CHAPTER III

CONCERNING THE INVESTIGATION OF DIFFERENTIAL EQUATIONS WHICH ARE RENDERED INTEGRABLE BY MULTIPLIERS OF A GIVEN FORM.

PROBLEM 65

493. *To define functions P and Q of x, in order that the differential equation*

$$Pydx+\left(y+Q\right)dy=0$$

is made integrable by the multiplier $\frac{1}{y^3+Myy+Ny}$ *, where M and N are functions of x.*

SOLUTION

Therefore it is by necessity that the differential of the factor *dx*, which is $\frac{Py}{y^3+Myy+Ny}$ from the variability arising from *y*, shall be equal to the differential of the factor of *dy*, which is $\frac{y+Q}{y^3+Myy+Ny}$, while *x* alone it taken as variable. The equality of these values with the common denominator ignored gives :

$$-2Py^3-PMy^2=\left(y^3+Myy+\mathrm{Ny}\right)\frac{dQ}{dx}-\left(y+Q\right)\frac{\left(yydM+ydN\right)}{dx}),$$

which following the ordered powers of *y* gives :

$$\begin{aligned}0=2Py^3dx&+PMy^2dx\\+y^3dQ&+\quad My^2dQ+NydQ\\-y^3dM&-\quad y^2dN\\&-\quad Qy^2dM-QydN\end{aligned}$$

from which the individual powers each are taken to zero, and in the first place we find

$NdQ-QdN=0$ or $\frac{dN}{N}=\frac{dQ}{Q}$, from the integration of which there follows $N=\alpha Q$. Then the two remaining conditions are :

$$\text{I.}\quad 2Pdx + dQ - dM = 0$$

and

$$\text{II.}\quad PMdx + MdQ - \alpha dQ - QdM = O,$$

from which $\text{I}\cdot M - \text{II}\cdot 2$ produces

$$-MdQ - MdM + 2\alpha dQ + 2QdM = 0$$

or

$$dQ + \frac{2QdM}{2\alpha - M} = \frac{MdM}{2\alpha - M},$$

which divided by $(2\alpha - M)^2$ and integrated gives [a factor of 2 has been ignored here, see below]

$$\frac{Q}{(2\alpha - M)^2} = \int \frac{MdM}{(2\alpha - M)^3} = -\int \frac{dM}{(2\alpha - M)^2} + 2\alpha \int \frac{dM}{(2\alpha - M)^3}$$

or

$$\frac{Q}{(2\alpha - M)^2} = \frac{-1}{2\alpha - M} + \frac{\alpha}{(2\alpha - M)^2} + \beta = \frac{M - \alpha}{(2\alpha - M)^2} + \beta$$

Hence there becomes

$$Q = M - \alpha + \beta (2a - M)^2$$

and hence

$$2Pdx = dM - dQ = +2\beta dM (2\alpha - M)$$

and thus any function of x is permitted to be taken for M. Hence there is taken $M = 2\alpha - X$; then

$$Pdx = -\beta XdX \quad \text{and} \quad Q = \alpha - X + \beta XX$$

and

$$N = \alpha\alpha - \alpha X + a\beta XX.$$

On account of which for this equation

$$-\beta yXdX + dy(\alpha - \mathrm{X} + \beta XX + y) = 0$$

we may have this multiplier

$$\frac{1}{y^3 + (2\alpha - X)yy + \alpha(\alpha - X + \beta XX)y},$$

by which that is returned integrable.

COROLLARY 1

494. This form can be attributed to the equation :

$$dy\left(y+A+BV+CVV\right)-\ CyVdV=O$$

and there is put in place

$$\alpha=A,\quad X=-BV,\quad \beta XX=\beta BBVV=CVV$$

hence $\beta=\frac{C}{BB}$, from which the multiplier becomes

$$\frac{1}{y^3+\left(2A+BV\right)yy+A\left(A+BV+CVV\right)y}.$$

COROLLARIUM 2

495. If here there is taken $V=\alpha+x$, an equation similar to that will be obtained, which we have integrated above § 488, and the multiplier also agrees with that which we have given here. But here the multiplier can be put more conveniently in this form :

$$\frac{1}{y\left(y+\mathrm{A}\right)^2+BVy\left(y+A\right)+ACVVy}.$$

COROLLARY 3

496. If we may put $y+A=z$, our equation will become

$$dz\left(z+BV+CVV\right)-C\left(z-A\right)VdV=0,$$

to which the multiplier agrees :

$$\frac{1}{\left(z-A\right)\left(zz+BVz+ACVV\right)},$$

thus so that it may be integrated by itself, this equation becomes :

$$\frac{dz\left(z+BV+CVV\right)-C\left(z-A\right)VdV}{\left(z-A\right)\left(zz+BVz+ACVV\right)}=0.$$

SCHOLIUM

497. Just as here we have assumed the multiplier of this equation $Pydx+\left(y+Q\right)dy=0$ to be equal to $\frac{y^{-1}}{yy+My+N}$, thus to suppose more generally in place of this, we can put $\frac{y^{n-1}}{yy+My+N}$ in order that this equation

$$\frac{Py^ndx+\left(y^n+Qy^{n-1}\right)dy}{yy+My+N}=0$$

must be integrable by itself; with which compared with the form $Rdx+Sdy=0$, so that there shall be $\left(\frac{dR}{dy}\right)=\left(\frac{dS}{dx}\right)$, we will have

$$(n-2)py^{n+1}+(n-1)PMy^{n}+nPNy^{n-1}$$
$$=\ (yy+My+N)y^{n-1}\tfrac{dQ}{dx}-\left(y^{n}+Qy^{n-1}\right)\left(\tfrac{ydM}{dx}+\tfrac{dN}{dx}\right)$$

or, from the ordered equation :

$$\left.\begin{array}{l}(n-2)Py^{n+1}dx+(n-1)PMy^{n}dx+nPNy^{n-1}dx\\ \quad -y^{n+1}dQ\qquad -My^{n}dQ-\quad Ny^{n-1}dQ\\ \quad +\ y^{n+1}dM+\qquad y^{n}dN+\ \ Qy^{n-1}dN\\ \qquad\qquad +\ Qy^{n}dM\end{array}\right\}=0,$$

from which, with the individual members reduced to zero, there becomes :

I. $(n-2)Pdx=dQ-dM$,
II. $(n-1)MPdx=MdQ-QdM-dN$,
III. $nNPdx=NdQ-QdN$.

Let $Pdx=dV$, and there becomes from the first equation $Q=A+M+(n-2)V$, with which value substituted in the second equation there is produced

$$MdV+(n-2)VdM+AdM+dN=0,$$

and the third becomes

$$2NdV+(n-2)VdN+MdN-NdM+AdN=0,$$

from which with dV eliminated there is found

$$(n-2)V+A=\tfrac{MMdN-MNdM-2NdN}{2NdM-MdN}.$$

Now if hence we wish to remove *V*, we might end up with a second order differential equation. Yet the case in which $n=2$ can be brought about.

EXAMPLE

498. *In the development of this case* n = 2, *there shall be this equation*

$$\tfrac{y(Pydx+(y+Q)dy)}{yy+My+N}=0,$$

in order that it ought to be integrable by itself.

And it is required in the first place that $Q=A+M$, for then

$$2ANdM-AMdN=M(MdN-NdM)-2NdN,$$

as hence we must integrate the equation ; which now is unrestrained, and it can be seen how it may be reduced in a more manageable way.
Hence there is put $M = Nu$, so that there becomes

$$MdN - NdM = -NNdu$$

and

$$2NdM - MdN = 2NNdu + NudN\ ,$$

hence

$$2ANNdu + ANudN + N^3udu + 2NdN = 0$$

or

$$\tfrac{2dN}{NN} + \tfrac{AudN}{NN} + \tfrac{2Adu}{N} + udu = 0\,,$$

again there is put in place $\frac{1}{N} = v$ or $N = \frac{1}{v}$; and there will be had

$$-2dv - Audv + 2Avdu + udu = 0$$

or

$$dv - \tfrac{2Avdu}{2+Au} = \tfrac{udu}{2+Au}\,,$$

where the variable v has a single dimension, and on that account it is evident that an integrable equation may be returned, if the equation is divided by $(2+Au)^2$, and there is produced

$$\frac{v}{(2+Au)^2} = \int \frac{udu}{(2+Au)^3} = \frac{C}{AA} - \frac{1+Au}{AA(2+Au)^2}$$

and thus

$$v = \frac{C(2+Au)^2 - 1 - Au}{AA}\,.$$

Hence on assuming for u some function of x then

$$N = \frac{AA}{C(2+Au)^2 - 1 - Au} \quad \text{and} \quad M = \frac{AAu}{C(2+Au)^2 - 1 - Au}$$

and

$$Q = \frac{AC(2+Au)^2 - A}{C(2+Au)^2 - 1 - Au}$$

Now from the third equation we come upon

$$2NPdx = NdQ - QdN \quad \text{or} \quad 2Pdx = Nd.\tfrac{Q}{N}$$

but

$$\tfrac{Q}{N} = \frac{C(2+Au)^2 - 1}{A}, \quad \text{from which} \quad d.\tfrac{Q}{N} = 2Cdu(2+Au)\,,$$

and thus

$$Pdx = \frac{AACdu(2+Au)}{C(2+Au)^2-1-Au}.$$

On account of which our equation by self integration is

$$\frac{AACyydu(2+Au)+ydy\left(C(2+Au)^2 y-(1+Au)y+AC(2+Au)^2-A\right)}{C(2+Au)^2 yy-(1+Au)yy+AAuy+AA} = 0\,,$$

which on putting $Au+2=t$ adopts this form

$$y.\frac{ACytdt+ydy(Ctt-t+1)+Ady(Ctt-1)}{Cttyy-(t-1)yy+A(t-2)y+AA} = 0.$$

Hence moreover on putting

$$A=\alpha,\quad C=\frac{\alpha\gamma}{\beta\beta}\ \text{ and }\ t=-\frac{\beta x}{\alpha}$$

we come upon

$$y\cdot\frac{\alpha\gamma xydx+ydy(\alpha+\beta x+\gamma xx)-ady(\alpha-\gamma xx)}{(\alpha+\beta x+\gamma xx)yy-\alpha(2\alpha+\beta x)y+\alpha^3} = 0.$$

COROLLARY 1

499. Therefore in this manner it is possible to integrate this equation :

$$\alpha\gamma xydx+ydy(a+\beta x+\gamma xx)-\alpha dy(\alpha-\gamma xx)=0,$$

which is not clear at once how it can be reduced to the separation of the variables. But a suitable multiplier is

$$\frac{y}{(a+\beta x+\gamma xx)yy-\alpha(2\alpha+\beta x)y+\alpha^3}.$$

COROLLARIUM 2

500. Here the multiplier can also be expressed in this manner, so that the denominator is resolved into factors :

$$(\alpha+\beta x+\gamma xx)y:\begin{Bmatrix}\left((\alpha+\beta x+\gamma xx)y-\alpha\left(\alpha+\tfrac{1}{2}\beta x\right)+ax\sqrt{\left(\tfrac{1}{4}\beta\beta-\alpha\gamma\right)}\right)\\ \left((\alpha+\beta x+\gamma xx)y-\alpha\left(\alpha+\tfrac{1}{2}\beta x\right)-ax\sqrt{\left(\tfrac{1}{4}\beta\beta-\alpha\gamma\right)}\right)\end{Bmatrix}$$

COROLLARY 3

501. Therefore is we put in place

$$\left(\alpha+\beta x+\gamma xx\right)y-\alpha\left(\alpha+\tfrac{1}{2}\beta x\right)=az\ ,$$

then the multiplier will be :

$$\frac{\alpha+\frac{1}{2}\beta x+z}{\left(z+x\sqrt{\left(\frac{1}{4}\beta\beta-\alpha\gamma\right)}\right)\left(z-x\sqrt{\left(\frac{1}{4}\beta\beta-\alpha\gamma\right)}\right)}.$$

But on account of $y=\frac{\alpha\alpha+\frac{1}{2}\alpha\beta x+\alpha z}{\alpha+\beta x+\gamma xx}$ our equation will be

$$\gamma xydx+dy\left(z+\tfrac{1}{2}\beta x+\gamma xx\right)=0\ .$$

But there is

$$dy=\frac{-\frac{1}{2}\alpha\left(\alpha\beta+4\alpha\gamma x+\beta xx\right)dx-\alpha zdx\left(\beta+2\gamma x\right)+\alpha dz\left(\alpha+\beta x+\gamma xx\right)}{\left(\alpha+\beta x+\gamma xx\right)^{2}};$$

and on the contrary, with this value substituted, there emerges an exceedingly complicated equation.

PROBLEM 66

502. *To find a differential equation of this form*

$$yPdx+\left(Qy+R\right)dy=0\ ,$$

in which P, Q and R shall be functions of X, in order that integrability prevails with that through this multiplier $\frac{y^{m}}{(1+Sy)^{n}}$, *where S is a certain function of x.*

SOLUTION

Because dx is multiplied by $\frac{y^{m+1}P}{(1+Sy)^{n}}$ and dy by $\frac{Qy^{m+1}+Ry^{m}}{(1+Sy)^{n}}$, it is required that

$$\left(m+1\right)Py^{m}\left(1+Sy\right)-nPSy^{m+1}=\frac{(1+Sy)\left(y^{m+1}dQ+y^{m}dR\right)-nydS\left(Qy^{m+1}+Ry^{m}\right)}{dx}\ ,$$

with which equation expanded there becomes

$$\left.\begin{array}{l}(m+1)Py^m dx+(m+1-n)PSy^{m+1}dx-Sy^{m+2}dQ\\ -y^m dR-y^{m+1}dQ+nQy^{m+2}dS\\ -Sy^{m+1}dR\\ +nRy^{m+1}dS\end{array}\right\}=0;$$

hence there becomes

$$Pdx=\tfrac{dR}{m+1} \text{ and } SdQ=nQdS$$

and thus

$$Q=AS^n \text{ and } dQ=nAS^{n-1}dS\,,$$

with which substituted into the middle member there is made :

$$\tfrac{m+1-n}{m+1}SdR-nAS^{n-1}dS-SdR+nRdS=0$$

or

$$-\tfrac{SdR}{m+1}-AS^{n-1}dS+RdS=0$$

and thus

$$dR-\tfrac{(m+1)RdS}{S}=-(m+1)AS^{n-2}dS\,,$$

which divided by S^{m+1} and integrated gives

$$\tfrac{R}{S^{m+1}}=B-\tfrac{(m+1)AS^{n-m-2}}{n-m-2}\,.$$

We may put $A=(m+2-n)C$, so that there shall be

$$Q=(m+2-n)CS^n \text{ and } R=BS^{m+1}+(m+1)CS^{n-1}$$

and thus

$$Pdx=BS^m dS+(n-1)CS^{n-2}dS\,.$$

On account of which we will have this equation

$$ydS\left(BS^m+(n-1)CS^{n-2}\right)+dy\left((m+2-n)CS^n y+BS^{m+1}+(m+1)CS^{n-1}\right)=0,$$

which multiplied by $\frac{y^m}{(1+Sy)^n}$ shall be made integrable, where it is allowed to take some function of *x* for the function *S*.

COROLLARY 1

503. Hence this equation can be integrated :

$$BySmdS + BS^{m+1}dy + (n-1)CyS^{n-2}dS$$
$$+(m+1)CS^{n-1}dy + (m+2-n)CS^{n}ydy = 0,$$

which is resolved at once into these two parts :

$$BS^{m}(ydS + Sdy) + CS^{n-2}\big((n-1)ydS + (m+1)Sdy + (m+2-n)S^{2}ydy\big) = 0,$$

each or which multiplied by $\frac{y^{m}}{(1+Sy)^{n}}$ shall be made integrable.

COROLLARY 2

504. The first part $BS^{m}(ydS + Sdy)$ is returned integrable by this multiplier $\frac{1}{S^{m}}\varphi{:}Sy;$ indeed this formula $B(ydS + Sdy)\varphi{:}Sy$ is integrable by itself [*i.e.* it is a total derivative]. From which the multiplier for this part shall be $S^{\lambda-m}y^{\lambda}(1+Sy)^{\mu}$, which certainly contains the assumed [multiplier] $\frac{y^{m}}{(1+Sy)^{n}}$ if indeed there is taken $\lambda = m$ and $\mu = -n$. For there is

$$\int \frac{y^{m}}{(1+Sy)^{n}} \cdot BS^{m}(ydS + Sdy) = B\int \frac{v^{m}dv}{(1+v)^{n}}$$

on putting $Sy = v$.

