CHAPTER VIII

CONCERNING THE EVALUATION OF INTEGRALS ON TAKING CERTAIN CASES ONLY

PROBLEM 38

330. *The value of the integral* $\int\frac{x^m dx}{\sqrt{(1-xx)}}$ *assigned, that it receives on putting* $x=1$ *, clearly with the integral thus determined so that it vanishes on putting* $x=0$.

SOLUTION

For the simplest cases, in which $m = 0$ or $m = 1$, we have after integration on putting $x = 1$

$$\int\frac{dx}{\sqrt{(1-xx)}}=\frac{\pi}{2}\quad\text{and}\quad\int\frac{xdx}{\sqrt{(1-xx)}}=1.$$

Then from above § 120 we have seen in general that

$$\int\frac{x^{m+1}dx}{\sqrt{(1-xx)}}=\frac{m}{m+1}\int\frac{x^{m-1}dx}{\sqrt{(1-xx)}}-\frac{1}{m+1}x^m\sqrt{(1-xx)}\ ;$$

hence in the case $x = 1$ there will be

$$\int\frac{x^{m+1}dx}{\sqrt{(1-xx)}}=\frac{m}{m+1}\int\frac{x^{m-1}dx}{\sqrt{(1-xx)}}$$

for which from the simplest to the greater values of the exponents m we obtain by progressing :

$$\int\frac{x^m dx}{\sqrt{(1-xx)}}=\frac{\pi}{2}$$
$$\int\frac{xxdx}{\sqrt{(1-xx)}}=\frac{1}{2}\cdot\frac{\pi}{2}$$
$$\int\frac{x^4dx}{\sqrt{(1-xx)}}=\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{\pi}{2}$$
$$\int\frac{x^6dx}{\sqrt{(1-xx)}}=\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\frac{\pi}{2}$$
$$\int\frac{x^8dx}{\sqrt{(1-xx)}}=\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\frac{7}{8}\cdot\frac{\pi}{2}$$
$$\vdots$$
$$\int\frac{x^{2n}dx}{\sqrt{(1-xx)}}=\frac{1\cdot3\cdot5\cdots(2n-1)}{2\cdot4\cdot6\cdots2n}\cdot\frac{\pi}{2}$$

$$\int\frac{xdx}{\sqrt{(1-xx)}}=1$$
$$\int\frac{x^3dx}{\sqrt{(1-xx)}}=\frac{2}{3}$$
$$\int\frac{x^5dx}{\sqrt{(1-xx)}}=\frac{2\cdot4}{3\cdot5}$$
$$\int\frac{x^7dx}{\sqrt{(1-xx)}}=\frac{2\cdot4\cdot6}{3\cdot5\cdot7}$$
$$\int\frac{x^9dx}{\sqrt{(1-xx)}}=\frac{2\cdot4\cdot6\cdot8}{3\cdot5\cdot7\cdot9}$$
$$\vdots$$
$$\int\frac{x^{2n+1}dx}{\sqrt{(1-xx)}}=\frac{2\cdot4\cdot6\cdots2n}{3\cdot5\cdot7\cdots(2n+1)}$$

COROLLARY 1

331. Hence the integral $\int \frac{x^m dx}{\sqrt{(1-xx)}}$ on putting $x = 1$ is expressed algebraically in the cases in which m is an odd whole number, but in the cases in which it is even, it involves the quadrature of the circle ; for π designates the periphery of the circle, of which the diameter is equal to 1.

COROLLARY 2

332. If we should multiply the two last formulas together, there is produced

$$\int \frac{x^{2n}dx}{\sqrt{(1-xx)}} \cdot \int \frac{x^{2n+1}dx}{\sqrt{(1-xx)}} = \frac{1}{(2n+1)} \cdot \frac{\pi}{2}$$

evidently on putting $x = 1$, since that is known to be true, even if n is not a whole number.
[This result can be shown generally from betta and gamma functions. See the *O.O.* edition for more details about this.]

COROLLARY 3

333. Hence this equality stands, if we should put $x = z^\nu$, with the same conditions, because on taking $x = 0$ or $z = 1$. Then there becomes :

$$\nu\nu \int \frac{z^{2n\nu+\nu-1}dz}{\sqrt{(1-z^{2\nu})}} \cdot \int \frac{z^{2n\nu+2\nu-1}dz}{\sqrt{(1-z^{2\nu})}} = \frac{1}{(2n+1)} \cdot \frac{\pi}{2},$$

and on putting $2n\nu+\nu-1=\mu$ it becomes on putting $z = 1$

$$\int \frac{z^{\mu}dz}{\sqrt{(1-z^{2\nu})}} \cdot \int \frac{z^{\mu+\nu}dz}{\sqrt{(1-z^{2\nu})}} = \frac{1}{\nu(\mu+1)} \cdot \frac{\pi}{2}.$$

SCHOLIUM 1

334. Since such a product of two integrals is able to be shown, consequently it is more worthy to be noted, as this equality stands, even if neither formula can be shown, either algebraically or through π. Just as if $\nu = 2$ and $\mu = 0$, there becomes

$$\int \frac{dz}{\sqrt{(1-z^4)}} \cdot \int \frac{zzdz}{\sqrt{(1-z^4)}} = \frac{1}{2} \cdot \frac{\pi}{2} = \frac{\pi}{4}$$

and in a similar manner,

$v=3, \quad \mu=0, \quad$ becomes $\int \frac{dz}{\sqrt{(1-z^6)}} \cdot \int \frac{z^3dz}{\sqrt{(1-z^6)}} = \frac{1}{3} \cdot \frac{\pi}{2} = \frac{\pi}{6},$

$v=3, \quad \mu=1, \quad$ becomes $\int \frac{zdz}{\sqrt{(1-z^6)}} \cdot \int \frac{z^4dz}{\sqrt{(1-z^6)}} = \frac{1}{6} \cdot \frac{\pi}{2} = \frac{\pi}{12},$

$v=4, \quad \mu=0, \quad$ becomes $\int \frac{dz}{\sqrt{(1-z^8)}} \cdot \int \frac{z^4dz}{\sqrt{(1-z^8)}} = \frac{1}{4} \cdot \frac{\pi}{2} = \frac{\pi}{8},$

$v=4, \quad \mu=2, \quad$ becomes $\int \frac{zzdz}{\sqrt{(1-z^8)}} \cdot \int \frac{z^6dz}{\sqrt{(1-z^8)}} = \frac{1}{12} \cdot \frac{\pi}{2} = \frac{\pi}{24},$

$v=5, \quad \mu=0, \quad$ becomes $\int \frac{dz}{\sqrt{(1-z^{10})}} \cdot \int \frac{z^5dz}{\sqrt{(1-z^{10})}} = \frac{1}{5} \cdot \frac{\pi}{2} = \frac{\pi}{10},$

$v=5, \quad \mu=1, \quad$ becomes $\int \frac{zdz}{\sqrt{(1-z^{10})}} \cdot \int \frac{z^6dz}{\sqrt{(1-z^{10})}} = \frac{1}{10} \cdot \frac{\pi}{2} = \frac{\pi}{20},$

$v=5, \quad \mu=2, \quad$ becomes $\int \frac{zzdz}{\sqrt{(1-z^{10})}} \cdot \int \frac{z^7dz}{\sqrt{(1-z^{10})}} = \frac{1}{15} \cdot \frac{\pi}{2} = \frac{\pi}{30},$

$v=5, \quad \mu=3, \quad$ becomes $\int \frac{z^3dz}{\sqrt{(1-z^{10})}} \cdot \int \frac{z^8dz}{\sqrt{(1-z^{10})}} = \frac{1}{20} \cdot \frac{\pi}{2} = \frac{\pi}{40},$

which theorems without doubt are all worthy of attention.

SCHOLIUM 2

335. Hence also the value of the integral $\int \frac{x^m dx}{\sqrt{(x-xx)}}$ can be deduced easily on putting $x = 1$; for if we write $x = zz$, the integral is made into this : $2\int \frac{z^{2m}dz}{\sqrt{(1-zz)}}$, concerning which for the case $x = 1$ we arrive at the following values :

$$\int \frac{dx}{\sqrt{(x-xx)}} = \pi, \qquad \int \frac{x^4dx}{\sqrt{(x-xx)}} = \frac{1\cdot3\cdot5\cdot7}{2\cdot4\cdot6\cdot8} \cdot \pi,$$

$$\int \frac{xdx}{\sqrt{(x-xx)}} = \frac{1}{2} \cdot \pi, \qquad \int \frac{x^5dx}{\sqrt{(x-xx)}} = \frac{1\cdot3\cdot5\cdot7\cdot9}{2\cdot4\cdot6\cdot8\cdot10} \cdot \pi,$$

$$\int \frac{xxdx}{\sqrt{(x-xx)}} = \frac{1\cdot3}{2\cdot4} \cdot \pi, \qquad \vdots$$

$$\int \frac{x^3dx}{\sqrt{(x-xx)}} = \frac{1\cdot3\cdot5}{2\cdot4\cdot6} \cdot \pi, \qquad \int \frac{x^m dx}{\sqrt{(x-xx)}} = \frac{1\cdot3\cdot5\cdots(2m-1)}{2\cdot4\cdot6\cdots2m} \cdot \pi.$$

Hence the formulas of this kind of integrals, involving more complicated values which they take on putting $x = 1$, can be succinctly expressed by series, the use of which we indicate with several examples.

EXAMPLE 1

336. *To show the value of the integral* $\int \frac{dx}{\sqrt{(1-x^4)}}$ *on putting* $x = 1$ *by a series.*

This form can be given to the integral [See E605 for this integral arising from an elastic curve]:

$$\int \frac{dx}{\sqrt{(1-xx)}} \cdot (1+xx)^{-\frac{1}{2}},$$

so that we have :

$$\int \frac{dx}{\sqrt{(1-x^4)}} = \int \frac{dx}{\sqrt{(1-xx)}} \left(1 - \tfrac{1}{2}xx + \tfrac{1\cdot 3}{2\cdot 4}x^4 - \tfrac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}x^6 + \tfrac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8}x^8 - \text{etc.}\right);$$

hence from the individual terms of the integral for the case $x = 1$ there arises :

$$\int \frac{dx}{\sqrt{(1-x^4)}} = \frac{\pi}{2}\left(1 - \tfrac{1}{4} + \tfrac{1\cdot 9}{4\cdot 16} - \tfrac{1\cdot 9\cdot 25}{4\cdot 16\cdot 36} + \tfrac{1\cdot 9\cdot 25\cdot 49}{4\cdot 16\cdot 36\cdot 64} - \text{etc.}\right).$$

COROLLARY

337. In a like manner in the same case $x = 1$ there is found :

$$\int \frac{xdx}{\sqrt{(1-x^4)}} = 1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} + \tfrac{1}{9} - \tfrac{1}{11} + \text{etc.} = \tfrac{\pi}{4},$$

$$\int \frac{xxdx}{\sqrt{(1-x^4)}} = \frac{\pi}{2}\left(\tfrac{1}{2} - \tfrac{1^2\cdot 3}{2^2\cdot 4} + \tfrac{1^2\cdot 3^2\cdot 5}{2^2\cdot 4^2\cdot 6} - \tfrac{1^2\cdot 3^2\cdot 5^2\cdot 7}{2^2\cdot 4^2\cdot 6^2\cdot 8} + \text{etc.}\right),$$

$$\int \frac{x^3dx}{\sqrt{(1-x^4)}} = \tfrac{2}{3} - \tfrac{4}{3\cdot 5} + \tfrac{6}{5\cdot 7} - \tfrac{8}{7\cdot 9} + \tfrac{10}{9\cdot 11} - \text{etc.};$$

but

$$\int \frac{x^3dx}{\sqrt{(1-x^4)}} = \tfrac{1}{2} - \tfrac{1}{2}\sqrt{(1-x^4)}$$

and thus $= \frac{1}{2}$ on putting $x = 1$, from which this final series is $= \frac{1}{2}$.

[Note that the first is Gregory's formula for $\frac{\pi}{4}$.]

EXAMPLE 2

338. *To show the value of the integral* $\int dx\sqrt{\frac{1+axx}{1-xx}}$ *in the case* $x = 1$ *by a series.*

Since there shall be

$$\sqrt{(1+axx)} = 1+\tfrac{1}{2}axx-\tfrac{1\cdot 1}{2\cdot 4}a^2x^4+\tfrac{1\cdot 1\cdot 3}{2\cdot 4\cdot 6}a^3x^6-\text{etc.},$$

it becomes on multiplying by $\int\frac{dx}{\sqrt{(1-xx)}}$ and on integrating :

$$\int dx\sqrt{\tfrac{1+axx}{1-xx}} = \tfrac{\pi}{2}\left(1+\tfrac{1\cdot 1}{2\cdot 2}a-\tfrac{1\cdot 1\cdot 1\cdot 3}{2\cdot 2\cdot 4\cdot 4}a^2+\tfrac{1\cdot 1\cdot 1\cdot 3\cdot 5}{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6}a^3-\text{etc.}\right),$$

from which the periphery of the ellipse can be recognised [See E154].

EXAMPLE 3

339. *To show the value of the integral* $\int\frac{dx}{\sqrt{x(1-xx)}}$ *in the case* $x = 1$ *by a series.*

This formula may be represented thus $\int\frac{dx(1+x)^{-\frac{1}{2}}}{\sqrt{(x-xx)}}$, so that it becomes :

$$\int\tfrac{dx}{\sqrt{(x-xx)}}\left(1-\tfrac{1}{2}x+\tfrac{1\cdot 3}{2\cdot 4}x^3-\tfrac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}x^3+\text{etc.}\right),$$

from which this series may be obtained :

$$\int\tfrac{dx}{\sqrt{x(x-xx)}} = \pi\left(1-\tfrac{1}{4}+\tfrac{1\cdot 9}{4\cdot 16}-\tfrac{1\cdot 9\cdot 25}{4\cdot 16\cdot 36}+\text{etc.}\right),$$

which does not differ from the first series; which is not surprising, since on putting $x = zz$ this formula is reduced to that.

PROBLEM 39

340. *To define the value of the integral* $\int x^{m-1}dx(1-xx)^{n-\frac{1}{2}}$ *which vanishes on putting* $x = 0$.

