CHAPTER VI

CONCERNING THE DEVELOPMENT OF INTEGRALS IN SERIES, PROGRESSING ACCORDING TO MULTIPLE ANGLES OF THE SINE OR COSINE

PROBLEM 32

272. *To express the integral of the formula* $\frac{d\varphi}{1+n\cos\varphi}$ *in a series progressing according to the sines of multiple angles.*

SOLUTION

Since it is the usual custom [to express the formula] by the series

$$\frac{d\varphi}{1+n\cos\varphi}=1-n\cos.\varphi+n^2\cos.^2\varphi-n^3\cos.^3\varphi+n^4\cos.^4\varphi-\text{etc.},$$

the powers of the cosine can be changed into the cosines of multiple angles with the help of the formulas treated in the *Introductione* [See Book I, Ch. XIV, and a number of papers in the *Opera Omnia*, Series I, vol. XVI.2, E747, etc.], and initially for the odd powers :

$$\cos.\varphi=\cos.\varphi,$$
$$\cos.^3\varphi=\tfrac{3}{4}\cos.\varphi+\tfrac{1}{4}\cos.3\varphi,$$
$$\cos.^5\varphi=\tfrac{10}{16}\cos.\varphi+\tfrac{5}{16}\cos.3\varphi+\tfrac{1}{16}\cos.5\varphi,$$
$$\cos.^7\varphi=\tfrac{35}{64}\cos.\varphi+\tfrac{21}{64}\cos.3p+\tfrac{7}{64}\cos.5p+\tfrac{1}{64}\cos.7\varphi,$$
$$\cos.^9\varphi=\tfrac{126}{256}\cos.\varphi+\tfrac{84}{256}\cos.3\varphi+\tfrac{36}{256}\cos.5\varphi+\tfrac{9}{256}\cos.7\varphi+\tfrac{1}{256}\cos.9\varphi,$$

where it is to be observed, if there is put in general:

$$\cos.^{2\lambda-1}\varphi=A\cos.\varphi+B\cos.3\varphi+C\cos.5\varphi+D\cos.7\varphi+E\cos.9\varphi+\text{etc.},$$

to be

$$A=2.\frac{1\cdot3\cdot5\cdots(2\lambda-1)}{2\cdot4\cdot6\cdots2\lambda}=\frac{2}{2^{2\lambda-1}}\cdot\frac{6}{2}\cdot\frac{10}{3}\cdot\frac{14}{4}\cdots\frac{4\lambda-2}{\lambda},$$

$$B=\frac{\lambda-1}{\lambda+1}A,\quad C=\frac{\lambda-2}{\lambda+2}B,\quad D=\frac{\lambda-3}{\lambda+3}C,\quad E=\frac{\lambda-4}{\lambda+4}D\quad\text{etc.}$$

Now for the odd powers there arises :

$$cos.^0\varphi = 1,$$
$$cos.^2\varphi = \tfrac{1}{2} + \tfrac{1}{2}cos.2\varphi,$$
$$cos.^4\varphi = \tfrac{3}{8} + \tfrac{4}{8}cos.2\varphi + \tfrac{1}{8}cos.4\varphi,$$
$$cos.^6\varphi = \tfrac{10}{32} + \tfrac{15}{32}cos.2\varphi + \tfrac{6}{32}cos.4\varphi + \tfrac{1}{32}cos.6\varphi,$$
$$cos.^8\varphi = \tfrac{35}{128} + \tfrac{56}{128}cos.2\varphi + \tfrac{28}{128}cos.4\varphi + \tfrac{8}{128}cos.6\varphi + \tfrac{1}{128}cos.8\varphi.$$

Moreover if in general there is put :

$$cos.^{2\lambda}\varphi = \mathfrak{A} + \mathfrak{B}cos.2\varphi + \mathfrak{C}cos.4\varphi + \mathfrak{D}cos.6\varphi + \mathfrak{E}cos.8\varphi + \text{etc.},$$

[the coefficients] become

$$\mathfrak{A} = \frac{1\cdot3\cdot5\cdots(2\lambda-1)}{2\cdot4\cdot6\cdots2\lambda} = \frac{1}{2^{2\lambda-1}}\cdot\frac{6}{2}\cdot\frac{10}{3}\cdot\frac{14}{4}\cdots\frac{4\lambda-2}{\lambda},$$

$$\mathfrak{B} = \tfrac{2\lambda}{\lambda+1}\mathfrak{A},\quad \mathfrak{C} = \tfrac{\lambda-1}{\lambda+2}\mathfrak{B},\quad \mathfrak{D} = \tfrac{\lambda-2}{\lambda+3}\mathfrak{C},\quad \mathfrak{E} = \tfrac{\lambda-3}{\lambda+4}\mathfrak{D}\quad \text{etc.}$$

Moreover if these values are now put in place, then there shall be

$$\frac{1}{1+n\,cos\varphi}$$

$=1$	$-n\,cos.\varphi$	$+\tfrac{1}{2}nn\,cos.2\varphi$	$-\tfrac{1}{4}n^3\,cos.3\varphi$	$+\tfrac{1}{8}n^4\,cos.4\varphi$	$-\tfrac{1}{16}n^5\,cos.5\varphi$	$+$ etc.,
$+\tfrac{1}{2}nn$	$-\tfrac{3}{4}n^3$	$+\tfrac{4}{8}n^4$	$-\tfrac{5}{16}n^5$	$+\tfrac{6}{32}n^6$	$-\tfrac{7}{64}n^7$	
$+\tfrac{3}{8}n^4$	$-\tfrac{10}{16}n^5$	$+\tfrac{15}{32}n^6$	$-\tfrac{21}{64}n^7$	$+\tfrac{28}{128}n^8$	$-\tfrac{36}{296}n^9$	
$+\tfrac{10}{32}n^6$	$-\tfrac{35}{64}n^7$	$+\tfrac{56}{128}n^8$	$-\tfrac{84}{256}n^9$			
$+\tfrac{35}{128}n^8$						

[note that in such tables, the power of the cosine is presumed from the first row;]
from which it is apparent, if there is put

$$\frac{1}{1+n\,cos\varphi} = A - B\,cos.\varphi + C\,cos.2\varphi - D\,cos.3\varphi + E\,cos.4\varphi - \text{etc.},$$

that then there becomes

$$A = 1 + \tfrac{1}{2}nn + \tfrac{3}{8}n^4 + \tfrac{10}{32}n^6 + \text{etc.},$$

or

$$A = 1 + \tfrac{1}{2}n^2 + \tfrac{1\cdot3}{2\cdot4}n^4 + \tfrac{1\cdot3\cdot5}{2\cdot4\cdot6}n^6 + \tfrac{1\cdot3\cdot5\cdot7}{2\cdot4\cdot6\cdot8}n^8 + \text{etc.},$$

and thus it is evident that

$$A=\frac{1}{\sqrt{(1-nn)}}.$$

In a similar way there arises

$$B=n+\tfrac{3}{4}n^3+\tfrac{10}{16}n^5+\text{etc.}=\tfrac{2}{n}\left(\tfrac{1}{2}n^2+\tfrac{1\cdot 3}{2\cdot 4}n^4+\tfrac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}n^6+\text{etc.}\right)$$

and thus

$$B=\tfrac{2}{n}\left(\frac{1}{\sqrt{(1-nn)}}-1\right).$$

Now it is permitted to define easily both this value and the following in this manner. Since there shall be :

$$\frac{1}{1+n\cos\varphi}=A-B\cos.\varphi+C\cos.2\varphi-D\cos.3\varphi+E\cos.4\varphi-\text{etc.},$$

it can be multiplied by $1+n\cos.\varphi$, and because

$$\cos.\varphi\ \cos.\lambda\varphi=\tfrac{1}{2}\cos.(\lambda-1)\varphi+\tfrac{1}{2}\cos.(\lambda+1)\varphi,$$

there becomes

$$\begin{array}{llllll} 1= & A & -B\cos.\varphi+ & C\cos.2\varphi- & D\cos.3\varphi+ & E\cos.4\varphi-\text{etc.},\\ & +An & -\tfrac{1}{2}Bn & +\tfrac{1}{2}Cn & -\tfrac{1}{2}Dn & \\ -\tfrac{1}{2}Bn & +\tfrac{1}{2}Cn & -\tfrac{1}{2}Dn & +\tfrac{1}{2}En & -\tfrac{1}{2}Fn & \end{array}$$

from which, since we have now defined A, the remaining coefficients are thus determined :

$$B=\tfrac{2}{n}(A-1),\quad C=\frac{2B-2An}{n},\quad D=\frac{2C-Bn}{n},$$
$$E=\frac{2D-Cn}{n},\quad F=\frac{2E-Dn}{n},\quad G=\frac{2F-En}{n},$$
etc.

Hence the integral can be assigned easily from these coefficients found; for since there shall be [the integral]

$$\int d\varphi\cos.\lambda\varphi=\tfrac{1}{\lambda}\sin.\lambda\varphi,$$

then we shall have

$$\int\frac{d\varphi}{1+n\cos\varphi}=A\varphi-B\sin.\varphi+\tfrac{1}{2}C\sin.2\varphi-\tfrac{1}{3}D\sin.3\varphi+\tfrac{1}{4}E\sin.4\varphi-\text{etc.},$$

which is the series progressing according to the sines of the angles $\varphi,\ 2\varphi,\ 3\varphi$ etc. as was desired.

COROLLARY 1

273. In the first place it is apparent that this resolution cannot be put in place unless *n* is a number less than one ; for if *n* > 1, individual imaginary coefficients emerge. But if there is *n* = 1, on account of $1+cos.\varphi = 2cos.^2 \tfrac{1}{2}\varphi$, then the integral

$$\int \frac{d\varphi}{1+cos.\varphi} = \int \frac{\tfrac{1}{2}d\varphi}{cos.^2 \tfrac{1}{2}\varphi} = \text{tang.}\tfrac{1}{2}\varphi.$$

COROLLARY 2

274. Since there arises

$$A = \frac{1}{\sqrt{(1-nn)}} \quad \text{and} \quad B = \frac{2}{n}\left(\frac{1}{\sqrt{(1-nn)}} - 1\right),$$

the remaining coefficients *C, D, E* etc. make a recurring series, thus in order that, if two neighbouring terms are *P* and *Q,* the following becomes $\frac{2}{n}Q - P$. [It is noted in the *O.O.* edition that since $C = \frac{2}{n}B - 2A$, the recurring series begins again with the term *D.* Thus in what now follows, heurestically, there is a two-fold operation to take us from *P* and *Q* to the next value, $\frac{2}{n}Q - P$; the intermediate operation *z* corresponds to a root of the two-fold operation z^2, and hence these operations are given by the roots of the equation that follows. In general any term is then a linear combination of the two possible powers of the roots found for fixed initial values; thus we have the foundations of the mathematics of linear recurring relations.] Hence, since the roots of the equation $zz = \frac{2}{n}z - 1$ are $\frac{1\pm\sqrt{(1-nn)}}{n}$, each term can be expressed in this form :

$$\alpha\left(\frac{1+\sqrt{(1-nn)}}{n}\right)^{\lambda} + \beta\left(\frac{1-\sqrt{(1-nn)}}{n}\right)^{\lambda}.$$

COROLLARY 3

275. But since in our rule it is not *A* but *2A* that is taken, in the position $\lambda = 0$, *2A* must be in place and thus

$$\alpha + \beta = \frac{2}{\sqrt{(1-nn)}};$$

from which on making $\lambda = 1$ there must become :

$$\frac{\alpha+\beta}{n}+\frac{(\alpha-\beta)\sqrt{(1-nn)}}{n}=\frac{2-2\sqrt{(1-nn)}}{n\sqrt{(1-nn)}},$$

from which

$$\alpha-\beta=-\frac{2}{\sqrt{(1-nn)}}.$$

Hence

$$\alpha=0 \quad \text{and} \quad \beta=\frac{2}{\sqrt{(1-nn)}}$$

and thus any term required in addition to A shall be equal to

$$\frac{2}{\sqrt{(1-nn)}}\left(\frac{1-\sqrt{(1-nn)}}{n}\right)^{\lambda}.$$

COROLLARY 4

276. Hence the coefficients to be established are to be had thus:

$$A=\frac{1}{\sqrt{(1-nn)}},$$

$$B=\frac{2-2\sqrt{(1-nn)}}{n\sqrt{(1-nn)}},$$

$$C=\frac{4-2nn-4\sqrt{(1-nn)}}{nn\sqrt{(1-nn)}},$$

$$D=\frac{8-6nn-2(4-nn)\sqrt{(1-nn)}}{n^3\sqrt{(1-nn)}},$$

$$E=\frac{16-16nn+2n^4-2(8-4nn)\sqrt{(1-nn)}}{n^4\sqrt{(1-nn)}},$$

$$F=\frac{32-40nn+10n^4-2\left(16-12nn+n^4\right)\sqrt{(1-nn)}}{n^5\sqrt{(1-nn)}},$$

$$G=\frac{64-96nn+36n^4-2n^6-2\left(32-32nn+6n^4\right)\sqrt{(1-nn)}}{n^6\sqrt{(1-nn)}}$$

etc.

COROLLARY 5

277. Because $n < 1$, these coefficients generally are more easily determined by the first series found, clearly

$$A = 1 + \tfrac{1}{2}n^2 + \tfrac{1\cdot 3}{2\cdot 4}n^4 + \tfrac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}n^6 + \tfrac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8}n^8 + \text{etc.},$$
$$B = n\left(1 + \tfrac{3}{4}n^2 + \tfrac{3\cdot 5}{4\cdot 6}n^4 + \tfrac{3\cdot 5\cdot 7}{4\cdot 6\cdot 8}n^6 + \tfrac{3\cdot 5\cdot 7\cdot 9}{4\cdot 6\cdot 8\cdot 10}n^8 + \text{etc.}\right),$$
$$C = \tfrac{1}{2}n^2\left(1 + \tfrac{3\cdot 4}{2\cdot 6}n^2 + \tfrac{3\cdot 4\cdot 5\cdot 6}{2\cdot 6\cdot 4\cdot 8}n^4 + \tfrac{3\cdot 4\cdot 5\cdot 6\cdot 7\cdot 8}{2\cdot 6\cdot 4\cdot 8\cdot 6\cdot 10}n^6 + \text{etc.}\right),$$
$$D = \tfrac{1}{4}n^3\left(1 + \tfrac{4\cdot 5}{2\cdot 8}n^2 + \tfrac{4\cdot 5\cdot 6\cdot 7}{2\cdot 8\cdot 4\cdot 10}n^4 + \tfrac{4\cdot 5\cdot 6\cdot 7\cdot 8\cdot 9}{2\cdot 8\cdot 4\cdot 10\cdot 6\cdot 12}n^6 + \text{etc.}\right),$$
$$E = \tfrac{1}{8}n^4\left(1 + \tfrac{5\cdot 6}{2\cdot 10}n^2 + \tfrac{5\cdot 6\cdot 7\cdot 8}{2\cdot 10\cdot 4\cdot 12}n^4 + \tfrac{5\cdot 6\cdot 7\cdot 8\cdot 9\cdot 10}{2\cdot 10\cdot 4\cdot 12\cdot 6\cdot 14}n^6 + \text{etc.}\right),$$
$$F = \tfrac{1}{16}n^5\left(1 + \tfrac{6\cdot 7}{2\cdot 12}n^2 + \tfrac{6\cdot 7\cdot 8\cdot 9}{2\cdot 12\cdot 4\cdot 14}n^4 + \tfrac{6\cdot 7\cdot 8\cdot 9\cdot 10\cdot 11}{2\cdot 12\cdot 4\cdot 14\cdot 6\cdot 16}n^6 + \text{etc.}\right)$$

etc.

