## CHAPTER V

# CONCERNING THE COMPARISON OF TRANSCENDENTAL QUANTITIES CONTAINED IN THE FORM $\int\frac{Pdx}{\sqrt{(A+2Bx+Cxx)}}$.

[See also E263 : *An example of a new method of finding both the quadrature and rectification of curves, and other transcending quantities to be compared among themselves* (Novi Com. Acad. Sc. Petrop. 7, 1758/9 : pub. in 1761, p.83; & in the *O. O.*, Series I, vol. 20, p.108). This Latin paper contains references to some of the history surrounding the integration of such problems by Bernoulli and Huygens, as are to be considered here, and in addition treats some simpler examples. Also see E818, with the translated title : *Concerning the comparison of unrectifiable arcs of curves*; Opera Postuma 1, Petropoli 1862, p.452; & in the O. O. Series I, vol. 21, p. 296.]

### PROBLEM 73

**580.** *From this proposed algebraic equation between x and y*

$$\alpha+2\beta(x+y)+\gamma(xx+yy)+2\delta xy=0,$$

*to find integral formulas of a prescribed form, which can be compared between themselves.*

### SOLUTION

The proposed equation may be differentiated, and from the differential of this

$$2\beta dx+2\beta dy+2\gamma xdx+2\gamma ydy+2\delta xdy+2\delta ydx=0,$$

this equation may be deduced :

$$dx(\beta+\gamma x+\delta y)+dy(\beta+\gamma y+\delta x)=0.$$

There is put in place

$$\beta+\gamma x+\delta y=p \quad \text{and} \quad \beta+\gamma y+\delta x=q$$

and from the first there arises

$$pp=\beta\beta+2\beta\gamma x+2\beta\delta y+\gamma\gamma xx+2\gamma\delta xy+\delta\delta yy,$$

from which there is taken the proposed equation multiplied by $\gamma$ :

$$0=\alpha\gamma+2\beta\gamma x+2\gamma\beta y+\gamma\gamma xx+\gamma\gamma yy+2\gamma xy$$

and there shall be

$$pp=\beta\beta-\alpha\gamma+2\beta(\delta-\gamma)y+(\delta\delta-\gamma\gamma)yy;$$

and in a similar manner there may be found from the other equation :

$$qq=\beta\beta-\alpha\gamma+2\beta(\delta-\gamma)x+(\delta\delta-\gamma\gamma)xx,$$

from which there shall be $pdx+qdy=0$. Now since $p$ shall be a function of $y$ and $q$ in a like manner a function of $x$, there may be put

$$\beta\beta-\alpha\gamma=\mathrm{A},\quad \beta(\delta-\gamma)=\mathrm{B}\quad\text{and}\quad \delta\delta-\gamma\gamma=\mathrm{C},$$

from which it is deduced

$$\delta-\gamma=\frac{B}{\beta}\quad\text{and}\quad \delta+\gamma=\frac{C}{\delta-\gamma}=\frac{\beta C}{B}$$

and hence

$$\delta=\frac{BB+\beta\beta C}{2B\beta}\quad\text{and}\quad \gamma=\frac{\beta\beta C-BB}{2B\beta};$$

now in the first place there is given

$$\alpha=\frac{\beta\beta-A}{\gamma}=\frac{2B\beta(\beta\beta-A)}{\beta\beta C-BB}.$$

With which values taken for $\alpha,\ \gamma,\delta$ the equation $\frac{dx}{q}+\frac{dy}{p}=0$ becomes

$$\frac{dx}{\sqrt{(A+2Bx+Cxx)}}+\frac{dy}{\sqrt{(A+2By+Cyy)}}=0,$$

hence the equation satisfying this differential equation shall be

$$\frac{2B\beta(\beta\beta-A)}{\beta\beta C-BB}+2\beta(x+y)+\frac{\beta\beta C-BB}{2B\beta}(xx+yy)+\frac{BB+\beta\beta C}{B\beta}xy=0;$$

which since it may retain the new constant $\beta$, thus the complete integral of the differential equation will be found.

Now neither is there a need, that the formulas for these are equal to the letters *A*, *B*, *C*, themselves, for it suffices that they shall be proportional to these, from which there arises

$$\frac{\beta\beta-\alpha\gamma}{\beta(\delta-\gamma)}=\frac{A}{B}\quad\text{and}\quad \frac{\delta+\gamma}{\beta}=\frac{C}{B}.$$

Hence

$$\delta=\frac{\beta C}{B}-\gamma \quad \text{and} \quad \alpha=\frac{\beta\beta}{\gamma}-\frac{\beta}{\gamma}\frac{A}{B}(\delta-\gamma) \quad \text{or} \quad \alpha=\frac{\beta\beta}{\gamma}-\frac{\beta\beta AC}{\gamma BB}+\frac{2\beta A}{B}.$$

Whereby the complete integral of the differential equation

$$\frac{dx}{\sqrt{(A+2Bx+Cxx)}}+\frac{dy}{\sqrt{(A+2By+Cyy)}}=0$$

becomes [on subsisting into $\alpha+2\beta(x+y)+\gamma(xx+yy)+2\delta xy=0$ ,]

$$\beta\beta(BB-AC)+2\beta\gamma AB+2\beta\gamma BB(x+y)+\gamma\gamma BB(xx+yy)$$
$$+\ 2\gamma B(\beta C-\gamma B)xy=0,$$

where the ratio $\frac{\beta}{\gamma}$ furnishes an arbitrary constant.

## COROLLARY 1

**581.** With a root extracted from the proposed equation there arises

$$-y=\frac{-\beta-\delta x+\sqrt{(\beta\beta+2\beta\delta x+\delta\delta xx-\alpha\gamma-2\beta\gamma x-\gamma\gamma xx)}}{\gamma},$$

or with the values of $\alpha$ and $\delta$ substituted in place

$$y=-\frac{\beta}{\gamma}-\frac{\beta C-\gamma B}{\gamma B}x+\sqrt{\frac{\beta\beta C-2\beta\gamma B}{\gamma\gamma BB}(A+2Bx+Cxx)}$$

[Note that the biquadratic equation can be solved for *x* in terms of *y* or for *y* in terms of *x*.]

## COROLLARY 2

**582.** Hence if $x=0$ , there becomes

$$y=-\frac{\beta}{\gamma}+\sqrt{\frac{\beta\beta AC-2\beta\gamma AB}{\gamma\gamma BB}}$$

this value may be put $=a$ , so that there becomes

$$\gamma Ba+\beta B=\sqrt{(\beta\beta AC-2\beta\gamma AB)},$$

from which with the squares taken there is produced

$$\gamma\gamma BBaa+2\beta\gamma BBa+\beta\beta BB=\beta\beta AC-2\beta\gamma AB$$

and hence

$$\frac{\gamma}{\beta}=\frac{-A-Ba+\sqrt{A(A+2Ba+Caa)}}{Baa}$$

or

$$\frac{\beta}{\gamma}=\frac{B\left(A+Ba+\sqrt{A(A+2Ba+Caa)}\right)}{AC-BB}.$$

## SCHOLIUM 1

**583.** In order that the assumed equation

$$\alpha+2\beta(x+y)+\gamma(xx+yy)+2\delta xy=0$$

satisfies the differential equation

$$\frac{dx}{\sqrt{(A+2Bx+Cxx)}}+\frac{dy}{\sqrt{(A+2By+Cyy)}}=0$$

it is necessary that there shall be

$$\beta\beta-\alpha\gamma=mA,\quad \beta\left(\delta-\gamma\right)=mB \quad \text{and} \quad \delta\delta-\gamma\gamma=mC\ ,$$

from which there becomes [see §581]

$$[q=]\ \beta+\gamma y+\delta x=\sqrt{m\left(A+2Bx+Cxx\right)}$$

and

$$[p=]\ \beta+\gamma x+\delta y=\sqrt{m\left(A+2By+Cyy\right)}.$$

But from the given *A*, *B*, *C* only three of the letters $\alpha,\beta,\ \gamma,\ \delta$ and *m* are defined ; whereby since two will remain indeterminate, the assumed equation, even if it should be divided by some constant, will contain only one new constant, from which that may be taken for the complete integral. Whereby even if neither part of the differential equation allows an algebraic integration, yet a complete integral is able to be found algebraically. In place of the arbitrary constant this value of *y* is possible to be introduced, which it takes on putting $x=0$; but since it is possible to arise, that this value becomes imaginary, it may be agreed to define this constant thus, so that on putting $x=a$ there becomes $y=b$, from which agreement the application can be made to all the cases. Hence there becomes

$$\frac{\beta+\gamma b+\delta a}{\beta+\gamma a+\delta b}=\sqrt{\frac{A+2Ba+Caa}{A+2Bb+Cbb}},$$

from which it is deduced

$$\beta=\frac{(\gamma a+\delta b)\sqrt{(A+2Ba+Caa)}-(\gamma b+\delta a)\sqrt{(A+2Bb+Cbb)}}{-\sqrt{(A+2Ba+Caa)}+\sqrt{(A+2Bb+Cbb)}}$$

and

$$\sqrt{m(A+2Ba+Caa)}=\frac{(\delta-\gamma)(b-a)\sqrt{(A+2Ba+Caa)}}{\sqrt{(A+2Bb+Cbb)}-\sqrt{(A+2Ba+Caa)}}$$

or

$$\sqrt{m}=\frac{(\delta-\gamma)(b-a)}{\sqrt{(A+2Bb+Cbb)}-\sqrt{(A+2Ba+Caa)}}.$$

For the sake of brevity there may be put :

$$\sqrt{(A+2Ba+Caa)}=\mathfrak{A} \quad \text{and} \quad \sqrt{(A+2Bb+Cbb)}=\mathfrak{B},$$

so that there becomes

$$\sqrt{m}=\frac{(\delta-\gamma)(b-a)}{\mathfrak{B}-\mathfrak{A}} \quad \text{and} \quad \beta=\frac{\mathfrak{A}(\gamma a+\delta b)-\mathfrak{B}(\gamma b+\delta a)}{\mathfrak{B}-\mathfrak{A}}$$

and the equation $\beta(\delta-\gamma)=mB$ adopts this form :

$$\mathfrak{A}(\gamma a+\delta b)-\mathfrak{B}(\gamma b+\delta a)=\frac{B(\delta-\gamma)(b-a)^2}{\mathfrak{B}-\mathfrak{A}},$$

from which there becomes

$$\left.\begin{array}{l}+\gamma\mathfrak{A}\mathfrak{B}-\gamma A-\gamma B(a+b)-\gamma C(aa-ab+bb)\\ +\delta\mathfrak{A}\mathfrak{B}-\delta \mathrm{A}-\delta \mathrm{B}(a+b)-\delta Cab\end{array}\right\}=0.$$

Hence there can be put in place

$$\gamma=n\mathfrak{A}\mathfrak{B}-nA-nB(a+b)-nCab,$$
$$\delta=nA+nB(a+b)+nC(aa-ab+bb)-n\mathfrak{A}\mathfrak{B},$$
$$\sqrt{m}=\frac{n(b-a)(\mathfrak{A}^2+\mathfrak{B}^2-2\mathfrak{A}\mathfrak{B})}{\mathfrak{B}-\mathfrak{A}}=n(b-a)(\mathfrak{B}-\mathfrak{A}),$$
$$\beta=nB(b-a)^2, \quad \text{hence} \quad \delta-\gamma=\frac{m}{n(b-a)^2},$$

from which there shall be $\alpha+\gamma=nC(b-a)^2$, and certainly there shall be $\delta\delta-\gamma\gamma=mC$. It remains, that there becomes $\alpha\gamma=\beta\beta-mA$, that is

$$\alpha\gamma=nnBB(b-a)^4-nnA(b-a)^2(\mathfrak{B}-\mathfrak{A})^2$$

or

$$\alpha\gamma=nn(b-a)^2\left(BB(b-a)^2-A(\mathfrak{B}-\mathfrak{A})^2\right).$$

Or since on putting $x=a$ there becomes $y=b$, then also there becomes

$$\alpha = -2\beta(a+b) - \gamma(aa+bb) - 2\delta ab$$

and hence

$$\alpha = n(a-b)^2\left(A - B(a+b) - Cab - \mathfrak{AB}\right),$$

from which our assumed equation becomes :

$$\begin{aligned}&(b-a)^2\left(A - B(a+b) - Cab - \mathfrak{AB}\right)\\&+2B(b-a)^2(x+y) - \left(A + B(a+b) + Cab - \mathfrak{AB}\right)(xx+yy).\\&+2\left(A + B(a+b) + C(aa-ab+bb) - \mathfrak{AB}\right)xy = 0.\end{aligned}$$

## SCHOLIUM 2

**584.** If there is put $\beta = 0$, so that the equation becomes

$$\alpha + \gamma(xx+yy) + 2\delta xy = 0,$$

then there becomes

$$y = \frac{-\delta x + \sqrt{\left(-\alpha\gamma + (\delta\delta - \gamma\gamma)xx\right)}}{\gamma}.$$