SOLUTION

The reductions given above in § 118 produce in this case :

$$\int x^{m-1}dx(1-xx)^{\frac{\mu}{2}+1} = \frac{x^m(1-xx)^{\frac{\mu}{2}+1}}{m+\mu+2} + \frac{\mu+2}{m+\mu+2}\int x^{m-1}dx(1-xx)^{\frac{\mu}{2}} ;$$

hence on taking $\mu = 2n-1$ there will be

$$\int x^{m-1}dx(1-xx)^{n+1} = \frac{2n+2}{m+2n+1}\int x^{m-1}dx(1-xx)^{n-\frac{1}{2}}$$

on putting $x = 1$. There since in the preceding problem the value $\int \frac{x^{m-1}dx}{\sqrt{(1-xx)}}$ can be assigned, as for the sake of brevity we put $= M$, hence we can progress to the following :

$$\int \frac{x^{m-1}dx}{\sqrt{(1-xx)}} = M,$$

$$\int x^{m-1}dx(1-xx)^{\frac{1}{2}} = \frac{1}{m+1}M,$$

$$\int x^{m-1}dx(1-xx)^{\frac{3}{2}} = \frac{1\cdot 3}{(m+1)(m+3)}M,$$

$$\int x^{m-1}dx(1-xx)^{\frac{5}{2}} = \frac{1\cdot 3\cdot 5}{(m+1)(m+3)(m+5)}M$$

and in general

$$\int x^{m-1}dx(1-xx)^{n-\frac{1}{2}} = \frac{1\cdot 3\cdot 5\cdots(2n-1)}{(m+1)(m+3)(m+5)\cdots(m+2n-1)}M.$$

Now there are two cases to be considered carefully, since $m-1$ is either an even or odd number ; for if $m-1$ is even, then

$$M = \frac{1\cdot 3\cdot 5\cdots(m-2)}{2\cdot 4\cdot 6\cdots(m-1)}\cdot\frac{\pi}{2};$$

but if $m-1$ is odd, then

$$M = \frac{2\cdot 4\cdot 6\cdots(m-2)}{3\cdot 5\cdot 7\cdots(m-1)}.$$

Hence the following values are deduced :

$$\int dx\sqrt{(1-xx)}=\tfrac{\pi}{4} \qquad \int xdx\sqrt{(1-xx)}=\tfrac{1}{3}$$
$$\int xxdx\sqrt{(1-xx)}=\tfrac{1}{4}\cdot\tfrac{\pi}{4} \qquad \int x^3dx\sqrt{(1-xx)}=\tfrac{1}{3}\cdot\tfrac{2}{5}$$
$$\int x^4dx\sqrt{(1-xx)}=\tfrac{1\cdot 3}{4\cdot 6}\cdot\tfrac{\pi}{4} \qquad \int x^5dx\sqrt{(1-xx)}=\tfrac{1}{3}\cdot\tfrac{2\cdot 4}{5\cdot 7}$$
$$\int x^6dx\sqrt{(1-xx)}=\tfrac{1\cdot 3\cdot 5}{4\cdot 6\cdot 8}\cdot\tfrac{\pi}{4} \qquad \int x^7dx\sqrt{(1-xx)}=\tfrac{1}{3}\cdot\tfrac{2\cdot 4\cdot 6}{5\cdot 7\cdot 9}$$

....

$$\int (1-xx)^{\frac{3}{2}}dx=\tfrac{3\pi}{16} \qquad \int xdx(1-xx)^{\frac{3}{2}}=\tfrac{1}{5}$$
$$\int xx(1-xx)^{\frac{3}{2}}dx=\tfrac{1}{6}\cdot\tfrac{3\pi}{16} \qquad \int x^3dx(1-xx)^{\frac{3}{2}}=\tfrac{1}{5}\cdot\tfrac{2}{7}$$
$$\int x^4(1-xx)^{\frac{3}{2}}dx=\tfrac{1\cdot 3}{6\cdot 8}\cdot\tfrac{3\pi}{16} \qquad \int x^5dx(1-xx)^{\frac{3}{2}}=\tfrac{1}{5}\cdot\tfrac{2\cdot 4}{7\cdot 9}$$
$$\int x^6(1-xx)^{\frac{3}{2}}dx=\tfrac{1\cdot 3\cdot 5}{6\cdot 8\cdot 10}\cdot\tfrac{3\pi}{16} \qquad \int x^7dx(1-xx)^{\frac{3}{2}}=\tfrac{1}{5}\cdot\tfrac{2\cdot 4\cdot 6}{7\cdot 9\cdot 11}$$

....

$$\int (1-xx)^{\frac{5}{2}}dx=\tfrac{5\pi}{32} \qquad \int xdx(1-xx)^{\frac{5}{2}}=\tfrac{1}{7}$$
$$\int x^2(1-xx)^{\frac{5}{2}}dx=\tfrac{1}{8}\cdot\tfrac{5\pi}{32} \qquad \int x^3dx(1-xx)^{\frac{5}{2}}=\tfrac{1}{7}\cdot\tfrac{2}{9}$$
$$\int x^4(1-xx)^{\frac{5}{2}}dx=\tfrac{1\cdot 3}{8\cdot 10}\cdot\tfrac{5\pi}{32} \qquad \int x^5dx(1-xx)^{\frac{5}{2}}=\tfrac{1}{7}\cdot\tfrac{2\cdot 4}{9\cdot 11}$$
$$\int x^6(1-xx)^{\frac{5}{2}}dx=\tfrac{1\cdot 3\cdot 5}{8\cdot 10\cdot 12}\cdot\tfrac{5\pi}{32} \qquad \int x^7dx(1-xx)^{\frac{5}{2}}=\tfrac{1}{7}\cdot\tfrac{2\cdot 4\cdot 6}{9\cdot 11\cdot 13}$$

etc.

PROBLEM 40

341. *To assign the values of the integrals* $\int\frac{x^m dx}{\sqrt[3]{(1-x^3)}}$ and $\int\frac{x^m dx}{\sqrt[3]{(1-x^3)^2}}$ *on putting* $x=1$.

SOLUTION

For the simplest cases we put :

$$\int\frac{dx}{\sqrt[3]{(1-x^3)}}=A,\quad \int\frac{xdx}{\sqrt[3]{(1-x^3)}}=B,\quad \int\frac{xxdx}{\sqrt[3]{(1-x^3)}}=C,$$
$$\int\frac{dx}{\sqrt[3]{(1-x^3)^2}}=A',\quad \int\frac{xdx}{\sqrt[3]{(1-x^3)^2}}=B',\quad \int\frac{xxdx}{\sqrt[3]{(1-x^3)^2}}=C'$$

from the first reduction § 118 on putting $a=1$ and $b=-1$ for the case $x=1$, we have

$$\int x^{m+n-1}dx\left(1-x^n\right)^{\frac{\mu}{\nu}}=\frac{m\nu}{m\nu+n\mu+n\nu}\int x^{m-1}dx\left(1-x^n\right)^{\frac{\mu}{\nu}}\ ;$$

hence for the former, when $n = 3,\ v = 3$ and $\mu = -1$,

$$\int x^{m+2}dx\left(1-x^3\right)^{-\frac{1}{3}} = \frac{m}{m+2}\int x^{m-1}dx\left(1-x^3\right)^{-\frac{1}{3}}$$

but for the latter, when $n = 3,\ v = 3$ and $\mu = -2$,

$$\int x^{m+2}dx\left(1-x^3\right)^{-\frac{2}{3}} = \frac{m}{m+1}\int x^{m-1}dx\left(1-x^3\right)^{-\frac{2}{3}};$$

hence we will obtain for the former integrals:

$\int\frac{dx}{\sqrt[3]{(1-x^3)}} = A$	$\int\frac{xdx}{\sqrt[3]{(1-x^3)}} = B$	$\int\frac{xxdx}{\sqrt[3]{(1-x^3)}} = C$
$\int\frac{x^3dx}{\sqrt[3]{(1-x^3)}} = \frac{1}{3}A$	$\int\frac{x^4dx}{\sqrt[3]{(1-x^3)}} = \frac{2}{4}B$	$\int\frac{x^5dx}{\sqrt[3]{(1-x^3)}} = \frac{3}{5}C$
$\int\frac{x^6dx}{\sqrt[3]{(1-x^3)}} = \frac{1\cdot 4}{3\cdot 6}A$	$\int\frac{x^7dx}{\sqrt[3]{(1-x^3)}} = \frac{2\cdot 5}{4\cdot 7}B$	$\int\frac{x^8dx}{\sqrt[3]{(1-x^3)}} = \frac{3\cdot 6}{5\cdot 8}C$
$\int\frac{x^9dx}{\sqrt[3]{(1-x^3)}} = \frac{1\cdot 4\cdot 7}{3\cdot 6\cdot 9}A$	$\int\frac{x^{10}dx}{\sqrt[3]{(1-x^3)}} = \frac{2\cdot 5\cdot 8}{4\cdot 7\cdot 10}B$	$\int\frac{x^{11}dx}{\sqrt[3]{(1-x^3)}} = \frac{3\cdot 6\cdot 9}{5\cdot 8\cdot 11}C$
$\int\frac{x^{12}dx}{\sqrt[3]{(1-x^3)}} = \frac{1\cdot 4\cdot 7\cdot 10}{3\cdot 6\cdot 9\cdot 12}A$	$\int\frac{x^{13}dx}{\sqrt[3]{(1-x^3)}} = \frac{2\cdot 5\cdot 8\cdot 11}{4\cdot 7\cdot 10\cdot 13}B$	$\int\frac{x^{14}dx}{\sqrt[3]{(1-x^3)}} = \frac{3\cdot 6\cdot 9\cdot 12}{5\cdot 8\cdot 11\cdot 14}C$

etc.,

but for the form of the latter integrals :

$\int\frac{dx}{\sqrt[3]{(1-x^3)^2}} = A'$	$\int\frac{xdx}{\sqrt[3]{(1-x^3)^2}} = B'$	$\int\frac{xxdx}{\sqrt[3]{(1-x^3)^2}} = C'$
$\int\frac{x^3dx}{\sqrt[3]{(1-x^3)^2}} = \frac{1}{2}A'$	$\int\frac{x^4dx}{\sqrt[3]{(1-x^3)^2}} = \frac{2}{3}B'$	$\int\frac{x^5dx}{\sqrt[3]{(1-x^3)^2}} = \frac{3}{4}C'$
$\int\frac{x^6dx}{\sqrt[3]{(1-x^3)^2}} = \frac{1\cdot 4}{2\cdot 5}A'$	$\int\frac{x^7dx}{\sqrt[3]{(1-x^3)^2}} = \frac{2\cdot 5}{3\cdot 6}B'$	$\int\frac{x^8dx}{\sqrt[3]{(1-x^3)^2}} = \frac{3\cdot 6}{4\cdot 7}C'$
$\int\frac{x^9dx}{\sqrt[3]{(1-x^3)^2}} = \frac{1\cdot 4\cdot 7}{2\cdot 5\cdot 8}A'$	$\int\frac{x^{10}dx}{\sqrt[3]{(1-x^3)^2}} = \frac{2\cdot 5\cdot 8}{3\cdot 6\cdot 9}B'$	$\int\frac{x^{11}dx}{\sqrt[3]{(1-x^3)^2}} = \frac{3\cdot 6\cdot 9}{4\cdot 7\cdot 10}C'$

$$\int\frac{x^{12}dx}{\sqrt[3]{\left(1-x^3\right)^2}}=\frac{1\cdot4\cdot7\cdot10}{2\cdot5\cdot8\cdot11}A' \quad\Big|\quad \int\frac{x^{13}dx}{\sqrt[3]{\left(1-x^3\right)^2}}=\frac{2\cdot5\cdot8\cdot11}{3\cdot6\cdot9\cdot12}B' \quad\Big|\quad \int\frac{x^{14}dx}{\sqrt[3]{\left(1-x^3\right)^2}}=\frac{3\cdot6\cdot9\cdot12}{4\cdot7\cdot10\cdot13}C'$$

etc.,

from which we may conclude to be generally :

$$\int\frac{x^{3n}dx}{\sqrt[3]{\left(1-x^3\right)}}=\frac{1\cdot4\cdot7\cdots(3n-2)}{3\cdot6\cdot9\cdots3n}A$$

$$\int\frac{x^{3n+1}dx}{\sqrt[3]{\left(1-x^3\right)}}=\frac{2\cdot5\cdot8\cdots(3n-1)}{4\cdot7\cdot10\cdots(3n+1)}B$$

$$\int\frac{x^{3n+2}dx}{\sqrt[3]{\left(1-x^3\right)}}=\frac{3\cdot6\cdot9\cdots3n}{5\cdot8\cdot11\cdots(3n+2)}C$$

$$\int\frac{x^{3n}dx}{\sqrt[3]{\left(1-x^3\right)^2}}=\frac{1\cdot4\cdot7\cdots(3n-2)}{2\cdot5\cdot8\cdots(3n-1)}A'$$

$$\int\frac{x^{3n+1}dx}{\sqrt[3]{\left(1-x^3\right)^2}}=\frac{2\cdot5\cdot8\cdots(3n-1)}{3\cdot6\cdot9\cdots3n}B'$$

$$\int\frac{x^{3n+2}dx}{\sqrt[3]{\left(1-x^3\right)^2}}=\frac{3\cdot6\cdot9\cdots3n}{4\cdot7\cdot10\cdots(3n+1)}C'$$

moreover it is to be observed that $C=\frac{1}{2}$ and $C'=1$.

