CHAPTER X

ON THE CONSTRUCTION OF SECOND ORDER DIFFERENTIAL EQUATIONS FROM THE QUADRATURES OF CURVES

PROBLEM 129

1017. *If there should be $y=\int Vdx$ with V denoting some function of the two quantities x and u, but of which this u is considered in the integration as constant, then upon integrating there is put x = a, in order that y is equal to a certain function of u, so that if now u is taken as variable, then the value of this function $\frac{dy}{du}$ can be investigated.*

SOLUTION

Since $\int Vdx$ furnishes a certain function of the two quantities *x* and *u*, the differential of this taken with *u* constant is equal to Vdx; thus as both *u* as well as *x* may be treated as variables, the differential of the equation $y=\int Vdx$ will have such a form $dy=Vdx+Udu$; which because it is a true differential, it is necessary [§ 443] that there should be $\left(\frac{dV}{du}\right)=\left(\frac{dU}{dx}\right)$. But since *V* shall be a given function of *x* and *u*, there may be put in place $dV=Pdx+Qdu$ and then there shall be $\left(\frac{dV}{du}\right)=Q$ and thus $\left(\frac{dU}{dx}\right)=Q$. Hence again with *u* considered as constant there shall be $dU=Qdx$ and $U=\int Qdx$, in which the integration may be considered for the variable *x* only. On account of which if we consider this value $\int Qdx$ as known, surely it is possible to assign that by quadrature, and there will be $dy= Vdx+du\int Qdx$. Moreover we seek that differential of *y*, which arises from the variability of *u* only; which since then there shall be $dy=du\int Qdx$, the value sought will be $\frac{dy}{du}=\int Qdx$, certainly if likewise we put *x* = *a* after integration.

[These translations are done usually according to the mathematical symbolism of the time; however, here it seems appropriate in order to put the matter on a sounder footing to introduce partial derivatives, at least for an outline of this part of this proposition. Thus, initially, we have $y=y(x,u)$, and at some point we consider the total derivative of

y, $Dy=\left(\frac{\partial y}{dx}\right)dx+\left(\frac{\partial y}{du}\right)du$, where
$\left(\frac{\partial y}{dx}\right)=V\left(x,u\right)$ and $\left(\frac{\partial y}{du}\right)=U\left(x,u\right)$, giving $Dy=Vdx+Udu$, and in the case that
du = 0, we have $Dy=\left(\frac{\partial y}{\partial x}\right)dx=Vdx$. Clearly, we require $\left(\frac{\partial^2 y}{dxdu}\right)=\frac{\partial V}{\partial u}$ and $\left(\frac{\partial^2 y}{dudx}\right)=\frac{\partial U}{dx}$,
and hence $\frac{\partial V}{\partial u}=\frac{\partial U}{dx}$. In turn, as $V=\frac{\partial y}{\partial x}=V\left(x,u\right)$ then
$DV=\left(\frac{\partial V}{\partial x}\right)dx+\left(\frac{\partial V}{\partial u}\right)du=Pdx+Qdu$, where $P=\left(\frac{\partial^2 y}{\partial x^2}\right)$ and $Q=\left(\frac{\partial^2 y}{\partial x\partial u}\right)$. With u constant,
so that du = 0, we have $\left(\frac{\partial^2 y}{dudx}\right)=\frac{\partial U}{dx}=Q$ and hence $U=\int Qdx$. Thus, the variable u is absent from the calculation. In what follows, Euler assumes that this integral is known, and works backwards to find the differential equations governed by V and y. Thus, continuing, we have $Dy=Vdx+Udu=Vdx+du\int Qdx$; from which $\frac{\partial y}{\partial u}=\int Qdx$, and the definite integral $\int_0^a Q\left(x,u\right)dx$ is also given, in which case $\frac{\partial y}{\partial u}$ is a function of u alone ; see also Corollary 3; in Euler's day there was no limits on integrals, and these were stated in words; no 'dummy' variable was used in integrands, and of course there was no special symbol for the partial derivative, so that one just had to understand what was going on, and to see more in the symbols than was written down.]

COROLLARY 1

1018. Since $y=\int Vdx$ shall be a function of the variables x and u themselves, moreover by the integration of the formula Vdx, in which u is regarded as constant, it is possible to add some function of u in place of the constant, and the function y by itself will be undefined, but it will be determined, and the integral $\int Vdx$ thus may be taken at once so that it vanishes, on putting $x=0$.

COROLLARY 2

1019. With this condition observed y will vanish on putting $x=0$, whatever value is attributed to the other quantity u ; hence there will become also $y+du\left(\frac{dy}{du}\right)=0$ on making $x=0$, hence also $\left(\frac{dy}{du}\right)=0$. From which it is apparent that $\int Qdx=\frac{dy}{du}$ thus also must be taken, so that it vanishes on putting $x=0$.

COROLLARY 3

1020. Since $y=\int Vdx$, there will be $\left(\frac{dy}{dx}\right)=V$, hence $\left(\frac{ddy}{dudx}\right)=\left(\frac{dV}{du}\right)$. But if there is put $\left(\frac{dy}{du}\right)=Z$, then also there will be $\left(\frac{ddy}{dudx}\right)=\left(\frac{dZ}{dx}\right)$, hence $\left(\frac{dZ}{dx}\right)=\left(\frac{dV}{du}\right)$. Whereby on regarding *u* as constant there will be $dZ=dx\left(\frac{dV}{du}\right)$ and $Z=\int dx\left(\frac{dV}{du}\right)$ and thus $\left(\frac{dy}{du}\right)=\int dx\left(\frac{dV}{du}\right)$.

COROLLARY 4

1021. But if hence after the integrations thus completed, so that the integrals vanish on putting $x=0$, there is put $x=a$, both the value $y=\int Vdx$ as well as $\left(\frac{dy}{du}\right)=\int dx\left(\frac{dV}{du}\right)$ will be a determined function of *u*.

COROLLARY 5

1022. In a similar manner on progressing further there will be $\frac{ddy}{du^2}=\int dx\left(\frac{ddV}{du^2}\right)$. Whereby if *L*, *M* and *N* denote some functions of *u*, there will be

$$\frac{Lddy}{du^2}+\frac{Mdy}{du}+Ny=\int dx\left(L\left(\frac{ddV}{du^2}\right)+M\left(\frac{dV}{du}\right)+NV\right)$$

and the whole transaction thus has been reduced, so that the formula is allowed to be integrated.

SCHOLION

1023. Evidently from the given functions of *u*, *L*, *M*, and *N*, a function *V* must be sought of the two variables *x* and *u*, thus so that with *u* regarded as constant the formula

$$\left(L\left(\frac{ddV}{du^2}\right)+M\left(\frac{dV}{du}\right)+NV\right)dx$$

becomes completely integrable; thus in order that the integral of this shall be determined, it is taken so that it vanishes on putting $x=0$. Then indeed there is put $x=a$, and if that integral vanishes also in this case, then there will be

$$\frac{Lddy}{du^2}+\frac{Mdy}{du}+Ny=0$$

and hence the value $y=\int Vdx$ satisfies the equation assumed by the rule indicated. But with the given functions *L*, *M* and *N*, the problem of finding the function *V* is largely indeterminate, and neither at this point can it be solved in general by a known method;

from which it is convenient to be treated in the inverse manner, so that by assuming a function *V* the other functions *L*, *M* and *N* may be tracked down. Hence we attend to the second order differential equations, the integrals of which we will prevail upon to assign, which if they cannot be treated by other methods, will at least supply a significant gain. But if that integral

$$\int\left(L\left(\tfrac{ddV}{du^2}\right)+M\left(\tfrac{dV}{du}\right)+NV\right)dx$$

does not vanish on putting $x=a$, but produces a given function *U* of u, the value $y=\int Vdx$ shall agree with this equation

$$L\left(\tfrac{ddy}{du^2}\right)+M\left(\tfrac{dy}{du}\right)+Ny=U\,,$$

which since it can be transformed in an infinite number of ways into other forms, the integrals of these also may become known, where likewise this conveniently comes about, so that , even if only a particular integral should be obtained, from that yet generally the complete integral can be deduced without difficulty.

PROBLEM 130

1024. *To find second order differential equations of the form*

$$\frac{Lddy}{du^2}+\frac{Mdy}{du}+Ny=U\,,$$

in order that L, M, N and U shall be functions of u, the element du of which is taken here as constant, and the integral of which can be shown with the aid of a construction by quadratures.

SOLUTION

Some function of the two variables *u* and *x* is taken, which shall be *V*, and the integral $\int Vdx$ is taken by regarding the quantity *u* as constant thus, so that it vanishes on putting $x=0$, then truly there is put $x=a$ with the quantity *a* denoting some constant quantity, so that now $\int Vdx$ expresses a certain function of *u* only, to which the quantity *y* is equal, so that there shall be $y=\int Vdx$. Since now there shall be

$$\tfrac{dy}{du}=\int dx\left(\tfrac{dV}{du}\right)\ \text{ and }\ \tfrac{ddy}{du^2}=\int dx\left(\tfrac{ddV}{du^2}\right)$$

with these integrals thus taken equally, so that they vanish on putting $x=0$, then indeed there is put in place $x=a$, the functions sought *L, M, N* of *u,* so that this formula

$$\int dx\left(L\left(\tfrac{ddV}{du^2}\right)+M\left(\tfrac{dV}{du}\right)+NV\right)$$

becomes completely integrable, and the integral of this thus can be determined , as on putting $x = a$, this is made $= U$. Because if that should be applied, it is evident that the second order differential equation

$$\frac{Lddy}{du^2} + \frac{Mdy}{du} + Ny = U$$

satisfies the assumed formula $y = \int Vdx$.

COROLLARY 1

1025. Hence the assumption of the function *V* is not allowed for us to be completely arbitrary, but it is to be considered chiefly as with such a form

$$\int dx\left(L\left(\frac{ddV}{du^2}\right) + M\left(\frac{dV}{du}\right) + NV\right)$$

that it becomes integrable by these.

COROLLARY 2

1026. Hence therefore at once infinitely many unsuitable curves are excluded from this goal, which are of the kind $V = UP$ with a function *U* of *u* and *P* of *x* arising only, because then there shall be

$$y = U\int Pdx,\ \frac{dy}{du} = \frac{dU}{du}\int Pdx \text{ and } \frac{ddy}{du^2} = \frac{ddU}{du^2}\int Pdx,$$

which clearly include the same integral, thus so that from these taken together the formula for complete integrability cannot be constructed. [*i. e.* the integral cancels out.]

EXAMPLE 1

1027. *Let there be* $V = x^n\sqrt{\frac{uu+xx}{cc-xx}}$ *and* $y = \int dxx^n\sqrt{\frac{uu+xx}{cc-xx}}$ *with the vanishing of the integral on putting* $x = 0$, *and then truly on making* $x = a$.

Hence there shall be

$$\left(\frac{dV}{du}\right) = x^n \cdot \frac{u}{\sqrt{(uu+xx)(cc-xx)}} \text{ and } \left(\frac{ddV}{du^2}\right) = x^n \cdot \frac{xx}{(uu+xx)^{\frac{3}{2}}\sqrt{(cc-xx)}}$$

and this formula must be rendered integrable

$$x^n dx\left(\frac{Lxx}{(uu+xx)^{\frac{3}{2}}\sqrt{(cc-xx)}}+\frac{Mu}{\sqrt{(uu+xx)(cc-xx)}}+N\sqrt{\frac{uu+xx}{cc-xx}}\right)$$

or

$$\frac{x^n dx}{(uu+xx)^{\frac{3}{2}}\sqrt{(cc-xx)}}\left(Lxx+Mu\left(uu+xx\right)+N\left(uu+xx\right)^2\right).$$

The integral is put in place

$$\frac{x^{n+1}\sqrt{(cc-xx)}}{\sqrt{(uu+xx)}}\ ;$$

since the differential of this shall be

$$\frac{(n+1)x^n(cc-xx)(uu+xx)-x^{n+2}(uu+xx)-x^{n+2}(cc-xx)}{(uu+xx)^{\frac{3}{2}}\sqrt{(cc-xx)}}dx$$

or

$$\frac{x^n dx}{(uu+xx)^{\frac{3}{2}}\sqrt{(cc-xx)}}\left\{\begin{matrix}(n+1)ccuu+(n+1)ccxx-(n+1)uuxx-(n+1)x^4\\ -ccxx-\qquad uuxx\end{matrix}\right\},$$

which since if it be compared with the proposed [*i.e.* on equating coefficients of powers of x], there becomes

$$Mu^3+Nu^4=(n+1)ccuu,\ \ L+Mu+2Nuu=ncc-(n+2)uu$$

and

$$N=-(n+1).$$

Hence there is elicited

$$Mu=(n+1)(cc+uu)\ \text{ or }\ M=\frac{(n+1)(cc+uu)}{u}$$

and

$$L=-(n+1)(cc+uu)+2(n+1)uu+ncc-(n+2)uu\ \text{ or }\ L=-cc-uu\,.$$

On account of which we shall have

$$-\frac{(cc+uu)ddy}{du^2}+\frac{(n+1)(cc+uu)dy}{udu}-(n+1)y=\frac{a^{n+1}\sqrt{(cc-aa)}}{\sqrt{(aa+uu)}},$$

to which the equation

$$y=\int x^{n}dx\sqrt{\tfrac{uu+xx}{cc-xx}}$$

gives satisfaction to the complete integration, as has been shown.

