THE INTEGRAL CALCULUS

SECOND VOLUME

IN WHICH
THE METHOD OF FINDING FUNCTIONS
OF ONE VARIABLE FROM A GIVEN RELATION OF THE DIFFERENTIALS OF THE SECOND OR HIGHER ORDER ARE TREATED.

FIRST SECTION
ON
THE RESOLUTION OF DIFFERENTIAL EQUATIONS OF THE SECOND ORDER ONLY

CHAPTER I

CONCERNING THE INTEGRATION OF SIMPLE DIFFERENTIAL FORMULAS OF THE SECOND ORDER

DEFINITION

706. *With the two variables x and y put in place, if there is called* $dy = pdx$ *and* $dp = qdx$ *, some equation defining the relation between the quantities x, y, p and q is called a differential equation of the second order between the variables x and y.*

COROLLARY 1

707. Therefore just as the letter *p* implies a ratio of the differentials of the first order, while there is $p = \frac{dy}{dx}$, thus the letter $q = \frac{dp}{dx}$ implies a ratio of the differentials of the second order. For on taking , as generally it is accustomed to happen, with the element *dx* constant then there will be $dp = \frac{ddy}{dx}$ and thus $q = \frac{ddy}{dx^2}$.

COROLLARY 2

708. Therefore as far as the letter *q* is present in the proposed equation, to that extent that is a differential equation of the second degree. But if *q* should be absent, on account of *p* only that would be only a differential equation of the first degree ; and if neither *p* nor *q* should be present, the equation between *x* and *y* becomes nothing more than what is being sought.

COROLLARY 3

709. Therefore a method is desired, for some proposed equation not only between the two variables *x* and *y* but also involving the quantities $p = \frac{dy}{dx}$ and $q = \frac{dp}{dx}$, for finding the relation between *x* and *y*, from which it would appear, that *y* should be such a function of *x* or vice versa.

SCHOLIUM 1

710. In this way the letter q can introduce higher order differential equations freed from that condition, by which any differential of the first order is accustomed to be constant. For since pure finite quantities are recalled, which express a ratio of the differentials of the first order, the consideration of constant differentials cannot have a place. [To date Euler has considered dx to be the constant differential, i. e. $ddx = 0$; this restriction is to be removed, and any of the differential quantities can be considered as the one which is constant, and to which the other differential refer.] Therefore when higher order differential equations are thus accustomed to be shown, as some differential shall be taken as constant, by introducing the letters $p=\frac{dy}{dx}$ and $q=\frac{dp}{dx}$ the kind of differentials is completely removed, while the equation only embraces finite quantities. And also in turn a proposed equation between finite quantities x, y, p, q can be reduced to a general form in innumerable ways, as one or another differential is taken as constant, to which still all the different kinds are in perfect agreement between themselves with the form [chosen]; also since with no constant differential assumed, the working can be set out in the customary form.

SCHOLIUM 2

711. Therefore briefly in the first place it shall be convenient to set out, how an equation expressed by differentials of the second degree in the customary form is able to be reduced to our form, whatever constant differential should be assumed. Let ds be this differential taken as constant, hence the ratio of this to dx on account of $\frac{dy}{dx}=p$ is given through p and perhaps the variables x and y ; therefore there is put $ds=vdx$, so that v is made a finite quantity. Now since ddx and ddy occur in the equation or at least on or the other, in place of ddx there may be written $ds\cdot d.\frac{dx}{ds}$, since on account of constant ds certainly shall be $ds\cdot d.\frac{dx}{ds}=ddx$.

Hence there shall be

$$ddx=ds\cdot d.\frac{1}{v}=-\frac{dsdv}{vv}.$$

In a similar manner in place of ddy on writing $ds\cdot d.\frac{dy}{ds}=ds\cdot d.\frac{p}{v}$ there becomes

$$ddy=\frac{ds(vdp-pdv)}{vv}.$$

Therefore since v may be given in terms of p, x and y, there shall be

$$dv=Mdx+Ndy+Pdp=dx(M+Np+Pq)$$

on account of $dp=qdx$ and [$ds=vdx$] thus there becomes

$$ddx=-\frac{dx^2}{v}(M+Np+Pq) \text{ and } ddy=\frac{dx^2}{v}\left(qv-Mp-Np^2-Ppq\right);$$

and these values substituted in place of *ddx* et *ddy* in the equation leave an equation of the first degree only, in which with everything reduced to *dx* the equation by division in short will be freed from differentials. Then in turn a proposed equation of this kind between x, y, p and q will be changed into the usual form on taking some constant element *ds*, if in the first place certainly for p there may be written $\frac{dy}{dx}$, moreover in place of q $\frac{1}{dx}d.\frac{dy}{dx}=\frac{dxddy-dyddx}{dx^3}$, where indeed at this stage the ratio of no constant element has been taken. But on account of ob $ds=vdx$ the above constant shall become $vddx+dvdx=0$ or on account of $dv=Mdx+Ndy+Pd.\frac{dy}{dx}$

$$vddx+Mdx^2+Ndxdy+\frac{P(dxddy-dxdyddx)}{dx}=0,$$

from which as it pleases either *ddx* or *ddy* can be elicited ; but by neither extraction can equivalent infinite forms be shown.

SCHOLIUM 3

712. Hence therefore the excellence of finite forms, to which we relate here differential equations of higher order, is considered outstanding compared to these shown in the customary way, since the same equation can be represented in the customary way in an infinite number of ways, just as one or another of the elements is taken to be constant, while according to our approach the same equation is always reduced to a single form. But if therefore by our approach different equations are produced, it is certain also that different relations are expressed between the variables x and y, since in contrast to the customary manner the most diverse differential equations of higher order are able to indicate the same relation, from which generally that is elicited with difficulty, which shall be of greatest benefit to the resolution. Therefore since here a method of this kind is required, with the help of which for any equation proposed between the four quantities x, y, p et q a relation can be defined between the two variables x and y, and since this question is seen to surpass human powers, it will be required to start from the simplest cases. But the most simple cases are without doubt, when there shall only be two quantities present in the proposed equation, clearly either only x and q, y and q, or p and q; that is, if q may be equal to a function of x or of y or of p only ; which cases we have decided to set out in this chapter.

DEFINITION

713. *The formula for a higher order differential equation is simple, when on putting $dy = pdx$ and $dp = qdx$ the quantity q is equal to a function either of x, y, or of p only.*

[Note : Throughout this work and in other places, Euler uses the phrase *differentio–differentiales*, which might be rendered literally as *the difference of the differentials*, for which the phrase *second* [or higher] *order differential(s)* means more or less the same thing, and an appropriate form of this will be used in the translations that follow.]

COROLLARY 1

714. Therefore we have three formulas for simple higher order differential equations, the solution of which it is convenient to instruct in this chapter, as the quantity q is determined either by a function of p or of x or of y only.

COROLLARY 2

715. Therefore if X denotes a function of x, Y of y and P of p only, the three kinds of simple formulas of these are 1) $q = X$, 2) $q = Y$, 3) $q = P$, in which the most simple case q = Const. is contained.

COROLLARY 3

716. If we wish to express these formulas in the usual manner, on account of $q = \frac{dp}{dx} = \frac{1}{dx} d.\frac{dy}{dx}$, with the element dx assumed constant there will be $q = \frac{ddy}{dx^2}$, with the element dy taken constant there will be $q = -\frac{dyddx}{dx^3}$, moreover with nothing taken constant there will be $q = \frac{dxddy - dyddx}{dx^3}$, from which the simplicity of these formulas is not in the least obscured.

COROLLARY 4

717. If the element $\sqrt{\left(dx^2 + dy^2\right)}$ is taken constant, which often happens, then there will be $dxddx + dyddy = 0$; from which the latter value of q or on account of $ddy = -\frac{dxddx}{dy}$ will change into $q = -\frac{\left(dx^2+dy^2\right)ddx}{dx^3dy}$ or on account of $ddx = -\frac{dyddy}{dx}$ it becomes $q = \frac{\left(dx^2+dy^2\right)ddy}{dx^4}$.

SCHOLIUM

718. Therefore with the usual account of expressing higher order differential equations completely rejected, clearly we may use here formulas simple enough in themselves and able to avoid being exceedingly complicated, established for that reason, and from which we may provide instruction on the resolution of formulas of this kind.