SCHOLIUM

278. Since there arises from these values:

$$\int \frac{d\varphi}{1+n\cos\varphi} = A\varphi - B\sin.\varphi + \tfrac{1}{2}C\sin.2\varphi - \tfrac{1}{3}D\sin.3\varphi + \tfrac{1}{4}E\sin.4\varphi - \text{etc.},$$

the first term in this series $A\varphi$ should be noted especially, since it increases continuously on the angle φ increasing, and that as far as infinity, while the remaining terms now increase and then decrease; yet they cannot exceed a certain limit, for $\sin\lambda\varphi$ cannot either increase beyond $+1$ nor decrease below -1. Then since the above integral can be found :

$$\frac{1}{\sqrt{(1-nn)}}\text{Ang.cos.}\frac{n+\cos.\varphi}{1+n\cos.\varphi},$$

that series is equal to this angle. Whereby if here the angle is called ω, so that there becomes

$$d\omega = \frac{d\varphi\sqrt{(1-nn)}}{1+n\cos.\varphi},$$

then

$$\cos.\omega = \frac{n+\cos.\varphi}{1+n\cos.\varphi}$$

and hence $n + \cos.\varphi - \cos.\omega - n\cos.\varphi\cos.\omega = 0$, from which in turn :

$$\cos.\varphi = \frac{\cos.\omega - n}{1 - n\cos.\omega};$$

which formula since it arises from that on taking n negative, becomes

$$d\varphi = \frac{d\omega\sqrt{(1-nn)}}{1-n\cos.\omega}$$

and

$$\frac{\varphi}{\sqrt{(1-nn)}} = A\omega + B\sin.\omega + \tfrac{1}{2}C\sin.2\omega + \tfrac{1}{3}D\sin.3\omega + \tfrac{1}{4}E\sin.4\omega + \text{etc.},$$

Now because there is here

$$\frac{\omega}{\sqrt{(1-nn)}} = A\varphi - B\sin.\varphi + \tfrac{1}{2}C\sin.2\varphi - \tfrac{1}{3}D\sin.3\varphi + \tfrac{1}{4}E\sin.4\varphi - \text{etc.},$$

on account of $\frac{1}{\sqrt{(1-nn)}} = A$ we have

$$0 = B\left(\sin.\omega - \sin.\varphi\right) + \tfrac{1}{2}C\left(\sin.2\omega + \sin.2\varphi\right) + \tfrac{1}{3}D\left(\sin.3\omega - \sin.3\varphi\right) + \text{etc.},$$

and it is helpful to observe relations of this kind.

PROBLEM 33

279. *To express the integral of the formula* $d\varphi(1+n\cos.\varphi)^{\nu}$ *in a series progressing in terms of the sines of multiple angles of* φ.

SOLUTION

Since there shall be :

$$(1+n\cos.\varphi)^{\nu} = 1 + \frac{\nu}{1}n\cos.\varphi + \frac{\nu(\nu-1)}{1\cdot 2}n^2\cos.^2\varphi + \frac{(\nu-1)(\nu-2)}{1\cdot 2\cdot 3}n^3\cos.^3\varphi + \text{etc.},$$

if we put

$$(1+n\cos.\varphi)^{\nu} = A + B\cos.\varphi + C\cos.2\varphi + D\cos.3\varphi + E\cos.4\varphi + \text{etc.},$$

from the above formulas indicated there shall be :

$$A = 1 + \frac{\nu(\nu-1)}{1\cdot 2}\cdot\frac{1}{2}n^2 + \frac{\nu(\nu-1)(\nu-2)(\nu-3)}{1\cdot 2\cdot 3\cdot 4}\cdot\frac{1\cdot 3}{2\cdot 4}n^4 + \frac{\nu(\nu-1)\cdots(\nu-5)}{1\cdot 2\cdot 3\cdots 6}\cdot\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}n^6 + \text{etc.},$$

$$B = 2n\left(\frac{v}{2} + \frac{v(v-1)(v-2)}{1\cdot2\cdot3}\cdot\frac{1\cdot3}{2\cdot4}n^2 + \frac{v(v-1)(v-2)(v-3)(v-4)}{1\cdot2\cdot3\cdot4\cdot5}\cdot\frac{1\cdot3\cdot5}{2\cdot4\cdot6}n^4 + \text{etc.}\right),$$

which series can be shown clearer thus :

$$A = 1 + \frac{v(v-1)}{2\cdot2}n^2 + \frac{v(v-1)(v-2)(v-3)}{2\cdot2\cdot4\cdot4}n^4 + \frac{v(v-1)\cdots(v-5)}{2\cdot2\cdot4\cdot4\cdot6\cdot6}n^6 + \text{etc.},$$
$$\frac{1}{2}B = \frac{v}{2}n + \frac{v(v-1)(v-2)}{2\cdot2\cdot4}n^3 + \frac{v(v-1)(v-2)(v-3)(v-4)}{2\cdot2\cdot3\cdot4\cdot6}n^5 + \text{etc.}$$

But with the two coefficients A and B found the remaining can be found more conveniently in the following manner. Since there is the equation

$$vl(1+n\cos.\varphi) = l(A + B\cos\varphi + C\cos.2\varphi + D\cos.3\varphi + E\cos.4\varphi + \text{etc.}),$$

the differentials are taken and on dividing by $-d\varphi$ there emerges :

$$\frac{vn\cos.\varphi}{1+n\cos.\varphi} = \frac{B\sin.\varphi + 2C\sin.2\varphi + 3D\sin.3\varphi + 4E\sin.4\varphi + \text{etc}}{A + B\cos.\varphi + C\cos.2\varphi + D\cos.3\varphi + E\cos.4\varphi + \text{etc.}}.$$

Now by cross multiplication on account of

$$\sin.\lambda\varphi\cos.\varphi = \tfrac{1}{2}\sin.(\lambda+1)\varphi + \tfrac{1}{2}\sin.(\lambda-1)\varphi$$

and

$$\sin.\varphi\cos.\lambda\varphi = \tfrac{1}{2}\sin.(\lambda+1)\varphi - \tfrac{1}{2}\sin.(\lambda-1)\varphi$$

this equation is come upon :

$$\begin{array}{llllll}
0 = & B\sin.\varphi & +2C\sin.2\varphi & +3D\sin.3\varphi & +4E\sin.4\varphi & +5F\sin.5\varphi + \text{etc.},\\
 & & +\frac{1}{2}Bn & +\frac{2}{2}Cn & +\frac{3}{2}Dn & +\frac{4}{2}En\\
 & +\frac{2}{2}Cn & +\frac{3}{2}Dn & +\frac{4}{2}En & +\frac{5}{2}Fn & +\frac{6}{2}Gn\\
 & -vAn & -\frac{v}{2}Bn & -\frac{v}{2}Cn & -\frac{v}{2}Dn & -\frac{v}{2}En\\
 & +\frac{v}{2}Cn & +\frac{v}{2}Dn & +\frac{v}{2}En & +\frac{v}{2}Fn & +\frac{v}{2}Gn
\end{array}$$

from which these determinations follow :

$$(v+2)Cn+2B-2vAn=0 \quad \Big| \quad C=\frac{2vAn-2B}{(v+2)n}$$

$$(v+3)Dn+4C-(v-1)Bn=0 \quad \Big| \quad D=\frac{(v-1)Bn-4C}{(v+3)n}$$

$$(v+4)En+6D-(v-2)Cn=0 \quad \Big| \quad E=\frac{(v-2)Cn-6D}{(v+4)n}$$

$$(v+5)Fn+8E-(v-3)Dn=0 \quad \Big| \quad F=\frac{(v-3)Dn-8E}{(v+5)n}$$

$$(v+6)Gn+10F-(v-4)En=0 \quad \Big| \quad G=\frac{(v-4)En-10F}{(v+6)n}$$

where if the above values for A and *B* are substituted, there is found

$$C=4n^2\left(\frac{1v(v-1)}{2\cdot 2\cdot 4}+\frac{2v(v-1)(v-2)(v-3)}{2\cdot 2\cdot 4\cdot 4\cdot 6}n^2+\frac{3v(v-1)\cdots(v-5)}{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6\cdot 8}n^4+\text{etc.}\right),$$

$$D=8n^3\left(\frac{1\cdot 2v(v-1)(v-2)}{2\cdot 2\cdot 4\cdot 4\cdot 6}+\frac{2\cdot 3v(v-1)(v-2)(v-3)(v-4)}{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6\cdot 8}n^2+\text{etc.}\right),$$

$$E=16n^4\left(\frac{1\cdot 2\cdot 3v(v-1)(v-2)(v-3)}{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6\cdot 8}+\frac{2\cdot 3\cdot 4v(v-1)\cdots(v-5)}{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6\cdot 8\cdot 8\cdot 10}n^2+\text{etc.}\right)$$

etc.,

from which the form of the following series is deduced. Moreover from these coefficients found, the integral sought becomes :

$$\int d\varphi(1+n\cos.\varphi)^v = A\varphi+B\sin.\varphi+\tfrac{1}{2}C\sin.2\varphi+\tfrac{1}{3}D\sin.3\varphi+\tfrac{1}{4}E\sin.4\varphi+\text{etc.}$$

COROLLARY 1

280. Also from the similarity of the terms of the given series for *C, D, E* etc., the value of *B* can thus be expressed :

$$B=2n\left(\frac{v}{2}+\frac{v(v-1)(v-2)}{2\cdot 2\cdot 4}n^2+\frac{(v-1)(v-2)(v-3)(v-4)}{2\cdot 2\cdot 4\cdot 4\cdot 6}n^4+etc.\right);$$

but a special form has been found for the series *A*, not understood from this rule.

COROLLARY 2

281. If the series *A* and *B* are compared to each other, it is possible to observe various relations between these, of which this is evident in the first place :

$$An+\tfrac{1}{2}B=\tfrac{v+2}{2}n\left(\begin{array}{l}1+\frac{v(v-1)}{2\cdot 4}n^2+\frac{v(v-1)(v-2)(v-3)}{2\cdot 4\cdot 4\cdot 6}n^4\\+\frac{v(v-1)\cdots(v-5)}{2\cdot 4\cdot 4\cdot 6\cdot 6\cdot 8}n^6+\text{etc.}\end{array}\right)$$

which only differs from the series *A* according to the denominators.

COROLLARY 3

282. If we put $\frac{2Ann+Bn}{v+2}=N$, in order that

$$N=n^2+\frac{v(v-1)}{2\cdot 4}n^4+\frac{v(v-1)(v-2)(v-3)}{2\cdot 4\cdot 4\cdot 6}n^6+\text{etc.},$$
$$A=1+\quad\frac{v(v-1)}{2\cdot 2}n^2+\frac{v(v-1)(v-2)(v-3)}{2\cdot 2\cdot 4\cdot 4}n^4+\text{etc.}$$

But if now *n* is treated as a variable, by differentiation there is produced :

$$\frac{dN}{ndn}=2+\frac{v(v-1)}{2}n^2+\frac{v(v-1)(v-2)(v-3)}{2\cdot 4\cdot 4}n^4+\text{etc.}=2A.$$

Hence since there becomes

$$dN=\frac{4Andn+Bdn+2nndA+ndB}{v+2}=2Andn,$$

then there shall be

$$2vAndn=2nndA+Bdn+ndB.$$

COROLLARY 4

283. Hence from the given coefficient *A*, the coefficient *B* can thus be found by integration, so that there becomes

$$Bn=2\int(vAndn-nndA),$$

or, there shall be also from that form

$$B=\frac{2(v+2)}{n}\int Andn-2An,$$

where it is to be noted that on putting $n=0$ the integral $\int Andn$ must vanish, because in this case *B* vanishes.

SCHOLIUM

284. The series found for the letters *B, C, D* etc. can also be expressed by continuous factors in the following manner :

$$B = vn\left(1+\frac{(v-1)(v-2)}{2\cdot 4}n^2+\frac{(v-3)(v-4)}{4\cdot 6}Pn^2+\frac{(v-5)(v-6)}{6\cdot 8}Pn^2+\text{etc.}\right),$$

$$C = \frac{v(v-1)}{1\cdot 2}\cdot\frac{n^2}{2}\left(1+\frac{(v-2)(v-3)}{2\cdot 6}n^2+\frac{(v-4)(v-5)}{4\cdot 8}Pn^2+\frac{(v-6)(v-7)}{6\cdot 10}Pn^2+\text{etc.}\right),$$

$$D = \frac{v\cdots(v-2)}{1\cdots 3}\cdot\frac{n^3}{4}\left(1+\frac{(v-3)(v-4)}{2\cdot 8}n^2+\frac{(v-5)(v-6)}{4\cdot 10}Pn^2+\frac{(v-7)(v-8)}{6\cdot 12}Pn^2+\text{etc.}\right),$$

$$E = \frac{v\cdots(v-3)}{1\cdots 4}\cdot\frac{n^4}{8}\left(1+\frac{(v-4)(v-5)}{2\cdot 10}n^2+\frac{(v-6)(v-7)}{4\cdot 12}Pn^2+\frac{(v-8)(v-9)}{6\cdot 14}Pn^2+\text{etc.}\right),$$

$$F = \frac{v\cdots(v-4)}{1\cdots 5}\cdot\frac{n^5}{16}\left(1+\frac{(v-5)(v-6)}{2\cdot 12}n^2+\frac{(v-7)(v-8)}{4\cdot 14}Pn^2+\frac{(v-9)(v-10)}{6\cdot 16}Pn^2+\text{etc.}\right),$$

etc.

where in any series the letter *P* denotes the whole preceding term. And with the help of such series the coefficients generally are found more easily than from the rule treated before, from which each is determined from the two preceding. But this rule suffers a defect also, because if *v* should be a whole negative number less than -1, certain coefficients clearly are not defined, which hence must be taken from these series [above]. Thus if there should be put $v=-2$, there will be $B = vAn = -2An$ and also

$$C = \frac{3}{1}\cdot\frac{n^2}{2}\left(1+\frac{4\cdot 5}{2\cdot 6}n^2+\frac{4\cdot 5\cdot 6\cdot 7}{2\cdot 6\cdot 4\cdot 8}n^4+\frac{4\cdot 5\cdot 6\cdot 7\cdot 8\cdot 9}{2\cdot 6\cdot 4\cdot 8\cdot 6\cdot 10}n^6+\text{etc.}\right);$$

if $v=-3$, then $C=-Bn$ and

$$D = \frac{4\cdot 5}{1\cdot 2}\cdot\frac{n^3}{4}\left(1+\frac{6\cdot 7}{2\cdot 8}n^2+\frac{6\cdot 7\cdot 8\cdot 9}{2\cdot 8\cdot 4\cdot 10}n^4+\frac{6\cdot 7\cdot 8\cdot 9\cdot 10\cdot 11}{2\cdot 8\cdot 4\cdot 10\cdot 6\cdot 12}n^4+\text{etc.}\right);$$

if $v=-4$, then $D=-Cn$ and

$$E = \frac{5\cdot 6\cdot 7}{1\cdot 2\cdot 3}\cdot\frac{n^4}{8}\left(1+\frac{8\cdot 9}{2\cdot 10}n^2+\frac{8\cdot 9\cdot 10\cdot 11}{2\cdot 10\cdot 4\cdot 12}n^4+\frac{8\cdot 9\cdot 10\cdot 11\cdot 12\cdot 13}{2\cdot 10\cdot 4\cdot 12\cdot 6\cdot 14}n^6+\text{etc.}\right);$$

if $v=-5$, then $E=-Dn$ and

$$F = \frac{6\cdot 7\cdot 8\cdot 9}{1\cdot 2\cdot 3\cdot 4}\cdot\frac{n^5}{16}\left(1+\frac{10\cdot 11}{2\cdot 12}n^2+\frac{10\cdot 11\cdot 12\cdot 13}{2\cdot 12\cdot 4\cdot 14}n^4+\frac{10\cdot 11\cdot 12\cdot 13\cdot 14\cdot 15}{2\cdot 12\cdot 4\cdot 14\cdot 6\cdot 16}n^6+\text{etc.}\right)$$

and thus for the others.

