1}{5^5}\right)$$
$$\text{etc.}$$

Quare si hae series substituantur et secundum potestates ipsius *lx* disponantur, integrale quaesitum exprimetur per has innumerabiles series infinitas

$$y=\int x^{nx}dx=x\left(1-\frac{nx}{2^2}\quad+\frac{n^2x^2}{3^3}-\frac{n^3x^3}{4^4}+\frac{n^4x^4}{5^5}-\text{etc.}\right)$$
$$+\frac{nx^2lx}{1}\left(\frac{1}{2^1}-\frac{nx}{3^2}+\frac{n^2x^2}{4^3}-\frac{n^3x^3}{5^4}+\frac{n^4x^4}{6^5}-\text{etc.}\right)$$
$$+\frac{n^2x^3(lx)^2}{1\cdot 2}\left(\frac{1}{3^1}-\frac{nx}{4^2}+\frac{n^2x^2}{5^3}-\frac{n^3x^3}{6^4}+\frac{n^4x^4}{7^5}-\text{etc.}\right)$$
$$+\frac{n^3x^4(lx)^2}{1\cdot 2\cdot 3}\left(\frac{1}{4^1}-\frac{nx}{5^2}+\frac{n^2x^2}{6^3}-\frac{n^3x^3}{7^4}+\frac{n^4x^4}{8^5}-\text{etc.}\right)$$
$$\text{etc.,}$$

quod integrale ita est sumtum, ut evanescat posito $x=0$.

COROLLARIUM

232. Hac ergo lege instituta integratione si ponatur $x=1$, valor integralis $\int x^{nx}dx$ huic seriei aequatur

$$1-\frac{n}{2^2}+\frac{n^2}{3^3}-\frac{n^3}{4^4}+\frac{n^4}{5^5}-\text{etc .},$$

quae ob concinnitatem terminorum omnino est notatu digna.

SCHOLION

233. Eodem modo reperitur integrale huius formulae

$$\int x^{nx}x^m dx=\int x^m dx\left(1+nxlx+\frac{n^2x^2(lx)^2}{1\cdot 2}+\frac{n^3x^3(lx)^3}{1\cdot 2\cdot 3}+\text{etc.}\right);$$

erit singulis terminis integrandis

$$\int x^m dx=\frac{x^{m+1}}{m+1},$$

$$\int x^{m+1}dxlx=x^{m+2}\left(\frac{lx}{m+2}-\frac{1}{(m+2)^2}\right),$$

$$\int x^{m+2}dx(lx)^2=x^{m+3}\left(\frac{(lx)^2}{m+3}-\frac{2lx}{(m+3)^2}+\frac{2\cdot 1}{(m+3)^3}\right),$$

$$\int x^{m+3}dx(lx)^3=x^{m+4}\left(\frac{(lx)^3}{m+4}-\frac{3(lx)^2}{(m+4)^2}+\frac{3\cdot 2lx}{(m+4)^3}-\frac{3\cdot 2\cdot 1}{(m+4)^4}\right)$$

etc.

Quodsi ergo integrale ita determinetur, ut evanescat posito $x=0$, tum vero statuatur $x=1$, pro hoc casu valor formulae integralis $\int x^{nx}x^m dx$ exprimetur hac serie satis memorabili

$$\frac{1}{m+1}-\frac{n}{(m+2)^2}+\frac{nn}{(m+3)^3}-\frac{n^3}{(m+4)^4}+\frac{n^4}{(m+5)^5}-\text{ etc.},$$

quae, uti manifestum est, locum habere nequit, quoties *m* est numerus integer negativus. Alia exempla formularum exponentialium non adiungo, quia plerumque integralia nimis inconcinne exprimuntur, methodus autem eas tractandi hic sufficienter est exposita. Interim tamen singularem attentionem merentur formulae integrationem absolute admittentes, quae in hac forma continentur $e^x(dP+Pdx)$, cuius integrale manifesto est e^xP. Huiusmodi autem casibus difficile est regulas tradere integrale inveniendi et coniecturae plerumque plurimum est tribuendum. Veluti si proponeretur haec formula

$$\frac{e^x xdx}{(1+x)^2}$$

facile est suspicari integrale, si datur, talem formam esse habiturum

$$\frac{e^{x}z}{1+x}$$

Huius ergo differentiale

$$\frac{e^{x}\left(dz(1+x)+xzdx\right)}{(1+x)^{2}}$$

cum illo comparatum dat

$$dz(1+x)+xzdx=xdx,$$

ubi statim patet esse $z=1$, quod, nisi per se pateret, ex regulis difficulter cognosceretur. Quare transeo ad alterum genus formularum transcendentium iam in Analysin receptarum, quae vel angulos vel sinus tangentesve angulorum complectuntur.