PROBLEMA 92

719. *On putting $dy = pdx$ and $dp = qdx$ if q is equal to some function of p, to find the relation between the variables x and y.*

SOLUTION

Let $q = P$ with *P* denoting some function of *p;* because therefore there is $q = \frac{dp}{dq}$, then $dp = Pdx$ and hence $dx = \frac{dp}{P}$ and $dy = pdx = \frac{pdp}{P}$. From which we follow with the integration

$$x = a + \int \frac{dp}{P} \quad \text{and} \quad y = b + \int \frac{pdp}{P},$$

thus as both *x* and *y* can be determined by the same new variable *p*. And since two new constants *a* and *b* shall be introduced by the two–fold integration, this integral can be taken as complete.

COROLLARY 1

720. The equation *q* = *P*, the integration of which we have treated here, if it should be resolved in the usual form with *dx* assumed constant, on account of $q = \frac{ddy}{dx^2}$ is changed into $ddy = dx^2 \; f : \frac{dy}{dx}$, which is a second order differential equation, in which the variables *x* and *y* do not themselves occur [Recall that $f : \frac{dy}{dx} \equiv f\left(\frac{dy}{dx}\right)$].

COROLLARY 2

721. Also a form of such a kind is produced, if the element *dy* or another expression of the differentials, in which *x* and *y* are not themselves present, such as $\sqrt{\left(dx^2 + dy^2\right)}$, is taken for the constant [differential]. Hence in this way every second order differential equations, in which the variables *x* and *y* are not present themselves, can be integrated.

COROLLARY 3

722. But if an element of this kind $ydx - xdy$ is assumed constant, as $yddx - xddy = 0$, on account of $q = \frac{1}{dx} d.\ \frac{dy}{dx} = \frac{dxddy - dyddx}{dx^3}$ becomes

$$q = \frac{(ydx - xdy)ddx}{x\ dx^3} = \frac{(ydx - xdy)ddy}{ydx^3},$$

which expression, if it is equal to a function of $p = \frac{dy}{dx}$, can be integrated.

COROLLARY 4

723. If P should be a constant quantity, so that $q = f$, then

$$x = a + \frac{p}{f} \quad \text{and} \quad y = b + \frac{pp}{2f},$$

from which there becomes

$$y = b + \frac{f}{2}(x - a)^2 \text{ or } y = \frac{1}{2} fxx - afx + \frac{1}{2} aaf + b$$

or with the form of the constants changed : $y = \frac{1}{2} fxx + Cx + D$.

SCHOLIUM

724. Since clearly a second order differential equation needs a two–fold integration, if with each whole extension put in place, two new arbitrary constants are introduced ; in which the criterion consists of whether or not an integral of this kind shall be complete. For just as the integration of differential equations of the first order implicates a single arbitrary constant, thus, if the differential equation should be of the second order, two new constants enter into the complete integral, moreover three and more, if the differential equation should be of the third order or higher. Moreover problems, the solution of which lead to the deduction of differential equations of higher degree, are to be prepared thus according to their nature, so that the determination of the solution requires just as many constants. Thus in the equation $q = f$ or on taking dx constant $ddy = fdx^2$, the equation of the complete integral $y = \frac{1}{2} fxx + Cx + D$ involves two new constants C and D, which will also be apparent in the adjoining examples.

EXAMPLE 1

725. *To find the complete integral of the second order differential equation* $addy = dxdy$ *, in which the element dx is assumed to be constant.*

On putting $dy = pdx$ and $dp = qdx$ there shall be $ddy = qdx^2$ and hence $aq = p$ and $P = \frac{p}{a}$.

On account of which the integration gives

$$x=\int\frac{adp}{p}=C+\ alp \quad \text{and} \quad y=\int adp=D+ap\,.$$

Therefore since $p=\frac{y-D}{a}$, then

$$x=C+al\frac{y-D}{a},$$

which is the complete integral of the equation involving the two constants *C* and *D*.

EXAMPLE 2

726. *On putting dx constant to find the equation between x and y, so that there becomes*

$$\frac{\left(dx^2+dy^2\right)\sqrt{\left(dx^2+dy^2\right)}}{-dxddy}=a.$$

On putting $dy = pdx$ on account of constant dx there will be $ddy=dpdx$ and thus our equation is $\frac{(1+pp)\sqrt{(1+pp)}}{-dp}dx=a$, from which there is made

$$dx=\frac{-adp}{(1+pp)^{\frac{3}{2}}} \quad \text{and} \quad dy=\frac{-apdp}{(1+pp)^{\frac{3}{2}}}.$$

Hence by integration we come upon [on writing $\left(-a-ap^2+ap^2\right)dp$ in the numerator to integrate the first term]

$$x=A-\frac{ap}{\sqrt{(1+pp)}} \quad \text{and} \quad y=B+\frac{a}{\sqrt{(1+pp)}},$$

from which we may conclude

$$\left(x-A\right)^2+\left(y-B\right)^2=aa\,.$$

COROLLARY

727. If *x* and *y* denote the rectangular coordinates of the curved line, the formula $\frac{\left(dx^2+dy^2\right)\sqrt{\left(dx^2+dy^2\right)}}{-dxddy}$ expresses the radius of osculation of this ; which hence as it is constant shall be equal to *a* , the equation of the integral found indicates a circle described with radius *a*.

EXAMPLE 3

728. *On putting* $ds=\sqrt{\left(dx^2+dy^2\right)}$ *and with that assumed constant, to find the equation between x and y so that there becomes* $\frac{dsdy}{ddx}=\frac{adx}{dy}$.

There is put $dy=pdx$; then $ds=dx\sqrt{(1+pp)}$ and on account of constant ds

$$ddx\sqrt{(1+pp)}+\frac{pdxdp}{\sqrt{(1+pp)}}=0 \quad \text{or} \quad ddx=\frac{-pdxdp}{1+pp},$$

from which the proposed equation will change into $\frac{pdx^2\sqrt{(1+pp)}}{-pdxdp}(1+pp)=\frac{a}{p}$; or

$$dx=\frac{-adp}{p(1+pp)^{\frac{3}{2}}} \quad \text{and} \quad dy=\frac{-adp}{(1+pp)^{\frac{3}{2}}},$$

hence

$$y=D-\frac{ap}{\sqrt{(1+pp)}}.$$

But for that other formula there is put in place $p=\frac{1}{r}$ and there becomes

$$dx=\frac{arrdr}{(rr+1)^{\frac{3}{2}}}=\frac{adr}{\sqrt{(rr+1)}}-\frac{adr}{(rr+1)^{\frac{3}{2}}},$$

from which there becomes on integration

$$x=C-\frac{ar}{\sqrt{(1+rr)}}+al\left(r+\sqrt{(1+rr)}\right)$$

or

$$x=C-\frac{a}{\sqrt{(1+pp)}}+al\frac{1+\sqrt{(1+pp)}}{p}.$$

EXAMPLE 4

729. *On putting* $ds = \sqrt{(dx^2 + dy^2)}$ *and with this element assumed constant, there is required to become*

$$\frac{dsdy}{ddx} = a\text{Ang.tang.}\frac{dy}{dx}.$$

[Note that $\text{Ang.tang.}\frac{dy}{dx}$ is equivalent to $\arctan\frac{dy}{dx}$ in modern notation.]

If there should be put in place $dy = pdx$ as before, this equation arises to be integrated

$$\frac{-dx(1+pp)^{\frac{3}{2}}}{dp} = \text{Ang.tang.}p$$

or

$$dx = \frac{-adp}{(1+pp)^{\frac{3}{2}}}\text{Ang.tang.}p \quad \text{et} \quad dy = \frac{-apdp}{(1+pp)^{\frac{3}{2}}}\text{Ang.tang.}p$$

Now since there shall be $d.\text{Ang. tang.}p = \frac{dp}{1+pp}$, then there becomes

$$x = \frac{-ap}{\sqrt{(1+pp)}}\text{Ang.tang.}p + a\int\frac{pdp}{(1+pp)^{\frac{3}{2}}}$$

and

$$y = \frac{p}{\sqrt{(1+pp)}}\text{Ang.tang.}p - a\int\frac{dp}{(1+pp)^{\frac{3}{2}}}$$

On account of which we deduce

$$x = C - \frac{a}{\sqrt{(1+pp)}} - \frac{ap}{\sqrt{(1+pp)}}\text{Ang.tang.}p$$

and

$$y = D - \frac{ap}{\sqrt{(1+pp)}} + \frac{a}{\sqrt{(1+pp)}}\text{Ang.tang.}p.$$

COROLLARY 1

730. If x should be the abscissa and y the applied line of the curve [i. e. the x and y axis], then the radius of osculation must be proportional to the angle, which the tangent of the curve makes with the axis ; from which it is apparent that this curve is a certain spiral unfolding itself about the origin of the abscissas.