EXAMPLE 1

285. *To set out the integral of the formula* $d\varphi(1+n\cos.\varphi)^{v}$, *if v is a positive whole number.*

On putting

$$(1+n\cos.\varphi)^{v} = A + B\cos.\varphi + C\cos.2\varphi + D\cos.3\varphi + E\cos.4\varphi + \text{etc.},$$

we have for the individual values of the exponent v :

$$1)\ \text{if } v=1: \quad A=1, \quad B=n, \quad C=0 \quad \text{etc.};$$

$$2)\ \text{if } v=2: \quad A=1+\tfrac{1}{2}n^2, \quad B=2n, \quad C=\tfrac{1}{2}n^2, \quad D=0 \quad \text{etc.};$$

$$3)\ \text{if } v=3: \quad A=1+\tfrac{3}{2}n^2, \quad B=3n\left(1+\tfrac{1}{4}n^2\right), \quad C=\tfrac{3}{2}n^2, \quad D=\tfrac{1}{4}n^2,$$
$$E=0 \quad \text{etc.};$$

$$4)\ \text{if } v=4: \quad A=1+\tfrac{6}{2}n^2+\tfrac{3}{8}n^4, \quad B=4n\left(1+\tfrac{3}{4}n^2\right), \quad C=3n^2\left(1+\tfrac{1}{6}n^2\right),$$
$$D=n^3, \quad E=\tfrac{1}{8}n^4, \quad F=0 \quad \text{etc.};$$

Moreover these cases present no difficulty. To be used, only the following first term A must be known absolutely :

$$\text{if } v=1: \quad A=1,$$

$$\text{if } v=2: \quad A=1+\tfrac{2\cdot1}{2\cdot2}n^2,$$

$$\text{if } v=3: \quad A=1+\tfrac{3\cdot2}{2\cdot2}n^2,$$

$$\text{if } v=4: \quad A=1+\tfrac{4\cdot3}{2\cdot2}n^2+\tfrac{4\cdot3\cdot2\cdot1}{2\cdot2\cdot4\cdot4}n^4,$$

$$\text{if } v=5: \quad A=1+\tfrac{5\cdot4}{2\cdot2}n^2+\tfrac{5\cdot4\cdot3\cdot2}{2\cdot2\cdot4\cdot4}n^4,$$

$$\text{if } v=6: \quad A=1+\tfrac{6\cdot5}{2\cdot2}n^2+\tfrac{6\cdot5\cdot4\cdot3}{2\cdot2\cdot4\cdot4}n^4+\tfrac{6\cdot5\cdot4\cdot3\cdot2\cdot1}{2\cdot2\cdot4\cdot4\cdot6\cdot6}n^6,$$

$$\text{if } v=7: \quad A=1+\tfrac{7\cdot6}{2\cdot2}n^2+\tfrac{7\cdot6\cdot5\cdot4}{2\cdot2\cdot4\cdot4}n^4+\tfrac{7\cdot6\cdot5\cdot4\cdot3\cdot2}{2\cdot2\cdot4\cdot4\cdot6\cdot6}n^6$$

etc.

EXAMPLE 2

286. *To set out the integral of the formula* $\frac{d\varphi}{(1+n\cos.\varphi)^{\mu}}$ *in a series.*

On putting

$$\frac{d\varphi}{(1+n\cos.\varphi)^{\mu}} = A + B\cos.\varphi + C\cos.2\varphi + D\cos.3\varphi + E\cos.4\varphi + \text{etc.}$$

from the preceding formulas on putting $\nu = -\mu$ then there becomes :

$$A = 1 + \frac{\mu(\mu+1)}{2\cdot 2}n^2 + \frac{\mu(\mu+1)(\mu+2)(\mu+3)}{2\cdot 2\cdot 4\cdot 4}n^4 + \frac{\mu(\mu+1)\cdots(\mu+5)}{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6}n^6 + \text{etc.}$$

$$B = -\mu n\left(1 + \frac{(\mu+1)(\mu+2)}{2\cdot 4}n^2 + \frac{(\mu+3)(\mu+4)}{4\cdot 6}Pn^2 + \frac{(\mu+5)(\mu+6)}{6\cdot 8}Pn^2 + \text{etc.}\right),$$

$$C = \frac{\mu(\mu+1)}{1\cdot 2}\cdot\frac{n^2}{2}\left(1 + \frac{(\mu+2)(\mu+3)}{2\cdot 6}n^2 + \frac{(\mu+4)(\mu+5)}{4\cdot 8}Pn^2 + \frac{(\mu+6)(\mu+7)}{6\cdot 10}Pn^2 + \text{etc.}\right),$$

$$D = -\frac{\mu(\mu+1)(\mu+2)}{1\cdot 2\cdot 3}\cdot\frac{n^3}{4}\left(1 + \frac{(\mu+3)(\mu+4)}{2\cdot 8}n^2 + \frac{(\mu+5)(\mu+6)}{4\cdot 10}Pn^2 + \text{etc.}\right),$$

etc.

where as before in any series P denotes the preceding term. But these coefficients thus depend on each other in turn, in order that if

$$B = \frac{-2(\mu-2)}{n}\int Andn - 2An$$

and

$$C = \frac{2B+2\mu An}{(\mu-2)n}, \quad D = \frac{4C+(\mu+1)Bn}{(\mu-3)n}, \quad E = \frac{6D+(\mu+2)Cn}{(\mu-4)n},$$

$$F = \frac{8E+(\mu+3)Dn}{(\mu-5)n}, \quad G = \frac{10F+(\mu+4)En}{(\mu-6)n}, \quad H = \frac{12G+(\mu+5)Fn}{(\mu-7)n}$$

Where with the disadvantage, when μ is a whole number, now the above remedy is to be applied. Therefore here we investigate especially, how the coefficients of each case are able to be determined from the preceding case, since thus it can be done. Since there shall be the equation,

$$\frac{d\varphi}{(1+n\cos.\varphi)^{\mu}} = A + B\cos.\varphi + C\cos.2\varphi + D\cos.3\varphi + E\cos.4\varphi + \text{etc.}$$

there is put

$$\frac{d\varphi}{(1+n\cos.\varphi)^{\mu+1}} = A' + B'\cos.\varphi + C'\cos.2\varphi + D'\cos.3\varphi + \text{etc.}$$

therefore this series multiplied by $1+n\cos.\varphi$ must be changed into that ; but there has been produced :

$$\begin{array}{llll} A' & +\ B'\cos.\varphi + & C'\cos.2\varphi + & D'\cos.3\varphi + \text{etc.}, \\ & +A'n & +\tfrac{1}{2}B'n & +\tfrac{1}{2}C'n \\ +\tfrac{1}{2}B'n & +\tfrac{1}{2}C'n & +\tfrac{1}{2}D'n & +\tfrac{1}{2}E'n \end{array}$$

from which we deduce

$$B' = 2\frac{(A-A')}{n}, \qquad C' = \frac{2(B-B')-2A'n}{n},$$
$$D' = \frac{2(C-C')-B'n}{n}, \quad E' = \frac{2(D-D')-C'n}{n} \quad \text{etc.;}$$

therefore provided the coefficient A' exists, we will have the following B', C', D' etc. Hence we may consider how A' can be determined from A ; because

$$A = 1 + \frac{\mu(\mu+1)}{2\cdot 2}n^2 + \frac{\mu(\mu+1)(\mu+2)(\mu+3)}{2\cdot 2\cdot 4\cdot 4}n^4 + \text{etc.},$$
$$A' = 1 + \frac{(\mu+1)(\mu+2)}{2\cdot 2}n^2 + \frac{(\mu+1)(\mu+2)(\mu+3)(\mu+4)}{2\cdot 2\cdot 4\cdot 4}n^4 + \text{etc.},$$

n is treated as a variable, and the first series multiplied by n^μ is to be differentiated, in order that there is produced :

$$\frac{d.An^\mu}{d.n} = \mu n^{\mu-1} + \frac{\mu(\mu+1)(\mu+2)}{2\cdot 2}n^{\mu+1} + \frac{\mu(\mu+1)(\mu+2)(\mu+3)(\mu+4)}{2\cdot 2\cdot 4\cdot 4}n^{\mu+3} + \text{etc.},$$

which clearly is the series equal to $\mu n^{\mu-1}A'$; on account of which A' is thus determine by A, so that there becomes

$$A' = \frac{d.An^\mu}{d.n^\mu} = A + \frac{ndA}{\mu dn}.$$

Therefore in the case $\mu = 1$ we find $A = \frac{1}{\sqrt{(1-nn)}}$, on account of $\frac{dA}{dn} = \frac{n}{(1-nn)^{\frac{3}{2}}}$, then

$$A' = \frac{1}{\sqrt{(1-nn)}} + \frac{nn}{(1-nn)^{\frac{3}{2}}} = \frac{1}{(1-nn)^{\frac{3}{2}}}$$

Now this is the value of A for $\mu = 2$, from which on account of $\frac{dA}{dn} = \frac{3n}{(1-nn)^{\frac{5}{2}}}$ there becomes for $\mu = 3$

$$A = \frac{1}{(1-nn)^{\frac{3}{2}}} + \frac{3nn}{2(1-nn)^{\frac{5}{2}}} = \frac{1+\frac{1}{2}nn}{(1-nn)^{\frac{5}{2}}}.$$

If we progress further in this manner, we find,

$$\text{if} \quad \mu = 2, \quad A = \frac{1}{(1-nn)\sqrt{(1-nn)}},$$

$$\text{if} \quad \mu = 3, \quad A = \frac{1+\frac{1}{2}nn}{(1-nn)^2\sqrt{(1-nn)}},$$

$$\text{if} \quad \mu = 4, \quad A = \frac{1+\frac{3}{2}nn}{(1-nn)^3\sqrt{(1-nn)}},$$

$$\text{if} \quad \mu = 5, \quad A = \frac{1+3nn+\frac{3}{8}n^4}{(1-nn)^4\sqrt{(1-nn)}}.$$

COROLLARIUM 1

287. In the same manner also the remaining coefficients B', C' etc. can be defined from the analogous B, C etc. and all these relations are similar to each other, clearly as there is the relation,

$$A' = \frac{d.An^{\mu}}{d.n^{\mu}} = A + \frac{ndA}{\mu dn},$$

thus there shall be,

$$B' = \frac{d.Bn^{\mu}}{d.n^{\mu}} = B + \frac{ndB}{\mu dn}, \quad C' = \frac{d.Cn^{\mu}}{d.n^{\mu}} = C + \frac{ndC}{\mu dn} \quad \text{etc.}$$

COROLLARY 2

288. But before we found $B' = \frac{2(A-A')}{n}$, from which there arises

$$B' = \frac{2dA}{\mu dn} = B + \frac{ndB}{\mu dn}$$

and hence $\mu Bdn + ndB + 2dA = 0$; this is multiplied by $n^{\mu-1}$, in order that there becomes

$$d.Bn^{\mu} + 2n^{\mu-1}dA = 0,$$

from which on integrating,

$$Bn^{\mu} = -2\int n^{\mu-1}dA = -2n^{\mu-1}A + 2(\mu-1)\int An^{\mu-2}dn$$

and hence

$$B = -\tfrac{2A}{n} + \tfrac{2(\mu-1)}{n^{\mu}}\int An^{\mu-2}dn$$

But before [§ 286] we had found

$$B = -2An + \tfrac{2(\mu-2)}{n}\int Andn.$$

COROLLARY 3

289. An equation between *A* and *n* is obtained from these equal values, from which the quantity *A* is determined through *n*; for there becomes

$$n^{-\mu}\int n^{\mu-1}dA = An + \tfrac{(\mu-2)}{n}\int Andn,$$

from which there is produced on differentiating twice :

$$(1-nn)ddA + \tfrac{dndA}{n} - 2(\mu+1)ndndA - \mu(\mu+1)Adn^2 = 0.$$

SCHOLIUM 1

290. If these values of A are to be compared with the above, where μ was a negative number, then we are surprised by the excellent agreement :

For the above values,	For these formulas,
if $v = 0$, $A=1$	if $\mu=1$, $A=\frac{1}{\sqrt{(1-nn)}}$
$v = 1$, $A=1$	$\mu=2$, $A=\frac{1}{(1-nn)\sqrt{(1-nn)}}$
$v = 2$, $A=1+\frac{1}{2}nn$	$\mu=3$, $A=\frac{1+\frac{1}{2}nn}{(1-nn)^2\sqrt{(1-nn)}}$
$v = 3$, $A=1+\frac{3}{2}nn$	$\mu=4$, $A=\frac{1+\frac{3}{2}nn}{(1-nn)^3\sqrt{(1-nn)}}$
$v = 4$, $A=1+3nn+\frac{3}{8}nn$	$\mu=5$, $A=\frac{1+3nn+\frac{3}{8}n^4}{(1-nn)^4\sqrt{(1-nn)}}$

etc.,

from which we conclude, if there should be

$$(1+n\cos.\varphi)^{\nu} = A + B\cos.\varphi + C\cos.2\varphi + \text{etc.},$$
$$(1+n\cos.\varphi)^{-\nu-1} = \mathfrak{A} + \mathfrak{B}\cos.\varphi + \mathfrak{C}\cos.2\varphi + \text{etc.},$$

then there becomes

$$\mathfrak{A} = \frac{A}{(1-nn)^{\nu}\sqrt{(1-nn)}}.$$

Whereby since for the cases, in which v is a positive whole number, the value of A can be defined easily, also for the cases, in which it is negative, we are able to assign it readily.

SCHOLIUM 2

291. Since for the case $\mu = 1$ the values of the individual letters A, B, C, D etc. have been found above, clearly for the sake of brevity putting $\frac{1-\sqrt{(1-nn)}}{n} = m$,

$$A = \frac{1}{\sqrt{(1-nn)}}, \quad B = \frac{2m}{\sqrt{(1-nn)}}, \quad C = \frac{2mm}{\sqrt{(1-nn)}}, \quad D = \frac{2m^3}{\sqrt{(1-nn)}},$$

and for any term generally,

$$N = \frac{2m^{\lambda}}{\sqrt{(1-nn)}},$$

if for the similar term in the case $\mu = 2$ we write N', then there shall be $N' = \frac{d.Nn}{dn}$. But now there is :

$$\frac{d.Nn}{dn} = \frac{2m^{\lambda}}{(1-nn)^{\frac{3}{2}}} + \frac{2\lambda n m^{\lambda-1} dm}{dn\sqrt{(1-nn)}},$$

then indeed $\frac{dm}{dn} = \frac{m}{n\sqrt{(1-nn)}}$, from which we deduce

$$N' = \frac{2m^{\lambda}}{(1-nn)^{\frac{3}{2}}} + \frac{2\lambda m^{\lambda}}{1-nn} = \frac{2m^{\lambda}\left(1+\lambda\sqrt{(1-nn)}\right)}{(1-nn)\sqrt{(1-nn)}}.$$

Whereby if we put in place :

$$\frac{1}{(1+n\cos.\varphi)^2} = A + B\cos.\varphi + C\cos.2\varphi + D\cos.3\varphi + E\cos.4\varphi + \text{etc.},$$

then there shall be

$$A=\frac{1}{(1-nn)^{\frac{3}{2}}},\quad B=\frac{2m\left(1+\sqrt{(1-nn)}\right)}{(1-nn)^{\frac{3}{2}}},\quad C=\frac{2m^2\left(1+2\sqrt{(1-nn)}\right)}{(1-nn)^{\frac{3}{2}}},$$

$$D=\frac{2m^3\left(1+3\sqrt{(1-nn)}\right)}{(1-nn)^{\frac{3}{2}}}\ \text{ etc.}$$

Now if the exponent μ were a fractional number, the coefficients *A, B, C, D, E* etc. are seen to be unable to be defined otherwise than by the above series. But initially *A* must be able to be defined in a particular manner from a nearby value, as we demonstrate in the following problem.

PROBLEMA 34

292. *For the expansion of the formula* $(1+n\cos.\varphi)^{\nu}$ *in a series of this kind*

$$A+B\cos.\varphi+C\cos.2\varphi+D\cos.3\varphi+E\cos.4\varphi+\text{etc.},$$

to define now the absolute term A approximately.

SOLUTION

Since by necessity there shall be $n<1$, then the above series found for *A* certainly converges, now if n falls a little short of unity, it is necessary for many terms in the series to be expanded out, before an exact enough value of *A* should be produced, especially if *ν* should be a number just a little more positive than negative. Yet because on putting the expansion of this series

$$(1+n\cos.\varphi)^{-\nu-1}=\mathfrak{A}+\mathfrak{B}\cos.\varphi+\mathfrak{C}\cos.2\varphi+\text{etc.}$$

with the term $\mathfrak{A}$ depending on that term A, in order that $A=(1-nn)^{\nu+\frac{1}{2}}\mathfrak{A}$, on finding this term *A* we have the two series :

$$A=1+\frac{\nu(\nu-1)}{2\cdot 2}n^2+\frac{\nu(\nu-1)(\nu-2)(\nu-3)}{2\cdot 2\cdot 4\cdot 4}n^4+\frac{\nu(\nu-1)(\nu-2)(\nu-3)(\nu-4)(\nu-5)}{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6}n^6+\text{etc.}$$

$$A=(1-nn)^{\nu+\frac{1}{2}}\left\{\begin{array}{l}1+\frac{\nu(\nu-1)}{2\cdot 2}n^2+\frac{\nu(\nu-1)(\nu-2)(\nu-3)}{2\cdot 2\cdot 4\cdot 4}n^4\\+\frac{\nu(\nu-1)(\nu-2)(\nu-3)(\nu-4)(\nu-5)}{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6}n^6+\text{etc.}\end{array}\right\};$$

in whichever case that must be taken, which converges more. Now since yet the remaining coefficients *B, C, D, E* etc. must finally converge, hence it is apparent that *A* can be approached other ways to its value.