Therefore putting $-\alpha\gamma = mA$ and $\delta\delta - \gamma\gamma = mC$, so that $\gamma y + \delta x = \sqrt{m(A+Cxx)}$, then the complete integral of this equation

$$\frac{dx}{\sqrt{(A+Cxx)}} + \frac{dy}{\sqrt{(A+Cyy)}} = 0,$$

will be that equation taken, for which there will be found
$\frac{C}{A} = \frac{\gamma\gamma - \delta\delta}{\alpha\gamma}$ or $\delta = \left(\gamma\gamma - \frac{\alpha\gamma C}{A}\right)$. But if on putting $x = 0$ there must become $y = b$, on account of $\gamma b = \sqrt{mA}$ then $\gamma = \frac{\sqrt{mA}}{b}$; while $\alpha = -b\sqrt{mA}$ and $\delta = \sqrt{\left(\frac{mA}{bb} + mC\right)}$.
Hence this equation will be found

$$\frac{y\sqrt{mA}}{b} + \frac{x\sqrt{m(A+Cbb)}}{b} = \sqrt{m(A+Cxx)}$$

which gives

$$y = -x\sqrt{\frac{A+Cbb}{A}} + b\sqrt{\frac{A+Cxx}{A}},$$

which is the complete integral of that differential equation. Whereby if *x* should be taken negative, then of this differential equation

$$\frac{dx}{\sqrt{(A+Cxx)}}=\frac{dy}{\sqrt{(A+Cyy)}}$$

the complete integral is

$$y=x\sqrt{\frac{A+Cbb}{A}}+b\sqrt{\frac{A+Cxx}{A}}.$$

But if the general calculation should be treated in a like manner, the complete integral of the differential equation

$$\frac{dx}{\sqrt{(A+2Bx+Cxx)}}+\frac{dy}{\sqrt{(A+2By+Cyy)}}=0,$$

if for the sake of brevity there is put $\sqrt{(A+2Bb+Cbb)}=\mathfrak{B}$, shall be

$$y\left(\sqrt{A}+\frac{Bb}{\sqrt{A}-\mathfrak{B}}\right)+x\left(\mathfrak{B}+\frac{Bb}{\sqrt{A}-\mathfrak{B}}\right)=\frac{Bbb}{\sqrt{A}-\mathfrak{B}}+b\sqrt{(A+2Bx+Cxx)},$$

from which the preceding case evidently follows, if there is put $B=0$. Now with the aid of trivial substitutions these formulas, where *B* is present, can be reduced to the case in which $B=0$.

## PROBLEM 74

**585.** *If* $\Pi : z$ *should signify that function of z, which arises from the integration of the formula* $\int\frac{dz}{\sqrt{(A+Czz)}}$ *, with this integral thus taken, so that it vanishes on putting z* = 0, *to establish a comparison between functions of this kind.*

## SOLUTION

This differential equation may be considered :

$$\frac{dx}{\sqrt{(A+Cxx)}}=\frac{dy}{\sqrt{(A+Cyy)}}$$

from which, since there shall be by hypothesis :

$$\int\frac{dx}{\sqrt{(A+Cxx)}}=\Pi : x \quad\text{and}\quad \int\frac{dy}{\sqrt{(A+Cyy)}}=\Pi : y$$

with each integral thus taken so that it vanishes on putting $x=0$, now here on putting $y=0$, the complete integral shall be

$$\Pi : y=\Pi : x+C.$$

But from above [§584] we have seen that the integral is :

$$y = x\sqrt{\frac{A+Cbb}{A}} + b\sqrt{\frac{A+Cxx}{A}}$$

where on putting $x=0$ there shall be $y=b$; whereby, since $\Pi:0=0$, then there becomes

$$\Pi:y = \Pi:x + \Pi:b,$$

hence this transcendental equation is satisfied algebraically by :

$$y = x\sqrt{\frac{A+Cbb}{A}} + b\sqrt{\frac{A+Cxx}{A}}$$

In a like manner on taking *b* negative this equation

$$\Pi:y = \Pi:x - \Pi:b$$

agrees with this

$$y = x\sqrt{\frac{A+Cbb}{A}} - b\sqrt{\frac{A+Cxx}{A}}$$

and thus both the sum and the difference of the two functions of the same kind are able to be expressed by a function of the same kind. Now here with no distinction between variable quantities and constants, provided that $\Pi: z$ signifies the determination of a function of *z*, clearly

$$\int \frac{dz}{\sqrt{(A+Czz)}} = \Pi: z,$$

which, as we have assumed, vanishes on putting $z=0$, as with the signs to be given in this way there becomes

$$\Pi:r = \Pi:p + \Pi:q,$$

which must be given by

$$r = p\sqrt{\frac{A+Cqq}{A}} + q\sqrt{\frac{A+Cpp}{A}};$$

so that now

$$\Pi:r = \Pi:p - \Pi:q,$$

must be given by

$$r = p\sqrt{\frac{A+Cqq}{A}} - q\sqrt{\frac{A+Cpp}{A}},$$

but each side with the irrationality removed produces this equation between $p,q,r$ :

$$p^4+q^4+r^4-2ppqq-2pprr-2qqrr=\frac{4Cppqqrr}{A},$$

the form of which supplies this property, so that, if $p,q,r$ should be the sides of a certain triangle and a circle being circumscribed to that, the diameter of which may be placed equal to $=T$ , there shall always be $A+4CTT=0$. [The above formula follows from the sine and cosine rules on eliminating the angles and obtaining a relation between the sides *p, q,* and *r* of a triangle.] But that equation on account of several roots which it embraces satisfies this relation :

$$\Pi:p\pm\Pi:q\pm\Pi:r=0.$$

### COROLLARY 1

**586.** Hence the comparison of the arcs of the circle is deduced at once by noting on putting $A=1$ and $C=-1$. Indeed there becomes

$$\Pi:z=\int\frac{dz}{\sqrt{(1-zz)}}=Ang.sin.z$$

and hence in order that

$$Ang.sin.r=Ang.sin.p+Ang.sin.q,$$

it is required that

$$r=p\sqrt{(1-qq)}+q\sqrt{(1-pp)},$$

and so that there shall be

$$Ang.sin.r=Ang.sin.p-Ang.sin.q,$$

there must be

$$r=p\sqrt{(1-qq)}-q\sqrt{(1-pp)},$$

as it is agreed.

### COROLLARY 2

**587.** If there should be $A=1$ and $C=1$, then

$$\Pi:z=\int\frac{dz}{\sqrt{(1+zz)}}=l\left(z+\sqrt{(1+zz)}\right);$$

from which so that there shall be

$$l\left(r+\sqrt{(1+rr)}\right)=l\left(p+\sqrt{(1+pp)}\right)+l\left(q+\sqrt{(1+qq)}\right),$$

then

$$r = p\sqrt{(1+qq)} + q\sqrt{(1+pp)}\ ;$$

but so that

$$l\left(r+\sqrt{(1+rr)}\right) = l\left(p+\sqrt{(1+pp)}\right) - l\left(q+\sqrt{(1+qq)}\right),$$

then

$$r = p\sqrt{(1+qq)} - q\sqrt{(1+pp)}\ ,$$

as follows at once from the nature of logarithms.

## COROLLARY 3

**588.** If we put $q = p$ in the first general formula, so that there becomes

$$\Pi : r = 2\Pi : p,$$

then

$$r = 2p\sqrt{\tfrac{A+Cpp}{A}}\ .$$

Hence again, if there is made $q = 2p\sqrt{\tfrac{A+Cpp}{A}}$, then $\Pi : r = \Pi : p + 2\Pi : p = 3\Pi : p$ on taking

$$r = p\sqrt{\tfrac{A+Cqq}{A}} + q\sqrt{\tfrac{A+Cpp}{A}}\ .$$

Now there is

$$\sqrt{\tfrac{A+Cqq}{A}} = \sqrt{\left(1+\tfrac{4Cpp}{A}\left(1+\tfrac{Cpp}{A}\right)\right)} = 1+\tfrac{2Cpp}{A}\ ,$$

from which, so that there shall be

$$\Pi : r = 3\Pi : p,$$

there is made

$$r = p\left(1+\tfrac{2Cpp}{A}\right) + 2p\left(1+\tfrac{Cpp}{A}\right) = 3p + \tfrac{4Cp^3}{A}\ .$$

## SCHOLIUM

**589.** In order that this multiplication is able to be continued easier, in addition the relation corresponding to the equation

$$\Pi:r=\Pi:p+\Pi:q\,,$$

which is

$$r=p\sqrt{\tfrac{A+Cqq}{A}}+q\sqrt{\tfrac{A+Cpp}{A}}\,,$$

the equation may be noted :

$$\Pi:p=\Pi:r-\Pi:q\,,$$

to which the relation corresponds

$$p=r\sqrt{\tfrac{A+Cqq}{A}}-q\sqrt{\tfrac{A+Crr}{A}}$$

from which there becomes

$$\sqrt{\tfrac{A+Crr}{A}}=\tfrac{r}{q}\sqrt{\tfrac{A+Cqq}{A}}-\tfrac{p}{q}=\tfrac{p}{q}\cdot\tfrac{A+Cqq}{A}+\sqrt{\left(\tfrac{A+Cpp}{A}\right)\left(\tfrac{A+Cqq}{A}\right)}-\tfrac{p}{q}$$

or

$$\sqrt{\tfrac{A+Crr}{A}}=\tfrac{Cpq}{A}+\sqrt{\left(\tfrac{A+Cpp}{A}\right)\left(\tfrac{A+Cqq}{A}\right)}$$

Whereby so that there shall be

$$\Pi:r=\Pi:p+\Pi:q\,,$$

we will have not only

$$r=p\sqrt{1+\tfrac{C}{A}qq}+q\sqrt{1+\tfrac{C}{A}pp}\,,$$

but also

$$\sqrt{1+\tfrac{C}{A}rr}=\tfrac{C}{A}pq+\sqrt{\left(1+\tfrac{C}{A}pp\right)\left(1+\tfrac{C}{A}qq\right)}.$$

For the sake of brevity we may put $\sqrt{1+\frac{C}{A}pp}=P$ and on taking $q=p$, so that there becomes

$$\Pi:r=2\Pi:p\,,$$

then

$$r=2Pp\quad\text{and}\quad\sqrt{\left(1+\tfrac{C}{A}rr\right)}=\tfrac{C}{A}pp+PP,$$

which value of $r$ taken for $q$ gives

$$\Pi:r=3\Pi:p$$

with the equations arising

$$r = \tfrac{C}{A}p^3 + 3PPp \quad \text{and} \quad \sqrt{\left(1+\tfrac{C}{A}rr\right)} = \tfrac{3C}{A}Ppp + P^3.$$

This value of *r* again taken for *q* gives

$$\Pi : r = 4\Pi : p$$

with the equations arising

$$r = \tfrac{4C}{A}Pp^3 + 4P^3p \quad \text{and} \quad \sqrt{\left(1+\tfrac{C}{A}rr\right)} = \tfrac{CC}{AA}p^4 + \tfrac{6C}{A}PPpp + P^4.$$

This value *q* can be substituted in place of *r*, in order that there is produced

$$\Pi : r = 5\Pi : p$$

with the equations arising

$$r = \tfrac{CC}{AA}p^5 + \tfrac{10C}{A}PPp^3 + 5P^4p \quad \text{and} \quad \sqrt{\left(1+\tfrac{C}{A}rr\right)} = \tfrac{5CC}{AA}Pp^4 + \tfrac{10C}{A}P^3pp + P^5.$$

And hence generally it is allowed to conclude, in order that

$$\Pi : r = n\Pi : p\,,$$

there must be

$$r\sqrt{\tfrac{C}{A}} = \tfrac{1}{2}\left(P + p\sqrt{\tfrac{C}{A}}\right)^n - \tfrac{1}{2}\left(P - p\sqrt{\tfrac{C}{A}}\right)^n$$

and

$$\sqrt{\left(1+\tfrac{C}{A}rr\right)} = \tfrac{1}{2}\left(P + p\sqrt{\tfrac{C}{A}}\right)^n + \tfrac{1}{2}\left(P - p\sqrt{\tfrac{C}{A}}\right)^n$$

or

$$r = \tfrac{\sqrt{A}}{2\sqrt{C}}\left(P + p\sqrt{\tfrac{C}{A}}\right)^n - \tfrac{\sqrt{A}}{2\sqrt{C}}\left(P - p\sqrt{\tfrac{C}{A}}\right)^n.$$

Therefore this relation between *p* and *r* will satisfy this differential equation

$$\frac{dr}{\sqrt{(A+Crr)}} = \frac{ndp}{\sqrt{(A+Cpp)}}\,,$$

while we bear in mind that $P = \sqrt{1+\frac{Cpp}{A}}$ .

