## CHAPTER IV

# CONCERNING THE PARTICULAR INTEGRATION OF DIFFERENTIAL EQUATIONS

### DEFINITION

**540.** *A particular integral of a differential equation is a relation of the variables satisfying the differential equation, which includes no new constant quantity within itself. Hence it opposes the complete integral, which includes a constant not present in the differential, and in which yet it is still necessary [for such a constant]to be present.*

[Thus, Euler declares that while the complete integral includes an unspecified constant: the particular integrals to be defined and investigated here may relate to the existence of solutions where the values of the added constant is zero or infinity, and in which cases the solution, perhaps found by inspection, degenerates into an asymptotic line, in which no added constant is apparent. Other situations to be shown arise in which an asymptotic line is evident as a solution, while some solutions may not be valid.]

### COROLLARY 1

**541.** Hence with a known complete integral, from that innumerable particular integrals can be shown, as other arbitrary constants are attributed to that and other values can be determined.

### COROLLARY 2

**542.** Hence with a proposed differential equation between the variables $x$ and $y$ all the functions of $x$, which substituted in place of $y$ satisfy the equation, will give particular integrals, unless perhaps they shall be complete.

### COROLLARY 3

**543.** Since each differential equation may be returned to this form $\frac{dy}{dx}=V$, with some function $V$ of $x$ and $y$ arising, if a relation of this kind should be set up between $x$ and $y$, from which equal values result for $\frac{dy}{dx}$ and $V$, then that may be taken as a particular integral.

### SCHOLIUM 1

**544.** Sometimes a particular integral is easily to come upon as if by prediction ; just as if this should be the proposed equation :

$$aady+yydx=aadx+xydx.$$

Clearly it is evident for that to be satisfied on putting $y=x$; which relation certainly is a particular integral, since not only does it involve no new constant but not even that

constant $a$ , which is present in the differential equation itself; and from which nothing about the complete integral can be deduced.
[This asymptotic line is found by considering values of $x$ and $y$ much greater than $a$.]
Quite often a certain known particular integral makes evident the way for finding the complete integral, just as comes about in using this example ; in which if we put $y = x + z$ , there becomes
$$aadx + aadz + xxdx + 2xzdx + zzdx = aadx + xxdx + xzdx$$
or
$$aadz + xzdx + zzdx = 0,$$
which equation on putting $z = \frac{aa}{v}$ changes into this $dv - \frac{xvdx}{aa} = dx$ , which multiplied by $e^{\int -\frac{xdx}{aa}} = e^{\frac{-xx}{2aa}}$ becomes integrable and gives
$$e^{\frac{-xx}{2aa}}v = \int e^{\frac{-xx}{2aa}}dx \quad \text{or} \quad v = e^{\frac{xx}{2aa}}\int e^{\frac{-xx}{2aa}}dx ,$$
which hence is especially transcendental, while still involving that most simple particular integral ; clearly if the constant taken within the integration $\int e^{\frac{-xx}{2aa}}dx$ is taken as infinite, there becomes $v = \infty$ and $z = 0$, from which $y = x$ .
But sometimes a particular integral is of little help in completing the investigation, as if this equation is had:
$$a^3dy + y^3dx = a^3dx + x^3dx,$$
evidently $y = x$ satisfies this equation; but on putting $y = x + z$ there is produced
$$a^3dz + 3xxzdx + 3xzzdx + z^3dx = 0,$$
the resolution of which is seen to be no easier than of that original equation.

## SCHOLIUM 2

**545.** In these examples the particular integral is at once clearly obvious ; but there are cases given, in which it is more difficult to be seen ; and hence though the way is seldom apparent how the complete integral may be come upon, yet on many occasions it is of great interest to know a particular integral, since from that sometimes the whole derivation can be made. Indeed now we turn our attention to all these problems, the solution of which is produced according to a differential equation, and an arbitrary constant is to be determined which is added through integration for each problem according to these conditions, thus so that there is always a need for a particular integral; whereby if it eventuates, that a particular known integral can itself be completed unaided, then the solution of the problem can be shown, even if the [complete] solution of the differential equation shall not be in force. Hence from which cases, a true solution is supposed to be found without integration, as since properly speaking no differential equation is to be considered to be integrated, unless the complete integral of this can be assigned. On account of which it will be useful to consider these cases, in which it is possible to show a particular integral.

### SCHOLIUM 3

**546.** But here the greatest attention must be paid to some equation not satisfying all the values of a particular integral for this to be possible. Just as if this equation should be had:

$$dy = \frac{dx}{\sqrt{(a-x)}} \quad \text{or} \quad \frac{dx}{dy} = \sqrt{(a-x)},$$

on putting $x = a$ there becomes both $\sqrt{(a-x)} = 0$ as well as $\frac{dx}{dy} = 0$, thus so that the equation $x = a$ satisfies the differential equation, while yet by no means shall it be a particular integral of this equation. For the complete integral is

$$y = C - 2\sqrt{(a-x)} \quad \text{or} \quad a - x = \tfrac{1}{4}(C-y)^2,$$

from which, whatever the value attributed to the constant *C*, under no circumstance does it follow that $a - x = 0$.
In a similar manner the finite equation $xx + yy = aa$ satisfies this equation

$$dy = \frac{xdx + ydy}{\sqrt{(xx+yy-aa)}}$$

which yet is not to be admitted among the particular integrals, on account of which not being contained in the complete integral

$$y = C + \sqrt{(xx+yy-aa)}.$$

Whereby it is not sufficient that a particular integral satisfies the differential equation for that, but in addition it is required to append this condition, that it must be contained in the complete integral; from which the investigation of particular integrals is rendered greatly uncertain, unless likewise the complete integral is known ; but with this known it may be superfluous for a particular method to be enquired about for particular integrals. But then it helps most in having to take recourse to the investigation of particular integrals, when the complete integral is not able to be elicited. From which therefore in order that we may grasp the reward, it is agreed that criteria are to be treated, from values satisfying some differential equation, and it is allowed to decide whether or not they shall be particular integrals. Clearly even if all the integrals are formed from values of the kind which satisfy the differential equation, yet in turn not all the values which satisfy the differential equation are integrals. As until now little attention has been given to this problem, I will work on this difficulty, so that I may set out this argument clearly.

### PROBLEM 70

**547.** *If in the differential equation* $dy = \frac{dx}{Q}$*; the function Q vanishes on putting x = a, to determine in which cases this equation x = a shall be a particular integral of the proposed differential equation.*

### SOLUTION

Since $Q = \frac{dx}{dy}$, on putting $x = a$ there becomes both $Q = 0$ as well as $\frac{dx}{dy} = 0$, from which this value $x = a$ certainly satisfies the proposed differential equation $dy = \frac{dx}{Q}$; yet it does not hence follow that it is a [particular] integral. Clearly this alone does not suffice, for in addition it is required, that the equation $x = a$ should be contained in the complete integral, if through integration a certain value should be given to the constant added. Hence we may put $P$ to be the integral of the formula $\frac{dx}{Q}$, so that the complete integral shall be $y = C + P$; which equation on putting $x = a$ cannot be satisfied, unless on putting $x = a$ there also becomes $P = \infty$; for then with the constant $C$ assumed equally infinite on placing $x = a$, the quantity $y$ remains indeterminate, and thus if there is put $x = a$ there becomes $P = \infty$; then at last the equation $x = a$ can be taken as a particular integral. Behold therefore the criterion, by which it is possible to distinguish, whether or not each value $x = a$ of the differential equation likewise satisfying $dy = \frac{dx}{Q}$ shall be a particular integral of this ; then at last evidently it will be an integral, if on putting $x = a$ not only is there $Q = 0$, but also the integral $P = \int \frac{dx}{Q}$ becomes infinite.
[e.g. geometrically, we could perhaps say that $x = a$ is a vertical asymptote to the curve.]

So that we may explain this more clearly, since on putting $x = a$ there becomes $Q = 0$, we may put $Q = (a - x)^n R$ with $n$ denoting some positive number, and while the equation

$$dy = \frac{dx}{Q} = \frac{dx}{(a-x)^n R}$$

is able to adopt this form

$$dy = \frac{\alpha dx}{(a-x)^n} + \frac{\beta dx}{(a-x)^{n-1}} + \frac{\gamma dx}{(a-x)^{n-2}} + \ldots\ldots \frac{Sdx}{R},$$

the reason for this infinite $P$ depends on the term $\int \frac{\alpha dx}{(a-x)^n}$; which on putting $x = a$ prevails as infinite, and also the integral $P = \int \frac{dx}{Q}$ will be infinite, whatever the values the rest of the members themselves. But there is $\int \frac{\alpha dx}{(a-x)^n} = \frac{\alpha}{(n-1)(a-x)^{n-1}}$ which expression becomes infinite on putting $x = a$, provided $n - 1$ shall be a positive number or also $n = 1$.

Whereby provided the exponent *n* shall not be less than one on putting $Q=(a-x)^n R$, the equation $x=a$ can be taken as a particular integral.

### COROLLARY 1

**548.** Hence as often as the exponent *n* is less than one on putting $Q=(a-x)^n R$, it does not follow that $x=a$ will be a particular integral for the equation $dy=\frac{dx}{Q}$, even if it satisfies the differential equation in this manner.

### COROLLARY 2

**549.** If the exponent *n* is less than one, the formula $\frac{dQ}{dx}$ becomes infinite on putting $x=a$, from which we come upon a new criterion. Clearly for the proposed equation $dy=\frac{dx}{Q}$, if on putting $x=a$ there becomes not only $Q=0$ but also $\frac{dQ}{dx}=\infty$, then the value $x=a$ is not an integral of this equation.

### COROLLARY 3

**550.** Therefore with these cases of the equation $dy=\frac{dx}{Q}$ excluded, where on putting $x=a$ there becomes $Q=0$, the particular integral will always be $x=a$, unless in the same case $x=a$ makes $\frac{dQ}{dx}=\infty$; that is , [it will be a particular integral] as long as the value of the formula $\frac{dQ}{dx}$ should be either finite or vanish.

[Roughly speaking, if *Q* is zero and $\frac{dQ}{dx}$ is finite at $x = a$, then *Q* only changes by a finite amount within an increment of *x*, and so the inverted quantity is still infinite in the same sense; an infinite change in $\frac{dQ}{dx}$ may upset this arrangement in some way.]

### SCHOLIUM 1

**551.** This conclusion, by leaning on the inverse of hypothetical propositions, can be viewed with suspicion, and as being against the rules of logic, but truly the whole reasoning of the rules is entirely consistent, since it may be inferred that [merely a change has been made in the order of the implications] from the removal of the consequence to the removal of the antecedent. For as long as the exponent *n* is less than one on putting $Q=(a-x)^n R$, then so also does $\frac{dQ}{dx}$ become $=\infty$ on putting $x=a$. Whereby if on putting $x = a$ there is not made $\frac{dQ}{dx}=\infty$ and thus the value of this is either finite or vanishes, then certainly the exponent *n* is not less than one ; hence it will be either greater than or equal to one itself, moreover in each case the integral $P=\int\frac{dx}{Q}$ on putting $x=a$ becomes infinite and thus the equation $x=a$ is a particular integral. Whereby if in the differential equation $dy = \frac{dx}{Q}$ on putting $x = a$ there becomes $Q = 0$, the value of $\frac{dQ}{dx}$ is examined for the case $x = a$; which if it should be either finite or vanish, then the equation $x = a$ is a particular integral; but if it should be infinite then that cannot be considered as a particular integral, even if it satisfies the differential equation. The same rule also can be treated as if the differential equation should be of this kind : $dy = \frac{Pdx}{Q}$ or $\frac{dy}{dx} = \frac{P}{Q}$ and on putting $x = a$ there becomes $Q = 0$, whatever function *P* should be of *x* and *y;* as also it is not necessary that *Q* should be a function of the variable *x* only, but likewise the other variable *y* can be involved in some manner.