For since these coefficients can be defined alternately by even and odd powers of *n*, on taking some angle α there will be :

$$\left(1+n\cos.\alpha\right)^{\nu} = A + B\cos.\alpha + C\cos.2\alpha + D\cos.3\alpha + E\cos.4\alpha + \text{etc.}$$

and

$$\left(1-n\cos.\alpha\right)^{\nu} = A - B\cos.\alpha + C\cos.2\alpha - D\cos.3\alpha + E\cos.4\alpha - \text{etc.}$$

Therefore with these added there is produced :

$$\tfrac{1}{2}\left(1+n\cos.\alpha\right)^{\nu} + \tfrac{1}{2}\left(1-n\cos.\alpha\right)^{\nu} = A + C\cos.2\alpha + E\cos.4\alpha + G\cos.6\alpha + \text{etc.};$$

where if for α we write $90^{\circ}-\alpha$, then

$$\tfrac{1}{2}\left(1+n\cos.\alpha\right)^{\nu} + \tfrac{1}{2}\left(1-n\sin.\alpha\right)^{\nu} = A - C\cos.2\alpha + E\cos.4\alpha - G\cos.6\alpha + \text{etc.},$$

from which with these added with half the terms removed anew. We can form many expressions of this kind and for the sake of brevity we put :

$$\tfrac{1}{4}\left(1+n\cos.\alpha\right)^{\nu} + \tfrac{1}{4}\left(1-n\cos.\alpha\right)^{\nu} + \tfrac{1}{4}\left(1+n\sin.\alpha\right)^{\nu} + \tfrac{1}{4}\left(1-n\sin.\alpha\right)^{\nu} = \mathfrak{A},$$
$$\tfrac{1}{4}\left(1+n\cos.\beta\right)^{\nu} + \tfrac{1}{4}\left(1-n\cos.\beta\right)^{\nu} + \tfrac{1}{4}\left(1+n\sin.\beta\right)^{\nu} + \tfrac{1}{4}\left(1-n\sin.\beta\right)^{\nu} = \mathfrak{B},$$
$$\tfrac{1}{4}\left(1+n\cos.\gamma\right)^{\nu} + \tfrac{1}{4}\left(1-n\cos.\gamma\right)^{\nu} + \tfrac{1}{4}\left(1+n\sin.\gamma\right)^{\nu} + \tfrac{1}{4}\left(1-n\sin.\gamma\right)^{\nu} = \mathfrak{C}$$

etc.

and for the coefficients *B, C, D, E* etc. we write respectively (1), (2), (3), (4) etc., by which we are able to represent easily the terms from some initial removed amount. Hence we have

$$\mathfrak{A} = A + (4)\cos.4\alpha + (8)\cos.8\alpha + (12)\cos.12\alpha + \text{etc.},$$
$$\mathfrak{B} = A + (4)\cos.4\beta + (8)\cos.8\beta + (12)\cos.12\beta + \text{etc.},$$
$$\mathfrak{C} = A + (4)\cos.4\gamma + (8)\cos.8\gamma + (12)\cos.12\gamma + \text{etc.}$$

etc.

And hence we obtain the approximate sequences.

I. If we take $4\alpha = \frac{\pi}{2}$ or $\alpha = \frac{\pi}{8}$, there is produced

$$\mathfrak{A} = A - (8) + (16) - (24) + \text{etc.}$$

Hence

$$A = \mathfrak{A} + (8) - (16) + (24) - \text{ etc.}$$

Whereby if the terms (8) and those following are able to be disregarded as insignificant, then exactly enough, $A = \mathfrak{A}$.

II. We take the two series and we put in place $4\alpha=\frac{\pi}{4}$ and $4\beta=\frac{3\pi}{4}$, so that $\alpha=\frac{\pi}{16}$ and $\beta=\frac{3\pi}{16}$; then there becomes

$$cos.4\alpha+cos.4\beta=0,\quad cos.8\alpha+cos.8\beta=0,\quad cos.12\alpha+cos.12\beta=0$$

and

$$cos.16\alpha+cos.16\beta=-2,$$

from which it follows that

$$\mathfrak{A}+\mathfrak{B}=2A-\ 2(16)+2(32)-2(48)+\text{etc.}$$

and thus

$$A=\tfrac{1}{2}(\mathfrak{A}+\mathfrak{B})+(16)-(32)+\text{etc.},$$

where the numbers (16), (32) generally as the are small, are able to be ignored.

III. We add the three series and put in place $4\alpha=\frac{\pi}{6},\ 4\beta=\frac{3\pi}{6},\quad 4\gamma=\frac{5\pi}{6}$, so that then there becomes $\alpha=\frac{\pi}{24},\ \beta=\frac{\pi}{8},\quad \gamma=\frac{5\pi}{24}$, and then

$$\begin{aligned}&cos.\ 4\alpha+cos.4\beta\ +cos.4\ \gamma=0,\quad cos.16\alpha+cos.16\beta+cos.16\ \gamma=0,\\&cos.\ 8\alpha+cos.8\beta\ +cos.8\ \gamma=0,\quad cos.20\alpha+cos.20\beta\ +cos.20\gamma=0,\\&cos.12\alpha+cos.12\beta+cos.12\gamma=0,\quad cos.24\alpha+cos.24\beta\ +cos.24\gamma=-3,\end{aligned}$$

from which there is deduced

$$A=\tfrac{1}{3}(\mathfrak{A}+\mathfrak{B}+\mathfrak{C})+(24)-(48)+\text{etc.}$$

IV. If this determination is not considered to be exact enough, the four expressions of this kind are added $\mathfrak{A},\mathfrak{B},\mathfrak{C},\mathfrak{D}$ and then $4\alpha=\frac{\pi}{8},\quad 4\beta=\frac{3\pi}{8},\quad 4\gamma=\frac{5\pi}{8},\quad 4\delta=\frac{7\pi}{8}$, and there is found

$$\mathfrak{A}+\mathfrak{B}+\mathfrak{C}+\mathfrak{D}=4A-4(32)+4(64)-\text{etc.},$$

hence more closely

$$A=\tfrac{1}{4}(\mathfrak{A}+\mathfrak{B}+\mathfrak{C}+\mathfrak{D}).$$

COROLLARY 1

293. But from the value found from *A, B* following is found readily enough, since there shall be

$$B=\frac{2(\nu+2)}{n}\int Andn-2An.$$

Hence in as much as there is present in *A* the member $(1\pm n\,cos.\alpha)^{\nu}$ or $(1+nf)^{\nu}$, while *f* embraces all those sines or cosines, from which there arises for *B*

$$\frac{2(v+2)}{n}\int ndn(1+nf)^{v}-2n(1+nf)^{v}=\frac{2-2(1-vnf)(1+nf)^{v}}{(v+1)nff}.$$

COROLLARY 2

294. Moreover with the coefficients A and B known just as we have shown above, all the sequences are able to be derived from these. Now from these found the integration of the formula by itself $d\varphi(1+n\cos.\varphi)^{v}$ has been shown.

PROBLEM 35

295. *To establish the integral of the formula* $d\varphi\, l(1+n\cos.\varphi)$ *by a series progressing according to the sines of the angles* $\varphi, 2\varphi, 3\varphi$ etc.

SOLUTION

Since there is the expansion

$$l(1+n\cos.\varphi)=n\cos.\varphi-\tfrac{1}{2}n^{2}\cos.^{2}\varphi+\tfrac{1}{3}n^{3}\cos.^{3}\varphi-\tfrac{1}{4}n^{4}\cos.^{4}\varphi+\text{etc.},$$

it will be reduced from these powers to simple cosines :

$$\begin{aligned}
l(1+n\cos.\varphi)=n\cos.\varphi&-\tfrac{1}{2}\cdot\tfrac{1}{2}n^{2}\cos.2\varphi+\tfrac{1}{3}\cdot\tfrac{1}{4}n^{3}\cos.3\varphi-\tfrac{1}{4}\cdot\tfrac{1}{8}n^{4}\cos.4\varphi+\text{etc.}\\
-\tfrac{1}{2}\cdot\tfrac{1}{2}n^{2}&+\tfrac{1}{3}\cdot\tfrac{1}{4}n^{3}\qquad -\tfrac{1}{4}\cdot\tfrac{1}{8}n^{4}\qquad +\tfrac{1}{5}\cdot\tfrac{1}{16}n^{5}\\
-\tfrac{1}{4}\cdot\tfrac{3}{8}n^{4}&+\tfrac{1}{5}\cdot\tfrac{10}{16}n^{5}\qquad -\tfrac{1}{6}\cdot\tfrac{15}{32}n^{6}\\
-\tfrac{1}{6}\cdot\tfrac{10}{32}n^{6}&+\tfrac{1}{7}\cdot\tfrac{35}{64}n^{7}\\
-\tfrac{1}{8}\cdot\tfrac{35}{128}n^{8}&
\end{aligned}$$

Whereby if we put

$$l(1+n\cos.\varphi)=-A+B\cos.\varphi-C\cos.2\varphi+D\cos.3\varphi-\text{etc.},$$

then

$$A=\tfrac{1}{2}\cdot\tfrac{n^{2}}{2}+\tfrac{1\cdot3}{2\cdot4}\cdot\tfrac{n^{4}}{4}+\tfrac{1\cdot3\cdot5}{2\cdot4\cdot6}\cdot\tfrac{n^{6}}{6}+\tfrac{1\cdot3\cdot5\cdot7}{2\cdot4\cdot6\cdot8}\cdot\tfrac{n^{8}}{8}+\text{etc.};$$

hence by considering the number n as variable there will be

$$\frac{ndA}{dn}=\tfrac{1}{2}nn+\tfrac{1\cdot3}{2\cdot4}n^{4}+\tfrac{1\cdot3\cdot5}{2\cdot4\cdot6}n^{6}+\text{etc.}=\frac{1}{\sqrt{(1-nn)}}-1.$$

Hence

$$dA=\frac{dn}{n\sqrt{(1-nn)}}-\frac{dn}{n},$$

from which by integration there is produced

$$A = l\frac{1-\sqrt{(1-nn)}}{n} - ln + C = l\frac{2-2\sqrt{(1-nn)}}{nn};$$

in this manner with n vanishing there becomes $A = l1 = 0$.
Now there then is present :

$$\tfrac{1}{2}B = \tfrac{1}{2}n + \tfrac{1\cdot 3}{2\cdot 4}\cdot\tfrac{n^3}{3} + \tfrac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\cdot\tfrac{n^5}{5} + \text{etc.},$$

from which on differentiating there is presented

$$\frac{nndB}{2dn} = \tfrac{1}{2}nn + \tfrac{1\cdot 3}{2\cdot 4}n^4 + \tfrac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}n^6 + \text{etc.} = \frac{1}{\sqrt{(1-nn)}} - 1,$$

hence

$$\tfrac{1}{2}dB = \frac{dn}{nn\sqrt{(1-nn)}} - \frac{dn}{nn}$$

and on integrating,

$$\tfrac{1}{2}B = \frac{-\sqrt{(1-nn)}}{n} + \frac{1}{n} + C = \frac{1-\sqrt{(1-nn)}}{n}$$

with the integral thus determined, so that it vanishes on putting $n = 0$. On account of which for the first two terms we have

$$A = l\frac{2-2\sqrt{(1-nn)}}{nn} \quad \text{and} \quad B = \frac{2-2\sqrt{(1-nn)}}{n},$$

in order that there becomes $A = l\frac{B}{n}$. But for the remaining terms we may differentiate the equation assumed

$$\frac{-nd\varphi \sin.\varphi}{1+n\cos.\varphi} = -Bd\varphi \sin.\varphi + 2Cd\varphi \sin.2\varphi - 3Dd\varphi \sin.3\varphi + 4Ed\varphi \sin.4\varphi - \text{etc.}$$

or

$$0 = \frac{n\sin.\varphi}{1+n\cos.\varphi} - B\sin.\varphi + 2C\sin.2\varphi - 3D\sin.3\varphi + 4E\sin.4\varphi - \text{etc.}$$

Whereby on multiplying by $2 + 2n\cos.\varphi$ there is produced :

$$\begin{aligned}0 = 2n\sin.\varphi - 2B\sin.\varphi &+ 4C\sin.2\varphi - 6D\sin.3\varphi + 8E\sin.4\varphi - \text{etc.}\\ & \quad - Bn \qquad\quad + 2Cn \qquad\quad - 3Dn\\ +2Cn &\quad - 3Dn \qquad\quad + 4En \qquad\quad - 5Fn\end{aligned}$$

from which we deduce

$$C=\tfrac{B-n}{n},\quad D=\tfrac{4C-Bn}{3n},\quad E=\tfrac{6D-2C}{4n},\quad F=\tfrac{8E-3Dn}{5n}.$$

Hence since $B=\frac{2-2\sqrt{(1-nn)}}{n}$, there then becomes

$$C=\frac{2-nn-2\sqrt{(1-nn)}}{n}\quad\text{or}\quad C=\left(\frac{1-\sqrt{(1-nn)}}{n}\right)^2$$

then indeed

$$D=\tfrac{2}{3}\left(\frac{1-\sqrt{(1-nn)}}{n}\right)^3,\quad E=\tfrac{2}{4}\left(\frac{1-\sqrt{(1-nn)}}{n}\right)^4,\quad F=\tfrac{2}{5}\left(\frac{1-\sqrt{(1-nn)}}{n}\right)^5\ \text{etc.}$$

Hence, if for the sake of brevity we put $\frac{1-\sqrt{(1-nn)}}{n}=m$, then there shall be

$$l\left(1+n\cos.\varphi\right)=-l\,\tfrac{2m}{n}+\tfrac{2}{1}m\cos.\varphi-\tfrac{2}{2}m^2\cos.2\varphi+\tfrac{2}{3}m^3\cos.3\varphi-\tfrac{2}{4}m^4\cos.4\varphi+\text{etc.}$$

and thus the integral sought :

$$\int d\varphi\, l\left(1+n\cos.\varphi\right)=\text{Const.}-\varphi\, l\,\tfrac{2m}{n}+\tfrac{2}{1}m\sin.\varphi-\tfrac{2}{4}m^2\sin.2\varphi$$
$$+\tfrac{2}{9}m^3\sin.3\varphi-\tfrac{2}{16}m^4\sin.4\varphi+\tfrac{2}{25}m^5\sin.5\varphi-\text{etc.}$$

COROLLARY

296. But if hence we put $n = 1$, then $m = 1$ and

$$l\left(1+\cos.\varphi\right)=-l2+\tfrac{2}{1}\cos.\varphi-\tfrac{2}{2}\cos.2\varphi+\tfrac{2}{3}\cos.3\varphi-\tfrac{2}{4}\cos.4\varphi+\text{etc.}$$

and

$$l\left(1-\cos.\varphi\right)=-l2-\tfrac{2}{1}\cos.\varphi-\tfrac{2}{2}\cos.2\varphi-\tfrac{2}{3}\cos.3\varphi-\tfrac{2}{4}\cos.4\varphi-\text{etc.}$$

Since now there is

$$1+\cos.\varphi=2\ \cos.\tfrac{1}{2}\varphi^2\ \text{ and }\ 1-\cos.\varphi=2\sin.\tfrac{1}{2}\varphi^2,$$

then

$$l\cos.\tfrac{1}{2}\varphi=-l2+\cos.\varphi-\tfrac{1}{2}\cos.2\varphi+\tfrac{1}{3}\cos.3\varphi-\tfrac{1}{4}\cos.4\varphi+\text{etc.}.$$

and

$$l\sin.\tfrac{1}{2}\varphi=-l2-\cos.\varphi-\tfrac{1}{2}\cos.2\varphi-\tfrac{1}{3}\cos.3\varphi-\tfrac{1}{4}\cos.4\varphi-\text{etc.}$$

hence,

$$l\ \text{tang}.\tfrac{1}{2}\varphi=-2\cos.\varphi-\tfrac{2}{3}\cos.3\varphi-\tfrac{2}{5}\cos.5\varphi-\tfrac{2}{7}\cos.7\varphi-\text{etc.}$$