COROLLARY 3

505. For the other part, which on putting $S = \frac{1}{v}$ becomes

$$\frac{C}{v^{n}}\big(-(n-1)ydv + (m+1)vdy + (m+2-n)ydy\big),$$

and we have :

$$-\frac{(n-1)Cy}{v^{n}}\left(dv - \frac{(m+1)vdy}{(n-1)y} - \frac{(m+2-n)dy}{n-1}\right)$$

$$= -\frac{(n-1)Cy^{\frac{m+n}{n-1}}}{v^{n}}\left(y^{\frac{-m-1}{n-1}}dv - \frac{m+1}{n-1}y^{\frac{-m-n}{n-1}}vdy - \frac{m+2-n}{n-1}y^{\frac{-m-1}{n-1}}dy\right)$$

$$= -\frac{(n-1)Cy^{\frac{m+n}{n-1}}}{v^{n}}d.\left(y^{\frac{-m-1}{n-1}}v + y^{\frac{n-m-2}{n-1}}\right).$$

Hence this other part is represented thus :

$$-(n-1)CS^n y^{\frac{m+n}{n-1}} d.\frac{1+Sy}{y^{\frac{m+1}{n-1}}S}.$$

Hence the multiplier returning this part integrable in general will be

$$\frac{1}{S^n y^{\frac{m+n}{n-1}}}\varphi : \frac{1+Sy}{Sy^{\frac{m+1}{n-1}}}$$

COROLLARY 4

506. Hence for the other part the multiplier shall be $\frac{(1+Sy)^{\mu}}{S^{n+\mu}y^{\frac{m+n+\mu(m+1)}{n-1}}}$, from which this part becomes

$$-(n-1)C\frac{(1+Sy)^{\mu}}{S^{\mu}y^{\frac{\mu(m+1)}{n-1}}}d.\frac{1+Sy}{y^{\frac{m+1}{n-1}}},$$

the integral of which is $-\frac{(n-1)CZ^{\mu+1}}{\mu+1}$ on putting $Z=\frac{1+Sy}{y^{\frac{m+1}{n-1}}S}$.

COROLLARY 5

507. Now I show how the multiplier for the first part $S^{\lambda-m}y^{\lambda}(1+Sy)^{\mu}$ may be returned agreeing with the multiplier of the other part, if there is taken $\lambda=m$ and $\mu=-n$, from which there results the common multiplier $\frac{y^m}{(1+Sy)^n}$, and hence on putting

$$Sy=v \text{ and } \frac{1+Sy}{y^{\frac{m+1}{n-1}}S}=z$$

the integral of our equation will be :

$$B\int\frac{v^m dv}{(1+v)^n}+Cz^{1-n}=D \quad \text{or} \quad B\int\frac{v^m dv}{(1+v)^n}+\frac{CS^{n-1}y^{m+1}}{(1+Sy)^{n-1}}=D\cdot$$

SCHOLIUM

508. Hence the equation, that we have learned how to integrate in this problem, can now be treated by the principles established above, and here we indicated a notable use of this method, provided that the multipliers are sought for both parts separately and these are returned to agree between each other.

Also we may put the multipliers to give this form $\frac{y^m}{(1+Sy+Tyy)^n}$, thus so that this equation

$$\frac{y^m\left(yPdx+(Qy+R)dy\right)}{(1+Sy+Tyy)^n}=0$$

should be integrable by itself, and from the calculation as established before we may find:

$$\left.\begin{array}{llll}(m+1)Py^m dx+ & (m+1-n)PSy^{m+1}dx+ & (m+1-2n)PTy^{m+2}dx- & Ty^{m+3}dQ\\ -\ y^m dR- & y^{m+1}dQ- & Sy^{m+2}dQ+ & nQy^{m+3}dT\\ & -Sy^{m+1}dR- & Ty^{m+2}dR & \\ & +\ nRy^{m+1}dS+ & nQy^{m+2}dS & \\ & + & nRy^{m+2}dT & \end{array}\right\}=0,$$

from which from the final member $-TdQ+nQdT=0$ we will conclude that $Q=AT^n$ and from the first $Pdx=\frac{dR}{m+1}$, which values substituted into the two middle parts give

$$RdS-\frac{SdR}{m+1}-AT^{n-1}dT=0$$

and

$$RdT-\frac{2TdR}{m+1}+AT^n dS-AST^{n-1}dT=0,$$

of which the former becomes integrable by itself, if $m=-2$, and indeed the latter can be integrated , if $m=2n-1$;
for there becomes

$$RdT-\frac{TdR}{n}+AT^{n-1}\left(TdS-SdT\right)=0$$

or

$$\frac{nRdT-TdR}{nT^{n-1}}+\frac{A(TdS-SdT)}{TT}=0,$$

the integral of which is

$$\frac{-R}{nT^n}+\frac{AS}{T}=\frac{-B}{n}$$

and hence $R=BT^n+nAT^{n-1}S$. Now in addition the case $m=-1$ is worth mentioning, which we set out with those in the following examples.

EXAMPLE 1

509. *To define this equation* $yPdx+(Qy+R)dy=0$ *, so that it becomes integrable by itself on being multiplied by* $\frac{1}{y(1+Sy+Tyy)^n}$.

On account of $m=-1$ we shall have at once $dR=0$ and thus $R=C$; then as before there is :
$Q=AT^n$ and $dQ=nAT^{n-1}dT$, from which the two remaining determinations shall be :

$$-PSdx-AT^{n-1}dT+CdS=0,$$
$$-2PTdx-AST^{n-1}dT+AT^{n}dS+CdT=0;$$

hence on eliminating *Pdx* there is produced :

$$ASST^{n-1}dT-2AT^{n}dT-AT^{n}SdS+2CTdS-CSdT=0.$$

Here there is put in place $SS=Tv$, so that there is produced

$$2TdS-SdT=TS\left(\tfrac{2dS}{S}-\tfrac{dT}{T}\right)=\tfrac{TSdv}{v}=\tfrac{Tdv\sqrt{T}}{\sqrt{v}},$$

and there shall be

$$\tfrac{1}{2}AT^{n}vdT-2AT^{n}dT-\tfrac{1}{2}AT^{n+1}dv+\tfrac{CTdv\sqrt{T}}{\sqrt{v}}=0$$

or in this way

$$-\tfrac{1}{2}AT^{n+2}d.\tfrac{v-4}{T}+\tfrac{CTdv\sqrt{T}}{\sqrt{v}}=0,$$

the first part of which is returned integrable by the multiplier $\frac{1}{T^{n+2}}\varphi:\ \frac{v-4}{T}$, and the latter part by $\frac{1}{T\sqrt{T}}\varphi:v$, from which the common multiplier shall be $\frac{1}{T(v-4)^{n+\frac{1}{2}}\sqrt{T}}$, and hence this equation of the integral is elicited :

$$\frac{AT^{n-\frac{1}{2}}}{(2n-1)(v-4)^{n-\frac{1}{2}}}+C\int\frac{dv}{(v-4)^{n+\frac{1}{2}}\sqrt{v}}=D,$$

from which *T* is defined by *v*; now then there becomes :

$$S=\sqrt{Tv},\ R=C,\ Q=AT^{n}\ \text{ and }\ Pdx=\tfrac{CdS-AT^{n-1}dT}{S}$$

COROLLARY 1

510. For the case, in which $n=\frac{1}{2}$, on account of $\frac{1}{0}z^0 = lz$, there is had

$$\tfrac{1}{2}Al\frac{T}{(v-4)\sqrt{v}}+C\int\frac{dv}{(v-4)^{n+\frac{1}{2}}\sqrt{v}}=\tfrac{1}{2}D \text{ or } \tfrac{1}{2}Al\frac{T}{(v-4)}-\tfrac{1}{2}Cl\frac{\sqrt{v}+2}{\sqrt{v}-2}=\tfrac{1}{2}D,$$

from which on putting $V=4uu$ and $C=lA$, there shall be

$$l\frac{T}{1-uu}-\lambda l\frac{1+u}{1-u}=\text{Const.}$$

or

$$T=E(1-uu)\left(\frac{1+u}{1-u}\right)^{\lambda}.$$

Hence again

$$S=2u\sqrt{T}=2u\left(\frac{1+u}{1-u}\right)^{\frac{\lambda}{2}}\sqrt{E(1-uu)} \text{ et } R=C=\lambda A;$$

then

$$Q=A\left(\frac{1+u}{1-u}\right)^{\frac{\lambda}{2}}\sqrt{E(1-uu)}$$

and

$$Pdx=\frac{\lambda Adu}{u}+\frac{\lambda AdT}{2T}-\frac{AdT}{2Tu}.$$

But there is

$$\frac{dT}{T}=\frac{-2udu+2\lambda du}{1-uu}$$

Hence

$$Pdx=\frac{Adu(1+\lambda\lambda-2\lambda u)}{1-uu}.$$

On account of which for this equation

$$\frac{Aydu(1+\lambda\lambda-2\lambda u)}{1-uu}+Ady\left(\lambda+y\left(\frac{1+u}{1-u}\right)^{\frac{\lambda}{2}}\sqrt{E(1-uu)}\right)=0$$

the multiplier will be

$$\frac{1}{y\sqrt{\left(1+2uy\left(\frac{1+u}{1-u}\right)^{\frac{\lambda}{2}}\sqrt{E(1-uu)}+Eyy(1-uu)\left(\frac{1+u}{1-u}\right)^{\frac{\lambda}{2}}\right)}}.$$

COROLLARY 2

511. In the case, for which $n=-\frac{1}{2}$, we have

$$-\frac{A(v-4)}{2T}+2C\sqrt{v}=-2D \text{ or } T=\frac{A(v-4)}{4D+4C\sqrt{v}}.$$

We may put $v = 4uu$, in order that there shall be $T = \frac{A(uu-1)}{D+2Cu}$; then there shall be

$$S = 2u\sqrt{T} = 2u\frac{A(uu-1)}{D+2Cu}, \quad R = C, \quad Q = \sqrt{\frac{A(D+2Cu)}{uu-1}}$$

and

$$Pdx = \frac{Cdu}{u} + \frac{CdT}{2T} - \frac{AdT}{2TTu} = \frac{Cdu}{u} + \frac{du(C+Du+Cuu)(Cu^3-3Cu-D)}{u(uu-1)^2(D+2Cu)},$$

from which both the equation as well as multiplier are defined.

EXAMPLE 2

512. *To define the equation* $yPdx + (Qy+R)dy = 0$, *in order that it shall become integrable by itself multiplied by* $\frac{1}{y^3(1+Sy+Tyy)^n}$.

On account of $m = -2$ from above [§ 508], we have

$$RS = \frac{A}{n}T^n + B \text{ or } R = \frac{AT^n}{nS} + \frac{B}{S}$$

which value substituted into the one equation gives

$$\frac{(2n+1)AT^n dT}{nS} - \frac{2AT^{n+1}dS}{nSS} + AT^n dS - AST^{n-1}dT + \frac{BdT}{S} - \frac{2BTdS}{SS} = 0$$

which is distinguished into these three parts :

$$\frac{AS}{nT^n}\left(\frac{(2n+1)T^{2n}dT}{S^2} - \frac{2AT^{n+1}dS}{S^3}\right) + AT^{n+1}\left(\frac{dS}{T} - \frac{SdT}{TT}\right)$$
$$+BS\left(\frac{dT}{SS} - \frac{2TdS}{S^3}\right) = 0$$

or

$$\frac{AS}{nT^n}d.\frac{T^{2n+1}}{SS} + AT^{n+1}d.\frac{S}{T} + BSd.\frac{T}{SS} = 0.$$

We put as an abbreviation

$$\frac{T^{2n+1}}{SS} = p, \quad \frac{S}{T} = q \quad \text{and} \quad \frac{T}{SS} = r$$

there is made $S = \frac{1}{qr}$, $T = \frac{1}{qqr}$, hence $p = \frac{1}{q^{4n}r^{2n-1}}$ and our equation becomes

$$\frac{A}{nq\sqrt{pr}}dp+\frac{A\sqrt{p}}{qqr\sqrt{r}}dq+\frac{B}{qr}dr=0 \text{ or } \frac{A\sqrt{r}}{n\sqrt{p}}dp+A\frac{\sqrt{p}}{q\sqrt{r}}dq+Bdr=0.$$

We shall consider these three parts separately and the first becomes integrable on multiplying by $\frac{\sqrt{p}}{\sqrt{r}}\varphi{:}p$, now the second by $\frac{q\sqrt{r}}{\sqrt{p}}\varphi{:}q$, and the third by $\varphi{:}r$. In order that the two first parts agree, there is put

$$\frac{\sqrt{p}}{\sqrt{r}}p^{\lambda}=\frac{q\sqrt{r}}{\sqrt{p}}q^{\mu}$$

or $p^{\lambda+1}=q^{\mu+1}r$, hence $p=q^{\frac{\mu+1}{\lambda+1}}r^{\frac{1}{\lambda+1}}=q^{-4n}r^{-2n+1}$. Hence there becomes

$$\lambda+1=-\tfrac{1}{2n-1} \text{ and } \mu+1=-4n(\lambda+1)=\tfrac{4n}{2n-1}$$

and thus

$$\mu=\tfrac{2n+1}{2n-1} \text{ et } \lambda=-\tfrac{2n}{2n-1}.$$

Hence the equation can be multiplied by

$$\frac{q^{\frac{4n}{2n-1}}\sqrt{r}}{\sqrt{p}}=q^{2n+\frac{4n}{2n-1}}r^{n}$$

[The editor of the O.O. edition has corrected a mistake made by Euler here, where r^{n+1} was used in the first edition rather than r^{n} ; hence the formulas from §512 to §514 have been altered to accommodate this change.]
and there will be produced

$$\tfrac{A}{n}p^{\lambda}dp+Aq^{\mu}dq+Bq^{2n+\frac{4n}{2n-1}}r^{n}dr=0$$

or

$$Ad.\left(\frac{p^{\lambda+1}}{n(\lambda+1)}+\frac{q^{\mu+1}}{\mu+1}\right)+Bq^{\frac{4nn+2n}{2n-1}}r^{n}dr=0$$

or

$$\frac{(2n-1)A}{4n}d.q^{\frac{4n}{2n-1}}(1-4r)+Bq^{\frac{4nn+2n}{2n-1}}r^{n}dr=0.$$

This is multiplied by $q^{\frac{4\nu n}{2n-1}}(1-4r)^{\nu}$ so that there arises

$$\frac{(2n-1)A}{4n}q^{\frac{4\nu n}{2n-1}}(1-4r)^{\nu}d.q^{\frac{4\nu n}{2n-1}}(1-4r)+Bq^{\frac{4nn+2n+4\nu n}{2n-1}}r^{n}dr(1-4r)^{\nu}=0.$$

Hence there becomes $4\nu+4n+2=0$ or $\nu=-n-\frac{1}{2}$ and both members are able to be integrated and there becomes

$$\frac{(2n-1)A}{4n(v+1)}q^{\frac{4n(v+1)}{2n-1}}(1-4r)^{v+1}+B\int r^n dr(1-4r)^v=\text{Const. };$$

but there is $v+1=-n+\frac{1}{2}=-\frac{2n-1}{2}$ and thus there is had

$$-\frac{A}{2n}q^{-2n}(1-4r)^{-\frac{2n-1}{2}}+B\int\frac{r^n dr}{(1-4r)^{\frac{2n+1}{2}}}=\text{Const.}$$

Hence q is given through r, and there shall be $S=\frac{1}{qr}$, $T=\frac{S}{q}$, then

$$R=\frac{AT^n}{nS}+\frac{B}{S},\quad Q=AT^n \text{ and } Pdx=-dR.$$

COROLLARY 1

513. If there is put $n=-\frac{1}{2}$, then there becomes

$$Aq(1-4r)+2B\sqrt{r}=C$$

$$q=\frac{C-2B\sqrt{r}}{A(1-4r)}\ ;$$

and hence

$$S=\frac{A(1-4r)}{r\left(C-2B\sqrt{r}\right)},\quad T=\frac{A^2(1-4r)^2}{r\left(C-2B\sqrt{r}\right)^2},\quad Q=\frac{\sqrt{r}\left(C-2B\sqrt{r}\right)}{(1-4r)}$$

and

$$R=\frac{Q+nB}{nS}=\frac{B-2Q}{S}=\frac{r\left(B-2C\sqrt{r}\right)\left(C-2B\sqrt{r}\right)}{A(1-4r)^2}$$

or

$$R=\frac{BCr-2\left(B^2+C^2\right)r\sqrt{r}+4BCr^2}{A(1-4r)^2}.$$

COROLLARIUM 2

514. In the same case we may put $r=\frac{1}{4}uu$; then there becomes

$$S=\frac{4A(1-uu)}{uu(C-Bu)},\quad T=\frac{4AA(1-uu)^2}{uu(C-Bu)^2},$$

$$Q=\frac{u(C-Bu)}{2(1-uu)},\qquad R=\frac{uu(B-Cu)(C-Bu)}{4A(1-uu)^2}$$

and hence

$$Pdx = \frac{\left(B^2+C^2\right)\left(3uu+u^4\right)-2BC\left(u+3u^3\right)}{4A(1-uu)^3}du$$

and the equation $yPdx+\left(Qy+R\right)dy=0$ will be integrable, if it is multiplied by

$$\frac{\sqrt{(1+Sy+Tyy)}}{yy}=\frac{1}{yy}\sqrt{\left(1+\frac{4A\left(1-uu\right)y}{uu\left(C-Bu\right)}+\frac{4AA\left(1-uu\right)^2 yy}{uu\left(C-Bu\right)^2}\right)}.$$

EXAMPLE 3

515. *To define the equation* $yPdx+\left(Qy+R\right)dy=0$, *which becomes integrable when multiplied by* $\frac{y^{2n-1}}{(1+Sy+Tyy)^n}$.

