CHAPTER XI

ON THE CONSTRUCTION OF SECOND ORDER DIFFERENTIAL EQUATIONS SOUGHT FROM THE RESOLUTION OF THESE BY INFINITE SERIES

PROBLEM 133

1059. *To express the sum of this proposed infinite series*

$$A+Bs+Cs^2+\ldots+Ms^{i-1}+Ns^i+\text{etc.},$$

by an integral formula , in which there shall be

$$B=\tfrac{0m+h}{1n+k}A,\quad C=\tfrac{1m+h}{2n+k}B,\quad D=\tfrac{2m+h}{3n+k}C$$

and in general

$$N=\tfrac{(i-1)m+h}{in+k}M.$$

.

SOLUTION

The sum sought is put equal to z, thus in order that

$$z=A+Bs+Cs^2+\ldots+Ms^{i-1}+Ns^i+\text{etc.},$$

and on differentiation there will be

$$\tfrac{sdz}{ds}=0A+1Bs+2Cs^2+3Ds^3+\ldots+(i-1)Ms^{i-1}+iNs^i+\text{etc.},$$

from which combined with the above series there arises

$$\tfrac{msdz}{ds}+hz=hA+(m+h)Bs+(2m+h)Cs^2+\ldots+((i-1)m+h)Ms^{i-1}+(im+h)Ns^i+\text{etc.}$$

Hence indeed also in a similar manner

$$\tfrac{nsdz}{ds}+kz=kA+(n+k)Bs+(2n+k)Cs^2+\ldots+((i-1)n+k)Ms^{i-1}+(in+k)Ns^i+\text{etc.};$$

hence on account of

$$(n+k)B = hA,\quad (2n+k)C = (m+h)B,\quad \text{etc.}$$

there will be

$$\tfrac{nsdz}{ds} + kz = kA + hAs + (m+h)Bs^2 + (2m+h)Cs^3 + \text{etc.},$$

from which evidently there is prepared

$$\tfrac{nsdz}{ds} + kz = kA + \tfrac{mssdz}{ds} + hsz \quad \text{or} \quad sdz(n-ms) + zds(k-hs) = kAds,$$

that is

$$dz + \frac{zds(k-hs)}{s(n-ms)} = \frac{kAds}{s(n-ms)}.$$

Now since there shall be

$$\frac{ds(k-hs)}{s(n-ms)} = \frac{kds}{ns} + \frac{(mk-nh)ds}{n(n-ms)},$$

this equation is made integrable on multiplying by $s^{\frac{k}{n}}(n-ms)^{\frac{nh-mk}{mn}}$ and there is produced

$$(n-ms)^{\frac{nh-mk}{mn}} s^{\frac{k}{n}} z = Ak\int s^{\frac{k}{n}-1} ds (n-ms)^{\frac{nh-mk}{mn}-1}$$

which integral must be taken thus, so that on putting $s=0$ there becomes $z=A$, with which observed we will have

$$z = Aks^{-\frac{k}{n}}(n-ms)^{\frac{mk-nh}{mn}}\int s^{\frac{k}{n}-1} ds (n-ms)^{\frac{nh-mk}{mn}-1}.$$

COROLLARY 1

1060. The case $m=0$ requires a special solution, in which there becomes

$$dz + \frac{zds(k-hs)}{ns} = \frac{Akds}{ns},$$

which multiplied by $s^{\frac{k}{n}}e^{-\frac{hs}{n}}$ gives

$$e^{-\frac{hs}{n}} s^{\frac{k}{n}} z = \frac{Ak}{n}\int e^{-\frac{hs}{n}} s^{\frac{k}{n}-1} ds \text{ and thus } z = \frac{Ak}{n} e^{\frac{hs}{n}} s^{\frac{-k}{n}} \int e^{-\frac{hs}{n}} s^{\frac{k}{n}-1} ds$$

thus with the integral taken thus, so that it becomes $z=A$ on putting $s=0$, [on applying L'Hôpital's Rule].