COROLLARY 1

1028. Hence on taking $a=c$ the formula of the integral $y=\int x^{n}dx\sqrt{\tfrac{uu+xx}{cc-xx}}$ put in place after the integration $x=c$ will show the integral of this equation

$$\text{[from } -\tfrac{(cc+uu)ddy}{du^{2}}+\tfrac{(n+1)(cc+uu)dy}{udu}-(n+1)y=\tfrac{a^{n+1}\sqrt{(cc-aa)}}{\sqrt{(aa+uu)}}\text{,]}$$

$$u(cc+uu)ddy-(n+1)(cc+uu)dudy+(n+1)uydu^{2}=0$$

or

$$ddy-\tfrac{(n+1)dudy}{u}+\tfrac{(n+1)ydu^{2}}{cc+uu}=0.$$

COROLLARY 2

1029. If there shall be $n=1$, by the integration there is found

$$\int xdx\sqrt{\tfrac{uu+xx}{cc-xx}}=\tfrac{1}{4}(cc+uu)\,\text{Ang.sin.}\tfrac{2xx-cc+uu}{cc+uu}$$

$$-\tfrac{1}{2}\sqrt{(ccuu+ccxx-uuxx-x^{4})}-\tfrac{1}{4}(cc+uu)\,\text{Ang.sin.}\tfrac{-cc+uu}{cc+uu}+\tfrac{1}{2}cu$$

and on putting $x=c$ there becomes

$$y=\tfrac{1}{4}(cc+uu)\,\text{Ang.cos.}\tfrac{uu-cc}{cc+uu}+\tfrac{1}{2}cu$$

and hence

$$\tfrac{dy}{du}=\tfrac{1}{2}u\ \text{Ang.cos.}\tfrac{uu-cc}{cc+uu}\quad\text{and}\quad\tfrac{ddy}{du^{2}}=\tfrac{1}{2}\text{Ang.cos.}\tfrac{uu-cc}{cc+uu}-\tfrac{cu}{cc+uu}$$

which formulas clearly satisfy the equation

$$ddy-\tfrac{2dudy}{u}+\tfrac{2ydu^{2}}{cc+uu}=0.$$

COROLLARY 3

1030. In this case the integral can also be expressed in this manner

$$y=\tfrac{1}{4}(cc+uu)\,\text{Ang.sin.}\tfrac{2cu}{cc+uu}+\tfrac{1}{2}cu\,,$$

or since as you please a multiple of this equally satisfies,

$$y=(cc+uu)\,\text{Ang.sin.}\tfrac{2cu}{cc+uu}+2cu\,.$$

Now also it is satisfied by $y=cc+uu$, from which the complete integral is

$$y=\alpha(cc+uu)\,\text{Ang.sin.}\tfrac{2cu}{cc+uu}+2\alpha cu+\beta(cc+uu)\,.$$

SCHOLIUM

1031. Because it is allowed to conclude from the integration that the value $y=cc+uu$ satisfies the integral; since indeed $\text{Ang.sin.}\tfrac{2cu}{cc+uu}$ is a many–valued function and can be increased by the term 2π, the integral itself can be increased by the term $2\pi(cc+uu)$. But in general the difference of the two integrals also is satisfied, hence also $y=2\pi(cc+uu)$ must be satisfactory and generally $y=2\beta(cc+uu)$. From this case it is seen more easily, how the value assumed of the general equation may be satisfied, even if it cannot be set out by integration. Moreover it is apparent that $n+1$ must be a positive number, since otherwise the condition of the integral, that it vanishes on putting $x=0$, cannot be fulfilled.

EXAMPLE 2

1032. *There is taken* $V=x^{n-1}(uu+xx)^{\mu}(cc-xx)^{\nu}$.

There will be

$$\left(\tfrac{dV}{du}\right)=2\mu ux^{n-1}(uu+xx)^{\mu-1}(cc-xx)^{\nu}$$

and

$$\left(\tfrac{ddV}{du^2}\right)=2\mu x^{n-1}(cc-xx)^{\nu}\left((uu+xx)^{\mu-1}+2(\mu-1)uu(uu+xx)^{\mu-2}\right)$$
$$=2\mu x^{n-1}(cc-xx)^{\nu}(uu+xx)^{\mu-2}\left(2(\mu-1)uu+xx\right).$$

Therefore the integrand must return this formula completely

$$\int x^{n-1}dx(cc-xx)(uu+xx)^{\mu-2}\begin{Bmatrix} 2\mu\big((2\mu-1)uu+xx\big)L \\ +2\mu u(uu+xx)M+(uu+xx)^2 N \end{Bmatrix}$$

or

$$\int x^{n-1}dx(cc-xx)^{\nu}(uu+xx)^{\mu-2}\begin{Bmatrix} 2\mu(2\mu-1)Luu+2\mu Lxx+Nx^4 \\ +2\mu Mu^3+2\mu Muxx \\ Nu^4+2Nuuxx \end{Bmatrix}.$$

The [evaluated] integral $x^n(uu+xx)^{\mu-1}(cc-xx)^{\nu+1}$ may be put in place ; the differential of which, since it shall be

$$x^{n-1}dx(uu+xx)^{\mu-2}(cc-xx)^{\nu}$$
$$\times n\big((uu+xx)(cc-xx)+2(\mu-1)xx(cc-xx)-2(\nu+1)xx(uu+xx)\big),$$

will become [on equating terms]

$$2\mu(2\mu-1)Luu+2\mu Mu^2+Nu^4=nccuu,$$
$$2\mu L+2\mu Mu+2Nuu=ncc-nuu+2(\mu-1)cc-2(\nu+1)uu,$$
$$N=-n-2(\mu-1)-2(\nu+1)=-n-2\mu-2\nu.$$

But the first $2\mu(2\mu-1)L+2\mu Mu+Nuu=ncc$ with the second taken gives

$$4\mu(\mu-1)L-Nuu=(n+2\nu+2)uu-2(\mu-1)cc$$

or $4\mu(\mu-1)L=-2(\mu-1)(uu+cc)$, hence

$$L=\tfrac{-cc-uu}{2\mu},$$

which value substituted into the first gives

$$-(2\mu-1)(cc+uu)+2\mu Mu-(n+2\mu+2\nu)uu=ncc$$

or

$$2\mu Mu=(n+2\mu-1)cc+(n+4\mu+2\nu-1)uu.$$

Hence

$$M=\tfrac{(n+2\mu-1)(cc+uu)}{2\mu u}+\tfrac{\mu+\nu}{\mu}u.$$

If $n > 0$, the above integral vanishes on putting $x=0$; whereby if we put $x=a$, this equation arises

$$-\frac{(cc+uu)ddy}{2\mu du^2}+\frac{(n+2\mu-1)(cc+uu)dy}{2\mu udu}+\frac{(\mu+v)udy}{\mu du}-(n+2\mu+2v)y$$
$$= a^n(aa+uu)^{\mu-1}(cc-aa)^{v+1},$$

the integral of which is

$$y=\int x^{n-1}dx(uu+xx)^{\mu}(cc-xx)^{v}$$

with the integral thus taken, so that it vanishes on putting $x=0$, then truly on putting $x=a$.

COROLLARY 1

1033. If there is taken $a=c$, so that the latter part becomes $=0$, if indeed the exponent $v+1$ shall be greater than zero, the formula

$$y=\int x^{n-1}dx(uu+xx)^{\mu}(cc-xx)^{v}$$

on putting $x=c$ after integrating thus arises on completion, so that in the case $x=0$ there becomes $y=0$, will be the integral of this equation

$$u(cc+uu)ddy-(n+2\mu-1)(cc+uu)dudy-2(\mu+v)uududy$$
$$+2\mu(n+2\mu+2v)uydu^2=0.$$

COROLLARY 2

1034. Let $n+2\mu-1=\alpha,\ n+4\mu+2v-1=\beta$; there becomes $2\mu=\alpha+1-n$ and $2v=\beta+1-n-2\alpha-2+2n=\beta-1+n-2a$ but the integral of the equation

$$u(cc+uu)ddy-(\alpha cc+\beta uu)dudy+(\alpha+1-n)(\beta-\alpha+n)uydu^2=0$$

will be

$$y=\int x^{n-1}dx(uu+xx)^{\frac{\alpha+1-n}{2}}(cc-xx)^{\frac{\beta-1+n-2\alpha}{2}}$$

on putting $x=c$, but only if there shall be $n>0$ and $\beta+1+n>2\alpha$.

SCHOLlUM

1035. This construction extends more widely to this equation

$$xx\left(a+bx^n\right)ddz+x\left(c+ex^n\right)dxdz+\left(f+gx^n\right)zdx^2=0.$$

For in the first place here without detriment to the fullness of the expression there can be taken $n=2$ on putting $x^n=uu$. Then truly, as we have seen above in § 997, on putting

$$z = x^{\frac{a-c}{a}+h}\left(a+bx^{n}\right)^{\frac{bc-ae}{nab}+1} y$$

the equation will change into this

$$xx\left(a+bx^{n}\right)ddy + x\left(2a-c+2ah+\left(2b-e+2nb+2bh\right)x^{n}\right)dxdy$$
$$+\left(f+ah-ch+ahh+\left(g+\left(b-e+nb+bh\right)\left(n+h\right)\right)x^{n}\right)ydx^{2}=0\,;$$

where if h is taken thus, so that there shall be $ahh+\left(a-c\right)h+f=0$, an equation of the form is produced, the construction of which we have given [§ 1033]. But in special cases difficulties can occur, which the following examples take care in overcoming.

EXAMPLE 3

1036. *Let* $V=e^{mux}x^{n}\left(c-x\right)^{v}$.

There will be

$$\left(\tfrac{dV}{du}\right)=me^{mux}x^{n+1}\left(c-x\right)^{v} \quad\text{and}\quad \left(\tfrac{ddV}{du^{2}}\right)=mme^{mux}x^{n+2}\left(c-x\right)^{v}.$$

The integrand therefore is required to return this formula

$$e^{mux}x^{n}dx\left(c-x\right)^{v}\left(mmLxx+mMx+N\right),$$

the integral of which is put $=e^{mux}x^{n+1}\left(c-x\right)^{v+1}$, therefore the differential of this must be equal to that formula ; which since there shall be

$$e^{mux}x^{n}dx\left(c-x\right)^{v}\left(mux\left(c-x\right)+\left(n+1\right)\left(c-x\right)-\left(v+1\right)x\right),$$

there will be

$$N=\left(n+1\right)c,\ mM=mcu-\left(n+v+2\right),\ mmL=-mu.$$

Now there is put in place $x=a$ and the formula

$$y=\int e^{mux}x^{n}dx\left(c-x\right)^{v}$$

will be the integral of this equation

$$-\frac{uddy}{mdu^{2}}+\frac{cudy}{du}-\frac{\left(n+v+2\right)dy}{mdu}+\left(n+1\right)cy=e^{mau}a^{n+1}\left(c-a\right)^{v+1}.$$

Here it is possible to put $m=1$ and on taking $c=a$ the integral of the equation

$$uddy - aududy + (n+\nu+2)dudy - (n+1)aydu^2 = 0$$

is

$$y = \int e^{ux} x^n dx (a-x)^\nu$$

on putting $x = a$ after the integration, while there shall be $\nu + 1 > 0$ and $n + 1 > 0$, so that the integral can be returned vanishing on putting $x = 0$.

COROLLARIUM 1

1037. If there is put here $y = e^{\int zdu}$, then there will be

$$udz + uzzdu - auzdu + (n+\nu+2)zdu - (n+1)adu = 0,$$

of which the integral is

$$z = \frac{dy}{ydu} = \frac{\int e^{ux} x^{n+1} dx (a-x)^\nu}{\int e^{ux} x^n dx (a-x)^\nu}.$$

But that equation on putting $z = \frac{1}{2}a + v$ is changed into this

$$udv + uvvdu + (n+\nu+2)vdu - \tfrac{1}{4}aaudu - \tfrac{1}{2}(n-\nu)adu = 0,$$

which on putting $v = u^{-n-\nu-2}s$ changes into this :

$$u^{-n-\nu-1}ds + u^{-2n-2\nu-3}ssdu - \tfrac{1}{4}aaudu - \tfrac{1}{2}(n-\nu)adu = 0.$$

COROLLARY 2

1038. Again if there is put $u^{-n-\nu-2}du = dt$ or $u^{-n-\nu-1} = -(n+\nu+1)t$, so that there becomes

$$ds + ssdt - \tfrac{1}{4}aau^{2n+2\nu+4}dt - \tfrac{1}{2}(n-\nu)u^{2n+2\nu+3}dt = 0,$$

which equation hence also can be constructed. Or if $-(n+\nu+1)t = r$; then there will be

$$ds - \frac{ssdr}{n+\nu+1} + \frac{aar^{\frac{-2n-2\nu-4}{n+\nu+1}}dr}{4(n+\nu+1)} + \frac{(n-\nu)r^{\frac{-2n-2\nu-3}{n+\nu+1}}dr}{2(n+\nu+1)} = 0,$$

which on putting $s = -(n+\nu+1)q$ becomes

$$dq + qqdr - \frac{aar^{\frac{-2n-2\nu-4}{n+\nu+1}}dr - 2(n-\nu)r^{\frac{-2n-2\nu-3}{n+\nu+1}}dr}{4(n+\nu+1)^2} = 0,$$

and here there is

$$u = r^{\frac{-1}{n+v+1}} \text{ et } z = \tfrac{1}{2}a - (n+v+1)r^{\frac{n+v+2}{n+v+1}}q.$$

SCHOLIUM

1039. Since the integral of the second order differential equation [§ 1036]

$$\frac{ddy}{du} - ady + \frac{(n+v+2)dy}{u} - \frac{(n+1)aydu}{u} = 0$$

shall be $y = \int e^{ux}x^n dx(a-x)^v$, we can see how this equation itself can be transformed into other forms.