## PROBLEM 75

**590.** *If there is put* $\int \frac{dz}{\sqrt{(A+Czz)}} = \Pi : z$ *for the integral thus taken, so that it vanishes on putting* $z = f$ *, from which* $\Pi : z$ *shall be a function determined of z, to put in place a comparison between all the functions of this kind.*

## SOLUTION

This differential equation may be considered :

$$\frac{dx}{\sqrt{(A+Cxx)}} + \frac{dy}{\sqrt{(A+Cyy)}} = 0,$$

from which on integration there becomes

$$\Pi : x + \Pi : y = \text{Const}.$$

But the integral shall be also [§ 584] :

$$\alpha + \gamma(xx + yy) + 2\delta xy = 0,$$

because so that it may have a place, by necessity there must be

$$-\alpha\gamma = Am \quad \text{and} \quad \delta\delta - \gamma\gamma = Cm;$$

then there now becomes :

$$\gamma x + \delta y = \sqrt{m(A + Cyy)} \text{ and } \gamma y + \delta x = \sqrt{m(A + Cxx)}.$$

We may put the constant arriving in the integration thus to be defined, so that on putting $x = a$ there becomes $y = b$, and the integral will be

$$\Pi : x + \Pi : y = \Pi : a + \Pi : b.$$

But for the algebraic form to be found there shall be put for brevity :

$$\sqrt{(A + Caa)} = \mathfrak{A} \quad \text{and} \quad \sqrt{(A + Cbb)} = \mathfrak{B}$$

and then there shall be

$$\gamma a + \delta b = \mathfrak{B}\sqrt{m} \text{ and } \gamma b + \delta a = \mathfrak{A}\sqrt{m},$$

from which there is deduced

$$\gamma = \frac{\mathfrak{A}b-\mathfrak{B}a}{bb-aa}\sqrt{m} \quad \text{and} \quad \delta = \frac{\mathfrak{B}b-\mathfrak{A}a}{bb-aa}\sqrt{m}.$$

On account of which the algebraic equation of the integral shall be

$$(\mathfrak{A}b-\mathfrak{B}a)x+(\mathfrak{B}b-\mathfrak{A}a)y=(bb-aa)\sqrt{(A+Cyy)}$$

or

$$(\mathfrak{A}b-\mathfrak{B}a)y+(\mathfrak{B}b-\mathfrak{A}a)x=(bb-aa)\sqrt{(A+Cxx)}.$$

Hence $y$ is thus defined through $x$, so that it becomes :

$$y=\frac{(\mathfrak{A}a-\mathfrak{B}b)x+(bb-aa)\sqrt{(A+Cxx)}}{\mathfrak{A}b-\mathfrak{B}a}$$

which fraction now being multiplied above and below by $\mathfrak{A}b-\mathfrak{B}a$ on account of

$$\mathfrak{A}\mathfrak{A}bb-\mathfrak{B}\mathfrak{B}aa=A(bb-aa)$$

and

$$(\mathfrak{A}a-\mathfrak{B}b)(\mathfrak{A}b+\mathfrak{B}a)=(\mathfrak{A}\mathfrak{A}-\mathfrak{B}\mathfrak{B})ab-\mathfrak{A}\mathfrak{B}(bb-aa)=-(bb-aa)(Cab+\mathfrak{A}\mathfrak{B})$$

becomes :

$$y=-\frac{(Cab+\mathfrak{A}\mathfrak{B})x}{A}+\frac{(\mathfrak{A}b+\mathfrak{B}a)\sqrt{(A+Cxx)}}{A}.$$

Hence again there is deduced :

$$(bb-aa)\sqrt{(A+Cyy)}=(\mathfrak{A}b-\mathfrak{B}a)x-\frac{(\mathfrak{B}b-\mathfrak{A}a)^2}{\mathfrak{A}b-\mathfrak{B}a}x+\frac{(\mathfrak{B}b-\mathfrak{A}a)(bb-aa)}{\mathfrak{A}b-\mathfrak{B}a}\sqrt{(A+Cxx)}$$

or

$$\sqrt{(A+Cyy)}=-\frac{C(bb-aa)}{\mathfrak{A}b-\mathfrak{B}a}x+\frac{\mathfrak{B}b-\mathfrak{A}a}{\mathfrak{A}b-\mathfrak{B}a}\sqrt{(A+Cxx)},$$

where again on multiplying above and below by $\mathfrak{A}b+\mathfrak{B}a$ there becomes

$$\sqrt{(A+Cyy)}=-\frac{C(\mathfrak{A}b+\mathfrak{B}a)}{A}x+\frac{(Cab+\mathfrak{A}\mathfrak{B})}{A}\sqrt{(A+Cxx)}.$$

But it is necessary that the value of the formula $\sqrt{(A+Cyy)}$ be defined in this way rather than from the extraction of the root, from which an ambiguity may be implicated.

On which account this transcending equation

$$\Pi{:}r + \Pi{:}s = \Pi{:}p + \Pi{:}q$$

gives the following algebraic determination, if for the sake of brevity we may put

$$\sqrt{(A + Cpp)} = P, \quad \sqrt{(A + Cqq)} = Q, \quad \sqrt{(A + Crr)} = R;$$

clearly so that there becomes $\Pi{:}s = \Pi{:}p + \Pi{:}q - \Pi{:}r$ , then

$$s = \frac{-PQr - Cpqr + PRq + QRp}{A}$$

and

$$\sqrt{(A + Css)} = \frac{-CPqr - CQpr + CRpq + PQR}{A}$$

or

$$\sqrt{(A + Css)} = \frac{PQR + C(Rpq - Pqr - Qpr)}{A}.$$

## COROLLARY 1

**591.** Because by hypothesis there is $\Pi{:}f = 0$, if for brevity we put $\sqrt{(A + Cff)} = F$ and $r = f$ , in order that there becomes $R = F$ , this equation

$$\Pi{:}s = \Pi{:}p + \Pi{:}q$$

gives

$$s = \frac{F(Pq + Qp) - PQf - Cfpq}{A}$$

and

$$\sqrt{(A + Css)} = \frac{FPQ + CFpq - Cf(Pq + Qp)}{A}).$$

## COROLLARY 2

**592.** If we put $q = f$ and $Q = F$, in order that there becomes $\Pi{:}q = 0$, this equation

$$\Pi{:}s = \Pi{:}p - \Pi{:}r .$$

gives

$$s = \frac{F(Rp - Pr) + fPR - Cfpr}{A}$$

and

$$\sqrt{(A + Css)} = \frac{FPR - CFpr + Cf(Rp - Pr)}{A}.$$

### COROLLARY 3

**593.** If there is put $C=0$ and $A=1$, then there becomes $\Pi: z=\int dz=z-f$, because the integral thus must be taken, so that it vanishes on putting $z=f$. Hence there will then be $P=1$, $Q=1$ and $\Pi=1$, from which, so that there becomes $\Pi{:}s=\Pi{:}p+I\Pi{:}q-\Pi{:}r$ or $s=p+q-r$, it is required that $s=-r+q+p$ and $\sqrt{(1+0ss)}=1$, as by itself is agreed.

### COROLLARY 4

**594.** If there is taken $A=1$ and $C=-1$ and there becomes $\Pi: z=Arc.\ cos.z$, so that there becomes $f=1$, then

$$Arc.cos.s=Arc.cos.p+Arc.cos.q-Arc.cos.r,$$

if there should be

$$s=pqr-PQr+PRq+QRp$$

and

$$\sqrt{(1-ss)}=PQR+Pqr+Qpr-Rpq,$$

from which on taking $r=1$, so that there becomes $R=0$ and $Arc.cos.r=0$, then

$$s=pq-PQ \quad \text{and} \quad \sqrt{(1-ss)}=Pq+Qp.$$

### SCHOLIUM

**595.** Hence our rules for cosines are deduced, which we shall not pursue further. Now the easiest case, in which $A=0$ and $C=1$ and hence there becomes

$$\Pi{:}z=\int\frac{dz}{z}=lz$$

with $f=1$, may be seen in the first place to be marked out with a difficulty on account of the expressions for $s$ and $\sqrt{(A+Css)}=s$ becoming infinite. To which inconvenience as it occurs, first indeed the number $A$ is considered as indefinitely small and then there shall be

$$P=\sqrt{(pp+A)}=p+\tfrac{A}{2p},\quad Q=p+\tfrac{A}{2q},\quad R=r+\tfrac{A}{2r}.$$

Whereby in order that there becomes $ls=lp+lq-lr$, there is found

$$As=-r\left(p+\tfrac{A}{2p}\right)\left(q+\tfrac{A}{2q}\right)-pqr+q\left(p+\tfrac{A}{2p}\right)\left(r+\tfrac{A}{2r}\right)+p\left(q+\tfrac{A}{2q}\right)\left(r+\tfrac{A}{2r}\right)$$

and from expanding out the individual terms,

$$As = -\frac{Aqr}{2p} - \frac{Apr}{2q} + \frac{Aqr}{2p} + \frac{Apq}{2r} + \frac{Apr}{2q} + \frac{Apq}{2r}$$

or $s = \frac{pq}{r}$, as demanded from the nature of logarithms.

The remaining multiplication of this kind of transcendental function is deduced without difficulty from the formulas found; just as there shall be $\Pi{:}y = n\Pi{:}x$, the relation between $x$ and $y$ can be assigned algebraically.

## PROBLEM 76

**596.** *If there is put* $\Pi{:}z = \int \frac{dz(L+Mzz)}{\sqrt{(A+Czz)}}$ *with this integral taken so that it vanishes on putting* $z = 0$, *to investigate the comparison between transcendent functions of this kind.*

## SOLUTION

That relation may be put between the two variables $x$ and $y$

$$\alpha + \gamma(xx + yy) + 2\delta xy = 0,$$

from which there shall be

$$y = \frac{-\delta x + \sqrt{(-\alpha\gamma + (\delta\delta - \gamma\gamma)xx)}}{\gamma}.$$

Putting $-\alpha\gamma = Am$ and $\gamma\gamma - \delta\delta = Cm$, so that there arises

$$\gamma y + \delta x = \sqrt{m(A + Cxx)} \text{ and } \gamma x + \delta y = \sqrt{m(A + Cyy)}.$$

But that equation on differentiation becomes

$$dx(yx + \delta \mathrm{y}) + dy(\gamma y + \delta x) = 0$$

or

$$\frac{dx}{\sqrt{(A+Cxx)}} + \frac{dy}{\sqrt{(A+Cyy)}} = 0.$$

Now there is put in place

$$\frac{dx(L+Mxx)}{\sqrt{(A+Cxx)}} + \frac{dy(L+Myy)}{\sqrt{(A+Cyy)}} = dV\sqrt{m},$$

so that on integrating it becomes

$$\Pi{:}x + \Pi{:}y = \text{Const.} + V\sqrt{m}.$$

Therefore since there is

$$\frac{dy}{\sqrt{(A+Cyy)}} = \frac{-dx}{\sqrt{(A+Cxx)}},$$

then

$$dV\sqrt{m} = \frac{Mdx(xx-yy)}{\sqrt{(A+Cxx)}}$$

and hence on account of $y = \frac{\sqrt{m(A+Cxx)}-\delta x}{\gamma}$ there becomes

$$xx-yy = \frac{1}{\gamma\gamma}\left(\gamma\gamma xx - mA - mCxx - \delta\delta xx + 2\delta x\sqrt{m(A+Cxx)}\right)$$

But $\gamma\gamma - yy = -mC$, hence

$$dV\sqrt{m} = \frac{Mdx\left(2\delta x\sqrt{m(A+Cxx)}-mA-2mCxx\right)}{\gamma\gamma\sqrt{(A+Cxx)}},$$

the integral of this can be taken conveniently, while there becomes

$$V\sqrt{m} = \frac{\delta Mxx\sqrt{m}}{\gamma\gamma} - \frac{Mmx}{\gamma\gamma}\sqrt{(A+Cxx)}$$

which formula on account of $\sqrt{m(A+Cxx)} = \gamma y + \delta x$ changes into

$$V\sqrt{m} = \frac{\delta Mxx - \gamma Mxy - \delta Mxx}{\gamma\gamma}\sqrt{m} = -\frac{Mxy}{\gamma}\sqrt{m}\ .$$

On account of which we have

$$\Pi\!: x + \Pi\!:y = \text{Const.} - \frac{Mxy}{\gamma}\sqrt{m}$$

with $\gamma y + \delta x = \sqrt{m(A+Cxx)}$ and $\gamma x + \delta y = \sqrt{m(A+Cyy)}$

and in addition

$$-\alpha\gamma = Am \quad \text{and} \quad \delta\delta - \gamma\gamma = Cm\,.$$

Towards defining the constant we may take $y = b$ on putting $x = 0$, so that there becomes

$$\Pi\!: x + \Pi\!:y = \Pi\!:b - \frac{Mxy}{\gamma}\sqrt{m}.$$

Moreover now there is $\gamma b = \sqrt{mA}$ and $\delta b = \sqrt{(mA + mCbb)}$, hence

$$\gamma = \frac{\sqrt{mA}}{b} \text{ and } \delta = \frac{\sqrt{(mA+mCbb)}}{b}.$$

Hence we may conclude therefore, if there should be

$$y\sqrt{A} + x\sqrt{(A + Cbb)} = b\sqrt{(A + Cxx)}$$

and, because that is reduced to the same ,

$$x\sqrt{A} + y\sqrt{(A + Cbb)} = b\sqrt{(A + Cyy)},$$

that there is

$$\Pi{:}x + \Pi{:}\,y = \Pi{:}b - \frac{Mbxy}{\sqrt{A}}$$

with $\Pi$ denoting a function of this kind of the suffix, so that there shall be

$$\Pi{:}z = \int \frac{dz(L+Mzz)}{\sqrt{(A+Czz)}}$$

with the integral thus taken, so that it vanishes on putting $z = 0$.