COROLLARY 2

1061. Also the case $n=0$ must be resolved separately ; for the equation

$$dz+zds\frac{hs-k}{mss}=-\frac{Akds}{mss}$$

has to be multiplied by $s^{\frac{h}{m}}e^{\frac{k}{ms}}$, and the integral is found

$$e^{\frac{k}{ms}}s^{\frac{h}{n}}z=-\frac{Ak}{n}\int e^{\frac{k}{ms}}s^{\frac{k}{m}-2}ds \text{ and thus } z=-\frac{Ak}{n}e^{\frac{-k}{ms}}s^{\frac{-h}{m}}\int e^{\frac{k}{ms}}s^{\frac{h}{m}-2}ds\,.$$

COROLLARY 3

1062. If there should be both $m=0$ and $n=0$, on account of $N=\frac{h}{k}M$ our series will be geometric, and our equation will be

$$zds(k-hs)=Akds \quad \text{or} \quad z=\frac{Ak}{k-hs},$$

as the nature of the expression shown demands.

SCHOLIUM

1063. Here especially the case is worth noting, in which $k=0$ and it is possible for the sum z to be expressed without a sign ; for if there shall be

$$(n-ms)^{\frac{h}{m}}z=\text{Const.},$$

and because, if $s=0$, there must become $z=A$, then $\text{Const.}=A^{\frac{h}{m}}$ and thus

$$z=A^{\frac{h}{m}}(n-ms)^{\frac{-h}{m}} \text{ or } z=A\left(\frac{n}{n-ms}\right)^{\frac{h}{m}} \text{ or also } z=A\left(1-\frac{ms}{n}\right)^{\frac{-h}{m}}.$$

But truly the integration also succeeds in the case in which $k=n$; for there shall be

$$(n-ms)^{\frac{h}{m}-1}sz=An\int ds(n-ms)^{\frac{h}{m}-2},$$

which integral is

$$\text{Const.}-\frac{An(n-ms)^{\frac{h}{m}-1}}{h-m},$$

and because on putting $s=0$ there will be made $z=A$, then

$$0=\text{Const.}-\frac{A}{h-m}n^{\frac{h}{m}}$$

and hence

$$z=\frac{An}{(h-m)s}\left(\left(\frac{n}{n-ms}\right)^{\frac{h}{m}-1}-1\right)=\frac{An}{(h-m)s}\left(\left(1-\frac{ms}{n}\right)^{1-\frac{h}{m}}-1\right).$$

Again it is evident that the integral can be set out in the case $k=2n$ *;* in which since there shall be

$$\left(n-ms\right)^{\frac{h}{m}-2}ssz=2An\int sds\left(n-ms\right)^{\frac{h}{m}-3},$$

this integral will be

$$=\text{Const.}-\tfrac{2Ans}{h-2m}\left(n-ms\right)^{\frac{h}{m}-2}+\tfrac{2An}{h-2m}\int ds\left(n-ms\right)^{\frac{h}{m}-2}$$

$$=\text{Const.}-\tfrac{2Ans}{h-2m}\left(n-ms\right)^{\frac{h}{m}-2}-\tfrac{2An\left(n-ms\right)^{\frac{h}{m}-1}}{\left(h-m\right)\left(h-2m\right)},$$

where

$$\text{Const.}=\tfrac{2An^{\frac{h}{m}}}{\left(h-m\right)\left(h-2m\right)}$$

and thus

$$z=\tfrac{2Ann}{\left(h-m\right)\left(h-2m\right)ss}\left(\left(\tfrac{n}{n-ms}\right)^{\frac{h}{m}-2}-1-\tfrac{\left(h-2m\right)s}{n}\right).$$

And in a like manner the integration can be resolved in the cases $k=3n,\ k=4n$ etc.

PROBLEM 134

1064. *With an infinite series of this kind arising*

$$A\mathfrak{A}+B\mathfrak{B}u+C\mathfrak{C}u^2+D\mathfrak{D}u^3+.....+M\mathfrak{M}u^{i-1}+N\mathfrak{N}u^i+\text{etc.}$$

with the law of the coefficients

$$B=\tfrac{0m+h}{1n+k}A,\quad C=\tfrac{1m+h}{2n+k}B,\quad D=\tfrac{2m+h}{3n+k}C,\quad ......\quad N=\tfrac{(i-1)m+h}{in+k}M,$$

$$\mathfrak{B}=\tfrac{0\mu+\eta}{1\nu+\theta}\mathfrak{A},\quad \mathfrak{C}=\tfrac{1\mu+\eta}{2\nu+\theta}\mathfrak{B},\quad \mathfrak{D}=\tfrac{2\mu+\eta}{3\nu+\theta}\mathfrak{C},\quad ......\quad \mathfrak{N}=\tfrac{(i-1)\mu+\eta}{i\nu+\theta}\mathfrak{M},$$

to express the sum of this series by an integral formula.