COROLLARY 2

731. If that angle, the tangent of which is equal to p, is put equal to φ, then $p = \text{tang}.\varphi$ and hence

$$x = C - a\cos.\varphi - a\varphi\sin\varphi \quad\text{and}\quad y = D - a\sin.\varphi + a\varphi\cos.\varphi,$$

from which it is deduced

$$x\cos.\varphi + y\sin.\varphi = C\cos.\varphi + D\sin.\varphi - a.$$

COROLLARY 3

732. As on assuming $\varphi = 0$ both x and y vanish, there must be assumed $C = a$ and $D = 0$ and there becomes

$$x = a - a\cos.\varphi - a\varphi\sin.\varphi \quad\text{and}\quad y = -a\sin.\varphi + a\varphi\cos.\varphi,$$

from which, as long as the angle φ is very small, there shall be

$$x = -\tfrac{1}{2}a\varphi\varphi + \tfrac{1}{8}a\varphi^4 \quad\text{and}\quad y = -\tfrac{1}{3}a\varphi^3 + \tfrac{1}{30}a\varphi^5;$$

and thus approximately,

$$\tfrac{x^3}{yy} = -\tfrac{9}{8}a \quad\text{or}\quad yy = -\tfrac{8x^3}{9a}.$$

PROBLEM 93

733. *On putting $dy = pdx$ and $dp = qdx$, if the quantity q is equal to a function of x, which shall be X, to define the relation between the two variables x and y.*

SOLUTION

Therefore since there shall be $q = X$, then there becomes $qdx = dp = Xdx$, from which on integrating we deduce

$p = \int Xdx + C$ and hence on account of $dy = pdx$ we will arrive at

$$y = \int dx\int Xdx + Cx + D.$$

But there the relation

$$\int dx\int Xdx = x\int Xdx - \int Xxdx,$$

as is apparent on taking the differentials at will. Whereby the complete equation of the integral containing the relation between the two variables *x* and *y* is

$$y = x\int Xdx - \int Xxdx + Cx + D$$

involving the two arbitrary constants *C* and *D*. Which therefore will be algebraic, if both the differential formulas *Xdx* and *Xxdx* admit to integration.

COROLLARY 1

734. But if therefore there shall be $q = 0$ or on assuming *dx* constant $ddy = 0$, as there shall be $X = 0$, and the equation of the complete integral shall be $y = Cx + D$.

COROLLARY 2

735. Therefore the second order differential equations, which it is permitted to integrate in this way, on assuming *dx* constant are contained in this form $ddy = Xdx^2$, from which the first integration gives

$$dy = dx\int Xdx + C \quad \text{and for other} \quad y = \int dx\int Xdx + Cx + D.$$

COROLLARIUM 3

736. But if the differential *dy* is taken constant, on account of $p = \frac{dy}{dx}$ there will be

$$dp = -\frac{dyddx}{dx^2} = qdx$$

and the form of the equations in this manner of integration will be $-dyddx = Xdx^3$.

COROLLARY 4

737. But if the element $ds = \sqrt{\left(dx^2 + dy^2\right)}$ shall be constant, on account of

$$dxddx + dyddy = 0$$

there will be

$$dp = \frac{dxddy - dyddx}{dx^2} = \frac{-ds^2ddx}{dx^2dy} = qdx.$$

Hence the form of the equations in this way of integrating is $-ds^2ddx = Xdx^3dy$.
Or since also there shall be $dp = qdx = +\frac{ds^2ddy}{dx^3}$, that will become $ds^2ddy = Xdx^4$.

SCHOLIUM

738. Here it is evident, however much the differential equations of higher orders may differ from the usual form, where some element is assumed constant, it is to be free from that condition, and to be reduced to the form established here. For if this equation is proposed $ds^2 ddy = Xdx^4$, in which the element $ds = \sqrt{\left(dx^2 + dy^2\right)}$ is assumed to be constant, it is not readily apparent, in what manner the integration of this should be undertaken. But by our method if we put $dy = pdx$, so that there becomes $ds = dx\sqrt{(1+pp)}$ and $ddy = pddx + dxdp$, this equation adopts this form :

$$dx^2(1+pp)(pddx+dxdp) = Xdx^4 \text{ or } (pddx+dxdp)(1+pp) = Xdx^2.$$

But because *ds* and hence also $ds^2 = dx^2(1+pp)$ is constant, there will be

$$ddx(1+pp) + pdxdp = 0, \text{ or } ddx = \tfrac{-pdxdp}{1+pp}$$

and thus

$$pddx + dxdp = \tfrac{dxdp}{1+pp},$$

thus so that there becomes $dp = Xdx$, which equation is treated most easily. Here clearly the aid must be called upon with that, which have been treated in the integration of differential formulas above [§ 733].

EXAMPLE 1

739. *On taking dx constant, if there should be* $ddy = \alpha x^n dx^2$*, to investigate the complete integral.*

Since there shall be $\frac{ddy}{dx} = \alpha x^n dx$, on account of *dx* constant there shall be on integration $\frac{dy}{dx} = \frac{\alpha}{n+1} x^{n+1} + C$, and hence on integrating again :

$$y = \tfrac{\alpha}{(n+1)(n+2)} x^{n+2} + Cx + D,$$

where the cases $n = -1$ and $n = -2$ are to be set out separately.

1. Hence if $n = -1$, there becomes $\frac{ddy}{dx} = \frac{\alpha dx}{x}$ and hence $\frac{dy}{dx} = \alpha lx + C$; from which since there shall be $dy = \alpha dxlx + Cdx$, there will be on integrating anew, $y = \alpha xlx - \alpha x + Cx + D$ or on writing *C* in place of $C - \alpha$ there will be had :

$$y = \alpha xlx + Cx + D.$$

II. If $n = -2$ and $\frac{ddy}{dx} = \frac{\alpha dx}{x^2}$, then $\frac{dy}{dx} = \frac{-\alpha}{x} + C$ and hence
$$y = -\alpha lx + Cx + D.$$

EXAMPLE 2

740. *On putting* $ds = \sqrt{\left(dx^2 + dy^2\right)}$ *constant if there should be* $\frac{ds^2 ddy}{dx^4} = \frac{1}{\alpha}\cos.\frac{x}{c}$, *to find the complete integral.*

From the above [§ 717] there is agreed to be $\frac{ds^2 ddy}{dx^4} = q$, thus to that the above shall become this equation $q = \frac{1}{\alpha}\cos.\frac{x}{c}$, from which there becomes $qdx = dp = dx\cos.\frac{x}{c}$, and on integrating $p = \frac{c}{\alpha}\sin.\frac{x}{c} + C = \frac{dy}{dx}$.
From which there is found
$$y = -\frac{cc}{\alpha}\cos.\frac{x}{c} + Cx + D,$$
which is the equation of the complete integral.

PROBLEM 94

741. *On putting* $dy = pdx$ *and* $dp = qdx$, *if the quantity q is equal to some function of y only, which shall be Y, to find the complete equation of the integral between x and y.*

SOLUTION

Since there shall be $q = Y = \frac{dp}{dx}$, then $dx = \frac{dp}{Y}$ and hence $pdx = dy = \frac{pdp}{Y}$, from which this separated equation can be constructed between p and y : $pdp = Ydy$, which on integration gives
$$\tfrac{1}{2}pp = \int Ydy + \tfrac{1}{2}C \quad \text{and} \quad p = \sqrt{\left(C + 2\int Ydy\right)} = \frac{dy}{dx}.$$
Hence again it may therefore be concluded :
$$x = \int \frac{dy}{\sqrt{\left(C + 2\int Ydy\right)}},$$
which integration again adopts an arbitrary constant, thus so that in this way the equation of the complete integral is obtained between x and y.

COROLLARY 1

742. Since there shall be $q=\frac{dp}{dx}$ and $dx=\frac{dy}{p}$, there shall be $q=\frac{pdp}{dy}$. Whereby since the proposed equation shall be $q=Y$, then $\frac{pdp}{dy}=Y$ and hence $pdp=Ydy$, from which the preceding integration is deduced at once.