CAPUT VI

DE EVOLUTIONE INTEGRALlUM PER SERIES SECUNDUM SINUS COSINUSVE ANGULORUM MULTIPLORUM PROGREDIENTES

PROBLEMA 32

272. *Integrale formulae* $\frac{d\varphi}{1+n\cos\varphi}$ *per seriem secundum sinus angulorum multiplorum progredientem exprimere.*

SOLUTIO

Cum sit more consueto per seriem

$$\frac{d\varphi}{1+n\cos\varphi}=1-n\cos.\varphi+n^2\cos.^2\varphi-n^3\cos.^3\varphi+n^4\cos.^4\varphi-\text{etc.},$$

potestates cosinus in cosinus angulorum multiplorum convertantur ope formularum in *Introductione* traditarum ac primo pro potestatibus imparibus:

$$\cos.\varphi=\cos.\varphi,$$

$$\cos.^3\varphi=\tfrac{3}{4}\cos.\varphi+\tfrac{1}{4}\cos.3\varphi,$$

$$\cos.^5\varphi=\tfrac{10}{16}\cos.\varphi+\tfrac{5}{16}\cos.3\varphi+\tfrac{1}{16}\cos.5\varphi,$$

$$\cos.^7\varphi=\tfrac{35}{64}\cos.\varphi+\tfrac{21}{64}\cos.3p+\tfrac{7}{64}\cos.5p+\tfrac{1}{64}\cos.7\varphi,$$

$$\cos.^9\varphi=\tfrac{126}{256}\cos.\varphi+\tfrac{84}{256}\cos.3\varphi+\tfrac{36}{256}\cos.5\varphi+\tfrac{9}{256}\cos.7\varphi+\tfrac{1}{256}\cos.9\varphi,$$

ubi notandum est, si ponatur in genere

$$\cos.^{2\lambda-1}\varphi=A\cos.\varphi+B\cos.3\varphi+C\cos.5\varphi+D\cos.7\varphi+E\cos.9\varphi+\text{etc.},$$

fore

$$A=2.\frac{1\cdot3\cdot5\cdots(2\lambda-1)}{2\cdot4\cdot6\cdots2\lambda}=\frac{2}{2^{2\lambda-1}}\cdot\frac{6}{2}\cdot\frac{10}{3}\cdot\frac{14}{4}\cdots\frac{4\lambda-2}{\lambda},$$

$$B=\frac{\lambda-1}{\lambda+1}A,\quad C=\frac{\lambda-2}{\lambda+2}B,\quad D=\frac{\lambda-3}{\lambda+3}C,\quad E=\frac{\lambda-4}{\lambda+4}D\quad\text{etc.}$$

Pro paribus vero potestatibus est

$$cos.^0\varphi = 1,$$
$$cos.^2\varphi = \tfrac{1}{2} + \tfrac{1}{2}cos.2\varphi,$$
$$cos.^4\varphi = \tfrac{3}{8} + \tfrac{4}{8}cos.2\varphi + \tfrac{1}{8}cos.4\varphi,$$
$$cos.^6\varphi = \tfrac{10}{32} + \tfrac{15}{32}cos.2\varphi + \tfrac{6}{32}cos.4\varphi + \tfrac{1}{32}cos.6\varphi,$$
$$cos.^8\varphi = \tfrac{35}{128} + \tfrac{56}{128}cos.2\varphi + \tfrac{28}{128}cos.4\varphi + \tfrac{8}{128}cos.6\varphi + \tfrac{1}{128}cos.8\varphi.$$

In genere autem si ponatur

$$cos.^{2\lambda}\varphi = \mathfrak{A} + \mathfrak{B}cos.2\varphi + \mathfrak{C}cos.4\varphi + \mathfrak{D}cos.6\varphi + \mathfrak{E}cos.8\varphi + \text{etc.},$$

erit

$$\mathfrak{A} = \tfrac{1\cdot3\cdot5\cdots(2\lambda-1)}{2\cdot4\cdot6\cdots2\lambda} = \tfrac{1}{2^{2\lambda-1}}\cdot\tfrac{6}{2}\cdot\tfrac{10}{3}\cdot\tfrac{14}{4}\cdots\tfrac{4\lambda-2}{\lambda},$$

$$\mathfrak{B} = \tfrac{2\lambda}{\lambda+1}\mathfrak{A},\quad \mathfrak{C} = \tfrac{\lambda-1}{\lambda+2}\mathfrak{B},\quad \mathfrak{D} = \tfrac{\lambda-2}{\lambda+3}\mathfrak{C},\quad \mathfrak{E} = \tfrac{\lambda-3}{\lambda+4}\mathfrak{D}\quad \text{etc.}$$

Quodsi nunc isti valores substituantur, erit

$$\begin{array}{llllll}
\tfrac{1}{1+n\,cos\varphi} & & & & & \\
=1 & -n\,cos.\varphi & +\tfrac{1}{2}nn\,cos.2\varphi & -\tfrac{1}{4}n^3cos.3\varphi & +\tfrac{1}{8}n^4cos.4\varphi & -\tfrac{1}{16}n^5cos.5\varphi + \text{etc.},\\
+\tfrac{1}{2}nn & -\tfrac{3}{4}n^3 & +\tfrac{4}{8}n^4 & -\tfrac{5}{16}n^5 & +\tfrac{6}{32}n^6 & -\tfrac{7}{64}n^7\\
+\tfrac{3}{8}n^4 & -\tfrac{10}{16}n^5 & +\tfrac{15}{32}n^6 & -\tfrac{21}{64}n^7 & +\tfrac{28}{128}n^8 & -\tfrac{36}{296}n^9\\
+\tfrac{10}{32}n^6 & -\tfrac{35}{64}n^7 & +\tfrac{56}{128}n^8 & -\tfrac{84}{256}n^9 & & \\
+\tfrac{35}{128}n^8 & & & & &
\end{array}$$

unde patet, si ponatur

$$\tfrac{1}{1+n\,cos\varphi} = A - B\,cos.\varphi + C\,cos.2\varphi - D\,cos.3\varphi + E\,cos.4\varphi - \text{etc.},$$

esse

$$A = 1 + \tfrac{1}{2}nn + \tfrac{3}{8}n^4 + \tfrac{10}{32}n^6 + \text{etc.},$$

seu

$$A = 1 + \tfrac{1}{2}n^2 + \tfrac{1\cdot3}{2\cdot4}n^4 + \tfrac{1\cdot3\cdot5}{2\cdot4\cdot6}n^6 + \tfrac{1\cdot3\cdot5\cdot7}{2\cdot4\cdot6\cdot8}n^8 + \text{etc.},$$

sicque evidens est esse

$$A = \tfrac{1}{\sqrt{(1-nn)}}.$$

Simili modo est

$$B = n + \frac{3}{4}n^3 + \frac{10}{16}n^5 + \text{etc.} = \frac{2}{n}\left(\frac{1}{2}n^2 + \frac{1\cdot 3}{2\cdot 4}n^4 + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}n^6 + \text{etc.}\right)$$

ideoque

$$B = \frac{2}{n}\left(\frac{1}{\sqrt{(1-nn)}} - 1\right).$$

Verum et hunc valorem et sequentes facilius hoc modo definire licet. Cum sit

$$\frac{1}{1+n\cos\varphi} = A - B\cos.\varphi + C\cos.2\varphi - D\cos.3\varphi + E\cos.4\varphi - \text{etc.},$$

multiplicetur per $1 + n\cos.\varphi$, et quia

$$\cos.\varphi\ \cos.\lambda\varphi = \frac{1}{2}\cos.(\lambda-1)\varphi + \frac{1}{2}\cos.(\lambda+1)\varphi,$$

fiet

$$\begin{array}{llllll} 1 = & A & - B\cos.\varphi + & C\cos.2\varphi - & D\cos.3\varphi + & E\cos.4\varphi - \text{etc.},\\ & +An & -\frac{1}{2}Bn & +\frac{1}{2}Bn & -\frac{1}{2}Dn & \\ & -\frac{1}{2}Bn & +\frac{1}{2}Cn & -\frac{1}{2}Dn & +\frac{1}{2}En & -\frac{1}{2}Fn \end{array}$$

unde, quia A iam definivimus, reliqui coefficientes ita determinantur

$$B = \frac{2}{n}(A-1), \quad C = \frac{2B-2An}{n}, \quad D = \frac{2C-Bn}{n},$$
$$E = \frac{2D-Cn}{n}, \quad F = \frac{2E-Dn}{n}, \quad G = \frac{2F-En}{n},$$
etc.

His igitur coefficientibus inventis integrale facile assignatur; nam cum sit

$$\int d\varphi\cos.\lambda\varphi = \frac{1}{\lambda}\sin.\lambda\varphi,$$

habebimus

$$\int\frac{d\varphi}{1+n\cos\varphi} = A\varphi - B\sin.\varphi + \frac{1}{2}C\sin.2\varphi - \frac{1}{3}D\sin.3\varphi + \frac{1}{4}E\sin.4\varphi - \text{etc.},$$

quae series secundum sinus angulorum $\varphi,\ 2\varphi,\ 3\varphi$ etc. progreditur, uti desiderabatur.

COROLLARIUM 1

273. Primo patet hanc resolutionem locum habere non posse, nisi n sit numerus unitate minor; si enim $n > 1$, singuli coefficientes prodeunt imaginarii. Sin autem sit $n = 1$, ob $1 + \cos.\varphi = 2\cos.^2\frac{1}{2}\varphi$ erit integrale

$$\int \frac{d\varphi}{1+\cos.\varphi} = \int \frac{\frac{1}{2}d\varphi}{\cos.^2 \frac{1}{2}\varphi} = \text{tang.}\frac{1}{2}\varphi.$$

COROLLARIUM 2

274. Cum sit

$$A = \frac{1}{\sqrt{(1-nn)}} \quad \text{et} \quad B = \frac{2}{n}\left(\frac{1}{\sqrt{(1-nn)}} - 1\right),$$

reliqui coefficientes *C*, *D*, *E* etc. seriem recurrentem constituunt, ita ut, si bini contigui sint *P* et *Q*, sequens futurus sit $\frac{2}{n}Q - P$. Hinc, cum aequationis $zz = \frac{2}{n}z - 1$ radices sint $\frac{1\pm\sqrt{(1-nn)}}{n}$, quisque terminus in hac forma continetur

$$\alpha\left(\frac{1+\sqrt{(1-nn)}}{n}\right)^{\lambda} + \beta\left(\frac{1-\sqrt{(1-nn)}}{n}\right)^{\lambda}.$$

COROLLARIUM 3

275. Quia autem in nostra lege non *A* sed *2A* sumitur, posito $\lambda = 0$ prodire debet *2A* ideoque

$$\alpha + \beta = \frac{2}{\sqrt{(1-nn)}};$$

deinde facto $\lambda = 1$ fieri debet

$$\frac{\alpha+\beta}{n} + \frac{(\alpha-\beta)\sqrt{(1-nn)}}{n} = \frac{2-2\sqrt{(1-nn)}}{n\sqrt{(1-nn)}},$$

unde

$$\alpha - \beta = -\frac{2}{\sqrt{(1-nn)}}.$$

Ergo

$$\alpha = 0 \quad \text{et} \quad \beta = \frac{2}{\sqrt{(1-nn)}}$$

sicque quilibet terminus praeter *A* erit

$$= \frac{2}{\sqrt{(1-nn)}}\left(\frac{1-\sqrt{(1-nn)}}{n}\right)^{\lambda}$$

COROLLARIUM 4

276. Coefficientes ergo evoluti ita se habebunt

$$A=\frac{1}{\sqrt{(1-nn)}},$$

$$B=\frac{2-2\sqrt{(1-nn)}}{n\sqrt{(1-nn)}},$$

$$C=\frac{4-2nn-4\sqrt{(1-nn)}}{nn\sqrt{(1-nn)}},$$

$$D=\frac{8-6nn-2(4-nn)\sqrt{(1-nn)}}{n^3\sqrt{(1-nn)}},$$

$$E=\frac{16-16nn+2n^4-2(8-4nn)\sqrt{(1-nn)}}{n^4\sqrt{(1-nn)}},$$

$$F=\frac{32-40nn+10n^4-2\left(16-12nn+n^4\right)\sqrt{(1-nn)}}{n^5\sqrt{(1-nn)}},$$

$$G=\frac{64-96nn+36n^4-2n^6-2\left(32-32nn+6n^4\right)\sqrt{(1-nn)}}{n^6\sqrt{(1-nn)}}$$

etc.

COROLLARIUM 5

277. Quia $n<1$, hi coefficientes plerumque facilius determinantur per series primum inventas, scilicet

$$A=1+\tfrac{1}{2}n^2+\tfrac{1\cdot3}{2\cdot4}n^4+\tfrac{1\cdot3\cdot5}{2\cdot4\cdot6}n^6+\tfrac{1\cdot3\cdot5\cdot7}{2\cdot4\cdot6\cdot8}n^8+\text{etc.},$$

$$B=n\left(1+\tfrac{3}{4}n^2+\tfrac{3\cdot5}{4\cdot6}n^4+\tfrac{3\cdot5\cdot7}{4\cdot6\cdot8}n^6+\tfrac{3\cdot5\cdot7\cdot9}{4\cdot6\cdot8\cdot10}n^8+\text{etc.}\right),$$

$$C=\tfrac{1}{2}n^2\left(1+\tfrac{3\cdot4}{2\cdot6}n^2+\tfrac{3\cdot4\cdot5\cdot6}{2\cdot6\cdot4\cdot8}n^4+\tfrac{3\cdot4\cdot5\cdot6\cdot7\cdot8}{2\cdot6\cdot4\cdot8\cdot6\cdot10}n^6+\text{etc.}\right),$$

$$D=\tfrac{1}{4}n^3\left(1+\tfrac{4\cdot5}{2\cdot8}n^2+\tfrac{4\cdot5\cdot6\cdot7}{2\cdot8\cdot4\cdot10}n^4+\tfrac{4\cdot5\cdot6\cdot7\cdot8\cdot9}{2\cdot8\cdot4\cdot10\cdot6\cdot12}n^6+\text{etc.}\right),$$

$$E=\tfrac{1}{8}n^4\left(1+\tfrac{5\cdot6}{2\cdot10}n^2+\tfrac{5\cdot6\cdot7\cdot8}{2\cdot10\cdot4\cdot12}n^4+\tfrac{5\cdot6\cdot7\cdot8\cdot9\cdot10}{2\cdot10\cdot4\cdot12\cdot6\cdot14}n^6+\text{etc.}\right),$$

$$F=\tfrac{1}{16}n^5\left(1+\tfrac{6\cdot7}{2\cdot12}n^2+\tfrac{6\cdot7\cdot8\cdot9}{2\cdot12\cdot4\cdot14}n^4+\tfrac{6\cdot7\cdot8\cdot9\cdot10\cdot11}{2\cdot12\cdot4\cdot14\cdot6\cdot16}n^6+\text{etc.}\right)$$

etc.

SCHOLION

278. Cum ex his valoribus sit

$$\int \frac{d\varphi}{1+n\cos\varphi} = A\varphi - B\sin.\varphi + \tfrac{1}{2}C\sin.2\varphi - \tfrac{1}{3}D\sin.3\varphi + \tfrac{1}{4}E\sin.4\varphi - \text{etc.},$$

in hac serie terminus primus $A\varphi$ imprimis est notandus, quod crescente angulo φ continuo crescat idque in infinitum usque, dum reliqui termini modo crescent modo decrescent; neque tamen certum limitem excedunt, nam $\sin\lambda\varphi$ neque supra +1 crescere neque infra –1 decrescere potest. Cum deinde hoc integrale supra inventum sit

$$\frac{1}{\sqrt{(1-nn)}}\text{Ang.cos.}\frac{n+\cos.\varphi}{1+n\cos.\varphi},$$

series illa huic angulo aequatur. Quare si hic angulus vocetur ω, ut sit

$$d\omega = \frac{d\varphi\sqrt{(1-nn)}}{1+n\cos.\varphi},$$

erit

$$\cos.\omega = \frac{n+\cos.\varphi}{1+n\cos.\varphi}$$

hincque $n+\cos.\varphi-\cos.\omega-n\cos.\varphi\cos.\omega = 0$, ex quo est vicissim

$$\cos.\varphi = \frac{\cos.\omega-n}{1-n\cos.\omega};$$

quae formula cum ex illa nascatur sumto *n* negativo, erit

$$d\varphi = \frac{d\omega\sqrt{(1-nn)}}{1-n\cos.\omega}$$

et

$$\frac{\varphi}{\sqrt{(1-nn)}} = A\omega + B\sin.\omega + \tfrac{1}{2}C\sin.2\omega + \tfrac{1}{3}D\sin.3\omega + \tfrac{1}{4}E\sin.4\omega + \text{etc.},$$

Quia vero est

$$\frac{\omega}{\sqrt{(1-nn)}} = A\varphi - B\sin.\varphi + \tfrac{1}{2}C\sin.2\varphi - \tfrac{1}{3}D\sin.3\varphi + \tfrac{1}{4}E\sin.4\varphi - \text{etc.},$$

ob $\frac{1}{\sqrt{(1-nn)}} = A$ habebimus

$$0 = B\left(\sin.\omega - \sin.\varphi\right) + \tfrac{1}{2}C\left(\sin.2\omega + \sin.2\varphi\right) + \tfrac{1}{3}D\left(\sin.3\omega - \sin.3\varphi\right) + \text{etc.},$$

cuiusmodi relationes notasse iuvabit.