SOLUTION

With the sum put in place

$$y=A\mathfrak{A}+B\mathfrak{B}u+C\mathfrak{C}u^2+D\mathfrak{D}u^3+E\mathfrak{E}u^4+\text{etc.}$$

a series formed in this manner can be considered

$$z = A + Bux + Cu^2x^2 + Du^3x^3 + Eu^4x^4 + \text{etc.}$$

of which the sum is, on putting $ux = s$, as we have found just now [§ 1059],

$$z = Aks^{-\frac{k}{n}}\left(n-ms\right)^{\frac{mk-nh}{mn}}\int s^{\frac{k}{n}-1}ds\left(n-ms\right)^{\frac{nh-mk}{mn}-1}$$

thus with the integral determined, so that on putting $s = 0$ there becomes $z = A$.

An integral formula of this kind may be formed

$$V = \int Pzdx = \int Pdx\left(A + Bux + Cu^2x^2 + Du^3x^3 + Eu^4x^4 + \text{etc.}\right),$$

in which *u* may be considered as constant, but a function of *x* of this kind may be taken for *P*, so that there becomes

$$\int Pxdx = \tfrac{\mathfrak{B}}{\mathfrak{A}}\int Pdx, \quad \int Px^2dx = \tfrac{\mathfrak{C}}{\mathfrak{B}}\int Pxdx, \quad \int Px^3dx = \tfrac{\mathfrak{D}}{\mathfrak{C}}\int Px^2dx, \quad \text{etc.},$$

then there will be

$$V = \left(A + \tfrac{B\mathfrak{B}}{\mathfrak{A}}u + \tfrac{C\mathfrak{C}}{\mathfrak{A}}u^2 + \tfrac{D\mathfrak{D}}{\mathfrak{A}}u^3 + \text{etc.}\right)\int Pdx,$$

from which it becomes apparent

$$y = \tfrac{\mathfrak{A}V}{\int Pdx} = \tfrac{\mathfrak{A}\int Pzdx}{\int Pdx}.$$

Whereby since the value of *z* shall be known, it remains only that the function *P* of *x* be investigated with the aforesaid conditions mentioned. But in general there is required to be

$$\int Px^idx = \tfrac{\mathfrak{N}}{\mathfrak{M}}\int Px^{i-1}dx = \tfrac{(i-1)\mu+\eta}{iv+\theta}\int Px^{i-1}dx;$$

which equality must stand, since clearly it suffices that only within a certain case [*i.e.* within a range of values of *x* that Euler will find. Hence an integration by parts is effected, where the function *P* is integrated to give *Q*, which is subsequently found by differentiation to fit the prescribed conditions :] is there a given value of *x*, thus we may put in place the equation in general to be

$$\left(iv+\theta\right)\int Px^idx = \left(\left(i-1\right)\mu+\eta\right)\int Px^{i-1}dx + x^iQ,$$

thus so that within the bounds of the integral there shall be $Q = 0$. Hence on differentiating we shall have

$$(i\nu+\theta)Px^{i}dx=(i\mu-\mu+\eta)Px^{i-1}dx+x^{i}dQ+ix^{i-1}Qdx$$

or on dividing by x^{i-1}

$$(i\nu+\theta)Pxdx=(i\mu-\mu+\eta)Pdx+xdQ+iQdx;$$

which equality must stand equally for all the values of *i*, hence we come upon these two equations

$$\nu Pxdx=\mu Pdx+Qdx \quad \text{and} \quad \theta Pxdx=(\eta-\mu)Pdx+xdQ,$$

from which in a two–fold manner we deduce

$$Pdx=\tfrac{Qdx}{\nu x-\mu} \quad \text{and} \quad Pdx=\tfrac{xdQ}{\theta x-(\eta-\mu)}$$

and thus on dividing the one value by the other

$$\tfrac{xdQ}{Qdx}=\tfrac{\theta x+\mu-\eta}{\nu x-\mu} \quad \text{or} \quad \tfrac{dQ}{Q}=\tfrac{dx(\theta x+\mu-\eta)}{x(\nu x-\mu)},$$

which is expanded out into

$$\tfrac{dQ}{Q}=\tfrac{\eta-\mu}{\mu x}dx+\tfrac{\mu\theta+\mu\nu-\eta\nu}{\mu(\nu x-\mu)}dx.$$