Here there is $m=2n-1$, $\quad Q=AT^n$ and $Pdx=\frac{dR}{2n}$, then from above [§ 508] $R=nAT^{n-1}S+BT^n$ and the equation remains

$$RdS-S\tfrac{dR}{2n}-AT^{n-1}dT=0,$$

which with the value found substituted in place of R changes into

$$\begin{gathered}\left(2n-1\right)AT^{n-1}SdS-\left(n-1\right)AT^{n-2}\ SSdT\\ -2AT^{n-1}dT+2BT^{n}dS-BT^{n-1}SdT=0\end{gathered}$$

or

$$\begin{gathered}\left(2n-1\right)ATSdS-\left(n-1\right)ASSdT\\ -2ATdT+2BTTdS-BTSdT=0.\end{gathered}$$

The first member on putting $SS=u$ becomes

$$\left(n-\tfrac{1}{2}\right)ATdu-\left(n-1\right)AudT-2ATdT$$

or

$$\left(n-\tfrac{1}{2}\right)AT\left(du-\frac{(n-1)udT}{\left(n-\frac{1}{2}\right)T}-\frac{2dT}{n-\frac{1}{2}}\right)$$

or

$$\tfrac{1}{2}\left(2n-1\right)AT^{\frac{4n-3}{2n-1}}\left(\frac{du}{T^{\frac{4n-2}{2n-1}}}-\frac{2(n-1)udT}{(2n-1)T^{\frac{4n-3}{2n-1}}}-\frac{4dT}{(2n-1)T^{\frac{4n-2}{2n-1}}}\right)$$

$$=\tfrac{1}{2}\left(2n-1\right)AT^{\frac{4n-3}{2n-1}}d.\left(\frac{u}{T^{\frac{2n-2}{2n-1}}}-4T^{\frac{1}{2n-1}}\right)$$

or

$$\tfrac{1}{2}(2n-1)AT^{\frac{4n-3}{2n-1}}d.T^{\frac{1}{2n-1}}\left(\tfrac{SS}{T}-4\right)+\tfrac{BT^3}{S}d.\tfrac{SS}{T}=0$$

or

$$(2n-1)AT^{\frac{-1}{2n-1}}d.T^{\frac{1}{2n-1}}\left(\tfrac{SS}{T}-4\right)+\tfrac{2BT}{S}d.\tfrac{SS}{T}=0.$$

There is put

$$\tfrac{SS}{T}=p \quad \text{and} \quad d.T^{\frac{1}{2n-1}}\left(\tfrac{SS}{T}-4\right)=q=T^{\frac{1}{2n-1}}(p-4),$$

so that $T^{\frac{1}{2n-1}}=\frac{q}{p-4}$, from which

$$T=\frac{q^{2n-1}}{(p-4)^{2n-1}} \quad \text{and} \quad S=\sqrt{\frac{pq^{2n-1}}{(p-4)^{2n-1}}}.$$

Hence

$$\frac{(2n-1)A(p-q)dq}{q}+\frac{2B\sqrt{q^{2n-1}}}{\sqrt{p(p-4)^{2n-1}}}=0$$

or

$$\frac{(2n-1)Adq}{q^{n+\frac{1}{2}}}+\frac{2Bdp:\sqrt{p}}{(p-4)^{n+\frac{1}{2}}}=0,$$

which integrated gives

$$\frac{-2A}{q^{n-\frac{1}{2}}}+2B\int\frac{dp:\sqrt{p}}{(p-4)^{n+\frac{1}{2}}}=2C,$$

and making $\frac{p}{p-4}=vv$ or $p=\frac{4vv}{vv-1}$ there becomes

$$\frac{-2A}{q^{n-\frac{1}{2}}}-\frac{B}{4^{n-1}}\int dv(vv-1)^{n-1}=C.$$

SCHOLIUM

516. I do not describe these in more detail, because above all in the end I have produced these examples in order that the above method of treating differential equations may be exercised; for in these not very hard cases present themselves, which thus it is clear can be resolved by resolving into parts, as suitable functions may be sought for the individual multipliers, and from these the common multiplier may be defined ; therefore we shall now investigate other kinds of equations which are able to be rendered integrable by multipliers.

PROBLEM 67

517. *To define the functions P, Q, R, S of x, in order that this equation*

$$(Py+Q)dx+ydy=0$$

is returned integrable by this multiplier $(yy+Ry+S)^n$.

SOLUTION

Therefore it is necessary that

$$\left(\frac{d.(Py+Q)(yy+Ry+S)^n}{dy}\right)=\left(\frac{d.y(yy+Ry+S)^n}{dx}\right),$$

from which it is deduced on dividing by $(yy+Ry+S)^{n-1}$

$$P(yy+Ry+S)+n(Py+Q)(2y+R)=\frac{ny(ydR+dS)}{dx}$$

or

$$\left.\begin{array}{l}(2n+1)Pyydx+(n+1)PRydx+PSdx\\ -\ nyydR+\qquad 2n\ Qydx+nQRdx\\ -\qquad\qquad nydS\end{array}\right\}=0.$$

Hence it is therefore inferred, that

$$Pdx=\frac{ndR}{2n+1}\ \text{and}\ \frac{(n+1)RdR}{2n+1}+2Qdx-dS=0,\quad \frac{SdR}{2n+1}+QRdx=0,$$

and again,

$$Qdx=\frac{-SdR}{(2n+1)R}=\frac{-(n+1)RdR}{2(2n+1)}+\frac{dS}{2},$$

hence

$$dS+\frac{2SdR}{(2n+1)R}=\frac{(n+1)RdR}{2n+1},$$

which multiplied by $R^{\frac{2}{2n+1}}$ and integrated gives

$$R^{\frac{2}{2n+1}}S=C+\tfrac{1}{4}R^{\frac{4n+4}{2n+1}},$$

and hence

$$S=\tfrac{1}{4}RR+CR^{\frac{-2}{2n+1}}$$

and

$$Qdx=\frac{-RdR}{4(2n+1)}-\frac{C}{2n+1}R^{\frac{-2n-3}{2n+1}}dR\quad \text{and}\quad Pdx=\frac{ndR}{2n+1},$$

from which equation we may obtain

$$\left(ny-\tfrac{1}{4}R-CR^{\frac{-2n-3}{2n+1}}\right)dR+(2n+1)ydy=0,,$$

which is returned integrable by this multiplier

$$\left(yy+Ry+\tfrac{1}{4}RR+CR^{\frac{-2}{2n+1}}\right)^{n}.$$

COROLLARIUM 1

518. In the case in which $n=-\frac{1}{2}$ there becomes $dR=0$ and $R=A$ and the remaining equations are

$$(n+1)APdx+2nQdx-ndS=0 \text{ and } PSdx+nAQdx=0.$$

Hence

$$Pdx=\frac{AQdx}{2S}=\frac{2Qdx-dS}{A} \text{ and thus } (AA-4S)Qdx=-2SdS$$

or

$$Qdx=\frac{-2SdS}{AA-4S} \text{ and } Pdx=\frac{-AdS}{AA-4S}$$

and thus this equation

$$\frac{(Ay+2S)dS}{4S-AA}+ydy=0$$

is returned integrable by this multiplier $\frac{1}{\sqrt{(yy+Ay+S)}}$.

COROLLARY 2

519. If here we put $A=2a$ et $S=x$, this equation

$$\frac{(ay+x)dx+2ydy(x-aa)}{(x-aa)\sqrt{(yy+2ay+x)}}=0$$

becomes integrable by itself, from which the integral can be found of this equation

$$xdx+aydx+2xydy-2aaydy=0,$$

which divided by $(x-aa)\sqrt{(yy+2ay+x)}$ is made integrable.

COROLLARY 3

520. Towards finding the integral first *x* is taken constant and the integral of the part

$$\frac{2ydy}{\sqrt{(yy+2ay+x)}}$$

is

$$2\sqrt{(yy+2ay+x)}+2al\left(a+y-\sqrt{(yy+2ay+x)}\right)+X\ ;$$

of which with the derivative taken with *y* constant

$$\frac{dx}{\sqrt{(yy+2ay+x)}}-\frac{adx:\sqrt{(yy+2ay+x)}}{a+y\ -\sqrt{(yy+2ay+x)}}+dX$$

if it is equated to the other part of the equation

$$\frac{(ax+y)dx}{(x-aa)\sqrt{(yy+2ay+x)}}\ ,$$

there is found $dX=\frac{adx}{aa-x}$ and $X=-al(aa-x)$. From which the complete integral will be

$$\sqrt{(yy+2ay+x)}+al\,\frac{a+y-\sqrt{(yy+2ay+x)}}{\sqrt{(aa-x)}}=C.$$

COROLLARY 4

521. Worthy of note also is the case $n=-1$, which on writing *a* in place of $C+\frac{1}{4}$ gives this equation $(y+aR)dR+ydy=0$, which divided by $yy+Ry+aRR$ becomes integrable ; moreover this equation is homogeneous.

SCHOLIUM

522. Also the multiplier of the equation $(Py+Q)dx+ydy=0$ is able to be put in place $(y+R)^m(y+S)^n$, and there must become

$$\left(\frac{d.(Py+Q)(y+R)^m(y+S)^n}{dy}\right)=\left(\frac{d.y(y+R)^m(y+S)^n}{dx}\right),$$

from which there is found

$$Pdx(y+R)(y+S)+mdx(Py+Q)(y+S)+ndx(Py+Q)(y+R)$$
$$=\ my(y+S)dR+ny(y+R)dS,$$

which expands out to :

$$\left.\begin{aligned}(m+n+1)Pyydx+(n+1)PRydx+PRSdx\\ -myydR+(m+1)PSydx+mQSdx\\ -\ nyydS+(m+n)Qydx+n\ QRdx\\ -mSydR\\ -nRydS\end{aligned}\right\}=\ 0,$$

from which it is deduced

$$Pdx=\tfrac{mdR+ndS}{m+n+1}\ \text{ and }\ Qdx=\tfrac{-PRSdx}{mS+nR}=\tfrac{-RS(mdR+ndS)}{(m+n+1)(mS+nR)}$$

and hence

$$\tfrac{(mdR+ndS)((n+1)R+(m+1)S)}{(m+n+1)}-\tfrac{(m+n)RS(mdR+ndS)}{(m+n+1)(mS+nR)}-mSdR-nRdS=0$$

or

$$\begin{aligned}&m(n+1)RdR-mnRdS+n(m+1)SdS-mnSdR\\ &-\tfrac{m(m+n)RSdR+n(m+n)RSdS}{mS+nR}=0,\end{aligned}$$

which is reduced to this form

$$\begin{aligned}&(n+1)RRdR+(m-n-1)RSdR\ -mSSdR\\ &+(m+1)SSdS+(n-m-1)RSdS-nRRdS=0\,;\end{aligned}$$

which since it shall be homogeneous, it can be divided by

$$(n+1)R^3+(m-2n-1)R^2S+(n-2m-1)RSS+(m+1)\mathrm{S}^3,$$

or by

$$(R-S)^2\left((n+1)R+(m+1)S\right),$$

in order to become integrable.
But with that equation divided by $R-S$ then it becomes

$$(n+1)RdR+mSdR-nRdS-(m+1)SdS=0\,.$$

It may be divided by $(R-S)\left((n+1)R+(m+1)S\right)$ and resolved into partial fractions

$$\frac{dR}{m+n+2}\left(\frac{m+n+1}{R-S}+\frac{n+1}{(n+1)R+(m+1)S}\right)$$
$$+\frac{dS}{m+n+2}\left(\frac{m+n+1}{S-R}+\frac{m+1}{(n+1)R+(m+1)S}\right)=0$$

or

$$\frac{(m+n+1)(dR-dS)}{R-S}+\frac{(n+1)dR+(m+1)dS}{(n+1)R+(m+1)S}=0$$

from which we may obtain on integration

$$(R-S)^{m+n+1}((n+1)R+(m+1)S)=C.$$

Let $R-S=u$; there becomes

$$(n+1)R+(m+1)S=\frac{C}{u^{m+n+1}}$$

and hence

$$R=\frac{(m+1)u}{m+n+2}+\frac{a}{u^{m+n+1}} \text{ and } S=-\frac{(n+1)u}{m+n+2}+\frac{a}{u^{m+n+1}},$$

then indeed

$$Pdx=\frac{(m-n)du}{m+n+2}-\frac{(m+n)adu}{u^{m+n+2}}$$

and

$$Qdx=\frac{du}{u}\left(\frac{a}{u^{m+n+1}}+\frac{(m+1)u}{m+n+2}\right)\left(\frac{a}{u^{m+n+1}}-\frac{(n+1)u}{m+n+2}\right).$$

COROLLARY 1

523. Hence it is therefore possible to integrate this equation

$$ydy+ydu\left(\frac{m-n}{m+n+2}-\frac{(m+n)a}{u^{m+n+2}}\right)$$
$$+\frac{du}{u}\left(\frac{aa}{u^{2m+2n+2}}+\frac{(m-n)a}{(m+n+2)u^{m+n}}-\frac{(m+1)(n+1)uu}{(m+n+2)^2}\right)=0,$$

certainly which shall be integrable by itself, if it is multiplied by

$$\left(y+\frac{a}{u^{m+n+1}}+\frac{(m+1)u}{m+n+2}\right)^m\left(y+\frac{a}{u^{m+n+1}}-\frac{(n+1)u}{m+n+2}\right)^n.$$

COROLLARIUM 2

524. Let $m=n$ and our equation will be

$$ydy-\frac{2naydu}{u^{2n+2}}+\frac{aadu}{u^{4n+3}}-\frac{1}{4}udu=0,$$

the multiplier of which is $\left(\left(y+\frac{a}{u^{2n+1}}\right)^2-\frac{1}{4}uu\right)^n$. Whereby if we put $y=z-\frac{a}{u^{2n+1}}$, it equation is produced

$$zdz-\frac{adz}{u^{2n+1}}+\frac{azdz}{u^{2n+2}}-\frac{1}{4}udu=0,$$

which becomes integrable multiplied by $\left(zz-\frac{1}{4}uu\right)^n$.

Or there is put $z=\frac{1}{2}y$ and $a=\frac{1}{2}b$; then there will be $ydy-udu-\frac{bdy}{u^{2n+1}}+\frac{bydu}{u^{2n+2}}=0$ and the multiplier $\left(yy-uu\right)^n$.

COROLLARY 3

525. If $m=-n$, this equation is produced :

$$ydy-nydu+\frac{aadu}{u^3}+\frac{1}{4}\left(nn-1\right)udu-\frac{nadu}{u}=0,$$

which multiplied by

$$\left(y+\frac{a}{u}-\frac{1}{2}\left(n+1\right)u\right)^n\left(y+\frac{a}{u}-\frac{1}{2}\left(n-1\right)u\right)^{-n}.$$

is returned integrable.

But on putting $y+\frac{a}{u}=z$ this equation is produced :

$$zdz-nzdu+\frac{1}{4}\left(nn-1\right)udu-\frac{adz}{u}+\frac{azdu}{uu}=0,$$

which here the multiplier returns integrable :

$$\left(y-\frac{1}{2}\left(n+1\right)u\right)^n\left(y-\frac{1}{2}\left(n-1\right)u\right)^{-n}.$$

COROLLARY 4

526. Here we may put $z = uv$ and this equation is had

$$uuvdv + udu\left(vv - nv + \tfrac{1}{4}(nn-1)\right) = adv\ ;$$

which if multiplied by $\left(\frac{v-\frac{1}{2}(nn+1)}{v-\frac{1}{2}(nn-1)}\right)^n$, each member becomes integrable.
For on putting

$$\frac{v-\frac{1}{2}(nn+1)}{v-\frac{1}{2}(nn-1)} = s \text{ or } v = \frac{n+1-(n-1)s}{2(1-s)}$$

there arises

$$\frac{s^{n+1}udu}{(1-s)^2} + \frac{n+1-(n-1)s}{2(1-s)^3}uus^n ds = \frac{as^n ds}{(1-s)^2}\ ,$$

of which the integral is

$$\frac{s^{n+1}uu}{2(1-s)^2} = a\int \frac{s^n ds}{(1-s)^2}.$$

SCHOLIUM

527. Since our equation in general may be returned neater, we may put

$$m = -\lambda - 1 + \mu \text{ and } n = -\lambda - 1 - \mu, \text{ so that it shall be } m + n + 2 = -2\lambda\ ,$$

and the equation becomes

$$ydy - ydu\left(\tfrac{\mu}{\lambda} - 2(\lambda+1)au^{2\lambda}\right) + udu\left(\tfrac{\mu\mu-\lambda\lambda}{4\lambda\lambda} - \tfrac{\mu}{\lambda}au^{2\lambda} + aau^{4\lambda}\right) = 0,$$

which is rendered integrable by this multiplier

$$\left(y + au^{2\lambda+1} - \tfrac{(\mu-\lambda)u}{2\lambda}\right)^{\mu-\lambda-1}\left(y + au^{2\lambda+1} - \tfrac{(\mu+\lambda)u}{2\lambda}\right)^{-\mu-\lambda-1}.$$

There can be put $y + au^{2\lambda+1} = uz$ and this equation arises

$$uzdz - au^{2\lambda+1}dz + du\left(zz - \tfrac{\mu}{\lambda}z + \tfrac{\mu\mu-\lambda\lambda}{4\lambda\lambda}\right) = 0,$$

to which the multiplier corresponds

$$u^{-2\lambda-1}\left(z + \tfrac{\lambda-\mu}{2\lambda}\right)^{\mu-\lambda-1}\left(z - \tfrac{\lambda+\mu}{2\lambda}\right)^{-\mu-\lambda-1}$$

Moreover the integral is found

$$C = a\int dz\left(z+\tfrac{\lambda-\mu}{2\lambda}\right)^{\mu-\lambda-1}\left(z-\tfrac{\lambda+\mu}{2\lambda}\right)^{-\mu-\lambda-1}+\tfrac{1}{2\lambda u^{2\lambda}}\left(z+\tfrac{\lambda-\mu}{2\lambda}\right)^{\mu-\lambda}\left(z-\tfrac{\lambda+\mu}{2\lambda}\right)^{-\mu-\lambda},$$

which hence agrees with this differential equation

$$zdz+\tfrac{du}{u}\left(z+\tfrac{\lambda-\mu}{2\lambda}\right)\left(z-\tfrac{\lambda+\mu}{2\lambda}\right)=au^{2\lambda}dz.$$

PROBLEM 68

528. *To define the functions P, Q, R and X of x, in order that this equation*

$$dy+yydx+Xdx=0$$

may be returned integrable by this multiplier $\frac{1}{Pyy+Qy+R}$.