With the nature of these functions established and with the difference between the variable and constant quantities removed there shall be

$$\Pi{:}r = \Pi{:}p + \Pi{:}q + \frac{Mpqr}{\sqrt{A}},$$

if there should be

$$q\sqrt{A} + p\sqrt{(A + Crr)} = r\sqrt{(A + Cpp)}$$

and

$$p\sqrt{A} + q\sqrt{(A + Crr)} = r\sqrt{(A + Cqq)}$$

from which there becomes

$$r = \frac{p\sqrt{(A+Cqq)}+q\sqrt{(A+Cpp)}}{\sqrt{A}}$$

and

$$\sqrt{(A + Crr)} = \frac{Cpq+\sqrt{(A+Cpp)(A+Cqq)}}{\sqrt{A}}$$

## COROLLARY 1

**597.** On taking *z* negative there is $\Pi{:}-z=-\Pi{:}z$, from which on taking the quantities *p* and *q* negative there becomes

$$\Pi{:}p+\Pi{:}q+\Pi{:}r=\frac{Mpqr}{\sqrt{A}},$$

if there should be either

$$p\sqrt{A}+q\sqrt{(A+Crr)}+r\sqrt{(A+Cqq)}=0,$$

or

$$q\sqrt{A}+p\sqrt{(A+Crr)}+r\sqrt{(A+Cpp)}=0,$$

or

$$r\sqrt{A}+p\sqrt{(A+Cqq)}+q\sqrt{(A+Cpp)}=0,$$

or

$$Cpq-\sqrt{A(A+Crr)}+\sqrt{(A+Cpp)(A+Cqq)}=0,$$

from which this relation is formed :

$$Cpqr+p\sqrt{(A+Cqq)(A+Crr)}+q\sqrt{(A+Cpp)(A+Crr)}+r\sqrt{(A+Cpp)(A+Cqq)}=0.$$

## COROLLARY 2

**598.** Hence this method can show three functions of this kind $\Pi{:}\,z$, the sum of which it is possible to express algebraically ; but it prevails also from the sum we have shown, that the third may be subtracted from the sum of two.

## COROLLARY 3

**599.** If we put $L=A$ and $M=C$, the proposed function

$$\Pi{:}z=\int dz\sqrt{(A+Czz)}$$

expresses the arc of a curve, the abscissa *z* of which agrees with the applied line [i.e. the y coordinate] $\sqrt{(A+Czz)}$, and the sum of the three arcs of this kind thus is given algebraically :

$$\Pi{:}p+\Pi{:}q+\Pi{:}r=\frac{Cpqr}{\sqrt{A}},$$

if the above relation is put in place between *p, q, r*.

## SCHOLIUM

**600.** From which this property thence has arisen, because the differential $dV$ gives rise to the integral. Since certainly there must be

$$dV\sqrt{m} = \frac{Mdx(xx-yy)}{\sqrt{(A+Cxx)}},$$

thus in order that

on account of $\sqrt{m(A+Cxx)} = \gamma y + \delta x$ then there becomes

$$dV = \frac{Mdx(xx-yy)}{\gamma y+\delta x},$$

the integral of which conveniently can be defined from the assumed equation

$$\alpha + \gamma(xx + yy) + 2\delta xy = 0 .$$

Indeed there is put

$$xx + yy = tt \quad \text{and} \quad xy = u\ ;$$

there shall be

$$\alpha + \gamma tt + 2\delta u = 0$$

and with the differentials taken:

$$xdx + ydy = tdt, \;\; xdy + ydx = du \quad \text{and } \gamma tdt + \delta du = 0\ ;$$

from the two equations before there is deduced

$$(xx - yy)dx = xtdt - ydu$$

and on account of $tdt = -\frac{\delta du}{\gamma}$ there shall be

$$(xx - yy)dx = -\frac{du}{\gamma}(\delta x + \gamma y),$$

thus in order that there becomes

$$\frac{dx(xx-yy)}{\gamma y+\delta x} = -\frac{du}{\gamma}$$

and hence $dV = -\frac{Mdu}{\gamma}$, from which it clearly follows that $V = -\frac{Mu}{\gamma} = -\frac{Mxy}{\gamma}$, as we have elicited more laboriously in the solution. Now by this convenient method as it is permitted in the following problem, where we are to consider more complex formulas.

## PROBLEM 77

**601.** *If there is put*

$$\Pi{:}z=\int\frac{dz\left(L+Mz^2+Nz^4+Oz^6\right)}{\sqrt{(A+Czz)}}$$

*for the integral of this taken so that it vanishes on putting* $z=0$, *to investigate the comparison between transcending functions of this kind.*

## SOLUTION

With this relation put as before between the variables $x$ and $y$

$$a+\gamma(xx+yy)+2\delta xy=0$$

there shall be $-\alpha\gamma=Am$ and $\delta\delta-\gamma\gamma=Cm$ and there becomes

$$\gamma y+\delta x=\sqrt{m(A+Cxx)} \text{ and } \gamma x+\delta y=\sqrt{m(A+Cyy)}$$

taken from the differentials

$$\frac{dx}{\sqrt{(A+Cxx)}}+\frac{dy}{\sqrt{(A+Cyy)}}=0.$$

Now there is put in place

$$\frac{dx\left(L+Mxx+Nx^4+Ox^6\right)}{\sqrt{(A+Cxx)}}+\frac{dy\left(L+Myy+Ny^4+Oy^6\right)}{\sqrt{(A+Cyy)}}=dV\sqrt{m},$$

so that there shall be

$$\Pi{:}\,x+\Pi{:}\,y=\text{Const.}+V\sqrt{m}.$$

But on account of $\frac{dy}{(A+Cyy)}=-\frac{dx}{(A+Cxx)}$ this equation changes into

$$\frac{dx\left(M(xx-yy)+N\left(x^4-y^4\right)+O\left(x^6-y^6\right)\right)}{\sqrt{(A+Cxx)}}=dV\sqrt{m}$$

and because $\sqrt{m(A+Cxx)}=\gamma y+\delta x$ into this :

$$\frac{dx(xx-yy)\left(M+N(xx+yy)+O\left(x^4+xxyy+y^4\right)\right)}{\gamma y+\delta x}=dV.$$

Now let there be $xx+yy=tt$ and $xy=u$, so that there becomes

$$a+\gamma tt+2\delta u=0 \text{ and } \gamma tdt+\delta du=0 \text{ or } tdt=-\tfrac{\delta du}{\gamma},$$

and on account of $xdx+ydy=tdt$ and $xdy+ydx=du$ we may deduce hence

$$(xx-yy)dx=xtdt-ydu=-\tfrac{du}{\gamma}(\gamma y+\delta x)$$

and thus

$$\frac{dx(xx-yy)}{\gamma y+\delta x}=-\frac{du}{\gamma}$$

from which we obtain:

$$dV=-\tfrac{du}{\gamma}\Big(M+N(xx+yy)+O\big(x^4+xxyy+y^4\big)\Big).$$

But there is

$$xx+yy=tt=\tfrac{-\alpha-2\delta u}{\gamma} \text{ and } x^4+xxyy+y^4=t^4-uu .$$

Moreover it is to be noted that $\frac{du}{\gamma}=-\frac{tdt}{\delta}$, from which we conclude

$$dV=-\tfrac{Mdu}{\gamma}+\tfrac{Nt^3dt}{\delta}+\tfrac{Ot^5}{\delta}dt+\tfrac{Ouudu}{\gamma},$$

and thus on integrating there is produced

$$V=-\tfrac{Mu}{\gamma}+\tfrac{Nt^4}{4\delta}+\tfrac{Ot^6}{6\delta}+\tfrac{Ou^3}{3\gamma}.$$

But if now we are able to make $y=b$, if $x=0$, then

$$\gamma=\tfrac{\sqrt{mA}}{b},\ \delta=\tfrac{\sqrt{m(A+Cbb)}}{b} \quad\text{and}\quad \alpha=b\sqrt{mA},$$

then indeed

$$y\sqrt{A}+x\sqrt{(A+Cbb)}=b\sqrt{(A+Cxx)},$$
$$x\sqrt{A}+y\sqrt{(A+Cbb)}=b\sqrt{(A+Cyy)}$$

and

$$b\sqrt{A}=x\sqrt{(A+Cyy)}+y\sqrt{(A+Cxx)}.$$

Hence, since there shall be

$$V=-\frac{Mbxy}{\sqrt{mA}}+\frac{Nb(xx+yy)^2}{4\sqrt{m(A+Cbb)}}+\frac{Ob(xx+yy)^3}{6\sqrt{m(A+Cbb)}}+\frac{Obx^3y^3}{3\sqrt{mA}}$$

our relation, to which the preceding determinations satisfy, will be between the transcending functions :

$$\varPi: x+\varPi: y=\varPi: bV-\frac{Mbxy}{\sqrt{A}}+\frac{Nb(xx+yy)^2}{4\sqrt{(A+Cbb)}}+\frac{Ob(xx+yy)^3}{6\sqrt{(A+Cbb)}}+\frac{Obx^3y^3}{3\sqrt{A}}-\frac{Nb^5}{4\sqrt{(A+Cbb)}}-\frac{Ob^7}{6\sqrt{(A+Cbb)}},$$

where it is to be observed in the calculations, that

$$-b\sqrt{A}+\frac{(xx+yy)\sqrt{A}}{b}+\frac{2xy\sqrt{(A+Cbb)}}{b}=0$$

or

$$xx+yy=bb-\frac{2xy\sqrt{(A+Cbb)}}{\sqrt{A}}.$$

Hence it may be deduced:

$$(xx+yy)^2-b^4=-\frac{4bbxy\sqrt{(A+Cbb)}}{\sqrt{A}}+\frac{4xxyy(A+Cbb)}{A}$$

and

$$(xx+yy)^3-b^6=-\frac{6b^4xy\sqrt{(A+Cbb)}}{\sqrt{A}}+\frac{12bbxxyy(A+Cbb)}{A}-\frac{8x^3y^3(A+Cbb)^{\frac{3}{2}}}{A\sqrt{A}},$$

and thus our equation shall be :

$$\varPi: x+\varPi: y=\varPi: b-\frac{Mbxy}{\sqrt{A}}-\frac{Nb^3xy}{\sqrt{A}}+\frac{Nbxx\,yy}{A}\sqrt{(A+Cbb)}-\frac{Ob^5xy}{\sqrt{A}}$$

$$+\ \frac{2Ob^3xxyy}{A}\sqrt{(A+Cbb)}-\frac{Obx^3y^3}{3A\sqrt{A}}(3A+4Cbb).$$

## COROLLARY 1

**602.** If we put $b=r$, $x=-p$, $y=-q$, our equation will be :

$$\varPi: p+\varPi: q+\varPi: r=\frac{pqr}{\sqrt{A}}\left(M+Nrr+Or^4\right)-\frac{ppqq\sqrt{(A+Crr)}}{A}\left(Nr+2Or^3\right)+\frac{Op^3q^3r}{3A\sqrt{A}}(3A+4Crr)$$

with the condition present

$$pp+qq=rr-\frac{2pq\sqrt{(A+Crr)}}{\sqrt{A}},$$

from which there becomes

$$\frac{\sqrt{(A+Crr)}}{\sqrt{A}}=\frac{rr-pp-qq}{2pq}.$$

### COROLLARY 2

**603.** On substituting this value for $\frac{\sqrt{(A+Crr)}}{\sqrt{A}}$ the following equation shall be obtained, in which the three quantities *p, q, r* enter equally,

$$\Pi:p+\Pi:q+\Pi:r=\frac{Mpqr}{\sqrt{A}}+\frac{Npqr}{2\sqrt{A}}(pp+qq+rr)$$
$$+\frac{Opqr}{3\sqrt{A}}\left(p^4+q^4+r^4+ppqq+pprr+qqrr\right),$$

which satisfy the formulas given above (§ 602), or this relation :

$$\frac{4Cppqqrr}{A}=p^4+q^4+r^4-2ppqq-2pprr-2qqrr.$$

### COROLLARY 3

**604.** If we should add the term $Pz^8$ to the numerator of the formula of the integral up to this point, so that there shall be :

$$\Pi:z=\int\frac{dz\left(L+Mz^2+NZ^4+Oz^6+Pz^8\right)}{\sqrt{(A+Czz)}}$$

to the equation found in this way at this point there must be added the term

$$\frac{Ppqr}{4A\sqrt{A}}\left(p^6+q^6+r^6+ppq^4+p^4qq+p^4rr+q^4rr+\tfrac{4}{3}ppqqrr\right).$$

### SCHOLIUM

**605.** These relations can also be derived from the above reductions [see §111and §113]; for since thence there shall be $\Pi:z=E\int\frac{dz}{\sqrt{(A+Czz)}}+$ *an algebraic quantity*, if here for *z* we put in place successively the quantities *p, q, r* thus depending on each other, as we have stated before, then

$$\int\frac{dp}{\sqrt{(A+Cpp)}}+\int\frac{dq}{\sqrt{(A+Cqq)}}+\int\frac{dr}{\sqrt{(A+Crr)}}=0,$$

from which we conclude

$$\Pi:p+\Pi:q+\Pi:r=f:p+f:q+f:r$$

with *f* denoting a certain algebraic function of the suffix quantity; and the sum of these three functions returns the expression found before, provided an account of the given relations between *p, q, r* may be had, clearly from which with the letter *C* required to be eliminated. But this reduction would require a great amount of work. Now in the first place this method which I have used, it is agreed to be considered that since it is completely straightforward, yet it is seen to offer deductions with more labour. Certainly the comparison of transcending functions, that I am about to treat in the following chapter, can be investigated with difficulty by another method, from which the usefulness of this method will be discerned in the following chapter.