## SCHOLIUM 2

**552.** Indeed from that a demonstration has been sought, so that the quantity *Q*, which has vanished on putting $x = a$, should involve a certain power of the factor $a - x$, which has been shown from algebraic functions. Now the same rule has a place with transcendental functions [*i.e.* the log and exponential], since with such higher qualities there are quantities equivalent to powers.
Just as if there may be

$$dy = \frac{dx}{lx - la}$$

where $Q = lx - la = l\frac{x}{a}$ and there becomes $Q = 0$ on putting $x = a$, there is sought $\frac{dQ}{dx} = \frac{1}{x}$; which formula since it is not made infinite on putting $x = a$, a particular integral shall be $x = a$. Which indeed prevails for the equation

$$dy = \frac{Pdx}{lx - la}$$

as long as *P* does not become $= 0$ on putting $x = a$. For let $P = \frac{1}{x}$; then it becomes on integration $y = C + l(lx - la)$ and $l\frac{x}{a} = e^{y-C}$. Now on taking the constant $C = \infty$ there becomes $l\frac{x}{a} = 0$ and thus $x = a$, which therefore is a particular integral. In a similar manner if there shall be

$$dy = Pdx : \left(e^{\frac{x}{a}} - e\right),$$

where $Q = e^{\frac{x}{a}} - e$ and thus on putting $x = a$ there becomes $Q = 0$, since $\frac{dQ}{dx} = \frac{1}{a}e^{\frac{x}{a}}$ and hence on putting $x = a$ there becomes $\frac{dQ}{dx} = \frac{e}{a}$, then also $x = a$ is a particular integral. There is taken $P = e^{\frac{x}{a}}$, in order that the integration may follow, and since $y = C + al\left(e^{\frac{x}{a}} - e\right)$ and hence $e^{\frac{x}{a}} = e + e^{\frac{y-C}{a}}$, and there is put $C = \infty$; then $e^{\frac{x}{a}} = e$ and thus $x = a$, which clearly is hence a particular integral.

### EXAMPLE 1

**553.** *With the proposed differential equation* $dy = \frac{Pdx}{\sqrt{S}}$*, in which S vanishes on putting* $x = a$*, to define the case, in which the equation* $x = a$ *is a particular integral of this.*

Since here there shall be $\sqrt{S} = Q$, then there becomes $dQ = \frac{dS}{2\sqrt{S}}$; hence in order that the particular integral shall be $x = a$, it is necessary, that on putting $x = a$ the quantity $\frac{dQ}{dx} = \frac{dS}{2dx\sqrt{S}}$ is made finite. Hence in the same case the quantity $\frac{dS^2}{Sdx^2}$ is finite, from which, since $S$ vanishes, also $\frac{dS^2}{dx^2}$ and hence $\frac{dS}{dx}$ must vanish. Then moreover on putting $x = a$ the value of that fraction is

$$\frac{2dSddS}{dSdx^2} = \frac{2ddS}{dx^2},$$

which hence is required to remain finite or be equal to zero.

Whereby in order that the equation $x = a$ shall be a particular integral of the proposed equation, these conditions are required, first so that on putting $x = a$ there becomes $S = 0$, secondly so that there becomes $\frac{dS}{dx} = 0$, and thirdly the value produced of this formula $\frac{ddS}{dx^2}$ is either finite or equal to zero, provided it does not become infinitely great. If $S$ shall be a rational function, these are reduced to that, so that $S$ has the factor $(a-x)^2$ or a higher power.

### SCHOLIUM

**554.** This resolution has a use in discerning the motion of a body attracted to some centre of force, whether it should become in a circle. For if the distance of the body from the centre is put equal to $x$ and the centripetal force agreeing with this distance is equal to $X$, for the time $t$ of such here is found the equation

$$dt = \frac{xdx}{\sqrt{\left(Exx - c^4 - 2\alpha xx\int Xdx\right)}},$$

where $E$ is a constant introduced from the preceding integration, the value of which is sought, in order that hence the value is satisfied by the equation $x = a$, in which case the body is revolving in a circle. Hence here there is $S = Exx - c^4 - 2\alpha xx\int Xdx$ or it is possible to take

$$S = E - \frac{c^4}{xx} - 2\alpha\int Xdx$$

Hence not only this quantity, but also the differential of this

$$\frac{dS}{dx} = \frac{2c^4}{x^3} - 2\alpha X$$

must vanish on putting $x=a$, yet the second order equation

$$\frac{ddS}{dx^2}=-\frac{6c^4}{x^4}-2\alpha\frac{dX}{dx}$$

must not become infinite. From which hence the constant *a* will be the value of *x* resulting from this equation $\alpha x^3X=c^4$, which is the radius of a circle, in which the body is able to revolve, provided the constant *E,* upon which the speed depends, shall be able to be compared, as on putting $x=a$ there becomes $E=\frac{c^4}{aa}+2\alpha\int Xdx$; unless perhaps in the same case the expression $\frac{6c^4}{x^4}+2\alpha\frac{dX}{dx}$ or rather this term $\frac{dX}{dx}$ becomes infinite. For if this comes about, the motion at *x* in a circle may be removes, in order that this may be shown we may put

$$X=b+\sqrt{(a-x)},$$

so that

$$\frac{dX}{dx}=-\frac{1}{2\sqrt{(a-x)}}$$

becomes infinite on putting $x=a$, and the equation $ax^3X=c^4$ gives $\alpha a^3b=c^4$. Then truly on account of

$$\int Xdx=bx-\tfrac{2}{3}(a-x)^{\frac{3}{2}}$$

there becomes $E=\alpha ab+2\alpha ab=3\alpha ab$ and our equation becomes

$$dt=\frac{xdx}{\sqrt{\left(3\alpha abxx-\alpha a^3b-2\alpha bx^3+\frac{4}{3}\alpha xx(a-x)^{\frac{3}{2}}\right)}}$$

to which the value $x=a$ certainly does not agree as an integral. For if there is made

$$S=\alpha(a-x)\left(-aab-abx+2bxx+\tfrac{4}{3}xx\sqrt{(a-x)}\right),$$

as the factor of this shall not be $(a-x)^2$, but only $(a-x)^{\frac{3}{2}}$, the particular integral $x=a$ is unable to have a place.

## EXEMPLUM 2

**555.** *From the proposed differential equation* $dy=\frac{Pdx}{\sqrt[n]{S^m}}$, *in which S vanishes on putting* $x=a$, *to find the cases in which it is the particular integral* $x=a$.

Since there becomes $S=0$ on putting $x=a$, it is permitted to consider $S=(a-x)^{\lambda}R$ and the denominator shall be

$$\sqrt[n]{S^m} = (a-x)^{\frac{\lambda m}{n}} R^{\frac{m}{n}},$$

from which it is apparent that the equation $x = a$ becomes a particular integral of the proposed equation, if $\frac{\lambda m}{n}$ should be a positive number greater than one or at any rate equal to one, that is, if there shall be either $\lambda = \frac{n}{m}$ vel $\lambda > \frac{n}{m}$, which judgement can be easily established, if $S$ should be an algebraic function. But if it should be a transcendental function, so that the exponent $\lambda$ cannot be shown in numbers, another rule is allowed to be used ; clearly since there shall be $\sqrt[n]{S^m} = Q$, then there becomes $\frac{dQ}{dx} = \frac{mS^{\frac{m-n}{n}} dS}{ndx}$, the value of which must be finite or zero on putting $x = a$, if indeed the integral shall become $x = a$. Therefore it is necessary too in this case that the quantity $\frac{S^{m-n} dS^n}{dx^n}$ shall be finite. Hence the value of this formula is sought in the case $x = a$, which if produces an infinite magnitude, then the equation $x = a$ will not be an integral, but if it becomes finite or zero, that certainly will be a particular integral of the proposed equation. Here the two cases are to be established, according to whether $m > n$ or $m < n$.

I. If $m > n$, since on putting $x = a$ there becomes $S^{m-n} = 0$, unless in the same case there becomes $\frac{dS}{dx} = \infty$, certainly $x = a$ is an integral. But if there becomes $\frac{dS}{dx} = \infty$, each can happen, so that it may be an integral or not. In order that this can be discerned there is put $\frac{dx}{dS} = T$, so that our formula $S^{m-n} : T^n$ prevails, both the numerator and denominator of which vanish on putting $x = a$, from which the value of this is reduced to

$$\frac{(m-n)S^{m-n-1}dS}{nT^{n-1}dT} = \frac{-(m-n)S^{m-n-1}dS^{n+2}}{ndx^n ddS}$$ [from L'Hôpital's Rule]

which if it should be either finite or zero, then the integral will be $x = a$. In a like manner it is possible to progress further by distinguishing the cases $m > n+1$ and $m < n+1$.

II. If $m < n$, our formula will be $\frac{dS^n}{S^{n-m} dx^n}$ the value of which so that it becomes finite by necessity shall be $\frac{dS}{dx} = 0$, and in addition, because the numerator and denominator vanish on putting $x = a$, the value of our formula will be

$$\frac{nS^{n-1}ddS}{(n-m)S^{n-m-1}dSdx^n} = \frac{ndS^{n-2}ddS}{(n-m)S^{n-m-1}ndx^n}$$

which is required to remain finite.

But the judgement is easily absolved at once by putting $x = a + \omega$; while indeed on putting $x = a$ there is made $S = 0$, by this substitution the quantity $S$ can always be resolved into a form in this way $P\omega^\alpha + Q\omega^\beta + R\omega^\gamma + \text{etc.}$, of which only one term

$P\omega^{\alpha}$ embracing the least power of $\omega$ may be considered; and if there should be either $\alpha=\frac{n}{m}$ or $\alpha>\frac{n}{m}$ then the equation $x=a$ certainly shall be a particular integral.

## SCHOLIUM

**556.** This last method is the safest and always also it can be applied with the best success in transcendental formulas. Clearly with the proposed equation $dy=\frac{Pdx}{Q}$, in which on putting $x=a$ there becomes $Q=0$ and neither also does the numerator $P$ vanish, there is established $x=a\pm\omega$ and the quantity $\omega$ may be considered as infinitely small, so that all the powers of this vanish before the smallest, and the quantity $Q$ will take a form of this kind $R\omega^{\lambda}$, from which it will be apparent, unless the exponent $\lambda$ should be less than one, that the equation $x=a$ certainly becomes a particular integral of the proposed equation. Just as if we should have

$$dy=\frac{dx}{\sqrt{\left(1+\cos.\frac{\pi x}{a}\right)}},$$

the denominator of which will vanish on assuming $x=a$ on account of $\cos.\pi=-1$, and we may put $x=a-\omega$ then there will be $\cos.\frac{\pi x}{a}=\cos.\left(\pi-\frac{\pi\omega}{a}\right)=-1+\frac{\pi\pi\omega\omega}{2aa}$ on account of $\omega$ being infinitely small, and hence the denominator of our equation shall become equal to $\frac{\pi\omega}{a\sqrt{2}}$, from which we infer that the particular integral certainly shall be $x=a$. But it shall not be the integral of this equation

$$dy=\frac{dx}{\sqrt[3]{\left(1+\cos.\frac{\pi x}{a}\right)}}.$$

## PROBLEM 71

**557.** *With a proposed differential equation, in which the variables have been separated from each other, to investigate the particular integrals of this.*

## SOLUTION

Let this be the proposed equation $\frac{dx}{X}=\frac{dy}{Y}$, in which $X$ shall be a function of $x$ only and $Y$ of $y$ only. And initially there is put $X=0$ and thence there are sought the values of $x$, and of which there shall be $x=a$, thus so that on putting $x=a$ there becomes $X=0$; then there may be examined the value of the formula $\frac{dX}{dx}$ on putting $x=a$, which unless it becomes infinite, certainly $x=a$ will be a particular integral of the proposed equation. Or there may be put $x=a\pm\omega$ by considering $\omega$ as an infinitely small quantity, and if it should give rise to $X=P\omega^{\lambda}$, then the exponent $\lambda$, unless it should be less than one, will indicate the integral $x=a$; but if it should be less than one, then the equation $x=a$ will not be had for an integral.

In a similar manner the denominator of the other part $Y$ may be examined; which if it vanishes on putting $y=b$ and in this case the formula $\frac{dY}{dy}$ does not become infinite, then the equation $y=b$ will be a particular integral; because hence it also comes about, if on putting $y=b\pm\omega$ there should be produced $y=Q\omega^{\lambda}$, where the exponent $\lambda$ should not be less than one.

## COROLLARY 1

**558.** Hence unless the separate members of the equation were fractions, of which the denominators vanish in certain cases, particular integrals of this kind are not given, unless perhaps in such an equation $Pdx=Qdy$ the factors $P$ and $Q$ in certain cases become infinite, but which are easily reduced to the preceding case.

## COROLLARY 2

**559.** Just as if there should be had

$$dx\,\text{tang.}\frac{\pi x}{2a}=\frac{dy}{b-y},$$

first a certain particular integral is $y=b$, then truly, because on putting $x=a$ there becomes $\text{tang.}\frac{\pi x}{2a}=\infty$, the first member may thus be shown as $dx{:}\cot.\frac{\pi x}{2a}=\infty$ the denominator of which on putting $x=a-\omega$ will become $\cot.\left(\frac{\pi}{2}-\frac{\pi\omega}{2a}\right)=\text{tang.}\frac{\pi\omega}{2a}=\frac{\pi\omega}{2a}$ where since the exponent of $\omega$ shall not be less than one, the equation $x=a$ also will be a particular integral.