PROBLEMA 33

279. *Integrale formulae* $d\varphi(1+n\cos.\varphi)^{\nu}$ *per seriem secundum sinus angulorum multiplorum ipsius* φ *progredientem exprimere.*

SOLUTIO

Cum sit

$$(1+n\cos.\varphi)^{\nu} = 1+\tfrac{\nu}{1}n\cos.\varphi+\tfrac{\nu(\nu-1)}{1\cdot 2}n^2\cos.^2\varphi+\tfrac{(\nu-1)(\nu-2)}{1\cdot 2\cdot 3}n^3\cos.^3\varphi+\text{etc.},$$

si ponamus

$$(1+n\cos.\varphi)^{\nu} = A+B\cos.\varphi+C\cos.2\varphi+D\cos.3\varphi+E\cos.4\varphi+\text{etc.},$$

erit per formulas supra indicatas

$$A = 1+\tfrac{\nu(\nu-1)}{1\cdot 2}\cdot\tfrac{1}{2}n^2+\tfrac{\nu(\nu-1)(\nu-2)(\nu-3)}{1\cdot 2\cdot 3\cdot 4}\cdot\tfrac{1\cdot 3}{2\cdot 4}n^4 + \tfrac{\nu(\nu-1)\cdots(\nu-5)}{1\cdot 2\cdot 3\cdots 6}\cdot\tfrac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}n^6+\text{etc.},$$

$$B = 2n\left(\tfrac{\nu}{2}+\tfrac{\nu(\nu-1)(\nu-2)}{1\cdot 2\cdot 3}\cdot\tfrac{1\cdot 3}{2\cdot 4}n^2+\tfrac{\nu(\nu-1)(\nu-2)(\nu-3)(\nu-4)}{1\cdot 2\cdot 3\cdot 4\cdot 5}\cdot\tfrac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}n^4+\text{etc.}\right),$$

quae series ita clarius exhibentur

$$A = 1+\tfrac{\nu(\nu-1)}{2\cdot 2}n^2+\tfrac{\nu(\nu-1)(\nu-2)(\nu-3)}{2\cdot 2\cdot 4\cdot 4}n^4+\tfrac{\nu(\nu-1)\cdots(\nu-5)}{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6}n^6+\text{etc.},$$
$$\tfrac{1}{2}B = \tfrac{\nu}{2}n+\tfrac{\nu(\nu-1)(\nu-2)}{2\cdot 2\cdot 4}n^3+\tfrac{\nu(\nu-1)(\nu-2)(\nu-3)(\nu-4)}{2\cdot 2\cdot 3\cdot 4\cdot 6}n^5+\text{etc.}$$

Inventis autem bis binis coefficientibus A et B reliqui ex bis sequenti modo commodius determinari poterunt. Cum sit

$$\nu l(1+n\cos.\varphi) = l(A+B\cos\varphi+C\cos.2\varphi+D\cos.3\varphi+E\cos.4\varphi+\text{etc.}),$$

sumantur differentialia ac per $-d\varphi$ dividendo prodit

$$\frac{\nu n\cos.\varphi}{1+n\cos.\varphi} = \frac{B\sin.\varphi+2C\sin.2\varphi+3D\sin.3\varphi+4E\sin.4\varphi+\text{etc}}{A+B\cos.\varphi+C\cos.2\varphi+D\cos.3\varphi+E\cos.4\varphi+\text{etc.}}.$$

Iam per crucem multiplicando ob

$$\sin.\lambda\varphi\cos.\varphi = \tfrac{1}{2}\sin.(\lambda+1)\varphi+\tfrac{1}{2}\sin.(\lambda-1)\varphi$$

et

$$sin.\varphi cos.\lambda\varphi = \tfrac{1}{2} sin.(\lambda+1)\varphi - \tfrac{1}{2} sin.(\lambda-1)\varphi$$

pervenietur ad hanc aequationem

$$\begin{array}{llllll} 0 = & B\,sin.\varphi & +2C\,sin.2\varphi & +3D\,sin.3\varphi & +4E\,sin.4\varphi & +5F\,sin.5\varphi + \text{etc.}, \\ & +\tfrac{1}{2}Bn & +\tfrac{2}{2}Cn & +\tfrac{3}{2}Dn & +\tfrac{4}{2}En & \\ +\tfrac{2}{2}Cn & +\tfrac{3}{2}Dn & +\tfrac{4}{2}En & +\tfrac{5}{2}Fn & +\tfrac{6}{2}Gn & \\ -vAn & -\tfrac{v}{2}Bn & -\tfrac{v}{2}Cn & -\tfrac{v}{2}Dn & -\tfrac{v}{2}En & \\ +\tfrac{v}{2}Cn & +\tfrac{v}{2}Dn & +\tfrac{v}{2}En & +\tfrac{v}{2}Fn & +\tfrac{v}{2}Gn & \end{array}$$

unde hae sequuntur determinationes

$$\begin{array}{l|l} (v+2)Cn + 2B - 2vAn = 0 & C = \frac{2vAn-2B}{(v+2)n} \\ (v+3)Dn + 4C - (v-1)Bn = 0 & D = \frac{(v-1)Bn-4C}{(v+3)n} \\ (v+4)En + 6D - (v-2)Cn = 0 & E = \frac{(v-2)Cn-6D}{(v+4)n} \\ (v+5)Fn + 8E - (v-3)Dn = 0 & F = \frac{(v-3)Dn-8E}{(v+5)n} \\ (v+6)Gn + 10F - (v-4)En = 0 & G = \frac{(v-4)En-10F}{(v+6)n} \end{array}$$

ubi si superiores valores pro A et *B* substituantur, reperitur

$$C = 4n^2\left(\frac{1v(v-1)}{2\cdot2\cdot4} + \frac{2v(v-1)(v-2)(v-3)}{2\cdot2\cdot4\cdot4\cdot6}n^2 + \frac{3v(v-1)\cdots(v-5)}{2\cdot2\cdot4\cdot4\cdot6\cdot6\cdot8}n^4 + \text{etc.}\right),$$
$$D = 8n^3\left(\frac{1\cdot2v(v-1)(v-2)}{2\cdot2\cdot4\cdot4\cdot6} + \frac{2\cdot3v(v-1)(v-2)(v-3)(v-4)}{2\cdot2\cdot4\cdot4\cdot6\cdot6\cdot8}n^2 + \text{etc.}\right),$$
$$E = 16n^4\left(\frac{1\cdot2\cdot3v(v-1)(v-2)(v-3)}{2\cdot2\cdot4\cdot4\cdot6\cdot6\cdot8} + \frac{2\cdot3\cdot4v(v-1)\cdots(v-5)}{2\cdot2\cdot4\cdot4\cdot6\cdot6\cdot8\cdot8\cdot10}n^2 + \text{etc.}\right)$$
$$\text{etc.},$$

unde forma sequentium serierum colligitur. His autem inventis coefficientibus erit integrale quaesitum

$$\int d\varphi(1+ncos.\varphi)^v = A\varphi + Bsin.\varphi + \tfrac{1}{2}C\,sin.2\varphi + \tfrac{1}{3}D\,sin.3\varphi + \tfrac{1}{4}E\,sin.4\varphi + \text{etc.}$$

COROLLARIUM 1

280. Ad similitudinem harum serierum pro *C, D, E* etc. datarum etiam valor ipsius *B* ita exprimi potest

$$B = 2n\left(\frac{v}{2} + \frac{v(v-1)(v-2)}{2\cdot 2\cdot 4}n^2 + \frac{(v-1)(v-2)(v-3)(v-4)}{2\cdot 2\cdot 4\cdot 4\cdot 6}n^4 + etc.\right);$$

series autem pro .*A* inventa formam habet singularem in hac lege non comprehensam.

COROLLARIUM 2

281. Si series *A* et *B* inter se comparemus, varias relationes inter eas observare licet, quarum haec primo se offert

$$An + \tfrac{1}{2}B = \frac{v+2}{2}n\left(\begin{array}{l}1 + \frac{v(v-1)}{2\cdot 4}n^2 + \frac{v(v-1)(v-2)(v-3)}{2\cdot 4\cdot 4\cdot 6}n^4 \\ + \frac{v(v-1)\cdots(v-5)}{2\cdot 4\cdot 4\cdot 6\cdot 6\cdot 8}n^6 + \text{etc.}\end{array}\right)$$

quae a serie *A* tantum secundum denominatores differt.

COROLLARIUM 3

282. Ponamus $\frac{2Ann+Bn}{v+2} = N$, ut sit

$$N = n^2 + \frac{v(v-1)}{2\cdot 4}n^4 + \frac{v(v-1)(v-2)(v-3)}{2\cdot 4\cdot 4\cdot 6}n^6 + \text{etc.},$$
$$A = 1 + \quad \frac{v(v-1)}{2\cdot 2}n^2 + \frac{v(v-1)(v-2)(v-3)}{2\cdot 2\cdot 4\cdot 4}n^4 + \text{etc.}$$

Quodsi iam *n* ut variabilis tractetur, differentiatio praebet

$$\frac{dN}{ndn} = 2 + \frac{v(v-1)}{2}n^2 + \frac{v(v-1)(v-2)(v-3)}{2\cdot 4\cdot 4}n^4 + \text{etc.} = 2A.$$

Cum igitur sit

$$dN = \frac{4Andn + Bdn + 2nndA + ndB}{v+2} = 2Andn,$$

erit

$$2vAndn = 2nndA + Bdn + ndB.$$

COROLLARIUM 4

283. Ex dato ergo coefficiente *A* coefficiens *B* ita per integrationem inveniri potest, ut sit

$$Bn = 2\int(vAndn - nndA),$$

vel erit etiam ex illa forma

$$B=\frac{2(v+2)}{n}\int Andn-2An,$$

ubi notandum est posito $n = 0$ integrale $\int Andn$ evanescere debere, quia hoc casu B evanescit.

SCHOLION

284. Series pro litteris B, C, D etc. inventas etiam sequenti modo per continuos factores exprimere licet

$$B=vn\left(1+\frac{(v-1)(v-2)}{2\cdot 4}n^2+\frac{(v-3)(v-4)}{4\cdot 6}Pn^2+\frac{(v-5)(v-6)}{6\cdot 8}Pn^2+\text{etc.}\right),$$

$$C=\frac{v(v-1)}{1\cdot 2}\cdot\frac{n^2}{2}\left(1+\frac{(v-2)(v-3)}{2\cdot 6}n^2+\frac{(v-4)(v-5)}{4\cdot 8}Pn^2+\frac{(v-6)(v-7)}{6\cdot 10}Pn^2+\text{etc.}\right),$$

$$D=\frac{v\cdots(v-2)}{1\cdots 3}\cdot\frac{n^3}{4}\left(1+\frac{(v-3)(v-4)}{2\cdot 8}n^2+\frac{(v-5)(v-6)}{4\cdot 10}Pn^2+\frac{(v-7)(v-8)}{6\cdot 12}Pn^2+\text{etc.}\right),$$

$$E=\frac{v\cdots(v-3)}{1\cdots 4}\cdot\frac{n^4}{8}\left(1+\frac{(v-4)(v-5)}{2\cdot 10}n^2+\frac{(v-6)(v-7)}{4\cdot 12}Pn^2+\frac{(v-8)(v-9)}{6\cdot 14}Pn^2+\text{etc.}\right),$$

$$F=\frac{v\cdots(v-4)}{1\cdots 5}\cdot\frac{n^5}{16}\left(1+\frac{(v-5)(v-6)}{2\cdot 12}n^2+\frac{(v-7)(v-8)}{4\cdot 14}Pn^2+\frac{(v-9)(v-10)}{6\cdot 16}Pn^2+\text{etc.}\right),$$

etc.

ubi in qualibet serie littera P terminum praecedentem integrum denotat. Atque ope serierum istarum coefficientes plerumque facilius inveniuntur quam ex lege ante tradita, qua quisque ex binis praecedentibus determinatur. Quin etiam haec lex defectu laborat, quod, si v fuerit numerus integer negativus praeter –1, quidam coefficientes plane non definiantur, quos ergo ex his seriebus desumi oportet. Ita si fuerit $v=-2$, erit et $B=vAn=-2An$ et

$$C=\frac{3}{1}\cdot\frac{n^2}{2}\left(1+\frac{4\cdot 5}{2\cdot 6}n^2+\frac{4\cdot 5\cdot 6\cdot 7}{2\cdot 6\cdot 4\cdot 8}n^4+\frac{4\cdot 5\cdot 6\cdot 7\cdot 8\cdot 9}{2\cdot 6\cdot 4\cdot 8\cdot 6\cdot 10}n^6+\text{etc.}\right);$$

si sit $v=-3$, erit $C=-Bn$ et

$$D=\frac{4\cdot 5}{1\cdot 2}\cdot\frac{n^3}{4}\left(1+\frac{6\cdot 7}{2\cdot 8}n^2+\frac{6\cdot 7\cdot 8\cdot 9}{2\cdot 8\cdot 4\cdot 10}n^4+\frac{6\cdot 7\cdot 8\cdot 9\cdot 10\cdot 11}{2\cdot 8\cdot 4\cdot 10\cdot 6\cdot 12}n^4+\text{etc.}\right);$$

si sit $v=-4$, erit $D=-Cn$ et

$$E=\frac{5\cdot 6\cdot 7}{1\cdot 2\cdot 3}\cdot\frac{n^4}{8}\left(1+\frac{8\cdot 9}{2\cdot 10}n^2+\frac{8\cdot 9\cdot 10\cdot 11}{2\cdot 10\cdot 4\cdot 12}n^4+\frac{8\cdot 9\cdot 10\cdot 11\cdot 12\cdot 13}{2\cdot 10\cdot 4\cdot 12\cdot 6\cdot 14}n^6+\text{etc.}\right);$$

si sit $v=-5$, erit $E=-Dn$ et

$$F=\tfrac{6\cdot7\cdot8\cdot9}{1\cdot2\cdot3\cdot4}\cdot\frac{n^5}{16}\left(1+\tfrac{10\cdot11}{2\cdot12}n^2+\tfrac{10\cdot11\cdot12\cdot13}{2\cdot12\cdot4\cdot14}n^4+\tfrac{10\cdot11\cdot12\cdot13\cdot14\cdot15}{2\cdot12\cdot4\cdot14\cdot6\cdot16}n^6+\text{etc.}\right)$$

et ita de reliquis.