## CAPUT V

# DE COMPARATIONE QUANTITATUM TRANSCENDENTIUM IN FORMA $\int \frac{Pdx}{\sqrt{(A+2Bx+Cxx)}}$ CONTENTARUM

### PROBLEMA 73

**580.** *Proposita inter x et y hac aequatione algebraica*

$$\alpha + 2\beta(x+y) + \gamma(xx+yy) + 2\delta xy = 0$$

*invenire formulas integrales formae praescriptae, quae inter se comparari queant.*

### SOLUTIO

Differentietur aequatio proposita et ex eius differentiali

$$2\beta dx + 2\beta dy + 2\gamma xdx + 2\gamma ydy + 2\delta xdy + 2\delta ydx = 0$$

colligetur haec aequatio

$$dx(\beta + \gamma x + \delta y) + dy(\beta + \gamma y + \delta x) = 0.$$

Statuatur

$$\beta + \gamma x + \delta y = p \quad \text{et} \quad \beta + \gamma y + \delta x = q$$

atque ex priori erit

$$pp = \beta\beta + 2\beta\gamma x + 2\beta\delta y + \gamma\gamma xx + 2\gamma\delta xy + \delta\delta yy\,,$$

a qua subtrahatur aequatio proposita per $\gamma$ multiplicata

$$0 = \alpha\gamma + 2\beta\gamma x + 2\gamma\beta y + \gamma\gamma xx + \gamma\gamma yy + 2\gamma xy$$

fietque

$$pp = \beta\beta - \alpha\gamma + 2\beta(\delta - \gamma)y + (\delta\delta - \gamma\gamma)yy\,;$$

similique modo reperietur

$$qq = \beta\beta - \alpha\gamma + 2\beta(\delta - \gamma)x + (\delta\delta - \gamma\gamma)xx\,,$$

unde erit $pdx + qdy = 0$. Cum iam sit *p* functio ipsius *y* et *q* similis functio ipsius *x*, ponatur

$$\beta\beta - \alpha\gamma = \mathrm{A}, \quad \beta(\delta - \gamma) = \mathrm{B} \quad \text{et} \quad \delta\delta - \gamma\gamma = \mathrm{C}\,,$$

unde colligitur

$$\delta-\gamma=\frac{B}{\beta} \quad \text{et} \quad \delta+\gamma=\frac{C}{\delta-\gamma}=\frac{\beta C}{B}$$

hincque

$$\delta=\frac{BB+\beta\beta C}{2B\beta} \quad \text{et} \quad \gamma=\frac{\beta\beta C-BB}{2B\beta}$$

prima vero dat

$$\alpha=\frac{\beta\beta-A}{\gamma}=\frac{2B\beta(\beta\beta-A)}{\beta\beta C-BB}.$$

Quibus valoribus pro $\alpha, \gamma, \delta$ assumtis aequatio $\frac{dx}{q}+\frac{dy}{p}=0$ abit in hanc

$$\frac{dx}{\sqrt{(A+2Bx+Cxx)}}+\frac{dy}{\sqrt{(A+2By+Cyy)}}=0,$$

cui ergo aequationi differentiali satisfacit aequatio

$$\frac{2B\beta(\beta\beta-A)}{\beta\beta C-BB}+2\beta(x+y)+\frac{\beta\beta C-BB}{2B\beta}(xx+yy)+\frac{BB+\beta\beta C}{B\beta}xy=0;$$

quae cum contineat constantem novam $\beta$, erit adeo integrale completum aequationis differentialis inventae.

Neque vero opus est, ut formulae illae ipsis litteris *A*, *B*, *C* aequentur, sed sufficit, ut ipsis sint proportionales, unde fit

$$\frac{\beta\beta-\alpha\gamma}{\beta(\delta-\gamma)}=\frac{A}{B} \quad \text{et} \quad \frac{\delta+\gamma}{\beta}=\frac{C}{B}.$$

Ergo

$$\delta=\frac{\beta C}{B}-\gamma \quad \text{et} \quad \alpha=\frac{\beta\beta}{\gamma}-\frac{\beta}{\gamma}\frac{A}{B}(\delta-\gamma) \quad \text{seu} \quad \alpha=\frac{\beta\beta}{\gamma}-\frac{\beta\beta AC}{\gamma BB}+\frac{2\beta A}{B}.$$

Quare aequationis differentialis

$$\frac{dx}{\sqrt{(A+2Bx+Cxx)}}+\frac{dy}{\sqrt{(A+2By+Cyy)}}=0$$

integrale completum est

$$\beta\beta(BB-AC)+2\beta\gamma AB+2\beta\gamma BB(x+y)+\gamma\gamma BB(xx+yy)$$
$$+\ 2\gamma B(\beta C-\gamma B)xy=0,$$

ubi ratio $\frac{\beta}{\gamma}$ constantem arbitrariam exhibet.

### COROLLARIUM 1

**581.** Ex aequatione proposita radicem extrahendo fit

$$-y=\frac{-\beta-\delta x+\sqrt{(\beta\beta+2\beta\delta x+\delta\delta xx-\alpha\gamma-2\beta\gamma x-\gamma\gamma xx)}}{\gamma}$$

seu loco $\alpha$ et $\delta$ substitutis valoribus

$$y=-\frac{\beta}{\gamma}-\frac{\beta C-\gamma B}{\gamma B}x+\sqrt{\frac{\beta\beta C-2\beta\gamma B}{\gamma\gamma BB}}\left(A+2Bx+Cxx\right)$$

### COROLLARIUM 2

**582.** Si ergo $x=0$, fit

$$y=-\frac{\beta}{\gamma}+\sqrt{\frac{\beta\beta AC-2\beta\gamma AB}{\gamma\gamma BB}}$$

ponatur hic valor $=a$, ut sit

$$\gamma Ba+\beta B=\sqrt{(\beta\beta AC-2\beta\gamma AB)},$$

unde sumtis quadratis oritur

$$\gamma\gamma BBaa+2\beta\gamma BBa+\beta\beta BB=\beta\beta AC-2\beta\gamma AB$$

hincque

$$\frac{\gamma}{\beta}=\frac{-A-Ba+\sqrt{A(A+2Ba+Caa)}}{Baa}$$

seu

$$\frac{\beta}{\gamma}=\frac{B\left(A+Ba+\sqrt{A(A+2Ba+Caa)}\right)}{AC-BB}.$$

### SCHOLION 1

**583.** Ut aequatio assumta

$$\alpha+2\beta(x+y)+\gamma(xx+yy)+2\delta xy=0$$

satisfaciat aequationi differentiali

$$\frac{dx}{\sqrt{(A+2Bx+Cxx)}}+\frac{dy}{\sqrt{(A+2By+Cyy)}}=0$$

necesse est, ut sit

$$\beta\beta-\alpha\gamma=mA,\quad \beta\left(\delta-\gamma\right)=mB\quad\text{et}\quad \delta\delta-\gamma\gamma=mC,$$

unde fit

$$\beta+\gamma y+\delta x=\sqrt{m(A+2Bx+Cxx)}$$

et

$$\beta+\gamma x+\delta y=\sqrt{m(A+2By+Cyy)}..$$

At ex datis *A, B, C* litterarum $\alpha, \beta, \gamma, \delta$ et *m* tres tantum definiuntur; quare cum binae maneant indeterminatae, aequatio assumta, etiamsi per quemvis coefficientium dividatur, unam tamen constantem continet novam, ex quo ea pro integrali completo erit habenda. Quare etsi aequationis differentialis neutra pars integrationem algebraice admittit, tamen integrale completum algebraice exhiberi potest. Loco constantis arbitrariae is valor ipsius *y* introduci potest, quem recipit posito $x=0$; cum autem evenire possit, ut hic valor fiat imaginarius, conveniet istam constantem ita definiri, ut posito $x=a$ fiat $y=b$, quo pacta ad omnes casus applicatio fieri poterit. Hinc erit

$$\frac{\beta+\gamma b+\delta a}{\beta+\gamma a+\delta b}=\sqrt{\frac{A+2Ba+Caa}{A+2Bb+Cbb}},$$

unde colligitur

$$\beta=\frac{(\gamma a+\delta b)\sqrt{(A+2Ba+Caa)}-(\gamma b+\delta a)\sqrt{(A+2Bb+Cbb)}}{-\sqrt{(A+2Ba+Caa)}+\sqrt{(A+2Bb+Cbb)}}$$

et

$$\sqrt{m(A+2Ba+Caa)}=\frac{(\delta-\gamma)(b-a)\sqrt{(A+2Ba+Caa)}}{\sqrt{(A+2Bb+Cbb)}-\sqrt{(A+2Ba+Caa)}}$$

seu

$$\sqrt{m}=\frac{(\delta-\gamma)(b-a)}{\sqrt{(A+2Bb+Cbb)}-\sqrt{(A+2Ba+Caa)}}.$$

Ponatur brevitatis gratia

$$\sqrt{(A+2Ba+Caa)}=\mathfrak{A} \quad \text{et} \quad \sqrt{(A+2Bb+Cbb)}=\mathfrak{B},$$

ut sit

$$\sqrt{m}=\frac{(\delta-\gamma)(b-a)}{\mathfrak{B}-\mathfrak{A}} \quad \text{et} \quad \beta=\frac{\mathfrak{A}(\gamma a+\delta b)-\mathfrak{B}(\gamma b+\delta a)}{\mathfrak{B}-\mathfrak{A}}$$

et aequatio $\beta(\delta-\gamma)=mB$ induet hanc formam

$$\mathfrak{A}(\gamma a+\delta b)-\mathfrak{B}(\gamma b+\delta a)=\frac{B(\delta-\gamma)(b-a)^2}{\mathfrak{B}-\mathfrak{A}},$$

unde fit

$$\left.\begin{aligned}&+\gamma\mathfrak{A}\mathfrak{B}-\gamma A-\gamma B(a+b)-\gamma C(aa-ab+bb)\\&+\delta\mathfrak{A}\mathfrak{B}-\delta A-\delta B(a+b)-\delta Cab\end{aligned}\right\}=0.$$

Statuatur ergo

$$\gamma = n\mathfrak{AB} - nA - nB(a+b) - nCab,$$
$$\delta = nA + nB(a+b) + nC(aa-ab+bb) - n\mathfrak{AB},$$
$$\sqrt{m} = \frac{n(b-a)(\mathfrak{A}^2+\mathfrak{B}^2-2\mathfrak{AB})}{\mathfrak{B}-\mathfrak{A}} = n(b-a)(\mathfrak{B}-\mathfrak{A}),$$
$$\beta = nB(b-a)^2, \quad \text{ergo} \quad \delta - \gamma = \frac{m}{n(b-a)^2},$$

unde cum sit $\alpha + \gamma = nC(b-a)^2$, erit utique $\delta\delta - \gamma\gamma = mC$. Superest, ut fiat $\alpha\gamma = \beta\beta - mA$, hoc est

$$\alpha\gamma = nnBB(b-a)^4 - nnA(b-a)^2(\mathfrak{B}-\mathfrak{A})^2$$

seu

$$\alpha\gamma = nn(b-a)^2\left(BB(b-a)^2 - A(\mathfrak{B}-\mathfrak{A})^2\right).$$

Vel cum posito $x = a$ fiat $y = b$, erit quoque

$$\alpha = -2\beta(a+b) - \gamma(aa+bb) - 2\delta ab$$

hincque

$$\alpha = n(a-b)^2\left(A - B(a+b) - Cab - \mathfrak{AB}\right),$$

unde aequatio nostra assumta est

$$(b-a)^2\left(A - B(a+b) - Cab - \mathfrak{AB}\right)$$
$$+2B(b-a)^2(x+y) - \left(A + B(a+b) + Cab - \mathfrak{AB}\right)(xx+yy).$$
$$+2\left(A + B(a+b) + C(aa-ab+bb) - \mathfrak{AB}\right)xy = 0.$$

## SCHOLIUM 2

**584.** Si ponatur $\beta = 0$, ut aequatio sit

$$\alpha + \gamma(xx+yy) + 2\delta xy = 0,$$

erit

$$y = \frac{-\delta x + \sqrt{\left(-\alpha\gamma + (\delta\delta - \gamma\gamma)xx\right)}}{\gamma}.$$

Posito ergo $-\alpha\gamma = mA$ et $\delta\delta - \gamma\gamma = mC$, ut sit $\gamma y + \delta x = \sqrt{m(A+Cxx)}$, erit

$$\frac{dx}{\sqrt{(A+Cxx)}} + \frac{dy}{\sqrt{(A+Cyy)}} = 0,$$

cuius aequationis integrale completum erit ipsa aequatio assumta, pro qua habebitur $\frac{C}{A} = \frac{\gamma\gamma - \delta\delta}{\alpha\gamma}$ seu $\delta = \left(\gamma\gamma - \frac{\alpha\gamma C}{A}\right)$. Sin autem posito $x = 0$ fieri debeat

$y=b$, ob $\gamma b=\sqrt{mA}$ erit $\gamma=\frac{\sqrt{mA}}{b}$; tum $\alpha=-b\sqrt{mA}$ et $\delta=\sqrt{\left(\frac{mA}{bb}+mC\right)}$.