## COROLLARIUM 3

**560.** Therefore it follows that sometimes two or more particular integrals can be assigned for the same equation. Just as for this equation

$$\frac{mdx}{a-x}=\frac{ndy}{b-y}$$

the particular integrals are $a-x=0$ and $b-y=0$, which also follow from the complete equation $(a-x)^m=C(b-y)^n$, the first on taking $C=0$, and the second on taking $C=\infty$.

## COROLLARIUM 4

**561.** In a similar manner four particular integrals are given for this equation

$$\frac{madx}{aa-xx}=\frac{nbdy}{bb-yy}:$$

$a+x=0,\quad a-x=0,\quad b+y=0,\quad b-y=0$. Now the complete integral is

$$\frac{m}{2}l\frac{a+x}{a-x}=\frac{1}{2}C+\frac{m}{2}l\frac{b+y}{b-y}\quad\text{or}\quad\left(\frac{a+x}{a-x}\right)^m=C\left(\frac{b+y}{b-y}\right)^n$$

or

$$(a+x)^m(b-y)^n = C(a-x)^m(b+y)^n$$

from which these follow at once.

### COROLLARY 5

**562.** Hence it is apparent, if there should be

$$dy = \frac{Pdx}{(a+x)^{\alpha}(b+x)^{\beta}(c+x)^{\gamma}},$$

then the particular integrals become $a+x=0, \quad b+x=0, \quad c+x=0$, but only if the exponents $\alpha,\beta,\gamma$ should not be less than one. Whereby if $Q$ shall be a rational function of $x$, with the proposed equation $dy=\frac{Pdx}{Q}$ all the factors of $Q$ put equal to zero produce particular integrals.

### SCHOLIUM 1

**563.** This also prevails for imaginary factors, even if we find little gain from this. For if the equation should be proposed

$$dy = \frac{adx}{aa+xx}$$

from the denominator $aa+xx$ there arises the particular integrals

$$x = a\sqrt{-1} \quad \text{and} \quad x = -a\sqrt{-1},$$

which from the complete integral, which is $y = C + \text{Ang.tang.}\frac{x}{a}$, may be less seen to follow . Now on putting $x = a\sqrt{-1}$ it is to be observed that $\text{Ang. tang.}\sqrt{-1} = \infty\sqrt{-1}$, from which if a similar form with the opposite sign is attributed to the constant $C$, with the other quantity $y$ remaining undefined, even if there is put $x = a\sqrt{-1}$, which arrangement therefore is required for the particular integral. For in general it is

$$\text{Ang.tang.}u\sqrt{-1} = \int \frac{du\sqrt{-1}}{1-uu} = \frac{\sqrt{-1}}{2} l \frac{1+u}{1-u}$$

from which on putting $u=+1$ or $u=-1$ there is produced $\infty\sqrt{-1}$, because the infinity is the reason why the assigned integrals can be treated.

On account of which in general it is allowed to affirm, if there should be $dy=\frac{Pdx}{Q}$ and the denominator $Q$ should have the factor $(a+x)^{\lambda}$, the exponent $\lambda$ of which is not less than one, then the equation $a+x=0$ always is a special integral. But if $\lambda$ is less than one, and if it should be positive, then $a+x=0$ is not a particular integral, even if on putting $x=-a$ it satisfies the differential equation.

## SCHOLIUM 2

**564.** There is no great mystery to anyone how great a warning must be agreed upon, since a value of this kind is able to satisfy the differential equation, but which yet is not an integral of this differential equation ; and thus it is scarcely apparent, how these are to be reconciled with the customary ideas of integrals. For as often as a proposed differential equation is allowed to show a relation between the variables, which substituted should satisfy or produce an identical equation, scarcely any doubt comes to mind, or at any rate that relation should be obtained for a particular integral of the differential equation, hence since still it easily falls into error. Just as if this finite equation $xx + yy = aa$ should satisfy this equation

$$dy\sqrt{(aa - xx - yy)} = xdx + ydy\,,$$

then still we will commit a great error, if we wish to have that for a particular integral, as that expression by no means is contained in the complete integral

$$y = C - \sqrt{(aa - xx - yy)}\,.$$

On account of which, if any differential equation must be satisfied, it is not yet permitted to infer in turn that any finite equation which satisfies that equation is to be an integral of this equation; truly it is therefore required, in order that a certain property shall be provided, and this which we have set out finally put in place, in order that such an integral must be contained in the complete integral. But this is not opposed to the true idea of integration, as we have established here, nor is it possible to fall into any doubts of this kind with integrations found by following certain rules, yet only in integrals of this kind is it to be treated, which we are able to pursue as if from guesswork. For on many occasions, when an integration has not been successful, it is customary to try many guesses ; therefore then it must be warned maximally, lest we offer some relation rashly satisfying the relation for a particular integral. Since now we shall follow with separated equations, we shall carefully investigate in all the differential equations of this kind how it is required to avoid such errors.

## PROBLEM 72

**565**. *If some relation between the two variables should satisfy a differential equation, to define whether or not each shall be a particular integral.*

## SOLUTION

Let $Pdx = Qdy$ be the proposed differential equation, where $P$ and $Q$ are some functions of $x$ and $y$, which is satisfied by some relation between $x$ and $y$, from which there becomes $y = X$ , clearly for a certain function of $x$, thus so that, if in place of $y$ everywhere there is written $X$, there may actually be produced $Pdx = Qdy$ or $\frac{dy}{dx} = \frac{P}{Q}$.
Hence it is inquired, whether or not this value $y = X$ can be had for an integral of the proposed equation.

Towards making this judgement there may be put $y = X + \omega$ and there becomes $\frac{dX}{dx} + \frac{d\omega}{dx} = \frac{P}{Q}$ , where it may be noted, if there should be $\omega = 0$, to become $\frac{dX}{dx} = \frac{P}{Q}$. Whereby on account of $\omega$ the expression $\frac{P}{Q}$ is reduced by this substitution to $\frac{dX}{dx}$ together with a quantity thus affected by $\omega$, so that it vanishes on putting $\omega = 0$. In this derivation it suffices that $\omega$ be considered as an infinitely small part, of which hence the higher powers are permitted to be ignored. Therefore we may put hence

$$\frac{P}{Q} = \frac{dX}{dx} + S\omega^{\lambda}$$

and there will be found $\frac{d\omega}{dx} = S\omega^{\lambda}$ or $\frac{d\omega}{\omega^{\lambda}} = Sdx$. Now from the above it is evident finally to be with the particular integral $y = X$ or $\omega = 0$, when the exponent $\lambda$ should be equal to or greater than one ; for here and above the reason is the same, from which it is required that \ $\int Sdx = \int \frac{d\omega}{\omega^{\lambda}}$ becomes infinite in the case proposed, in which $\omega = 0$; but this does not come about unless $\lambda$ shall be greater than or equal to one.

But hence if the value $y = X$ should satisfy the equation $Pdx = Qdy$ or $\frac{dy}{dx} = \frac{P}{Q}$, there may be established $y = X + \omega$ with $\omega$ considered as an infinitely small part, and hence this form can be investigated:

$$\frac{P}{Q} = \frac{dX}{dx} + S\omega^{\lambda},$$

from which, unless $\lambda < 1$, it may be concluded that the value $y = X$ is a particular integral of the proposed differential equation.

## SCHOLIUM

**566.** Since $\omega$ may be treated as an infinitely small quantity, the value of $\frac{P}{Q}$ on putting $y = X + \omega$ is seen to be found most conveniently by differentiation. For since $\frac{P}{Q}$ shall be a function of *x* and *y*, we may put in place $d.\frac{P}{Q} = Mdx + Ndy$, from which on putting $y = X$ the fraction $\frac{P}{Q}$ will change into $\frac{dX}{dx}$; by hypothesis, if in place of *y* there is written $X + \omega$, that will pass into $\frac{dX}{dx} + N\omega$, from which on account of the unit exponent of $\omega$ there is the following equation $y = X$ which is always a particular integral, since it is still possible to become otherwise. From which it is apparent that differentiation cannot be used in place of substitution ; so that this can be shown more clearly, we put $\frac{P}{Q} = \sqrt{(y - X)} + \frac{dX}{dx}$, from which on putting $y = X + \omega$ evidently there arises $\frac{P}{Q} = \frac{dX}{dx} + \sqrt{\omega}$. But using differentiation, on putting $d.\frac{P}{Q} = Mdx + Ndy$ there becomes

$N = \frac{1}{2\sqrt{(y-X)}}$ and hence $\frac{P}{Q} = \frac{dX}{dx} + N\omega$, which expression differs from that. Evidently with that number removed from the equation $y = X$ of the integrations, this now is seen to be allowed. Truly and this is to be noted the quantity $N$ itself involves a negative power of $\omega$, from which the power $\omega$ may be suppressed. Whereby lest this is considered to be worth the effort, always truly the method by substitution is to be preferred with differentiation set aside. From this observation it will not be difficult to judge all the values which satisfy a certain differential equation, whether each shall be a true integral or not.

## EXAMPLE 1

**567.** *Since evidently $y = x$ may satisfy this differential $dx\left(1-y^m\right)^n = dy\left(1-x^m\right)^n$, to define whether each shall be a particular integral or not.*

There may be put $y = x + \omega$ and with $\omega$ observed as a minimal quantity there is

$$y^m = x^m + mx^{m-1}\omega$$

and

$$\left(1-y^m\right)^n = \left(1-x^m - mx^{m-1}\omega\right)^n = \left(1-x^m\right)^n - mnx^{m-1}\omega\left(1-x^m\right)^{n-1},$$

from which the equation $\frac{dy}{dx} = \frac{\left(1-y^m\right)^n}{\left(1-x^m\right)^n}$ will change into

$$1 + \frac{d\omega}{dx} = 1 - \frac{mnx^{m-1}\omega}{1-x^m} \text{ or } \frac{d\omega}{\omega} = -\frac{mnx^{m-1}dx}{1-x^m}$$

where since $\omega$ should have the dimension of a whole number, the equation $y = x$ certainly is the particular integral of the proposed differential equation.

## EXAMPLE 2

**568.** *Since the value $y = x$ satisfies this equation $ady - adx = dx\sqrt{(yy-xx)}$, to investigate whether or not this shall be a particular integral of this.*

There may be put $y = x + \omega$ and assuming $\omega$ to be an infinitely small quantity, since there shall be $\sqrt{(yy-xx)} = \sqrt{2x\omega}$, then $ad\omega = dx\sqrt{2x\omega}$ or $\frac{ad\omega}{\sqrt{\omega}} = \sqrt{2x}$. Therefore because here $d\omega$ is divided by a power of $\omega$, the exponent of which is less than one, it follows that the value $y = x$ is not a particular integral of the proposed equation, even if it should satisfy that equation. Evidently if the complete integral of this could be shown, it would be apparent, in whatever manner the arbitrary constant of integration entering were defined, in that the equation $y = x$ is not going to be contained.

### SCHOLION

**569.** Hence a new reason is understood, on which account the judgement of the integral depends on the exponent of $\omega$. For since in the proposed example on making $y = x + \omega$ there appears $\frac{ad\omega}{\sqrt{\omega}} = \sqrt{2x}$, then on integration $2a\sqrt{\omega} = C + \frac{2}{3}x\sqrt{2x}$. Now by the hypothesis $\omega$ is an infinitely small quantity, but hence in whatever manner the constant $C$ may be defined, the quantity $\omega$ gets a finite value, which thus however great it is able to prevail; since that is contrary to the hypothesis, by necessity it follows that the equation $y = x$ cannot be an integral and this must always come about, as often as the $d\omega$ appears divided by a power of $\omega$, the exponent of which is less than unity. Now it is apparent otherwise, with the substitution made set forth there emerges $\frac{d\omega}{\omega} = Rdx$, that on putting $\int Rdx = S$ becomes $l\omega = lC + lS$ or $\omega = CS$, with the constant $C$ assumed to vanish whenever the quantity $\omega$ itself is to vanish; which likewise comes about, if there should arise $\frac{d\omega}{\omega^{\lambda}} = Rdx$ taking $\lambda > 1$. For then $\frac{1}{(\lambda-1)\omega^{\lambda-1}} = C - S$ or $(\lambda-1)\omega^{\lambda-1} = \frac{1}{C-S}$, from which on taking $C = \infty$ the quantity $\omega$ actually becomes vanishing, as the hypothesis proposes.

The remaining equation of this example on putting $x = pp - qq$ and $y = pp + qq$ is free from irrationality and becomes

$$4aqdq = 4pq(pdp - qdq) \text{ or } adq = ppdp - pqdq,$$

which it is seen cannot be treated in any way ; and neither can the complete integral of this be shown. Since $x = y$ or $q = 0$ no further satisfies that equation, hence also it must be concluded that the value $x = y$ is not a particular integral.