Hence on integrating there is elicited

$$Q=x^{\frac{\eta}{\mu}-1}(\nu x-\mu)^{\frac{\theta\mu-\eta\nu}{\mu\nu}+1} \quad \text{by a constant.}$$

or

$$Q=-x^{\frac{\eta}{\mu}-1}(\mu-\nu x)^{\frac{\theta\mu-\eta\nu}{\mu\nu}+1},$$

from which there becomes

$$Pdx=x^{\frac{\eta}{\mu}-1}dx(\mu-\nu x)^{\frac{\theta\mu-\eta\nu}{\mu\nu}}.$$

Therefore since there shall be

$$(i\nu+\theta)\int Px^{i}dx=((i-1)\mu+\eta)\int Px^{i-1}dx-x^{i+\frac{\eta}{\mu}-1}(\mu-\nu x)^{\frac{\theta\mu-\eta\nu}{\mu\nu}+1},$$

if these integrals are taken thus, so that they vanish on putting $x=0$, then truly there is put in place $x=\frac{\mu}{\nu}$, and there becomes as our hypothesis demands,

$$\int Px^{i}dx=\frac{(i-1)\mu+\eta}{i\nu+\theta}\int Px^{i-1}dx\ ;$$

but truly towards this end it is necessary that there shall be

$$i+\frac{\eta}{\mu}-1>0 \quad \text{and} \quad \frac{\theta\mu-\eta\nu}{\mu\nu}+1>0.$$

If this condition is considered to be in place, the sum of the proposed series may be expressed thus, so that there shall be

$$y\int x^{\frac{\eta}{\mu}-1}dx\left(\mu-\nu x\right)^{\frac{\theta\mu-\eta\nu}{\mu\nu}}=\mathfrak{A}\int x^{\frac{\eta}{\mu}-1}zdx\left(\mu-\nu x\right)^{\frac{\theta\mu-\eta\nu}{\mu\nu}}$$

with this integral thus taken

$$z=Aks^{\frac{-k}{n}}\left(n-ms\right)^{\frac{mk-nh}{mn}}\int s^{\frac{k}{n}-1}ds\left(n-ms\right)^{\frac{nh-mk}{mn}-1}$$

so that it becomes $z=A$ on putting $s=0$. But in this integral found there is written *ux* for *s* and with this value of *z* substituted into that formula it is required to treat the quantity *u* as constant, as long as these integrations should be completed by the prescribed rule; indeed then for *y* there will be produced a function of *u* expressing the sum of the proposed series.

COROLLARY 1

1065. Because the law of progression of our series is assumed in twin coefficients, the individual series *A, B, C, D* etc. and $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$ etc . are allowed to be interchanged between themselves, from which by this method a twofold formula expressing the sum of the series may be obtained.

COROLLARY 2

1066. But if the function *Q* does not enter into the calculation, yet that is required to be known, since from the nature of this the limits of the integration must be put in place, thus so that for each there becomes $Q=0$. Clearly these limits are $x=0$ and $x=\frac{\mu}{\nu}$, as long as there should be $i+\frac{\eta}{\nu}-1>0$ and $\frac{\theta\mu-\eta\nu}{\mu\nu}+1>0$, where *i* is a positive whole number.