Initially let $u = \alpha t^{\lambda}$ and thus $du = \alpha\lambda t^{\lambda-1}dt$, from which there becomes

$$\tfrac{1}{\alpha\lambda}d.\frac{dy}{t^{\lambda-1}dt} - ady + \frac{(n+v+2)dy}{\alpha t^{\lambda}} - \frac{\lambda(n+1)aydt}{t} = 0 \ ;$$

now the element *dt* is assumed to be constant and there shall be

$$\frac{ddy}{\alpha\lambda t^{\lambda-1}dt} - \frac{(\lambda-1)dy}{\alpha\lambda t^{\lambda}} - ady + \frac{(n+v+2)dy}{\alpha t^{\lambda}} - \frac{\lambda(n+1)aydt}{t} = 0$$

or

$$ddy - \alpha\lambda a t^{\lambda-1}dtdy + \frac{(\lambda n+\lambda v+\lambda+1)dtdy}{t} - \alpha\lambda\lambda(n+1)at^{\lambda-2}ydt^2 = 0,$$

and the integral of this is

$$y = \int e^{\alpha t^{\lambda}x}x^n dx(a-x)^v.$$

Again on putting $\frac{dy}{y} = Pdt + \frac{dz}{z}$, so that there shall be $z = e^{-\int Pdt}y$ *;* then there becomes [§ 993] :

$$ddz + 2Pdtdz - \alpha\lambda a^{\lambda-1}dtdz + (\lambda n + \lambda v + \lambda + 1)\tfrac{dtdz}{t} + zdtdP$$
$$+zdt^2\left(PP - \alpha\lambda a^{\lambda-1}P + \frac{(\lambda n+\lambda v+\lambda+1)P}{t} - \alpha\lambda\lambda(n+1)at^{\lambda-2}\right) = 0.$$

To remove the terms influenced by the element *dz* there is put

$$P = \tfrac{1}{2}\alpha\lambda a^{\lambda-1} - \frac{\lambda n+\lambda v+\lambda+1}{2t}$$

and there becomes [$z = e^{-\frac{1}{2}\alpha a t^{\lambda}}t^{\frac{\lambda n+\lambda v+\lambda+1}{2}}y$] and this equation will be produced :

$$ddz - zdt^2\left(\frac{(\lambda n+\lambda v+\lambda+1)^2-1}{4tt}+\frac{1}{2}\alpha\lambda\lambda(n-v)at^{\lambda-2}+\frac{1}{4}\alpha^2\lambda^2a^2t^{2\lambda-2}\right)=0,$$

and therefore the integral of this is

$$z = e^{-\frac{1}{2}\alpha at^{\lambda}}t^{\frac{\lambda n+\lambda v+\lambda+1}{2}}\int e^{\alpha t^{\lambda}x}x^n dx(a-x)^v.$$

But if there should be hence

$$v=n,\ \lambda\lambda(2n+1)^2-1=0 \text{ or } \lambda=\tfrac{\pm1}{2n+1} \text{ and } \alpha=\pm\tfrac{2}{\lambda}=\pm2(2n+1),$$

this equation can be considered :

$$ddz - aazt^{\frac{\pm2}{2n+1}-2}dt^2=0,$$

the integral of which is :

$$z=e^{\pm(2n+1)\alpha t^{\frac{\pm1}{2n+1}}}t^{\pm\frac{1}{2}+\frac{1}{2}}\int e^{\pm2(2n+1)t^{\frac{\pm1}{2n+1}}x}x^n dx(a-x)^n.$$

Or the integral of this equation

$$ddz - aat^{2\lambda-2}zdt^2=0$$

is

$$z=e^{-\frac{\alpha}{\lambda}t^{\lambda}}t\int e^{\frac{2x}{\lambda}t^{\lambda}}x^{\pm\frac{1}{2\lambda}-\frac{1}{2}}dx(a-x)^{\pm\frac{1}{2\lambda}-\frac{1}{2}},$$

from which we seize the opportunity of investigating more general equations of this kind.

EXAMPLE 4

1040. *If P and Q shall be some functions of u and there is taken*

$$y=P\int e^{Qx}x^{n-1}dx(a-x)^{v-1}$$

clearly on putting $x=a$ after the integration, this will be the value of the integral y of any second order differential equation

$$\frac{Lddy}{du^2}+\frac{Mdy}{du}+Ny=0,$$

which is sought.

Towards assembling the calculation we put $dP = P'du$, $dP' = P''du$, likewise

$dQ = Q'du$ and $dQ' = Q''du$. Hence there shall be

$$\frac{dy}{du} = P'\int e^{Qx}x^{n-1}dx(a-x)^{v-1} + PQ'\int e^{Qx}x^{n}dx(a-x)^{v-1}$$

and

$$\frac{ddy}{du^2} = P''\int e^{Qx}x^{n-1}dx(a-x)^{v-1} + 2P'Q'\int e^{Qx}x^{n}dx(a-x)^{v-1}$$
$$+2PQ''\int e^{Qx}x^{n}dx(a-x)^{v-1} + PQ'Q'\int e^{Qx}x^{n+1}dx(a-x)^{v-1},$$

from which there is deduced

$$\frac{Lddy}{du^2} + \frac{Mdy}{du} + Ny = \int e^{Qx}x^{n-1}dx(a-x)^{v-1}\left\{\begin{matrix} LP'' + 2LP'Q'x + LPQ''x + LPQ'Q'xx \\ +MP' + MPQ'x + NP \end{matrix}\right\},$$

because the integral is put in place equal to $e^{Qx}x^{n}(a-x)^{v}$, thus so that it vanishes on putting $x = a$, while there shall be $v > 0$, as also it vanishes in the case $x = 0$, only if $n > 0$. Therefore since the differential of this formula shall be

$$e^{Qx}x^{n-1}dx(a-x)^{v-1}\big(Qx(a-x) + na - (n+v)x\big),$$

the comparison of this with the form found gives

$$LP'' + MP' + NP = na,$$
$$2LP'Q' + LPQ'' + MPQ' = aQ - (n+v) \text{ and } LPQ'Q' = -Q,$$

hence

$$L = \frac{-Q}{PQ'Q''},$$

hence

$$M = \frac{aQ}{PQ'} - \frac{n+v}{PQ'} + \frac{2P'Q}{PPQ'Q'} + \frac{QQ''}{PQ'^3} \text{ and } N = \frac{na}{P} + \frac{P''Q}{PPQ'Q'} - \frac{MP'}{P};$$

and thus the equation for the second order differential equation will be known.

COROLLARY 1

1041. If we wish that there becomes $M = 0$, then there will be

$$aQ - (n+v) + \frac{2P'Q}{PQ'} + \frac{QQ''}{Q'Q'} = 0,$$

which multiplied by $\frac{Q'du}{Q}$ changes into this

$$\frac{2dP}{P} + adQ - \frac{(n+v)dQ}{Q} + \frac{dQ'}{Q'} = 0\,,$$

the integral of which is

$$\frac{e^{aQ}P^2Q'}{Q^{n+v}} = \text{Const. or } P = Ce^{-\frac{1}{2}\alpha Q}Q^{\frac{n+v}{2}}\sqrt{\frac{du}{dQ}}.$$

COROLLARY 2

1042. Let $Q = 2\alpha u^{\lambda}$, then there will be $Q' = 2\alpha\lambda u^{\lambda-1}$ and $P = Ce^{-\alpha au^{\lambda}}u^{\frac{\lambda(n+v-1)+1}{2}}$, hence on taking $C = 1$

$$L = -\frac{1}{2\alpha\lambda\lambda}e^{\alpha\alpha u^{\lambda}}u^{\frac{-\lambda(n+v+1)+3}{2}} \text{ and } N = \frac{na}{P} + \frac{QddP}{PPdQ^2}.$$

But there is

$$\frac{Q}{dQ^2} = \frac{u^{-\lambda+2}}{2\alpha\lambda\lambda du^2}$$

and on account of

$$\frac{dP}{P} = -\alpha\lambda au^{\lambda-1}du + \frac{\lambda(n+v-1)+1}{2}\cdot\frac{du}{u}$$

there will be

$$\frac{ddP}{P} = -\alpha\lambda(\lambda-1)au^{\lambda-2}du^2 - \frac{\lambda(n+v-1)+1}{2}\cdot\frac{du^2}{uu} + \alpha\alpha\lambda\lambda aau^{2\lambda-2}du^2$$
$$-\alpha\lambda a\big(\lambda(n+v-1)+1\big)u^{\lambda-2}du^2 + \frac{\big(\lambda(n+v-1)+1\big)^2}{4}\cdot\frac{du^2}{uu}$$

or

$$\frac{ddP}{P} = \alpha\alpha\lambda\lambda aau^{2\lambda-2} - \alpha\lambda\lambda(n+v)\alpha u^{\lambda-2}du^2 + \frac{\lambda\lambda(n+v-1)^2-1}{4}\cdot\frac{du^2}{uu}\,,$$

hence

$$na + \frac{Q}{dQ^2}\cdot\frac{ddP}{P} = \frac{1}{2}\alpha aau^{\lambda} + \frac{1}{2}(n-v\ )a + \frac{\lambda\lambda(n+v-1)^2-1}{8\alpha\lambda\lambda}u^{-\lambda}$$

and

$$N = e^{\alpha au^{\lambda}} u^{\frac{-\lambda(n+\nu-1)-1}{2}} \left(\tfrac{1}{2}\alpha aau^{\lambda} + \tfrac{1}{2}(n-\nu)a + \frac{\lambda\lambda(n+\nu-1)^2-1}{8\alpha\lambda\lambda} u^{-\lambda} \right).$$

COROLLARY 3

1043. Hence there shall be

$$\frac{N}{L} = -2\alpha\lambda\lambda u^{\lambda-2} \left(\tfrac{1}{2}\alpha aau^{\lambda} + \tfrac{1}{2}(n-\nu)a + \frac{\lambda\lambda(n+\nu-1)^2-1}{8\alpha\lambda\lambda} u^{-\lambda} \right)$$

and the integral of this equation

$$\frac{ddy}{du^2} = y\left(\alpha\alpha\lambda\lambda aau^{2\lambda-2} + \alpha\lambda\lambda(n-\nu)au^{\lambda-2} + \frac{\lambda\lambda(n+\nu-1)^2-1}{4uu} \right)$$

is

$$y = e^{-\alpha au^{\lambda}} u^{\frac{\lambda(n+\nu-1)+1}{2}} \int e^{2\alpha u^{\lambda} x} x^{n-1} dx (a-x)^{\nu-1}.$$

We can put

$$\alpha = \tfrac{1}{\lambda},\ \ \lambda(n-\nu) = f \ \text{ or } \ \nu = n - \tfrac{f}{\lambda} \ \text{ and } \ \frac{\lambda\lambda(n+\nu-1)^2-1}{4} = g,$$

from which there becomes

$$n = \frac{f+\lambda+\sqrt{(1+4g)}}{2\lambda} \ \text{ and } \ \nu = \frac{-f+\lambda+\sqrt{(1+4g)}}{2\lambda},$$

and the integral of this equation

$$ddy = ydu^2\left(aau^{2\lambda-2} + afu^{\lambda-2} + gu^{-2}\right)$$

is

$$y = e^{\frac{-\alpha}{\lambda}u^{\lambda}} u^{\frac{\lambda(n+\nu-1)+1}{2}} \int e^{\frac{2x}{\lambda}u^{\lambda}} x^{n-1} dx (a-x)^{\nu-1}$$

or

$$y = e^{\frac{-\alpha}{\lambda}u^{\lambda}} u^{\frac{1+\sqrt{(1+4g)}}{2}} \int e^{\frac{2x}{\lambda}u^{\lambda}} x^{\frac{f-\lambda+\sqrt{(1+4g)}}{2\lambda}} dx (a-x)^{\frac{-f-\lambda+\sqrt{(1+4g)}}{2\lambda}}.$$

COROLLARY 4

1044. If we put

$$\alpha = \tfrac{-1}{\lambda},\ \lambda(n-v) = -f \quad \text{and} \quad \tfrac{\lambda\lambda(n+v-1)^2-1}{4} = g,,$$

there will be

$$n = \tfrac{-f+\lambda+\sqrt{(1+4g)}}{2\lambda} \quad \text{and} \quad v = \tfrac{f+\lambda+\sqrt{(1+4g)}}{2\lambda},$$

from which the integral of this equation, which as it agrees with the preceding,

$$ddy = ydu^2\left(aau^{2\lambda-2} + afu^{\lambda-2} + gu^{-2}\right)$$

will be

$$y = e^{\frac{\alpha}{\lambda}u^{\lambda}} u^{\frac{1+\sqrt{(1+4g)}}{2}} \int e^{\frac{-2x}{\lambda}u^{\lambda}} x^{\frac{-f-\lambda+\sqrt{(1+4g)}}{2\lambda}} dx(a-x)^{\frac{f-\lambda+\sqrt{(1+4g)}}{2\lambda}},$$

where it is necessary that $n > 0$ and $v > 0$.

EXEMPLUM 5

1045. *If we put*

$$y = \int dx(aa-xx)^{v-1}\cos.\alpha u^{\lambda}x$$

on putting $x = a$ *after integrating, so that y is equal to a certain function of* u, *to find the second order differential equation which satisfies this condition.*

Since there shall be

$$\frac{dy}{du} = -\alpha\lambda u^{\lambda-1}\int xdx(aa-xx)^{v-1}\sin.\alpha u^{\lambda}x$$

and

$$\frac{ddy}{du^2} = \int xdx(aa-xx)^{v-1}\left(-\alpha\lambda(\lambda-1)u^{\lambda-2}\sin.\alpha u^{\lambda}x - \alpha\alpha\lambda\lambda u^{2\lambda-2}x\cos.\alpha u^{\lambda}x\right),$$

hence there shall be

$$\frac{Lddy}{du^2} + \frac{Mdy}{du} + Ny$$
$$= \int dx(aa-xx)^{v-1}\left\{\begin{matrix} N\cos.\alpha u^{\lambda}x - \alpha\lambda Mu^{\lambda-1}x\sin.\alpha u^{\lambda}x - \alpha\lambda(\lambda-1)Lu^{\lambda-2}x\sin.\alpha u^{\lambda}x \\ -\alpha\alpha\lambda\lambda Lu^{2\lambda-2}xx\cos.\alpha u^{\lambda}x \end{matrix}\right\}.$$

An integral equal to $(aa-xx)^{\nu}\sin.au^{\lambda}x$ is put in place, which vanishes both on putting $x=0$ as well as $x=a$, and there is found on comparing with what is in place :

$$L=\tfrac{u^{-\lambda+2}}{\alpha\lambda\lambda},\quad M=\tfrac{2\lambda\nu-\lambda+1}{\alpha\lambda\lambda}u^{-\lambda+1},\quad N=\alpha aau^{\lambda}.,$$

Whereby the integral of this equation

$$\tfrac{ddy}{du^{2}}+(2\lambda\nu-\lambda+1)\tfrac{dy}{udu}+\alpha\alpha\lambda\lambda aau^{2\lambda-2}y=0$$

is

$$y=\int dx(aa-xx)^{\nu-1}\cos.\alpha u^{\lambda}x.$$

COROLLARIUM 1

1046. Therefore if there shall be $\nu=\tfrac{\lambda-1}{2\lambda}$ and $\alpha=\tfrac{1}{\lambda}$, then the integral of this equation

$$\tfrac{ddy}{du^{2}}+aau^{2\lambda-2}y=0$$

is

$$y=\int dx(aa-xx)^{\frac{-\lambda-1}{2\lambda}}\cos.\tfrac{1}{\lambda}u^{\lambda}x,$$

if indeed after the integration $x=a$ is put in place, with the integral thus taken, so that it vanishes on putting $x=0$.