### EXAMPLE 3

**570.** *While the value $y = x$ satisfies this equation $aady - aadx = dx(yy - xx)$, to investigate whether or not it is a particular integral of this.*

There may be put $y = x + \omega$ on regarding $\omega$ as an infinitely small quantity and on account of $yy - xx = 2x\omega$ our equation hence adopts the form $aad\omega = 2x\omega dx$ or $\frac{aad\omega}{\omega} = 2xdx$. Therefore because here $d\omega$ is divided by the first power of $\omega$, the equation $y = x$ certainly will be a particular integral of the proposed equation and thus is contained in the complete integral. Indeed this is found on putting $y = x - \frac{aa}{u}$, from which there becomes

$$\frac{a^4du}{uu} = dx\left(\frac{a^4}{uu} - \frac{2aax}{u}\right) \text{ or } du + \frac{2uxdx}{aa} = dx.$$

This can be multiplied by $e^{\frac{xx}{aa}}$ and the integral produced

$$e^{\frac{xx}{aa}}u = C + \int e^{\frac{xx}{aa}}dx \text{ and hence } y = x - aae^{\frac{xx}{aa}} : \left(C + \int e^{\frac{xx}{aa}}dx\right).$$

Hence if an infinite constant $C$ is taken, there becomes $y = x$.

## SCHOLION

**571.** If in this equation as above there is put $x = pp - qq$ and $y = pp + qq$, there arises $aadq = 2ppq(pdp - qdq)$, which $q = 0$ satisfies, from which the case $y = x$ is produced. But from this transformation it is hardly apparent, how the integral of this is required to be found. If indeed we carefully assess the above reduction, we may understand that the equation is reduced to integrability, if it is multiplied by $e^{(pp-qq)^2:aa}:q^3$; because by itself it is not readily apparent, the decision will be from this substitution to be used $pp - qq = rr$, which becomes $pp = qq + rr$ and $pdp - qdq = rdr$, from which the equation becomes

$$aadq = 2qrdr(qq + rr) \text{ or } \frac{aadq}{2q^3} = rdr + \frac{r^3}{qq},$$

which on putting $\frac{1}{qq} = s$ is integrated easily.

Hence as often as it is permitted to deduce a relation of this kind between the variables, which satisfies the differential equation, in this manner it is possible to judge, whether or not this relation can be had as a particular integral. But the rules for finding particular integrals of this kind are scarcely touched on ; for which rules that may be had appear the same as the rules for finding complete integrals. Thus which above we have observed concerning separate equations, on account of that itself, since they have been separated, a similar way to finding the complete integral has been found. In a like manner if another method succeeds by factors, generally from these factors, by which the equation is rendered integrable, particular integrals are able to be inferred ; just as we will declare in the following propositions.

### THEOREM

**572.** *If the differential equation $Pdx + Qdy = 0$ multiplied by the function M is rendered integrable, the particular integral will be $M = 0$, unless in the same case P or Q become infinite.*

### DEMONSTRATION

We may put *u* to be a factor of $M$ and it must be shown that the equation $u = 0$ is a particular integral of the proposed equation. Since *u* is equal to a certain function of *x* and *y*, thence the other variable *y* may be defined, in order that an equation may be produced between the variables *x* and *u*, which shall be $Rdx + Sdu = 0$, from which on putting the multiplier $M = Nu$, this form will be integrable

$$NRudx + NSudu = 0.$$

But if now neither *R* nor *S* is divided by *u*, in which case *u* = 0 and neither *P* nor *Q* becomes infinite, then the integral certainly will be divisible by *u*. For that is deduced from the term $NRudx$ on regarding *u* as constant or from the term $NSudu$ on regarding *x* constant, the integral produces an integral implying the factor *u*, if indeed in the integration the constant should be omitted. From which we conclude that the complete integral shall be had of this form $Vu = C$. Whereby if this constant *C* is taken equal to zero, the particular integral will be $u = 0$, clearly with these cases excepted, in which the functions *R* and *S* now are to be divided by *u* and thus our reasoning may lose its strength. Hence with these cases excluded, as often as the equation $Pdx + Qdy = 0$ shall be multiplied by the function *M* it shall become integrable by itself and that function *M* shall have a factor *u*, for which the particular integral shall be $u = 0$, which likewise prevails from the individual factors of the function $M$.

### SCHOLIUM

**573.** It is completely necessary that a limitation should be added, since with that ignored the general reasoning may be defective. So that this can be easier understood, we shall consider this equation

$$\frac{adx}{y-x} + dy - dx = 0,$$

which multiplied by $y - x$ evidently becomes integrable ; hence we may put this multiplier
$y - x = u$ or $y = x + u$, from which our equation becomes $\frac{adx}{u} + du = 0$,
which multiplied by *u* changes into $adx + udu = 0$; where since the part *adx* shall not be multiplied by *u*, neither is it allowed to conclude that the integral shall be divisible by *u*, evidently which is $ax + \frac{1}{2}uu$. Hence it is apparent, only if the part *dx* should be multiplied by *u*, even if the other part *du* with the factor *u* should be missing, yet the integral is divisible by *u*, just as comes about in $udx + xdu$, the integral of which *xu* certainly has the

factor *u*. From which it is understood, if the formula $Pudx + Qdu$ were integrable by itself, provided that *Q* should not be divided by *u* or by the first power of this or higher, also the integral with the constant omitted becomes divisible by *u*.

## THEOREM

**574**. *If the differential equation* $Pdx + Qdy = 0$ *divided by the function M becomes integrable by itself, the particular shall be* $M = 0$, *unless on putting* $M = 0$ *either* *P or Q should vanish.*

## DEMONSTRATION

The divisor *M* may have the factor *u*, so that there shall be $M = Nu$, and it is required to be shown that the particular integral shall become $u = 0$, that which is to be understood from the individual factors of the divisor of *M*, if indeed it should have several. Therefore since *u* shall be a function of *x* and *y*, thence the other *y* may be defined by *x* and *u*, so that the equation $Rdx + Sdu = 0$ is produced, which hence divided by *Nu* will itself become integrable. Therefore it is required to find the integral of the formula $\frac{Rdx}{Nu} + \frac{Sdu}{Nu}$, where we have assumed neither *R* nor *S* to be multiplied by *u* nor in this way the factor *u* to have been removed from the denominator. Now if this integral is deduced from the member $\frac{Rdx}{Nu}$ only by regarding *u* as constant, there will be produced $\frac{1}{u}\int\frac{Rdx}{N} + f:u$ ;[recall here Euler's early function notation *f:u* rather than *f(u)*] ; but if with the other member taken $\frac{Sdu}{Nu}$ with *x* constant there is deduced, because *S* does not have the factor *u*, that always thus will be comparable, as on putting $u = 0$ it is made infinite. From which the integral, which shall be $V$, thus shall be comparable, as on putting $u = 0$ it becomes $= \infty$ ; whereby since the complete integral shall become $V = C$, this equation is satisfied with the constant *C* infinite on putting $u = 0$. Hence we conclude, if the divisor $M = Nu$ reduces the differential equation $Pdx + Qdy = 0$ to be integrable by itself, which for some factor *u* of the divisor *M* there may be obtained the particular integral $u = 0$, unless perhaps on putting $u = 0$ , the quantities *P* and *Q* or *R* and *S* vanish.

## COROLLARY 1

**575.** If the equation $Pdx + Qdy = 0$ were homogeneous, that, as we have seen above, is rendered integrable, if it is divided by $Px + Qy$, whereby the particular integral of this shall be $Px + Qy = 0$. Which equation since also it shall be homogeneous, will have factors of the form $\alpha x + \beta y$, of which any equated to zero will give a particular integral.

**COROLLARY 2**

**576.** For this equation

$$ydx(c+nx)-dy(y+a+bx+nxx)=0$$

the divisor, by which it is rendered integrable, we have shown above (§ 488), from which the particular integral is inferred $y=0$, then truly

$$nyy+(2na-bc)y+n(b-2c)xy+(na+cc-bc)(a+bx+nxx)=0,$$

the roots of which are

$$ny=\tfrac{1}{2}bc-na+n\left(c-\tfrac{1}{2}b\right)x\pm(c+nx)\sqrt{\left(\tfrac{1}{4}bb-na\right)}.$$

**COROLLARY 3**

**577.** For this differential equation

$$\frac{ndx(1+yy)\sqrt{(1+yy)}}{\sqrt{(1+xx)}}+(x-y)dy=0$$

the divisor, by which it is rendered integrable, we have deduced above (§ 489), from which we infer the particular integral

$$x-y+n\sqrt{(1+xx)(1+yy)}=0$$

or

$$yy-2xy+xx=nn+nnxx+nnyy+nnxxyy,$$

from which again there becomes

$$y=\frac{x\pm n(1+xx)(1-nn)\sqrt{(1-nn)}}{1-nn(1+xx)}.$$

**COROLLARY 4**

**578.** For this differential equation

$$dy+yydx-\frac{adx}{x^4}=0$$

we have found the multiplier above $\frac{xx}{xx(1-xy)^2-a}$ (§ 491), from which we infer the particular integral $xx(1-xy)^2-a=0$ and hence

$$x(1-xy)=\pm\sqrt{a}\quad \text{or}\quad y=\frac{1}{x}\pm\frac{\sqrt{a}}{xx},$$

thus so that we have two particular integrals, but which become imaginary, if *a* should be a negative quantity.

## SCHOLIUM

**579.** These nearly are all that have been investigated about the treatment of differential equations up to this time, yet several aids in the working out of differential equations of the second order will be supplied below. But here these can be conveniently referred to, which concern the construction of certain transcendental formulas that have been investigated recently. For just as logarithms and arcs of circles, although they are transcending quantities, are thus to be compared among themselves and treated equally along with algebraic quantities in calculations, thus a similar comparison can be put in place between certain transcending quantities of a higher kind, which clearly are contained in the formula here

$$\int \frac{dx}{\left(A+Bx+Cx^2+Dx^3+Ex^4\right)},$$

where also a numerator of rationals can be added as you wish $\mathfrak{A}+\mathfrak{B}x+\mathfrak{C}x^2+\text{etc.}$ Because the argument shall be especially hard and thus the forces of analysis may be seen to be overcome, unless by a certain reason it may be arranged in analysis thence not to be scorned in which the extra parts are made redundant ; moreover in the first place the resolution of a not very difficult differential equations may be seen to be completed. For since an equation of this kind shall be proposed

$$\frac{dx}{\sqrt{\left(A+Bx+Cx^2+Dx^3+Ex^4\right)}}=\frac{dy}{\sqrt{\left(A+Bx+Cx^2+Dx^3+Ex^4\right)}},$$

indeed at once it is apparent that a particular integral of this shall be $x=y$, truly the complete integral is seen especially to be transcending, since each formula by itself is unable to be reduced either to logarithms or to the arcs of circles. Whereby with that the more to be wondered at, as the complete integral can thus be shown by an algebraic equation between *x* and *y*. But in order that the method leading to this high position shall be seen more clearly, we will apply that first to the transcending quantity you may note contained in this formula

$$\int \frac{dx}{\sqrt{(A+Bx+Cxx)}},$$

then the use of this in more complex formulas will be shown.

## CAPUT IV

# DE INTEGRATIONE PARTICULARI AEQUATIONUM DIFFERENTIALIUM

### DEFINITIO

**540.** *Integrale particulare aequationis differentialis est relatio variabilium aequationi satisfaciens, quae nullam novam quantitatem constantem in se complectitur. Opponitur ergo integrali completo, quod constantem in differentiali non contentam involvit, in quo tamen contineatur necesse est.*

### COROLLARIUM 1

**541.** Cognito ergo integrali completo ex eo innumerabilia integralia particularia exhiberi possunt, prout constanti illi arbitrariae alii atque alii valores determinati tribuuntur.

### COROLLARY 2

**542.** Proposita ergo aequatione differentiali inter variabiles $x$ et $y$ omnes functiones ipsius $x$, quae loco $y$ substitutae aequationi satisfaciunt, dabunt integralia particularia, nisi forte sint completa.

### COROLLARIUM 3

**543.** Cum omnis aequatio differentialis ad hanc formam $\frac{dy}{dx}=V$ revocetur existente $V$ functione quacunque ipsarum $x$ et $y$, si eiusmodi constet relatio inter $x$ et $y$, unde pro $\frac{dy}{dx}$ et $V$ resultent valores aequales, ea pro integrali particulari erit habenda.