SOLUTION

Hence there must be

$$\tfrac{1}{dy}d.\tfrac{yy+X}{Pyy+Qy+R}=\tfrac{1}{dx}d.\tfrac{1}{Pyy+Qy+R}$$

and hence

$$2y(Pyy+Qy+R)-(yy+X)(2Py+Q)=\tfrac{-yydP-ydQ-dR}{dx}\ ;$$

hence there must become :

$$\left.\begin{array}{l} Qyydx\ +2Rydx-QXdx \\ +\ yydP-2PXydx+dR \\ \quad +\ ydQ \end{array}\right\}=0.$$

Whereby there is had

$$Q=-\tfrac{dP}{dx}=\tfrac{dR}{Xdx}\quad\text{and}\quad X=-\tfrac{dR}{dP}.$$

Hence on taking dx constant, there is $dQ=-\frac{ddP}{dx}$, from which there is required to become :

$$2Rdx+\tfrac{2PdRdx}{dP}-\tfrac{ddP}{dx}=0\ \text{ or }\ RdP+PdR=\tfrac{dPddP}{2dx^2},$$

the integration of this gives :

$$PR=\tfrac{dP^2}{4dx^2}+C,$$

hence

$$R=\tfrac{dP^2}{4Pdx^2}+\tfrac{C}{P},\ \text{then}\ Q=-\tfrac{dP}{dx}\ \text{ and }\ X=\tfrac{C}{PP}+\tfrac{dP^2}{4PPdx^2}-\tfrac{ddP}{2Pdx^2}.$$

We may put $P = SS$, so that S shall be some function of x, and we will obtain

$$P = SS,\; Q = -\frac{2SdS}{dx},\; R = \frac{C}{SS} + \frac{dS^2}{dx^2} \quad \text{et} \quad X = \frac{C}{S^4} - \frac{ddS}{Sdx^2},$$

with which values taken for self integrability, there will be this equation

$$\frac{dy + yydx + Xdx}{Pyy + Qy + R} = 0.$$

SCHOLIUM

529. This more convenient solution can be put in place, if this form $\frac{P}{yy+2Qy+R}$ is attributed to the multiplier so that there must be

$$\frac{1}{dy} d.\frac{P(yy+X)}{Pyy+2Qy+R} = \frac{1}{dx} d.\frac{P}{Pyy+2Qy+R}$$

from which there arises

$$\left.\begin{array}{r} 2PQyydx + 2PRydx - 2PQXdx \\ -\, yydP - 2PXydx - \quad RdP \\ -\, 2QydP + \quad PdR \\ +\; 2PydQ \qquad\qquad \end{array}\right\} = 0,$$

where from the individual equations $\frac{dP}{P}$ is defined conveniently, clearly

$$\frac{dP}{P} = 2Qdx = \frac{Rdx - Xdx + dQ}{Q} = \frac{dR - 2QXdx}{R}.$$

Hence there is deduced $2Q(R + X)dx = dR$, from which now we may define the element dx itself : $dx = \frac{dR}{2Q(R+X)}$ with which value substituted we obtain

$$\frac{QdR}{R+X} = \frac{(R-X)dR}{2Q(R+X)} + dQ$$

or

$$2QQdR = RdR - XdR + 2QRdQ + 2QXdQ,$$

from which we deduce

$$X = \frac{2QQdR - 2QRdQ - RdR}{2QdQ - dR} \text{ and } R + X = \frac{2(QQ-R)dR}{2QdQ - dR}$$

hence

$$dx = \frac{2QdQ - dR}{4Q(QQ-R)} \text{ and } \frac{dP}{P} = \frac{2QdQ - dR}{4(QQ-R)}$$

and thus $P = A\sqrt{(QQ-R)}$.
Making $QQ - R = S$ and there is found

$$dx = \frac{dS}{4QS}, \quad X = \frac{4QSdQ}{dS} - QQ - S, \quad R = QQ - S$$

and $P = A\sqrt{S}$. On account of which we have this equation :

$$dy + \frac{yydS}{4QS} + dQ - \frac{(QQ+S)dS}{4QS} = 0,$$

which is returned integrable by this multiplier

$$\frac{\sqrt{S}}{yy + 2Qy + QQ - S} = \frac{\sqrt{S}}{(y+Q)^2 - S}.$$

Towards finding the integral of this Q and S are taken as constant and there is produced

$$\int \frac{dy\sqrt{S}}{(y+Q)^2 - S} = \frac{1}{2} l \frac{y+Q-\sqrt{S}}{y+Q+\sqrt{S}} + V$$

with a certain function V present of S or Q. Now this form may be differentiated with y assumed constant and there is produced

$$\frac{dQ\sqrt{S} - \frac{(Q+y)dS}{2\sqrt{S}}}{(y+Q)^2 - S} + dV = \frac{yydS + 4QSdQ - QQdS - SdS}{4Q\left((y+Q)^2 - S\right)\sqrt{S}}$$

and thus

$$dV = \frac{yydS + 2QydS + QQdS - SdS}{4Q\left((y+Q)^2 - S\right)\sqrt{S}} = \frac{dS}{4Q\sqrt{S}}.$$

From which equation our integral is

$$\frac{1}{2} l \frac{y+Q-\sqrt{S}}{y+Q+\sqrt{S}} + \frac{1}{4}\int \frac{dS}{Q\sqrt{S}} = C.$$

COROLLARY 1

530. There is a singular case, in which $R = QQ$; for there arises [from the original expansion in the Scholium §529]

$$\frac{dP}{P} = 2Qdx = \frac{QQdx - Xdx + dQ}{Q} = \frac{2dQ - 2Xdx}{Q},$$

from which we can elicit these two equations

$$QQdx + Xdx - dQ = 0 \text{ and } QQdx + Xdx - dQ = 0;$$

which since they agree between themselves, there shall be

$$Xdx = dQ - QQdx \text{ and } lP = 2\int Qdx.$$

COROLLARY 2

531. Hence with Q taken negative, so that we may have this equation [on substituting the above form of Xdx into the original problem in §528] :

$$dy + yydx - dQ - QQdx = 0,$$

this is returned integrable by this multiplier $\frac{e^{-2\int Qdx}}{(y-Q)^2}$,

[arising from $\frac{P}{yy+2Qy+R}$ on putting $lP = 2\int Qdx$ with Q made negative, and $R = QQ$.]

And the integral will be

$$\frac{-1}{y-Q}e^{-2\int Qdx} + V = \text{Const.},$$

where V is a function of x, towards defining which this equation is differentiated with y assumed constant :

$$\frac{-dQ}{(y-Q)^2}e^{-2\int Qdx} + \frac{2Qdx}{y-Q}e^{-2\int Qdx} + dV = \frac{yydx - dQ - QQdx}{(y-Q)^2}e^{-2\int Qdx},$$

from which there becomes

$$V = \int e^{-2\int Qdx}dx,$$

thus in order that the integral shall be

$$\int e^{-2\int Qdx}dx - \frac{e^{-2\int Qdx}}{y-Q} = C.$$

COROLLARY 3

532. Hence with the proposed equation $dy + yydx + Xdx = 0$ if a certain particular integral agrees with $y = Q$, so that there shall be $dQ + QQdx + Xdx = 0$ and thus $dy + yydx - dQ - QQdx = 0$, the multiplier for that shall be $\frac{1}{(y-Q)^2}e^{-2\int Qdx}$ and the complete integral

$$Ce^{2\int Qdx} + \frac{1}{y-Q} = e^{2\int Qdx}\int e^{-2\int Qdx}dx.$$

SCHOLIUM

533. Moreover the equation found in the last scholium [§ 529]

$$dy + \frac{yydS}{4QS} + dQ - \frac{(QQ+S)dS}{4QS} = 0$$

does not have a great deal on going back ; for on putting $Y + Q = z$ there is produced

$$dz - \frac{zdS}{2S} + \frac{dS(zz-S)}{4QS} = 0$$

in which in order that both the first terms may be contained in one , there is put $z = v\sqrt{S}$ and there is found

$$dv\sqrt{S} + \frac{vvdS}{4Q} - \frac{dS}{4Q} = 0 \quad \text{or} \quad \frac{dv}{vv-1} + \frac{dS}{4Q\sqrt{S}} = 0\,;$$

which since it can be separated, the integral will be

$$\frac{1}{2}l\,\frac{1+v}{1-v} = \frac{1}{4}\int\frac{dS}{Q\sqrt{S}},$$

where there is $v = \frac{y+Q}{\sqrt{S}}$.

But the equation found in this solution [§ 528]

$$dy + yydx + \frac{Cdx}{S^4} - \frac{ddS}{Sdx} = 0,$$

where S is some function of x and $\frac{ddS}{dx} = d:\frac{dS}{dx}$, is considered more arduous, while it becomes integrable by itself, if divided by

$$SSyy - \frac{2SydS}{dx} + \frac{dS^2}{dx^2} + \frac{C}{SS} = \left(Sy - \frac{dS}{dx}\right)^2 + \frac{C}{SS}\,.$$

But on taking x constant the integral is found :

$$\frac{1}{\sqrt{C}}\text{Arc. tang.}\frac{SSydx-SdS}{dx\sqrt{C}}+V=\text{Const.};$$

hence now towards finding the function *V*, the differential is taken with *y* put constant, which is

$$\frac{2SydS-\frac{SddS}{dx}-\frac{dS^2}{dx}}{SS\left(Sy-\frac{dS}{dx}\right)^2+C}+dV$$

and it must be equal to the other part

$$\frac{\frac{Cdx}{S^4}-\frac{ddS}{Sdx}+yydx}{\left(Sy-\frac{dS}{dx}\right)^2+\frac{C}{SS}}=\frac{\frac{Cdx}{SS}-\frac{SddS}{dx}+SSyydx}{SS\left(Sy-\frac{dS}{dx}\right)^2+C}.$$

Hence

$$dV=\frac{SSyydx-2SydS+\frac{dS^2}{dx}+\frac{Cdx}{SS}}{SS\left(Sy-\frac{dS}{dx}\right)^2+C}=\frac{dx}{SS}.$$

On account of which the complete integral is

$$\frac{1}{\sqrt{C}}\text{Arc. tang.}\frac{SSydx-SdS}{dx\sqrt{C}}+\int\frac{dx}{SS}=D.$$

But if we assume $S=x$, the complete integral of this equation [§ 491]

$$dy+yydx+\frac{Cdx}{x^4}=0$$

is

$$\frac{1}{\sqrt{C}}\text{Arc.tang.}\frac{xxy-x}{\sqrt{C}}-\frac{1}{x}=D.$$

Moreover if there arises $S=x^n$, on account of $\frac{dS}{dx}=nx^{n-1}$ and $d.\frac{dS}{dx}=n(n-1)x^{n-2}dx$ this equation can be integrated :

$$dy+yydx+\frac{Cdx}{x^{4n}}-\frac{n(n-1)dx}{xx}=0;$$

indeed the integral shall be :

$$\frac{1}{\sqrt{C}}\text{Arc.tang.}\frac{x^{2n}y-nx^{2n-1}}{\sqrt{C}}-\frac{1}{(2n-1)x^{2n-1}}=D.$$

But above [§436] we have found this equation

$$dy+yydx+Cx^m dx=0$$

able to be reduced to separation, as often as $m=\frac{-4i}{2i\pm1}$; hence from the same cases it is allowed to assign a function *S*, so that there becomes

$$\frac{C}{S^4}-\frac{ddS}{Sdx^2}=Cx^m;$$

which since it may relate to differential equations of the second grade, we will not touch on here.

PROBLEM 69

534. *To define functions P and Q of both the variables x and y, in order that the differential equation* $Pdx+Qdy=0$ *divided by* $Px+Qy$ *is made integrable by itself.*

SOLUTION

Since the formula $\frac{Pdx+Qdy}{Px+Qy}$ must be integrable, we may put $Q=PR$, so that we have $\frac{dx+Rdy}{x+Ry}$, and let $dR=Mdx+Ndy$. Whereby it is required to become

$$\frac{1}{dy}d.\frac{1}{x+Ry}=\frac{1}{dx}d.\frac{R}{x+Ry},$$

from which we arrive at

$$\frac{-R-Ny}{(x+Ry)^2}=\frac{Mx-R}{(x+Ry)^2}$$

or $N=-\frac{Mx}{y}$; hence there shall be

$$dR=Mdx-\frac{Mxdy}{y}=My\frac{ydx-xdy}{yy};$$

which formula since it must be integrable, it is necessary that the function *My* shall be of $\frac{x}{y}$, because $\frac{ydx-xdy}{yy}=d.\frac{x}{y}$, and from this integration there is produced $R=\text{funct}.\frac{x}{y}$ or, because it returns with the same, R is a function of zero dimensions of *x* and *y*. On account of which since $\frac{Q}{P}=R$, it is evident for this condition to be satisfied, if *P* and *Q* were homogeneous functions of the same number of dimensions of *x* and *y;* hence in this way we are to follow the same integration of homogeneous equations that we instructed in the above chapter [§ 477].

COROLLARY 1

535. Therefore since $\frac{dt+Rdu}{t+Ru}$ shall be integrable, if there should be $R=f:\frac{t}{u}$ [recall that we have taken notation to mean that *R* is some function f of $\frac{t}{u}$], or $R=\frac{t}{u}f:\frac{t}{u}$ also this formula

$$\frac{\frac{dt}{t}+\frac{du}{u}f:\frac{t}{u}}{1+f:\frac{t}{u}}$$

shall be integrable, which can be represented thus

$$\frac{\frac{dt}{t}+\frac{du}{u}f:\left(\int\frac{dt}{t}-\int\frac{du}{u}\right)}{1+f:\left(\int\frac{dt}{t}-\int\frac{du}{u}\right)}$$

if everywhere the letter f denotes some function of the adjoined quantity.

COROLLARY 2

536. There may be put $\frac{dt}{t}=\frac{dx}{X}$ and $\frac{du}{u}=\frac{dy}{Y}$ and this formula

$$\frac{\frac{dx}{X}+\frac{dy}{Y}f:\left(\int\frac{dx}{X}-\int\frac{dy}{Y}\right)}{1+f:\left(\int\frac{dx}{X}-\int\frac{dy}{Y}\right)}=\frac{dx+\frac{Xdy}{Y}f:\left(\int\frac{dx}{X}-\int\frac{dy}{Y}\right)}{X+Xf:\left(\int\frac{dx}{X}-\int\frac{dy}{Y}\right)}$$

will be integrable by itself [*i.e.* a complete differential]. Whereby on putting

$$R=\frac{X}{Y}f:\left(\int\frac{dx}{X}-\int\frac{dy}{Y}\right)$$

this formula $\frac{dx+Rdy}{X+RY}$ is integrable by itself, X being some function of x and Y of y.

COROLLARY 3

537. Whereby if the functions P and Q are sought, in order that the equation $Pdx+Qdy=0$ becomes integrable, if divided by $PX+QY$ with some function X of x and Y of y, there must be the relation :

$$\frac{Q}{P}=\frac{X}{Y}\text{funct.}\left(\int\frac{dx}{X}-\int\frac{dy}{Y}\right).$$

COROLLARY 4

538. Whereby if the signs φ and ψ indicate some functions, and there should be

$$P=\frac{V}{X}\varphi:\left(\int\frac{dx}{X}-\int\frac{dy}{Y}\right)\quad\text{and}\quad Q=\frac{V}{Y}\psi:\left(\int\frac{dx}{X}-\int\frac{dy}{Y}\right),$$

this equation $Pdx+Qdy=0$ may be returned integrable, if it should be divided by $PX+QY$.

SCHOLIUM

539. Hence innumerable equations can be brought forwards, which can be integrate, even if in general it may be appear with most difficulty, how these are able to be reduced to the separation of the variables. Now this investigation must properly be referred to the second book of the *Integral Calculus*, now the outstanding examples of this are to be had here ; indeed we have defined a function R of the two variables x and y with a certain condition proposed between M and N, evidently $Mx+Ny=0$ or, $x\left(\frac{dR}{dx}\right)+y\left(\frac{dR}{dy}\right)=0$ this is from a certain condition of the differentials.