Habebitur ergo haec aequatio

$$\frac{y\sqrt{mA}}{b}+\frac{x\sqrt{m(A+Cbb)}}{b}=\sqrt{m(A+Cxx)}$$

quae praebet

$$y=-x\sqrt{\frac{A+Cbb}{A}}+b\sqrt{\frac{A+Cxx}{A}},$$

quae est integrale completum aequationis illius differentialis. Quare si $x$ capiatur negative, huius aequationis differentialis

$$\frac{dx}{\sqrt{(A+Cxx)}}=\frac{dy}{\sqrt{(A+Cyy)}}$$

integrale completum est

$$y=x\sqrt{\frac{A+Cbb}{A}}+b\sqrt{\frac{A+Cxx}{A}}.$$

Quodsi simili modo calculus in genere tractetur, aequationis differentialis

$$\frac{dx}{\sqrt{(A+2Bx+Cxx)}}+\frac{dy}{\sqrt{(A+2By+Cyy)}}=0$$

si brevitatis gratia ponatur $\sqrt{(A+2Bb+Cbb)}=\mathfrak{B}$, erit integrale completum

$$y\left(\sqrt{A}+\frac{Bb}{\sqrt{A}-\mathfrak{B}}\right)+x\left(\mathfrak{B}+\frac{Bb}{\sqrt{A}-\mathfrak{B}}\right)=\frac{Bbb}{\sqrt{A}-\mathfrak{B}}+b\sqrt{(A+2Bx+Cxx)},$$

unde casus praecedens manifesto sequitur, si ponatur $B=0$. Verum ope levis substitutionis hae formulae, ubi adest $B$, ad illum casum, ubi $B=0$, reduci possunt.

## PROBLEMA 74

**585.** *Si* $\Pi: z$ *significet eam functionem ipsius z, quae oritur ex integratione formulae* $\int \frac{dz}{\sqrt{(A+Czz)}}$, *integrali hoc ita sumto, ut evanescat posito z* = 0, *comparationem inter huiusmodi functiones instituere.*

## SOLUTIO

Consideretur haec aequatio differentialis

$$\frac{dx}{\sqrt{(A+Cxx)}} = \frac{dy}{\sqrt{(A+Cyy)}}$$

unde, cum sit per hypothesin

$$\int \frac{dx}{\sqrt{(A+Cxx)}} = \Pi: x \quad \text{et} \quad \int \frac{dy}{\sqrt{(A+Cyy)}} = \Pi: y$$

utroque integrali ita sumto, ut evanescat illud posito $x = 0$, hoc vero posito $y = 0$, integrale completum erit

$$\Pi: y = \Pi: x + C\,.$$

Ante autem vidimus hoc integrale esse

$$y = x\sqrt{\frac{A+Cbb}{A}} + b\sqrt{\frac{A+Cxx}{A}}$$

ubi posito $x = 0$ fit $y = b$; quare, cum $\Pi: 0 = 0$, erit

$$\Pi: y = \Pi: x + \Pi: b\,,$$

cui ergo aequationi transcendentali satisfacit haec algebraica

$$y = x\sqrt{\frac{A+Cbb}{A}} + b\sqrt{\frac{A+Cxx}{A}}$$

Simili modo sumto *b* negativo haec aequatio

$$\Pi: y = \Pi: x - \Pi: b$$

convenit cum hac

$$y = x\sqrt{\frac{A+Cbb}{A}} - b\sqrt{\frac{A+Cxx}{A}}$$

sicque tam summa quam differentia duarum huiusmodi functionum per similem functionem exprimi potest. Hic iam nullo habito discrimine inter quantitates variabiles et constantes, dum $\Pi : z$ functionem determinatam ipsius $z$ significat, scilicet

$$\int \frac{dz}{\sqrt{(A+Czz)}} = \Pi : z,$$

quae, ut assumsimus, evanescat posito $z = 0$, ut hoc signandi modo recepto sit

$$\Pi : r = \Pi : p + \Pi : q\,,$$

debet esse

$$r = p\sqrt{\frac{A+Cqq}{A}} + q\sqrt{\frac{A+Cpp}{A}}\,;$$

ut vero sit

$$\Pi : r = \Pi : p - \Pi : q\,,$$

debet esse

$$r = p\sqrt{\frac{A+Cqq}{A}} - q\sqrt{\frac{A+Cpp}{A}}\,,$$

utrinque autem sublata irrationalitate prodit inter $p, q, r$ haec aequatio

$$p^4 + q^4 + r^4 - 2ppqq - 2pprr - 2qqrr = \frac{4Cppqqrr}{A}\,,$$

cuius forma hanc suppeditat proprietatem, ut, si $p, q, r$ sint latera cuiusdam trianguli eique circumscribatur circulus, cuius diameter vocetur $= T$ , semper sit $A + 4CTT = 0$. Illa autem aequatio ob plures quas complectitur radices satisfacit huic relationi

$$\Pi : p \pm \Pi : q \pm \Pi : r = 0\,.$$

## COROLLARIUM 1

**586.** Hinc statim deducitur nota arcuum circularium comparatio ponendo $A = 1$ et $C = -1$. Tum enim fit

$$\Pi : z = \int \frac{dz}{\sqrt{(1-zz)}} = Ang.sin.z$$

hincque ut sit

$$Ang.sin.r = Ang.sin.p + Ang.sin.q\,,$$

oportet esse

$$r = p\sqrt{(1-qq)} + q\sqrt{(1-pp)},$$

et ut sit

$$Ang.sin.r = Ang.sin.p - Ang.sin.q,$$

debet esse

$r = p\sqrt{(1-qq)} - q\sqrt{(1-pp)}$, uti constat.

**COROLLARIUM 2**

**587.** Si sit $A=1$ et $C=1$, erit

$$\Pi : z=\int \frac{dz}{\sqrt{(1+zz)}}=l\left(z+\sqrt{(1+zz)}\right);$$

unde ut sit

$$l\left(r+\sqrt{(1+rr)}\right)=l\left(p+\sqrt{(1+pp)}\right)+l\left(q+\sqrt{(1+qq)}\right),$$

erit

$$r=p\sqrt{(1+qq)}+q\sqrt{(1+pp)};$$

ut autem sit

$$l\left(r+\sqrt{(1+rr)}\right)=l\left(p+\sqrt{(1+pp)}\right)-l\left(q+\sqrt{(1+qq)}\right),$$

erit

$$r=p\sqrt{(1+qq)}-q\sqrt{(1+pp)},$$

uti ex indole logarithmorum sponte liquet.

**COROLLARIUM 3**

**588.** Si ponamus in priori formula generali $q=p$, ut sit

$$\Pi : r=2\Pi : p,$$

erit

$$r=2p\sqrt{\frac{A+Cpp}{A}}.$$

Hinc porro, si fiat $q=2p\sqrt{\frac{A+Cpp}{A}}$, erit $\Pi : r=\Pi : p+2\Pi : p=3\Pi : p$ sumto

$$r=p\sqrt{\frac{A+Cqq}{A}}+q\sqrt{\frac{A+Cpp}{A}}.$$

Est vero

$$\sqrt{\frac{A+Cqq}{A}}=\sqrt{\left(1+\frac{4Cpp}{A}\left(1+\frac{Cpp}{A}\right)\right)}=1+\frac{2Cpp}{A},$$

unde, ut sit

$$\Pi : r=3\Pi : p,$$

fit

$$r=p\left(1+\frac{2Cpp}{A}\right)+2p\left(1+\frac{Cpp}{A}\right)=3p+\frac{4Cp^3}{A}.$$

### SCHOLION

**589.** Quo haec multiplicatio facilius continuari queat, praeter relationem aequationi

$$\Pi:r = \Pi:p + \Pi:q$$

respondentem, quae est

$$r = p\sqrt{\tfrac{A+Cqq}{A}} + q\sqrt{\tfrac{A+Cpp}{A}},$$

notetur aequatio

$$\Pi:p = \Pi:r - \Pi:q,$$

cui respondet relatio

$$r = p\sqrt{1+\tfrac{C}{A}qq} + q\sqrt{1+\tfrac{C}{A}pp}$$

unde fit

$$\sqrt{\tfrac{A+Crr}{A}} = \tfrac{r}{q}\sqrt{\tfrac{A+Cqq}{A}} - \tfrac{p}{q} = \tfrac{p}{q}\cdot\tfrac{A+Cqq}{A} + \sqrt{\left(\tfrac{A+Cpp}{A}\right)\left(\tfrac{A+Cqq}{A}\right)} - \tfrac{p}{q}$$

seu

$$\sqrt{\tfrac{A+Crr}{A}} = \tfrac{Cpq}{A} + \sqrt{\left(\tfrac{A+Cpp}{A}\right)\left(\tfrac{A+Cqq}{A}\right)}$$

Quare ut sit

$$\Pi:r = \Pi:p + \Pi:q,$$

habemus non solum

$$r = p\sqrt{\tfrac{A+Cqq}{A}} + q\sqrt{\tfrac{A+Cpp}{A}},$$

sed etiam

$$\sqrt{1+\tfrac{C}{A}rr} = \tfrac{C}{A}pq + \sqrt{\left(1+\tfrac{C}{A}pp\right)\left(1+\tfrac{C}{A}qq\right)}.$$

Ponamus brevitatis gratia $\sqrt{1+\tfrac{C}{A}pp} = P$ et sumto $q = p$, ut sit

$$\Pi:r = 2\Pi:p,$$

erit

$$r = 2Pp \quad \text{et} \quad \sqrt{\left(1+\tfrac{C}{A}rr\right)} = \tfrac{C}{A}pp + PP,$$

qui valor ipsius $r$ pro $q$ sumtus dabit

$$\Pi:r = 3\Pi:p$$

existente

$$r = \tfrac{C}{A}p^3 + 3PPp \quad \text{et} \quad \sqrt{\left(1+\tfrac{C}{A}rr\right)} = \tfrac{3C}{A}Ppp + P^3.$$

Hic valor ipsius $r$ denuo pro $q$ sumtus dabit

$$\Pi:r = 4\Pi:p$$

existente

$$r=\tfrac{4C}{A}Pp^3+4P^3p \quad \text{et} \quad \sqrt{\left(1+\tfrac{C}{A}rr\right)}=\tfrac{CC}{AA}p^4+\tfrac{6C}{A}PPpp+P^4.$$

Loco $q$ substituatur hic valor ipsius $r$, ut prodeat

$$\Pi:r=5\Pi:p$$

existente

$$r=\tfrac{CC}{AA}p^5+\tfrac{10C}{A}PPp^3+5P^4p \quad \text{et} \quad \sqrt{\left(1+\tfrac{C}{A}rr\right)}=\tfrac{5CC}{AA}Pp^4+\tfrac{10C}{A}P^3pp+P^5.$$

Atque hinc generatim concludere licet, ut sit

$$\Pi:r=n\Pi:p\ ,$$

esse debere

$$r\sqrt{\tfrac{C}{A}}=\tfrac{1}{2}\left(P+p\sqrt{\tfrac{C}{A}}\right)^n-\tfrac{1}{2}\left(P-p\sqrt{\tfrac{C}{A}}\right)^n$$

et

$$\sqrt{\left(1+\tfrac{C}{A}rr\right)}=\tfrac{1}{2}\left(P+p\sqrt{\tfrac{C}{A}}\right)^n+\tfrac{1}{2}\left(P-p\sqrt{\tfrac{C}{A}}\right)^n$$

seu

$$r=\tfrac{\sqrt{A}}{2\sqrt{C}}\left(P+p\sqrt{\tfrac{C}{A}}\right)^n-\tfrac{\sqrt{A}}{2\sqrt{C}}\left(P-p\sqrt{\tfrac{C}{A}}\right)^n.$$

Haec igitur relatio inter $p$ et $r$ satisfaciet huic aequationi differentiali

$$\frac{dr}{\sqrt{(A+Crr)}}=\frac{ndp}{\sqrt{(A+Cpp)}}\,,$$

dum meminerimus esse $P=\sqrt{1+\frac{Cpp}{A}}$ .