### SCHOLION 1

**544.** Interdum facile est integrale particulare quasi divinatione colligere; veluti si proposita sit haec aequatio

$$aady+yydx=aadx+xydx.$$

Statim liquet ei satisfieri ponendo $y=x$; quae relatio cum non solum nullam novam constantem, sed ne eam quidem $a$, quae in ipsa aequatione differentiali continetur, implicet, utique est integrale particulare; unde nihil pro integrali completo colligere licet. Saepenumero quidem cognitio integralis particularis ad inventionem completi viam patefacit, quemadmodum in hoc ipso exemplo usu venit; in quo si statuamus $y=x+z$, fit

$$aadx+aadz+xxdx+2xzdx+zzdx=aadx+xxdx+xzdx$$

seu

$$aadz+xzdx+zzdx=0,$$

quae aequatio posito $z = \frac{aa}{v}$ abit in hanc $dv - \frac{xvdx}{aa} = dx$ , quae per $e^{\int -\frac{xdx}{aa}} = e^{\frac{-xx}{2aa}}$ multiplicata fit integrabilis et dat

$$e^{\frac{-xx}{2aa}} v = \int e^{\frac{-xx}{2aa}} dx \quad \text{seu} \quad v = e^{\frac{xx}{2aa}} \int e^{\frac{-xx}{2aa}} dx ,$$

quod ergo est maxime transcendens, cum tamen simplicissimum illud particulare involvat; scilicet si constans integratione $\int e^{\frac{-xx}{2aa}} dx$ invecta sumatur infinita, fit $v = \infty$ et $z = 0$, unde $y = x$.

Interdum autem integrale particulare parum iuvat ad completum investigandum, veluti si habeatur haec aequatio

$$a^3 dy + y^3 dx = a^3 dx + x^3 dx,$$

cui manifesto satisfacit $y = x$ ; posito autem $y = x + z$ prodit

$$a^3 dz + 3xxzdx + 3xzzdx + z^3 dx = 0,$$

cuius resolutio haud facilior videtur quam illius.

## SCHOLION 2

**545.** In his exemplis integrale particulare statim in oculos incurrit; dantur autem casus, quibus difficilius perspicitur; et quanquam raro inde via pateat ad integrale completum perveniendi, tamen saepenumero plurimum interest integrale particulare nosse, cum eo nonnunquam totum negotium confici possit. Iam enim animadvertimus in omnibus problematibus, quorum solutio ad aequationem differentialem perducitur, constantem arbitrariam per integrationem invectam ex ipsis conditionibus cuique problemati adiunctis determinari, ita ut semper integrali tantum particulari sit opus; quare si eveniat, ut hoc ipsum integrale particulare cognosci possit sine subsidio completi, solutio problematis exhiberi poterit, etiamsi integratio aequationis differentialis non sit in potestate. Quibus ergo casibus sine integratione vera solutio inveniri est censenda, propterea quod proprie loquendo nulla aequatio differentialis integrari existimatur, nisi eius integrale completum assignetur. Quocirca utile erit eos casus perpendere, quibus integrale particulare exhibere licet.

## SCHOLION 3

**546.** Maximi autem est momenti hic animadvertisse non omnes valores aequationi cuipiam differentiali satisfacientes pro eius integrali particulari haberi posse. Veluti si habeatur haec aequatio

$$dy = \frac{dx}{\sqrt{(a-x)}} \quad \text{seu} \quad \frac{dx}{dy} = \sqrt{(a-x)} ,$$

posito $x=a$ fit tam $\sqrt{(a-x)}=0$ quam $\frac{dx}{dy}=0$, ita ut aequatio $x=a$ illi differentiali satisfaciat, cum tamen nequaquam eius sit integrale particulare.
Integrale namque completum est

$$y=C-2\sqrt{(a-x)} \quad \text{seu} \quad a-x=\tfrac{1}{4}(C-y)^2,$$

unde, quicunque valor constanti $C$ tribuatur, nunquam sequitur $a-x=0$.
Simili modo huic aequationi

$$dy=\frac{xdx+ydy}{\sqrt{(xx+yy-aa)}}$$

satisfacit haec aequatio finita $xx+yy=aa$, quae tamen inter integralia particularia admitti nequit, propterea quod in integrali completo

$$y=C+\sqrt{(xx+yy-aa)}$$

neutiquam continetur.
Quare ad integrale particulare non sufficit, ut eo aequationi differentiali satisfiat, sed insuper hanc conditionem adiungi oportet, ut in integrali completo contineatur; ex quo investigatio integralium particularium maxime est lubrica, nisi simul integrale completum innotescat; hoc autem cognito supervacuum esset methodo peculiari in integralia particularia inquirere. Tum enim potissimum iuvat ad investigationem integralium particularium confugere, quando integrale completum elicere non licet. Quo igitur hinc fructum percipere queamus, criteria tradi conveniet, ex quibus valores, qui aequationi cuipiam differentiali satisfaciunt, diiudicare liceat, utrum sint integratia parlicularia necne. Etiamsi scilicet omnia integralia sint eiusmodi valores, qui aequationi differentiali satisfaciant, tamen non vicissim omnes valores, qui satisfaciunt, sunt integralia. Quod cum parum adhuc sit animadversum, operam dabo, ut hoc argumentum dilucide evolvam.

## PROBLEMA 70

**547.** *Si in aequatione differentiali $dy=\frac{dx}{Q}$; functio Q evanescat posito x = a, determinare, quibus casibus haec aequatio x = a sit integrate particulare aequationis differentialis propositae.*

## SOLUTIO

Cum sit $Q=\frac{dx}{dy}$, posito $x=a$ fit tam $Q=0$ quam $\frac{dx}{dy}=0$, unde hic valor $x=a$ aequationi differentiali propositae $dy=\frac{dx}{Q}$; utique satisfacit; neque tamen hinc sequitur eum esse integrale. Hoc solum scilicet non sufficit, sed insuper requiritur, ut aequatio $x=a$ in integrali completo contineatur, si quidem constanti per integrationem invectae certus quidam valor tribuatur. Ponamus ergo $P$ esse integrale formulae $\frac{dx}{Q}$, ut integrale completum sit $y = C + P$; cui aequationi ponendo $x = a$ satisfieri nequit, nisi posito $x = a$ fiat $P = \infty$; tum enim sumta constante *C* pariter infinita positione $x = a$ quantitas *y* manet indeterminata, ideoque si posito $x = a$ fiat $P = \infty$ ; tum demum aequatio $x = a$ pro integrali particulari erit habenda. En ergo criterium, ex quo dignoscere licet, utrum valor $x = a$ aequationi differentiali $dy = \frac{dx}{Q}$ satisfaciens simul sit eius integrale particulare necne; scilicet tum demum erit integrale, si posito $x = a$ non solum fiat $Q = 0$, sed etiam integrale $P = \int \frac{dx}{Q}$ abeat in infinitum.

Quod quo clarius exponamus, quoniam posito $x = a$ fit $Q = 0$, ponamus $Q = (a - x)^n R$ denotante *n* numerum quemcunque positivum, et cum aequatio

$$dy = \frac{dx}{Q} = \frac{dx}{(a-x)^n R}$$

induere queat hanc formam

$$dy = \frac{\alpha dx}{(a-x)^n} + \frac{\beta dx}{(a-x)^{n-1}} + \frac{\gamma dx}{(a-x)^{n-2}} + \ldots\ldots \frac{Sdx}{R},$$

ratio illius infiniti *P* pendebit a termino $\int \frac{\alpha dx}{(a-x)^n}$; qui si posito $x = a$ evadat infinitus, etiam integrale $P = \int \frac{dx}{Q}$ erit infinitum, utcunque se habeant reliqua membra. At est $\int \frac{\alpha dx}{(a-x)^n} = \frac{\alpha}{(n-1)(a-x)^{n-1}}$ quae expressio fit infinita posito $x = a$, dummodo $n - 1$ sit numerus positivus vel etiam $n = 1$. Quare dummodo exponens *n* non sit unitate minor posito $Q = (a - x)^n R$, aequatio $x = a$ pro integraIi particulari erit habenda.

## COROLLARIUM 1

**548.** Quoties ergo posito $Q = (a - x)^n R$ exponens *n* est unitate minor, aequationi $dy = \frac{dx}{Q}$ non convenit integrale particulare $x = a$, etiamsi hoc modo aequationi differentiali satisfiat.

## COROLLARIUM 2

**549.** Si exponens *n* est unitate minor, formula $\frac{dQ}{dx}$ fit infinita posito $x = a$, unde novum criterium adipiscimur. Scilicet proposita aequatione $dy = \frac{dx}{Q}$ si posito $x = a$ fiat quidem $Q = 0$, at $\frac{dQ}{dx} = \infty$, tum valor $x = a$ non est integrale particulare illius aequationis.

## COROLLARIUM 3

**550.** His igitur casibus exclusis aequationis $dy = \frac{dx}{Q}$, ubi posito $x = a$ fit $Q = 0$, integrale particulare semper erit $x = a$, nisi eodem casu $x = a$ fiat $\frac{dQ}{dx} = \infty$; hoc est, quoties valor formulae $\frac{dQ}{dx}$ fuerit vel finitus vel evanescat.

### SCHOLION 1

**551.** Haec conclusio inversioni propositionum hypotheticarum innixa licet videri queat suspecta ac regulis Logicae adversa, verum totum ratiocinium regulis apprime est consentaneum, cum a sublatione consequentis ad sublationem antecedentis concludat. Quoties enim posito $Q = (a - x)^n R$ exponens $n$ est unitate minor, toties $\frac{dQ}{dx}$ fit $= \infty$ posito $x = a$. Quare si posito $x = a$ non fiat $\frac{dQ}{dx} = \infty$ ideoque eius valor vel finitus vel evanescat, tum certe exponens $n$ non est unitate minor; erit ergo vel maior unitate vel ipsi aequalis, utroque autem casu integrale $P = \int \frac{dx}{Q}$; posito $x = a$ fit infinitum ideoque aequatio $x = a$ est integrale particulare. Quare si in aequatione differentiali $dy = \frac{dx}{Q}$ posito $x = a$ fiat $Q = 0$, examinatur valor $\frac{dQ}{dx}$ pro casu $x = a$; qui si fuerit vel finitus vel evanescat, aequatio $x = a$ est integrale particulare; sin autem is sit infinitus, ea inter integralia locum non habet, etiamsi aequationi differentiali satisfiat. Eadem regula quoque locum habet, si aequatio diffierentialis fuerit huiusmodi $dy = \frac{Pdx}{Q}$ seu $\frac{dy}{dx} = \frac{P}{Q}$, ac posito $x = a$ fiat $Q = 0$, quaecunque fuerit $P$ functio ipsarum $x$ et $y$; quin etiam necesse non est, ut $Q$ sit functio solius variabilis $x$, sed simul alteram $y$ utcunque implicare potest.

### SCHOLION 2

**552.** Demonstratio quidem inde est petita, quod quantitas $Q$, quae posito $x = a$ evanescit, factorem implicet potestatem quampiam ipsius $a - x$, quod in functionibus algebraicis est manifestum. Verum in functionibus transcendentibus eadem regula locum habet, cum potestate talibus dignitatibus aequivaleant.
Veluti si sit

$$dy = \frac{dx}{lx - la}$$

ubi $Q = lx - la = l\frac{x}{a}$ fitque $Q = 0$ posito $x = a$, quaeratur $\frac{dQ}{dx} = \frac{1}{x}$; quae formula cum non fiat infinita posito $x = a$, integrale particulare erit $x = a$. Quod etiam valet pro aequatione

$$dy = \frac{Pdx}{lx - la}$$

dummodo $P$ non fiat $= 0$ posito $x = a$. Sit enim $P = \frac{1}{x}$; erit integrando $y = C + l(lx - la)$ et $l\frac{x}{a} = e^{y-C}$. Sumta iam constante $C = \infty$ fit $l\frac{x}{a} = 0$ ideoque $x = a$, quod ergo est integrale particulare. Simili modo si sit

$$dy = Pdx:\left(e^{\frac{x}{a}} - e\right),$$

ubi $Q = e^{\frac{x}{a}} - e$ ideoque posito $x = a$ fit $Q = 0$, quia $\frac{dQ}{dx} = \frac{1}{a}e^{\frac{x}{a}}$ hincque posito $x = a$ fit $\frac{dQ}{dx} = \frac{e}{a}$, erit $x = a$ etiam integrale particulare. Sumatur $P = e^{\frac{x}{a}}$, ut integratio succedat, et quia $y = C + al\left(e^{\frac{x}{a}} - e\right)$ hincque $e^{\frac{x}{a}} = e + e^{\frac{y-C}{a}}$, statuatur $C = \infty$; erit $e^{\frac{x}{a}} = e$ ideoque $x = a$, quod ergo manifesto est integrale particulare.