CAPUT III

DE INVESTIGATIONE AEQUATIONUM DIFFERENTIALIUM QUAE PER MULTIPLICATORES DATAE FORMAE INTEGRABILES REDDANTUR

PROBLEMA 65

493. *Definire functiones P et Q ipsius x, ut aequatio differentialis*

$$Pydx+\left(y+Q\right)dy=0$$

per multiplicatorem $\frac{1}{y^3+Myy+Ny}$ *, ubi M et N sunt functiones ipsius x, fiat integrabilis.*

SOLUTIO

Necesse igitur est, ut factoris ipsius *dx*, qui est $\frac{Py}{y^3+Myy+Ny}$ differentiale ex variabilitate ipsius *y* natum aequale sit differentiali factoris ipsius *dy*, qui est $\frac{y+Q}{y^3+Myy+Ny}$ dum sola *x* variabilis sumitur. Horum valorum aequalium neglecto denominatore communi aequalitas dat

$$-2Py^3-PMy^2=\left(y^3+Myy+\mathrm{Ny}\right)\frac{dQ}{dx}-\left(y+Q\right)\frac{\left(yydM+ydN\right)}{dx}),$$

quae secundum potestates ipsius *y* ordinata praebet

$$\begin{aligned}0=2Py^3dx&+PMy^2dx\\+y^3dQ&+My^2dQ+NydQ\\-y^3dM&-y^2dN\\&-Qy^2dM-QydN\end{aligned}$$

unde singulis potestatibus seorsim ad nihilum perductis nanciscimur primo $NdQ-QdN=0$ seu $\frac{dN}{N}=\frac{dQ}{Q}$, ex cuius integratione sequitur $N=\alpha Q$. Tum binae reliquae conditiones sunt

$$\text{I.}\quad 2Pdx+dQ-dM=0$$

et

$$\text{II.}\quad PMdx+MdQ-\alpha dQ-QdM=O,$$

unde $\mathrm{I}\cdot M-\mathrm{II}\cdot 2$ suppeditat

$$-MdQ-MdM+2\alpha dQ+2QdM=0$$

seu

$$dQ+\frac{2QdM}{2\alpha-M}=\frac{MdM}{2\alpha-M},$$

quae per $(2\alpha-M)^2$ divisa et integrata dat

$$\frac{Q}{(2\alpha-M)^2}=\int\frac{MdM}{(2\alpha-M)^3}=-\int\frac{dM}{(2\alpha-M)^2}\;+2\alpha\int\frac{dM}{(2\alpha-M)^3}$$

seu

$$\frac{Q}{(2\alpha-M)^2}=\frac{-1}{2\alpha-M}+\frac{\alpha}{(2\alpha-M)^2}+\beta=\frac{M-\alpha}{(2\alpha-M)^2}+\beta$$

Erit ergo

$$Q=M-\alpha+\beta(2a-M)^2$$

hincque

$$2Pdx=dM-dQ=+2\beta dM(2\alpha-M)$$

sicque pro M functionem quamcunque ipsius x sumere licet. Capiatur ergo $M=2\alpha-X$; erit

$$Pdx=-\beta XdX \quad \text{et} \quad Q=\alpha-X+\beta XX$$

atque

$$N=\alpha\alpha-\alpha X+a\beta XX.$$

Quocirca pro hac aequatione

$$-\beta yXdX+dy(\alpha-\mathrm{X}+\beta XX+y)=0$$

habemus hunc multiplicatorem

$$\frac{1}{y^3+(2\alpha-X)yy+\alpha(\alpha-X+\beta XX)y},$$

quo ea integrabilis redditur.

COROLLARIUM 1

494. Tribuatur aequationi haec forma

$$dy(y+A+BV+CVV)-CyVdV=O$$

eritque

$$\alpha=A,\quad X=-BV,\quad \beta XX=\beta BBVV=CVV$$

ergo $\beta=\frac{C}{BB}$, unde multiplicator fiet

$$\frac{1}{y^3+(2A+BV)yy+A(A+BV+CVV)y}.$$

COROLLARIUM 2

495. Si hic sumatur $V=\alpha+x$, obtinebitur aequatio similis illi, quam supra § 488 integravimus, et multiplicator quoque cum eo, quem ibi dedimus, convenit. Hic autem multiplicator commodius hac forma exhibetur

$$\frac{1}{y(y+\mathrm{A})^2+BVy(y+A)+ACVVy}.$$

COROLLARIUM 3

496. Si ponamus $y + A = z$, nostra aequatio erit

$$dz\left(z + BV + CVV\right) - C\left(z - A\right)VdV = 0,$$

cui convenit multiplicator

$$\frac{1}{(z-A)(zz+BVz+ACVV)},$$

ita ut per se integrabilis sit haec aequatio

$$\frac{dz(z+BV+CVV)-C(z-A)VdV}{(z-A)(zz+BVz+ACVV)} = 0.$$

SCHOLION

497. Quemadmodum hic aequationis $Pydx + \left(y + Q\right)dy = 0$ multiplicatorem assumsimus $= \frac{y^{-1}}{yy+My+N}$ ita generalius eius loco sumere poterimus $\frac{y^{n-1}}{yy+My+N}$, ut haec aequatio

$$\frac{Py^n dx+\left(y^n+Qy^{n-1}\right)dy}{yy+My+N} = 0$$

per se debeat esse integrabilis, qua comparata cum forma $Rdx + Sdy = 0$,
ut sit $\left(\frac{dR}{dy}\right) = \left(\frac{dS}{dx}\right)$, habebimus

$$\begin{aligned}&\left(n-2\right)py^{n+1} + \left(n-1\right)PMy^n + nPNy^{n-1}\\ =\ &\left(yy + My + N\right)y^{n-1}\frac{dQ}{dx} - \left(y^n + Qy^{n-1}\right)\left(\frac{ydM}{dx} + \frac{dN}{dx}\right)\end{aligned}$$

sive ordinata aequatione

$$\left.\begin{array}{l}\left(n-2\right)Py^{n+1}dx + \left(n-1\right)PMy^n dx + nPNy^{n-1}dx\\ \quad -y^{n+1}dQ \qquad\qquad -My^n dQ - \quad Ny^{n-1}dQ\\ \quad +\ y^{n+1}dM + \qquad\qquad y^n dN + \quad Qy^{n-1}dN\\ \qquad\qquad\qquad\qquad +\ Qy^n dM\end{array}\right\} = 0,$$

unde singulis membris ad nihilum reductis fit

I. $\left(n-2\right)Pdx = dQ - dM$,
II. $\left(n-1\right)MPdx = MdQ - QdM - dN$,
III. $nNPdx = NdQ - QdN$.

Sit $Pdx = dV$ eritque ex prima $Q = A + M + \left(n-2\right)V$, quo valore in secunda substituto prodit

$$MdV+(n-2)VdM+AdM+dN=0,$$

et tertia fit

$$2NdV+(n-2)VdN+MdN-NdM+AdN=0,$$

unde eliminando dV reperitur

$$(n-2)V+A=\frac{MMdN-MNdM-2NdN}{2NdM-MdN}.$$

Verum si hinc vellemus V elidere, in aequationem differentio-differentialem illaberemur. Casus tamen, quo $n=2$, expediri potest.

EXEMPLUM

498. *Sit in evolutione huius casus* $n=2$, *ut per se integrabilis esse debeat haec aequatio*

$$\frac{y(Pydx+(y+Q)dy)}{yy+My+N}=0.$$

Ac primo esse oportet $Q=A+M$, tum vero

$$2ANdM-AMdN=M(MdN-NdM)-2NdN,$$

quam ergo aequationem integrare debemus; quae cum in nulla iam tractatarum contineatur, videndum est, quomodo tractabilior reddi queat.
Ponatur ergo $M=Nu$, ut fiat

$$MdN-NdM=-NNdu$$

et

$$2NdM-MdN=2NNdu+NudN,$$

hinc

$$2ANNdu+ANudN+N^3udu+2NdN=0$$

sive

$$\frac{2dN}{NN}+\frac{AudN}{NN}+\frac{2Adu}{N}+udu=0,$$

statuatur porro $\frac{1}{N}=v$ seu $N=\frac{1}{v}$, habebitur

$$-2dv-Audv+2Avdu+udu=0$$

seu

$$dv-\frac{2Avdu}{2+Au}=\frac{udu}{2+Au},$$

ubi varibilis v unicam habet dimensionem, et hanc ob rem patet hanc aequationem integrabilem reddi, si dividatur per $(2+Au)^2$, prodibitque

$$\frac{v}{(2+Au)^2}=\int\frac{udu}{(2+Au)^3}=\frac{C}{AA}-\frac{1+Au}{AA(2+Au)^2}$$

ideoque

$$v=\frac{C(2+Au)^2-1-Au}{AA}.$$

Sumta ergo pro u functione quaeunque ipsius x erit

$$N=\frac{AA}{C(2+Au)^2-1-Au} \quad \text{et} \quad M=\frac{AAu}{C(2+Au)^2-1-Au}$$

atque

$$Q=\frac{AC(2+Au)^2-A}{C(2+Au)^2-1-Au}$$

Iam ex tertia aequatione adipiscimur

$$2NPdx=NdQ-QdN \quad \text{seu} \quad 2Pdx=Nd.\tfrac{Q}{N}$$

at

$$\frac{Q}{N}=\frac{C(2+Au)^2-1}{A}, \quad \text{unde} \quad d.\tfrac{Q}{N}=2Cdu(2+Au),$$

ideoque

$$Pdx=\frac{AACdu(2+Au)}{C(2+Au)^2-1-Au}.$$

Quocirca aequatio nostra per se integrabilis est

$$\frac{AACyydu(2+Au)+ydy\left(C(2+Au)^2y-(1+Au)y+AC(2+Au)^2-A\right)}{C(2+Au)^2yy-(1+Au)yy+AAuy+AA}=0,$$

quae posito $Au+2=t$ induet hanc formam

$$y.\frac{ACytdt+ydy(Ctt-t+1)+Ady(Ctt-1)}{Cttyy-(t-1)yy+A(t-2)y+AA}=0.$$

Hinc autem posito

$$A=\alpha, \quad C=\frac{\alpha\gamma}{\beta\beta} \quad \text{et} \quad t=-\frac{\beta x}{\alpha}$$

invenimus

$$y\cdot\frac{\alpha\gamma xydx+ydy(\alpha+\beta x+\gamma xx)-ady(\alpha-\gamma xx)}{(\alpha+\beta x+\gamma xx)yy-\alpha(2\alpha+\beta x)y+\alpha^3}=0.$$

COROLLARIUM 1

499. Hoc igitur modo integrari potest haec aequatio

$$\alpha\gamma xydx+ydy(a+\beta x+\gamma xx)-\alpha dy(\alpha-\gamma xx)=0,$$

quae quomodo ad separationem reduci debeat, non statim patet. Est autem multiplicator idoneus

$$\frac{y}{(a+\beta x+\gamma xx)yy-\alpha(2\alpha+\beta x)y+\alpha^3}.$$

COROLLARIUM 2

500. Hic multiplicator etiam hoc modo exprimi potest, ut eius denominator in factores resolvatur

$$(\alpha+\beta x+\gamma xx)y:\begin{cases}\left((\alpha+\beta x+\gamma xx)y-\alpha\left(\alpha+\frac{1}{2}\beta x\right)+ax\sqrt{\left(\frac{1}{4}\beta\beta-\alpha\gamma\right)}\right)\\\left((\alpha+\beta x+\gamma xx)y-\alpha\left(\alpha+\frac{1}{2}\beta x\right)-ax\sqrt{\left(\frac{1}{4}\beta\beta-\alpha\gamma\right)}\right)\end{cases}$$

COROLLARIUM 3

501. Si ergo ponamus

$$(\alpha+\beta x+\gamma xx)y-\alpha\left(\alpha+\tfrac{1}{2}\beta x\right)=az\,,$$

erit multiplicator

$$\frac{\alpha+\frac{1}{2}\beta x+z}{\left(z+x\sqrt{\left(\frac{1}{4}\beta\beta-\alpha\gamma\right)}\right)\left(z-x\sqrt{\left(\frac{1}{4}\beta\beta-\alpha\gamma\right)}\right)}.$$

At ob $y=\frac{\alpha\alpha+\frac{1}{2}\alpha\beta x+\alpha z}{\alpha+\beta x+\gamma xx}$ aequation nostra erit

$$\gamma xydx+dy\left(z+\tfrac{1}{2}\beta x+\gamma xx\right)=0\,.$$

At est

$$dy=\frac{-\frac{1}{2}\alpha(\alpha\beta+4\alpha\gamma x+\beta xx)dx-\alpha zdx(\beta+2\gamma x)+\alpha dz(\alpha+\beta x+\gamma xx)}{(\alpha+\beta x+\gamma xx)^2};$$

hoc autem valore substituto prodit aequatio nimis complicata.

PROBLEMA 66

502. *Invenire aequationem differentialem huius formae*

$$yPdx+(Qy+R)dy=0\,,$$

in qua P, Q et R sint functiones ipsius X, ut ea integrabilis evadat per hunc multiplicatorem $\frac{y^m}{(1+Sy)^n}$, *ubi S est etiam functio ipsius x.*

SOLUTIO

Quia dx per $\frac{y^{m+1}P}{(1+Sy)^n}$ et dy per $\frac{Q^{m+1}+Ry^m}{(1+Sy)^n}$ multiplicatur, oportet sit

$$(m+1)Py^m(1+Sy)-nPSy^{m+1}=\frac{(1+Sy)(y^{m+1}dQ+y^m dR)-nydS(Qy^{m+1}+Ry^m)}{dx},$$

qua evoluta aequatione erit

$$\left.\begin{array}{r}(m+1)Py^m dx+(m+1-n)PSy^{m+1}dx-\quad Sy^{m+2}dQ\\ -\ y^m dR-\qquad\qquad y^{m+1}dQ+nQy^{m+2}dS\\ -Sy^{m+1}dR\\ +\ nRy^{m+1}dS\end{array}\right\}=0;$$

hinc fit

$$Pdx=\frac{dR}{m+1}\ \text{ et }\ SdQ=nQdS$$

ideoque

$$Q=AS^n\ \text{ et }\ dQ=nAS^{n-1}dS\,,$$

quibus in membro medio substitutis fit

$$\frac{m+1-n}{m+1}SdR-nAS^{n-1}dS-SdR+nRdS=0$$

seu

$$-\frac{SdR}{m+1}-AS^{n-1}dS+\ RdS=0$$

ideoque

$$dR-\frac{(m+1)RdS}{S}=-(m+1)AS^{n-2}dS$$

quae per S^{m+1} divisa et integrata praebet

$$\frac{R}{S^{m+1}}=B-\frac{(m+1)AS^{n-m-2}}{n-m-2}.$$

Ponamus $A=(m+2-n)C$, ut sit

$$Q=(m+2-n)CS^n\ \text{ et }\ R=BS^{m+1}+(m+1)CS^{n-1}$$

ideoque

$$Pdx=BS^m dS+(n-1)CS^{n-2}dS\,.$$

Quocirca habebimus hanc aequationem

$$ydS\left(BS^m+(n-1)CS^{n-2}\right)+dy\left((m+2-n)CS^n y+BS^{m+1}+(m+1)CS^{n-1}\right)=0,.$$

quae multiplicata per $\frac{y^m}{(1+Sy)^n}$ fit integrabilis, ubi pro S functionem quamcunque ipsius x capere licet.

COROLLARIUM 1

503. Integrari ergo poterit haec aequatio

$$ByS^m dS + BS^{m+1}dy + (n-1)CyS^{n-2}dS$$
$$+(m+1)CS^{n-1}dy + (m+2-n)CS^n ydy = 0,$$

quae sponte resolvitur in has duas partes

$$BS^m(ydS + Sdy) + CS^{n-2}\left((n-1)ydS + (m+1)Sdy + (m+2-n)S^2 ydy\right) = 0,$$

quarum utraque seorsim per $\frac{y^m}{(1+Sy)^n}$ multiplicata fit integrabilis.

COROLLARIUM 2

504. Prior pars $BS^m(ydS + Sdy)$ integrabilis redditur per hunc multiplicatorem $\frac{1}{S^m}\varphi{:}Sy$; est enim haec formula $B(ydS + Sdy)\varphi{:}Sy$ per se integrabilis. Unde pro hac parte multiplicator erit $S^{\lambda-m}y^{\lambda}(1+Sy)^{\mu}$, qui utique continet assumtum $\frac{y^m}{(1+Sy)^n}$ si quidem capiatur $\lambda = m$ et $\mu = -n$. Est vero

$$\int \frac{y^m}{(1+Sy)^n}\cdot BS^m(ydS + Sdy) = B\int \frac{v^m dv}{(1+v)^n}$$

posito $Sy = v$.

COROLLARIUM 3

505. Pro altera parte, quae posito $S = \frac{1}{v}$ abit in

$$\frac{C}{v^n}\left(-(n-1)ydv + (m+1)vdy + (m+2-n)ydy\right),$$

habebimus

$$-\frac{(n-1)Cy}{v^n}\left(dv-\frac{(m+1)vdy}{(n-1)y}-\frac{(m+2-n)dy}{n-1}\right)$$
$$=-\frac{(n-1)Cy^{\frac{m+n}{n-1}}}{v^n}\left(y^{\frac{-m-1}{n-1}}dv-\frac{m+1}{n-1}y^{\frac{-m-n}{n-1}}vdy-\frac{m+2-n}{n-1}y^{\frac{-m-1}{n-1}}dy\right)$$
$$=-\frac{(n-1)Cy^{\frac{m+n}{n-1}}}{v^n}d.\left(y^{\frac{-m-1}{n-1}}v+y^{\frac{n-m-2}{n-1}}\right).$$

Ideoque haec altera pars ita repraesentabitur

$$-(n-1)CS^n y^{\frac{m+n}{n-1}}d.\frac{1+Sy}{y^{\frac{m+1}{n-1}}S}.$$

Multiplicator ergo hanc partem integrabilem reddens erit in genere

$$\frac{1}{S^n y^{\frac{m+n}{n-1}}}\varphi:\frac{1+Sy}{Sy^{\frac{m+1}{n-1}}}$$

COROLLARIUM 4

506. altera ergo parte multiplicator erit $\frac{(1+Sy)^\mu}{S^{n+\mu}y^{\frac{m+n+\mu(m+1)}{n-1}}}$, quo haec pars fit

$$-(n-1)C\frac{(1+Sy)^\mu}{S^\mu y^{\frac{\mu(m+1)}{n-1}}}d.\frac{1+Sy}{y^{\frac{m+1}{n-1}}},$$

cuius integrale est $-\frac{(n-1)CZ^{\mu+1}}{\mu+1}$ posito $Z=\frac{1+Sy}{y^{\frac{m+1}{n-1}}S}$.