COROLLARY 1

342. These formulas are able to be combined in various ways, so that outstanding theorems arise thus; clearly there shall be :

$$\int\frac{x^{3n}dx}{\sqrt[3]{\left(1-x^3\right)}}\cdot\int\frac{x^{3n+2}dx}{\sqrt[3]{\left(1-x^3\right)^2}}=\frac{AC'}{3n+1}=\frac{1}{3n+1}\int\frac{dx}{\sqrt[3]{\left(1-x^3\right)}},$$

$$\int\frac{x^{3n+1}dx}{\sqrt[3]{\left(1-x^3\right)}}\cdot\int\frac{x^{3n}dx}{\sqrt[3]{\left(1-x^3\right)^2}}=\frac{A'B}{3n+1}=\frac{1}{3n+1}\int\frac{dx}{\sqrt[3]{\left(1-x^3\right)^2}}\cdot\int\frac{xdx}{\sqrt[3]{\left(1-x^3\right)}},$$

$$\int\frac{x^{3n+2}dx}{\sqrt[3]{\left(1-x^3\right)}}\cdot\int\frac{x^{3n+1}dx}{\sqrt[3]{\left(1-x^3\right)^2}}=\frac{2B'C}{3n+2}=\frac{1}{3n+2}\int\frac{xdx}{\sqrt[3]{\left(1-x^3\right)^2}}.$$

COROLLARY 2

343. Now since the reckoning of the exponent does not progress beyond three in the computation , then generally

$$\int\frac{x^{\lambda-1}dx}{\sqrt[3]{\left(1-x^3\right)}}\cdot\int\frac{x^{\lambda+1}dx}{\sqrt[3]{\left(1-x^3\right)^2}}=\frac{1}{\lambda}\int\frac{dx}{\sqrt[3]{\left(1-x^3\right)}},$$

$$\int\frac{x^{\lambda}dx}{\sqrt[3]{\left(1-x^3\right)}}\cdot\int\frac{x^{\lambda-1}dx}{\sqrt[3]{\left(1-x^3\right)^2}}=\frac{1}{\lambda}\int\frac{xdx}{\sqrt[3]{\left(1-x^3\right)}}\cdot\int\frac{dx}{\sqrt[3]{\left(1-x^3\right)^2}},$$

$$\int\frac{x^{\lambda}dx}{\sqrt[3]{\left(1-x^3\right)}}\cdot\int\frac{x^{\lambda-1}dx}{\sqrt[3]{\left(1-x^3\right)^2}}=\frac{1}{\lambda}\int\frac{xdx}{\sqrt[3]{\left(1-x^3\right)^2}},$$

whereby from the last two we follow with :

$$\int \frac{xdx}{\sqrt[3]{\left(1-x^3\right)}} \cdot \int \frac{dx}{\sqrt[3]{\left(1-x^3\right)^2}} = \int \frac{xdx}{\sqrt[3]{\left(1-x^3\right)^2}}.$$

COROLLARY 3

344. There is put $x = z^n$ and $\lambda n = m$ and our theorems adopt the following forms:

$$\int \frac{z^{m-1}dz}{\sqrt[3]{\left(1-z^{3n}\right)}} \cdot \int \frac{x^{m+2n-1}dz}{\sqrt[3]{\left(1-z^{3n}\right)^2}} = \frac{1}{m}\int \frac{z^{m-1}dz}{\sqrt[3]{\left(1-z^{3n}\right)}},$$

$$\int \frac{z^{m+n-1}dz}{\sqrt[3]{\left(1-z^{3n}\right)}} \cdot \int \frac{z^{m-1}dz}{\sqrt[3]{\left(1-z^{3n}\right)^2}} = \frac{n}{m}\int \frac{z^{2n-1}dz}{\sqrt[3]{\left(1-z^{3n}\right)}} \cdot \int \frac{z^{n-1}dz}{\sqrt[3]{\left(1-z^{3n}\right)^2}} = \frac{1}{m}\int \frac{z^{2n-1}dz}{\sqrt[3]{\left(1-z^{3n}\right)^2}}$$

PROBLEM 41

345. *With the integral* $\int \frac{x^{m-1}dx}{\left(1-x^n\right)^{\frac{n-k}{n}}}$ *given, to assign the integral of this formula* $\int \frac{x^{m+\lambda n-1}dx}{\left(1-x^n\right)^{\frac{n-k}{n}}}$ *on putting* $x = 1$.

SOLUTIO

In order that the integral shall be finite, it is necessary that *m* and *k* shall be positive numbers. Therefore since by the general reduction there shall be :

$$\int x^{m+n-1}dx\left(1-x^n\right)^{\frac{\mu}{\nu}} = \frac{m\nu}{m\nu+n(\mu+\nu)}\int x^{m-1}dx\left(1-x^n\right)^{\frac{\mu}{\nu}},$$

putting $\nu = n$ and $\mu = k-n$, so that there becomes $\mu+\nu = k$; then

$$\int \frac{x^{m+n-1}dx}{\left(1-x^n\right)^{\frac{n-k}{n}}} = \frac{m}{m+k}\int \frac{x^{m-1}dx}{\left(1-x^n\right)^{\frac{n-k}{n}}}.$$

Therefore there is put the value of this formula, since it is given equal to *A*, and this reduction repeated continually gives on putting for brevity *P* for $\left(1-x^n\right)^{\frac{n-k}{n}}$

$$\int \frac{x^{m-1}dx}{P} = A,$$

$$\int \frac{x^{m+n-1}dx}{P} = \frac{m}{m+k}A,$$

$$\int \frac{x^{m+2n-1}dx}{P} = \frac{m(m+n)}{(m+k)(m+n+k)}A,$$

$$\int \frac{x^{m+3n-1}dx}{P} = \frac{m(m+n)(m+2n)}{(m+k)(m+n+k)(m+2n+k)}A$$

and generally

$$\int \frac{x^{m+\alpha n-1}dx}{P} = \frac{m(m+n)(m+2n)\cdots(m+(\alpha-1)n)}{(m+k)(m+n+k)(m+2n+k)\cdots(m+(\alpha-1)n+k)}A.$$

COROLLARY 1

346. If in a like manner the other [latter] formula becomes

$$\int \frac{x^{p-1}dx}{(1-x^n)^{\frac{n-q}{n}}} = B$$

on putting $x=1$, but for brevity there is written Q for $(1-x^n)^{\frac{n-q}{n}}$, then we have

$$\int \frac{x^{m+\alpha n-1}dx}{Q} = \frac{p(p+n)(p+2n)\cdots(p+(\alpha-1)n)}{(p+q)(p+n+q)(p+2n+q)\cdots(p+(\alpha-1)n+q)}B,$$

and which contains just as many factors with that other.

COROLLARY 2

347. Now there can be put in place $p=m+k$ in order that the latter numerator becomes equal to the first denominator, and the product of the two formulas is

$$\frac{m(m+n)(m+2n)\cdots(m+(\alpha-1)n)}{(m+k+q)(m+n+k+q)(m+2n+k+q)\cdots(m+(\alpha-1)n+k+q)}AB;$$

again there can be made $m+k+q=m+n$ or $q=n-k$; this product is equal to $\frac{m}{m+\alpha n}AB$ and thus

$$\int \frac{x^{m+\alpha n-1}dx}{(1-x^n)^{\frac{n-k}{n}}}\cdot\int \frac{x^{m+k+\alpha n-1}dx}{(1-x^n)^{\frac{k}{n}}} = \frac{m}{m+\alpha n}\int \frac{x^{m-1}dx}{(1-x^n)^{\frac{n-k}{n}}}\cdot\int \frac{x^{m+k-1}dx}{(1-x^n)^{\frac{k}{n}}},$$

which is a the theorem worthy of attention, since here it is no longer necessary that *a* should be a whole number.

COROLLARY 3

348. Whereby in place of $m+\alpha n$ we can write μ; then

$$\mu\int\frac{x^{\mu-1}dx}{\left(1-x^n\right)^{\frac{n-k}{n}}}\cdot\int\frac{x^{\mu+k-1}dx}{\left(1-x^n\right)^{\frac{k}{n}}}=m\int\frac{x^{m-1}dx}{\left(1-x^n\right)^{\frac{n-k}{n}}}\cdot\int\frac{x^{m+k-1}dx}{\left(1-x^n\right)^{\frac{k}{n}}}.$$

Hence if we assume $m+k=n$ or $m=n-k$, on account of

$$\int\frac{x^{n-1}dx}{\left(1-x^n\right)^{\frac{k}{n}}}=\frac{1-\left(1-x^n\right)^{\frac{n-k}{n}}}{n-k}=\frac{1}{n-k}$$

on putting $x = 1$ then [§ 352]

$$\int\frac{x^{\mu-1}dx}{\left(1-x^n\right)^{\frac{n-k}{n}}}\cdot\int\frac{x^{\mu+k-1}dx}{\left(1-x^n\right)^{\frac{k}{n}}}=\frac{1}{\mu}\int\frac{x^{n-k-1}dx}{\left(1-x^n\right)^{\frac{n-k}{n}}}=\frac{\pi}{\mu n\sin.\frac{k\pi}{n}}.$$

And on putting $x=z^{\nu}$, then indeed $\mu\nu=p,\ \nu n=q$ and $k=\lambda n$ and there is had

$$\int\frac{z^{p-1}dx}{\left(1-z^q\right)^{1-\lambda}}\cdot\int\frac{z^{p+\lambda q-1}dz}{\left(1-z^q\right)^{\lambda}}=\frac{\nu}{p}\int\frac{z^{(1-\lambda)q-1}dz}{\left(1-z^q\right)^{1-\lambda}}.$$

SCHOLION 1

349. The particular theorems which hence follow, thus themselves may be found :

I. $n=2,k=1;\quad \int\frac{x^{\mu-1}dx}{\sqrt{(1-xx)}}\cdot\int\frac{x^{\mu}dx}{\sqrt{(1-xx)}}=\frac{1}{\mu}\int\frac{dx}{\sqrt{(1-xx)}}=\frac{\pi}{2\mu}$

II. $n=3,k=1;\quad \int\frac{x^{\mu-1}dx}{\sqrt[3]{\left(1-x^3\right)^2}}\cdot\int\frac{x^{\mu}dx}{\sqrt[3]{\left(1-x^3\right)}}=\frac{1}{\mu}\int\frac{dx}{\sqrt[3]{\left(1-x^3\right)^2}}=\frac{2\pi}{3\mu\sqrt{3}}$

$n=3,k=2;\quad \int\frac{x^{\mu-1}dx}{\sqrt[3]{\left(1-x^3\right)}}\cdot\int\frac{x^{\mu+1}dx}{\sqrt[3]{\left(1-x^3\right)^2}}=\frac{1}{\mu}\int\frac{dx}{\sqrt[3]{\left(1-x^3\right)}}=\frac{2\pi}{3\mu\sqrt{3}}$

III. $n=4,k=1;\quad \int\frac{x^{\mu-1}dx}{\sqrt[4]{\left(1-x^4\right)^3}}\cdot\int\frac{x^{\mu}dx}{\sqrt[4]{\left(1-x^4\right)}}=\frac{1}{\mu}\int\frac{dx}{\sqrt[4]{\left(1-x^4\right)^3}}=\frac{\pi}{2\mu\sqrt{2}}$

$n=4,k=2;\quad \int\frac{x^{\mu-1}dx}{\sqrt{\left(1-x^4\right)}}\cdot\int\frac{x^{\mu+1}dx}{\sqrt{\left(1-x^4\right)}}=\frac{1}{\mu}\int\frac{dx}{\sqrt{\left(1-x^4\right)}}=\frac{\pi}{2\mu\sqrt{2}}$

$n=4,k=3;\quad \int\frac{x^{\mu-1}dx}{\sqrt[4]{\left(1-x^4\right)}}\cdot\int\frac{x^{\mu+2}dx}{\sqrt[4]{\left(1-x^4\right)}}=\frac{1}{\mu}\int\frac{dx}{\sqrt[4]{\left(1-x^4\right)}}=\frac{\pi}{2\mu\sqrt{2}}$

etc.

Where it is to be observed that the formula $\int \frac{x^{n-k-1}dx}{(1-x^n)^{\frac{n-k}{n}}}$ can be reduced to rationality. For on putting $\frac{x^n}{1-x^n} = z^n$ or $x^n = \frac{z^n}{1+z^n}$, from which $\frac{dx}{x} = \frac{dz}{z(1+z^n)}$. Whereby since our formula shall be $= \int \left(\frac{x^n}{1-x^n}\right)^{\frac{n-k}{n}} \cdot \frac{dx}{x}$, that becomes equal to $\int \frac{z^{n-k-1}dz}{1+z^n}$, the integral of which thus must be determined, so that it vanishes on putting $x = 0$ and thus $z = 0$; then on now putting $x = 1$, that is $z = \infty$, it gives the value which we use here. But soon [§ 352] we will show the value of this integral $\int \frac{z^{n-k-1}dz}{1+z^n}$ on putting $z = \infty$ thus, and of this $\int \frac{x^{n-k-1}dx}{(1-x^n)^{\frac{n-k}{n}}}$ it is possible to be expressed by angles, the values of which I have put here at once. Then also this transformation of the formula $\int \frac{x^{m-1}dx}{(1-x^n)^{\frac{n-k}{n}}}$ deserves to be noted which is derived on putting $1-x^n = z^n$, which provides $-\int \frac{z^{k-1}dz}{(1-z^n)^{\frac{n-m}{n}}}$ thus to be integrated, so that it vanishes on putting $x = 0$ or $z = 1$; then there must now be put in place $x = 1$ or $z = 0$. Since the same is returned, if by changing the sign this formula thus is integrated $\int \frac{z^{k-1}dz}{(1-z^n)^{\frac{n-m}{n}}}$, so that it vanishes on putting $z = 0$, then truly there is put $z = 1$. Since now nothing stands in the way we may write x in place of z, then we have this conspicuous theorem

$$\int \frac{x^{m-1}dx}{(1-x^n)^{\frac{n-k}{n}}} = \int \frac{x^{k-1}dx}{(1-x^n)^{\frac{n-m}{n}}},$$

thus so that in a formula of this kind the exponents m et k are allowed to be interchanged, clearly for the case $x = 1$. Thus for the preceding formula to be reduced to rationality, where $m = n-k$, then

$$\int \frac{x^{n-k-1}dx}{(1-x^n)^{\frac{n-k}{n}}} = \int \frac{x^{k-1}dx}{(1-x^n)^{\frac{k}{n}}},$$

from which it also follows on putting $z = \infty$

$$\int \frac{z^{n-k-1}dz}{1+z^n} = \int \frac{z^{k-1}dz}{1+z^n}.$$

SCHOLIUM 2

350. Hence also the integrals of more composite can be expressed concisely in series for the case $x = 1$. For since in the above reduction on putting $m+k=\mu$ or $k=\mu-m$ sit

$$\int\frac{x^{m+n-1}dx}{\left(1-x^n\right)^{\frac{m+n-\mu}{n}}}=\frac{m}{\mu}\int\frac{x^{m-1}dx}{\left(1-x^n\right)^{\frac{m+n-\mu}{n}}},$$

if there should be a formula of the differential of this kind :

$$dy=\frac{x^{m-1}dx}{\left(1-x^n\right)^{\frac{m+n-\mu}{n}}}\left(A+Bx^n+Cx^{2n}+Dx^{3n}+\text{etc.}\right),$$

that thus it should be required to integrate, in order that y vanishes on putting $x = 0$, and the value of this y is required in the case $x=1$, then in this case if we are able to put

$$\int\frac{x^{m+n-1}dx}{\left(1-x^n\right)^{\frac{m+n-\mu}{n}}}=O,$$

that value is equal to

$$O\left(A+\frac{m}{\mu}B+\frac{m(m+n)}{\mu(\mu+n)}C+\frac{m(m+n)(m+2n)}{\mu(\mu+n)(\mu+2n)}D+\text{etc.}\right).$$

Hence in turn for this proposed series

$$A+\frac{m}{\mu}B+\frac{m(m+n)}{\mu(\mu+n)}C+\frac{m(m+n)(m+2n)}{\mu(\mu+n)(\mu+2n)}D+\text{etc.}$$

the sum of this is equal to this formula to be integrated :

$$\frac{1}{O}\int\frac{x^{m-1}dx}{\left(1-x^n\right)^{\frac{m+n-\mu}{n}}}\left(A+Bx^n+Cx^{2n}+Dx^{3n}+\text{etc.}\right),$$

if after integration there is put $x=1$. Hence if that comes about, so that the sum of this series $A+Bx^n+Cx^{2n}+Dx^{3n}+\text{etc.}$ is to be assigned and hence the integration can be completed, then the sum of this series will be obtained.

PROBLEM 42

351. *Thus having determined the integral of this formula* $\frac{x^{m-1}dx}{1+x^n}$ *, so that it vanishes on putting* $x = 0$*, to assign the value in the case when* $x=\infty$.