COROLLARY 2

743. With the element *dx* assumed constant since there shall be $q=\frac{ddy}{dx^2}$, these equations will have this form $ddy=Ydx^2$, the integral of which is evident, if is it multiplied by dy; for it becomes $\frac{1}{2}dy^2=dx^2\int Ydy+\frac{1}{2}Cdx^2$ on account of constant *dx*, and hence $dx=\frac{dy}{\sqrt{(C+2\int Ydy)}}$ as before.

SCHOLIUM

744. Hence behold a specimen of differential equations, which are rendered integrable by suitable multiplication, from which it is understood that this method can also have a use in those equations ; but then the position will be that the method be further improved, obviously the use of this particularly in differential equations of higher order is indicated, where the separation of the variables brings no help. And now for this reason above [§ 447] we have recommended this method by multipliers of the integrand and we have preferred the other method by separation for a long time before.

EXAMPLE 1

745. *On putting dx constant if there should be* $aaddy=ydx^2$*, to find the complete integral.*

The proposed equation is multiplied by *2dy*, so that there is produced

$$2aadyddy=2ydydx^2,$$

which on account of constant *dx* the integral is obtained $aady^2=yydx^2+Cdx^2$, from which there is deduced

$$dx=\frac{ady}{\sqrt{(yy+C)}},$$

which gives on integrating again

$$x=al\left(y+\sqrt{(yy+C)}\right)-alb,$$

[In this case note that

$$\frac{ady\left(y+\sqrt{(yy+C)}\right)}{\sqrt{(yy+C)}\left(y+\sqrt{(yy+C)}\right)}=\frac{ady\left(y+\sqrt{(yy+C)}\right)}{yy+C+y\sqrt{(yy+C)}}=\frac{a}{y+\sqrt{(yy+C)}}\times dy\left(\frac{y}{\sqrt{(yy+C)}}+1\right)$$
$$=\frac{a}{y+\sqrt{(yy+C)}}d\left(y+\sqrt{(yy+C)}\right).\ ]$$

from which conclude on taking e for the number, the logarithm of which is equal to 1, to be
$be^{\frac{x}{a}}=y+\sqrt{(yy+C)}$ and with the irrationality removed $bbe^{\frac{2x}{a}}-2bye^{\frac{x}{a}}=C$, thus so that there shall be

$$y=\tfrac{1}{2}be^{\frac{x}{a}}-\tfrac{C}{2b}e^{-\frac{x}{a}};$$

but with the form of the constants C and b changed there is obtained

$$y=Ce^{\frac{x}{a}}+De^{-\frac{x}{a}},$$

which is the equation of the complete integral.

EXAMPLE 2

746. *On putting dx constant if there should be* $aaddy+ydx^2=0$, *to find the complete integral.*

With the multiplication of the equation by *2dy* made $2aadyddy+2ydydx^2=0$ the integral is

$$aady^2+yydx^2=ccdx^2,$$

from which we deduce

$$dx=\frac{ady}{\sqrt{(cc-yy)}},$$

with which integrated again gives

$$x=a\text{Ang.sin.}\tfrac{y}{c}+b.$$

Hence there becomes

$$\tfrac{y}{c}=\sin.\tfrac{x-b}{a}=\cos.\tfrac{b}{a}\sin.\tfrac{x}{a}-\sin.\tfrac{b}{a}\cos.\tfrac{x}{a}$$

or with the constants b and c changed, thus so that there becomes
$c\cos.\frac{b}{a}=C$ and $-c\sin.\frac{b}{a}=D$, there shall be

$$y=C\sin.\tfrac{x}{a}+D\cos.\tfrac{x}{a}$$

Or on retaining the first form, we have $y=c\sin.\left(\frac{x}{a}+\alpha\right)$.

COROLLARY

747. This example can be resolved from the preceding, since there shall be

$$e^{u\sqrt{-1}} = \cos.u + \sqrt{-1}\cdot\sin.u \quad \text{and} \quad e^{-u\sqrt{-1}} = \cos.u - \sqrt{-1}\cdot\sin.u$$

and in turn

$$\cos.u\sqrt{-1} = \tfrac{1}{2}e^{u} + \tfrac{1}{2}e^{-u} \quad \text{and} \quad \sin.u\sqrt{-1} = \tfrac{1}{2\sqrt{-1}}e^{u} - \tfrac{1}{2\sqrt{-1}}e^{-u}.$$

EXAMPLE 3

748. *On putting dx constant if there should be* $ddy\sqrt{ay} = dx^2$, *to find the complete integral.*

Therefore since there shall be $2dyddy = \frac{2dy}{\sqrt{ay}}\cdot dx^2$, there becomes on integrating

$$dy^2 = 4\frac{dx^2\sqrt{y}}{\sqrt{a}} + 4ndx^2 = \frac{4dx^2\left(\sqrt{y}+n\sqrt{a}\right)}{\sqrt{a}},$$

from which we deduce

$$2dx = \frac{dy\sqrt[4]{a}}{\sqrt{\left(\sqrt{y}+n\sqrt{a}\right)}}.$$

For convenience put $n\sqrt{a} = b$ and $\sqrt{y} = z$, so that there becomes *dy = 2zdz* and

$$\frac{dx\sqrt{n}}{\sqrt{b}} = \frac{zdz}{\sqrt{(b+z)}},$$

the integral of which is

$$\frac{x\sqrt{n}}{\sqrt{b}} = \tfrac{2}{3}(z-2b)\sqrt{(b+z)} + C$$

or on restoring

$$\frac{x}{\sqrt[4]{a}} = \tfrac{2}{3}\left(\sqrt{y}-2\sqrt{c}\right)\sqrt{\left(\sqrt{y}+\sqrt{c}\right)} + C,$$

where *c* and *C* are two arbitrary constants. Hence there becomes

$$\frac{3(x+f)}{2\sqrt[4]{a}} = \left(\sqrt{y}-2\sqrt{c}\right)\sqrt{\left(\sqrt{y}+\sqrt{c}\right)}$$

on putting $C = \frac{-f}{\sqrt[4]{a}}$ and on taking the squares,

$$\frac{9(x+f)^2}{4\sqrt{a}} = y\sqrt{y} - 3y\sqrt{c} + 4c\sqrt{c}$$

SCHOLIUM

749. Therefore the usual form of the equations in this method of integration with the element *dx* assumed constant is $ddy = Ydx^2$, which multiplied by *dy* evidently shall become integrable. But if the element *dy* is taken constant, on account of $q = \frac{dp}{dx}$ and $p = \frac{dy}{dx}$ there becomes $q = -\frac{dyddx}{dx^3}$ and hence the usual form $dyddx = -Ydx^3$. Again on taking the element $ds = \sqrt{(dx^2 + dy^2)}$ constant, so that $dxddx + dyddy = 0$, on account of $dp = \frac{dxddy - dyddx}{dx^2}$ there will be either $q = -\frac{ds^2 ddx}{dx^3 dy}$ or $q = \frac{ds^2 ddy}{dx^4}$, from which there arises either this form $-\frac{ds^2 ddx}{dx^3 dy} = Y$ or $\frac{ds^2 ddy}{dx^4} = Y$, which also becomes integrable on multiplying by *y*, even if this now is less apparent. In a like manner if the element *ydx* it taken constant, so that there becomes $yddx + dxdy = 0$ and $ddx = -\frac{dxdy}{y}$, on account of $dp = \frac{ddy}{dx} + \frac{dy^2}{ydx}$ this form arises : $yddy + dy^2 = Yydx^2$ the first part of which is rendered integrable if it is multiplied by some function of *ydy* and *ydx*, and therefore also by $\frac{ydy}{yydx^2}$, by which multiplier likewise the other part $Yydx^2$ is rendered integrable.

Therefore from these simplest cases of second order differential equations set out, which lest they labour under a certain difficulty, we progress to more difficult and indeed at first to these equations, in which either of the two variables *x* and *y* is not present itself, thus so that the equation only contains the three letters *x, p* and *q,* or *y, p* and *q;* and indeed the account of each has been prepared almost in the same manner.

CALCULI INTEGRALIS

VOLUME SECUNDUM

IN QUO METHODUS INVENIENDI FUNCTIONES
UNIUS VARIABILIS EX DATA RELATIONE DIFFERENTIALIUM SECUNDI ALTIORISVE GRADUS.