COROLLARY 3

1067. The cases for the function *Q*, in which either $\mu = 0$ or $v = 0$, are to be set out separately. In the first case, where $\mu = 0$, there is

$$\frac{dQ}{Q} = \frac{dx(\theta x - \eta)}{vxx} = \frac{\theta dx}{vx} - \frac{\eta dx}{vxx},$$

from which here becomes

$$Q = e^{\frac{\eta}{vx}} x^{\frac{\theta}{v}}.$$

For the latter case, where $v = 0$, there is

$$\frac{dQ}{Q} = \frac{dx(\theta x + \mu - \eta)}{-\mu x} = -\frac{\theta dx}{\mu} + \frac{\eta - \mu}{\mu} \cdot \frac{dx}{x}$$

and thus

$$Q = e^{-\frac{\theta x}{\mu}} x^{\frac{\eta}{\mu} - 1}.$$

SCHOLIUM

1068. The constructions to be set out in this manner therefore are similar to these, which we have treated in the previous chapter, since the calculation also must be reduced to an integral of this kind $\int V dx$, in which *V* is a function of the two variables *u* and *x*, but that variable *u* is considered constant in the integration itself, and after the integration a certain value is assigned to the given value of *x*. Yet truly this construction is not satisfied to be extended to the cases in the above method, since it can come about that the quantity *z* involves greatly transcending functions. But in turn we see that the preceding method can be applied to equations of this kind, which are unable to be treated by series such as we have set out here, from which resource it may be considered possible that worthwhile advances in analysis be made.

PROBLEM 135

1069. *With the second order differential equation proposed*

$$xx\left(a+bx^{n}\right)ddy+x\left(c+ex^{n}\right)dxdy+\left(f+gx^{n}\right)ydx^{2}=0,$$

to construct the value of y by an integral formula.

SOLUTION I

Thus we set out this above equation (§ 967) in a series, so that on putting

$$y=x^{\lambda}\left(A+Bx^{n}+Cx^{2n}+Dx^{3n}+Ex^{4n}+\text{etc.}\right)$$

in the first place the root of this equation $\lambda(\lambda-1)a+\lambda c+f=0$ must be given to the exponent λ; then truly for the sake of brevity $\lambda(\lambda-1)b+\lambda e+g=h$ there shall become

$$B=\frac{-h}{n\left(na+(2\lambda-1)a+c\right)}A,\ \ C=\frac{-nnb-(2\lambda-1)nb-ne-h}{2n\left(2na+(2\lambda-1)a+c\right)}B,\ \ D=\frac{-4nnb-2(2\lambda-1)nb-2ne-h}{3n\left(3na+(2\lambda-1)a+c\right)}C$$

and thus on putting the two neighbouring indefinite terms of this series to be $Mx^{(i-1)n}+Nx^{in}$, generally

$$N=\frac{-(i-1)^{2}nnb-(2\lambda-1)(i-1)nb-(i-1)ne-h}{in\left(ina+(2\lambda-1)a+c\right)}M$$

where since the denominator now has factors, such as we assumed before [§ 1064], we can resolve the numerator too into factors, with which on equating to zero here is found :

$$(i-1)n=-\tfrac{1}{2}(2\lambda-1)-\tfrac{e}{2b}\pm\sqrt{\left(\tfrac{1}{4}(2\lambda-1)^{2}+\tfrac{(2\lambda-1)e}{2b}+\tfrac{ee}{4bb}-\tfrac{h}{b}\right)}$$

or

$$(i-1)n=-\tfrac{1}{2}(2\lambda-1)-\tfrac{e}{2b}\pm\tfrac{1}{2b}\sqrt{\left((b-e)^{2}-4bg\right)}.$$

For brevity there is put

$$\sqrt{\left((b-e)^{2}-4bg\right)}=q,$$

so that there becomes

$$(i-1)n=\frac{-(2\lambda-1)b-e\pm q}{2b},$$

and our relation becomes

$$N=\frac{-\left((i-1)^2nb+\frac{1}{2}(2\lambda-1)b+\frac{1}{2}e-\frac{1}{2}q\right)\left((i-1)nb+\frac{1}{2}(2\lambda-1)b+\frac{1}{2}e+\frac{1}{2}q\right)}{inb\left(ina+(2\lambda-1)a+c\right)}M\,.$$