SOLUTION

We have shown the integral of this formula above now in § 77 [Ch. I, Example 2] and indeed thus to be determined so that it should vanish on putting $x = 0$, because on putting for brevity $\frac{\pi}{n}=\omega$ thus there can be established :

$$-\tfrac{2}{n}\cos.m\omega\, l\sqrt{(1-2x\cos.\omega+xx)}\ +\tfrac{2}{n}\sin.m\omega\cdot \text{Arc. tang.}\tfrac{x\sin.\omega}{1-x\cos.\omega}$$
$$-\tfrac{2}{n}\cos.3m\omega\, l\sqrt{(1-2x\cos.3\omega+xx)}\ +\tfrac{2}{n}\sin.3m\omega\cdot \text{Arc. tang.}\tfrac{x\sin.3\omega}{1-x\cos.3\omega}$$
$$-\tfrac{2}{n}\cos.5m\omega\, l\sqrt{(1-2x\cos.5\omega+xx)}\ +\tfrac{2}{n}\sin.5m\omega\cdot \text{Arc. tang.}\tfrac{x\sin.5\omega}{1-x\cos.5\omega}$$
$$\vdots$$
$$-\tfrac{2}{n}\cos.\lambda m\omega\, l\sqrt{(1-2x\cos.\lambda\omega+xx)}\ +\tfrac{2}{n}\sin.\lambda m\omega\cdot \text{Arc. tang.}\tfrac{x\sin.\lambda\omega}{1-x\cos.\lambda\omega},$$

where λ denotes the greatest odd number less than the exponent n, and if n itself should be odd, the integral agrees with the even part, $\pm\frac{1}{n}l(1+x)$, according as m should be either an odd or even number ; clearly in that case the + sign, and in this case the – sign prevails. Hence here the value of this integral is sought, which it produces on putting $x=\infty$. Hence we expand out the first part involving logarithms, and since on account of $x=\infty$ then

$$l\sqrt{(1-2x\cos.\lambda\omega+xx)}=l\left(x-\cos.\lambda\omega\right)=lx+\left(1-\tfrac{\cos.\lambda\omega}{x}\right)=lx$$

on account of $\frac{\cos.\lambda\omega}{x}=0$, from which the parts of the logarithm give

$$-\ \tfrac{2lx}{n}\left(\cos.m\omega+\cos.3m\omega+\cos.5m\omega+...+\cos.\lambda m\omega\right)\ \left(\pm\tfrac{lx}{n},\text{if } n \text{ is odd}\right),$$

and we put this series of cosines

$$\cos.m\omega+\cos.3m\omega+\cos.5m\omega+...+\cos.\lambda m\omega=s$$

and then on multiplying by $2\sin.m\omega$

$$2s\,sin.m\omega = sin.2m\omega + sin.4m\omega + sin.6m\omega + ... + sin.(\lambda+1)m\omega,$$
$$-\ sin.2m\omega - sin.4m\omega - sin.6m\omega - ......$$

from which there becomes

$$s = \frac{sin.(\lambda+1)m\omega}{2\,sin.m\omega}.$$

Whereby if *n* is an even number, then $\lambda = n-1$ and thus the parts of the logarithm become

$$-\frac{lx}{n}\cdot\frac{sin.nm\omega}{sin.m\omega} = -\frac{lx}{n}\cdot\frac{sin.m\pi}{sin.m\omega}$$

on account of $n\omega = \pi$. But as *m* is a whole number, then $sin.m\pi = 0$, from which these parts vanish. But if *n* should be an odd number, then $\lambda = n-2$ and the sum of the parts of the logarithms becomes

$$-\frac{lx}{n}\cdot\frac{sin.(n-1)m\omega}{sin.m\omega} \pm \frac{lx}{n}\ ;$$

but $sin.(n-1)m\omega = sin.(m\pi - m\omega) = \pm sin.m\omega$, where the upper sign prevails, if *m* is an odd number, otherwise the lower sign, because the same is to be understood from the other by ambiguity, thus so that we shall have

$$\mp\frac{lx}{n}\cdot\frac{sin.m\omega}{sin.m\omega} \pm \frac{lx}{n} = 0.$$

Hence the logarithmic parts always cancel each other out; since hence it is evident, that otherwise the integral becomes infinite, since yet clearly it must be finite.

Hence only the angles are left, which we gather into one sum ; hence there is considered Arc.tang $\frac{x\,sin.\lambda\omega}{1-x\,cos.\lambda\omega}$, which arc vanishes in the case $x = 0$, then truly in the case $x = \frac{1}{cos.\lambda\omega}$ the angle becomes a quadrant, hence on increasing *x* further it will be the next quadrant, then on making $x = \infty$ the tangent of this becomes $= -\frac{sin.\lambda\omega}{cos.\lambda\omega} = -\text{tang.}\lambda\omega = \text{tang.}(\pi - \lambda\omega)$ and thus the arc itself is equal to $\pi - \lambda\omega$, from which these arcs jointly taken give

$$\frac{2}{n}\left((\pi-\omega)sin.m\omega + (\pi-3\omega)sin.3mw + (\pi-5\omega)sin.5m\omega + \ ... \ + (\pi-\lambda\omega)sin.\lambda m\omega\right),$$

from which we obtain the two series :

$$\frac{2\pi}{n}\ \left(sin.m\omega + sin.3m\omega + sin.5m\omega + ... + \quad sin.\lambda m\omega\right) = \frac{2\pi}{n}\,p,$$
$$\frac{-2\omega}{n}\left(sin.m\omega + 3\,sin.3m\omega + 5\,sin.5m\omega + ... + \lambda\,sin.\lambda m\omega\right) = \frac{-2\omega}{n}\,q,$$

which we must investigate separately. And for the latter indeed since before we were having :

$$cos.m\omega + cos.3m\omega + cos.5m\omega + ... + cos.\lambda m\omega = s = \frac{sin.(\lambda+1)m\omega}{2sin.m\omega},$$

if we look upon the angle ω as a variable, on differentiation it gives

$$-md\omega\left(sin.m\omega + 3sin.3m\omega + 5sin.5m\omega + ... + \lambda sin.\lambda m\omega\right)$$
$$= \frac{(\lambda+1)md\omega cos.(\lambda+1)m\omega}{2sin.m\omega} - \frac{md\omega sin.(\lambda+1)m\omega cos.m\omega}{2sin.^2 m\omega};$$

hence

$$-q = \frac{(\lambda+1)cos.(\lambda+1)m\omega}{2sin.m\omega} - \frac{sin.(\lambda+1)m\omega cos.m\omega}{2sin.^2 m\omega}$$

or

$$-q = \frac{\lambda cos.(\lambda+1)m\omega}{2sin.m\omega} - \frac{sin.\lambda m\omega}{2sin.^2 m\omega}.$$

For the other series

$$p = sin.m\omega + sin.3m\omega + sin.5m\omega + ... + sin.\lambda m\omega$$

we multiply each by $2sin.m\omega$ and there becomes

$$2p\,sin.m\omega = 1 - cos.2m\omega - cos.4m\omega - cos.6m\omega - ..... - cos.(\lambda+1)m\omega$$
$$+ cos.2m\omega + cos.4m\omega + cos.6m\omega + ...$$

and thus there becomes

$$p = \frac{1-cos.(\lambda+1)m\omega}{2sin.m\omega}.$$

But if now n is an even number, then $\lambda = n-1$ and thus

$$cos.(\lambda+1)m\omega = cos.m\omega = cos.m\pi \quad \text{and} \quad sin.(\lambda+1)m\omega = sin.m\pi = 0,$$

hence

$$p = \frac{1-cos.m\pi}{2sin.m\omega} \quad \text{and} \quad -q = \frac{n\,cos.m\pi}{2sin.m\omega}$$

and hence all the arcs jointly taken give

$$\frac{2\pi}{n}\cdot\frac{1-cos.m\pi}{2sin.m\omega} + \frac{2\omega}{n}\cdot\frac{n\,cos.m\pi}{2sin.m\omega} = \frac{\pi}{n\,sin.m\omega}$$

on account of $n\omega = \pi$.
Now if n is an odd number; then $\lambda = n-2$ and hence

$$cos.(\lambda+1)m\omega = cos.(m\pi - m\omega) \text{ and } sin.(\lambda+1)m\omega = sin.(m\pi - m\omega)$$

or

$$cos.(\lambda+1)m\omega = cos.m\pi\, cos.m\omega \text{ et } sin.(\lambda+1)m\omega = -cos.m\pi\, sin.m\omega,$$

hence

$$p = \frac{1-cos.m\pi\, cos.m\omega}{2\, sin.m\omega} \quad \text{and} \quad -q = \frac{(n-1)cos.m\pi\, cos.m\omega}{2\, sin.m\omega} + \frac{cos.m\pi\, cos.m\omega}{2\, sin.m\omega},$$

from which the sum of all the angles becomes

$$\frac{\pi(1-cos.m\pi\, cos.m\omega)}{n\, sin.m\omega} + \frac{\omega(n-1)cos.m\pi\, cos.m\omega}{n\, sin.m\omega} + \frac{\omega\, cos.m\pi\, cos.m\omega}{n\, sin.m\omega},$$

which on account of $n\omega = \pi$ is reduced to $\frac{\pi}{n\, sin.m\omega}$.
Hence if the exponent *n* shall be even or odd, on putting $x = \infty$ we have

$$\int \frac{x^{m-1}dx}{1+x^n} = \frac{\pi}{n\, sin.m\omega} = \frac{\pi}{n\, sin.\frac{m\pi}{n}}.$$

COROLLARY 1

352. Hence the above formula will be recalled (§ 349)

$$\int \frac{z^{n-k-1}dz}{1+z^n} = \int \frac{z^{k-1}dz}{1+z^n} = \frac{\pi}{n\, sin.\frac{(n-k)\pi}{n}} = \frac{\pi}{n\, sin.\frac{k\pi}{n}}$$

on putting $z = \infty$. From which also there follows the following formula, that we have shown to be equal to this equation :

$$\int \frac{x^{n-k-1}dx}{(1-x^n)^{\frac{(n-k)}{n}}} = \int \frac{x^{k-1}dx}{(1-x^n)^{\frac{k}{n}}} = \frac{\pi}{n\, sin.\frac{k\pi}{n}}$$

on putting $x = 1$.

COROLLARY 2

353. We can run through the simpler cases for each kind of formula on putting $z = \infty$ and $x = 1$:

$$\int \frac{dz}{1+zz} = \int \frac{dx}{\sqrt{(1-xx)}} = \frac{\pi}{2\, sin.\frac{1}{2}\pi} = \frac{\pi}{2},$$

$$\int \frac{dz}{1+z^3} = \int \frac{zdz}{1+z^3} = \int \frac{dx}{\sqrt[3]{(1-x^3)}} = \int \frac{xdx}{\sqrt[3]{(1-x^3)^2}} = \frac{\pi}{3 sin.\frac{1}{3}\pi} = \frac{2\pi}{3\sqrt{3}},$$

$$\int \frac{dz}{1+z^4} = \int \frac{zzdz}{1+z^4} = \int \frac{dx}{\sqrt[4]{(1-x^4)}} = \int \frac{xxdx}{\sqrt[4]{(1-x^4)^3}} = \frac{\pi}{4 sin.\frac{1}{4}\pi} = \frac{\pi}{2\sqrt{2}},$$

$$\int \frac{dz}{1+z^6} = \int \frac{z^4dz}{1+z^6} = \int \frac{dx}{\sqrt[6]{(1-x^6)}} = \int \frac{x^4dx}{\sqrt[6]{(1-x^6)^5}} = \frac{\pi}{6 sin.\frac{1}{6}\pi} = \frac{\pi}{3}.$$

COROLLARY 3

354. Since there shall be

$$\frac{1}{\left(1-x^n\right)^{\frac{k}{n}}}=1+\frac{k}{n}x^n+\frac{k(k+n)}{n\cdot 2n}x^{3n}+\frac{k(k+n)(k+2n)}{n\cdot 2n\cdot 3n}\ x^{3n}+\text{etc.},$$

then on multiplying by $x^{k-1}dx$, then on integrating and on putting $x=1$

$$\frac{\pi}{n\sin.\frac{k\pi}{n}}=\frac{1}{k}+\frac{k}{n(k+n)}+\frac{k(k+n)}{n\cdot 2n(k+2n)}+\frac{k(k+n)(k+2n)}{n\cdot 2n\cdot 3n(k+3n)}+\text{etc.}$$

and on writing $n-k$ in place of k there also becomes

$$\frac{\pi}{n\sin.\frac{k\pi}{n}}=\frac{1}{n-k}+\frac{n-k}{n(2n-k)}+\frac{(n-k)(2n-k)}{n\cdot 2n(3n-k)}+\frac{(n-k)(2n-k)(3n-k)}{n\cdot 2n\cdot 3n(4n-k)}+\text{etc.}$$

SCHOLIUM

355. We have now set out above, those integrations for formulas containing transcending quantities taking particular values, while indeed a certain value is attributed to the variable [as the upper limit of the integrand], thus so that there shall be no further need to examine formulas of this kind here. Hence moreover it is understood that these values of the integral $\int Xdx$ are more noteworthy than others, and which generally can be expressed much more succinctly, corresponding to values of the variable *x* of this kind, for which the function *X* either becomes infinite or goes to zero. Thus the integrations of the formulas $\int\frac{x^{m-1}dx}{\left(1-x^n\right)^{\frac{\mu}{\nu}}}$ and $\int\frac{z^{m-1}dz}{1+z^n}$ receive values that are outstanding before others, if there is made $x=1$ and $z=\infty$, where the denominator of that will vanish, indeed the integral of this becomes infinite. Otherwise everything is worthy of attention, that we have shown here, the value of the formulas of the integrals $\int\frac{z^{m-1}dz}{1+z^n}$ in the case $z=\infty$ as expressed concisely, so that it becomes $\frac{\pi}{n\sin.\frac{m\pi}{n}}$, and the demonstration of this, as it shall be built up by all circuitous routes, deservedly arouses some suspicion, can be constructed in a much easier way, even if the manner is not yet clear. Indeed it has been shown that this demonstration can be reached from an account of the sines of multiple angles ; and since in the *Introductione* [Book I, Ch. XI, § 184] the $\sin.\frac{m\pi}{n}$ can be expressed by a product of an infinite number of factors, soon we will show that the same truth can be deduced in a much easier manner, even if indeed I may not be willing to consider this as the most natural way.

But I have sent these investigations to the following chapter, in which the values of the integrals, which as in this chapter take a certain case, I will show to be expressed by infinite products or from innumerable constant factors ; as hence the considerable aids to analysis are even more bounteous, and many more developments thus can be expected.

CAPUT VIII

DE VALORIBUS INTEGRALIUM QUOS CERTIS TANTUM CASIBUS RECIPIUNT

PROBLEMA 38

330. *Integralis* $\int \frac{x^m dx}{\sqrt{(1-xx)}}$ *valorem, quem posito* $x=1$ *recepit, assignare, integrali scilicet ita determinato, ut evanescat posito* $x=0$.