SECTIO PRIOR
DE
RESOLUTIONE AEQUATIONUM DIFFERENTIALIUM SECUNDUM GRADUS DUAS TANTUM

CAPUT I

DE INTEGRATIONE FORMULARUM DIFFERENTIALIUM SECUNDI GRADUS SIMPLICIUM

DEFINITIO

706. *Positis binis variabilibus x et y si vocetur* $dy = pdx$ *et* $dp = qdx$ *, aequatio quaecunque relationem inter quantitates x, y, p et q definiens vocatur aequatio differentialis secundi gradus inter binas variabiles x et y.*

COROLLARIUM 1

707. Quemadmodum ergo littera p implicat rationem differentialium primi gradus, dum est $p = \frac{dy}{dx}$, ita littera $q = \frac{dp}{dx}$ implicat rationem differentialium secundi gradus. Sumto enim, ut vulgo fieri solet, elemento dx constante erit $dp = \frac{ddy}{dx}$ ideoque $q = \frac{ddy}{dx^2}$.

COROLLARIUM 2

708. Quatenus ergo in aequatione proposita littera q inest, eatenus ea est differentialis secundi gradus. Si enim q abesset, ob solam p esset tantum differentialis primi gradus; ac si neque p neque q inesset, aequatio foret inter x et y neque quicquam praeterea quaereretur.

COROLLARIUM 3

709. Methodus ergo desideratur, proposita aequatione quacunque praeter binas variabiles x et y etiam quantitates $p = \frac{dy}{dx}$ et $q = \frac{dp}{dx}$ involvente, inveniendi relationem inter ipsas x et y, unde pateat, qualis y sit functio ipsius x seu vicissim.

SCHOLION 1

710. Hoc modo litteram q introducendo aequationes differentio–differentiales a conditione illa, qua quodpiam differentiale primi gradus pro constante assumi solet, liberantur. Cum enim ad meras quantitates finitas revocentur, quae rationem differentialium primi gradus exprimunt, consideratio differentialis constantis ne locum quidem habere potest. Quando ergo aequationes differentio–differentiales more solito ita exhibentur, ut quodpiam differentiale constans sit assumtum, introducendo litteras $p = \frac{dy}{dx}$ et $q = \frac{dp}{dx}$ species differentialium penitus tollitur, dum aequatio tantum quantitates finitas complectitur. Atque etiam vicissim proposita aequatione inter quantitates finitas x, y, p, q ea ad formam vulgarem infinitis modis reduci potest, prout aliud atque aliud differentiale pro constante assumitur, quae tamen omnes formae specie diversae inter se perfecte conveniunt; quin etiam nullo differentiali constante assumto evolutio in formam solitam fieri potest.

SCHOLION 2

711. Primum igitur breviter exponi conveniet, quomodo aequatio more solito per differentialia secundi gradus expressa ad formam nostram reduci queat, quodcunque differentiale constans fuerit assumtum. Sit *ds* hoc differentiale pro constante sumtum, cuius ergo ratio ad *dx* ob $\frac{dy}{dx}=p$ per *p* et forte ipsas variabiles *x* et *y* datur; ponatur ergo $ds=vdx$, ut *v* fiat quantitas finita. Iam cum in aequatione occurrant *ddx* et *ddy* vel alterutrum saltem, loco *ddx* scribatur $ds\cdot d.\frac{dx}{ds}$, quia ob *ds* constans fit utique

$ds\cdot d.\frac{dx}{ds}=ddx$.

Erit ergo

$$ddx=ds\cdot d.\frac{1}{v}=-\frac{dsdv}{vv}.$$

Simili modo loco *ddy* scribendo $ds\cdot d.\frac{dy}{ds}=ds\cdot d.\frac{p}{v}$ fiet

$$ddy=\frac{ds(vdp-pdv)}{vv}.$$

Cum igitur *v* per *p, x* et *y* detur, erit

$$dv=Mdx+Ndy+Pdp=dx(M+Np+Pq)$$

ob $dp=qdx$ sicque fiet

$$ddx=-\frac{dx^2}{v}(M+Np+Pq) \text{ et } ddy=\frac{dx^2}{v}\left(qv-Mp-Np^2-Ppq\right);$$

hique valores loco *ddx* et *ddy* substituti in aequatione tantum differentialia primi gradus relinquent, quibus omnibus ad *dx* reductis aequatio per divisionem prorsus a differentialibus liberabitur. Deinde vicissim huiusmodi aequatio inter *x, y, p* et *q* proposita in formam solitam sumto quopiam elemento *ds* constante evolvetur, si primo pro *p* ubique scribatur $\frac{dy}{dx}$, loco *q* autem $\frac{1}{dx}d.\frac{dy}{dx}=\frac{dxddy-dyddx}{dx^3}$, ubi quidem nullius adhuc elementi constantis ratio est habita. At ob $ds=vdx$ constans insuper erit $vddx+dvdx=0$ seu ob $dv=Mdx+Ndy+Pd.\frac{dy}{dx}$

$$vddx+Mdx^2+Ndxdy+\frac{P(dxddy-dxdyddx)}{dx}=0,$$

unde pro lubitu vel *ddx* vel *ddy* elidi potest; neutro autem eliso infinitae formae aequivalentes exhiberi possunt.

SCHOLION 3

712. Hinc ergo praestantia formae finitae, ad quam hic aequationes differentio–differentiales revocamus, prae more solito eas exhibendi luculenter perspicitur, cum eadem aequatio more solito infinitis modis, prout aliud atque aliud elementum constans assumitur, repraesentari possit, dum nostro more eadem aequatio semper ad unicam formam reducitur. Quodsi ergo nostro more aequationes prodeant diversae, certum est iis quoque diversas relationes inter variabiles x et y exprimi, cum contra solito more diversissimae aequationes differentio–differentiales eandem relationem indicare queant, ex quibus plerumque difficile est eam eligere, quae ad resolutionem maxime sit accommodata. Cum igitur hic eiusmodi methodus requiratur, cuius ope proposita quacunque aequatione inter quaternas quantitates x, y, p et q relatio inter binas variabiles x et y definiri queat, quoniam haec quaestio vires humanas superare videtur, a casibus simplicissimis erit exordiendum. Casus autem simplicissimi sine dubio sunt, quando in aequatione proposita duae tantum insunt quantitates, scilicet vel x et q tantum, vel y et q, vel p et q; hoc est, si q aequetur functioni vel ipsius x vel ipsius y vel ipsius p tantum; quos casus in hoc capite evolvere constituimus.

DEFINITIO

713. *Formula differentio–differentialis simplex est, quando posito $dy = pdx$ et $dp = qdx$ quantitas q aequatur functioni vel ipsius x vel ipsius y vel ipsius p tantum.*

COROLLARIUM 1

714. Triplices ergo habemus formulas differentio–differentiales simplices, quarum resolutionem in hoc capite doceri convenit, prout quantitas q vel per functionem ipsius p vel ipsius x vel ipsius y tantum determinatur.

COROLLARIUM 2

715. Si ergo X denotet functionem ipsius x, Y ipsius y et P ipsius p tantum, terna genera harum formularum simplicium sunt 1) $q = X$, 2) $q = Y$, 3) $q = P$, in quibus continetur casus simplicissimus q = Const.

COROLLARIUM 3

716. Si has formulas more solito exprimere velimus, ob $q = \frac{dp}{dx} = \frac{1}{dx} d.\frac{dy}{dx}$ sumto elemento dx constante erit $q = \frac{ddy}{dx^2}$, sumto elemento dy constante erit $q = -\frac{dyddx}{dx^3}$, nullo autem sumto constante erit $q = \frac{dxddy - dyddx}{dx^3}$, quibus simplicitas earum formularum haud mediocriter offuscatur.

COROLLARIUM 4

717. Si elementum $\sqrt{\left(dx^2+dy^2\right)}$, quod saepe fit, constans accipiatur, erit $dxddx+dyddy=0$; unde postremus valor ipsius q vel ob $ddy=-\frac{dxddx}{dy}$ abit in $q=-\frac{\left(dx^2+dy^2\right)ddx}{dx^3dy}$ vel ob $ddx=-\frac{dyddy}{dx}$ abit in $q=\frac{\left(dx^2+dy^2\right)ddy}{dx^4}$.

SCHOLION

718. Repudiata ergo penitus vulgari ratione aequationes differentio–differentiales exprimendi, quippe qua formulae in se satis simplices vehementer complicatae evadere possent, ratione hic stabilita utamur indeque resolutionem huiusmodi formularum simplicium doceamus.