COROLLARIUM 5

507. Iam multiplicator pro prima parte $S^{\lambda-m}y^\lambda(1+Sy)^\mu$ congruens reddetur cum multiplicatore alterius partis modo exhibito, si sumatur $\lambda=m$ et $\mu=-n$, unde resultat multiplicator communis $\frac{y^m}{(1+Sy)^n}$, hincque posito

$$Sy=v \text{ et } \frac{1+Sy}{y^{\frac{m+1}{n-1}}S}=z \text{ nostrae aequationis integrale erit}$$

$$B\int\frac{v^m dv}{(1+v)^n}+Cz^{1-n}=D \text{ sive } B\int\frac{v^m dv}{(1+v)^n}+\frac{CS^{n-1}y^{m+1}}{(1+Sy)^{n-1}}=D\cdot$$

SCHOLION

508. Aequatio ergo, quam hoc problemate integrare didicimus, per principia iam supra stabilita tractari potest, dum pro binis eius partibus seorsim multiplicatores quaeruntur iique inter se congruentes redduntur, cuius methodi hic insignem usum declaravimus.

Possemus etiam multiplicatori hanc formam dare $\frac{y^m}{(1+Sy+Tyy)^n}$, ita ut haec aequatio

$$\frac{y^m\left(yPdx+(Qy+R)dy\right)}{(1+Sy+Tyy)^n}=0$$

per se debeat esse integrabilis, et calculo ut ante instituto inveniemus

$$\left.\begin{array}{llll}(m+1)Py^m dx+ & (m+1-n)PSy^{m+1}dx+ & (m+1-2n)PTy^{m+2}dx- & Ty^{m+3}dQ\\ -\ y^m dR- & y^{m+1}dQ- & Sy^{m+2}dQ+ & nQy^{m+3}dT\\ & -Sy^{m+1}dR- & Ty^{m+2}dR & \\ & +\ nRy^{m+1}dS+ & nQy^{m+2}dS & \\ & + & nRy^{m+2}dT & \end{array}\right\}=0,$$

unde ex ultimo membro $-TdQ+nQdT=0$ concludimus $Q=AT^n$ et ex primo $Pdx=\frac{dR}{m+1}$, qui valores in binis mediis substituti praebent

$$RdS-\tfrac{SdR}{m+1}-AT^{n-1}dT=0$$

et

$$RdT-\tfrac{2TdR}{m+1}+AT^n dS-AST^{n-1}dT=0,$$

quarum illa fit integrabilis per se, si $m=-2$, haec vero integrari potest, si $m=2n-1$; fit enim

$$RdT-\tfrac{TdR}{n}+AT^{n-1}(TdS-SdT)=0$$

seu

$$\tfrac{nRdT-TdR}{nT^{n-1}}+\tfrac{A(TdS-SdT)}{TT}=0,$$

cuius integrale est

$$\tfrac{-R}{nT^n}+\tfrac{AS}{T}=\tfrac{-B}{n}$$

hincque $R=BT^n+nAT^{n-1}S$. Praeterea vera notari meretur casus $m=-1$, quem cum illis in subiunctis exemplis evolvamus.

EXEMPLUM 1

509. *Definire hanc equationem* $yPdx+(Qy+R)dy=0$, *ut multiplicata per* $\frac{1}{y(1+Sy+Tyy)^n}$, *fiat per se integrabilis.*

Ob $m=-1$ habemus statim $dR=0$ ideoque $R=C$; tum est ut ante $Q=AT^n$ et $dQ=nAT^{n-1}dT$, unde binae reliquae determinationes erunt

$$-PSdx-AT^{n-1}dT+CdS=0,$$
$$-2PTdx-AST^{n-1}dT+AT^{n}dS+CdT=0;$$

hinc eliminando Pdx prodit

$$ASST^{n-1}dT-2AT^{n}dT-AT^{n}SdS+2CTdS-CSdT=0.$$

Statuatur hic $SS=Tv$, ut fiat

$$2TdS-SdT=TS\left(\frac{2dS}{S}-\frac{dT}{T}\right)=\frac{TSdv}{v}=\frac{Tdv\sqrt{T}}{\sqrt{v}},$$

eritque

$$\frac{1}{2}AT^{n}vdT-2AT^{n}dT-\frac{1}{2}AT^{n+1}dv+\frac{CTdv\sqrt{T}}{\sqrt{v}}=0$$

seu hoc modo

$$-\frac{1}{2}AT^{n+2}d.\frac{v-4}{T}+\frac{CTdv\sqrt{T}}{\sqrt{v}}=0,$$

cuius prior pars integrabilis redditur per multiplicatorem $\frac{1}{T^{n+2}}\varphi:\frac{v-4}{T}$, posterior vero per $\frac{1}{T\sqrt{T}}\varphi:v$, unde communis multiplicator erit $\frac{1}{T(v-4)^{n+\frac{1}{2}}\sqrt{T}}$, hincque aequatio elicitur integralis haec

$$\frac{AT^{n-\frac{1}{2}}}{(2n-1)(v-4)^{n-\frac{1}{2}}}+C\int\frac{dv}{(v-4)^{n+\frac{1}{2}}\sqrt{v}}=D,$$

unde T definitur per v; tum vero est

$$S=\sqrt{Tv},\ R=C,\ Q=AT^{n}\ \text{ et }\ Pdx=\frac{CdS-AT^{n-1}dT}{S}$$

COROLLARIUM 1

510. Casu, quo est $n=\frac{1}{2}$, ob $\frac{1}{0}z^0=lz$, habetur

$$\tfrac{1}{2}Al\frac{T}{(v-4)\sqrt{v}}+C\int\frac{dv}{(v-4)^{n+\frac{1}{2}}\sqrt{v}}=\tfrac{1}{2}D \text{ seu } \tfrac{1}{2}Al\frac{T}{(v-4)}-\tfrac{1}{2}Cl\frac{\sqrt{v}+2}{\sqrt{v}-2}=\tfrac{1}{2}D,$$

unde posito $V=4uu$ et $C=lA$ erit

$$l\frac{T}{1-uu}-\lambda l\frac{1+u}{1-u}=\text{Const.}$$

seu

$$T=E(1-uu)\left(\frac{1+u}{1-u}\right)^{\lambda}.$$

Hinc porro

$$S=2u\sqrt{T}=2u\left(\frac{1+u}{1-u}\right)^{\frac{\lambda}{2}}\sqrt{E(1-uu)} \text{ et } R=C=\lambda A;$$

tum

$$Q=A\left(\frac{1+u}{1-u}\right)^{\frac{\lambda}{2}}\sqrt{E(1-uu)}$$

atque

$$Pdx=\frac{\lambda Adu}{u}+\frac{\lambda AdT}{2T}-\frac{AdT}{2Tu}.$$

At est

$$\frac{dT}{T}=\frac{-2udu+2\lambda du}{1-uu}$$

Ergo

$$Pdx=\frac{Adu(1+\lambda\lambda-2\lambda u)}{1-uu}.$$

Quocirca pro hac aequatione

$$\frac{Aydu(1+\lambda\lambda-2\lambda u)}{1-uu}+Ady\left(\lambda+y\left(\frac{1+u}{1-u}\right)^{\frac{\lambda}{2}}\sqrt{E(1-uu)}\right)=0$$

multiplicator erit

$$\frac{1}{y\sqrt{\left(1+2uy\left(\frac{1+u}{1-u}\right)^{\frac{\lambda}{2}}\sqrt{E(1-uu)}+Eyy(1-uu)\left(\frac{1+u}{1-u}\right)^{\frac{\lambda}{2}}\right)}}.$$

COROLLARIUM 2

511. Casu, quo $n=-\frac{1}{2}$, habemus

$$-\frac{A(v-4)}{2T}+2C\sqrt{v}=-2D \text{ seu } T=\frac{A(v-4)}{4D+4C\sqrt{v}}.$$

Ponamus $v = 4uu$, ut sit $T = \frac{A(uu-1)}{D+2Cu}$; tum sit

$$S = 2u\sqrt{T} = 2u\frac{A(uu-1)}{D+2Cu}, \quad R = C, \quad Q = \sqrt{\frac{A(D+2Cu)}{uu-1}}$$

et

$$Pdx = \frac{Cdu}{u} + \frac{CdT}{2T} - \frac{AdT}{2TTu} = \frac{Cdu}{u} + \frac{du(C+Du+Cuu)(Cu^3-3Cu-D)}{u(uu-1)^2(D+2Cu)},$$

unde tam aequatio quam multiplicator definitur.

EXEMPLUM 2

512. *Definire aequationem* $yPdx + (Qy+R)dy = 0$ *, ut multiplicata per* $\frac{1}{y^3(1+Sy+Tyy)^n}$ *fiat per se integrabilis.*

Ob $m = -2$ ex superioribus [§ 508] habemus

$$RS = \frac{A}{n}T^n + B \text{ seu } R = \frac{AT^n}{nS} + \frac{B}{S}$$

qui valor in altera aequatione substitutus praebet

$$\frac{(2n+1)AT^n dT}{nS} - \frac{2AT^{n+1}dS}{nSS} + AT^n dS - AST^{n-1}dT + \frac{BdT}{S} - \frac{2BTdS}{SS} = 0$$

quae in has tres partes distinguatur

$$\frac{AS}{nT^n}\left(\frac{(2n+1)T^{2n}dT}{S^2} - \frac{2AT^{n+1}dS}{S^3}\right) + AT^{n+1}\left(\frac{dS}{T} - \frac{SdT}{TT}\right)$$
$$+BS\left(\frac{dT}{SS} - \frac{2TdS}{S^3}\right) = 0$$

seu

$$\frac{AS}{nT^n}d.\frac{T^{2n+1}}{SS} + AT^{n+1}d.\frac{S}{T} + BSd.\frac{T}{SS} = 0.$$

Statuamus ad abbreviandum

$$\frac{T^{2n+1}}{SS} = p, \quad \frac{S}{T} = q \quad \text{et} \quad \frac{T}{SS} = r$$

fiet $S = \frac{1}{qr}$, $T = \frac{1}{qqr}$, hinc $p = \frac{1}{q^{4n}r^{2n-1}}$ nostraque aequatio ita se habebit

$$\frac{A}{nq\sqrt{pr}}dp + \frac{A\sqrt{p}}{qqr\sqrt{r}}dq + \frac{B}{qr}dr = 0 \text{ seu } \frac{A\sqrt{r}}{n\sqrt{p}}dp + A\frac{\sqrt{p}}{q\sqrt{r}}dq + Bdr = 0.$$

Quas tres partes seorsim consideremus ac prima fit integrabilis multiplicata per $\frac{\sqrt{p}}{\sqrt{r}}\varphi{:}p$, secunda vero per $\frac{q\sqrt{r}}{\sqrt{p}}\varphi{:}q$, tertia vero per $\varphi{:}r$. Ut bini primi conveniant, ponatur

$$\frac{\sqrt{p}}{\sqrt{r}}p^{\lambda}=\frac{q\sqrt{r}}{\sqrt{p}}q^{\mu}$$

seu $p^{\lambda+1}=q^{\mu+1}r$, hinc $p=q^{\frac{\mu+1}{\lambda+1}}r^{\frac{1}{\lambda+1}}=q^{-4n}r^{-2n+1}$. Fit ergo

$$\lambda+1=-\tfrac{1}{2n-1}\text{ et }\mu+1=-4n(\lambda+1)=\tfrac{4n}{2n-1}$$

sicque

$$\mu=\tfrac{2n+1}{2n-1}\text{ et }\lambda=-\tfrac{2n}{2n-1}.$$

Multiplicetur ergo aequatio per

$$\frac{q^{\frac{4n}{2n-1}}\sqrt{r}}{\sqrt{p}}=q^{2n+\frac{4n}{2n-1}}r^{n}$$

ac prodibit

$$\tfrac{A}{n}p^{\lambda}dp+Aq^{\mu}dq+Bq^{2n+\frac{4n}{2n-1}}r^{n}dr=0$$

seu

$$Ad.\left(\frac{p^{\lambda+1}}{n(\lambda+1)}+\frac{q^{\mu+1}}{\mu+1}\right)+Bq^{\frac{4nn+2n}{2n-1}}r^{n}dr=0$$

vel

$$\frac{(2n-1)A}{4n}d.q^{\frac{4n}{2n-1}}(1-4r)+Bq^{\frac{4nn+2n}{2n-1}}r^{n}dr=0.$$

Multiplicetur per $q^{\frac{4\nu n}{2n-1}}(1-4r)^{\nu}$ ut prodeat

$$\frac{(2n-1)A}{4n}q^{\frac{4\nu n}{2n-1}}(1-4r)^{\nu}d.q^{\frac{4\nu n}{2n-1}}(1-4r)+Bq^{\frac{4nn+2n+4\nu n}{2n-1}}r^{n}dr(1-4r)^{\nu}=0.$$

Fiat ergo $4\nu+4n+2=0$ seu $\nu=-n-\frac{1}{2}$ et ambo membra integrali poterunt eritque

$$\frac{(2n-1)A}{4n(\nu+1)}q^{\frac{4n(\nu+1)}{2n-1}}(1-4r)^{\nu+1}+B\int r^{n}dr(1-4r)^{\nu}=\text{Const.}$$

at est $\nu+1=-n+\frac{1}{2}=-\frac{2n-1}{2}$ sicque habebitur

$$-\tfrac{A}{2n}q^{-2n}(1-4r)^{-\frac{2n-1}{2}}+B\int\frac{r^{n}dr}{(1-4r)^{\frac{2n+1}{2}}}=\text{Const.}$$

Dabitur ergo q per r eritque $S=\frac{1}{qr}$, $T=\frac{S}{q}$, tum

$$R = \frac{AT^n}{nS} + \frac{B}{S}, \quad Q = AT^n \text{ et } Pdx = -dR.$$

COROLLARIUM 1

513. Si sit $n = -\frac{1}{2}$, erit

$$Aq(1-4r) + 2B\sqrt{r} = C$$

$$q = \frac{C-2B\sqrt{r}}{A(1-4r)} \;;$$

hincque

$$S = \frac{A(1-4r)}{r\left(C-2B\sqrt{r}\right)}, \quad T = \frac{A^2(1-4r)^2}{r\left(C-2B\sqrt{r}\right)^2}, \quad Q = \frac{\sqrt{r}\left(C-2B\sqrt{r}\right)}{(1-4r)}$$

et

$$R = \frac{Q+nB}{nS} = \frac{B-2Q}{S} = \frac{r\left(B-2C\sqrt{r}\right)\left(C-2B\sqrt{r}\right)}{A(1-4r)^2}$$

seu

$$R = \frac{BCr-2\left(B^2+C^2\right)r\sqrt{r}+4BCr^2}{A(1-4r)^2}.$$

COROLLARIUM 2

514. Ponamus eodem casu $r = \frac{1}{4}uu$; erit

$$S = \frac{4A(1-uu)}{uu(C-Bu)}, \quad T = \frac{4AA(1-uu)^2}{uu(C-Bu)^2},$$

$$Q = \frac{u(C-Bu)}{2(1-uu)}, \qquad R = \frac{uu(B-Cu)(C-Bu)}{4A(1-uu)^2}$$

hincque

$$Pdx = \frac{\left(B^2+C^2\right)\left(3uu+u^4\right)-2BC\left(u+3u^3\right)}{4A(1-uu)^3}du$$

eritque aequatio $yPdx + (Qy+R)dy = 0$ integrabilis, si multiplicetur per

$$\frac{\sqrt{(1+Sy+Tyy)}}{yy} = \frac{1}{yy}\sqrt{\left(1+\frac{4A(1-uu)y}{uu(C-Bu)}+\frac{4AA(1-uu)^2 yy}{uu(C-Bu)^2}\right)}.$$