COROLLARY 2

1047. Therefore if there shall be $\tfrac{-\lambda-1}{2\lambda}=i$ with a whole number or $\lambda=\tfrac{-1}{2i+1}$, the integral of which equation

$$ddy+aau^{\frac{-4i-4}{2i+1}}ydu^{2}=0$$

is

$$y=\int dx(aa-xx)^{i}\cos.\tfrac{1}{\lambda}u^{\lambda}x,$$

which in fact can be shown. Clearly they give rise to the integrable cases indicated above [§ 951].

SCHOLIUM

1048. Since we have put in place $y=\int Vdx$ for some function V of u and x present, but of which in this integration, only x has been treated as a variable , thus there is no need to determine the integral absolutely, in order that it vanishes on putting $x=0$, but it is sufficient so that certainly it vanishes in a certain case $x=b$, with which done if again there is put $x=a$, so that y is equal to some function of u, in order that it can be assigned by quadratures, since here we may postulate justly the integration of simple formulas to be conceded. And this value of y in terms of u expresses the integral of a certain second order differential equation

$$Lddy+Mdudy+Nydu^2=Udu^2,$$

but where it is necessary, that this expression

$$\int dx\left(L\left(\tfrac{ddV}{du^2}\right)+M\left(\tfrac{dV}{du}\right)+NV\right)$$

actually can be integrated, as likewise thus the integral is taken, so that it vanishes on putting $x=b$, and then indeed on putting $x=a$ it becomes $=U$.

PROBLEM 131

1049. *If P and Q should be functions of x, but K a function of u and there is put*

$$y=\int Pdx\left(K+Q\right)^n$$

with the integral thus taken, so that it vanishes in the case $x=b$, then there is put in place $x=a$, in order that a function of u is produced for y, to find the second order differential equation between y and u, which satisfies that value of y.

SOLUTION

Let $dK=K'du$ and $dK'=K''du$ and on account of $y=\int\left(K+Q\right)^n Pdx$ there will be

$$\tfrac{dy}{du}=\int nK'\left(K+Q\right)^{n-1}Pdx$$

and on differentiating again,

$$\tfrac{ddy}{du^2}=\int\left(nK''\left(K+Q\right)^{n-1}+n\left(n-1\right)K'K'\left(K+Q\right)^{n-2}\right)Pdx,$$

from which, if L, M, N denote functions of u, there will be this expression

$$\frac{Lddy}{du^2}+\frac{Mdy}{du}+Ny$$
$$=\int Pdx(K+Q)^{n-2}\left\{\begin{array}{l}N(K+Q)^2+nMK'(K+Q)\\+nLK''(K+Q)+n(n-1)LK'K'\end{array}\right\}$$
$$=\int Pdx(K+Q)^{n-2}\left\{\begin{array}{l}NKK+nMKK'+nLKK'+n(n-1)LK'K'\\+2NKQ+nMK'Q+nLK''Q+NQQ\end{array}\right\};$$

which since it must be integrable, the integral is put $=R(K+Q)^{n-1}+\text{Const.}$,
thus in order that it vanishes on putting as before $x=b$, where R shall be a function of x only.
The differential [w.r.t. *x*] of this form which is

$$(K+Q)^{n-2}\big(KdR+QdR+(n-1)RdQ\big),$$

is required in order that there shall be

$$\begin{array}{l}\big(NKK+nMKK'+nLKK''+n(n-1)LK'K'\big)Pdx\\+\big(2NK+nMK'+nLK''\big)PQdx+NPQQdx=KdR+QdR+(n-1)RdQ.\end{array}$$

Hence here there must be present terms of two kinds, some clearly free from *u*, others truly influenced by the function *K*, which hence separately will be agreed to be equal. Hence in the end we put

$$\begin{array}{l}NKK+nMKK'+nLKK''+n(n-1)LK'K'=A+\alpha K,\\2NK+nMK'+nLK''=B+\beta K\quad\text{et}\ \ N=C+\gamma K.\end{array}$$

From the two in the first place on removing *M* there is deduced

$$-NKK+n(n-1)LK'K=A+\alpha K-BK-\beta KK,$$

from which on account of $N=C+\gamma K$ there is concluded

$$L=\frac{A+(\alpha-B)K-(\beta-C)KK+\gamma K^3}{n(n-1)K'K'}\ \text{and hence}\ M=\frac{B+\beta K-2NK-nLK''}{nK'},,$$

thus so that from the function *K* the letters *L*, *M* and *N* may be determined, while $A,\ \alpha,\ B,\ \beta,\ C,\ \gamma$ denote some constants.

But now there remains, in order that there may be brought about

$$(A+\alpha K)Pdx+(B+\beta K)PQdx+(C+\gamma K)PQQdx = KdR+QdR+(n-1)RdQ\,,$$

from which there shall be made equal terms of two kinds

$$Pdx(A+BQ+CQQ)=QdR+(n-1)RdQ,$$
$$Pdx(\alpha+\beta Q+\gamma QQ)=dR$$

and thus

$$\frac{A+BQ+CQQ}{\alpha+\beta Q+\gamma QQ}=Q+\frac{(n-1)RdQ}{dR}$$

or

$$\frac{(n-1)RdQ}{dR}=\frac{A+(B-\alpha)Q+(C-\beta)QQ-\gamma Q^3}{\alpha+\beta Q+\gamma QQ}$$

hence

$$\frac{dR}{R}=\frac{(n-1)dQ(\alpha+\beta Q+\gamma QQ)}{A+(B-\alpha)Q+(C-\beta)QQ-\gamma Q^3}\,,$$

from which the function *R* is defined from the function *Q*; then indeed there shall be

$$Pdx=\frac{(n-1)RdQ}{A+(B-\alpha)Q+(C-\beta)QQ-\gamma Q^3}$$

Now that integral $R(K+Q)^{n-1}+\text{Const.}$ may be changed into the function *U* on putting $x=a$ and the initial value assumed, then

$$y=\int\frac{(n-1)RdQ(K+Q)^n}{A+(B-\alpha)Q+(C-\beta)QQ-\gamma Q^3}$$

will be the integral of this second order differential equation

$$Lddy+Mdudy+Nydu^2=Udu^2.$$

COROLLARIUM 1

1050. Since some function of *x* can be taken for *Q* , nothing stands in the way, why we should not take *Q* = *x*. Therefore it is required then to find *R* from this equation

$$\frac{dR}{R}=\frac{(n-1)dx(\alpha+\beta x+\gamma xx)}{A+(B-\alpha)x+(C-\beta)xx-\gamma x^3}$$

and there will be taken for *K* some function of *u*

$$y=(n-1)\int\frac{Rdx(K+x)^{n}}{A+(B-\alpha)x+(C-\beta)xx-\gamma x^{3}},$$

in which with the integral thus taken, so that it vanishes on putting $x=b$, then there must be put in place $x=a$.

COROLLARIUM 2

1051. But from the function *K* a second order differential equation is formed, so that there shall be

$$L=\frac{A-(B-\alpha)K+(C-\beta)KK+\gamma K^{3}}{du^{2}n(n-1)dK^{2}},$$

$$M=\frac{B-(2C-\beta)K-2\gamma KK}{ndK}du-\frac{LddK}{dudK} \text{ and } N=C+\gamma K.$$

Thereupon in the expression $R(K+x)^{n-1}+\text{Const.}$ thus put in place, so that it vanishes on putting $x=b$, there is put $x=a$ and the function of *u* resulting from this is called *U* and there shall be the second order differential equation

$$Lddy+Mdudy+Nydu^{2}=Udu^{2}.$$

COROLLARY 3

1052. If the expression $R(K+x)^{n-1}+\text{Const.}$ shall be constructed thus, so that it vanishes either in the case $x=b$ or $x=a$, or rather these limits of integration thus are put in place, so that this comes about, that the formula for *y* assumed satisfies this equation

$$Lddy+Mdudy+Nydu^{2}=0;$$

which if it may be transformed henceforth into other forms, the integrals of these will be assigned also.

PROBLEM 132

1053. *If there should be the functions P, Q of x, but K a function of u and there is put*

$$y=\int e^{KQ}Pdx$$

thus with the integral taken, so that it vanishes in the case $x=b$, *then there is put* $x=a$, *y will become equally a function of u, which satisfies a certain second order differential equation, which it is required to find.*

SOLUTION

Since there shall be $y=\int e^{KQ}Pdx$, then there will be

$$\frac{dy}{du}=\int e^{KQ}K'PQdx \text{ and } \frac{ddy}{du^{2}}=\int e^{KQ}Pdx\left(K''Q+K'K'QQ\right),,$$

from which there becomes

$$\frac{Lddy}{du^2}+\frac{Mdy}{dy}+Ny=\int e^{KQ}Pdx\left(N+MK'Q+LK''Q+LK'K'QQ\right),$$

the integral of which is put in place $e^{KQ}R+\text{Const.}$, which expression vanishes on putting $x=b$, and there is required to become [on diff. w.r.t. *x*]

$$dR+KRdQ=Pdx\left(N+\left(MK'+LK''\right)Q+LK'K'QQ\right)$$

and on account of the reasons raised before [§ 1049] we can make

$$LK'K'=A+\alpha K,\ MK'+LK''=B+\beta K,\ N=C+\gamma K$$

and there will be

$$L=\frac{A+\alpha K}{K'K'} \text{ and } M=\frac{B+\beta K}{K'}-\frac{LK''}{K'}$$

and we obtain these equations

$$dR=Pdx\left(C+BQ+AQQ\right),\quad RdQ=Pdx\left(\gamma+\beta Q+\alpha QQ\right),$$

from which there is deduced

$$\frac{dR}{R}=\frac{dQ\left(C+BQ+AQQ\right)}{\gamma+\beta Q+\alpha QQ},$$

and with the function *R* found, there will be

$$Pdx=\frac{RdQ}{\gamma+\beta Q+\alpha QQ},$$

thus in order that

$$y=\int e^{KQ}\frac{RdQ}{\gamma+\beta Q+\alpha QQ}.$$

If now on putting $x=a$ the expression $e^{KQ}R+\text{Const.}$ changes into the function *U*, the second order differential equation, with which this integral agrees, will be

$$Lddy+Mdudy+Nydu^2=Udu^2.$$

COROLLARY 1

1054. Here for Q it is permissible to write *x* as before, from which there becomes

$$\frac{dR}{R}=\frac{dx(C+Bx+Axx)}{\gamma+\beta x+\alpha xx} \text{ and } y=\int e^{Kx}\frac{Rdx}{\gamma+\beta x+\alpha xx},$$

and *U* arises from the form $e^{Kx}R+\text{Const.}$ on putting $x=a$. Moreover the value of R by reason of the coefficients α,β,γ is able to adopt various forms.

COROLLARY 2

1055. But some function of *u* can be taken for *K*, on which the innate form of the second order differential equation depends. But there will be

$$L=\frac{A+\alpha K}{dK^2}du^2,\ M=\frac{B+\beta K}{dK}du-\frac{(A+\alpha K)duddK}{dK^3} \text{ and } N=C+\gamma K,$$

from which the second order differential equation is

$$\frac{(A+aK)ddy}{dK^2}+\frac{(B+\beta K)dy}{dK}-\frac{(A+aK)ddKdy}{dK^3}+(C+\gamma K)y=U.$$

COROLLARY 3

1056. Since here also *u* departs from the calculation, in the same way a function of such a kind can be assumed for *K*; then why not also without further detriment put $K=u$, provided an account of the element is considered, which is assumed constant [*i.e.* $dK = du$ is constant].

SCHOLIUM 1

1057. Hence if there is taken $K=u$ and the element *du* is assumed constant, so that there becomes $ddK=0$, hence this equation can be constructed

$$\frac{(A+\alpha u)ddy}{du^2}+\frac{(B+\beta u)dy}{du}+(C+\gamma u)y=U$$

with *U* a function of this kind present, as we have described. But in a similar manner from the preceding problem [§1051] it is possible to construct this equation :

$$\left(A-(B-\alpha)u+(C-\beta)uu+\gamma u^3\right)\frac{ddy}{du^2}+(n-1)\left(B-(2C-\beta)u-2\gamma uu\right)\frac{dy}{du}$$
$$+n(n-1)(C+\gamma u)y=U,$$

which equally can be considered more widely, and if we should write some function of u in place of K . Hence indeed in place of u on writing some function of t and on taking dt for the constant [element] all these forms can be derived. From which this equation can be extended wider than that, which we have resolved generally by an infinite series above [§ 967, 992]. But generally these equations are to be prepared thus, so that the integration of these by other methods cannot be performed, on account of which this method is considered entirely worthy, to the further development of which all the strengths of the Geometers may be exerted.

SCHOLIUM 2

1058. Thus I have applied myself to the investigation of constructions of this kind, so that initially as if by conjecture a certain differential formula $\int Vdx = y$, in which V was a certain function of u and x, and where u was seen as a constant, and from that value of x given I have come upon a second order differential equation between u and y, which satisfies that formula assumed. But here it is to be observed that the integral formula evidently does not depend on our choice, but must be defined from the innate character provided, so that from the development made the calculation leads to a differential equation of the second order. But as long as we have been allowed to make this choice, very few formulas come to mind, which lead to the proposed goal proposed, and many less can be hoped for, so that in this way at any time we can arrive at a given second order differential equation, and chiefly they are to be attributed to the cases that may be considered in the constructions which we have treated here. Therefore since at this stage we may be far removed from a solution to the problem, in which a certain second order differential equation is sought from a proposed formula supporting an integration from that formula, because a problem or the solution at any time should be forthcoming, may be considered very uncertain, for that more work is to be summoned, so that at least for particular cases we may attempt to derive the nature of the formulas of the integrands of the proposed equation, and thus in some manner we may prepare a way to the direct solution. Moreover towards this end, infinite series can be usefully summoned, by which above we have taught how to resolve equations of this kind; from which in the following chapter, I set out the solution of any second order differential equation from an infinite series containing that integral formula to be investigated.