## EXEMPLUM 1

**553.** *Proposita aequatione differentiali* $dy = \frac{Pdx}{\sqrt{S}}$, *in qua S evanescat posito* $x = a$, *definire casus, quibus aequatio* $x = a$ *est eius integrale particulare.*

Cum hic sit $\sqrt{S} = Q$, erit $dQ = \frac{dS}{2\sqrt{S}}$; ergo ut integrale particulare sit $x = a$, necesse est, ut posito $x = a$ fiat $\frac{dQ}{dx} = \frac{dS}{2dx\sqrt{S}}$ quantitas finita. Hinc eodem casu quantitas $\frac{dS^2}{Sdx^2}$ fieri debet finita, unde, cum *S* evanescat, etiam $\frac{dS^2}{dx^2}$ ac proinde $\frac{dS}{dx}$ evanescere debet. Tum autem posito $x = a$ illius fractionis valor est

$$\frac{2dSddS}{dSdx^2} = \frac{2ddS}{dx^2},$$

quem ergo finitum esse oportet vel $= 0$.

Quare ut aequatio $x = a$ sit integrale particulare aequationis propositae, hae conditiones requiruntur, primo ut posito $x = a$ fiat $S = 0$, secundo ut fiat $\frac{dS}{dx} = 0$, ac tertio ut huius formulae $\frac{ddS}{dx^2}$ valor prodeat vel finitus vel $= 0$, dummodo ne fiat infinite magnus. Si *S* sit functio rationalis, haec eo redeunt, ut *S* factorem habeat $(a - x)^2$ vel potestatem altiorem.

## SCHOLION

**554.** Haec resolutio usum habet in motu corporis ad centrum virium attracti dignoscendo, num in circulo fiat. Si enim distantia corporis a centro ponatur $= x$ et vis centripeta huic distantiae conveniens $= X$, pro tempore *t* talis reperitur aequatio

$$dt = \frac{xdx}{\sqrt{\left(Exx - c^4 - 2\alpha xx\int Xdx\right)}},$$

ubi *E* est constans per praecedentem integrationem ingressa, cuius valor quaeritur, ut hinc aequationi satisfaciat valor $x = a$, quo casu corpus in circulo revolvetur. Hic ergo est $S = Exx - c^4 - 2\alpha xx\int Xdx$ vel sumi potest

$$S = E - \frac{c^4}{xx} - 2\alpha\int Xdx$$

Non solum ergo haec quantitas, sed etiam eius differentiale

$$\frac{dS}{dx} = \frac{2c^4}{x^3} - 2\alpha X$$

evanescere debet posito $x = a$, neque tamen differentio-differentiale

$$\frac{ddS}{dx^2} = -\frac{6c^4}{x^4} - 2\alpha\frac{dX}{dx}$$

in infinitum abire debet. Inde ergo constans $a$ erit valor ipsius $x$ ex hac aequatione $\alpha x^3 \mathrm{X} = c^4$ resultans, qui est radius circuli, in quo corpus revolvi poterit, dummodo constans *E*, a qua celeritas pendet, ita fuerit comparata, ut posito $x = a$ fiat $E = \frac{c^4}{aa} + 2\alpha\int Xdx$; nisi forte eodem casu expressio $\frac{6c^4}{x^4} + 2\alpha\frac{dX}{dx}$ seu saltem haec $\frac{dX}{dx}$ fiat infinita. Hoc enim si eveniret, motus in *x* circulo tolleretur; ad quod ostendendum ponamus

$$X = b + \sqrt{(a-x)},$$

ut

$$\frac{dX}{dx} = -\frac{1}{2\sqrt{(a-x)}}$$

fiat infinitum posito $x = a$, et aequatio $ax^3 X = c^4$ dabit $\alpha a^3 b = c^4$. Tum vero ob

$$\int Xdx = bx - \tfrac{2}{3}(a-x)^{\frac{3}{2}}$$

erit $E = \alpha ab + 2\alpha ab = 3\alpha ab$ nostraque aequatio fit

$$dt = \frac{xdx}{\sqrt{\left(3\alpha abxx - \alpha a^3 b - 2\alpha bx^3 + \frac{4}{3}\alpha xx(a-x)^{\frac{3}{2}}\right)}}$$

cui valor $x = a$ certe non convenit tanquam integrale. Fit enim

$$S = \alpha(a-x)\left(-aab - abx + 2bxx + \tfrac{4}{3}xx\sqrt{(a-x)}\right),$$

cuius factor cum non sit $(a-x)^2$, sed tantum $(a-x)^{\frac{3}{2}}$, integrale particulare $x = a$ locum habere nequit.

## EXEMPLUM 2

**555.** *Proposita aequatione differentiali* $dy = \frac{Pdx}{\sqrt[n]{S^m}}$, *in qua* $S$ *evanescat posito* $x = a$, *invenire casus, quibus integrale particulare est* $x = a$.

Cum fiat $S=0$ posito $x=a$, concipere licet $S=(a-x)^{\lambda}R$ eritque denominator

$$\sqrt[n]{S^m}=(a-x)^{\frac{\lambda m}{n}}R^{\frac{m}{n}},$$

unde patet aequationem $x=a$ fore integrale particulare aequationis propositae, si fuerit $\frac{\lambda m}{n}$ numerus positivus unitate maior seu saltem unitati aequalis, hoc est, si sit vel $\lambda=\frac{n}{m}$ vel $\lambda>\frac{n}{m}$, quae diiudicatio, si *S* sit functio algebraica, facillime instituitur. Sin autem sit transcendens, ut exponens $\lambda$ in numeris exhiberi nequeat, uti licebit altera regula; scilicet cum sit $\sqrt[n]{S^m}=Q$, erit $\frac{dQ}{dx}=\frac{mS^{\frac{m-n}{n}}dS}{ndx}$, cuius valor debet esse finitus vel nullus posito $x=a$, siquidem integrale sit $x=a$. Sit igitur quoque necesse est hoc casu quantitas $\frac{S^{m-n}dS^n}{dx^n}$ finita. Quaeratur ergo huius formulae valor casu $x=a$, qui si prodeat infinite magnus, aequatio $x=a$ non erit integrale, sin autem sit vel finitus vel nullus, erit ea certe integrale particulare aequationis propositae. Hic duo constituendi sunt casus, prout fuerit vel $m>n$ vel $m<n$.

1. Si $m>n$, quia posito $x=a$ fit $S^{m-n}=0$, nisi eodem casu fiat $\frac{dS}{dx}=\infty$, certe erit $x=a$ integrale. Sin autem fiat $\frac{dS}{dx}=\infty$, utrumque evenire potest, ut sit integrale et ut non sit. Ad quod dignoscendum ponatur $S^{m-n}:T^n$, ut nostra formula evadat $S^{m-n}:T$, cuius tam numerator quam denominator evanescit posito $x=a$, ex quo eius valor reducitur ad

$$\frac{(m-n)S^{m-n-1}dS}{nT^{n-1}dT}=\frac{-(m-n)S^{m-n-1}dS^{n+2}}{ndx^n ddS}$$

qui si sit vel finitus vel nullus, integrale erit $x=a$. Simili modo ulterius progredi licet distinguendo casus $m>n+1$ et $m<n+1$.

ll. Si $m<n$, formula nostra erit $\frac{dS^n}{S^{n-m}dx^n}$ cuius valor ut fiat finitus, necesse est, ut sit $\frac{dS}{dx}=0$, ac praeterea, quia numerator ac denominator posito $x=a$ evanescit, formulae nostrae valor erit

$$\frac{nS^{n-1}ddS}{(n-m)S^{n-m-1}dSdx^n}=\frac{ndS^{n-2}ddS}{(n-m)S^{n-m-1}ndx^n}$$

quem finitum esse oportet.

Facillime autem iudicium absolvetur ponendo statim $x=a+\omega$; cum enim posito $x=a$ fiat $\mathrm{S}=0$, hac substitutione quantitas *S* semper resolvi poterit in huiusmodi formam $P\omega^{\alpha}+Q\omega^{\beta}+R\omega^{\gamma}+\text{etc.}$, cuius tantum unus terminus $P\omega^{\alpha}$ infimam potestatem ipsius $\omega$ complectens spectetur; ac si fuerit vel $\alpha=\frac{n}{m}$ vel $\alpha>\frac{n}{m}$ aequatio $x=a$ certe erit integrale particulare.

### SCHOLION

**556.** Haec ultima methodus est tutissima ac semper etiam in formulis transcendentibus optimo successu adhiberi potest. Scilicet proposita aequatione $dy=\frac{Pdx}{Q}$, in qua posito $x=a$ fiat $Q=0$ neque vero etiam numerator $P$ evanescat, statuatur $x=a\pm\omega$ et quantitas $\omega$ spectetur ut infinite parva, ut omnes eius potestates prae infima evanescant, atque quantitas $Q$ huiusmodi formam $R\omega^{\lambda}$ accipiet, ex qua patebit, nisi exponens $\lambda$ unitate fuerit minor, aequationem $x=a$ certe fore integrale particulare aequationis propositae.Veluti si habeamus

$$dy=\frac{dx}{\sqrt{\left(1+\cos.\frac{\pi x}{a}\right)}},$$

cuius denominator evanescit sumto $x=a$ ob $\cos.\pi=-1$, ponamus $x=a-\omega$ erit $\cos.\frac{\pi x}{a}=\cos.\left(\pi-\frac{\pi\omega}{a}\right)=-1+\frac{\pi\pi\omega\omega}{2aa}$ ob $\omega$ finite parvum, hinc nostrae aequationis denominator fiet $=\frac{\pi\omega}{a\sqrt{2}}$, unde concludimus integrale particulare utique esse $x=a$. Non autem foret integrale huius aequationis

$$dy=\frac{dx}{\sqrt[3]{\left(1+\cos.\frac{\pi x}{a}\right)}}.$$

### PROBLEMA 71

**557.** *Proposita aequatione differentiali, in qua variabiles sunt a se invicem separatae, investigare eius integralia particularia.*

### SOLUTIO

Sit proposita haec aequatio $\frac{dx}{X}=\frac{dy}{Y}$, in qua $X$ sit functio ipsius $x$ et $Y$ ipsius $y$ tantum. Ac primo ponatur $X=0$ indeque quaerantur valores ipsius $x$, quorum quisque sit $x=a$, ita ut posito $x=a$ fiat $X=0$; tum examinetur valor formulae $\frac{dX}{dx}$ posito $x=a$, qui nisi fiat infinitus, aequationis propositae integrale particulare certe erit $x=a$. Vel ponatur $x=a\pm\omega$ spectando $\omega$ ut quantitatem infinite parvam, ac si prodeat $X=P\omega^{\lambda}$, exponens $\lambda$, nisi sit unitate minor, indicabit integrale $x=a$; sin autem sit unitate minor, aequatio $x=a$ pro integrali non erit habenda.

Simili modo examinetur alterius partis denominator $Y$; qui si evanescat posito $y=b$ hocque casu formula $\frac{dY}{dy}$ non fiat infinita, aequatio $y=b$ erit integrale particulare; quod ergo etiam evenit, si posito $y=b\pm\omega$ prodeat $y=Q\omega^{\lambda}$, ubi exponens $\lambda$ unitate non sit minor.

### COROLLARIUM 1

**558.** Nisi ergo membra aequationis separatae fuerint fractiones, quarum denominatores certis casibus evanescant, huiusmodi integralia particularia non dantur, nisi forte in tali aequatione $Pdx = Qdy$ factores $P$ et $Q$ certis casibus fiant infiniti, qui autem casus ad praecedentem facile reducitur.

### COROLLARIUM 2

**559.** Veluti si habeatur

$$dx \operatorname{tang.} \tfrac{\pi x}{2a} = \tfrac{dy}{b-y}$$

primo quidem integrale particulare est $y = b$, tum vero, quia posito $x = a$ fit $\operatorname{tang.} \tfrac{\pi x}{2a} = \infty$, prius membrum ita exhibeatur $dx{:}\operatorname{cot.} \tfrac{\pi x}{2a} = \infty$ cuius denominator posito $x = a - \omega$ fit $\operatorname{cot.}\left(\tfrac{\pi}{2} - \tfrac{\pi\omega}{2a}\right) = \operatorname{tang.} \tfrac{\pi\omega}{2a} = \tfrac{\pi\omega}{2a}$ ubi cum exponens ipsius $\omega$ unitate non sit minor, aequatio $x = a$ erit quoque integrale particulare.

### COROLLARIUM 3

**560.** Hinc ergo interdum pro eadem aequatione duo plurave integralia particularia assignari possunt. Veluti pro hac aequatione

$$\tfrac{mdx}{a-x} = \tfrac{ndy}{b-y}$$

integralia particularia sunt $a - x = 0$ et $b - y = 0$, quae etiam ex integrali completo $(a-x)^m = C(b-y)^n$ consequuntur, illud sumendo $C = 0$, hoc vero sumendo $C = \infty$.