## PROBLEMA 75

**590.** *Si ponatur* $\int \frac{dz}{\sqrt{(A+Czz)}} = \Pi : z$ *integrali ita sumto, ut evanescat posito* $z = f$, *unde* $\Pi : z$ *fit functio determinata ipsius z, comparationem inter huiusmodi functiones instituere.*

## SOLUTIO

Consideretur haec aequatio differentialis

$$\frac{dx}{\sqrt{(A+Cxx)}} + \frac{dy}{\sqrt{(A+Cyy)}} = 0,$$

unde integrando fit

$$\Pi : x + \Pi : y = \text{Const}.$$

Integrale autem sit quoque [§ 584]

$$\alpha + \gamma(xx+yy) + 2\delta xy = 0,$$

quod ut locum habeat, necesse est sit

$$-\alpha\gamma = Am \quad \text{et} \quad \delta\delta - \gamma\gamma = Cm;$$

tum vero erit

$$\gamma x + \delta y = \sqrt{m(A+Cyy)} \ \text{ et } \ \gamma y + \delta x = \sqrt{m(A+Cxx)}.$$

Ponamus constantem integratione ingressam ita definiri, ut posito $x = a$ fiat $y = b$, et integrale erit

$$\Pi : x + \Pi : y = \Pi : a + \Pi : b.$$

Pro forma autem algebraica invenienda sit brevitatis gratia

$$\sqrt{(A+Caa)} = \mathfrak{A} \quad \text{et} \quad \sqrt{(A+Cbb)} = \mathfrak{B}$$

eritque

$$\gamma a + \delta b = \mathfrak{B}\sqrt{m} \ \text{ et } \ \gamma b + \delta a = \mathfrak{A}\sqrt{m},$$

unde colligitur

$$\gamma = \frac{\mathfrak{A}b - \mathfrak{B}a}{bb-aa}\sqrt{m} \quad \text{et} \quad \delta = \frac{\mathfrak{B}b - \mathfrak{A}a}{bb-aa}\sqrt{m}.$$

Quocirca aequatio integralis algebraica erit

$$(\mathfrak{A}b - \mathfrak{B}a)x + (\mathfrak{B}b - \mathfrak{A}a)y = (bb-aa)\sqrt{(A+Cyy)}$$

seu

$$(\mathfrak{A}b - \mathfrak{B}a)y + (\mathfrak{B}b - \mathfrak{A}a)x = (bb-aa)\sqrt{(A+Cxx)}.$$

Hinc $y$ per $x$ ita definitur, ut sit

$$y=\frac{(\mathfrak{A}a-\mathfrak{B}b)x+(bb-aa)\sqrt{(A+Cxx)}}{\mathfrak{A}b-\mathfrak{B}a}$$

quae fractio supra et infra per $\mathfrak{A}b-\mathfrak{B}a$ multiplicando ob

$$\mathfrak{A}\mathfrak{A}bb-\mathfrak{B}\mathfrak{B}aa=A(bb-aa)$$

et

$$(\mathfrak{A}a-\mathfrak{B}b)(\mathfrak{A}b+\mathfrak{B}a)=(\mathfrak{A}\mathfrak{A}-\mathfrak{B}\mathfrak{B})ab-\mathfrak{A}\mathfrak{B}(bb-aa)=-(bb-aa)(Cab+\mathfrak{A}\mathfrak{B})$$

abit in

$$y=-\frac{(Cab+\mathfrak{A}\mathfrak{B})x}{A}+\frac{(\mathfrak{A}b+\mathfrak{B}a)\sqrt{(A+Cxx)}}{A}.$$

Hinc porro colligitur

$$(bb-aa)\sqrt{(A+Cyy)}=(\mathfrak{A}b-\mathfrak{B}a)x-\frac{(\mathfrak{B}b-\mathfrak{A}a)^2}{\mathfrak{A}b-\mathfrak{B}a}x+\frac{(\mathfrak{B}b-\mathfrak{A}a)(bb-aa)}{\mathfrak{A}b-\mathfrak{B}a}\sqrt{(A+Cxx)}$$

seu

$$\sqrt{(A+Cyy)}=-\frac{C(bb-aa)}{\mathfrak{A}b-\mathfrak{B}a}x+\frac{(\mathfrak{B}b-\mathfrak{A}a)}{\mathfrak{A}b-\mathfrak{B}a}\sqrt{(A+Cxx)},$$

ubi iterum supra et infra multiplicando per $\mathfrak{A}b+\mathfrak{B}a$ fit

$$\sqrt{(A+Cyy)}=-\frac{C(\mathfrak{A}b+\mathfrak{B}a)}{A}x+\frac{(Cab+\mathfrak{A}\mathfrak{B})}{A}\sqrt{(A+Cxx)}.$$

Necesse autem est valorem formulae $\sqrt{(A+Cyy)}$ hoc modo potius definiri quam extractione radicis, qua ambiguitas implicaretur.

Quocirca haec aequatio transcendens

$$\Pi{:}r+\Pi{:}s=\Pi{:}p+\Pi{:}q$$

praebet sequentem determinationem algebraicam, si quidem brevitatis gratia ponamus

$$\sqrt{(A+Cpp)}=P,\quad \sqrt{(A+Cqq)}=Q,\quad \sqrt{(A+Crr)}=R;$$

scilicet ut sit $\Pi{:}s=\Pi{:}p+\Pi{:}q-\Pi{:}r$, erit

$$s=\frac{-PQr-Cpqr+PRq+QRp}{A}$$

et

$$\sqrt{(A+Css)} = \frac{-CPqr-CQpr+CRpq+PQR}{A}$$

seu

$$\sqrt{(A+Css)} = \frac{PQR+C(Rpq-Pqr-Qpr)}{A}.$$

## COROLLARIUM 1

**591.** Quoniam est per hypothesin $\Pi:f=0$, si ponamus brevitatis gratia $\sqrt{(A+Cff)}=F$ et $r=f$, ut sit $R=F$, haec aequatio

$$\Pi:s=\Pi:p+\Pi:q$$

praebet

$$s=\frac{F(Pq+Qp)-PQf-Cfpq}{A}$$

et

$$\sqrt{(A+Css)} = \frac{FPQ+CFpq-Cf(Pq+Qp)}{A}).$$

## COROLLARIUM 2

**592.** Si ponamus $q=f$ et $Q=F$, ut sit $\Pi:q=0$, haec aequatio

$$\Pi:s=\Pi:p-\Pi:r.$$

praebet

$$s=\frac{F(Rp-Pr)+fPR-Cfpr}{A}$$

et

$$\sqrt{(A+Css)} = \frac{FPR-CFpr+Cf(Rp-Pr)}{A}.$$

## COROLLARIUM 3

**593.** Si sit $C=0$ et $A=1$, erit $\Pi:z=\int dz=z-f$, quia integrale ita capi debet, ut evanescat posito $z=f$. Tum ergo erit $P=1$, $Q=1$ et $\Pi=1$,unde, ut sit $\Pi:s=\Pi:p+I\Pi:q-\Pi:r$ seu $s=p+q-r$, oportet esse $s=-r+q+p$ et $\sqrt{(1+0ss)}=1$, uti per se constat.

## COROLLARIUM 4

**594.** Si sumatur $A=1$ et $C=-1$ fiatque $\Pi:z=Arc.\cos.z$, ut sit $f=1$, erit

$$Arc.\cos.s=Arc.\cos.p+Arc.\cos.q-Arc.\cos.r,$$

si fuerit

$$s=pqr-PQr+PRq+QRp$$

et

$$\sqrt{(1-ss)} = PQR + Pqr + Qpr - Rpq,$$

unde sumto $r=1$, ut sit $R=0$ et $Arc.cos.r=0$, erit

$$s = pq - PQ \quad \text{et} \quad \sqrt{(1-ss)} = Pq + Qp.$$

**SCHOLION**

**595.** Hinc notae regulae pro cosinibus deducuntur, quas fusius non prosequor.
Verum casus facillimus, quo $A=0$ et $C=1$ hincque fit

$$\Pi{:}z = \int \tfrac{dz}{z} = lz$$

existente $f=1$, insigni difficultate premi videtur ob expressiones pro $s$ et $\sqrt{(A+Css)} = s$ in infinitum abeuntes. Cui incommodo ut occurratur, primo quidem numerus $A$ ut infinite parvus spectetur eritque

$$P = \sqrt{(pp+A)} = p + \tfrac{A}{2p}, \quad Q = p + \tfrac{A}{2q}, \quad R = r + \tfrac{A}{2r}.$$

Quare ut fiat $ls = lp + lq - lr$, reperitur

$$As = -r\left(p+\tfrac{A}{2p}\right)\left(q+\tfrac{A}{2q}\right) - pqr + q\left(p+\tfrac{A}{2p}\right)\left(r+\tfrac{A}{2r}\right) + p\left(q+\tfrac{A}{2q}\right)\left(r+\tfrac{A}{2r}\right)$$

ac singulis membris evolutis

$$As = -\tfrac{Aqr}{2p} - \tfrac{Apr}{2q} + \tfrac{Aqr}{2p} + \tfrac{Apq}{2r} + \tfrac{Apr}{2q} + \tfrac{Apq}{2r}$$

seu $s = \frac{pq}{r}$, uti natura logarithmorum exigit.

Caeterum ex formulis inventis haud difficulter multiplicatio huiusmodi functionum transcendentium colligitur; veluti ut sit $\Pi{:}y = n\Pi{:}x$, relatio \inter $x$ et $y$ algebraice assignari poterit.

## PROBLEMA 76

**596.** *Si ponatur* $\Pi{:}z=\int\frac{dz(L+Mzz)}{\sqrt{(A+Czz)}}$ *sumto hoc integrali ita, ut evanescat posito* $z=0$, *comparationem inter huiusmodi functiones transcendentes investigare.*

## SOLUTIO

Statuatur inter binas variabiles $x$ et $y$ ista relatio

$$\alpha+\gamma(xx+yy)+2\delta xy=0\,,$$

unde fit

$$y=\frac{-\delta x+\sqrt{(-\alpha\gamma+(\delta\delta-\gamma\gamma)xx)}}{\gamma}\,.$$

Ponatur $-\alpha\gamma=Am$ et $\gamma\gamma-\delta\delta=Cm$, ut sit

$$\gamma y+\delta x=\sqrt{m(A+Cxx)} \text{ et } \gamma x+\delta y=\sqrt{m(A+Cyy)}\,.$$

At illam aequationem differentiando fit

$$dx(yx+\delta \mathrm{y})+dy(\gamma y+\delta x)=0$$

seu

$$\frac{dx}{\sqrt{(A+Cxx)}}+\frac{dy}{\sqrt{(A+Cyy)}}=0\,.$$

Iam statuatur

$$\frac{dx(L+Mxx)}{\sqrt{(A+Cxx)}}+\frac{dy(L+Myy)}{\sqrt{(A+Cyy)}}=dV\sqrt{m}\,,$$

ut sit integrando

$$\Pi{:}\,x+\Pi{:}y=\text{Const.}+V\sqrt{m}\,.$$

Cum igitur sit

$$\frac{dy}{\sqrt{(A+Cyy)}}=\frac{-dx}{\sqrt{(A+Cxx)}}\,,$$

erit

$$dV\sqrt{m}=\frac{Mdx(xx-yy)}{\sqrt{(A+Cxx)}}$$

hincque ob $y=\frac{\sqrt{m(A+Cxx)}-\delta x}{\gamma}$ erit

$$xx-yy=\frac{1}{\gamma\gamma}\left(\gamma\gamma xx-mA-mCxx-\delta\delta xx+2\delta x\sqrt{m(A+Cxx)}\right)$$

At $\gamma\gamma-yy=-mC$, ergo

$$dV\sqrt{m} = \frac{Mdx\left(2\delta x\sqrt{m(A+Cxx)} - mA - 2mCxx\right)}{\gamma\gamma\sqrt{(A+Cxx)}},$$

cuius integrale commode capi potest, dum fit

$$V\sqrt{m} = \frac{\delta Mxx\sqrt{m}}{\gamma\gamma} - \frac{Mmx}{\gamma\gamma}\sqrt{(A+Cxx)}$$

quae formula ob $\sqrt{m(A+Cxx)} = \gamma y + \delta x$ abit in

$$V\sqrt{m} = \frac{\delta Mxx - \gamma Mxy - \delta Mxx}{\gamma\gamma}\sqrt{m} = -\frac{Mxy}{\gamma}\sqrt{m}\ .$$

Quocirca habebimus

$$\Pi : x + \Pi : y = \text{Const.} - \frac{Mxy}{\gamma}\sqrt{m}$$

existente $\gamma y + \delta x = \sqrt{m(A+Cxx)}$ et $\gamma x + \delta y = \sqrt{m(A+Cyy)}$

ac praeterea

$$-\alpha\gamma = Am \quad \text{et} \quad \delta\delta - \gamma\gamma = Cm.$$

Ad constantem definiendam sumamus posito $x = 0$ fieri $y = b$, ut sit

$$\Pi : x + \Pi : y = \Pi : b - \frac{Mxy}{\gamma}\sqrt{m}.$$

Tum vero est $\gamma b = \sqrt{mA}$ et $\delta b = \sqrt{(mA + mCbb)}$, ergo

$$\gamma = \frac{\sqrt{mA}}{b} \quad \text{et} \quad \delta = \frac{\sqrt{(mA+mCbb)}}{b}.$$

Hinc ergo concludimus, si fuerit

$$y\sqrt{A} + x\sqrt{(A+Cbb)} = b\sqrt{(A+Cxx)}$$

et, quod eodem redit,

$$x\sqrt{A} + y\sqrt{(A+Cbb)} = b\sqrt{(A+Cyy)},$$

fore

$$\Pi : x + \Pi : y = \Pi : b - \frac{Mbxy}{\sqrt{A}}$$

denotante $\Pi$ eiusmodi functionem quantitatis suffixae, ut sit

$$\Pi{:}z = \int \frac{dz(L+Mzz)}{\sqrt{(A+Czz)}}$$

integrali hoc ita sumto, ut evanescat posito $z = 0$.