SOLUTIO

Pro casibus simplicissimis, quibus $m = 0$ vel $m = 1$, habemus posito $x = 1$ post integrationem

$$\int \frac{dx}{\sqrt{(1-xx)}} = \frac{\pi}{2} \quad \text{et} \quad \int \frac{xdx}{\sqrt{(1-xx)}} = 1.$$

Deinde supra § 120 vidimus esse in genere

$$\int \frac{x^{m+1}dx}{\sqrt{(1-xx)}} = \frac{m}{m+1}\int \frac{x^{m-1}dx}{\sqrt{(1-xx)}} - \frac{1}{m+1}x^m\sqrt{(1-xx)}\ ;$$

casu ergo $x = 1$ erit

$$\int \frac{x^{m+1}dx}{\sqrt{(1-xx)}} = \frac{m}{m+1}\int \frac{x^{m-1}dx}{\sqrt{(1-xx)}}$$

unde a simplicissimis ad maiores exponentis m valores progrediendo obtinebimus

$$\begin{aligned}
&\int \frac{x^m dx}{\sqrt{(1-xx)}} = \frac{\pi}{2} \\
&\int \frac{xxdx}{\sqrt{(1-xx)}} = \frac{1}{2}\cdot\frac{\pi}{2} \\
&\int \frac{x^4 dx}{\sqrt{(1-xx)}} = \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{\pi}{2} \\
&\int \frac{x^6 dx}{\sqrt{(1-xx)}} = \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\frac{\pi}{2} \\
&\int \frac{x^8 dx}{\sqrt{(1-xx)}} = \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\frac{7}{8}\cdot\frac{\pi}{2} \\
&\quad\vdots
\end{aligned}$$

$$\begin{aligned}
&\int \frac{xdx}{\sqrt{(1-xx)}} = 1 \\
&\int \frac{x^3 dx}{\sqrt{(1-xx)}} = \frac{2}{3} \\
&\int \frac{x^5 dx}{\sqrt{(1-xx)}} = \frac{2\cdot4}{3\cdot5} \\
&\int \frac{x^7 dx}{\sqrt{(1-xx)}} = \frac{2\cdot4\cdot6}{3\cdot5\cdot7} \\
&\int \frac{x^9 dx}{\sqrt{(1-xx)}} = \frac{2\cdot4\cdot6\cdot8}{3\cdot5\cdot7\cdot9} \\
&\quad\vdots
\end{aligned}$$

$$\int\frac{x^{2n}dx}{\sqrt{(1-xx)}}=\frac{1\cdot3\cdot5\cdots(2n-1)}{2\cdot4\cdot6\cdots2n}\cdot\frac{\pi}{2} \qquad \int\frac{x^{2n+1}dx}{\sqrt{(1-xx)}}=\frac{2\cdot4\cdot6\cdots2n}{3\cdot5\cdot7\cdots(2n+1)}$$

COROLLARIUM 1

331. Integrale ergo $\int\frac{x^{m}dx}{\sqrt{(1-xx)}}$ posito $x = 1$ algebraice exprimitur casibus, quibus exponens m est numerus integer impar, casibus autem, quibus est par, quadraturam circuli involvit; semper enim π designat peripheriam circuli, cuius diameter = 1.

COROLLARIUM 2

332. Si binas postremas formulas in se multiplicemus, prodit

$$\int\frac{x^{2n}dx}{\sqrt{(1-xx)}}\cdot\int\frac{x^{2n+1}dx}{\sqrt{(1-xx)}}=\frac{1}{(2n+1)}\cdot\frac{\pi}{2}$$

posito scilicet $x = 1$, quam veram esse patet, etiamsi n non sit numerus integer.

COROLLARIUM 3

333. Haec ergo aequalitas subsistet, si ponamus $x=z^{\nu}$, iisdem conditionibus, quia sumto $x = 0$ vel $z = 1$. Erit ergo

$$\nu\nu\int\frac{z^{2n\nu+\nu-1}dz}{\sqrt{(1-z^{2\nu})}}\cdot\int\frac{z^{2n\nu+2\nu-1}dz}{\sqrt{(1-z^{2\nu})}}=\frac{1}{(2n+1)}\cdot\frac{\pi}{2}$$

et posito $2n\nu+\nu-1=\mu$ fiet posito $z = 1$

$$\int\frac{z^{\mu}dz}{\sqrt{(1-z^{2\nu})}}\cdot\int\frac{z^{\mu+\nu}dz}{\sqrt{(1-z^{2\nu})}}=\frac{1}{\nu(\mu+1)}\cdot\frac{\pi}{2}.$$

SCHOLION 1

334. Quod tale productum binorum integralium exhiberi queat, eo magis est notatu dignum, quod aequalitas haec subsistit, etiamsi neutra formula neque algebraice neque per π exhiberi queat. Veluti si $\nu = 2$ et $\mu=0$, fit

$$\int\frac{dz}{\sqrt{(1-z^{4})}}\cdot\int\frac{zzdz}{\sqrt{(1-z^{4})}}=\frac{1}{2}\cdot\frac{\pi}{2}=\frac{\pi}{4}$$

similique modo,

$$v=3,\quad \mu=0,\quad \text{fit}\quad \int\frac{dz}{\sqrt{(1-z^6)}}\cdot\int\frac{z^3dz}{\sqrt{(1-z^6)}}=\frac{1}{3}\cdot\frac{\pi}{2}=\frac{\pi}{6},$$

$$v=3,\quad \mu=1,\quad \text{fit}\quad \int\frac{zdz}{\sqrt{(1-z^6)}}\cdot\int\frac{z^4dz}{\sqrt{(1-z^6)}}=\frac{1}{6}\cdot\frac{\pi}{2}=\frac{\pi}{12},$$

$$v=4,\quad \mu=0,\quad \text{fit}\quad \int\frac{dz}{\sqrt{(1-z^8)}}\cdot\int\frac{z^4dz}{\sqrt{(1-z^8)}}=\frac{1}{4}\cdot\frac{\pi}{2}=\frac{\pi}{8},$$

$$v=4,\quad \mu=2,\quad \text{fit}\quad \int\frac{zzdz}{\sqrt{(1-z^8)}}\cdot\int\frac{z^6dz}{\sqrt{(1-z^8)}}=\frac{1}{12}\cdot\frac{\pi}{2}=\frac{\pi}{24},$$

$$v=5,\quad \mu=0,\quad \text{fit}\quad \int\frac{dz}{\sqrt{(1-z^{10})}}\cdot\int\frac{z^5dz}{\sqrt{(1-z^{10})}}=\frac{1}{5}\cdot\frac{\pi}{2}=\frac{\pi}{10},$$

$$v=5,\quad \mu=1,\quad \text{fit}\quad \int\frac{zdz}{\sqrt{(1-z^{10})}}\cdot\int\frac{z^6dz}{\sqrt{(1-z^{10})}}=\frac{1}{10}\cdot\frac{\pi}{2}=\frac{\pi}{20},$$

$$v=5,\quad \mu=2,\quad \text{fit}\quad \int\frac{zzdz}{\sqrt{(1-z^{10})}}\cdot\int\frac{z^7dz}{\sqrt{(1-z^{10})}}=\frac{1}{15}\cdot\frac{\pi}{2}=\frac{\pi}{30},$$

$$v=5,\quad \mu=3,\quad \text{fit}\quad \int\frac{z^3dz}{\sqrt{(1-z^{10})}}\cdot\int\frac{z^8dz}{\sqrt{(1-z^{10})}}=\frac{1}{20}\cdot\frac{\pi}{2}=\frac{\pi}{40},$$

quae theoremata sine dubio omni attentione sunt digna.

SCHOLION 2

335. Facile hinc etiam colligitur valor integralis $\int\frac{x^m dx}{\sqrt{(x-xx)}}$ posito $x = 1$; si enim scribamus $x = zz$, fiat hoc iutegrale $2\int\frac{z^{2m}dz}{\sqrt{(1-zz)}}$, quocirca pro casu $x = 1$ nanciscimur sequentes valores

$$\int\frac{dx}{\sqrt{(x-xx)}}=\pi,\qquad \int\frac{x^4dx}{\sqrt{(x-xx)}}=\frac{1\cdot3\cdot5\cdot7}{2\cdot4\cdot6\cdot8}\cdot\pi,$$

$$\int\frac{xdx}{\sqrt{(x-xx)}}=\frac{1}{2}\cdot\pi,\qquad \int\frac{x^5dx}{\sqrt{(x-xx)}}=\frac{1\cdot3\cdot5\cdot7\cdot9}{2\cdot4\cdot6\cdot8\cdot10}\cdot\pi,$$

$$\int\frac{xxdx}{\sqrt{(x-xx)}}=\frac{1\cdot3}{2\cdot4}\cdot\pi,\qquad \vdots$$

$$\int\frac{x^3dx}{\sqrt{(x-xx)}}=\frac{1\cdot3\cdot5}{2\cdot4\cdot6}\cdot\pi,\qquad \int\frac{x^m dx}{\sqrt{(x-xx)}}=\frac{1\cdot3\cdot5\cdots(2m-1)}{2\cdot4\cdot6\cdots2m}\cdot\pi.$$

Hinc ergo integralium huiusmodi formulas involventium, quae magis sunt complicata, valores, quos posito $x = 1$ recipiunt, per series succincte exprimi possunt, quem usum aliquot exemplis declaremus.

EXEMPLUM 1

336. *Valorem integralis* $\int \frac{dx}{\sqrt{(1-x^4)}}$ *posito* $x = 1$ *per seriem exhibere.*

Integrali detur haec forma

$$\int \frac{dx}{\sqrt{(1-xx)}} \cdot (1+xx)^{-\frac{1}{2}},$$

ut habeamus

$$\int \frac{dx}{\sqrt{(1-x^4)}} = \int \frac{dx}{\sqrt{(1-xx)}} \left(1 - \frac{1}{2}xx + \frac{1\cdot 3}{2\cdot 4}x^4 - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}x^6 + \frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8}x^8 - \text{etc.}\right);$$

singulis ergo terminis pro casu $x = 1$ integratis orietur

$$\int \frac{dx}{\sqrt{(1-x^4)}} = \frac{\pi}{2}\left(1 - \frac{1}{4} + \frac{1\cdot 9}{4\cdot 16} - \frac{1\cdot 9\cdot 25}{4\cdot 16\cdot 36} + \frac{1\cdot 9\cdot 25\cdot 49}{4\cdot 16\cdot 36\cdot 64} - \text{etc.}\right).$$

COROLLARIUM

337. Simili modo pro eodem casu $x = 1$ reperitur

$$\int \frac{xdx}{\sqrt{(1-x^4)}} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \text{etc.} = \frac{\pi}{4},$$

$$\int \frac{xxdx}{\sqrt{(1-x^4)}} = \frac{\pi}{2}\left(\frac{1}{2} - \frac{1^2\cdot 3}{2^2\cdot 4} + \frac{1^2\cdot 3^2\cdot 5}{2^2\cdot 4^2\cdot 6} - \frac{1^2\cdot 3^2\cdot 5^2\cdot 7}{2^2\cdot 4^2\cdot 6^2\cdot 8} + \text{etc.}\right),$$

$$\int \frac{x^3dx}{\sqrt{(1-x^4)}} = \frac{2}{3} - \frac{4}{3\cdot 5} + \frac{6}{5\cdot 7} - \frac{8}{7\cdot 9} + \frac{10}{9\cdot 11} - \text{etc.};$$

est autem

$$\int \frac{x^3dx}{\sqrt{(1-x^4)}} = \frac{1}{2} - \frac{1}{2}\sqrt{(1-x^4)}$$

ideoque $= \frac{1}{2}$ posito $x = 1$, unde haec postrema series est $= \frac{1}{2}$.

EXEMPLUM 2

338. *Valorem integralis* $\int dx\sqrt{\frac{1+axx}{1-xx}}$ *casu* $x = 1$ *per seriem exhibere.*

Cum sit

$$\sqrt{(1+axx)} = 1 + \frac{1}{2}axx - \frac{1\cdot 1}{2\cdot 4}a^2x^4 + \frac{1\cdot 1\cdot 3}{2\cdot 4\cdot 6}a^3x^6 - \text{etc.},$$

erit per $\int \frac{dx}{\sqrt{(1-xx)}}$ multiplicando et integrando

$$\int dx\sqrt{\frac{1+axx}{1-xx}} = \frac{\pi}{2}\left(1+\frac{1\cdot 1}{2\cdot 2}a-\frac{1\cdot 1\cdot 1\cdot 3}{2\cdot 2\cdot 4\cdot 4}a^2+\frac{1\cdot 1\cdot 1\cdot 3\cdot 5}{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6}a^3-\text{etc.}\right),$$

unde peripheriam ellipsis cognoscere licet.

EXEMPLUM 3

339. *Valorem integralis* $\int \frac{dx}{\sqrt{x(1-xx)}}$ *casu* $x = 1$ *per seriem exhibere.*

Repraesentetur haec formula ita $\int \frac{dx(1+x)^{-\frac{1}{2}}}{\sqrt{(x-xx)}}$, ut sit

$$\int \frac{dx}{\sqrt{(x-xx)}}\left(1-\frac{1}{2}x+\frac{1\cdot 3}{2\cdot 4}x^3-\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}x^3+\text{etc.}\right),$$

unde series haec obtinetur

$$\int \frac{dx}{\sqrt{x(x-xx)}} = \pi\left(1-\frac{1}{4}+\frac{1\cdot 9}{4\cdot 16}-\frac{1\cdot 9\cdot 25}{4\cdot 16\cdot 36}+\text{etc.}\right),$$

quae ab exemplo primo haud differt; quod non mirum, cum posito $x = zz$ haec formula ad illam reducatur.