### COROLLARIUM 4

**561.** Simili modo huius aequationis

$$\tfrac{madx}{aa-xx} = \tfrac{nbdy}{bb-yy}$$

quatuor dantur integralia particularia $a + x = 0$, $a - x = 0$, $b + y = 0$, $b - y = 0$. Integrale completum vero est

$$\tfrac{m}{2} l \tfrac{a+x}{a-x} = \tfrac{1}{2} C + \tfrac{m}{2} l \tfrac{b+y}{b-y} \quad \text{seu} \quad \left(\tfrac{a+x}{a-x}\right)^m = C\left(\tfrac{b+y}{b-y}\right)^n$$

vel

$$(a+x)^m (b-y)^n = C(a-x)^m (b+y)^n$$

unde illa sponte fluunt.

### COROLLARIUM 5

**562.** Hinc patet, si fuerit

$$dy = \frac{Pdx}{(a+x)^{\alpha}(b+x)^{\beta}(c+x)^{\gamma}},$$

integraIia particularia fore $a+x=0, \quad b+x=0, \quad c+x=0$, si modo exponentes $\alpha, \beta, \gamma$ non fuerint unitate minores. Quare si $Q$ sit functio rationalis ipsius $x$, proposita aequatione $dy = \frac{Pdx}{Q}$ omnes factores ipsius $Q$ nihilo aequales positi praebent integralia particularia.

### SCHOLION 1

**563.** Hoc etiam pro factoribus imaginariis valet, etiamsi inde parum lucri nanciscamur. Si enim proposita sit aequatio

$$dy = \frac{adx}{aa+xx}$$

ex denominatore $aa+xx$ oriuntur integralia particularia

$$x = a\sqrt{-1} \quad \text{et} \quad x = -a\sqrt{-1},$$

quae ex integrali completo, quod est $y = C + \text{Ang.tang.}\frac{x}{a}$ minus sequi videntur Verum posito $x = a\sqrt{-1}$ notandum est esse Ang. tang.$\sqrt{-1} = \infty\sqrt{-1}$, unde si constanti $C$ similis forma signo contrario affecta tribuatur, altera quantitas $y$ manet indeterminata, etiamsi ponatur $x = a\sqrt{-1}$, quae positio propterea pro integrali particulari est habenda. Est enim in genere

$$\text{Ang.tang.}u\sqrt{-1} = \int \frac{du\sqrt{-1}}{1-uu} = \frac{\sqrt{-1}}{2} l \frac{1+u}{1-u}$$

unde posito $u = +1$ vel $u = -1$ prodit $\infty\sqrt{-1}$, quod infinitum in causa est, ut integralia assignata locum habeant.

Quocirca in genere affirmare licet, si fuerit $dy = \frac{Pdx}{Q}$ denominatorque $Q$ factorem habeat $(a+x)^{\lambda}$, cuius exponens $\lambda$ unitate non sit minor, semper aequationem $a+x=0$ fore integrale particulare. Sin autem $\lambda$ sit unitate minor, etsi positivus, non erit $a+x=0$ integrale particulare, etiamsi posito $x=-a$ aequationi differentiali satisfaciat.

### SCHOLION 2

**564.** Insigne hoc est paradoxon a nemine adhuc, quantum mihi quidem constat, observatum, quod aequationi differentiali eiusmodi valor satisfacere queat, qui tamen eius non sit integrale; atque adeo vix patet, quomodo haec cum solita integralium idea conciliari possint. Quoties enim proposita aequatione differentiali eiusmodi relationem variabilium exhibere licet, quae ibi substituta satisfaciat seu aequationem identicam producat, vix cuiquam in mentem venit dubitare, an illa relatio pro integrali saltem particulari sit habenda, cum tamen hinc proclive sit in errorem delabi. Veluti etiamsi huic aequationi

$$dy\sqrt{(aa-xx-yy)}=xdx+ydy$$

satisfaciat haec aequatio finita $xx+yy=aa$, tamen enormem errorem committeremus, si eam pro integrali particulari habere vellemus, propterea quod ea in integrali completo

$$y=C-\sqrt{(aa-xx-yy)}$$

neutiquam continetur. Quamobrem etsi omne integrale aequationi differentiali satisfacere debet, tamen non vicissim concludere licet omnem aequationem finitam, quae satisfaciat, eius esse integrale; verum praeterea requiritur, ut ea certa quadam proprietate sit praedita, cuiusmodi hie exposuimus et qua demum efficitur, ut in integrali completo contineatur. Hoc autem minime adversatur verae integralium notioni, quam hic stabilivimus, neque huiusmodi dubium unquam in integralia per certas regulas inventa cadere potest, sed tantum in eiusmodi integralibus, quae divinando quasi sumus assecuti, locum habet. Saepenumero autem, quando integratio non succedit, divinationi plurimum tribui solet; tum igitur maxime cavendum est, ne relationem quampiam satisfacientem temere pro integrali particulari proferamus. Quod cum iam in aequationibus separatis simus assecuti, quomodo in omnibus aequationibus differentialibus huiusmodi errores vitari oporteat, sedulo investigemus.

## PROBLEMA 72

**565**. *Si quaepiam relatio inter binas variabiles satisfaciat aequationi differentiali, definire, utrum ea sit integrale particulare necne.*

## SOLUTIO

Sit $Pdx=Qdy$ aequatio differentialis proposita, ubi *P* et *Q* sint functiones quaecunque ipsarum *x* et *y*, cui satisfaciat relatio quaepiam inter *x* et *y*, ex qua fiat $y=X$, functioni scilicet cuidam ipsius *x*, ita ut, si loco *y* ubique scribatur *X*, revera prodeat $Pdx=Qdy$ seu $\frac{dy}{dx}=\frac{P}{Q}$. Quaeritur ergo, utrum hic valor $y=X$ pro integrali aequationis propositae haberi possit necne.

Ad hoc diiudicandum ponatur $y=X+\omega$ fiatque $\frac{dX}{dx}+\frac{d\omega}{dx}=\frac{P}{Q}$, ubi notetur, si esset $\omega=0$, fore $\frac{dX}{dx}=\frac{P}{Q}$. Quare ob $\omega$ expressio $\frac{P}{Q}$ hac substitutiona reducetur ad $\frac{dX}{dx}$ una cum quantitate ita per $\omega$ affecta, ut evanescat posito $\omega=0$. In hoc negotio sufficit $\omega$ ut particulam infinite parvam spectasse, cuius ergo potestates altiores prae infima negligere liceat. Ponamus igitur hinc fieri

$$\frac{P}{Q}=\frac{dX}{dx}+S\omega^{\lambda}$$

habebiturque $\frac{d\omega}{dx}=S\omega^{\lambda}$ seu $\frac{d\omega}{\omega^{\lambda}}=Sdx$. Ex superioribus iam perspicuum est tum demum fore $y=X$ integrale particulare seu $\omega=0$, cum exponans $\lambda$ fuerit unitati aequalis vel maior; similis enim hic est ratio ac supra, qua requiritur, ut integrale $\int Sdx=\int\frac{d\omega}{\omega^{\lambda}}$ fiat infinitum casu proposito, quo $\omega=0$; hoc autem non evenit, nisi $\lambda$ sit unitati aequalis vel > 1.

Quodsi ergo aequationi $Pdx=Qdy$ seu $\frac{dy}{dx}=\frac{P}{Q}$ satisfaciat valor $y=X$, statuatur $y=X+\omega$ spectata particula $\omega$ infinite parva et investigetur hinc forma

$$\frac{P}{Q}=\frac{dX}{dx}+S\omega^{\lambda},$$

ex qua, nisi sit $\lambda<1$, concludetur illum valorem $y=X$ esse integrale particulare aequationis propositae.

## SCHOLION

**566.** Cum $\omega$ tractetur ut quantitas infinite parva, valor ipsius $\frac{P}{Q}$ posito $y=X+\omega$ per differentiationem commodissime inveniri posse videtur. Cum enim $\frac{P}{Q}$ sit functio ipsarum *x* et *y*, statuamus $d.\frac{P}{Q}=Mdx+Ndy$, et quia posito $y=X$ fractio $\frac{P}{Q}$ abit in $\frac{dX}{dx}$; per hypothesin, si loco *y* scribatur $X+\omega$, ea in $\frac{dX}{dx}+N\omega$ transibit, unde ob exponentem ipsius $\omega$ unitatem sequeretur aequationem $y=X$ semper esse integrale particulare, quod tamen secus evenire potest. Ex quo patet differentiationem loco substitutionis adhiberi non posse; quod quo clarius ostendatur, ponamus esse $\frac{P}{Q}=\sqrt{(y-X)}+\frac{dX}{dx}$, unde posito $y=X+\omega$ manifesto oritur $\frac{P}{Q}=\frac{dX}{dx}+\sqrt{\omega}$. At differentiatione utentes ponendo $d.\frac{P}{Q}=Mdx+Ndy$ fiet $N=\frac{1}{2\sqrt{(y-X)}}$ hincque $\frac{P}{Q}=\frac{dX}{dx}+N\omega$, quae expressio ab illa discrepat. Illa scilicet aequationem $y=X$ ex integralium numero removet, haec vero admittere videtur. Verum et hic notandum est quantitatem *N* ipsam potestatem ipsius $\omega$ negative involvere, unde potestas $\omega$ deprimatur. Quare ne hanc rationem spectare opus sit, semper praestat vera substitutione uti with differentiatione seposita. Hoc observato haud difficile erit omnes valores, qui aequationi cuipiam differentiali satisfaciunt, diiudicare, utrum sint vera integralia necne.

### EXEMPLUM 1

**567.** *Cum huic aequationi* $dx\left(1-y^m\right)^n = dy\left(1-x^m\right)^n$ *manifesto satisfaciat* $y=x$, *utrum sit eius integrale particulare necne, definire.*

Ponatur $y=x+\omega$ et spectato $\omega$ ut quantitate minima est

$$y^m = x^m + mx^{m-1}\omega$$

et

$$\left(1-y^m\right)^n = \left(1-x^m - mx^{m-1}\omega\right)^n = \left(1-x^m\right)^n - mnx^{m-1}\omega\left(1-x^m\right)^{n-1},$$

unde aeqnatio $\frac{dy}{dx} = \frac{\left(1-y^m\right)^n}{\left(1-x^m\right)^n}$ abit in

$$1+\frac{d\omega}{dx} = 1-\frac{mnx^{m-1}\omega}{1-x^m} \text{ seu } \frac{d\omega}{\omega} = -\frac{mnx^{m-1}dx}{1-x^m}$$

ubi cum $\omega$ habeat dimensionem integram, aequatio $y=x$ certe est integrale particulare aequationis differentialis propositae.

### EXEMPLUM 2

**568.** *Cum huic aequationi* $ady - adx = dx\sqrt{(yy-xx)}$ *satisfaciat valor* $y=x$, *investigare, utrum is sit eius integrale particulare necne.*

Ponatur $y=x+\omega$ et sumta $\omega$ quantitate infinite parva, cum sit $\sqrt{(yy-xx)} = \sqrt{2x\omega}$, erit $ad\omega = dx\sqrt{2x\omega}$ seu $\frac{ad\omega}{\sqrt{\omega}} = \sqrt{2x}$. Quoniam igitur hic $d\omega$ dividitur per potestatem ipsius $\omega$, cuius exponens est unitate minor, sequitur valorem $y=x$ non esse integrale particulare aequationis propositae, etiamsi ei satisfaciat. Scilicet si eius integrale completum exhibere liceret, pateret, quomodocunque constans arbitraria per integrationem ingressa definiretur, in ea aequationem $y=x$ non contentum iri.

### SCHOLION

**569.** Hinc nova ratio intelligitur, cur diiudicatio integralis ab exponente ipsius $\omega$ pendeat. Cum enim in exemplo proposito facto $y=x+\omega$ prodeat $\frac{ad\omega}{\sqrt{\omega}} = \sqrt{2x}$, erit integrando $2a\sqrt{\omega} = C + \frac{2}{3}x\sqrt{2x}$. Verum per hypothesin $\omega$ est quantitas infinite parva, hinc autem, utcunque definiatur constans *C*, quantitas $\omega$ obtinet valorem finitum, qui adeo quantumvis magnus evadere potest; quod cum hypothesi adversetur, necessario sequitur aequationem $y=x$ integrale esse non posse hocque semper evenire debere , quoties $d\omega$

prodit divisum per potestatem ipsius $\omega$, cuius exponens unitate est minor. Contra vero patet, facta substitutione exposita prodeat $\frac{d\omega}{\omega} = Rdx$, ut posito $\int Rdx = S$ fiat $l\omega = lC + lS$ seu $\omega = CS$, sumta constante $C$ evanescente utique ipsam quantitatem $\omega$ evanescere; quod idem evenit, si prodeat $\frac{d\omega}{\omega^\lambda} = Rdx$ existente $\lambda > 1$. Erit enim $\frac{1}{(\lambda-1)\omega^{\lambda-1}} = C - S$ seu $(\lambda-1)\omega^{\lambda-1} = \frac{1}{C-S}$, unde sumto $C = \infty$ quantitas $\omega$ revera fit evanescens, ut hypothesis exigit.