EXEMPLUM 3

515. *Definire aequationem* $yPdx+(Qy+R)dy=0$, *quae multiplicata per*

$$\frac{y^{2n-1}}{(1+Sy+Tyy)^n}$$

fiat per se integrabilis

Hic est $m=2n-1$, $Q=AT^n$ et $Pdx=\frac{dR}{2n}$, tum vero ex superioribus [§ 508] $R=nAT^{n-1}S+BT^n$ ac superest aequatio

$$RdS-S\frac{dR}{2n}-AT^{n-1}dT=0,$$

quae loco R substituto valore invento abit in

$$(2n-1)AT^{n-1}SdS-(n-1)AT^{n-2}SSdT$$
$$-2AT^{n-1}dT+2BT^n dS-BT^{n-1}SdT=0$$

seu

$$(2n-1)ATSdS-(n-1)ASSdT$$
$$-2ATdT+2BTTdS-BTSdT=0.$$

Prius membrum posito $SS=u$ abit in

$$\left(n-\tfrac{1}{2}\right)ATdu-(n-1)AudT-2ATdT$$

seu

$$\left(n-\tfrac{1}{2}\right)AT\left(du-\frac{(n-1)udT}{\left(n-\frac{1}{2}\right)T}-\frac{2dT}{n-\frac{1}{2}}\right)$$

sive

$$\tfrac{1}{2}(2n-1)AT^{\frac{4n-3}{2n-1}}\left(\frac{du}{T^{\frac{4n-2}{2n-1}}}-\frac{2(n-1)udT}{(2n-1)T^{\frac{4n-3}{2n-1}}}-\frac{4dT}{(2n-1)T^{\frac{4n-2}{2n-1}}}\right)$$

$$=\tfrac{1}{2}(2n-1)AT^{\frac{4n-3}{2n-1}}d.\left(\frac{u}{T^{\frac{2n-2}{2n-1}}}-4T^{\frac{1}{2n-1}}\right)$$

vel

$$\tfrac{1}{2}(2n-1)AT^{\frac{4n-3}{2n-1}}d.T^{\frac{1}{2n-1}}\left(\frac{SS}{T}-4\right)+\frac{BT^3}{S}d.\frac{SS}{T}=0$$

seu

$$(2n-1)AT^{\frac{-1}{2n-1}}d.T^{\frac{1}{2n-1}}\left(\frac{SS}{T}-4\right)+\frac{2BT}{S}d.\frac{SS}{T}=0.$$

Ponatur

$$\frac{SS}{T}=p \quad \text{et} \quad d.T^{\frac{1}{2n-1}}\left(\frac{SS}{T}-4\right)=q=T^{\frac{1}{2n-1}}(p-4), \text{ut sit } T$$

ut sit $T^{\frac{1}{2n-1}}=\frac{q}{p-4}$, unde

$$T=\frac{q^{2n-1}}{(p-4)^{2n-1}} \quad \text{et} \quad S=\sqrt{\frac{pq^{2n-1}}{(p-4)^{2n-1}}}.$$

Ergo

$$\frac{(2n-1)A(p-q)dq}{q}+\frac{2B\sqrt{q^{2n-1}}}{\sqrt{p(p-4)^{2n-1}}}=0$$

sive

$$\frac{(2n-1)Adq}{q^{n+\frac{1}{2}}}+\frac{2Bdp:\sqrt{p}}{(p-4)^{n+\frac{1}{2}}}=0,$$

quae integrata praebet

$$\frac{-2A}{q^{n-\frac{1}{2}}}+2B\int\frac{dp:\sqrt{p}}{(p-4)^{n+\frac{1}{2}}}=2C,$$

et facto $\frac{p}{p-4}=vv$ seu $p=\frac{4vv}{vv-1}$ fiet

$$\frac{-2A}{q^{n-\frac{1}{2}}}-\frac{B}{4^{n-1}}\int dv(vv-1)^{n-1}=C.$$

SCHOLION

516. Haec fusius non prosequor, quia ista exempla eum in finem potissimum attuli, ut methodus supra tradita aequationes differentiales tractandi exerceretur; in his enim exemplis casus non parum difficiles se obtulerunt, quos ita per partes resolvere licuit, ut pro singulis multiplicatores idonei quaererentur, ex iisque multiplicator communis definiretur; nunc igitur alia aequationum genera, quae per multiplicatores integrabiles reddi queant, investigemus.

PROBLEMA 67

517. *Ipsius x functiones P, Q, B, S definire, ut haec aequatio*

$$(Py+Q)dx+ydy=0$$

per hunc multiplicatorem $(yy+Ry+S)^n$ *integrabilis reddatur.*

SOLUTIO

Necesse igitur est sit

$$\left(\frac{d.(Py+Q)(yy+Ry+S)^n}{dy}\right)=\left(\frac{d.y(yy+Ry+S)^n}{dx}\right),$$

unde colligitur per $(yy+Ry+S)^{n-1}$ dividendo

$$P(yy+By+S)+n(Py+Q)(2y+R)=\frac{ny(ydR+dS\ )}{dx}$$

seu

$$\left.\begin{array}{r}(2n+1)Pyydx+(n+1)PRydx+PSdx\\ -\ nyydR+\qquad 2n\ Qydx+nQRdx\\ -\qquad\qquad nydS\end{array}\right\}=0.$$

Hinc ergo concluditur

$$Pdx=\tfrac{ndR}{2n+1}\text{ et }\tfrac{(n+1)RdR}{2n+1}+2Qdx-dS=0,\quad \tfrac{SdR}{2n+1}+QRdx=0\,,$$

porroque

$$Qdx=\tfrac{-SdR}{(2n+1)R}\ =\tfrac{-(n+1)RdR}{2(2n+1)}+\tfrac{dS}{2}\,,$$

ergo

$$dS+\tfrac{2SdR}{(2n+1)R}=\tfrac{(n+1)RdR}{2n+1}\,,$$

quae per $R^{\frac{2}{2n+1}}$ multiplicata et integrata dat

$$R^{\frac{2}{2n+1}}S=C+\tfrac{1}{4}R^{\frac{4n+4}{2n+1}}\,,$$

hincque

$$S=\tfrac{1}{4}RR+CR^{\frac{-2}{2n+1}}$$

atque

$$Qdx=\tfrac{-RdR}{4(2n+1)}-\tfrac{C}{2n+1}R^{\frac{-2n-3}{2n+1}}dR\quad\text{et}\quad Pdx=\tfrac{ndR}{2n+1}\,,$$

unde aequationem obtinemus

$$\left(ny-\tfrac{1}{4}R-CR^{\frac{-2n-3}{2n+1}}\right)dR+(2n+1)ydy=0,,$$

quae integrabilis redditur per hunc multiplicatorem

$$\left(yy+Ry+\tfrac{1}{4}RR+CR^{\frac{-2}{2n+1}}\right)^{n}.$$

COROLLARIUM 1

518. Casu, quo $n=-\frac{1}{2}$ fit $dR=0$ et $R=A$ et reliquae aequationes sunt

$$(n+1)APdx+2nQdx-ndS=0 \text{ et } PSdx+nAQdx=0.$$

Ergo

$$Pdx=\frac{AQdx}{2S}=\frac{2Qdx-dS}{A} \text{ ideoque } (AA-4S)Qdx=-2SdS$$

seu

$$Qdx=\frac{-2SdS}{AA-4S} \text{ et } Pdx=\frac{-AdS}{AA-4S}$$

sicque haec aequatio

$$\frac{(Ay+2S)dS}{4S-AA}+ydy=0$$

integrabilis redditur per hunc multiplicatorem $\frac{1}{\sqrt{(yy+Ay+S)}}$.

COROLLARIUM 2

519. Si hic ponamus $A=2a$ et $S=x$, haec aequatio

$$\frac{(ay+x)dx+2ydy(x-aa)}{(x-aa)\sqrt{(yy+2ay+x)}}=0$$

per se est integrabilis, unde integrale inveniri potest huius aequationis

$$xdx+aydx+2xydy-2aaydy=0,$$

quae divisa per $(x-aa)\sqrt{(yy+2ay+x)}$ fit integrabilis.

COROLLARIUM 3

520. Ad integrale inveniendum sumatur primo x constans et partis

$$\frac{2ydy}{\sqrt{(yy+2ay+x)}}$$

integrale est

$$2\sqrt{(yy+2ay+x)}+2al\left(a+y-\sqrt{(yy+2ay+x)}\right)+X \;;$$

cuius differentiale sumto y constante

$$\frac{dx}{\sqrt{(yy+2ay+x)}}-\frac{adx:\sqrt{(yy+2ay+x)}}{a+y-\sqrt{(yy+2ay+x)}}+dX$$

si alteri aequationis parti

$$\frac{(ax+y)dx}{(x-aa)\sqrt{(yy+2ay+x)}}$$

aequetur, reperitur $dX = \frac{adx}{aa-x}$ et $X = -al(aa-x)$. Ex quo integrale completum erit

$$\sqrt{(yy+2ay+x)} + al\,\frac{a+y-\sqrt{(yy+2ay+x)}}{\sqrt{(aa-x)}} = C.$$

COROLLARIUM 4

521. Memoratu dignus est etiam casus $n = -1$, qui scripto a loco $C + \frac{1}{4}$ praebet hanc aequationem $(y+aR)dR + ydy = 0$, quae divisa per $yy + Ry + aRR$ fit integrabilis; haec autem aequatio est homogenea.

SCHOLION

522. Potest etiam aequationis $(Py+Q)dx + ydy = 0$ multiplicator statui $(y+R)^m(y+S)^n$ fierique debet

$$\left(\frac{d.(Py+Q)(y+R)^m(y+S)^n}{dy}\right) = \left(\frac{d.y(y+R)^m(y+S)^n}{dx}\right),$$

unde reperitur

$$Pdx(y+R)(y+S) + mdx(Py+Q)(y+S) + ndx(Py+Q)(y+R)$$
$$= my(y+S)dR + ny(y+R)dS,$$

quae evolvitur in

$$\left.\begin{array}{r} (m+n+1)Pyydx + (n+1)PRydx + PRSdx \\ -myydR + (m+1)PSydx + mQSdx \\ -nyydS + (m+n)Qydx + n\,QRdx \\ -mSydR \\ -nRydS \end{array}\right\} = 0,$$

unde colligitur

$$Pdx = \frac{mdR+ndS}{m+n+1} \quad \text{et} \quad Qdx = \frac{-PRSdx}{mS+nR} = \frac{-RS(mdR+ndS)}{(m+n+1)(mS+nR)}$$

hincque

$$\frac{(mdR+ndS)((n+1)R+(m+1)S)}{(m+n+1)} - \frac{(m+n)RS(mdR+ndS)}{(m+n+1)(mS+nR)} - mSdR - nRdS = 0$$

seu

$$m(n+1)RdR - mnRdS + n(m+1)SdS - mnSdR$$
$$-\frac{m(m+n)RSdR + n(m+n)RSdS}{mS+nR} = 0,$$

quae reducitur ad hanc formam

$$(n+1)RRdR+(m-n-1)RSdR \ -mSSdR$$
$$+(m+1)SSdS+(n-m-1)RSdS-nRRdS=0;$$

quae cum sit homogenea, dividatur per

$$(n+1)R^3+(m-2n-1)R^2S+(n-2m-1)RSS+(m+1)S^3,$$

seu per

$$(R-S)^2((n+1)R+(m+1)S),$$

ut fiat integrabilis.
At ipsa illa aequatio per $R-S$ divisa erit

$$(n+1)RdR+mSdR-nRdS-(m+1)SdS=0.$$

Dividatur per $(R-S)((n+1)R+(m+1)S)$ et resolvatur in fractiones partiales

$$\frac{dR}{m+n+2}\left(\frac{m+n+1}{R-S}+\frac{n+1}{(n+1)R+(m+1)S}\right)$$
$$+\frac{dS}{m+n+2}\left(\frac{m+n+1}{S-R}+\frac{m+1}{(n+1)R+(m+1)S}\right)=0$$

seu

$$\frac{(m+n+1)(dR-dS)}{R-S}+\frac{(n+1)dR+(m+1)dS}{(n+1)R+(m+1)S}=0$$

unde integrando obtinemus

$$(R-S)^{m+n+1}((n+1)R+(m+1)S)=C.$$

Sit $R-S=u$; erit

$$(n+1)R+(m+1)S=\frac{C}{u^{m+n+1}}$$

hincque

$$R=\frac{(m+1)u}{m+n+2}+\frac{a}{u^{m+n+1}} \quad \text{et} \quad S=-\frac{(n+1)u}{m+n+2}+\frac{a}{u^{m+n+1}},$$

tum vero

$$Pdx=\frac{(m-n)du}{m+n+2}-\frac{(m+n)adu}{u^{m+n+2}}$$

et

$$Qdx=\frac{du}{u}\left(\frac{a}{u^{m+n+1}}+\frac{(m+1)u}{m+n+2}\right)\left(\frac{a}{u^{m+n+1}}-\frac{(n+1)u}{m+n+2}\right).$$

COROLLARIUM 1

523. Hinc ergo integrari potest ista aequatio

$$ydy + ydu\left(\frac{m-n}{m+n+2}-\frac{(m+n)a}{u^{m+n+2}}\right)$$
$$+\frac{du}{u}\left(\frac{aa}{u^{2m+2n+2}}+\frac{(m-n)a}{(m+n+2)u^{m+n}}-\frac{(m+1)(n+1)uu}{(m+n+2)^2}\right)=0,$$

quippe quae per se fit integrabilis, si multiplicetur per

$$\left(y+\frac{a}{u^{m+n+1}}+\frac{(m+1)u}{m+n+2}\right)^m\left(y+\frac{a}{u^{m+n+1}}-\frac{(n+1)u}{m+n+2}\right)^n.$$

COROLLARIUM 2

524. Sit $m=n$ et aequatio nostra erit

$$ydy-\frac{2naydu}{u^{2n+2}}+\frac{aadu}{u^{4n+3}}-\frac{1}{4}udu=0,$$

cuius multiplicator est $\left(\left(y+\frac{a}{u^{2n+1}}\right)^2-\frac{1}{4}uu\right)^n$. Quare si ponamus $y=z-\frac{a}{u^{2n+1}}$, aequatio prodit

$$zdz-\frac{adz}{u^{2n+1}}+\frac{azdz}{u^{2n+2}}-\frac{1}{4}udu=0,$$

quae integrabilis fit multiplicata per $\left(zz-\frac{1}{4}uu\right)^n$.

Vel ponatur $z=\frac{1}{2}y$ et $a=\frac{1}{2}b$; erit $ydy-udu-\frac{bdy}{u^{2n+1}}+\frac{bydu}{u^{2n+2}}=0$

et multiplicator $(yy-uu)^n$.

COROLLARIUM 3

525. Si $m=-n$, prodit haec aequatio

$$ydy-nydu+\frac{aadu}{u^3}+\frac{1}{4}(nn-1)udu-\frac{nadu}{u}=0,$$

quae integrabilis redditur multiplicata per

$$\left(y+\frac{a}{u}-\frac{1}{2}(n+1)u\right)^n\left(y+\frac{a}{u}-\frac{1}{2}(n-1)u\right)^{-n}.$$

Posito autem $y+\frac{a}{u}=z$ prodit haec aequatio

$$zdz-nzdu+\frac{1}{4}(nn-1)udu-\frac{adz}{u}+\frac{azdu}{uu}=0,$$

quam integrabilem reddit hic multiplicator

$$\left(y-\tfrac{1}{2}(n+1)u\right)^{n}\left(y-\tfrac{1}{2}(n-1)u\right)^{-n}.$$

COROLLARIUM 4

526. Ponamus hic $z = uv$ et habebitur ista aequatio

$$uuvdv + udu\left(vv - nv + \tfrac{1}{4}(nn-1)\right) = adv\ ;$$

quae si multiplicetur per $\left(\frac{v-\frac{1}{2}(nn+1)}{v-\frac{1}{2}(nn-1)}\right)^{n}$, utrumque membrum fiet integrabile.
Posito enim

$$\frac{v-\frac{1}{2}(nn+1)}{v-\frac{1}{2}(nn-1)} = s \text{ seu } v = \frac{n+1-(n-1)s}{2(1-s)}$$

oritur

$$\frac{s^{n+1}udu}{(1-s)^{2}} + \frac{n+1-(n-1)s}{2(1-s)^{3}}uus^{n}ds = \frac{as^{n}ds}{(1-s)^{2}}\ ,$$

cuius integrale est

$$\frac{s^{n+1}uu}{2(1-s)^{2}} = a\int\frac{s^{n}ds}{(1-s)^{2}}.$$

SCHOLION

527. Quo nostram aequationem in genere concinniorem reddamus, ponamus
$m = -\lambda - 1 + \mu$ et $n = -\lambda - 1 - \mu$, ut sit $m + n + 2 = -2\lambda$,
fietque aequatio

$$ydy - ydu\left(\tfrac{\mu}{\lambda} - 2(\lambda+1)au^{2\lambda}\right) + udu\left(\tfrac{\mu\mu-\lambda\lambda}{4\lambda\lambda} - \tfrac{\mu}{\lambda}au^{2\lambda} + aau^{4\lambda}\right) = 0,$$

quae per hunc multiplicatorem integrabilis redditur

$$\left(y + au^{2\lambda+1} - \tfrac{(\mu-\lambda)u}{2\lambda}\right)^{\mu-\lambda-1}\left(y + au^{2\lambda+1} - \tfrac{(\mu+\lambda)u}{2\lambda}\right)^{-\mu-\lambda-1}.$$

Ponatur $y + au^{2\lambda+1} = uz$ et orietur haec aequatio

$$uzdz - au^{2\lambda+1}dz + du\left(zz - \tfrac{\mu}{\lambda}z + \tfrac{\mu\mu-\lambda\lambda}{4\lambda\lambda}\right) = 0,$$

cui respondet multiplicator

$$u^{-2\lambda-1}\left(z + \tfrac{\lambda-\mu}{2\lambda}\right)^{\mu-\lambda-1}\left(z - \tfrac{\lambda+\mu}{2\lambda}\right)^{-\mu-\lambda-1}$$

Reperitur autem integrale

$$C=a\int dz\left(z+\tfrac{\lambda-\mu}{2\lambda}\right)^{\mu-\lambda-1}\left(z-\tfrac{\lambda+\mu}{2\lambda}\right)^{-\mu-\lambda-1}+\tfrac{1}{2\lambda u^{2\lambda}}\left(z+\tfrac{\lambda-\mu}{2\lambda}\right)^{\mu-\lambda}\left(z-\tfrac{\lambda+\mu}{2\lambda}\right)^{-\mu-\lambda},$$

quod ergo convenit huic aequationi differentiali

$$zdz+\tfrac{du}{u}\left(z+\tfrac{\lambda-\mu}{2\lambda}\right)\left(z-\tfrac{\lambda+\mu}{2\lambda}\right)=au^{2\lambda}dz.$$

PROBLEMA 68

528. *Ipsius x functiones P, Q, R et X definire, ut haec aequatio*

$$dy+yydx+Xdx=0$$

integrabilis reddatur per hunc multiplicatorem $\frac{1}{Pyy+Qy+R}$.