CAPUT X

DE CONSTRUCTIONE AEQUATIONUM DIFFERENTIO-DIFFERENTIALIUM PER QUADRATURAS CURVARUM

PROBLEMA 129

1017. *Si fuerit* $y=\int Vdx$ *denotante V functionem quamcunque binarum quantitatum x et u, quarum autem haec u in integratione ut constans spectatur, post integrationem vero statuatur x = a, ut y aequetur functioni cuidam ipsius u, quodsi iam u variabilis sumatur, investigare valorem ipsius* $\frac{dy}{du}$.

SOLUTIO

Cum $\int Vdx$ exhibeat functionem quandam binarum quantitatum x et u, cuius differentiale sumta u constante est $=Vdx$, si tam u quam x ut variabiles tractentur, differentiate aequationis $y=\int Vdx$ talem habebit formam $dy=Vdx+Udu$; quae quia est differentiale verum, necesse est [§ 443] sit $\left(\frac{dV}{du}\right)=\left(\frac{dU}{dx}\right)$. At cum V sit functio data ipsarum x et u, ponatur $dV=Pdx+Qdu$ eritque $\left(\frac{dV}{du}\right)=Q$ ideoque $\left(\frac{dU}{dx}\right)=Q$. Hinc considerata iterum u ut constante erit $dU=Qdx$ et $U=\int Qdx$, in qua integratione sola x pro variabili habetur. Quocirca si hunc valorem $\int Qdx$ ut cognitum spectemus, quippe quem per quadraturas assignare licet, erit $dy= \ Vdx+du\int Qdx$. Quaerimus autem id ipsius y differentiale, quod ex variabilitate ipsius u tantum nascitur; quod cum sit $dy=du\int Qdx$, erit valor quaesitus $\frac{dy}{du}=\int Qdx$, si nempe post integrationem itidem ponatur $x=a$.

COROLLARIUM 1

1018. Cum sit $y=\int Vdx$ functio ipsarum x et u, per integrationem autem formulae Vdx, in qua u constans spectatur, functio quaecunque ipsius u loco constantis accedere possit, functio y per se erit indeterminata, determinabitur autem, statim atque integrale $\int Vdx$ ita accipiatur, ut evanescat posito $x=0$.

COROLLARIUM 2

1019. Hac conditione observata evanescet *y* posito $x=0$, quicunque valor alteri quantitati *u* tribuatur; erit ergo etiam $y+du\left(\frac{dy}{du}\right)=0$ facto $x=0$, ergo etiam $\left(\frac{dy}{du}\right)=0$. Unde patet $\int Qdx=\frac{dy}{du}$ ita quoque accipi debere, ut posito $x=0$ evanescat.

COROLLARIUM 3

1020. Cum $y=\int Vdx$, erit $\left(\frac{dy}{dx}\right)=V$, hinc $\left(\frac{ddy}{dudx}\right)=\left(\frac{dV}{du}\right)$. At si ponatur $\left(\frac{dy}{du}\right)=Z$, erit quoque $\left(\frac{ddy}{dudx}\right)=\left(\frac{dZ}{dx}\right)$, ergo $\left(\frac{dZ}{dx}\right)=\left(\frac{dV}{du}\right)$. Quare spectata *u* ut constante erit $dZ=dx\left(\frac{dV}{du}\right)$ et $Z=\int dx\left(\frac{dV}{du}\right)$ ideoque $\left(\frac{dy}{du}\right)=\int dx\left(\frac{dV}{du}\right)$.

COROLLARIUM 4

1021. Quodsi ergo post integrationes ita absolutas, ut integralia evanescant posito $x=0$, ponatur $x=a$, tam valor $y=\int Vdx$ quam $\left(\frac{dy}{du}\right)=\int dx\left(\frac{dV}{du}\right)$ erit functio determinata ipsius *u*.

COROLLARIUM 5

1022. Simili modo ulterius progrediendo erit $\frac{ddy}{du^2}=\int dx\left(\frac{ddV}{du^2}\right)$. Quare si *L, M* et *N* denotent functiones quascunque ipsius *u*, erit

$$\frac{Lddy}{du^2}+\frac{Mdy}{du}+Ny=\int dx\left(L\left(\frac{ddV}{du^2}\right)+M\left(\frac{dV}{du}\right)+NV\right)$$

totumque negotium huc redit, ut ista formula integrationem admittat.

SCHOLION

1023. Datis scilicet ipsius *u* functionibus *L, M, N* quaeri debet functio *V* binarum variabilium *x* et *u*, ita ut spectata *u* constante formula

$$\left(L\left(\frac{ddV}{du^2}\right)+M\left(\frac{dV}{du}\right)+NV\right)dx$$

absolute fiat integrabilis; cuius integrale ut sit determinatum, ita capiatur, ut posito $x=0$ evanescat. Tum vero statuatur $x=a$, ac si illud integrale etiam hoc casu evanescat, erit

$$\frac{Lddy}{du^2}+\frac{Mdy}{du}+Ny=0$$

hincque aequationi satisfacit valor $y=\int Vdx$ lege indicata sumtus. Problema autem datis functionibus *L, M* et *N* investigandi functionem *V* maxime est indeterminatum neque methodis adhuc cognitis in genere resolvi potest; ex quo conveniet id inverso modo tractari, ut sumta functione *V* alterae *L, M*et *N* indagentur. Hinc aequationes differentio-differentiales consequemur, quarum integralia hoc modo assignare valebimus, quae si aliis methodis tractari nequeant, insigne lucrum suppeditant. Quodsi integrale illud

$$\int\left(L\left(\frac{ddV}{du^2}\right)+M\left(\frac{dV}{du}\right)+NV\right)dx$$

posito $x=a$ non evanescat, sed datam ipsius *u* functionem *U* exhibeat, valor $y=\int Vdx$ conveniet huic aequationi

$$L\left(\frac{ddy}{du^2}\right)+M\left(\frac{dy}{du}\right)+Ny=U\ ,$$

quae cum infinitis modis in alias formas transmutari possit, etiam harum integralia innotescent, ubi simul hoc commode evenit, ut, etiamsi integrale tantum particulare obtineatur, inde tamen plerumque integrale completum haud difficulter colligi queat.

PROBLEMA 130

1024. *Invenire aequationes differentio-differentiales formae*

$$\frac{Lddy}{du^2}+\frac{Mdy}{du}+Ny=U\ ,$$

ut L, M, N et U sint functiones ipsius u, euius elementum du hic pro constante accipitur, quarum integrale ope constructionis per quadraturas exhiberi possit.

SOLUTIO

Sumatur functio quaecunque binarum variabilium *u* et *x*, quae sit *V*, capiaturque integrale $\int Vdx$ spectata quantitate *u* ut constante ita, ut posito $x=0$ evanescat, tum vero fiat $x=a$ denotante *a* quantitatem quamcunque constantem, ut iam $\int Vdx$ exprimat functionem quandam ipsius *u* tantum, cui quantitas *y* aequetur, ut sit $y=\int Vdx$. Cum iam sit

$$\frac{dy}{du}=\int dx\left(\frac{dV}{du}\right)\ \text{ et }\ \frac{ddy}{du^2}=\int dx\left(\frac{ddV}{du^2}\right)$$

his integralibus pariter ita sumtis, ut posito $x=0$ evanescant, tum vero statuatur $x=a$, quaerantur functiones *L, M, N* ipsius *u*, ut haec formula

$$\int dx\left(L\left(\frac{ddV}{du^2}\right)+M\left(\frac{dV}{du}\right)+NV\right)$$

fiat absolute integrabilis, eiusque integrale ita determinetur, ut posito $x = a$ fiat id $= U$. Quod si fuerit praestitum, evidens est aequationi differentio-differentiali

$$\frac{Lddy}{du^2} + \frac{Mdy}{du} + Ny = U$$

satisfacere formulam assumtam $y = \int V dx$.

COROLLARIUM 1

1025. Assumtio ergo functionis V non penitus arbitrio nostro permittitur, sed ad hoc potissimum est spectandum, ut talis forma

$$\int dx \left(L\left(\frac{ddV}{du^2}\right) + M\left(\frac{dV}{du}\right) + NV \right)$$

per se fiat integrabilis.

COROLLARIUM 2

1026. Infinitae ergo hinc statim excluduntur formae ad hunc scopum ineptae, euiusmodi sunt $V = UP$ existente U functione ipsius u et P ipsius x tantum, quia tum foret

$$y = U\int Pdx, \ \frac{dy}{du} = \frac{dU}{du}\int Pdx \ \text{ et } \ \frac{ddy}{du^2} = \frac{ddU}{du^2}\int Pdx,$$

quippe quae idem integrale complectuntur, ita ut ex earum coniunctione formula absolute integrabilis confici nequeat.

EXEMPLUM 1

1027. *Sit* $V = x^n\sqrt{\frac{uu+xx}{cc-xx}}$ *et* $y = \int dx x^n\sqrt{\frac{uu+xx}{cc-xx}}$ *integrali evanescente posito* $x = 0$, *tum vero facto* $x = a$.

Erit ergo

$$\left(\frac{dV}{du}\right) = x^n \cdot \frac{u}{\sqrt{(uu+xx)(cc-xx)}} \quad \text{et} \quad \left(\frac{ddV}{du^2}\right) = x^n \cdot \frac{xx}{(uu+xx)^{\frac{3}{2}}\sqrt{(cc-xx)}}$$

et integrabilis reddi debet haec formula

$$x^n dx\left(\frac{Lxx}{(uu+xx)^{\frac{3}{2}}\sqrt{(cc-xx)}} + \frac{Mu}{\sqrt{(uu+xx)(cc-xx)}} + N\sqrt{\frac{uu+xx}{cc-xx}} \right)$$

seu

$$\frac{x^n dx}{(uu+xx)^{\frac{3}{2}}\sqrt{(cc-xx)}}\left(Lxx+Mu(uu+xx)+N(uu+xx)^2\right).$$

Statuatur integrale

$$\frac{x^{n+1}\sqrt{(cc-xx)}}{\sqrt{(uu+xx)}}\ ;$$

euius differentiale cum sit

$$\frac{(n+1)x^n(cc-xx)(uu+xx)-x^{n+2}(uu+xx)-x^{n+2}(cc-xx)}{(uu+xx)^{\frac{3}{2}}\sqrt{(cc-xx)}}dx$$

seu

$$\frac{x^n dx}{(uu+xx)^{\frac{3}{2}}\sqrt{(cc-xx)}}\left\{\begin{matrix}(n+1)ccuu+(n+1)ccxx-(n+1)uuxx-(n+1)x^4\\ -ccxx- \qquad uuxx\end{matrix}\right\},$$

cum qua si proposita comparetur, fiet

$$Mu^3+Nu^4=(n+1)ccuu,\ \ L+Mu+2Nuu=ncc-(n+2)uu$$

et

$$N=-(n+1).$$

Hinc elicitur

$$Mu=(n+1)(cc+uu)\ \text{ seu }\ M=\frac{(n+1)(cc+uu)}{u}$$

et

$$L=-(n+1)(cc+uu)+2(n+1)uu+ncc-(n+2)uu\ \text{ seu }\ L=-cc-uu\,.$$

Quamobrem habebimus

$$-\frac{(cc+uu)ddy}{du^2}+\frac{(n+1)(cc+uu)dy}{udu}-(n+1)y=\frac{a^{n+1}\sqrt{(cc-aa)}}{\sqrt{(aa+uu)}},$$

cui aequationi satisfacit

$$y=\int x^n dx\sqrt{\frac{uu+xx}{cc-xx}}$$

integratione absoluta, ut est indicatum.

COROLLARIUM 1

1028. Sumto ergo $a=c$ formula integralis $y=\int x^n dx\sqrt{\frac{uu+xx}{cc-xx}}$ posito post integrationem $x=c$ exhibebit integrale huius aequationis

$$u(cc+uu)ddy-(n+1)(cc+uu)dudy+(n+1)uydu^2=0$$

seu

$$ddy-\frac{(n+1)dudy}{u}+\frac{(n+1)ydu^2}{cc+uu}=0.$$

COROLLARIUM 2

1029. Si sit $n=1$, per integrationem invenitur

$$\int xdx\sqrt{\frac{uu+xx}{cc-xx}}=\frac{1}{4}(cc+uu)\text{Ang.sin.}\frac{2xx-cc+uu}{cc+uu}$$
$$-\frac{1}{2}\sqrt{(ccuu+ccxx-uuxx-x^4)}-\frac{1}{4}(cc+uu)\text{Ang.sin.}\frac{-cc+uu}{cc+uu}+\frac{1}{2}cu$$

et posito $x=c$ fit

$$y=\frac{1}{4}(cc+uu)\text{Ang.cos.}\frac{uu-cc}{cc+uu}+\frac{1}{2}cu$$

hincque

$$\frac{dy}{du}=\frac{1}{2}u\ \text{Ang.cos.}\frac{uu-cc}{cc+uu}\quad\text{et}\quad\frac{ddy}{du^2}=\frac{1}{2}\text{Ang.cos.}\frac{uu-cc}{cc+uu}-\frac{cu}{cc+uu}$$

quae formulae evidenter satisfaciunt aequationi

$$ddy-\frac{2dudy}{u}+\frac{2ydu^2}{cc+uu}=0.$$

COROLLARIUM 3

1030. Hoc casu integrale etiam hoc modo exprimi potest

$$y=\frac{1}{4}(cc+uu)\text{Ang.sin.}\frac{2cu}{cc+uu}+\frac{1}{2}cu\,,$$

seu cum eius multiplum quodvis aeque satisfaciat,

$$y=(cc+uu)\text{Ang.sin.}\frac{2cu}{cc+uu}+2cu\,.$$

Satisfacit vero etiam $y=cc+uu$, unde integrale completum est

$$y=\alpha(cc+uu)\text{Ang.sin.}\frac{2cu}{cc+uu}+2\alpha cu+\beta(cc+uu).$$

SCHOLION

1031. Quod valor $y=cc+uu$ satisfaciat, ex integrali invento concludere licet; quia enim Ang.sin.$\frac{2cu}{cc+uu}$ est functio multiplex et termino 2π augeri potest, integrale ipsum augeri potest termino $2\pi(cc+uu)$. At in genere differentia binorum integralium quoque satisfacit, ergo etiam satisfacere debet $y=2\pi(cc+uu)$ et generatim $y=2\beta(cc+uu)$. Ex hoc casu facilius perspicitur, quomodo valor assumtus aequationi generali satisfaciat, etiamsi is per integrationem evolvi nequeat. Patet autem $n+1$ esse debere numerum positivum, quia alioquin conditio integralis, ut posito $x=0$ evanescat, impleri nequit.