PROBLEMA 92

719. *Posito* $dy=pdx$ et $dp=qdx$ *si q aequetur functioni cuicunque ipsius p, invenire relationem inter ipsas variabiles x et y.*

SOLUTIO

Sit ergo $q=P$ denotante *P* functionem quamcunque ipsius *p;* quoniam igitur est $q=\frac{dp}{dq}$, erit $dp=Pdx$ hincque $dx=\frac{dp}{P}$ et $dy=pdx=\frac{pdp}{P}$. Ex quo consequimur integrando

$$x=a+\int\frac{dp}{P}\quad \text{et}\quad y=b+\int\frac{pdp}{P},$$

ita ut tam *x* quam *y* per eandem novam variabilem *p* determinetur. Atque cum duae novae constantes *a* et *b* per duplicem integrationem sint introductae, hoc integrale pro completo erit habendum.

COROLLARIUM 1

720. Aequatio $q = P$, cuius integrationem hic tradidimus, si in formam consuetam sumto *dx* constante resolvatur, ob $q=\frac{ddy}{dx^2}$; transmutabitur in $ddy=dx^2\ f:\frac{dy}{dx}$, quae est aequatio differentio–differentialis, in qua ipsae variabiles *x* et *y* non occurrunt.

COROLLARlUM 2

721. Talis quoque forma prodit, si elementum *dy* vel alia expressio differentialis, in quam ipsae *x* et *y* non ingrediuntur, veluti $\sqrt{\left(dx^2+dy^2\right)}$, pro constante sumatur. Hoc ergo modo omnis aequatio differentio–differentialis, in quam ipsae variabiles *x* et *y* non ingrediuntur, integrari poterit.

COROLLARIUM 3

722. Sin autem huiusmodi elementum $ydx - xdy$ constans assumatur, ut $yddx - xddy = 0$, ob $q = \frac{1}{dx} d. \frac{dy}{dx} = \frac{dxddy - dyddx}{dx^3}$ fiet

$$q = \frac{(ydx - xdy)ddx}{x\,dx^3} = \frac{(ydx - xdy)ddy}{ydx^3},$$

quae expressio, si aequetur functioni ipsius $p = \frac{dy}{dx}$, integrari poterit.

COROLLARIUM 4

723. Si fuerit *P* quantitas constans, ut sit $q = f$, erit

$$x = a + \frac{p}{f} \quad \text{et} \quad y = b + \frac{pp}{2f},$$

unde fit

$$y = b + \frac{f}{2}(x - a)^2 \text{ seu } y = \frac{1}{2} fxx - afx + \frac{1}{2} aaf + b$$

seu mutatata forma constantium $y = \frac{1}{2} fxx + Cx + D$.

SCHOLION

724. Cum scilicet aequatio differentio–differentialis duplici integratione indigeat, si utraque omni extensione instituatur, duae novae constantes arbitrariae introducuntur; in quo criterium, num huiusmodi integrale sit completum, consistit. Quemadmodum enim aequationum differentialium primi gradus integratio completa unam constantem arbitrariam implicat, ita, si aequatio differentialis fuerit secundi gradus, binae constantes novae in integrale completum ingredientur, ternae autem ac plures, si aequatio differentialis fuerit tertii altiorisve gradus. Problemata autem, quorum resolutio ad huiusmodi aequationes differentiales altiorum graduum deducunt, natura sua ita sunt comparata, ut solutionis determinatio totidem constantes requirat. Ita in aequatione $q = f$ seu sumto dx constante $ddy = fdx^2$ aequatio integralis completa $y = \frac{1}{2} fxx + Cx + D$ duas constantes novas *C* et *D* involvit, quod etiam in subiunctis exemplis patebit.

EXEMPLUM 1

725. *Aequationis differentio–differentialis* $addy = dxdy$ *, in qua elementum dx constans est sumtum, integrale completum invenire.*

Posito $dy = pdx$ et $dp = qdx$ erit $ddy = qdx^2$ hincque $aq = p$ et $P = \frac{p}{a}$. Quocirca integratio praebet

$$x = \int \frac{adp}{p} = C + alp \quad \text{et} \quad y = \int adp = D + ap.$$

Cum igitur sit $p = \frac{y-D}{a}$, erit

$$x = C + al\frac{y-D}{a},$$

quae est aequatio integralis completa binas constantes C et D involvens.

EXEMPLUM 2

726. *Posito dx constante invenire aequationem inter x et y, ut fiat*

$$\frac{(dx^2+dy^2)\sqrt{(dx^2+dy^2)}}{-dxddy} = a.$$

Posito $dy = pdx$ ob dx constans erit $ddy = dpdx$ sicque nostra aequatio est $\frac{(1+pp)\sqrt{(1+pp)}}{-dp}dx = a$, unde fit

$$dx = \frac{-adp}{(1+pp)^{\frac{3}{2}}} \quad \text{et} \quad dy = \frac{-apdp}{(1+pp)^{\frac{3}{2}}}.$$

Per integrationem ergo nanciscimur

$$x = A - \frac{ap}{\sqrt{(1+pp)}} \quad \text{et} \quad y = B + \frac{a}{\sqrt{(1+pp)}},$$

unde concludimus

$$(x-A)^2 + (y-B)^2 = aa.$$

COROLLARIUM

727. Si x et y denotent coordinatas rectangulas lineae curvae, formula $\frac{(dx^2+dy^2)\sqrt{(dx^2+dy^2)}}{-dxddy}$ exprimit eius radium osculi; qui ergo ut sit constans $= a$, aequatio integralis inventa cireulum radio a describendum indicat.

EXEMPLUM 3

728. *Posito* $ds = \sqrt{(dx^2 + dy^2)}$ *eoque sumto constante invenire aequationem inter x et y ut fiat* $\frac{dsdy}{ddx} = \frac{adx}{dy}$.

Ponatur $dy = pdx;$ erit $ds = dx\sqrt{(1+pp)}$ et ob ds constans

$$ddx\sqrt{(1+pp)} + \frac{pdxdp}{\sqrt{(1+pp)}} = 0 \quad \text{seu} \quad ddx = \frac{-pdxdp}{1+pp},$$

unde aequatio proposita abit in $\frac{pdx^2\sqrt{(1+pp)}}{-pdxdp}(1+pp) = \frac{a}{p}$; seu

$$dx = \frac{-adp}{p(1+pp)^{\frac{3}{2}}} \quad \text{et} \quad dy = \frac{-adp}{(1+pp)^{\frac{3}{2}}},$$

ergo

$$y = D - \frac{ap}{\sqrt{(1+pp)}}.$$

At pro illa formula statuatur $p = \frac{1}{r}$ eritque

$$dx = \frac{arrdr}{(rr+1)^{\frac{3}{2}}} = \frac{adr}{\sqrt{(rr+1)}} - \frac{adr}{(rr+1)^{\frac{3}{2}}},$$

unde fit integrando

$$x = C - \frac{ar}{\sqrt{(1+rr)}} + al\left(r + \sqrt{(1+rr)}\right)$$

seu

$$x = C - \frac{a}{\sqrt{(1+pp)}} + al\frac{1+\sqrt{(1+pp)}}{p}.$$

EXEMPLUM 4

729. *Posito* $ds = \sqrt{(dx^2 + dy^2)}$ *hocque elemento sumto constante fieri oporteat* $\frac{dsdy}{ddx} = a\text{Ang.tang.}\frac{dy}{dx}$.

Si fiat ut ante $dy = pdx$, orietur haec aequatio integranda

$$\frac{-dx(1+pp)^{\frac{3}{2}}}{dp} = \text{Ang.tang.}p$$

seu

$$dx = \frac{-adp}{(1+pp)^{\frac{3}{2}}}\text{Ang.tang.}p \quad \text{et} \quad dy = \frac{-apdp}{(1+pp)^{\frac{3}{2}}}\text{Ang.tang.}p$$

Cum nunc sit $d.\text{Ang. tang.}p = \frac{dp}{1+pp}$, erit

$$x = \frac{-ap}{\sqrt{(1+pp)}}\text{Ang.tang.}p + a\int\frac{pdp}{(1+pp)^{\frac{3}{2}}}$$

et

$$y = \frac{p}{\sqrt{(1+pp)}}\text{Ang.tang.}p - a\int\frac{dp}{(1+pp)^{\frac{3}{2}}}$$

Quamobrem colligimus

$$x = C - \frac{a}{\sqrt{(1+pp)}} - \frac{ap}{\sqrt{(1+pp)}}\text{Ang.tang.}p$$

et

$$y = D - \frac{ap}{\sqrt{(1+pp)}} + \frac{a}{\sqrt{(1+pp)}}\text{Ang.tang.}p.$$

COROLLARIUM 1

730. Si x sit abscissa et y applicata curvae, radius osculi proportionalis esse debet angulo, quem curvae tangens cum axe constituit; unde patet hanc curvam fore quandam spiralem circa originem abscissarum se evolventem.