Now we put $x^n=u$ and thus we can represent the series found :

$$\frac{y}{x^\lambda}=A\mathfrak{A}+B\mathfrak{B}u+C\mathfrak{C}u^2+....+M\mathfrak{M}u^{i-1}+N\mathfrak{N}u^i+\text{etc.}$$

and of which the law of the two-fold coefficients may thus itself be considered

$$N=\frac{(i-1)nb+\frac{1}{2}(2\lambda-1)b+\frac{1}{2}e-\frac{1}{2}q}{-inb}M$$

and

$$\mathfrak{N}=\frac{(i-1)nb+\frac{1}{2}(2\lambda-1)b+\frac{1}{2}e+\frac{1}{2}q}{ina+(2\lambda-1)a+c}\mathfrak{M}.$$

Therefore since this series shall be similar to that which we have constructed before, we can put a comparison in place and we will have :

$$\begin{aligned}&m=nb,\quad h=\tfrac{1}{2}(2\lambda-1)b+\tfrac{1}{2}e-\tfrac{1}{2}q,\\&n=-nb\quad\text{and}\quad k=0,\\&\mu=nb,\quad \eta=\tfrac{1}{2}(2\lambda-1)b+\tfrac{1}{2}e+\tfrac{1}{2}q,\\&\nu=na,\quad\text{and}\quad \theta=(2\lambda-1)a+c.\end{aligned}$$

Therefore at first we look for the quantity *z*, and lest the letter *x* should create an ambiguity, in place of the letter *x* used in the preceding problem we may use the letter t and there shall be $ut=s$, and because there is $k=0$, then by § 1063 there will be

$$z=A(1+s)^{\frac{-(2\lambda-1)b-e+q}{2nb}}=A(1+ut)^{\frac{-(2\lambda-1)b-e+q}{2nb}}\,.$$

With this value found the quantity *u* may be treated as constant in the following integrations, and since, which above [§ 1064] there was *y*, here there shall be $\frac{y}{x^\lambda}$ and t, which above was *x*, there will be

$$\frac{y}{x^\lambda}\int t^{\frac{\eta}{\mu}-1}dt(\mu-\nu t)^{\frac{\theta\mu-\eta\nu}{\mu\nu}}=\mathfrak{A}\int t^{\frac{\eta}{\mu}-1}zdt(\mu-\nu t)^{\frac{\theta\mu-\eta\nu}{\mu\nu}}\,;$$

where since there shall be $u = x^n$, here the value for *u* can be written down at once, so that it becomes

$$z = A\left(1 + x^n t\right)^{\frac{-(2\lambda-1)b-e+q}{2nb}},$$

and in these integrations the letter *x* must be regarded as constant. But if there should be $i + \frac{\eta}{\mu} - 1 > 0$ and $\frac{\theta\mu-\eta\nu}{\mu\nu} + 1 > 0$, these integral must be taken thus, so that they vanish on putting $t = 0$, with which done the value of t must be attributed $t = \frac{\mu}{\nu} = \frac{b}{a}$.
Since one shall be the smallest value of *i*, it is sufficient that there shall be

$$\frac{(2\lambda-1)b+e+q}{2nb} > 0 \text{ , then indeed } \frac{(2\lambda-1)ab+2bc-ae-aq}{2nab} + 1 > 0.$$

Now truly since $\int t^{\frac{\eta}{\mu}-1} dt\left(\mu - \nu t\right)^{\frac{\theta\mu-\eta\nu}{\mu\nu}}$ becomes a constant quantity, for *y* moreover some multiple of our equation is equally satisfying, with the integral of this expressed thus :

$$y = Cx^{\lambda}\int t^{\frac{\eta}{\mu}-1} z dt\left(\mu - \nu t\right)^{\frac{\theta\mu-\eta\nu}{\mu\nu}}$$

with $z = \left(1 + x^n t\right)^{\frac{-(2\lambda-1)b-e+q}{2nb}}$ arising.