EXEMPLUM 2

1032. *Sumatur* $V=x^{n-1}(uu+xx)^{\mu}(cc-xx)^{\nu}$.

Erit

$$\left(\tfrac{dV}{du}\right)=2\mu ux^{n-1}(uu+xx)^{\mu-1}(cc-xx)^{\nu}$$

et

$$\left(\tfrac{ddV}{du^2}\right)=2\mu x^{n-1}(cc-xx)^{\nu}\left((uu+xx)^{\mu-1}+2(\mu-1)uu(uu+xx)^{\mu-2}\right)$$
$$=2\mu x^{n-1}(cc-xx)^{\nu}(uu+xx)^{\mu-2}\left(2(\mu-1)uu+xx\right).$$

Integrabilis igitur reddi debet absolute haec formula

$$\int x^{n-1}dx(cc-xx)(uu+xx)^{\mu-2}\left\{\begin{matrix}2\mu\left((2\mu-1)uu+xx\right)L\\ +2\mu u(uu+xx)M+(uu+xx)^2N\end{matrix}\right\}$$

seu

$$\int x^{n-1}dx(cc-xx)^{\nu}(uu+xx)^{\mu-2}\left\{\begin{matrix}2\mu(2\mu-1)Luu+2\mu Lxx+Nx^4\\ +2\mu Mu^3+2\mu Muxx\\ Nu^4+2Nuuxx\end{matrix}\right\}.$$

Statuatur integrale $x^n(uu+xx)^{\mu-1}(cc-xx)^{\nu+1}$; cuius differentiale cum sit

$$x^{n-1}dx(uu+xx)^{\mu-2}(cc-xx)^{\nu}$$
$$\times n\left((uu+xx)(cc-xx)+2(\mu-1)xx(cc-xx)-2(\nu+1)xx(uu+xx)\right),$$

erit

$$2\mu(2\mu-1)Luu+2\mu Mu^2+Nu^4=nccuu,$$
$$2\mu L+2\mu Mu+2Nuu=ncc-nuu+2(\mu-1)cc-2(\nu+1)uu,$$
$$N=-n-2(\mu-1)-2(\nu+1)=-n-2\mu-2\nu.$$

At prima $2\mu(2\mu-1)L+2\mu Mu+Nuu=ncc$ demta secunda dat

$$4\mu(\mu-1)L-Nuu=(n+2v+2)uu-2(\mu-1)cc$$

seu $4\mu(\mu-1)L=-2(\mu-1)(uu+cc)$, hinc

$$L=\tfrac{-cc-uu}{2\mu},$$

qui valor in prima substitutus dat

$$-(2\mu-1)(cc+uu)+2\mu Mu-(n+2\mu+2v)uu=ncc$$

seu

$$2\mu Mu=(n+2\mu-1)cc+(n+4\mu\ +2v-1)uu.$$

Ergo

$$M=\tfrac{(n+2\mu-1)(cc+uu)}{2\mu u}+\tfrac{\mu+v}{\mu}u.$$

Si $n>0$, superius integrale evanescit posito $x=0$; quare si ponamus $x=a$, orietur haec aequatio

$$-\tfrac{(cc+uu)ddy}{2\mu du^2}+\tfrac{(n+2\mu-1)(cc+uu)dy}{2\mu udu}+\tfrac{(\mu+v)udy}{\mu du}-(n+2\mu+2v)y$$
$$=a^n(aa+uu)^{\mu-1}(cc-aa)^{v+1},$$

cuius integrale est

$$y=\int x^{n-1}dx(uu+xx)^{\mu}(cc-xx)^{v}$$

integrali hoc ita sumto, ut evanescat posito $x=0$, tum vero posito $x=a$.

COROLLARIUM 1

1033. Si capiatur $a=c$, ut postrema pars fiat $=0$, siquidem exponens $v+1$ sit nihilo maior, formula

$$y=\int x^{n-1}dx(uu+xx)^{\mu}(cc-xx)^{v}$$

posito $x=c$ post integrationem ita peractam, ut casu $x=0$ fiat $y=0$, erit integrale huius aequationis

$$u(cc+uu)ddy-(n+2\mu-1)(cc+uu)dudy-2(\mu+\nu)uududy$$
$$+2\mu(n+2\mu+2\nu)uydu^2=0.$$

COROLLARIUM 2

1034. Sit $n+2\mu-1=\alpha$, $n+4\mu+2\nu-1=\beta$; fiet $2\mu=\alpha+1-n$ et $2\nu=\beta+1-n-2\alpha-2+2n=\beta-1+n-2a$ at aequationis

$$u(cc+uu)ddy-(\alpha cc+\beta uu)dudy+(\alpha+1-n)(\beta-\alpha+n)uydu^2=0$$

integrale erit

$$y=\int x^{n-1}dx(uu+xx)^{\frac{\alpha+1-n}{2}}(cc-xx)^{\frac{\beta-1+n-2\alpha}{2}}$$

posito $x=c$, si modo sit $n>0$ et $\beta+1+n>2\alpha$.

SCHOLlON

1035. Haec constructio latissime ad hanc aequationem patet

$$xx\left(a+bx^n\right)ddz+x\left(c+ex^n\right)dxdz+\left(f+gx^n\right)zdx^2=0.$$

Primo enim hic sine detrimento amplitudinis sumi potest $n=2$ ponendo $x^n=uu$. Tum vero, uti supra § 997 vidimus, ponendo

$$z=x^{\frac{a-c}{a}+h}\left(a+bx^n\right)^{\frac{bc-ae}{nab}+1}y$$

aequatio abit in hanc

$$xx\left(a+bx^n\right)ddy+x\left(2a-c+2ah+(2b-e+2nb+2bh)x^n\right)dxdy$$
$$+\left(f+ah-ch+ahh+\left(g+(b-e+nb+bh)(n+h)\right)x^n\right)ydx^2=0;$$

ubi si h ita accipiatur, ut sit $ahh+(a-c)h+f=0$, prodit aequatio formae, cuius constructionem [§ 1033] dedimus. In casibus autem specialibus difficultates occurrere possunt, quibus superandis sequentia exempla inserviunt.

EXEMPLUM 3

1036. *Sit* $V=e^{mux}x^n(c-x)^\nu$.

Erit

$$\left(\frac{dV}{du}\right)=me^{mux}x^{n+1}(c-x)^\nu \quad \text{et} \quad \left(\frac{ddV}{du^2}\right)=mme^{mux}x^{n+2}(c-x)^\nu.$$

Integrabilem ergo reddi oportet hanc formulam

$$e^{mux}x^n dx(c-x)^\nu (mmLxx+mMx+N),$$

cuius integrale ponatur $=e^{mux}x^{n+1}(c-x)^{\nu+1}$, cuius propterea differentiale illi formulae aequari debet; quod eum sit

$$e^{mux}x^n dx(c-x)^\nu \left(mux(c-x)+(n+1)(c-x)-(\nu+1)x\right),$$

erit

$$N=(n+1)c,\ mM=mcu-(n+\nu+2),\ mmL=-mu.$$

Statuatur nunc $x=a$ et formula

$$y=\int e^{mux}x^n dx(c-x)^\nu$$

erit integrale huius aequationis

$$-\frac{uddy}{mdu^2}+\frac{cudy}{du}-\frac{(n+\nu+2)dy}{mdu}+(n+1)cy=e^{mau}a^{n+1}(c-a)^{\nu+1}.$$

Hic poni potest $m=1$ ac sumto $c=a$ aequationis

$$uddy-aududy+(n+\nu+2)dudy-(n+1)aydu^2=0$$

integrale est

$$y=\int e^{ux}x^n dx(a-x)^\nu$$

posito post integrationem $x=a$, dum sit $\nu+1>0$ et $n+1>0$, ut integrale evanescens reddi possit posito $x=0$.

COROLLARIUM 1

1037. Si hic ponatur $y=e^{\int zdu}$, erit

$$udz+uzzdu-auzdu+(n+\nu+2)zdu-(n+1)adu=0,$$

cuius integrale est

$$z=\frac{dy}{ydu}=\frac{\int e^{ux}x^{n+1}dx(a-x)^\nu}{\int e^{ux}x^n dx(a-x)^\nu}.$$

Illa aequatio autem posito $z=\frac{1}{2}a+v$ transmutatur in hanc

$$udv + uvvdu + (n+v+2)vdu - \tfrac{1}{4}aaudu - \tfrac{1}{2}(n-v)adu = 0,$$

quae ponendo $v = u^{-n-v-2}s$ abit in hanc

$$u^{-n-v-1}ds + u^{-2n-2v-3}ssdu - \tfrac{1}{4}aaudu - \tfrac{1}{2}(n-v)adu = 0.$$

COROLLARIUM 2

1038. Sit porro $u^{-n-v-2}du = dt$ seu $u^{-n-v-1} = -(n+v+1)t$, ut fiat

$$ds + ssdt - \tfrac{1}{4}aau^{2n+2v+4}dt - \tfrac{1}{2}(n-v)u^{2n+2v+3}dt = 0,$$

quae ergo aequatio etiam construi potest. Vel sit $-(n+v+1)t = r$; erit

$$ds - \frac{ssdr}{n+v+1} + \frac{aar^{\frac{-2n-2v-4}{n+v+1}}dr}{4(n+v+1)} + \frac{(n-v)r^{\frac{-2n-2v-3}{n+v+1}}dr}{2(n+v+1)} = 0,$$

quae posito $s = -(n+v+1)q$ abit in

$$dq + qqdr - \frac{aar^{\frac{-2n-2v-4}{n+v+1}}dr - 2(n-v)r^{\frac{-2n-2v-3}{n+v+1}}dr}{4(n+v+1)^2} = 0,$$

hicque est

$$u = r^{\frac{-1}{n+v+1}} \quad \text{et} \quad z = \tfrac{1}{2}a - (n+v+1)r^{\frac{n+v+2}{n+v+1}}q.$$

SCHOLION

1039. Cum aequationis differentio-differentialis [§ 1036]

$$\frac{ddy}{du} - ady + \frac{(n+v+2)dy}{u} - \frac{(n+1)aydu}{u} = 0$$

integrale sit $y = \int e^{ux}x^n dx(a-x)^v$, videamus, quomodo haec ipsa aequatio in alias formas transfundi possit.

Sit primo $u = \alpha t^{\lambda}$ ideoque $du = \alpha\lambda t^{\lambda-1}dt$, unde fit

$$\frac{1}{\alpha\lambda}d.\frac{dy}{t^{\lambda-1}dt} - ady + \frac{(n+v+2)dy}{\alpha t^{\lambda}} - \frac{\lambda(n+1)aydt}{t} = 0 \;;$$

sumatur iam elementum dt constans eritque

$$\frac{ddy}{\alpha\lambda t^{\lambda-1}dt}-\frac{(\lambda-1)dy}{\alpha\lambda t^{\lambda}}-ady+\frac{(n+v+2)dy}{\alpha t^{\lambda}}-\frac{\lambda(n+1)aydt}{t}=0$$

seu

$$ddy-\alpha\lambda at^{\lambda-1}dtdy+\frac{(\lambda n+\lambda v+\lambda+1)dtdy}{t}-\alpha\lambda\lambda(n+1)at^{\lambda-2}ydt^{2}=0,$$

cuius integrale est

$$y=\int e^{\alpha t^{\lambda}x}x^{n}dx(a-x)^{v}.$$

Ponatur porro $\frac{dy}{y}=Pdt+\frac{dz}{z}$, ut sit $z=e^{-\int Pdt}y$; erit [§ 993]

$$ddz+2Pdtdz-\alpha\lambda a^{\lambda-1}dtdz+(\lambda n+\lambda v+\lambda+1)\frac{dtdz}{t}+zdtdP$$
$$+zdt^{2}\left(PP-\alpha\lambda a^{\lambda-1}P+\frac{(\lambda n+\lambda v+\lambda+1)P}{t}-\alpha\lambda\lambda(n+1)at^{\lambda-2}\right)=0.$$

Ad terminos elemento *dz* adfectos tollendos statuatur

$$P=\tfrac{1}{2}\alpha\lambda a^{\lambda-1}-\frac{\lambda n+\lambda v+\lambda+1}{2t}$$

fietque [$z=e^{-\frac{1}{2}\alpha at^{\lambda}}t^{\frac{\lambda n+\lambda v+\lambda+1}{2}}y$] ac prodibit haec aequatio

$$ddz-zdt^{2}\left(\frac{(\lambda n+\lambda v+\lambda+1)^{2}-1}{4tt}+\tfrac{1}{2}\alpha\lambda\lambda(n-v)at^{\lambda-2}+\tfrac{1}{4}\alpha^{2}\lambda^{2}a^{2}t^{2\lambda-2}\right)=0,$$

cuius propterea integrale est

$$z=e^{-\frac{1}{2}\alpha at^{\lambda}}t^{\frac{\lambda n+\lambda v+\lambda+1}{2}}\int e^{\alpha t^{\lambda}x}x^{n}dx(a-x)^{v}.$$

Quodsi ergo sit

$$v=n,\ \lambda\lambda(2n+1)^{2}-1=0 \text{ seu } \lambda=\frac{\pm1}{2n+1} \text{ et } \alpha=\pm\frac{2}{\lambda}=\pm2(2n+1),$$

habebitur haec aequatio

$$ddz-aazt^{\frac{\pm2}{2n+1}-2}dt^{2}=0,$$

cuius integrale est

$$z = e^{\pm(2n+1)\alpha t^{\frac{\pm 1}{2n+1}}} t^{\pm\frac{1}{2}+\frac{1}{2}} \int e^{\pm 2(2n+1) t^{\frac{\pm 1}{2n+1}} x} x^n dx (a-x)^n.$$

Vel huius aequationis

$$ddz - aat^{2\lambda-2} zdt^2 = 0$$

integrale est

$$z = e^{-\frac{\alpha}{\lambda} t^\lambda} t \int e^{\frac{2x}{\lambda} t^\lambda} x^{\pm\frac{1}{2\lambda}-\frac{1}{2}} dx (a-x)^{\pm\frac{1}{2\lambda}-\frac{1}{2}},$$

unde occasionem arripimus huiusmodi integrationes generalius investigandi.