PROBLEMA 39

340. *Valorem, integralis* $\int x^{m-1}dx(1-xx)^{n-\frac{1}{2}}$, *quod posito* $x = 0$ *evanescat*,[*casu* $x = 1$] *definire.*

SOLUTIO

Reductiones supra § 118 datae praebent pro hoc casu

$$\int x^{m-1}dx(1-xx)^{\frac{\mu}{2}+1} = \frac{x^m(1-xx)^{\frac{\mu}{2}+1}}{m+\mu+2}+\frac{\mu+2}{m+\mu+2}\int x^{m-1}dx(1-xx)^{\frac{\mu}{2}} \;;$$

sumto ergo $\mu = 2n-1$ erit

$$\int x^{m-1}dx(1-xx)^{n+1} = \frac{2n+2}{m+2n+1}\int x^{m-1}dx(1-xx)^{n-\frac{1}{2}} \;;$$

posito $x = 1$. Cum igitur in praecedente problemate valor $\int \frac{x^{m-1}dx}{\sqrt{(1-xx)}}$ sit assignatus, quam brevitatis gratia ponamus = M, hinc ad sequentes progrediamur

$$\int\frac{x^{m-1}dx}{\sqrt{(1-xx)}}=M,$$

$$\int x^{m-1}dx(1-xx)^{\frac{1}{2}}=\frac{1}{m+1}M,$$

$$\int x^{m-1}dx(1-xx)^{\frac{3}{2}}=\frac{1\cdot 3}{(m+1)(m+3)}M,$$

$$\int x^{m-1}dx(1-xx)^{\frac{5}{2}}=\frac{1\cdot 3\cdot 5}{(m+1)(m+3)(m+5)}M$$

et in genere

$$\int x^{m-1}dx(1-xx)^{n-\frac{1}{2}}=\frac{1\cdot 3\cdot 5\cdots(2n-1)}{(m+1)(m+3)(m+5)\cdots(m+2n-1)}M.$$

Iam duo casus sunt perpendendi, prout $m-1$ est vel numerus par vel impar; si enim $m-1$ sit par, erit

$$M=\frac{1\cdot 3\cdot 5\cdots(m-2)}{2\cdot 4\cdot 6\cdots(m-1)}\cdot\frac{\pi}{2};$$

m - 1 sit impar, erit

$$M=\frac{2\cdot 4\cdot 6\cdots(m-2)}{3\cdot 5\cdot 7\cdots(m-1)}.$$

Hinc sequentes deducuntur valores

$$\int dx\sqrt{(1-xx)}=\frac{\pi}{4}$$

$$\int xxdx\sqrt{(1-xx)}=\frac{1}{4}\cdot\frac{\pi}{4}$$

$$\int x^4dx\sqrt{(1-xx)}=\frac{1\cdot 3}{4\cdot 6}\cdot\frac{\pi}{4}$$

$$\int x^6dx\sqrt{(1-xx)}=\frac{1\cdot 3\cdot 5}{4\cdot 6\cdot 8}\cdot\frac{\pi}{4}$$

$$\int xdx\sqrt{(1-xx)}=\frac{1}{3}$$

$$\int x^3dx\sqrt{(1-xx)}=\frac{1}{3}\cdot\frac{2}{5}$$

$$\int x^5dx\sqrt{(1-xx)}=\frac{1}{3}\cdot\frac{2}{5}\cdot\frac{4}{7}$$

$$\int x^7dx\sqrt{(1-xx)}=\frac{1}{3}\cdot\frac{2\cdot 4\cdot 6}{5\cdot 7\cdot 9}$$

$$\int(1-xx)^{\frac{3}{2}}dx=\frac{3\pi}{16}$$

$$\int xx(1-xx)^{\frac{3}{2}}dx=\frac{1}{6}\cdot\frac{3\pi}{16}$$

$$\int x^4(1-xx)^{\frac{3}{2}}dx=\frac{1\cdot 3}{6\cdot 8}\cdot\frac{3\pi}{16}$$

$$\int x^6(1-xx)^{\frac{3}{2}}dx=\frac{1\cdot 3\cdot 5}{6\cdot 8\cdot 10}\cdot\frac{3\pi}{16}$$

$$\int xdx(1-xx)^{\frac{3}{2}}=\frac{1}{5}$$

$$\int x^3dx(1-xx)^{\frac{3}{2}}=\frac{1}{5}\cdot\frac{2}{7}$$

$$\int x^5dx(1-xx)^{\frac{3}{2}}=\frac{1}{5}\cdot\frac{2\cdot 4}{7\cdot 9}$$

$$\int x^7dx(1-xx)^{\frac{3}{2}}=\frac{1}{5}\cdot\frac{2\cdot 4\cdot 6}{7\cdot 9\cdot 11}$$

$$\int (1-xx)^{\frac{5}{2}}dx = \frac{5\pi}{32}$$
$$\int x^2(1-xx)^{\frac{5}{2}}dx = \frac{1}{8}\cdot\frac{5\pi}{32}$$
$$\int x^4(1-xx)^{\frac{5}{2}}dx = \frac{1\cdot 3}{8\cdot 10}\cdot\frac{5\pi}{32}$$
$$\int x^6(1-xx)^{\frac{5}{2}}dx = \frac{1\cdot 3\cdot 5}{8\cdot 10\cdot 12}\cdot\frac{5\pi}{32}$$

$$\int xdx(1-xx)^{\frac{5}{2}} = \frac{1}{7}$$
$$\int x^3dx(1-xx)^{\frac{5}{2}} = \frac{1}{7}\cdot\frac{2}{9}$$
$$\int x^5dx(1-xx)^{\frac{5}{2}} = \frac{1}{7}\cdot\frac{2\cdot 4}{9\cdot 11}$$
$$\int x^7dx(1-xx)^{\frac{5}{2}} = \frac{1}{7}\cdot\frac{2\cdot 4\cdot 6}{9\cdot 11\cdot 13}$$

etc.

PROBLEMA 40

341. *Valores integralium* $\int \frac{x^m dx}{\sqrt[3]{(1-x^3)}}$ et $\int \frac{x^m dx}{\sqrt[3]{(1-x^3)^2}}$ *determinatorum, ut posito* $x=1$ *assignare.*

SOLUTIO

Ponamus pro casibus simplicissimis

$$\int \frac{dx}{\sqrt[3]{(1-x^3)}} = A, \quad \int \frac{xdx}{\sqrt[3]{(1-x^3)}} = B, \quad \int \frac{xxdx}{\sqrt[3]{(1-x^3)}} = C,$$
$$\int \frac{dx}{\sqrt[3]{(1-x^3)^2}} = A', \quad \int \frac{xdx}{\sqrt[3]{(1-x^3)^2}} = B', \quad \int \frac{xxdx}{\sqrt[3]{(1-x^3)^2}} = C'$$

ex reductione prima § 118 posito $a=1$ et $b=-1$ pro casu $x=1$ habemus

$$\int x^{m+n-1}dx\left(1-x^n\right)^{\frac{\mu}{\nu}} = \frac{m\nu}{m\nu+n\mu+n\nu}\int x^{m-1}dx\left(1-x^n\right)^{\frac{\mu}{\nu}} \;;$$

ergo pro priori, ubi $n=3$, $\nu=3$ et $\mu=-1$,

$$\int x^{m+2}dx\left(1-x^3\right)^{-\frac{1}{3}} = \frac{m}{m+2}\int x^{m-1}dx\left(1-x^3\right)^{-\frac{1}{3}}$$

at pro posteriori, ubi $n=3$, $\nu=3$ et $\mu=-2$,

$$\int x^{m+2}dx\left(1-x^3\right)^{-\frac{2}{3}} = \frac{m}{m+1}\int x^{m-1}dx\left(1-x^3\right)^{-\frac{2}{3}};$$

hinc obtinemus pro forma priori

$$\int\frac{dx}{\sqrt[3]{(1-x^3)}}=A \qquad \int\frac{xdx}{\sqrt[3]{(1-x^3)}}=B \qquad \int\frac{xxdx}{\sqrt[3]{(1-x^3)}}=C$$

$$\int\frac{x^3dx}{\sqrt[3]{(1-x^3)}}=\frac{1}{3}A \qquad \int\frac{x^4dx}{\sqrt[3]{(1-x^3)}}=\frac{2}{4}B \qquad \int\frac{x^5dx}{\sqrt[3]{(1-x^3)}}=\frac{3}{5}C$$

$$\int\frac{x^6dx}{\sqrt[3]{(1-x^3)}}=\frac{1\cdot 4}{3\cdot 6}A \qquad \int\frac{x^7dx}{\sqrt[3]{(1-x^3)}}=\frac{2\cdot 5}{4\cdot 7}B \qquad \int\frac{x^8dx}{\sqrt[3]{(1-x^3)}}=\frac{3\cdot 6}{5\cdot 8}C$$

$$\int\frac{x^9dx}{\sqrt[3]{(1-x^3)}}=\frac{1\cdot 4\cdot 7}{3\cdot 6\cdot 9}A \qquad \int\frac{x^{10}dx}{\sqrt[3]{(1-x^3)}}=\frac{2\cdot 5\cdot 8}{4\cdot 7\cdot 10}B \qquad \int\frac{x^{11}dx}{\sqrt[3]{(1-x^3)}}=\frac{3\cdot 6\cdot 9}{5\cdot 8\cdot 11}C$$

$$\int\frac{x^{12}dx}{\sqrt[3]{(1-x^3)}}=\frac{1\cdot 4\cdot 7\cdot 10}{3\cdot 6\cdot 9\cdot 12}A \qquad \int\frac{x^{13}dx}{\sqrt[3]{(1-x^3)}}=\frac{2\cdot 5\cdot 8\cdot 11}{4\cdot 7\cdot 10\cdot 13}B \qquad \int\frac{x^{14}dx}{\sqrt[3]{(1-x^3)}}=\frac{3\cdot 6\cdot 9\cdot 12}{5\cdot 8\cdot 11\cdot 14}C$$

etc.,

at pro forma posteriori

$$\int\frac{dx}{\sqrt[3]{(1-x^3)^2}}=A' \qquad \int\frac{xdx}{\sqrt[3]{(1-x^3)^2}}=B' \qquad \int\frac{xxdx}{\sqrt[3]{(1-x^3)^2}}=C'$$

$$\int\frac{x^3dx}{\sqrt[3]{(1-x^3)^2}}=\frac{1}{2}A' \qquad \int\frac{x^4dx}{\sqrt[3]{(1-x^3)^2}}=\frac{2}{3}B' \qquad \int\frac{x^5dx}{\sqrt[3]{(1-x^3)^2}}=\frac{3}{4}C'$$

$$\int\frac{x^6dx}{\sqrt[3]{(1-x^3)^2}}=\frac{1\cdot 4}{2\cdot 5}A' \qquad \int\frac{x^7dx}{\sqrt[3]{(1-x^3)^2}}=\frac{2\cdot 5}{3\cdot 6}B' \qquad \int\frac{x^8dx}{\sqrt[3]{(1-x^3)^2}}=\frac{3\cdot 6}{4\cdot 7}C'$$

$$\int\frac{x^9dx}{\sqrt[3]{(1-x^3)^2}}=\frac{1\cdot 4\cdot 7}{2\cdot 5\cdot 8}A' \qquad \int\frac{x^{10}dx}{\sqrt[3]{(1-x^3)^2}}=\frac{2\cdot 5\cdot 8}{3\cdot 6\cdot 9}B' \qquad \int\frac{x^{11}dx}{\sqrt[3]{(1-x^3)^2}}=\frac{3\cdot 6\cdot 9}{4\cdot 7\cdot 10}C'$$

$$\int\frac{x^{12}dx}{\sqrt[3]{(1-x^3)^2}}=\frac{1\cdot 4\cdot 7\cdot 10}{2\cdot 5\cdot 8\cdot 11}A' \qquad \int\frac{x^{13}dx}{\sqrt[3]{(1-x^3)^2}}=\frac{2\cdot 5\cdot 8\cdot 11}{3\cdot 6\cdot 9\cdot 12}B' \qquad \int\frac{x^{14}dx}{\sqrt[3]{(1-x^3)^2}}=\frac{3\cdot 6\cdot 9\cdot 12}{4\cdot 7\cdot 10\cdot 13}C'$$

etc.,

unde concludimus fore generaliter

$$\int\frac{x^{3n}dx}{\sqrt[3]{(1-x^3)}}=\frac{1\cdot 4\cdot 7\cdots(3n-2)}{3\cdot 6\cdot 9\cdots 3n}A \qquad \int\frac{x^{3n}dx}{\sqrt[3]{(1-x^3)^2}}=\frac{1\cdot 4\cdot 7\cdots(3n-2)}{2\cdot 5\cdot 8\cdots(3n-1)}A'$$

$$\int\frac{x^{3n+1}dx}{\sqrt[3]{(1-x^3)}}=\frac{2\cdot 5\cdot 8\cdots(3n-1)}{4\cdot 7\cdot 10\cdots(3n+1)}B \qquad \int\frac{x^{3n+1}dx}{\sqrt[3]{(1-x^3)^2}}=\frac{2\cdot 5\cdot 8\cdots(3n-1)}{3\cdot 6\cdot 9\cdots 3n}B'$$

$$\int\frac{x^{3n+2}dx}{\sqrt[3]{(1-x^3)}}=\frac{3\cdot 6\cdot 9\cdots 3n}{5\cdot 8\cdot 11\cdots(3n+2)}C \qquad \int\frac{x^{3n+2}dx}{\sqrt[3]{(1-x^3)^2}}=\frac{3\cdot 6\cdot 9\cdots 3n}{4\cdot 7\cdot 10\cdots(3n+1)}C'$$

notandum autem est esse $C=\frac{1}{2}$ et $C'=1$.

COROLLARIUM 1

342. Hae formulae variis modis combinari possunt, ut egregia theoremata inde oriantur; erit scilicet

$$\int\frac{x^{3n}dx}{\sqrt[3]{(1-x^3)}}\cdot\int\frac{x^{3n+2}dx}{\sqrt[3]{(1-x^3)^2}}=\frac{AC'}{3n+1}=\frac{1}{3n+1}\int\frac{dx}{\sqrt[3]{(1-x^3)}},$$

$$\int\frac{x^{3n+1}dx}{\sqrt[3]{(1-x^3)}}\cdot\int\frac{x^{3n}dx}{\sqrt[3]{(1-x^3)^2}}=\frac{A'B}{3n+1}=\frac{1}{3n+1}\int\frac{dx}{\sqrt[3]{(1-x^3)^2}}\cdot\int\frac{xdx}{\sqrt[3]{(1-x^3)}},$$

$$\int\frac{x^{3n+2}dx}{\sqrt[3]{(1-x^3)}}\cdot\int\frac{x^{3n+1}dx}{\sqrt[3]{(1-x^3)^2}}=\frac{2B'C}{3n+2}=\frac{1}{3n+2}\int\frac{xdx}{\sqrt[3]{(1-x^3)^2}}.$$

COROLLARIUM 2

343. Quia nunc ratio exponentium ad ternarium non amplius in computum ingreditur, erit generaliter

$$\int\frac{x^{\lambda-1}dx}{\sqrt[3]{(1-x^3)}}\cdot\int\frac{x^{\lambda+1}dx}{\sqrt[3]{(1-x^3)^2}}=\frac{1}{\lambda}\int\frac{dx}{\sqrt[3]{(1-x^3)}},$$

$$\int\frac{x^{\lambda}dx}{\sqrt[3]{(1-x^3)}}\cdot\int\frac{x^{\lambda-1}dx}{\sqrt[3]{(1-x^3)^2}}=\frac{1}{\lambda}\int\frac{xdx}{\sqrt[3]{(1-x^3)}}\cdot\int\frac{dx}{\sqrt[3]{(1-x^3)^2}},$$

$$\int\frac{x^{\lambda}dx}{\sqrt[3]{(1-x^3)}}\cdot\int\frac{x^{\lambda-1}dx}{\sqrt[3]{(1-x^3)^2}}=\frac{1}{\lambda}\int\frac{xdx}{\sqrt[3]{(1-x^3)^2}},$$

quare ex binis postremis consequimur

$$\int\frac{xdx}{\sqrt[3]{(1-x^3)}}\cdot\int\frac{dx}{\sqrt[3]{(1-x^3)^2}}=\int\frac{xdx}{\sqrt[3]{(1-x^3)^2}}.$$

COROLLARIUM 3

344. Ponatur $x=z^n$ et $\lambda n=m$ et nostra theoremata sequentes induent formas

$$\int\frac{z^{m-1}dz}{\sqrt[3]{(1-z^{3n})}}\cdot\int\frac{x^{m+2n-1}dz}{\sqrt[3]{(1-z^{3n})^2}}=\frac{1}{m}\int\frac{z^{m-1}dz}{\sqrt[3]{(1-z^{3n})}},$$

$$\int\frac{z^{m+n-1}dz}{\sqrt[3]{(1-z^{3n})}}\cdot\int\frac{z^{m-1}dz}{\sqrt[3]{(1-z^{3n})^2}}=\frac{n}{m}\int\frac{z^{2n-1}dz}{\sqrt[3]{(1-z^{3n})}}\cdot\int\frac{z^{n-1}dz}{\sqrt[3]{(1-z^{3n})^2}}=\frac{1}{m}\int\frac{z^{2n-1}dz}{\sqrt[3]{(1-z^{3n})^2}}$$

PROBLEMA 41

345. *Dato integrali* $\int \frac{x^{m-1}dx}{\left(1-x^n\right)^{\frac{n-k}{n}}}$ *assignare integrale huius formulae* $\int \frac{x^{m+\lambda n-1}dx}{\left(1-x^n\right)^{\frac{n-k}{n}}}$ *posito* $x=1$.