SOLUTION 2

1071. [Wrongly labeled as 1070 in the first ed.] If the doubled coefficients are interchanged between themselves, so that there shall be :

$$\begin{aligned}
&m = nb, \quad h = \tfrac{1}{2}(2\lambda-1)b + \tfrac{1}{2}e + \tfrac{1}{2}q,\\
&n = na \quad k = (2\lambda-1)a + c,\\
&\mu = nb, \quad \eta = \tfrac{1}{2}(2\lambda-1)b + \tfrac{1}{2}e - \tfrac{1}{2}q,\\
&\nu = -nb, \quad \theta = 0,
\end{aligned}$$

and λ is taken from the equation $\lambda(\lambda-1)a + \lambda c + f = 0$, at first there is put $x^n t = s$ and *z* is sought, so that there shall be

$$z = Aks^{\frac{-k}{n}}\left(n - ms\right)^{\frac{mk-nh}{mn}}\int s^{\frac{k}{n}-1} ds\left(n - ms\right)^{\frac{nh-mk}{mn}-1}$$

thus with the definition of the integral thus taken, so that on putting $s = 0$ there becomes $z = A$, which value *A* is indeed arbitrary ; then on considering *x* as constant there will be

$$y = Cx^{\lambda}\int t^{\frac{\eta}{\mu}-1} z dt\left(\mu - \nu t\right)^{\frac{\theta\mu-\eta\nu}{\mu\nu}}$$

with the integral thus taken, so that it vanishes on putting $t=0$, then truly on making $t=\frac{\mu}{\nu}$, but if there should be $\frac{\eta}{\mu}>0$ and $1-\frac{\eta}{\mu}>0$ on account of $\theta=0$; where it is to be noted that *z* be defined by this differential equation

$$\frac{dz}{ds}=\frac{Ak-z(k-hs)}{s(n-ms)}.$$

SOLUTION 3

1072. Thus we have resolved the proposed equation [§ 967] by a descending series , so that on putting

$$y=x^{\lambda}\left(A+Bx^{-n}+Cx^{-2n}+Dx^{-3n}+\text{ etc.}\right)$$

the exponent λ has to be defined from this equation $\lambda(\lambda-1)b+\lambda e+g=0$, then truly on putting $\lambda(\lambda-1)a+\lambda c+f=h$ there shall be

$$B=\frac{-h}{n(nb-(2\lambda-1)b-e)}A,\quad C=\frac{-nna+(2\lambda-1)na+nc-h}{n(nb-(2\lambda-1)b-e)}B$$

and generally

$$N=\frac{-(i-1)^2nna+(2\lambda-1)(i-1)na+(i-1)nc-h}{in(inb-(2\lambda-1)b-e)}M\ ,$$

which equality on putting $\sqrt{\left((a-c)^2-4af\right)}=p$ may be shown thus by factors

$$N=\frac{-\left((i-1)na-\frac{1}{2}(2\lambda-1)a-\frac{1}{2}c-\frac{1}{2}p\right)\left((i-1)na-\frac{1}{2}(2\lambda-1)a-\frac{1}{2}c+\frac{1}{2}p\right)}{ina(inb+(2\lambda-1)b-e)}M$$

But if now we put $x^{-n}=u$ and we set up such a series :

$$\frac{y}{x^{\lambda}}=A\mathfrak{A}+B\mathfrak{B}u+C\mathfrak{C}u^2+....+M\mathfrak{M}u^{i-1}+N\mathfrak{N}u^{i}+\text{etc.},$$

there will be

$$N=\frac{(i-1)na-\frac{1}{2}(2\lambda-1)a-\frac{1}{2}c-\frac{1}{2}p}{-ina}M$$

and

$$\mathfrak{N} = \frac{(i-1)na - \frac{1}{2}(2\lambda-1)a - \frac{1}{2}c + \frac{1}{2}p}{inb - (2\lambda-1)b - e}\mathfrak{M};$$

and we will have by the comparison established with the general construction :

$$m = na, \quad h = -\tfrac{1}{2}(2\lambda-1)a - \tfrac{1}{2}c - \tfrac{1}{2}p,$$
$$n = -na \quad \text{and} \quad k = 0,$$
$$\mu = nb, \quad \eta = -\tfrac{1}{2}(2\lambda-1)a - \tfrac{1}{2}c + \tfrac{1}{2}p,$$
$$v = nb, \quad \theta = -(2\lambda-1)b - e.$$