EXEMPLUM 1

285. *Formulae* $d\varphi(1+n\cos.\varphi)^v$ *integrale evolvere, si v sit numerus integer positivus.*

Posito

$$(1+n\cos.\varphi)^v=A+B\cos.\varphi+C\cos.2\varphi+D\cos.3\varphi+E\cos.4\varphi+\text{etc.}$$

pro singulis valoribus exponentis v habebimus,

1) si $v=1$: $A=1,\quad B=n,\quad C=0\quad$ etc.;

2) si $v=2$: $A=1+\tfrac{1}{2}n^2,\quad B=2n,\quad C=\tfrac{1}{2}n^2,\quad D=0\quad$ etc.;

3) si $v=3$: $A=1+\tfrac{3}{2}n^2,\quad B=3n\left(1+\tfrac{1}{4}n^2\right),\quad C=\tfrac{3}{2}n^2,\quad D=\tfrac{1}{4}n^2,$
$E=0\quad$ etc.;

4) si $v=4$: $A=1+\tfrac{6}{2}n^2+\tfrac{3}{8}n^4,\quad B=4n\left(1+\tfrac{3}{4}n^2\right),\quad C=3n^2\left(1+\tfrac{1}{6}n^2\right),$
$D=n^3,\quad E=\tfrac{1}{8}n^4,\quad F=0\quad$ etc.;

Hi autem casus nihil habent difficultatis. Ad usum sequentem tantum iuvabit primum terminum absolutum A notasse:

si $v=1$: $A=1,$

si $v=2$: $A=1+\tfrac{2\cdot1}{2\cdot2}n^2,$

si $v=3$: $A=1+\tfrac{3\cdot2}{2\cdot2}n^2,$

si $v=4$: $A=1+\tfrac{4\cdot3}{2\cdot2}n^2+\tfrac{4\cdot3\cdot2\cdot1}{2\cdot2\cdot4\cdot4}n^4,$

si $v=5$: $A=1+\tfrac{5\cdot4}{2\cdot2}n^2+\tfrac{5\cdot4\cdot3\cdot2}{2\cdot2\cdot4\cdot4}n^4,$

si $v=6$: $A=1+\tfrac{6\cdot5}{2\cdot2}n^2+\tfrac{6\cdot5\cdot4\cdot3}{2\cdot2\cdot4\cdot4}n^4+\tfrac{6\cdot5\cdot4\cdot3\cdot2\cdot1}{2\cdot2\cdot4\cdot4\cdot6\cdot6}n^6,$

si $v=7$: $A=1+\tfrac{7\cdot6}{2\cdot2}n^2+\tfrac{7\cdot6\cdot5\cdot4}{2\cdot2\cdot4\cdot4}n^4+\tfrac{7\cdot6\cdot5\cdot4\cdot3\cdot2}{2\cdot2\cdot4\cdot4\cdot6\cdot6}n^6$

etc.

EXEMPLUM 2

286. *Formulae* $\frac{d\varphi}{(1+n\cos.\varphi)^{\mu}}$ *integrale per seriem evolvere.*

Posito

$$\frac{d\varphi}{(1+n\cos.\varphi)^{\mu}} = A + B\cos.\varphi + C\cos.2\varphi + D\cos.3\varphi + E\cos.4\varphi + \text{etc.}$$

ex praecedentibus formulis ponendo $v = -\mu$ erit

$$A = 1 + \frac{\mu(\mu+1)}{2\cdot 2}n^2 + \frac{\mu(\mu+1)(\mu+2)(\mu+3)}{2\cdot 2\cdot 4\cdot 4}n^4 + \frac{\mu(\mu+1)\cdots(\mu+5)}{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6}n^6 + \text{etc.}$$

$$B = -\mu n\left(1 + \frac{(\mu+1)(\mu+2)}{2\cdot 4}n^2 + \frac{(\mu+3)(\mu+4)}{4\cdot 6}Pn^2 + \frac{(\mu+5)(\mu+6)}{6\cdot 8}Pn^2 + \text{etc.}\right),$$

$$C = \frac{\mu(\mu+1)}{1\cdot 2}\cdot\frac{n^2}{2}\left(1 + \frac{(\mu+2)(\mu+3)}{2\cdot 6}n^2 + \frac{(\mu+4)(\mu+5)}{4\cdot 8}Pn^2 + \frac{(\mu+6)(\mu+7)}{6\cdot 10}Pn^2 + \text{etc.}\right),$$

$$D = -\frac{\mu(\mu+1)(\mu+2)}{1\cdot 2\cdot 3}\cdot\frac{n^3}{4}\left(1 + \frac{(\mu+3)(\mu+4)}{2\cdot 8}n^2 + \frac{(\mu+5)(\mu+6)}{4\cdot 10}Pn^2 + \text{etc.}\right),$$

etc.

ubi ut ante in quaque serie P terminum praecedentem denotat. Hi autem coefficientes ita a se invicem pendent, ut sit

$$B = \frac{-2(\mu-2)}{n}\int Andn - 2An$$

et

$$C = \frac{2B+2\mu An}{(\mu-2)n}, \quad D = \frac{4C+(\mu+1)Bn}{(\mu-3)n}, \quad E = \frac{6D+(\mu+2)Cn}{(\mu-4)n},$$

$$F = \frac{8E+(\mu+3)Dn}{(\mu-5)n}, \quad G = \frac{10F+(\mu+4)En}{(\mu-6)n}, \quad H = \frac{12G+(\mu+5)Fn}{(\mu-7)n}$$

Ubi incommodo, quando μ est numerus integer, supra iam remedium est allatum. Hic igitur praecipue investigamus, quomodo coefficientes cuiusque casus ex casu praecedente determinari queant, quod ita fieri poterit. Cum sit

$$\frac{d\varphi}{(1+n\cos.\varphi)^{\mu}} = A + B\cos.\varphi + C\cos.2\varphi + D\cos.3\varphi + E\cos.4\varphi + \text{etc.}$$

ponatur

$$\frac{d\varphi}{(1+n\cos.\varphi)^{\mu+1}} = A' + B'\cos.\varphi + C'\cos.2\varphi + D'\cos.3\varphi + \text{etc.}$$

haec igitur series per $1 + n\cos.\varphi$ multiplicata in illam abire debet; est autem productum

$$\begin{array}{llll} A' \;+ & B'cos.\varphi + & C'cos.2\varphi + & D'cos.3\varphi + \text{etc.,} \\ & +A'n & +\tfrac{1}{2}B'n & +\tfrac{1}{2}C'n \\ +\tfrac{1}{2}B'n & +\tfrac{1}{2}C'n & +\tfrac{1}{2}D'n & +\tfrac{1}{2}E'n \end{array}$$

unde colligimus

$$B' = 2\frac{(A-A')}{n}, \qquad C' = \frac{2(B-B')-2A'n}{n},$$
$$D' = \frac{2(C-C')-B'n}{n}, \quad E' = \frac{2(D-D')-C'n}{n} \quad \text{etc.;}$$

dummodo ergo coefficiens A' constaret, sequentes B', C', D' etc. haberemus. Videamus igitur, quomodo A' ex A determinari possit; quia est

$$A = 1 + \frac{\mu(\mu+1)}{2\cdot 2}n^2 + \frac{\mu(\mu+1)(\mu+2)(\mu+3)}{2\cdot 2\cdot 4\cdot 4}n^4 + \text{etc.,}$$
$$A' = 1 + \frac{(\mu+1)(\mu+2)}{2\cdot 2}n^2 + \frac{(\mu+1)(\mu+2)(\mu+3)(\mu+4)}{2\cdot 2\cdot 4\cdot 4}n^4 + \text{etc.,}$$

tractetur n ut variabilis ac prior series per n^μ multiplicata differentietur, ut prodeat

$$\frac{d.An^\mu}{d.n} = \mu n^{\mu-1} + \frac{\mu(\mu+1)(\mu+2)}{2\cdot 2}n^{\mu+1} + \frac{\mu(\mu+1)(\mu+2)(\mu+3)(\mu+4)}{2\cdot 2\cdot 4\cdot 4}n^{\mu+3} + \text{etc.,}$$

quae series manifesto est $\mu n^{\mu-1}A'$; quocirca A' ita per A determinatur, ut sit

$$A' = \frac{d.An^\mu}{d.n^\mu} = A + \frac{ndA}{\mu dn}.$$

Cum igitur pro casu $\mu = 1$ invenerimus $A = \frac{1}{\sqrt{(1-nn)}}$, ob $\frac{dA}{dn} = \frac{n}{(1-nn)^{\frac{3}{2}}}$ erit

$$A' = \frac{1}{\sqrt{(1-nn)}} + \frac{nn}{(1-nn)^{\frac{3}{2}}} = \frac{1}{(1-nn)^{\frac{3}{2}}}$$

Hic iam est valor ipsius A pro $\mu = 2$, unde ob $\frac{dA}{dn} = \frac{3n}{(1-nn)^{\frac{5}{2}}}$ fiet pro $\mu = 3$

$$A = \frac{1}{(1-nn)^{\frac{3}{2}}} + \frac{3nn}{2(1-nn)^{\frac{5}{2}}} = \frac{1+\frac{1}{2}nn}{(1-nn)^{\frac{5}{2}}}.$$

Hoc modo si ulterius progrediamur, reperiemus,

$$\text{si}\quad \mu=2,\quad A=\frac{1}{(1-nn)\sqrt{(1-nn)}},$$

$$\text{si}\quad \mu=3,\quad A=\frac{1+\frac{1}{2}nn}{(1-nn)^2\sqrt{(1-nn)}},$$

$$\text{si}\quad \mu=4,\quad A=\frac{1+\frac{3}{2}nn}{(1-nn)^3\sqrt{(1-nn)}},$$

$$\text{si}\quad \mu=5,\quad A=\frac{1+3nn+\frac{3}{8}n^4}{(1-nn)^4\sqrt{(1-nn)}}.$$

COROLLARIUM 1

287. Eodem modo etiam reliqui coefficientes B', C' etc. ex analogis B, C etc. definientur eruntque omnes istae relationes inter se similes, scilicet uti est

$$A'=\frac{d.An^{\mu}}{d.n^{\mu}}=A+\frac{ndA}{\mu dn},$$

ita erit

$$B'=\frac{d.Bn^{\mu}}{d.n^{\mu}}=B+\frac{ndB}{\mu dn},\quad C'=\frac{d.Cn^{\mu}}{d.n^{\mu}}=C+\frac{ndC}{\mu dn}\quad \text{etc.}$$

COROLLARIUM 2

288. At ante invenimus $B'=\frac{2(A-A')}{n}$, unde fiet

$$B'=\frac{2dA}{\mu dn}=B+\frac{ndB}{\mu dn}$$

hincque $\mu Bdn+ndB+2dA=0;$ multiplicetur per $n^{\mu-1}$, ut sit

$$d.Bn^{\mu}+2n^{\mu-1}dA=0,$$

unde integrando

$$Bn^{\mu}=-2\int n^{\mu-1}dA=-2n^{\mu-1}A+2(\mu-1)\int An^{\mu-2}dn$$

ideoque

$$B=-\frac{2A}{n}+\frac{2(\mu-1)}{n^{\mu}}\int An^{\mu-2}dn$$

At ante [§ 286] habueramus

$$B=-2An+\frac{2(\mu-2)}{n}\int Andn.$$

COROLLARIUM 3

289. His valoribus aequatis obtinetur aequatio inter A et n, qua quantitas A per n determinatur; erit enim

$$n^{-\mu}\int n^{\mu-1}dA = An + \frac{(\mu-2)}{n}\int Andn,$$

unde per duplicem differentiationem prodit

$$(1-nn)ddA + \frac{dndA}{n} - 2(\mu+1)ndndA - \mu(\mu+1)Adn^2 = 0.$$

SCHOLION 1

290. Si hos valores ipsius A cum superioribus, ubi μ erat numerus integer negativus, inter se comparemus, eximiam convenientiam deprehendemus:

Pro superioribus	Pro his formulis
si $v = 0$, $A=1$	si $\mu=1$, $A=\frac{1}{\sqrt{(1-nn)}}$
$v = 1$, $A=1$	$\mu=2$, $A=\frac{1}{(1-nn)\sqrt{(1-nn)}}$
$v = 2$, $A=1+\frac{1}{2}nn$	$\mu=3$, $A=\frac{1+\frac{1}{2}nn}{(1-nn)^2\sqrt{(1-nn)}}$
$v = 3$, $A=1+\frac{3}{2}nn$	$\mu=4$, $A=\frac{1+\frac{3}{2}nn}{(1-nn)^3\sqrt{(1-nn)}}$
$v = 4$, $A=1+3nn+\frac{3}{8}nn$	$\mu=5$, $A=\frac{1+3nn+\frac{3}{8}n^4}{(1-nn)^4\sqrt{(1-nn)}}$

etc.,

unde concludimus, si fuerit

$$(1+n\cos.\varphi)^{v} = A + B\cos.\varphi + C\cos.2\varphi + \text{etc.},$$
$$(1+n\cos.\varphi)^{-v-1} = \mathfrak{A} + \mathfrak{B}\cos.\varphi + \mathfrak{C}\cos.2\varphi + \text{etc.},$$

fore

$$\mathfrak{A} = \frac{A}{(1-nn)^{v}\sqrt{(1-nn)}}.$$

Quare cum pro casibus, quibus v est numerus integer positivus, valor ipsius A facile definiatur, etiam pro casibus, quibus est negativus, inde expedite assignabitur.

SCHOLION 2

291. Cum pro casu $\mu=1$ supra valores singularum litterarum *A, B, C, D* etc. sint inventi, scilicet posito brevitatis gratia $\frac{1-\sqrt{(1-nn)}}{n}=m$

$$A=\frac{1}{\sqrt{(1-nn)}},\quad B=\frac{2m}{\sqrt{(1-nn)}},\quad C=\frac{2mm}{\sqrt{(1-nn)}},\quad D=\frac{2m^3}{\sqrt{(1-nn)}}$$

et in genere pro termino quocunque

$$N=\frac{2m^\lambda}{\sqrt{(1-nn)}},$$

si pro simili termino casu $\mu=2$ scribamus N', erit $N'=\frac{d.Nn}{dn}$. Nunc autem est

$$\frac{d.Nn}{dn}=\frac{2m^\lambda}{(1-nn)^{\frac{3}{2}}}+\frac{2\lambda nm^{\lambda-1}dm}{dn\sqrt{(1-nn)}},$$

tum vero $\frac{dm}{dn}=\frac{m}{n\sqrt{(1-nn)}}$, unde colligimus

$$N'=\frac{2m^\lambda}{(1-nn)^{\frac{3}{2}}}+\frac{2\lambda m^\lambda}{1-nn}=\frac{2m^\lambda\left(1+\lambda\sqrt{(1-nn)}\right)}{(1-nn)\sqrt{(1-nn)}}.$$

Quare si statuamus

$$\frac{1}{(1+n\cos.\varphi)^2}=A+B\cos.\varphi+C\cos.2\varphi+D\cos.3\varphi+E\cos.4\varphi+\text{etc.},$$

erit

$$A=\frac{1}{(1-nn)^{\frac{3}{2}}},\quad B=\frac{2m\left(1+\sqrt{(1-nn)}\right)}{(1-nn)^{\frac{3}{2}}},\quad C=\frac{2m^2\left(1+2\sqrt{(1-nn)}\right)}{(1-nn)^{\frac{3}{2}}},$$

$$D=\frac{2m^3\left(1+3\sqrt{(1-nn)}\right)}{(1-nn)^{\frac{3}{2}}}\quad \text{etc.}$$

Verum si exponens μ fuerit numerus fractus, coefficientes *A, B, C, D, E* etc. haud aliter ac per series supra datas definiri posse videntur. Primus autem *A* modo peculiari vero proxime assignari potest, quemadmodum in problemate sequente docemus.

PROBLEMA 34

292. *Pro evolutione formulae* $(1+n\cos.\varphi)^{\nu}$ *in huiusmodi seriem*

$$A+B\cos.\varphi+C\cos.2\varphi+D\cos.3\varphi \;+E\cos.4\varphi+\text{etc.},$$

terminum absolutum A vero proxime definire.

SOLUTIO

Cum necessario sit *n* < 1, series quidem supra inventa pro *A* convergit, verum si *n* parum ab unitate deficiat, permultos terminos actu evolvi oportet, antequam valor ipsius *A* satis exacte prodeat, praecipue si *v* fuerit numerus mediocriter magnus tam positivus quam negativus. Quoniam tamen posita evolutione huius formulae

$$(1+n\cos.\varphi)^{-\nu-1}=\mathfrak{A}+\mathfrak{B}\cos.\varphi+\mathfrak{C}\cos.2\varphi+\text{etc.}$$

a termino $\mathfrak{A}$ ille *A* ita pendet, ut sit $A=(1-nn)^{\nu+\frac{1}{2}}\mathfrak{A}$, pro hoc termino *A* inveniendo duplicem habebimus seriem

$$A=1+\frac{\nu(\nu-1)}{2\cdot 2}n^2+\frac{\nu(\nu-1)(\nu-2)(\nu-3)}{2\cdot 2\cdot 4\cdot 4}n^4+\frac{\nu(\nu-1)(\nu-2)(\nu-3)(\nu-4)(\nu-5)}{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6}n^6+\text{etc.}$$

$$A=(1-nn)^{\nu+\frac{1}{2}}\left\{\begin{array}{l}1+\frac{\nu(\nu-1)}{2\cdot 2}n^2+\frac{\nu(\nu-1)(\nu-2)(\nu-3)}{2\cdot 2\cdot 4\cdot 4}n^4\\+\frac{\nu(\nu-1)(\nu-2)(\nu-3)(\nu-4)(\nu-5)}{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6}n^6+\text{etc.}\end{array}\right\};$$

quovis casu ea usurpari potest, quae magis convergit. Verum tamen quia reliqui coefficientes *B, C, D, E* etc. tandem convergere debent, hinc alia via ad valorem ipsius *A* appropinquandi patet.