EXEMPLUM 4

1040. *Si sint P et Q functiones quaecunque ipsius u et capiatur*

$$y = P \int e^{Qx} x^{n-1} dx (a-x)^{\nu-1}$$

posito scilicet post integrationem $x = a$ *, erit hic valor ipsius y integrale cuiuspiam aequationis differentio–differentialis*

$$\frac{Lddy}{du^2} + \frac{Mdy}{du} + Ny = 0,$$

quae quaeritur.

Ad calculum contrahendum ponamus $dP = P'du$, $dP' = P''du$, item $dQ = Q'du$ et $dQ' = Q''du$. Hinc erit

$$\frac{dy}{du} = P' \int e^{Qx} x^{n-1} dx (a-x)^{\nu-1} + PQ' \int e^{Qx} x^n dx (a-x)^{\nu-1}$$

et

$$\frac{ddy}{du^2} = P'' \int e^{Qx} x^{n-1} dx (a-x)^{\nu-1} + 2P'Q' \int e^{Qx} x^n dx (a-x)^{\nu-1}$$
$$+2PQ'' \int e^{Qx} x^n dx (a-x)^{\nu-1} + PQ'Q' \int e^{Qx} x^{n+1} dx (a-x)^{\nu-1},$$

unde colligitur

$$\frac{Lddy}{du^2} + \frac{Mdy}{du} + Ny = \int e^{Qx} x^{n-1} dx (a-x)^{\nu-1} \left\{ \begin{array}{l} LP'' + 2LP'Q'x + LPQ''x + LPQ'Q'xx \\ +MP' + MPQ'x + NP \end{array} \right\},$$

quod integrale statuatur $= e^{Qx}x^{n}\left(a-x\right)^{\nu}$, ita ut evanescat posito $x=a$, dum sit $\nu>0$, uti etiam evanescit casu $x=0$, si modo $n>0$. Cum igitur huius formulae differentiale sit

$$e^{Qx}x^{n-1}dx\left(a-x\right)^{\nu-1}\left(Qx\left(a-x\right)+na-\left(n+\nu\right)x\right),$$

cuius comparatio cum forma inventa praebet

$$LP''+MP'+NP=na,$$
$$2LP'Q'+LPQ''+MPQ'=aQ-\left(n+\nu\right) \text{ et } LPQ'Q'=-Q,$$

ergo

$$L=\tfrac{-Q}{PQ'Q''},$$

hinc

$$M=\tfrac{aQ}{PQ'}-\tfrac{n+\nu}{PQ'}+\tfrac{2P'Q}{PPQ'Q'}+\tfrac{QQ''}{PQ'^{3}} \text{ et } N=\tfrac{na}{P}+\tfrac{P''Q}{PPQ'Q'}-\tfrac{MP'}{P};$$

sicque aequatio differentio–differentialis erit cognita.

COROLLARIUM 1

1041. Si velimus, ut sit $M=0$, erit

$$aQ-\left(n+\nu\right)+\tfrac{2P'Q}{PQ'}+\tfrac{QQ''}{Q'Q'}=0,$$

quae per $\tfrac{Q'du}{Q}$ multiplicata abit in hanc

$$\tfrac{2dP}{P}+adQ-\tfrac{\left(n+\nu\right)dQ}{Q}+\tfrac{dQ'}{Q'}=0,$$

cuius integrale est

$$\tfrac{e^{aQ}P^{2}Q'}{Q^{n+\nu}}=\text{Const. sive } P=Ce^{-\frac{1}{2}\alpha Q}Q^{\frac{n+\nu}{2}}\sqrt{\tfrac{du}{dQ}}.$$

COROLLARIUM 2

1042. Sit $Q=2\alpha u^{\lambda}$, erit $Q'=2\alpha\lambda u^{\lambda-1}$ et $P=Ce^{-\alpha au^{\lambda}}u^{\frac{\lambda(n+\nu-1)+1}{2}}$, hinc [sumto $C=1$]

$$L=-\tfrac{1}{2\alpha\lambda\lambda}e^{\alpha\alpha u^{\lambda}}u^{\frac{-\lambda(n+\nu+1)+3}{2}} \text{ et } N=\tfrac{na}{P}+\tfrac{QddP}{PPdQ^{2}}.$$

At est

$$\frac{Q}{dQ^2}=\frac{u^{-\lambda+2}}{2\alpha\lambda\lambda du^2}$$

et ob

$$\frac{dP}{P}=-\alpha\lambda au^{\lambda-1}du+\frac{\lambda(n+v-1)+1}{2}\cdot\frac{du}{u}$$

erit

$$\frac{ddP}{P}=-\alpha\lambda(\lambda-1)au^{\lambda-2}du^2-\frac{\lambda(n+v-1)+1}{2}\cdot\frac{du^2}{uu}+\alpha\alpha\lambda\lambda aau^{2\lambda-2}du^2$$
$$-\alpha\lambda a\left(\lambda(n+v-1)+1\right)u^{\lambda-2}du^2+\frac{\left(\lambda(n+v-1)+1\right)^2}{4}\cdot\frac{du^2}{uu}$$

seu

$$\frac{ddP}{P}=\alpha\alpha\lambda\lambda aau^{2\lambda-2}-\alpha\lambda\lambda(n+v)\alpha u^{\lambda-2}du^2+\frac{\lambda\lambda(n+v-1)^2-1}{4}\cdot\frac{du^2}{uu},$$

hinc

$$na+\frac{Q}{dQ^2}\cdot\frac{ddP}{P}=\frac{1}{2}\alpha aau^{\lambda}+\frac{1}{2}(n-v\ )a+\frac{\lambda\lambda(n+v-1)^2-1}{8\alpha\lambda\lambda}u^{-\lambda}$$

et

$$N=e^{aau^{\lambda}}u^{\frac{-\lambda(n+v-1)-1}{2}}\left(\frac{1}{2}\alpha aau^{\lambda}+\frac{1}{2}(n-v)a+\frac{\lambda\lambda(n+v-1)^2-1}{8\alpha\lambda\lambda}u^{-\lambda}\right).$$

COROLLARIUM 3

1043. Hinc erit

$$\frac{N}{L}=-2\alpha\lambda\lambda u^{\lambda-2}\left(\frac{1}{2}\alpha aau^{\lambda}+\frac{1}{2}(n-v)a+\frac{\lambda\lambda(n+v-1)^2-1}{8\alpha\lambda\lambda}u^{-\lambda}\right)$$

et huius aequationis

$$\frac{ddy}{du^2}=y\left(\alpha\alpha\lambda\lambda aau^{2\lambda-2}+\alpha\lambda\lambda(n-v)au^{\lambda-2}+\frac{\lambda\lambda(n+v-1)^2-1}{4uu}\right)$$

integrale est

$$y=e^{-\alpha au^{\lambda}}u^{\frac{\lambda(n+v-1)+1}{2}}\int e^{2\alpha u^{\lambda}x}x^{n-1}dx(a-x)^{v-1}.$$

Ponamus

$$\alpha=\frac{1}{\lambda},\ \ \lambda(n-v)=f\ \text{ seu }\ v=n-\frac{f}{\lambda}\ \text{ et }\ \frac{\lambda\lambda(n+v-1)^2-1}{4}=g,$$

unde fit

$$n=\frac{f+\lambda+\sqrt{(1+4g)}}{2\lambda}\ \text{ et }\ v=\frac{-f+\lambda+\sqrt{(1+4g)}}{2\lambda},$$

et huius aequationis

$$ddy=ydu^2\left(aau^{2\lambda-2}+afu^{\lambda-2}+gu^{-2}\right)$$

integrale est

$$y = e^{\frac{-\alpha}{\lambda}u^{\lambda}} u^{\frac{\lambda(n+\nu-1)+1}{2}} \int e^{\frac{2x}{\lambda}u^{\lambda}} x^{n-1} dx (a-x)^{\nu-1}$$

seu

$$y = e^{\frac{-\alpha}{\lambda}u^{\lambda}} u^{\frac{1+\sqrt{(1+4g)}}{2}} \int e^{\frac{2x}{\lambda}u^{\lambda}} x^{\frac{f-\lambda+\sqrt{(1+4g)}}{2\lambda}} dx (a-x)^{\frac{-f-\lambda+\sqrt{(1+4g)}}{2\lambda}}.$$

COROLLARIUM 4

1044. Si ponamus

$$\alpha = \frac{-1}{\lambda},\ \ \lambda(n-\nu) = -f \quad \text{et} \quad \frac{\lambda\lambda(n+\nu-1)^2 - 1}{4} = g,,$$

erit

$$n = \frac{-f+\lambda+\sqrt{(1+4g)}}{2\lambda} \quad \text{et} \quad \nu = \frac{f+\lambda+\sqrt{(1+4g)}}{2\lambda},$$

unde huius aequationis, quae cum praecedente convenit,

$$ddy = ydu^2\left(aau^{2\lambda-2} + afu^{\lambda-2} + gu^{-2}\right)$$

integraIe erit

$$y = e^{\frac{\alpha}{\lambda}u^{\lambda}} u^{\frac{1+\sqrt{(1+4g)}}{2}} \int e^{\frac{-2x}{\lambda}u^{\lambda}} x^{\frac{-f-\lambda+\sqrt{(1+4g)}}{2\lambda}} dx (a-x)^{\frac{f-\lambda+\sqrt{(1+4g)}}{2\lambda}},$$

ubi necesse est sit $n > 0$ et $\nu > 0$.

EXEMPLUM 5

1045. *Si ponamus*

$$y = \int dx(aa-xx)^{\nu-1}\cos.\alpha u^{\lambda}x$$

posito post integrationem $x = a$ *, ut y aequetur certae functioni ipsius u, invenire aequationem differentio–differentialem, cui ea satisfaciat.*

Cum sit

$$\frac{dy}{du} = -\alpha\lambda u^{\lambda-1}\int xdx(aa-xx)^{\nu-1}\sin.\alpha u^{\lambda}x$$

et

$$\frac{ddy}{du^2} = \int xdx(aa-xx)^{\nu-1}\left(-\alpha\lambda(\lambda-1)u^{\lambda-2}\sin.\alpha u^{\lambda}x - \alpha\alpha\lambda\lambda u^{2\lambda-2}x\cos.\alpha u^{\lambda}x\right),$$

hinc erit

$$\frac{Lddy}{du^2}+\frac{Mdy}{du}+Ny$$
$$=\int dx\left(aa-xx\right)^{\nu-1}\left\{\begin{matrix}N\cos.\alpha u^{\lambda}x-\alpha\lambda Mu^{\lambda-1}x\sin.\alpha u^{\lambda}x-\alpha\lambda\left(\lambda-1\right)Lu^{\lambda-2}x\sin.\alpha u^{\lambda}x\\-\alpha\alpha\lambda\lambda Lu^{2\lambda-2}xx\cos.\alpha u^{\lambda}x\end{matrix}\right\}.$$

Fingatur integrale $=\left(aa-xx\right)^{\nu}\sin.au^{\lambda}x$, quod evanescit posito tam $x=0$ quam $x=a$, reperiturque comparatione instituta

$$L=\frac{u^{-\lambda+2}}{\alpha\lambda\lambda},\quad M=\frac{2\lambda\nu-\lambda+1}{\alpha\lambda\lambda}u^{-\lambda+1},\quad N=\alpha aau^{\lambda}.,$$

Quare huius aequationis

$$\frac{ddy}{du^2}+\left(2\lambda\nu-\lambda+1\right)\frac{dy}{udu}+\alpha\alpha\lambda\lambda aau^{2\lambda-2}y=0$$

integrale est

$$y=\int dx\left(aa-xx\right)^{\nu-1}\cos.\alpha u^{\lambda}x.$$

COROLLARIUM 1

1046. Si ergo sit $\nu=\frac{\lambda-1}{2\lambda}$ et $\alpha=\frac{1}{\lambda}$, huius equationis

$$\frac{ddy}{du^2}+aau^{2\lambda-2}y=0$$

integrale est

$$y=\int dx\left(aa-xx\right)^{\frac{-\lambda-1}{2\lambda}}\cos.\frac{1}{\lambda}u^{\lambda}x,$$

siquidem post integrationem statuatur $x=a$, integrali ita sumto, ut evanescat posito $x=0$.

COROLLARIUM 2

1047. Si igitur sit $\frac{-\lambda-1}{2\lambda}=i$ numero integro seu $\lambda=\frac{-1}{2i+1}$, huius aequationis

$$ddy+aau^{\frac{-4i-4}{2i+1}}ydu^2=0$$

integrale est

$$y=\int dx\left(aa-xx\right)^{i}\cos.\frac{1}{\lambda}u^{\lambda}x,$$

quod revera exhiberi potest. Prodeunt scilicet casus integrabiles supra indicati [§ 951].