Natura harum functionum stabilita ac sublato discrimine inter quantitates constantes ac variabiles erit

$$\Pi{:}r = \Pi{:}p + \Pi{:}q + \frac{Mpqr}{\sqrt{A}},$$

si fuerit

$$q\sqrt{A} + p\sqrt{(A+Crr)} = r\sqrt{(A+Cpp)}$$

et

$$p\sqrt{A} + q\sqrt{(A+Crr)} = r\sqrt{(A+Cqq)}$$

unde fit

$$r = \frac{p\sqrt{(A+Cqq)}+q\sqrt{(A+Cpp)}}{\sqrt{A}}$$

et

$$\sqrt{(A+Crr)} = \frac{Cpq+\sqrt{(A+Cpp)(A+Cqq)}}{\sqrt{A}}$$

## COROLLARIUM 1

**597.** Sumto *z* negativo est $\Pi{:}-z = -\Pi{:}z$, unde capiendo quantitates *p* et *q* negative fiet

$$\Pi{:}p + \Pi{:}q + \Pi{:}r = \frac{Mpqr}{\sqrt{A}},$$

si fuerit

$$p\sqrt{A} + q\sqrt{(A+Crr)} + r\sqrt{(A+Cqq)} = 0$$

seu

$$q\sqrt{A} + p\sqrt{(A+Crr)} + r\sqrt{(A+Cpp)} = 0$$

seu

$$r\sqrt{A} + p\sqrt{(A+Cqq)} + q\sqrt{(A+Cpp)} = 0$$

vel

$$Cpq - \sqrt{A(A+Crr)} + \sqrt{(A+Cpp)(A+Cqq)} = 0,,$$

ex qua formatur haec relatio

$$Cpqr + p\sqrt{(A+Cqq)(A+Crr)} + q\sqrt{(A+Cpp)(A+Crr)}$$
$$+r\sqrt{(A+Cpp)(A+Cqq)} = 0.$$

## COROLLARIUM 2

**598.** Hac ergo methodo tres huiusmodi functiones $\Pi : z$ exhiberi possunt, quarum summam algebraice exprimere licet; quod autem de summa ostendimus, valet quoque de summa binarum demta tertia.

## COROLLARIUM 3

**599.** Si ponamus $L = A$ et $M = C$, functio proposita

$$\Pi : z = \int dz\sqrt{(A+Czz)}$$

exprimit arcum curvae, cuius abscissae *z* convenit applicata $\sqrt{(A+Czz)}$, et summa trium huiusmodi arcuum ita algebraice dabitur

$$\Pi : p + \Pi : q + \Pi : r = \frac{Cpqr}{\sqrt{A}},$$

si inter *p, q, r* superior relatio statuatur.

## SCHOLION

**600.** Haec proprietas inde est nata, quod differentiale *dV* integrationem admisit. Cum nempe esset

$$dV\sqrt{m} = \frac{Mdx(xx-yy)}{\sqrt{(A+Cxx)}},$$

ita ut sit

ob $\sqrt{m(A+Cxx)} = \gamma y + \delta x$ erit

$$dV = \frac{Mdx(xx-yy)}{\gamma y+\delta x},$$

cuius integrale commode ex aequatione assumta $\alpha + \gamma(xx+yy) + 2\delta xy = 0$ definiri potest. Ponatur enim

$$xx + yy = tt \quad \text{et} \quad xy = u\,;$$

erit

$$\alpha + \gamma tt + 2\delta u = 0$$

et differentialibus sumendis

$$xdx + ydy = tdt, \quad xdy + ydx = du \quad \text{et} \quad \gamma tdt + \delta du = 0\,;$$

ex binis prioribus colligitur

$$(xx-yy)dx = xtdt - ydu$$

et ob $tdt = -\frac{\delta du}{\gamma}$ erit

$$(xx-yy)dx = -\frac{du}{\gamma}(\delta x + \gamma y),$$

ita ut sit

$$\frac{dx(xx-yy)}{\gamma y + \delta x} = -\frac{du}{\gamma}$$

hincque $dV = -\frac{Mdu}{\gamma}$, unde manifiesto sequitur $V = -\frac{Mu}{\gamma} = -\frac{Mxy}{\gamma}$, uti in solutione operosius eruimus. Verum hac operatione commode uti licebit in sequente problemate, ubi formulas magis complexas sumus contemplaturi.

## PROBLEMA 77

**601.** *Si ponatur*

$$\Pi : z = \int \frac{dz(L + Mz^2 + Nz^4 + Oz^6)}{\sqrt{(A + Czz)}}$$

*integrali hoc ita sumto, ut evanescat posito* $z = 0$, *comparationem inter huiusmodi functiones transcendentes investigare.*

## SOLUTIO

Posita ut ante inter variabiles *x* et *y* hac relatione

$$a + \gamma(xx + yy) + 2\delta xy = 0$$

sit $-\alpha\gamma = Am$ et $\delta\delta - \gamma\gamma = Cm$ fietque

$$\gamma y + \delta x = \sqrt{m(A + Cxx)} \quad \text{et} \quad \gamma x + \delta y = \sqrt{m(A + Cyy)}$$

sumtisque differentialibus

$$\frac{dx}{\sqrt{(A + Cxx)}} + \frac{dy}{\sqrt{(A + Cyy)}} = 0.$$

Iam statuatur

$$\frac{dx(L + Mxx + Nx^4 + Ox^6)}{\sqrt{(A + Cxx)}} + \frac{dy(L + Myy + Ny^4 + Oy^6)}{\sqrt{(A + Cyy)}} = dV\sqrt{m},$$

ut sit

$$\Pi : x + \Pi : y = \text{Const.} + V\sqrt{m}.$$

At ob $\frac{dy}{(A + Cyy)} = -\frac{dx}{(A + Cxx)}$ ista aequatio abit in

$$\frac{dx\left(M\left(xx-yy\right)+N\left(x^4-y^4\right)+O\left(x^6-y^6\right)\right)}{\sqrt{\left(A+Cxx\right)}}=dV\sqrt{m}$$

et ob $\sqrt{m\left(A+Cxx\right)}=\gamma y+\delta x$ in hanc

$$\frac{dx\left(xx-yy\right)\left(M+N\left(xx+yy\right)+O\left(x^4+xxyy+y^4\right)\right)}{\gamma y+\delta x}=dV.$$

Sit nunc $xx+yy=tt$ et $xy=u$, ut habeatur

$$a+\gamma tt+2\delta u=0 \text{ et } \gamma tdt+\delta du=0 \text{ seu } tdt=-\tfrac{\delta du}{\gamma},$$

atque ob $xdx+ydy=tdt$ et $xdy+ydx=du$ hinc colligimus

$$\left(xx-yy\right)dx=xtdt-ydu=-\tfrac{du}{\gamma}\left(\gamma y+\delta x\right) \text{l'}$$

ideoque

$$\frac{dx\left(xx-yy\right)}{\gamma y+\delta x}=-\frac{du}{\gamma}$$

unde habebimus

$$dV=-\tfrac{du}{\gamma}\left(M+N\left(xx+yy\right)+O\left(x^4+xxyy+y^4\right)\right).$$

At est

$$xx+yy=tt=\tfrac{-\alpha-2\delta u}{\gamma} \text{ et } x^4+xxyy+y^4=t^4-uu.$$

Notetur autem esse $\frac{du}{\gamma}=-\frac{tdt}{\delta}$, unde concludimus

$$dV=-\tfrac{Mdu}{\gamma}+\tfrac{Nt^3dt}{\delta}+\tfrac{Ot^5}{\delta}dt+\tfrac{Ouudu}{\gamma},$$

sicque prodit integrando

$$V=-\tfrac{Mu}{\gamma}+\tfrac{Nt^4}{4\delta}+\tfrac{Ot^6}{6\delta}+\tfrac{Ou^3}{3\gamma}.$$

Quodsi iam ponamus fieri $y=b,$ si $x=0$, erit

$$\gamma=\tfrac{\sqrt{mA}}{b},\ \delta=\tfrac{\sqrt{m\left(A+Cbb\right)}}{b} \quad \text{et} \quad \alpha=b\sqrt{mA},$$

tum vero

$$y\sqrt{A}+x\sqrt{\left(A+Cbb\right)}=b\sqrt{\left(A+Cxx\right)},$$
$$x\sqrt{A}+y\sqrt{\left(A+Cbb\right)}=b\sqrt{\left(A+Cyy\right)}$$

et

$$b\sqrt{A}=x\sqrt{\left(A+Cyy\right)}+y\sqrt{\left(A+Cxx\right)}.$$

Hinc, cum sit

$$V=-\frac{Mbxy}{\sqrt{mA}}+\frac{Nb(xx+yy)^2}{4\sqrt{m(A+Cbb)}}+\frac{Ob(xx+yy)^3}{6\sqrt{m(A+Cbb)}}+\frac{Obx^3y^3}{3\sqrt{mA}}$$

nostra relatio, cui satisfaciunt praecedentes determinationes, inter functiones transcendentes erit

$$\Pi : x+\Pi : y = \Pi : bV-\frac{Mbxy}{\sqrt{A}}+\frac{Nb(xx+yy)^2}{4\sqrt{(A+Cbb)}}+\frac{Ob(xx+yy)^3}{6\sqrt{(A+Cbb)}}$$
$$+\frac{Obx^3y^3}{3\sqrt{A}}-\frac{Nb^5}{4\sqrt{(A+Cbb)}}-\frac{Ob^7}{6\sqrt{(A+Cbb)}},$$

ubi notandum est esse in rationalibus

$$-b\sqrt{A}+\frac{(xx+yy)\sqrt{A}}{b}+\frac{2xy\sqrt{(A+Cbb)}}{b}=0$$

seu

$$xx+yy=bb-\frac{2xy\sqrt{(A+Cbb)}}{\sqrt{A}}.$$

Hinc colligitur

$$(xx+yy)^2-b^4=-\frac{4bbxy\sqrt{(A+Cbb)}}{\sqrt{A}}+\frac{4xxyy(A+Cbb)}{A}$$

et

$$(xx+yy)^3-b^6=-\frac{6b^4xy\sqrt{(A+Cbb)}}{\sqrt{A}}+\frac{12bbxxyy(A+Cbb)}{A}-\frac{8x^3y^3(A+Cbb)^{\frac{3}{2}}}{A\sqrt{A}},$$

ita ut nostra aequatio sit

$$\Pi : x+\Pi : y=\Pi : b-\frac{Mbxy}{\sqrt{A}}-\frac{Nb^3xy}{\sqrt{A}}+\frac{Nbxx\,yy}{A}\sqrt{(A+Cbb)}-\frac{Ob^5xy}{\sqrt{A}}$$
$$+\ \frac{2Ob^3xxyy}{A}\sqrt{(A+Cbb)}-\frac{Obx^3y^3}{3A\sqrt{A}}(3A+4Cbb).$$

## COROLLARIUM 1

**602.** Si ponamus $b=r$, $x=-p$, $y=-q$, erit nostra aequatio

$$\Pi : p+\Pi : q+\Pi : r=\frac{pqr}{\sqrt{A}}\left(M+Nrr+Or^4\right)-\frac{ppqq\sqrt{(A+Crr)}}{A}\left(Nr+2Or^3\right)+\frac{Op^3q^3r}{3A\sqrt{A}}(3A+4Crr)$$

existence

$$pp+qq=rr-\frac{2pq\sqrt{(A+Crr)}}{\sqrt{A}},$$

unde fit

$$\frac{\sqrt{(A+Crr)}}{\sqrt{A}}=\frac{rr-pp-qq}{2pq}.$$

## COROLLARIUM 2

**603.** Substituto hoc valore pro $\frac{\sqrt{(A+Crr)}}{\sqrt{A}}$ sequens obtinebitur aequatio, in quam ternae quantitates *p, q, r* aequaliter ingrediuntur,

$$\Pi{:}p+\Pi{:}q+\Pi{:}r=\frac{Mpqr}{\sqrt{A}}+\frac{Npqr}{2\sqrt{A}}\left(pp+qq+rr\right)$$
$$+\frac{Opqr}{3\sqrt{A}}\left(p^4+q^4+r^4+ppqq+pprr+qqrr\right),$$

cui satisfaciunt formulae supra (§ 602) datae vel haec rationalis

$$\frac{4Cppqqrr}{A}=p^4+q^4+r^4-2ppqq-2pprr-2qqrr.$$

## COROLLARIUM 3

**604.** Si numeratori formulae integralis adhuc adiecissemus terminum $Pz^8$,
ut esset

$$\Pi{:}\,z=\int\frac{dz\left(L+Mz^2+NZ^4+Oz^6+Pz^8\right)}{\sqrt{(A+Czz)}}$$

ad aequationem modo inventam adhuc accessisset terminus

$$\frac{Ppqr}{4A\sqrt{A}}\left(p^6+q^6+r^6+ppq^4+p^4qq+p^4rr+q^4rr+\tfrac{4}{3}ppqqrr\right).$$

## SCHOLION

**605.** Istae relationes, quoque ex superioribus reductionibus derivari possunt;
cum enim inde sit $\Pi{:}\,z=E\int\frac{dz}{\sqrt{(A+Czz)}}+$ *quantitate algebraica*, si hic pro *z* successive quantitates *p, q, r* substituamus ita a se invicem pendentes, ut ante declaravimus, erit

$$\int\frac{dp}{\sqrt{(A+Cpp)}}+\int\frac{dq}{\sqrt{(A+Cqq)}}+\int\frac{dr}{\sqrt{(A+Crr)}}=0,$$

unde concludimus

$$\Pi{:}p+\Pi{:}q+\Pi{:}r=f{:}\,p+f{:}\,q+f{:}\,r$$

denotante *f* functionem quandam algebraicam quantitatis suffixae; atque summa harum trium functionum rediret ad expressionem ante inventam, si modo relationis inter *p, q, r* datae ratio habeatur, scilicet inde littera *C* eliminari deberet. Haec autem reductio ingentem laborem requireret. Hic vero imprimis methodum, qua hic sum usus, spectari convenit, quae cum sit prorsus singularis, ad magis arduam deducere videtur. Certe comparatio functionum transcendentium, quam in capite sequente sum traditurus, vix alia methodo investigari posse videtur, unde huius methodi utilitas in sequenti capite potissimum cernetur.