Quoniam enim hi coefficientes alternatim per pares et impares potestates ipsius *n* definiuntur, sumto angulo quocunque α erit

$$(1+n\cos.\alpha)^{\nu}=A+B\cos.\alpha+C\cos.2\alpha+D\cos.3\alpha+E\cos.4\alpha+\text{etc.}$$

et

$$(1-n\cos.\alpha)^{\nu}=A-B\cos.\alpha+C\cos.2\alpha-D\cos.3\alpha+E\cos.4\alpha-\text{etc.}$$

His igitur additis prodit

$$\tfrac{1}{2}(1+n\cos.\alpha)^{\nu}+\tfrac{1}{2}(1-n\cos.\alpha)^{\nu}=A+C\cos.2\alpha+E\cos.4\alpha+G\cos.6\alpha+\text{etc.};$$

ubi si pro α scribamus $90^\circ-\alpha$, erit

$$\tfrac{1}{2}(1+n\cos.\alpha)^{\nu}+\tfrac{1}{2}(1-n\sin.\alpha)^{\nu}=A-C\cos.2\alpha+E\cos.4\alpha-G\cos.6\alpha+\text{etc.},$$

unde his additis semissis terminorum denuo tollitur. Formemus plures huiusmodi expressiones ac ponamus brevitatis gratia

$$\tfrac{1}{4}(1+n\cos.\alpha)^{\nu}+\tfrac{1}{4}(1-n\cos.\alpha)^{\nu}+\tfrac{1}{4}(1+n\sin.\alpha)^{\nu}+\tfrac{1}{4}(1-n\sin.\alpha)^{\nu}=\mathfrak{A},$$
$$\tfrac{1}{4}(1+n\cos.\beta)^{\nu}+\tfrac{1}{4}(1-n\cos.\beta)^{\nu}+\tfrac{1}{4}(1+n\sin.\beta)^{\nu}+\tfrac{1}{4}(1-n\sin.\beta)^{\nu}=\mathfrak{B},$$
$$\tfrac{1}{4}(1+n\cos.\gamma)^{\nu}+\tfrac{1}{4}(1-n\cos.\gamma)^{\nu}+\tfrac{1}{4}(1+n\sin.\gamma)^{\nu}+\tfrac{1}{4}(1-n\sin.\gamma)^{\nu}=\mathfrak{C}$$
etc.

et pro coefficientibus *B, C, D, E* etc. scribamus respective (1), (2), (3), (4) etc., quo facilius terminos ab initio quantumvis remotos repraesentare possimus. Habebimus ergo

$$\mathfrak{A}=A+(4)\cos.4\alpha+(8)\cos.8\alpha+(12)\cos.12\alpha+\text{etc.},$$
$$\mathfrak{B}=A+(4)\cos.4\beta+(8)\cos.8\beta+(12)\cos.12\beta+\text{etc.},$$
$$\mathfrak{C}=A+(4)\cos.4\gamma+(8)\cos.8\gamma+(12)\cos.12\gamma+\text{etc.}$$
etc.

Atque hinc sequentes approximationes adipiscimur.

I. Si capiamus $4\alpha=\frac{\pi}{2}$ seu $\alpha=\frac{\pi}{8}$, prodit

$$\mathfrak{A}=A-(8)+(16)-(24)+\text{etc.}$$

Ergo

$$A=\mathfrak{A}+(8)-(16)+(24)-\text{ etc.}$$

Quare si termini (8) et sequentes ob parvitatem contemni queant, erit satis exacte $A=\mathfrak{A}$.

II. Sumamus duas series ac statuamus $4\alpha=\frac{\pi}{4}$ et $4\beta=\frac{3\pi}{4}$, ut sit $\alpha=\frac{\pi}{16}$ et $\beta=\frac{3\pi}{16}$; erit

$$\cos.4\alpha+\cos.4\beta=0,\quad \cos.8\alpha+\cos.8\beta=0,\quad \cos.12\alpha+\cos.12\beta=0$$

et

$$\cos.16\alpha+\cos.16\beta=-2,$$

unde sequitur

$$\mathfrak{A}+\mathfrak{B}=2A-2(16)+2(32)-2(48)+\text{etc.}$$

ideoque

$$A=\tfrac{1}{2}(\mathfrak{A}+\mathfrak{B})+(16)-(32)+\text{etc.},$$

ubi numeri (16), (32) plerumque tam erunt parvi, ut negligi queant.

III. Addamus tres series ac statuamus $4\alpha = \frac{\pi}{6}$, $4\beta = \frac{3\pi}{6}$, $4\gamma = \frac{5\pi}{6}$, ut sit $\alpha = \frac{\pi}{24}$, $\beta = \frac{\pi}{8}$, $4\gamma = \frac{5\pi}{24}$, eritque

$$cos.\,4\alpha + cos.4\beta + cos.4\,\gamma = 0, \quad cos.16\alpha + cos.16\beta + cos.16\,\gamma = 0,$$
$$cos.\,8\alpha + cos.8\beta + cos.8\,\gamma = 0, \quad cos.20\alpha + cos.20\beta + cos.20\gamma = 0,$$
$$cos.12\alpha + cos.12\beta + cos.12\gamma = 0, \quad cos.24\alpha + cos.24\beta + cos.24\gamma = -3,$$

unde colligitur

$$A = \tfrac{1}{3}(\mathfrak{A} + \mathfrak{B} + \mathfrak{C}) + (24) - (48) + \text{etc.}$$

IV. Si haec determinatio non satis exacta videatur, addantur quatuor euismodi expressiones $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D}$ sitque $4\alpha = \frac{\pi}{8}$, $4\beta = \frac{3\pi}{8}$, $4\gamma = \frac{5\pi}{8}$, $4\delta = \frac{7\pi}{8}$
ac reperietur

$$\mathfrak{A} + \mathfrak{B} + \mathfrak{C} + \mathfrak{D} = 4A - 4(32) + 4(64) - \text{etc.},$$

ergo multo propius

$$A = \tfrac{1}{4}(\mathfrak{A} + \mathfrak{B} + \mathfrak{C} + \mathfrak{D}).$$

COROLLARIUM 1

293. Ex invento autem valore A sequens B satis expedite reperitur, cum sit

$$B = \frac{2(v+2)}{n}\int Andn - 2An.$$

Quatenus ergo in A ingreditur membrum $(1 \pm n\,cos.\alpha)^v$ vel $(1 + nf)^v$, dum f omnes illos sinus et cosinus complectitur, inde pro B oritur

$$\frac{2(v+2)}{n}\int ndn(1+nf)^v - 2n(1+nf)^v = \frac{2-2(1-vnf)(1+nf)^v}{(v+1)nff}.$$

COROLLARIUM 2

294. Cognitis autem coefficientibus A et B quemadmodum sequentes omnes ex illis derivari possint, supra ostendimus. Iis vero inventis integratio formulae $d\varphi(1 + n\,cos.\varphi)^v$ per se est manifesta.

PROBLEMA 35

295. *Integrale formulae* $d\varphi l(1+n\cos.\varphi)$ *per seriem secundum sinus angulorum* $\varphi, 2\varphi, 3\varphi$ etc. *progredientem evolvere.*

SOLUTIO

Cum sit

$$l(1+n\cos.\varphi)=n\cos.\varphi-\tfrac{1}{2}n^2\cos.^2\varphi+\tfrac{1}{3}n^3\cos.^3\varphi-\tfrac{1}{4}n^4\cos.^4\varphi+\text{etc.},$$

erit his potestatibus ad simplices cosinus reductis

$$\begin{array}{llll} l(1+n\cos.\varphi)=n\cos.\varphi-\tfrac{1}{2}\cdot\tfrac{1}{2}n^2\cos.2\varphi+\tfrac{1}{3}\cdot\tfrac{1}{4}n^3\cos.3\varphi-\tfrac{1}{4}\cdot\tfrac{1}{8}n^4\cos.4\varphi+\text{etc.} & & & \\ -\tfrac{1}{2}\cdot\tfrac{1}{2}n^2+\tfrac{1}{3}\cdot\tfrac{1}{4}n^3 & -\tfrac{1}{4}\cdot\tfrac{1}{8}n^4 & +\tfrac{1}{5}\cdot\tfrac{1}{16}n^5 & \\ -\tfrac{1}{4}\cdot\tfrac{3}{8}n^4+\tfrac{1}{5}\cdot\tfrac{10}{16}n^5 & -\tfrac{1}{6}\cdot\tfrac{15}{32}n^6 & & \\ -\tfrac{1}{6}\cdot\tfrac{10}{32}n^6+\tfrac{1}{7}\cdot\tfrac{35}{64}n^7 & & & \\ -\tfrac{1}{8}\cdot\tfrac{35}{128}n^8 & & & \end{array}$$

Quare si ponamus

$$l(1+n\cos.\varphi)=-A+B\cos.\varphi-C\cos.2\varphi+D\cos.3\varphi-\text{etc.},$$

erit

$$A=\tfrac{1}{2}\cdot\tfrac{n^2}{2}+\tfrac{1\cdot3}{2\cdot4}\cdot\tfrac{n^4}{4}+\tfrac{1\cdot3\cdot5}{2\cdot4\cdot6}\cdot\tfrac{n^6}{6}+\tfrac{1\cdot3\cdot5\cdot7}{2\cdot4\cdot6\cdot8}\cdot\tfrac{n^8}{8}+\text{etc.};$$

considerato ergo numero *n* ut variabili erit

$$\tfrac{ndA}{dn}=\tfrac{1}{2}nn+\tfrac{1\cdot3}{2\cdot4}n^4+\tfrac{1\cdot3\cdot5}{2\cdot4\cdot6}n^6+\text{etc.}=\tfrac{1}{\sqrt{(1-nn)}}-1.$$

Hinc

$$dA=\tfrac{dn}{n\sqrt{(1-nn)}}-\tfrac{dn}{n},$$

unde integratio praebet

$$A=l\tfrac{1-\sqrt{(1-nn)}}{n}-ln+C=l\tfrac{2-2\sqrt{(1-nn)}}{nn};$$

hoc enim modo evanescente *n* fit $A=l1=0$.
Tum vero erit

$$\tfrac{1}{2}B=\tfrac{1}{2}n+\tfrac{1\cdot3}{2\cdot4}\cdot\tfrac{n^3}{3}+\tfrac{1\cdot3\cdot5}{2\cdot4\cdot6}\cdot\tfrac{n^5}{5}+\text{etc.},$$

unde differentiatio praebet

$$\frac{nndB}{2dn} = \frac{1}{2}nn + \frac{1\cdot3}{2\cdot4}n^4 + \frac{1\cdot3\cdot5}{2\cdot4\cdot6}n^6 + \text{etc.} = \frac{1}{\sqrt{(1-nn)}} - 1,$$

ergo

$$\frac{1}{2}dB = \frac{dn}{nn\sqrt{(1-nn)}} - \frac{dn}{nn}$$

et integrando

$$\frac{1}{2}B = \frac{-\sqrt{(1-nn)}}{n} + \frac{1}{n} + C = \frac{1-\sqrt{(1-nn)}}{n}$$

integrali ita determinato, ut evanescat posito $n = 0$. Quocirca pro binis primis terminis habemus

$$A = l\,\frac{2-2\sqrt{(1-nn)}}{nn} \quad \text{et} \quad B = \frac{2-2\sqrt{(1-nn)}}{n},$$

ut sit $A = l\frac{B}{n}$. At pro reliquis differentiemus aequationem assumtam

$$\frac{-nd\varphi\, sin.\varphi}{1+n\,cos.\varphi} = -Bd\varphi\, sin.\varphi + 2Cd\varphi\, sin.2\varphi - 3Dd\varphi\, sin.3\varphi + 4Ed\varphi\, sin.4\varphi - \text{etc.}$$

seu

$$0 = \frac{n\,sin.\varphi}{1+n\,cos.\varphi} - B\,sin.\varphi + 2C\,sin.2\varphi - 3D\,sin.3\varphi + 4E\,sin.4\varphi - \text{etc.}$$

Quare per $2 + 2n\,cos.\varphi$ multiplicando prodit

$$\begin{aligned} 0 = 2n\,sin.\varphi - 2B\,sin.\varphi &+ 4C\,sin.2\varphi - 6D\,sin.3\varphi + 8E\,sin.4\varphi - \text{etc.} \\ &- Bn \qquad + 2Cn \qquad - 3Dn \\ + 2Cn &- 3Dn \qquad + 4En \qquad - 5Fn \end{aligned}$$

unde colligimus

$$C = \frac{B-n}{n}, \quad D = \frac{4C-Bn}{3n}, \quad E = \frac{6D-2C}{4n}, \quad F = \frac{8E-3Dn}{5n}.$$

Cum igitur sit $B = \frac{2-2\sqrt{(1-nn)}}{n}$ erit

$$C = \frac{2-nn-2\sqrt{(1-nn)}}{n} \quad \text{seu} \quad C = \left(\frac{1-\sqrt{(1-nn)}}{n}\right)^2$$

tum vero

$$D = \tfrac{2}{3}\left(\tfrac{1-\sqrt{(1-nn)}}{n}\right)^3, \quad E = \tfrac{2}{4}\left(\tfrac{1-\sqrt{(1-nn)}}{n}\right)^4, \quad F = \tfrac{2}{5}\left(\tfrac{1-\sqrt{(1-nn)}}{n}\right)^5 \text{ etc.}$$

Hinc, si brevitatis gratia ponamus $\frac{1-\sqrt{(1-nn)}}{n} = m$, erit

$$l(1+n\cos.\varphi) = -l\tfrac{2m}{n} + \tfrac{2}{1}m\cos.\varphi - \tfrac{2}{2}m^2\cos.2\varphi + \tfrac{2}{3}m^3\cos.3\varphi - \tfrac{2}{4}m^4\cos.4\varphi + \text{etc.}$$

ideoque integrale quaesitum

$$\int d\varphi l(1+n\cos.\varphi) = \text{Const.} - \varphi l\tfrac{2m}{n} + \tfrac{2}{1}m\sin.\varphi - \tfrac{2}{4}m^2\sin.2\varphi$$
$$+\tfrac{2}{9}m^3\sin.3\varphi - \tfrac{2}{16}m^4\sin.4\varphi + \tfrac{2}{25}m^5\sin.5\varphi - \text{etc.}$$

COROLLARIUM

296. Quodsi ergo ponamus $n = 1$, erit $m = 1$ et

$$l(1+\cos.\varphi) = -l2 + \tfrac{2}{1}\cos.\varphi - \tfrac{2}{2}\cos.2\varphi + \tfrac{2}{3}\cos.3\varphi - \tfrac{2}{4}\cos.4\varphi + \text{etc.}$$

et

$$l(1-\cos.\varphi) = -l2 - \tfrac{2}{1}\cos.\varphi - \tfrac{2}{2}\cos.2\varphi - \tfrac{2}{3}\cos.3\varphi - \tfrac{2}{4}\cos.4\varphi - \text{etc.}$$

Cum iam sit

$$1+\cos.\varphi = 2\ \cos.\tfrac{1}{2}\varphi^2 \quad \text{et} \quad 1-\cos.\varphi = 2\sin.\tfrac{1}{2}\varphi^2,$$

erit

$$l\cos.\tfrac{1}{2}\varphi = -l2 + \cos.\varphi - \tfrac{1}{2}\cos.2\varphi + \tfrac{1}{3}\cos.3\varphi - \tfrac{1}{4}\cos.4\varphi + \text{etc.}.$$

et

$$l\sin.\tfrac{1}{2}\varphi = -l2 - \cos.\varphi - \tfrac{1}{2}\cos.2\varphi - \tfrac{1}{3}\cos.3\varphi - \tfrac{1}{4}\cos.4\varphi - \text{etc.}$$

hinc

$$l\text{tang.}\tfrac{1}{2}\varphi = -2\cos.\varphi - \tfrac{2}{3}\cos.3\varphi - \tfrac{2}{5}\cos.5\varphi - \tfrac{2}{7}\cos.7\varphi - \text{etc.}$$