Caeterum aequatio huius exempli posito $x = pp - qq$ et $y = pp + qq$ ab irrationalitate liberatur fitque

$$4aqdq = 4pq(pdp - qdq) \text{ sive } adq = ppdp - pqdq,$$

quae nullo modo tractari posse videtur; neque ergo eius integrale completum exhiberi potest. Cui aequationi cum non amplius satisfacit $x = y$ seu $q = 0$, hinc quoque concludendum est valorem $x = y$ non esse integrale particulare.

## EXEMPLUM 3

**570.** *Dum huic aequationi* $aady - aadx = dx(yy - xx)$ *satisfaciat valor* $y = x$, *investigare, utrum is sit eius integrale particulare necne.*

Ponatur $y = x + \omega$ spectata $\omega$ ut quantitate infinite parva et ob $yy - xx = 2x\omega$ aequatio nostra hanc induet formam $aad\omega = 2x\omega dx$ seu $\frac{aad\omega}{\omega} = 2xdx$. Quia igitur hic $d\omega$ dividitur per potestatem primam ipsius $\omega$, aequatio $y = x$ utique erit integrale particulare aequationis propositae atque adeo etiam in integrali completo continetur. Hoc enim invenitur ponendo $y = x - \frac{aa}{u}$, quo fit

$$\frac{a^4du}{uu} = dx\left(\frac{a^4}{uu} - \frac{2aax}{u}\right) \text{ seu } du + \frac{2uxdx}{aa} = dx.$$

Multiplicetur per $e^{\frac{xx}{aa}}$ et integrale prodit

$$e^{\frac{xx}{aa}}u = C + \int e^{\frac{xx}{aa}}dx \text{ hincque } y = x - aae^{\frac{xx}{aa}} : \left(C + \int e^{\frac{xx}{aa}}dx\right).$$

Quodsi ergo constans $C$ capiatur infinita, fit $y = x$.

## SCHOLION

**571.** Si in hac aequatione ut supra ponatur $x = pp - qq$ et $y = pp + qq$, oritur $aadq = 2ppq(pdp - qdq)$, cui satisfacit $q = 0$, unde casus $y = x$ nascitur. At facta hac transformatione difficulter patet, quomodo eius integrale inveniri oporteat. Si quidem superiorem reductionem perpendamus, intelligemus hanc aequationem integrabilem reddi, si multiplicetur per $e^{(pp-qq)^2:aa}:q^3$; quod cum per se haud facile pateat, consultum erit hac substitutione uti $pp-qq=rr$, qua fit $pp=qq+rr$ et $pdp-qdq=rdr$, unde aequatio abit in

$$aadq=2qrdr\left(qq+rr\right) \text{ seu } \frac{aadq}{2q^3}=rdr+\frac{r^3}{qq},$$

quae posito $\frac{1}{qq}=s$ facile integratur.

Quoties ergo licet eiusmodi relationem inter variabiles colligere, quae aequationi differentiali satisfaciat, hoc modo iudicari poterit, utrum ea relatio pro integrali particulari sit habenda necne. Pro inventione autem huiusmodi integralium particularium regulae vix tradi possunt; quae enim habentur regulae aeque ad integralia completa invenienda patent. Ita quae supra circa aequationes separatas observavimus, ob id ipsum, quod sunt separatae, via simul ad integrale completum est patefacta. Simili modo si altera methodus per factores succedat, plerumque ex ipsis factoribus, quibus aequatio integrabilis redditur, integralia particularia concludi possunt; quemadmodum in sequentibus propositionibus declarabimus.

## THEOREMA

**572.** *Si aequatio differentialis $Pdx+Qdy=0$ per functionem M multiplicata reddatur integrabilis, integrale particulare erit $M=0$, nisi eodem casu P vel Q abeat in infinitum.*

## DEMONSTRATIO

Ponamus *u* esse factorem ipsius *M* et ostendendum est aequationem $u=0$ esse integrale particulare aequationis propositae. Cum *u* aequetur certae functioni ipsarum *x* et *y*, definiatur inde altera variabilis *y*, ut aequatio prodeat inter binas variabiles *x* et *u*, quae sit $Rdx+Sdu=0$, unde posito multiplicatore $M=Nu$ integrabilis erit haec forma

$$NRudx+NSudu=0.$$

Quodsi iam neque *R* neque *S* per *u* dividatur, quo casu posito *u* = 0 neque *P* neque *Q* abit in infinitum, integrale utique per *u* erit divisibile. Nam sive id colligatur ex termino $NRudx$ spectata *u* ut constante sive ex termino $NSudu$ spectata *x* constante, integrale prodit factorem *u* implicans, si quidem in integratione constans omittatur. Unde concludimus integrale completum huiusmodi formam esse habiturum $Vu=C$. Quare si haec constans *C* nihilo aequalis capiatur, integrale particulare erit $u=0$, iis scilicet casibus exceptis, quibus functiones *R* et *S* iam ipsae per *u* essent divisae ideoque ratiocinium nostrum vim suam amitteret. His ergo casibus exclusis, quoties aequatio $Pdx+Qdy=0$ per functionem *M* multiplicata fit per se integrabilis eaque functio *M* factorem habeat *u*, integrale particulare erit $u=0$, quod similiter de singulis factoribus functionis *M* valet.

## SCHOLION

**573.** Limitatio adiecta absolute est necessaria, cum ea neglecta universum ratiocinium claudicet. Quod quo facilius intelligatur, consideremus hanc aequationem

$$\frac{adx}{y-x}+dy-dx=0,$$

quae per $y-x$ multiplicata manifesto fit integrabilis; ponamus ergo hunc muItiplicatorem $y-x=u$ seu $y=x+u$, unde nostra aequatio erit $\frac{adx}{u}+du=0$,
quae per *u* multiplicata abit in $adx+udu=0$; ubi cum pars *adx* non per *u* sit multiplicata, neutiquam concludere licet integrale per *u* fore divisibile, quippe quod est $ax+\frac{1}{2}uu$. Hinc patet, si modo pars *dx* per *u* esset multiplicata, etiamsi altera pars *du* factore *u* careret, tamen integrale per *u* divisibile fore, veluti evenit in $udx+xdu$, cuius integrale *xu* utique factorem habet *u*. Ex quo intelligitur, si formula $Pudx+Qdu$ fuerit per se integrabills, dummodo *Q* non dividatur per *u* vel per potestatem eius prima altiorem, etiam integrale omissa scilicet constante fore per *u* divisibile.

## THEOREMA

**574**. *Si aequatio differentialis $Pdx+Qdy=0$ per functionem M divisa evadat per se integrabilis, integrate particulare erit $M=0$, nisi posito $M=0$ vel P vel Q evanescat.*

## DEMONSTRATIO

Habeat divisor *M* factorem *u*, ut sit $M=Nu$, et ostendi oportet integrale particulare futurum $u=0$, id quod de singulis factoribus divisoris *M*, siquidem plures habeat, est tenendum. Cum igitur *u* sit functio ipsarum *x* et *y*, definiatur inde altera *y* per *x* et *u*, ut prodeat huiusmodi aequatio $Rdx+Sdu=0$, quae ergo per *Nu* divisa per se erit integrabilis. Quaeri igitur oportet integale formulae $\frac{Rdx}{Nu}+\frac{Sdu}{Nu}$, ubi assumimus neque *R* neque *S* per *u* multiplicari neque hoc modo factorem *u* ex denominatore tolli. Quodsi iam hoc integrale ex solo membro $\frac{Rdx}{Nu}$ colligatur spectando *u* ut constantem, prodit id $\frac{1}{u}\int\frac{Rdx}{N}+f:u$; sin autem ex altero membro $\frac{Sdu}{Nu}$ sumta *x* constante colligatur, quia *S* non factorem habet *u*, id semper ita erit comparatum, ut posito $u=0$ fiat infinitum. Ex quo integrale, quod sit *V*, ita erit comparatum, ut fiat $=\infty$ posito $u=0$; quare cum integrale eompletum futurum sit $V=C$, huic aequationi sumta constante *C* infinita satisfit ponendo $u=0$. Concludimus itaque, si divisor $M=Nu$ reddat aequationem differentialem $Pdx+Qdy=0$ per se integrabilem, ex quolibet divisoris *M* factore *u* obtineri integrale particulare $u=0$, nisi forte posito $u=0$ quantitates *P* et *Q* vel *R* et *S* evanescant.

## COROLLARIUM 1

**575.** Si aequatio $Pdx+Qdy=0$ fuerit homogenea, ea, ut supra vidimus, integrabilis redditur, si dividatur per $Px+Qy$, quare integrale eius particulare erit $Px+Qy=0$. Quae aequatio cum etiam sit homogenea, factores habebit formae $\alpha x+\beta y$, quorum quisque nihilo aequatus dabit integrale particulare.

## COROLLARIUM 2

**576.** Pro hac aequatione

$$ydx(c+nx)-dy(y+a+bx+nxx)=0$$

divisorem, quo integrabilis redditur, supra (§ 488) exhibuimus, unde integrale particulare concluditur $y=0$, tum vero

$$nyy+(2na-bc)y+n(b-2c)xy+(na+cc-bc)(a+bx+nxx)=0,$$

cuius radices sunt

$$ny=\tfrac{1}{2}bc-na+n\left(c-\tfrac{1}{2}b\right)x\pm(c+nx)\sqrt{\left(\tfrac{1}{4}bb-na\right)}.$$

## COROLLARIUM 3

**577.** Pro hac aequatione differentiali

$$\frac{ndx(1+yy)\sqrt{(1+yy)}}{\sqrt{(1+xx)}}+(x-y)dy=0$$

divisorem, quo integrabilis redditur, supra (§ 489) dedimus, unde integrale particulare concludimus $x-y+n\sqrt{(1+xx)(1+yy)}=0$ seu

$$yy-2xy+xx=nn+nnxx+nnyy+nnxxyy,$$

ex quo porro fit

$$y=\frac{x\pm n(1+xx)(1-nn)\sqrt{(1-nn)}}{1-nn(1+xx)}.$$

## COROLLARIUM 4

**578.** Pro hac aequatione differentiali

$$dy+yydx-\frac{adx}{x^4}=0$$

multiplicatorem supra (§ 491) invenimus $\frac{xx}{xx(1-xy)^2-a}$, unde integrale particulare concludimus $xx(1-xy)^2-a=0$ hincque

$$x(1-xy)=\pm\sqrt{a} \quad \text{seu} \quad y=\frac{1}{x}\pm\frac{\sqrt{a}}{xx},$$

ita ut bina habeamus integralia particularia, quae autem imaginaria evadunt, si $a$ fuerit quantitas negativa.

## SCHOLION

**579.** Haec fere sunt, quae circa tractationem aequationum differentialium adhuc sunt explorata, nonnulla tamen subsidia evolutio aequationum differentialium secundi gradus infra suppeditabit. Huc autem commode referri possunt, quae circa comparationem certarum formularum transcendentium haud ita pridem sunt investigata. Quemadmodum enim logarithmi et arcus circulares, etsi sunt quantitates transcendentes, inter se comparari atque adeo aeque ac quantitates algebraicae in calculo tractari possunt, ita similem comparationem inter certas quantitates transcendentes altioris generis instituere licet, quae scilicet continentur in formula hac

$$\int\frac{dx}{\left(A+Bx+Cx^2+Dx^3+Ex^4\right)}$$

ubi etiam numerator rationalis veluti $\mathfrak{A}+\mathfrak{B}x+\mathfrak{C}x^2+\text{etc.}$ addi potest. Quod argumentum cum sit maxime arduum atque adeo vires Analyseos superare videatur, nisi certa ratione expediatur, in Analysin inde haud spernenda incrementa redundant; imprimis autem resolutio aequationum differentialium non mediocriter perfici videtur. Cum enim proposita fuerit huiusmodi aequatio

$$\frac{dx}{\sqrt{\left(A+Bx+Cx^2+Dx^3+Ex^4\right)}}=\frac{dy}{\sqrt{\left(A+Bx+Cx^2+Dx^3+Ex^4\right)}},$$

statim quidem patet eius integrale particulare $x=y$, verum integrale completum maxime transcendens fore videtur, cum utraque formula per se neque ad logarithmos neque ad arcus circulares reduci queat. Quare eo magis erit mirandum, quod integrale completum per aequationem adeo algebraicam inter $x$ et $y$ exhiberi possit. Quo autem methodus ad haec sublimia ducens clarius perspiciatur, eam primo ad quantitates transcendentes notas hac formula

$$\int\frac{dx}{\sqrt{(A+Bx+Cxx)}}$$

contentas applicemus, deinceps eius usum in formulis illis magis complexis ostensuri.