SCHOLION

1048. Cum posuerimus $y=\int Vdx$ existente V functione quacunque ipsarum u et x, quarum autem in hac integratione sola x ut variabilis tractatur, non opus est absolute integrale ita determinari, ut evanescat posito $x=0$, sed sufficit, ut certo quodam casu $x=b$ evanescat, quo facto si porro ponatur $x=a$, ut y aequetur functioni cuipiam ipsius u, quam per quadraturas assignare licet, quandoquidem hic integrationem formularum simplicium nobis concedi iure postulamus. Atque hic valor ipsius y per u expressus integrale exhibebit cuiusdam aequationis differentio–differentialis

$$Lddy+Mdudy+Nydu^2=Udu^2,$$

ubi autem necesse est, ut haec formula

$$\int dx\left(L\left(\tfrac{ddV}{du^2}\right)+M\left(\tfrac{dV}{du}\right)+NV\right)$$

integrari actu possit, quod integrale itidem ita est capiendnm, ut evanescat posito $x=b$, tum vero posito $x=a$ id fiat $=U$.

PROBLEMA 131

1049. *Si fuerint P et Q functiones ipsius x, at K functio ipsius u ac ponatur*

$$y=\int Pdx(K+Q)^n$$

integrali ita sumto, ut evanescat casu $x=b$, tum vero statuatur $x=a$, ut pro y prodeat functio ipsius u, invenire aequationem differentio–differentialem inter y et u, cui ille valor ipsius y satisfaciat.

SOLUTIO

Sit $dK=K'du$ et $dK'=K''du$ et ob $y=\int(K+Q)^n Pdx$ erit

$$\tfrac{dy}{du}=\int nK'(K+Q)^{n-1}Pdx$$

ac denuo differentiando

$$\tfrac{ddy}{du^2}=\int\left(nK''(K+Q)^{n-1}+n(n-1)K'K'(K+Q)^{n-2}\right)Pdx,$$

unde, si L, M, N denotent functiones ipsius u, erit haec expressio

$$\frac{Lddy}{du^2}+\frac{Mdy}{du}+Ny$$
$$=\int Pdx(K+\mathrm{Q})^{n-2}\left\{\begin{array}{l}N(K+Q)^2+nMK'(K+Q)\\+nLK''(K+Q)+n(n-1)LK'K'\end{array}\right\}$$
$$=\int Pdx(K+\mathrm{Q})^{n-2}\left\{\begin{array}{l}NKK+nMKK'+nLKK'+n(n-1)LK'K'\\+2NKQ+nMK'Q+nLK''Q+NQQ\end{array}\right\};$$

quae cum debeat esse integrabilis, statuatur integrale $=R(K+Q)^{n-1}+\text{Const.}$,
ita ut evanescat posito ut ante $x=b$, ubi R sit functio ipsius x tantum.
Cuius formae differentiale quia est

$$(K+Q)^{n-2}\left(KdR+QdR+(n-1)RdQ\right),$$

oportet sit

$$\left(NKK+nMKK'+nLKK''+n(n-1)LK'K'\right)Pdx$$
$$+\left(2NK+nMK'+nLK''\right)PQdx+NPQQdx=KdR+QdR+(n-1)RdQ.$$

Hic ergo duplicis generis termini adesse debent, alii ab u plane liberi, alii vero functione K affecti, quos deinceps seorsim aequari conveniet.
Hunc in finem ponamus

$$NKK+nMKK'+nLKK''+n(n-1)LK'K'=A+\alpha K,$$
$$2NK+nMK'+nLK''=B+\beta K \quad \text{et} \quad N=C+\gamma K.$$

Ex binis prioribus elidendo M colligitur

$$-NKK+n(n-1)LK'K=A+\alpha K-BK-\beta KK,$$

unde ob $N=C+\gamma K$ concluditur

$$L=\frac{A+(\alpha-B)K-(\beta-C)KK+\gamma K^3}{n(n-1)K'K'} \text{ hincque } M=\frac{B+\beta K-2NK-nLK''}{nK'},,$$

ita ut ex funetione K litterae L, M et N determinentur, dum $A,\ \alpha,\ B,\ \beta,\ C,\ \gamma$ constantes quascunque denotant.

Nunc autem superest, ut efficiatur

$$(A+\alpha K)Pdx+(B+\beta K)PQdx+(C+\gamma K)PQQdx= KdR+QdR+(n-1)RdQ\,,$$

unde duplicis generis terminos seorsim aequando fit

$$Pdx(A+BQ+CQQ)=QdR+(n-1)RdQ,$$
$$Pdx(\alpha+\beta Q+\gamma QQ)=dR$$

ideoque

$$\frac{A+BQ+CQQ}{\alpha+\beta Q+\gamma QQ}=Q+\frac{(n-1)RdQ}{dR}$$

seu

$$\frac{(n-1)RdQ}{dR}=\frac{A+(B-\alpha)Q+(C-\beta)QQ-\gamma Q^3}{\alpha+\beta Q+\gamma QQ}$$

ergo

$$\frac{dR}{R}=\frac{(n-1)dQ(\alpha+\beta Q+\gamma QQ)}{A+(B-\alpha)Q+(C-\beta)QQ-\gamma Q^3}\,,$$

unde ex functione Q functio R definitur; tum vero erit

$$Pdx=\frac{(n-1)RdQ}{A+(B-\alpha)Q+(C-\beta)QQ-\gamma Q^3}$$

Abeat iam integrale illud $R(K+Q)^{n-1}+\text{Const.}$ in functionem U posito $x=a$ ac valor initio assumtus

$$y=\int\frac{(n-1)RdQ(K+Q)^n}{A+(B-\alpha)Q+(C-\beta)QQ-\gamma Q^3}$$

erit integrale huius aequationis differentio-differentialis

$$Lddy+Mdudy+Nydu^2=Udu^2$$

COROLLARIUM 1

1050. Cum pro Q functio quaecunque ipsius x accipi possit, nihil impedit, quominus sumamus $Q = x$. Tum igitur quaeri oportet R ex hac aequatione

$$\frac{dR}{R}=\frac{(n-1)dx(\alpha+\beta x+\gamma xx)}{A+(B-\alpha)x+(C-\beta)xx-\gamma x^3}$$

eritque pro K functione quacunque ipsius u assumta

$$y=(n-1)\int\frac{Rdx(K+x)^n}{A+(B-\alpha)x+(C-\beta)xx-\gamma x^3}$$

in quo integrali ita sumto, ut posito $x=b$ evanescat, deinceps statui debet $x=a$.

COROLLARIUM 2

1051. Ex functione autem *K* aequatio differentio–differentialis ita formatur, ut sit

$$L=\frac{A-(B-\alpha)K+(C-\beta)KK+\gamma K^3}{du^2n(n-1)dK^2},$$

$$M=\frac{B-(2C-\beta)K-2\gamma KK}{ndK}du-\frac{LddK}{dudK} \quad \text{et} \quad N=C+\gamma K.$$

Deinde in expressione $R(K+x)^{n-1}+\text{Const.}$ ita constituta, ut posito $x=b$ evanescat, ponatur $x=a$ et functio ipsius *u* inde resultans vocetur *U* eritque aequatio differentio–differentialis

$$Lddy+Mdudy+Nydu^2=Udu^2.$$

COROLLARIUM 3

1052. Si expressio $R(K+x)^{n-1}+\text{Const.}$ ita sit comparata, ut utroque casu $x=b$ et $x=a$ evanescat, seu potius hi termini integrationis ita constituantur, ut hoc eveniat, formula pro *y* assumta satisfaciet huic aequationi

$$Lddy+Mdudy+Nydu^2=0;$$

quae si deinceps in alias formas transmutetur, earum quoque integralia assignari poterunt.

PROBLEMA 132

1053. *Si fuerint P, Q functiones ipsius x, at K functio ipsius u ac ponatur*

$$y=\int e^{KQ}Pdx$$

integrali ita sumto, ut evanescat casu $x=b$, *tum vero ponatur* $x=a$, *et* y *aequabitur functioni ipsius u, quae satisfaciet cuipiam aequationi differentio–differentiali, quam invenire oportet.*

SOLUTIO

Cum sit $y=\int e^{KQ}Pdx$, erit

$$\frac{dy}{du}=\int e^{KQ}K'PQdx \quad \text{et} \quad \frac{ddy}{du^2}=\int e^{KQ}Pdx(K''Q+K'K'QQ),,$$

unde fit

$$\frac{Lddy}{du^2}+\frac{Mdy}{dy}+Ny=\int e^{KQ}Pdx\left(N+MK'Q+LX''Q+LK'K'QQ\right),$$

cuius integrale statuatur $e^{KQ}R+\text{Const.}$, quae expressio evanescat posito $x=b$, fierique oportet

$$dR+KRdQ=Pdx\left(N+\left(MK'+LK''\right)Q+LK'K'QQ\right)$$

et ob rationes ante [§ 1049] allegatas faciamus

$$LK'K'=A+\alpha K,\ MK'+LK''=B+\beta K,\ N=C+\gamma K$$

eritque

$$L=\frac{A+\alpha K}{K'K'}\ \text{ et }\ M=\frac{B+\beta K}{K'}-\frac{LK''}{K'}$$

atque obtinebimus has aequationes

$$dR=Pdx\left(C+BQ+AQQ\right),\quad RdQ=Pdx\left(\gamma+\beta Q+\alpha QQ\right),$$

unde colligitur

$$\frac{dR}{R}=\frac{dQ\left(C+BQ+AQQ\right)}{\gamma+\beta Q+\alpha QQ},$$

inventaque functione *R* erit

$$Pdx=\frac{RdQ}{\gamma+\beta Q+\alpha QQ},$$

ita ut sit

$$y=\int e^{KQ}\frac{RdQ}{\gamma+\beta Q+\alpha QQ}.$$

Si iam expressio $e^{KQ}R+\text{Const.}$ posito $x=a$ abeat in functionem *U*, aequatio differentio–differentialis, cui hoc integrate convenit, erit

$$Lddy+Mdudy+Nydu^2=Udu^2.$$

COROLLARIUM 1

1054. Hic pro *Q* scribere licet *x* ut ante, unde fit

$$\frac{dR}{R}=\frac{dx\left(C+Bx+Axx\right)}{\gamma+\beta x+\alpha xx}\ \text{ et }\ y=\int e^{Kx}\frac{Rdx}{\gamma+\beta x+\alpha xx},$$

et *U* oritur ex forma $e^{Kx}R+\text{Const.}$ posito $x=a$. Valor autem ipsius *R* pro ratione coefficientium α,β,γ varias formas induere potest.

COROLLARIUM 2

1055. Pro *K* autem quaecunque functio ipsius *u* accipi potest, a cuius indole aequatio differentio–differentialis pendet. Erit autem

$$L=\frac{A+\alpha K}{dK^2}du^2,\ M=\frac{B+\beta K}{dK}du-\frac{(A+\alpha K)duddK}{dK^3}\quad \text{et}\quad N=C+\gamma K,$$

unde aequatio differentio–differentialis est

$$\frac{(A+aK)ddy}{dK^2}+\frac{(B+\beta K)dy}{dK}-\frac{(A+aK)ddKdy}{dK^3}+(C+\gamma K)y=U.$$

COROLLARIUM 3

1056. Cum hic etiam *u* ex calculo excedat, perinde est, cuiusmodi functio eius pro *K* assumatur; quin etiam sine detrimento amplitudinis poni potest $K=u$, dummodo ratio elementi, quod constans assumitur, habeatur.

SCHOLION 1

1057. Si ergo sumatur $K=u$ atque elementum *du* sumatur constans, ut fiat $ddK=0$, hinc ista aequatio construi potest

$$\frac{(A+\alpha u)ddy}{du^2}+\frac{(B+\beta u)dy}{du}+(C+\gamma u)y=U$$

existente *U* eiusmodi functione ipsius *u*, quam descripsimus. Simili autem modo ex praecedente problemate [§1051] construi potest haec aequatio

$$\left(A-(B-\alpha)u+(C-\beta)uu+\gamma u^3\right)\frac{ddy}{du^2}+(n-1)\left(B-(2C-\beta)u-2\gamma uu\right)\frac{dy}{du}$$
$$+n(n-1)(C+\gamma u)y=U,$$

quae aeque late patere est censenda, ac si functionem quamcunque ipsius *u* loco *K* scripsissemus. Hinc enim loco *u* scribendo functionem quamcunque ipsius *t* ac *dt* pro constante sumendo omnes illae formae derivari possunt. Ex quo haec aequatio multo latius patet illa, quam supra [§ 967, 992] in genere per series infinitas resolvimus. Plerumque autem hae aequationes ita sunt comparatae, ut earum integratio aliis methodis expediri haud possit, quocirca haec methodus omnino digna videtur, ad quam ulterius excolendam Geometrae omnes vires intendant.

SCHOLION 2

1058. In investigatione huiusmodi constructionum ita sum versatus, ut primo quasi per coniecturam formulam quandam differentialem $\int Vdx = y$,in qua *V* erat certa functio ipsarum *u* et *x*, ubi autem *u* ut constans spectabatur, assumserim indeque dato ipsi *x* valore tributo pertigerim ad aequationem differentio–differentialem inter *u* et *y*, cui formula illa assumta satisfaceret. Hic autem observandum est illam formulam integralem non prorsus ab arbitrio nostro pendere, sed certa quadam indole praeditam esse debere, ut evolutione facta res perdueatur ad aequationem differentialem secundi gradus. Quamdiu autem hanc electionem soli coniecturae permittimus, perpaucae huiusmodi formulae menti se offerunt, quae ad scopum propositum perducant, multoque minus sperare licet, ut hoc modo unquam ad datam aequationem differentio–differentialem perveniamus, casuique potissimum tribuendae videntur constructiones, quas hic tradidimus. Cum igitur longissime adhuc simus remoti a solutione problematis, quo proposita quadam aequatione differentio–differentiali quaeritur formula illa eius integrationem suppeditans, quod problema an unquam solutionem sit naeturum, admodum incertum videtur, eo magis opera est adhibenda, ut saltem pro casibus particularibus investigationem formulae integrantis ex indole aequationis propositae derivare conemur sicque quodam modo viam ad solutionem directam paremus. Ad hoc autem series infinitae, per quas huiusmodi aequationes supra resolvere docuimus, utiliter adhiberi possunt; unde in sequenti capite methodum exponam ex serie infinita solutionem cuiuspiam aequationis differentio–differentialis continente formulam illam integralem investigandi.