COROLLARIUM 2

731. Si angulus ille, cuius tangens = p, ponatur $=\varphi$, erit $p = \text{tang.}\varphi$ hincque

$$x = C - a\cos.\varphi - a\varphi\sin\varphi \text{ et } y = D - a\sin.\varphi + a\varphi\cos.\varphi,$$

unde colligitur

$$x\cos.\varphi + y\sin.\varphi = C\cos.\varphi + D\sin.\varphi - a.$$

COROLLARIUM 3

732. Ut sumto $\varphi = 0$ ambae x et y evanescant, sumi debet $C = a$ et $D = 0$ eritque

$$x = a - a\cos.\varphi - a\varphi\sin.\varphi \text{ et } y = -a\sin.\varphi + a\varphi\cos.\varphi,$$

unde, quamdiu angulus φ est minimus, erit

$$x = -\tfrac{1}{2}a\varphi\varphi + \tfrac{1}{8}a\varphi^4 \quad \text{et} \quad y = -\tfrac{1}{3}a\varphi^3 + \tfrac{1}{30}a\varphi^5;$$

ideoque proxime

$$\frac{x^3}{yy} = -\frac{9}{8}a \quad \text{seu} \quad yy = -\frac{8x^3}{9a}.$$

PROBLEMA 93

733. *Posito $dy = pdx$ et $dp = qdx$ si quantitas q aequetur functioni ipsius x, quae sit X, definire relationem inter binas variabiles x et y.*

SOLUTIO

Cum ergo sit $q = X$, erit $qdx = dp = Xdx$, unde integrando colligimus $p = \int Xdx + C$ atque hinc ob $dy = pdx$ adipiscemur

$$y = \int dx \int Xdx + Cx + D.$$

At est

$$\int dx \int Xdx = x\int Xdx - \int Xxdx,$$

uti sumendis differentialibus sponte patet. Quare aequatio integralis completa relationem inter binas variabiles x et y continens est

$$y = x\int Xdx - \int Xxdx + Cx + D$$

duas constantes arbitrarias C et D involvens. Quae ergo erit algebraica, si ambae formulae differentiales Xdx et $Xxdx$ integrationem admittant.

COROLLARIUM 1

734. Quodsi ergo sit $q = 0$ seu sumto dx constante $ddy = 0$, ut sit $X = 0$, erit aequatio integralis completa $y = Cx + D$.

COROLLARIUM 2

735. Aequationes ergo differentio–differentiales, quas hoc modo integrare licet, sumto dx constante in hac forma $ddy = Xdx^2$ continentur, unde prima integratio praebet

$$dy = dx\int Xdx + C \quad \text{et altera} \quad y = \int dx \int Xdx + Cx + D.$$

COROLLARIUM 3

736. Sin autem differentiale dy capiatur constans, ob $p=\frac{dy}{dx}$ erit

$$dp=-\frac{dyddx}{dx^2}=qdx$$

et forma aequationum hoc modo integrandarum erit $-dyddx=Xdx^3$.

COROLLARIUM 4

737. Quodsi elementum $ds=\sqrt{\left(dx^2+dy^2\right)}$ sit constans, ob

$$dxddx+dyddy=0$$

erit

$$dp=\frac{dxddy-dyddx}{dx^2}=\frac{-ds^2ddx}{dx^2dy}=qdx.$$

Hinc forma aequationum hoc modo integrandarum est $-ds^2ddx=Xdx^3dy$.
Vel cum etiam sit $dp=qdx=+\frac{ds^2ddy}{dx^3}$, ea erit $ds^2ddy=Xdx^4$.

SCHOLION

738. Hic manifestum est, quantum intersit aequationes differentio–differentiales a forma solita, ubi elementum quodpiam constans est assumtum, ab hac conditione liberare et ad formam hic stabilitam reducere. Si enim proponatur haec aequatio $ds^2ddy=Xdx^4$, in qua elementum $ds=\sqrt{\left(dx^2+dy^2\right)}$ constans sit assumtum, haud facile patet, quomodo eius integratio sit suscipienda. Nostra autem methodo si ponamus $dy=pdx$, ut sit $ds=dx\sqrt{(1+pp)}$ et $ddy=pddx+dxdp$, induit ista aequatio hanc formam

$$dx^2(1+pp)(pddx+dxdp)=Xdx^4 \quad \text{seu} \quad (pddx+dxdp)(1+pp)=Xdx^2.$$

At quia ds ac proinde quoque $ds^2=dx^2(1+pp)$ est constans, erit

$$ddx(1+pp)+pdxdp=0 \text{ seu } ddx=\frac{-pdxdp}{1+pp}$$

ideoque

$$pddx+dxdp=\frac{dxdp}{1+pp},$$

ita ut fiat $dp=Xdx$, quae aequatio iam facillime tractatur. Hic scilicet in subsidium vocari debent ea, quae supra [§ 733] de integratione formularum differentialium simplicium sunt tradita.

EXEMPLUM 1

739. *Sumto dx constante si fuerit* $ddy = \alpha x^n dx^2$, *integrale completum investigare.*

Cum sit $\frac{ddy}{dx} = \alpha x^n dx$, ob dx constans erit integrando $\frac{dy}{dx} = \frac{\alpha}{n+1} x^{n+1} + C$ hincque denuo integrando

$$y = \frac{\alpha}{(n+1)(n+2)} x^{n+2} + Cx + D,$$

ubi casus $n = -1$ et $n = -2$ seorsim sunt evolvendi.

1. Ergo si $n = -1$, erit $\frac{ddy}{dx} = \frac{\alpha dx}{x}$ hincque $\frac{dy}{dx} = \alpha lx + C$; unde cum sit $dy = \alpha dxlx + Cdx$, erit denuo integrando $y = \alpha xlx - \alpha x + Cx + D$ seu loco $C - \alpha$ scribendo C habebitur

$$y = \alpha xlx + Cx + D.$$

II. Si $n = -2$ et $\frac{ddy}{dx} = \frac{\alpha dx}{x^2}$, erit $\frac{dy}{dx} = \frac{-\alpha}{x} + C$ hincque

$$y = -\alpha lx + Cx + D.$$

EXEMPLUM 2

740. *Posito* $ds = \sqrt{(dx^2 + dy^2)}$ *constante si fuerit* $\frac{ds^2 ddy}{dx^4} = \frac{1}{\alpha} \cos. \frac{x}{c}$, *invenire integrale completum.*

Ex superioribus [§ 717] constat fore $\frac{ds^2 ddy}{dx^4} = q$, ita ut proposita sit haec aequatio $q = \frac{1}{\alpha} \cos. \frac{x}{c}$, unde fit $qdx = dp = dx \cos. \frac{x}{c}$, et integrando $p = \frac{c}{\alpha} \sin. \frac{x}{c} + C = \frac{dy}{dx}$.
Quare obtinebitur

$$y = -\frac{cc}{\alpha} \cos. \frac{x}{c} + Cx + D,$$

quae est aequatio integralis completa.

PROBLEMA 94

741. *Posito* $dy = pdx$ *et* $dp = qdx$ *si quantitas q aequetur functioni cuicunque ipsius y tantum, quae sit Y, invenire aequationem integralem completam inter x et y.*

SOLUTIO

Cum sit $q = Y = \frac{dp}{dx}$, erit $dx = \frac{dp}{Y}$ hincque $pdx = dy = \frac{pdp}{Y}$, unde conficitur haee aequatio inter p et y separata $pdp = Ydy$, quae integrata praebet

$$\tfrac{1}{2}pp = \int Ydy + \tfrac{1}{2}C \quad \text{et} \quad p = \sqrt{\left(C + 2\int Ydy\right)} = \tfrac{dy}{dx}.$$

Hinc ergo porro concluditur

$$x = \int \tfrac{dy}{\sqrt{(C+2\int Ydy)}},$$

quae integratio denuo constantem arbitrariam inducit, ita ut hoc modo aequatio integralis completa inter x et y obtineatur.