SOLUTIO

Ut integrale sit finitum, necesse est, ut *m* et *k* sint numeri positivi. Cum igitur per reductionem generalem sit

$$\int x^{m+n-1}dx\left(1-x^n\right)^{\frac{\mu}{\nu}} = \frac{m\nu}{m\nu+n(\mu+\nu)}\int x^{m-1}dx\left(1-x^n\right)^{\frac{\mu}{\nu}},$$

ponatur $\nu=n$ et $\mu=k-n$, ut sit $\mu+\nu=k$; erit

$$\int \frac{x^{m+n-1}dx}{\left(1-x^n\right)^{\frac{n-k}{n}}} = \frac{m}{m+k}\int \frac{x^{m-1}dx}{\left(1-x^n\right)^{\frac{n-k}{n}}}.$$

Ponatur ergo huius formulae valor, quia datur, = *A* haecque reductio repetita continuo dabit posito brevitatis gratia *P* pro $\left(1-x^n\right)^{\frac{n-k}{n}}$

$$\int \frac{x^{m-1}dx}{P} = A,$$

$$\int \frac{x^{m+n-1}dx}{P} = \frac{m}{m+k}A,$$

$$\int \frac{x^{m+2n-1}dx}{P} = \frac{m(m+n)}{(m+k)(m+n+k)}A,$$

$$\int \frac{x^{m+3n-1}dx}{P} = \frac{m(m+n)(m+2n)}{(m+k)(m+n+k)(m+2n+k)}A$$

et generaliter

$$\int \frac{x^{m+\alpha n-1}dx}{P} = \frac{m(m+n)(m+2n)\cdots(m+(\alpha-1)n)}{(m+k)(m+n+k)(m+2n+k)\cdots(m+(\alpha-1)n+k)}A.$$

COROLLARIUM 1

346. Si simili modo alia formula sit

$$\int \frac{x^{p-1}dx}{\left(1-x^n\right)^{\frac{n-q}{n}}} = B$$

posito $x=1$, at brevitatis gratia scribatur *Q* pro $\left(1-x^n\right)^{\frac{n-q}{n}}$, habebimus

$$\int \frac{x^{m+\alpha n-1}dx}{Q} = \frac{p(p+n)(p+2n)\cdots(p+(\alpha-1)n)}{(p+q)(p+n+q)(p+2n+q)\cdots(p+(\alpha-1)n+q)}B,$$

quae totidem atque illa continet factores.

COROLLARIUM 2

347. Statuatur nunc $p=m+k$ ut posterior numerator aequalis fiat priori denominatori, et productum harum duarum formularum est

$$\frac{m(m+n)(m+2n)\cdots(m+(\alpha-1)n)}{(m+k+q)(m+n+k+q)(m+2n+k+q)\cdots(m+(\alpha-1)n+k+q)}AB;$$

fiat porro $m+k+q=m+n$ seu $q=n-k$; erit hoe productum $=\frac{m}{m+\alpha n}AB$ ideoque

$$\int\frac{x^{m+\alpha n-1}dx}{(1-x^n)^{\frac{n-k}{n}}}\cdot\int\frac{x^{m+k+\alpha n-1}dx}{(1-x^n)^{\frac{k}{n}}}=\frac{m}{m+\alpha n}\int\frac{x^{m-1}dx}{(1-x^n)^{\frac{n-k}{n}}}\cdot\int\frac{x^{m+k-1}dx}{(1-x^n)^{\frac{k}{n}}},$$

quod est theorema omni attentione dignum, cum hic non amplius opus sit,
ut a sit numerus integer.

COROLLARIUM 3

348. Quare loco $m+\alpha n$ scribamus μ; erit

$$\mu\int\frac{x^{\mu-1}dx}{(1-x^n)^{\frac{n-k}{n}}}\cdot\int\frac{x^{\mu+k-1}dx}{(1-x^n)^{\frac{k}{n}}}=m\int\frac{x^{m-1}dx}{(1-x^n)^{\frac{n-k}{n}}}\cdot\int\frac{x^{m+k-1}dx}{(1-x^n)^{\frac{k}{n}}}.$$

Hinc si sumamus $m+k=n$ seu $m=n-k$, ob

$$\int\frac{x^{n-1}dx}{(1-x^n)^{\frac{k}{n}}}=\frac{1-(1-x^n)^{\frac{n-k}{n}}}{n-k}=\frac{1}{n-k}$$

posito $x=1$ erit

$$\int\frac{x^{\mu-1}dx}{(1-x^n)^{\frac{n-k}{n}}}\cdot\int\frac{x^{\mu+k-1}dx}{(1-x^n)^{\frac{k}{n}}}=\frac{1}{\mu}\int\frac{x^{n-k-1}dx}{(1-x^n)^{\frac{n-k}{n}}}=\frac{\pi}{\mu n\sin.\frac{k\pi}{n}}.$$

Ac posito $x=z^{\nu}$, tum vero $\mu\nu=p$, $\nu n=q$ et $k=\lambda n$ habebitur

$$\int\frac{z^{p-1}dx}{(1-z^q)^{1-\lambda}}\cdot\int\frac{z^{p+\lambda q-1}dz}{(1-z^q)^{\lambda}}=\frac{\nu}{p}\int\frac{z^{(1-\lambda)q-1}dz}{(1-z^q)^{1-\lambda}}.$$

SCHOLION 1

349. Theoremata particularia, quae hinc consequuntur, ita se habebunt:

$$\text{I. } n=2, k=1;\quad \int\frac{x^{\mu-1}dx}{\sqrt{(1-xx)}}\cdot\int\frac{x^{\mu}dx}{\sqrt{(1-xx)}}=\frac{1}{\mu}\int\frac{dx}{\sqrt{(1-xx)}}=\frac{\pi}{2\mu}$$

$$\text{II. } n=3, k=1;\quad \int\frac{x^{\mu-1}dx}{\sqrt[3]{(1-x^3)^2}}\cdot\int\frac{x^{\mu}dx}{\sqrt[3]{(1-x^3)}}=\frac{1}{\mu}\int\frac{dx}{\sqrt[3]{(1-x^3)^2}}=\frac{2\pi}{3\mu\sqrt{3}}$$

$$n=3, k=2;\quad \int\frac{x^{\mu-1}dx}{\sqrt[3]{(1-x^3)}}\cdot\int\frac{x^{\mu+1}dx}{\sqrt[3]{(1-x^3)^2}}=\frac{1}{\mu}\int\frac{dx}{\sqrt[3]{(1-x^3)}}=\frac{2\pi}{3\mu\sqrt{3}}$$

$$\text{III. } n=4, k=1;\quad \int\frac{x^{\mu-1}dx}{\sqrt[4]{(1-x^4)^3}}\cdot\int\frac{x^{\mu}dx}{\sqrt[4]{(1-x^4)}}=\frac{1}{\mu}\int\frac{dx}{\sqrt[4]{(1-x^4)^3}}=\frac{\pi}{2\mu\sqrt{2}}$$

$$n=4, k=2;\quad \int\frac{x^{\mu-1}dx}{\sqrt{(1-x^4)}}\cdot\int\frac{x^{\mu+1}dx}{\sqrt{(1-x^4)}}=\frac{1}{\mu}\int\frac{dx}{\sqrt{(1-x^4)}}=\frac{\pi}{2\mu\sqrt{2}}$$

$$n=4, k=3;\quad \int\frac{x^{\mu-1}dx}{\sqrt[4]{(1-x^4)}}\cdot\int\frac{x^{\mu+2}dx}{\sqrt[4]{(1-x^4)}}=\frac{1}{\mu}\int\frac{dx}{\sqrt[4]{(1-x^4)}}=\frac{\pi}{2\mu\sqrt{2}}$$

etc.

Ubi notandum est formulam $\int\frac{x^{n-k-1}dx}{(1-x^n)^{\frac{n-k}{n}}}$ ad rationalitatem reduci posse. Ponatur enim $\frac{x^n}{1-x^n}=z^n$ seu $x^n=\frac{z^n}{1+z^n}$, unde $\frac{dx}{x}=\frac{dz}{z(1+z^n)}$. Quare cum formula nostra sit $=\int\left(\frac{x^n}{1-x^n}\right)^{\frac{n-k}{n}}\cdot\frac{dx}{x}$, evadet ea $=\int\frac{z^{n-k-1}dz}{1+z^n}$, cuius integrale ita determinari debet, ut evanescat posito $x = 0$ ideoque $z = 0$; tum vero posito $x = 1$, hoc est $z=\infty$, dabit valorem, quo hic utimur. Mox [§ 352] autem ostendemus valorem huius integralis $\int\frac{z^{n-k-1}dz}{1+z^n}$ posito $z=\infty$ ideoque et huius $\int\frac{x^{n-k-1}dx}{(1-x^n)^{\frac{n-k}{n}}}$ per angulos exprimi posse, quorum valores hic statim apposui. Deinde etiam notari meretur formulae $\int\frac{x^{m-1}dx}{(1-x^n)^{\frac{n-k}{n}}}$ haec transformatio oriunda posito $1-x^n=z^n$, quae praebet $-\int\frac{z^{k-1}dz}{(1-z^n)^{\frac{n-m}{n}}}$ ita integranda, ut evanescat posito $x = 0$ seu $z = 1$; tum vero statui debet $x = 1$ seu $z=0$. Quod eodem redit, ac si mutato signo haec formula $\int\frac{z^{k-1}dz}{(1-z^n)^{\frac{n-m}{n}}}$ ita integretur, ut evanescat posito $z=0$, tum vero ponatur $z = 1$. Cum iam nihil impediat, quominus loco z scribamus x, habebimus hoc insigne theorema

$$\int\frac{x^{n-k-1}dx}{(1-x^n)^{\frac{n-k}{n}}}=\int\frac{x^{k-1}dx}{(1-x^n)^{\frac{k}{n}}},$$

ita ut in huiusmodi formula exponentes *m* et *k* inter se commutare liceat, pro casu scilicet $x = 1$. Ita pro praecedente formula ad rationalitatem reducibili, ubi $m = n - k$, erit

$$\int \frac{x^{m-k-1}dx}{(1-x^n)^{\frac{n-k}{n}}} = \int \frac{x^{k-1}dx}{(1-x^n)^{\frac{k}{n}}},$$

unde sequitur etiam fore posito $z = \infty$

$$\int \frac{z^{n-k-1}dz}{1+z^n} = \int \frac{z^{k-1}dz}{1+z^n}.$$

SCHOLION 2

350. Hinc etiam formularum magis compositarum integralia pro casu $x = 1$ per series concinnas exprimi possunt. Cum enim in reductione superiori posito $m + k = \mu$ seu $k = \mu - m$ sit

$$\int \frac{x^{m+n-1}dx}{(1-x^n)^{\frac{m+n-\mu}{n}}} = \frac{m}{\mu}\int \frac{x^{m-1}dx}{(1-x^n)^{\frac{m+n-\mu}{n}}},$$

si habeatur huiusmodi formula differentialis

$$dy = \frac{x^{m-1}dx}{(1-x^n)^{\frac{m+n-\mu}{n}}}\left(A + Bx^n + Cx^{2n} + Dx^{3n} + \text{etc.}\right),$$

quam ita integrari oporteat, ut *y* evanescat posito $x = 0$, ac requiratur valor ipsius *y* casu $x = 1$, erit, si hoc casu fieri ponamus

$$\int \frac{x^{m+n-1}dx}{(1-x^n)^{\frac{m+n-\mu}{n}}} = O,$$

iste valor

$$= O\left(A + \frac{m}{\mu}B + \frac{m(m+n)}{\mu(\mu+n)}C + \frac{m(m+n)(m+2n)}{\mu(\mu+n)(\mu+2n)}D + \text{etc.}\right).$$

Vicissim ergo proposita hac serie

$$A + \frac{m}{\mu}B + \frac{m(m+n)}{\mu(\mu+n)}C + \frac{m(m+n)(m+2n)}{\mu(\mu+n)(\mu+2n)}D + \text{etc.}$$

eius summa aequabitur huic formulae integrali

$$\frac{1}{O}\int \frac{x^{m-1}dx}{(1-x^n)^{\frac{m+n-\mu}{n}}}\left(A + Bx^n + Cx^{2n} + Dx^{3n} + \text{etc.}\right),$$

si post integrationem ponatur $x=1$. Quodsi ergo eveniat, ut huius seriei $A+Bx^n+Cx^{2n}+Dx^{3n}+\text{etc.}$ summa assignari indeque integratio absolvi queat, obtinebitur summa illius seriei.