SOLUTIO

Debet ergo esse

$$\tfrac{1}{dy}d.\tfrac{yy+X}{Pyy+Qy+R}=\tfrac{1}{dx}d.\tfrac{1}{Pyy+Qy+R}$$

hincque

$$2y\left(Pyy+Qy+R\right)-\left(yy+X\right)\left(2Py+Q\right)=\tfrac{-yydP-ydQ-dR}{dx};$$

ergo fieri debet

$$\left.\begin{aligned}&Qyydx+2Rydx-QXdx\\&+yydP-2PXydx+dR\\&\quad+ydQ\end{aligned}\right\}=0.$$

Quare habetur

$$Q=-\tfrac{dP}{dx}=\tfrac{dR}{Xdx}\quad\text{et}\quad X=-\tfrac{dR}{dP}.$$

Sumto ergo dx constante est $dQ=-\frac{ddP}{dx}$, unde fieri oportet

$$2Rdx+\tfrac{2PdRdx}{dP}-\tfrac{ddP}{dx}=0\ \text{ seu }\ RdP+PdR=\tfrac{dPddP}{2dx^2},$$

cuius integratio praebet

$$PR=\tfrac{dP^2}{4dx^2}+C,$$

hinc

$$R=\tfrac{dP^2}{4Pdx^2}+\tfrac{C}{P},\ \text{tum}\ Q=-\tfrac{dP}{dx}\ \text{ et }\ X=\tfrac{C}{PP}+\tfrac{dP^2}{4PPdx^2}-\tfrac{ddP}{2Pdx^2}.$$

Ponamus $P=SS$, ut S sit functio quaecunque ipsius x, obtinebimusque

$$P = SS,\ Q = -\frac{2SdS}{dx},\ R = \frac{C}{SS} + \frac{dS^2}{dx^2} \quad \text{et} \quad X = \frac{C}{S^4} - \frac{ddS}{Sdx^2},$$

quibus sumtis valoribus per se integrabilis erit haec aequatio

$$\frac{dy + yydx + Xdx}{Pyy + Qy + R} = 0.$$

SCHOLION

529. Haec solutio commodius institui poterit, si multiplicatori tribuatur haec forma $\frac{P}{yy+2Qy+R}$ ut fieri debeat

$$\frac{1}{dy} d.\frac{P(yy+X)}{Pyy+2Qy+R} = \frac{1}{dx} d.\frac{P}{Pyy+2Qy+R}$$

unde oritur

$$\left.\begin{array}{r} 2PQyydx + 2PRydx - 2PQXdx \\ -yydP - 2PXydx - \quad RdP \\ -2QydP + \quad PdR \\ +\ 2PydQ \qquad\quad \end{array}\right\} = 0,$$

ubi ex singulis commode definitur $\frac{dP}{P}$, scilicet

$$\frac{dP}{P} = 2Qdx = \frac{Rdx - Xdx + dQ}{Q} = \frac{dR - 2QXdx}{R}.$$

Hinc colligitur $2Q(R+X)dx = dR$, unde nunc ipsum elementum dx definiamus $dx = \frac{dR}{2Q(R+X)}$ quo valore substituto adipiscimur

$$\frac{QdR}{R+X} = \frac{(R-X)dR}{2Q(R+X)} + dQ$$

seu

$$2QQdR = RdR - XdR + 2QRdQ + 2QXdQ,$$

unde colligimus

$$X = \frac{2QQdR - 2QRdQ - RdR}{2QdQ - dR} \quad \text{et} \quad R + X = \frac{2(QQ-R)dR}{2QdQ - dR}$$

hinc

$$dx = \frac{2QdQ - dR}{4Q(QQ-R)} \quad \text{atque} \quad \frac{dP}{P} = \frac{2QdQ - dR}{4(QQ-R)}$$

ideoque $P = A\sqrt{(QQ-R)}$.

Fiat $QQ - R = S$ ac reperietur

$$dx = \frac{dS}{4QS},\ X = \frac{4QSdQ}{dS} - QQ - S,\ R = QQ - S$$

atque $P = A\sqrt{S}$. Quocirca habebimus hanc aequationem

$$dy + \frac{yydS}{4QS} + dQ - \frac{(QQ+S)dS}{4QS} = 0,$$

quae integrabilis redditur per hunc multiplicatorem

$$\frac{\sqrt{S}}{yy+2Qy+QQ-S} = \frac{\sqrt{S}}{(y+Q)^2-S}.$$

Ad eius integrale inveniendum sumantur Q et S constantes prodibitque

$$\int \frac{dy\sqrt{S}}{(y+Q)^2-S} = \frac{1}{2} l \frac{y+Q-\sqrt{S}}{y+Q+\sqrt{S}} + V$$

existente V certa functione ipsius S vel Q. Iam differentietur haec forma sumta y constante proditque

$$\frac{dQ\sqrt{S} - \frac{(Q+y)dS}{2\sqrt{S}}}{(y+Q)^2-S} + dV = \frac{yydS+4QSdQ-QQdS-SdS}{4Q\left((y+Q)^2-S\right)\sqrt{S}}$$

ideoque

$$dV = \frac{yydS+2QydS+QQdS-SdS}{4Q\left((y+Q)^2-S\right)\sqrt{S}} = \frac{dS}{4Q\sqrt{S}}.$$

Ex quo aequationis nostrae integrale est

$$\frac{1}{2} l \frac{y+Q-\sqrt{S}}{y+Q+\sqrt{S}} + \frac{1}{4}\int \frac{dS}{Q\sqrt{S}} = C.$$

COROLLARIUM 1

530. Singularis est casus, quo $R = QQ$; fit enim

$$\frac{dP}{P} = 2Qdx = \frac{QQdx-Xdx+dQ}{Q} = \frac{2dQ-2Xdx}{Q},$$

unde has duas aequationes elicimus

$$QQdx + Xdx - dQ = 0 \ \text{ et } \ QQdx + Xdx - dQ = 0;$$

quae cum inter se conveniant, erit

$$Xdx = dQ - QQdx \ \text{ et } \ lP = 2\int Qdx.$$

COROLLARIUM 2

531. Sumto ergo Q negativo, ut habeamus hanc aequationem

$$dy + yydx - dQ - QQdx = 0,$$

haec integrabilis reddtur per hunc multiplicatorem $\frac{e^{-2\int Qdx}}{(y-Q)^2}$. Et integraIe erit

$$\frac{-1}{y-Q} e^{-2\int Qdx} + V = \text{Const.},$$

ubi V est functio ipsius x, ad quam definiendam differentietur sumta y constante

$$\frac{-dQ}{(y-Q)^2} e^{-2\int Qdx} + \frac{2Qdx}{y-Q} e^{-2\int Qdx} + dV = \frac{yydx - dQ - QQdx}{(y-Q)^2} e^{-2\int Qdx},$$

unde fit

$$V = \int e^{-2\int Qdx} dx,$$

ita ut integrale sit

$$\int e^{-2\int Qdx} dx - \frac{e^{-2\int Qdx}}{y-Q} = C.$$

COROLLARIUM 3

532. Proposita ergo aequatione $dy + yydx + Xdx = 0$ si eius integrale particulare quoddam constet $y = Q$, ut sit $dQ + QQdx + Xdx = 0$ ideoque $dy + yydx - dQ - QQdx = 0$, multiplicator pro ea erit $\frac{1}{(y-Q)^2} e^{-2\int Qdx}$ et integrale completum

$$Ce^{2\int Qdx} + \frac{1}{y-Q} = e^{2\int Qdx} \int e^{-2\int Qdx} dx.$$

SCHOLION

533. Aequatio autem in praecedente scholio [§ 529] inventa

$$dy + \frac{yydS}{4QS} + dQ - \frac{(QQ+S)dS}{4QS} = 0$$

non multum habet in recessu; posito enim $Y + Q = z$ prodit

$$dz - \frac{zdS}{2S} + \frac{dS(zz-S)}{4QS} = 0$$

in qua ut bini priores termini in unum contrahantur, ponatur $z = v\sqrt{S}$ reperieturque

$$dv\sqrt{S} + \frac{vvdS}{4Q} - \frac{dS}{4Q} = 0 \quad \text{seu} \quad \frac{dv}{vv-1} + \frac{dS}{4Q\sqrt{S}} = 0;$$

quae cum sit separata, integrale erit

$$\frac{1}{2}l\frac{1+v}{1-v}=\frac{1}{4}\int\frac{dS}{Q\sqrt{S}},$$

ubi est $v=\frac{y+Q}{\sqrt{S}}$.

Aequatio autem in ipsa solutione [§ 528] inventa

$$dy+yydx+\frac{Cdx}{S^4}-\frac{ddS}{Sdx}=0,$$

ubi S est functio quaecunque ipsius x et $\frac{ddS}{dx}=d:\frac{dS}{dx}$, magis ardua videtur, dum per se fit integrabilis, si dividatur per

$$SSyy-\frac{2SydS}{dx}+\frac{dS^2}{dx^2}+\frac{C}{SS}=\left(Sy-\frac{dS}{dx}\right)^2+\frac{C}{SS}.$$

At sumto x constante integrale reperitur

$$\frac{1}{\sqrt{C}}\text{Arc. tang.}\frac{SSydx-SdS}{dx\sqrt{C}}+V=\text{Const.};$$

nunc ergo ad functionem V inveniendam sumatur differentiale posita y constante, quod est

$$\frac{2SydS-\frac{SddS}{dx}-\frac{dS^2}{dx}}{SS\left(Sy-\frac{dS}{dx}\right)^2+C}+dV$$

et aequari debet alteri parti

$$\frac{\frac{Cdx}{S^4}-\frac{ddS}{Sdx}+yydx}{\left(Sy-\frac{dS}{dx}\right)^2+\frac{C}{SS}}=\frac{\frac{Cdx}{SS}-\frac{SddS}{dx}+SSyydx}{SS\left(Sy-\frac{dS}{dx}\right)^2+C}.$$

Ergo

$$dV=\frac{SSyydx-2SydS+\frac{dS^2}{dx}+\frac{Cdx}{SS}}{SS\left(Sy-\frac{dS}{dx}\right)^2+C}=\frac{dx}{SS}.$$

Quocirca integrale completum est

$$\frac{1}{\sqrt{C}}\text{Arc. tang.}\frac{SSydx-SdS}{dx\sqrt{C}}+\int\frac{dx}{SS}=D.$$

Quodsi sumamus $S=x$, huius aequationis [§ 491]

$$dy+yydx+\frac{Cdx}{x^4}=0$$

integrale completum est

$$\frac{1}{\sqrt{C}}\text{Arc.tang.}\frac{xxy-x}{\sqrt{C}}-\frac{1}{x}=D.$$

Sin autem sit $S=x^n$, ob $\frac{dS}{dx}=nx^{n-1}$ et $d.\frac{dS}{dx}=n(n-1)x^{n-2}dx$ integrari poterit haec aequatio

$$dy+yydx+\frac{Cdx}{x^{4n}}-\frac{n(n-1)dx}{xx}=0;$$

integrale enim erit

$$\frac{1}{\sqrt{C}}\text{Arc.tang.}\frac{x^{2n}y-nx^{2n-1}}{\sqrt{C}}-\frac{1}{(2n-1)x^{2n-1}}=D.$$

Supra autem [§ 436] invenimus hanc aequationem

$$dy+yydx+Cx^m dx=0$$

ad separationem reduci posse, quoties fuerit $m=\frac{-4i}{2i\pm1}$; iisdem ergo casibus functionem S assignare licebit, ut fiat

$$\frac{C}{S^4}-\frac{ddS}{Sdx^2}=Cx^m;$$

quod cum ad aequationes differentiales secundi gradus pertineat, hic non attingemus.

PROBLEMA 69

534. *Definire functiones P et Q ambarum variabilium x et y, ut aequatio differentialis* $Pdx+Qdy=0$ *divisa per* $Px+Qy$ *fiat per se integrabilis.*

SOLUTIO

Cum formula $\frac{Pdx+Qdy}{Px+Qy}$ debeat esse integrabilis, statuamus $Q=PR$, ut habeamus $\frac{dx+Rdy}{x+Ry}$, sitque $dR=Mdx+Ndy$. Quare fieri oportet

$$\frac{1}{dy}d.\frac{1}{x+Ry}=\frac{1}{dx}d.\frac{R}{x+Ry},$$

unde nanciscimur

$$\frac{-R-Ny}{(x+Ry)^2}=\frac{Mx-R}{(x+Ry)^2}$$

seu $N=-\frac{Mx}{y}$; hinc fit

$$dR=Mdx-\frac{Mxdy}{y}=My\frac{ydx-xdy}{yy};$$

quae formula cum debeat esse integrabilis, necesse est sit My functio ipsius $\frac{x}{y}$, quia $\frac{ydx-xdy}{yy}=d.\frac{x}{y}$, atque ex hac integratione prodit $R=\text{funct.}\frac{x}{y}$ seu, quod eodem redit, R erit functio nullius dimensionis ipsarum x et y. Quocirca cum $\frac{Q}{P}=R$, manifestum est huic conditioni satisfieri, si P et Q fuerint functiones homogeneae eiusdem dimensionum numeri ipsarum x et y; hoc ergo modo eandem integrationem aequationum homogenearum sumus assecuti, quam in capite superiori [§ 477] docuimus.

COROLLARIUM 1

535. Cum igitur $\frac{dt+Rdu}{t+Ru}$ sit integrabile, si fuerit $R=f:\frac{t}{u}$ seu $R=\frac{t}{u}f:\frac{t}{u}$ etiam haec formula

$$\frac{\frac{dt}{t}+\frac{du}{u}f:\frac{t}{u}}{1+f:\frac{t}{u}}$$

integrabilis, quae ita repraesentari potest

$$\frac{\frac{dt}{t}+\frac{du}{u}f:\left(\int\frac{dt}{t}-\int\frac{du}{u}\right)}{1+f:\left(\int\frac{dt}{t}-\int\frac{du}{u}\right)}$$

si fuerit ubi littera f denotat functionem quamcunque quantitatis suffixae.

COROLLARIUM 2

536. Ponatur $\frac{dt}{t}=\frac{dx}{X}$ et $\frac{du}{u}=\frac{dy}{Y}$ atque haec formula

$$\frac{\frac{dx}{X}+\frac{dy}{Y}f:\left(\int\frac{dx}{X}-\int\frac{dy}{Y}\right)}{1+f:\left(\int\frac{dx}{X}-\int\frac{dy}{Y}\right)}=\frac{dx+\frac{Xdy}{Y}f:\left(\int\frac{dx}{X}-\int\frac{dy}{Y}\right)}{X+Xf:\left(\int\frac{dx}{X}-\int\frac{dy}{Y}\right)}$$

erit per se integrabilis. Quare posito

$$R=\frac{X}{Y}f:\left(\int\frac{dx}{X}-\int\frac{dy}{Y}\right)$$

haec formula $\frac{dx+Rdy}{X+RY}$ per se integrabilis, quaecunque functio sit X ipsius x et Y ipsius y.

COROLLARIUM 3

537. Quare si quaerantur functiones P et Q, ut haec aequatio $Pdx+Qdy=0$ fiat integrabilis, si dividatur per $PX+QY$ existente X functione quacunque ipsius x et Y ipsius y, debet esse

$$\frac{Q}{P}=\frac{X}{Y}\,\text{funct.}\left(\int\frac{dx}{X}-\int\frac{dy}{Y}\right).$$

COROLLARIUM 4

538. Quare si signa φ et ψ functiones quascunque indicent fueritque

$$P=\frac{V}{X}\varphi:\left(\int\frac{dx}{X}-\int\frac{dy}{Y}\right)\quad\text{et}\quad Q=\frac{V}{Y}\psi:\left(\int\frac{dx}{X}-\int\frac{dy}{Y}\right),$$

haec aequatio $Pdx+Qdy=0$ integrabilis reddetur, si dividatur per $PX+QY$.

SCHOLION

539. Hinc ergo innumerabiles aequationes proferri possunt, quas integrare licebit, etiamsi alioquin difficillime pateat, quomodo eae ad separationem variabilium reduci queant. Verum haec investigatio proprie ad librum secundum *Calculi Integralis* est referenda, cuius iam egregia specimina hic habentur; definivimus enim functionem *R* binarum variabilium *x* et *y* ex certa conditione inter *M* et *N* proposita, scilicet $Mx + Ny = 0$ seu, $x\left(\frac{dR}{dx}\right) + y\left(\frac{dR}{dy}\right) = 0$ hoc est ex certa differentialium conditione.