COROLLARIUM 1

742. Cum sit $q = \frac{dp}{dx}$ et $dx = \frac{dy}{p}$, erit $q = \frac{pdp}{dy}$. Quare cum aequatio proposita sit $q = Y$, erit $\frac{pdp}{dy} = Y$ hincque $pdp = Ydy$, unde praecedens integratio sponte deducitur.

COROLLARIUM 2

743. Sumto elemento dx constanta cum sit $q = \frac{ddy}{dx^2}$, aequationes hic integratae habebunt hanc formam $ddy = Ydx^2$, cuius integratio, si per dy multiplicatur, est manifesta; fit enim. $\frac{1}{2}dy^2 = dx^2\int Ydy + \frac{1}{2}Cdx^2$ ob dx constans hincque $dx = \frac{dy}{\sqrt{(C+2\int Ydy)}}$ ut ante.

SCHOLION

744. En ergo specimen aequationum differentialium, quae per idoneum multiplicatorem integrabiles redduntur, ex quo intelligitur hanc methodum etiam in his aequationibus usum habere posse; deinceps autem locus erit hane methodum uberius excolendi, cuius quippe usus praecipue in aequationibus differentialibus altiorum graduum est insignis, ubi variabilium separatio nihil subsidii affert. Atque hanc ob causam iam supra [§ 447] hanc methodum per multiplicatores integrandi commendavimus alterique per separationem procedenti longe antetulimus.

EXEMPLUM 1

745. *Posito dx constante si fuerit* $aaddy = ydx^2$, *invenire integrale completum.*

Multiplicetur aequatio proposita per *2dy*, ut prodeat

$$2aadyddy = 2ydydx^2,$$

quae ob dx constans habebit integrale $aady^2 = yydx^2 + Cdx^2$, unde colligitur

$$dx = \frac{ady}{\sqrt{(yy+C)}},$$

quae denuo integrata dat

$$x = al\left(y + \sqrt{(yy+C)}\right) - alb,$$

unde concludimus sumto e pro numero, cuius logarithmus est = 1, fore $be^{\frac{x}{a}} = y + \sqrt{(yy+C)}$ et irrationalitatem tollendo $bbe^{\frac{2x}{a}} - 2bye^{\frac{x}{a}} = C$, ita ut sit

$$y = \tfrac{1}{2}be^{\frac{x}{a}} - \tfrac{C}{2b}e^{-\frac{x}{a}};$$

at forma constantium C et b mutata habebitur

$$y = Ce^{\frac{x}{a}} + De^{-\frac{x}{a}},$$

quae est aequatio integralis completa.

EXEMPLUM 2

746. *Posito dx constante si fuerit* $aaddy + ydx^2 = 0$, *invenire integrale completum.*

Multiplicatione per *2dy* facta aequationis $2aadyddy + 2ydydx^2 = 0$ integrale est

$$aady^2 + yydx^2 = ccdx^2,$$

unde deducimus

$$dx = \frac{ady}{\sqrt{(cc-yy)}},$$

quae denuo integrata dat

$$x = a\text{Ang.sin.}\tfrac{y}{c} + b.$$

Erit ergo

$$\tfrac{y}{c} = \sin.\tfrac{x-b}{a} = \cos.\tfrac{b}{a}\sin.\tfrac{x}{a} - \sin.\tfrac{b}{a}\cos.\tfrac{x}{a}$$

vel mutatis constantibus b et c, ita ut sit $c\cos.\frac{b}{a} = C$ et $-c\sin.\frac{b}{a} = D$, erit

$$y = C\sin.\tfrac{x}{a} + D\cos.\tfrac{x}{a}$$

Vel retenta prima forma habemus

$$y = c\sin.\left(\tfrac{x}{a} + \alpha\right).$$

COROLLARIUM

747. Hoc exemplum ex praecedente resolvi potuisset, cum sit

$$e^{u\sqrt{-1}} = \cos.u + \sqrt{-1}\cdot\sin.u \quad \text{et} \quad e^{-u\sqrt{-1}} = \cos.u - \sqrt{-1}\cdot\sin.u$$

ac vicissim

$$\cos.u\sqrt{-1} = \tfrac{1}{2}e^{u} + \tfrac{1}{2}e^{-u} \quad \text{et} \quad \sin.u\sqrt{-1} = \tfrac{1}{2\sqrt{-1}}e^{u} - \tfrac{1}{2\sqrt{-1}}e^{-u}.$$

EXEMPLUM 3

748. *Posito dx constante si fuerit* $ddy\sqrt{ay} = dx^2$, *integrale completum invenire.*

Cum ergo sit $2dyddy = \frac{2dy}{\sqrt{ay}} \cdot dx^2$, erit integrando

$$dy^2 = 4\frac{dx^2\sqrt{y}}{\sqrt{a}} + 4ndx^2 = \frac{4dx^2\left(\sqrt{y}+n\sqrt{a}\right)}{\sqrt{a}},$$

unde colligimus

$$2dx = \frac{dy\sqrt[4]{a}}{\sqrt{\left(\sqrt{y}+n\sqrt{a}\right)}}.$$

Sit commoditatis gratia $n\sqrt{a} = b$ et $\sqrt{y} = z$, ut fiat $dy = 2zdz$ et

$$\frac{dx\sqrt{n}}{\sqrt{b}} = \frac{zdz}{\sqrt{(b+z)}},$$

cuius integrale est

$$\frac{x\sqrt{n}}{\sqrt{b}} = \frac{2}{3}(z-2b)\sqrt{(b+z)} + C$$

seu restituendo

$$\frac{x}{\sqrt[4]{a}} = \frac{2}{3}\left(\sqrt{y}-2\sqrt{c}\right)\sqrt{\left(\sqrt{y}+\sqrt{c}\right)} + C$$

ubi c et C sunt binae constantes arbitrariae. Erit ergo

$$\frac{3(x+f)}{2\sqrt[4]{a}} = \left(\sqrt{y}-2\sqrt{c}\right)\sqrt{\left(\sqrt{y}+\sqrt{c}\right)}$$

posito $C = \frac{-f}{\sqrt[4]{a}}$ et sumtis quadratis

$$\frac{9(x+f)^2}{4\sqrt{a}} = y\sqrt{y} - 3y\sqrt{c} + 4c\sqrt{c}$$

SCHOLION

749. Forma ergo vulgaris aequationum hoc modo integrandarum sumto elemento dx constante est $ddy = Ydx^2$, quae per dy multiplicata manifesto fit integrabilis. Sin autem elementum dy capiatur constans, ob $q = \frac{dp}{dx}$ et $p = \frac{dy}{dx}$ erit $q = -\frac{dyddx}{dx^3}$ hincque forma vulgaris $dyddx = -Ydx^3$. Porro sumto elemento $ds = \sqrt{\left(dx^2 + dy^2\right)}$ constante, ut sit

$dxddx + dyddy = 0$, ob $dp = \frac{dxddy - dyddx}{dx^2}$ erit vel $q = -\frac{ds^2 ddx}{dx^3 dy}$ vel $q = \frac{ds^2 ddy}{dx^4}$, unde nascitur haec forma $-\frac{ds^2 ddx}{dx^3 dy} = Y$ vel $\frac{ds^2 ddy}{dx^4} = Y$, quae etiam per *y* multiplicatae integrabiles evadunt, etiamsi hoc iam minus pateat. Simili modo si elementum *ydx* sumatur constans, ut sit $yddx + dxdy = 0$ et $ddx = -\frac{dxdy}{y}$, ob $dp = \frac{ddy}{dx} + \frac{dy^2}{ydx}$ orietur haec forma $yddy + dy^2 = Yydx^2$ cuius membrum prius integrabile redditur, si per functionem quamcunque ipsarum *ydy* et *ydx* multiplicetur, ergo etiam per $\frac{ydy}{yydx^2}$, quo multiplicatore simul alterum membrum $Yydx^2$ redditur integrabile.

His igitur casibus simplicissimis aequationum differentio– differentialium expeditis, qui ne ulla quidem difficultate laborant, ad difficiliores progrediamur ac primo quidem ad eas aequationes, in quibus altera binarum variabilium *x* et *y* ipsa non inest, ita ut aequatio proposita ternas tantum contineat litteras *x*, *p* et *q*, vel *y*, *p* et *q;* utriusque enim ratio fere perinde est comparata.