PROBLEMA 42

351. *Integralis huius formulae* $\frac{x^{m-1}dx}{1+x^n}$ *ita determinatum, ut posito* $x = 0$ *evanescat, valorem casu* $x=\infty$ *assignare.*

SOLUTIO

Huius formulae integrale iam supra § 77 exhibuimus et quidem ita determinatum, ut posito $x = 0$ evanescat, quod posito brevitatis gratia $\frac{\pi}{n}=\omega$ ita se habet:

$$-\tfrac{2}{n}\cos.m\omega\, l\sqrt{(1-2x\cos.\omega+xx)} + \tfrac{2}{n}\sin.m\omega\cdot \text{Arc. tang.}\tfrac{x\sin.\omega}{1-x\cos.\omega}$$
$$-\tfrac{2}{n}\cos.3m\omega\, l\sqrt{(1-2x\cos.3\omega+xx)} + \tfrac{2}{n}\sin.3m\omega\cdot \text{Arc. tang.}\tfrac{x\sin.3\omega}{1-x\cos.3\omega}$$
$$-\tfrac{2}{n}\cos.5m\omega\, l\sqrt{(1-2x\cos.5\omega+xx)} + \tfrac{2}{n}\sin.5m\omega\cdot \text{Arc. tang.}\tfrac{x\sin.5\omega}{1-x\cos.5\omega}$$
$$\vdots$$
$$-\tfrac{2}{n}\cos.\lambda m\omega\, l\sqrt{(1-2x\cos.\lambda\omega+xx)} + \tfrac{2}{n}\sin.\lambda m\omega\cdot \text{Arc. tang.}\tfrac{x\sin.\lambda\omega}{1-x\cos.\lambda\omega},$$

ubi λ denotat maximum numerum imparem exponente n minorem, ac si n fuerit ipse numerus impar, insuper accedit pars $\pm\frac{1}{n}l(1+x)$, prout m fuerit vel numerus impar vel par; illo scilicet casu signum +, hoc vero signum – valet. Hic igitur quaeritur istius integralis valor, qui prodit posito $x=\infty$. Primo ergo partes logarithmos implicantes expendamus, et quia ob $x=\infty$ est

$$l\sqrt{(1-2x\cos.\lambda\omega+xx)} = l(x-\cos.\lambda\omega) = lx + \left(1-\frac{\cos.\lambda\omega}{x}\right) = lx$$

ob $\frac{\cos.\lambda\omega}{x}=0$, unde partes logarithmicae praebent

$$-\ \frac{2lx}{n}(\cos.m\omega+\cos.3m\omega+\cos.5m\omega+...+\cos.\lambda m\omega)\ \left(\pm\frac{lx}{n}, \text{si } n \text{ impar}\right),$$

ponamus hanc seriem cosinuum

$$\cos.m\omega+\cos.3m\omega+\cos.5m\omega+...+\cos.\lambda m\omega = s$$

eritque per $2\sin.m\omega$ multiplicando

$$2s\, sin.m\omega = sin.2m\omega + sin.4m\omega + sin.6m\omega + ... + sin.(\lambda+1)m\omega,$$
$$- \ sin.2m\omega - sin.4m\omega - sin.6m\omega - ......$$

unde fit

$$s = \frac{sin.(\lambda+1)m\omega}{2\, sin.m\omega}.$$

Quare si n sit numerus par, erit $\lambda = n-1$sicque partes logarithmicae fiunt

$$-\frac{lx}{n}\cdot\frac{sin.nm\omega}{sin.m\omega} = -\frac{lx}{n}\cdot\frac{sin.m\pi}{sin.m\omega}$$

ob $n\omega = \pi$. At propter m numerum integrum est $sin.m\pi = 0$, unde hae partes evanescunt. Sin autem sit n numerus impar, est $\lambda = n-2$ et summa partium logarithmicarum fit

$$-\frac{lx}{n}\cdot\frac{sin.(n-1)m\omega}{sin.m\omega} \pm \frac{lx}{n} \ ;$$

at $sin.(n-1)m\omega = sin.(m\pi - m\omega) = \pm sin.m\omega$, ubi signum superius valet, si m sit numerus impar, contra vero inferius, quod idem de altera ambiguitate est tenendum, ita ut habeamus

$$\mp\frac{lx}{n}\cdot\frac{sin.m\omega}{sin.m\omega} \pm \frac{lx}{n} = 0.$$

Perpetuo ergo partes logarithmicae se mutuo tollunt; quod etiam inde est perspicuum, quod alioquin integrale foret infinitum, cum tamen manifesto debeat esse finitum.

Relinquuntur ergo soli anguli, quos in unam summam colligamus; consideretur ergo Arc.tang $\frac{x\, sin.\lambda\omega}{1-x\, cos.\lambda\omega}$, qui arcus casu $x = 0$ evanescit, tum vero casu $x = \frac{1}{cos.\lambda\omega}$ fit quadrans, ulterius ergo aucta x quadrantem superabit, donec facto $x = \infty$ eius tangens fiat $= -\frac{sin.\lambda\omega}{cos.\lambda\omega} = -\text{tang.}\lambda\omega = \text{tang.}(\pi - \lambda\omega)$ ideoque ipse arcus $= \pi - \lambda\omega$, ex quo hi arcus iunctim sumti dabunt

$$\frac{2}{n}\left((\pi-\omega)sin.m\omega + (\pi-3\omega)sin.3mw + (\pi-5\omega)sin.5m\omega + \ ... \ + (\pi-\lambda\omega)sin.\lambda m\omega\right),,$$

unde duas series adipiscimur

$$\frac{2\pi}{n}\ \left(sin.m\omega + sin.3m\omega + sin.5m\omega + ... + \quad sin.\lambda m\omega\right) = \frac{2\pi}{n}p,$$
$$\frac{-2\omega}{n}\left(sin.m\omega + 3\, sin.3m\omega + 5\, sin.5m\omega + ... + \lambda\, sin.\lambda m\omega\right) = \frac{-2\omega}{n}q,$$

quas seorsim investigemus. Ac pro posteriori quidem cum ante habuissemus

$$cos.m\omega + cos.3m\omega + cos.5m\omega + ... + cos.\lambda m\omega = s = \frac{sin.(\lambda+1)m\omega}{2\, sin.m\omega},$$

si angulum ω ut variabilem spectemus, differentiatio praebet

$$-md\omega\left(\sin.m\omega+3\sin.3m\omega+5\sin.5m\omega+...+\lambda\sin.\lambda m\omega\right)$$
$$=\frac{(\lambda+1)md\omega\cos.(\lambda+1)m\omega}{2\sin.m\omega}-\frac{md\omega\sin.(\lambda+1)m\omega\cos.m\omega}{2\sin.^2 m\omega}\ ;$$

ergo

$$-q=\frac{(\lambda+1)\cos.(\lambda+1)m\omega}{2\sin.m\omega}-\frac{\sin.(\lambda+1)m\omega\cos.m\omega}{2\sin.^2 m\omega}$$

seu

$$-q=\frac{\lambda\cos.(\lambda+1)m\omega}{2\sin.m\omega}-\frac{\sin.\lambda m\omega}{2\sin.^2 m\omega}.$$

Pro altera serie

$$p=\sin.m\omega+\sin.3m\omega+\sin.5m\omega+.....+\sin.\lambda m\omega$$

multiplicemus utrinque per $2\sin.m\omega$ fietque

$$2p\sin.m\omega=1-\cos.2m\omega-\cos.4m\omega-\cos.6m\omega-.....-\cos.(\lambda+1)m\omega$$
$$+\cos.2m\omega+\cos.4m\omega+\cos.6m\omega+....$$

sicque erit

$$p=\frac{1-\cos.(\lambda+1)m\omega}{2\sin.m\omega}.$$

Quodsi iam fuerit *n* numerus par, erit $\lambda=n-1$ indeque

$$\cos.(\lambda+1)m\omega=\cos.m\omega=\cos.m\pi \quad \text{et} \quad \sin.(\lambda+1)m\omega=\sin.m\pi=0,$$

ergo

$$p=\frac{1-\cos.m\pi}{2\sin.m\omega} \quad \text{et} \quad -q=\frac{n\cos.m\pi}{2\sin.m\omega}$$

hincque omnes arcus iunctim sumti

$$\frac{2\pi}{n}\cdot\frac{1-\cos.m\pi}{2\sin.m\omega}+\frac{2\omega}{n}\cdot\frac{n\cos.m\pi}{2\sin.m\omega}=\frac{\pi}{n\sin.m\omega}$$

ob $n\omega=\pi$.
Sit nunc *n* numerus impar; erit $\lambda=n-2$ indeque

$$\cos.(\lambda+1)m\omega=\cos.(m\pi-m\omega) \ \text{ et } \ \sin.(\lambda+1)m\omega=\sin.(m\pi-m\omega)$$

seu

$$\cos.(\lambda+1)m\omega=\cos.m\pi\cos.m\omega \ \text{ et } \ \sin.(\lambda+1)m\omega=-\cos.m\pi\sin.m\omega,$$

ergo

$$p=\frac{1-\cos.m\pi\cos.m\omega}{2\sin.m\omega} \quad \text{et} \quad -q=\frac{(n-1)\cos.m\pi\cos.m\omega}{2\sin.m\omega}+\frac{\cos.m\pi\cos.m\omega}{2\sin.m\omega},$$

unde summa omnium angulorum

$$\frac{\pi(1-\cos.m\pi\cos.m\omega)}{n\sin.m\omega}+\frac{\omega(n-1)\cos.m\pi\cos.m\omega}{n\sin.m\omega}+\frac{\omega\cos.m\pi\cos.m\omega}{n\sin.m\omega},$$

quae ob $n\omega=\pi$ reducitur ad $\frac{\pi}{n\sin.m\omega}$.
Sive ergo exponens *n* sit par sive impar, posito $x=\infty$ habemus

$$\int\frac{x^{m-1}dx}{1+x^n}=\frac{\pi}{n\sin.m\omega}=\frac{\pi}{n\sin.\frac{m\pi}{n}}.$$

COROLLARIUM 1

352. Hinc ergo erit formula supra memorata (§ 349)

$$\int\frac{z^{n-k-1}dz}{1+z^n}=\int\frac{z^{k-1}dz}{1+z^n}=\frac{\pi}{n\sin.\frac{(n-k)\pi}{n}}=\frac{\pi}{n\sin.\frac{k\pi}{n}}$$

posito $z=\infty$. Unde sequitur fore etiam formulam, cui hanc aequari ostendimus,

$$\int\frac{x^{n-k-1}dx}{(1-x^n)^{\frac{(n-k)}{n}}}=\int\frac{x^{k-1}dx}{(1-x^n)^{\frac{k}{n}}}=\frac{\pi}{n\sin.\frac{k\pi}{n}}$$

posito $x=1$.

COROLLARIUM 2

353. Percurramus casus simpliciores pro utroque formularum genere
posito $z=\infty$ et $x=1$:

$$\int\frac{dz}{1+zz}=\int\frac{dx}{\sqrt{(1-xx)}}=\frac{\pi}{2\sin.\frac{1}{2}\pi}=\frac{\pi}{2},$$

$$\int\frac{dz}{1+z^3}=\int\frac{zdz}{1+z^3}=\int\frac{dx}{\sqrt[3]{(1-x^3)}}=\int\frac{xdx}{\sqrt[3]{(1-x^3)^2}}=\frac{\pi}{3\sin.\frac{1}{3}\pi}=\frac{2\pi}{3\sqrt{3}},$$

$$\int\frac{dz}{1+z^4}=\int\frac{zzdz}{1+z^4}=\int\frac{dx}{\sqrt[4]{(1-x^4)}}=\int\frac{xxdx}{\sqrt[4]{(1-x^4)^3}}=\frac{\pi}{4\sin.\frac{1}{4}\pi}=\frac{\pi}{2\sqrt{2}},$$

$$\int\frac{dz}{1+z^6}=\int\frac{z^4dz}{1+z^6}=\int\frac{dx}{\sqrt[6]{(1-x^6)}}=\int\frac{x^4dx}{\sqrt[6]{(1-x^6)^5}}=\frac{\pi}{6\sin.\frac{1}{6}\pi}=\frac{\pi}{3}.$$

COROLLARIUM 3

354. Cum sit

$$\frac{1}{\left(1-x^n\right)^{\frac{k}{n}}}=1+\frac{k}{n}x^n+\frac{k(k+n)}{n\cdot 2n}x^{3n}+\frac{k(k+n)(k+2n)}{n\cdot 2n\cdot 3n}\ x^{3n}+\text{etc.},$$

erit per $x^{k-1}dx$ multiplicando, tum integrando ac $x=1$ ponendo

$$\frac{\pi}{n\,sin.\frac{k\pi}{n}}=\frac{1}{k}+\frac{k}{n(k+n)}+\frac{k(k+n)}{n\cdot 2n(k+2n)}+\frac{k(k+n)(k+2n)}{n\cdot 2n\cdot 3n(k+3n)}+\text{etc.}$$

et loco *k* scribendo $n-k$ erit quoque

$$\frac{\pi}{n\,sin.\frac{k\pi}{n}}=\frac{1}{n-k}+\frac{n-k}{n(2n-k)}+\frac{(n-k)(2n-k)}{n\cdot 2n(3n-k)}+\frac{(n-k)(2n-k)(3n-k)}{n\cdot 2n\cdot 3n(4n-k)}+\text{etc.}$$

SCHOLION

355. Pro formulis quantitates transcendentes continentibus supra iam praecipuos valores, quos integralia, dum variabili certus quidam valor tribuitur, recipiunt, evolvimus, ita ut non opus sit huiusmodi formulas hic denuo examinare. Hinc autem intelligitur eos valores integralis $\int Xdx$ prae reliquis esse notatu dignos ac plerumque multo succinctius exprimi posse, qui eiusmodi valoribus variabilis *x* respondent, quibus functio *X* vel fit infinita vel in nihilum abit. Ita integralia formularum $\int\frac{x^{m-1}dx}{\left(1-x^n\right)^{\frac{\mu}{\nu}}}$ et $\int\frac{z^{m-1}dz}{1+z^n}$ valores prae reliquis memorabiles recipiunt, si fiat $x=1$ et $z=\infty$, ubi illius denominator evanescit, huius vero fit infinitus. Caeterum omni attentione dignum est, quod hic ostendimus, formulae integralis $\int\frac{z^{m-1}dz}{1+z^n}$ valorem casu $z=\infty$ tam concinne exprimi, ut sit $\frac{\pi}{n\,sin.\frac{m\pi}{n}}$, cuius demonstratio cum per tot ambages sit adstructa, merito suspicionem excitat eam via multo faciliori confici posse, etiamsi modus nondum perspiciatur. Id quidem manifestum est hanc demonstrationem ex ratione sinuum angulorum multiplorum peti oportere; et quoniam in *Introductione* $sin.\frac{m\pi}{n}$ per productum infinitorum factorum expressi, mox videbimus inde eandem veritatem multo facilius deduci posse, etiamsi ne hanc quidem viam pro maxime naturali haberi velim.

Sequens autem caput huiusmodi investigationi destinavi, quo valores integralium, quos uti in hoc capite certo quodam casu recipiunt, per producta infinita seu ex innumeris factoribus constantia exprimere docebo; quandoquidem hinc insignia subsidia in Analysin redundant pluraque alia incrementa inde expectari possunt.