Hence on putting $s = ut = x^{-n}t$ there will be

$$z = A(1+s)^{\frac{-h}{m}} = A\left(1+x^{-n}t\right)^{\frac{-h}{m}};$$

from which value found if now there is considered only the quantity t as variable, this construction arises :

$$y = Cx^{\lambda}\int t^{\frac{\eta}{\mu}-1}zdt\,(\mu - vt)^{\frac{\theta\mu-\eta v}{\mu v}},$$

where the limits of integration are thus to be put in place, so that for each there becomes

$$t^{\frac{\eta}{\mu}}(\mu - vt)^{\frac{\theta\mu-\eta v}{\mu v}+1} = 0.$$

SOLUTION 4

1073. Here also the twin coefficients can be interchanged , so that there shall be

$$m = na, \quad h = -\tfrac{1}{2}(2\lambda-1)a - \tfrac{1}{2}c + \tfrac{1}{2}p,$$
$$n = nb \quad k = -(2\lambda-1)b - e,$$
$$\mu = na, \quad \eta = -\tfrac{1}{2}(2\lambda-1)a - \tfrac{1}{2}c - \tfrac{1}{2}p,$$
$$v = -na, \quad \theta = 0,$$

as before [§ 1072] with λ taken from the equation $\lambda(\lambda-1)b + \lambda e + g = 0$ and on putting $\sqrt{\left((a-c)^2 - 4af\right)} = p$ there shall be $s = x^{-n}t$ and z is sought from the equation

$$\frac{dz}{ds} = \frac{Ak - z(k-hs)}{s(n-ms)},$$

thus so that on putting $s=0$ there becomes $z=A$, from which there shall be made

$$z=Aks^{\frac{-k}{n}}\left(n-ms\right)^{\frac{mk-nh}{mn}}\int s^{\frac{k}{n}-1}ds\left(n-ms\right)^{\frac{nh-mk}{mn}-1};$$

then indeed on considering *x* as constant there will be

$$y=Cx^{\lambda}\int t^{\frac{\eta}{\mu}-1}zdt\left(\mu-\nu t\right)^{\frac{\theta\mu-\eta\nu}{\mu\nu}},$$

where the two limits of integration thus are to be taken, so that in each there becomes

$$t^{\frac{\eta}{\mu}}\left(\mu-\nu t\right)^{\frac{\theta\mu-\eta\nu}{\mu\nu}+1}=0.$$

SCHOLIUM

1074. Generally these constructions are able to be shown in many more ways, since not only λ is able to have two values, but also the roots of the formulas *p* and *q* are able to rejoice in the ambiguity of signs. But these constructions and the preceding can be made to work in another way ; so that which may be made more clear, we may consider the equation [§ 978]

$$xx\left(1-xx\right)ddy-x\left(1+xx\right)dxdy+xxydx^{2}=0,$$

thus so that there shall be $a=1,\ b=-1,\ c=-1,\ e=-1,\ f=0$ and $g=1$ as well as $n=2$, from which for the two first constructions there may be considered $\lambda\left(\lambda-1\right)-\lambda=0$, hence either $\lambda=0$ or $\lambda=2$, then truly $q=\pm 2$. Hence the first construction gives

$$\lambda=0,\ m=-2,\ h=\mp 1,\ n=2,\ k=0,\ \mu=-2,\ \eta=\pm 1,\ \nu=2,\ \theta=-2,$$

from which there is deduced

$$z=A\left(1+xxt\right)^{\mp\frac{1}{2}}\ \text{ and }\ y=C\int t^{\mp\frac{1}{2}-1}zdt\left(1+t\right)^{\pm\frac{1}{2}-1}.$$

The lower signs may prevail, so that there shall be

$$z=\sqrt{\left(1+xxt\right)}\ \text{ and }\ y=C\int\frac{zdt}{\left(1+t\right)\sqrt{t\left(1+t\right)}};$$

which satisfy the formulas too, as we may observe. Certainly on taking *x* only to be variable, there shall become

$$\frac{dz}{dx}=\frac{xt}{\sqrt{\left(1+xxt\right)}}\ \text{ and }\ \frac{ddz}{dx^{2}}=\frac{t}{\left(1+xxt\right)^{\frac{3}{2}}},$$

hence

$$\frac{dy}{dx}=C\int\frac{xtdt}{t^{\frac{1}{2}}(1+t)^{\frac{3}{2}}(1+xxt)^{\frac{1}{2}}} \text{ and } \frac{ddy}{dx^2}=C\int\frac{tdt}{t^{\frac{1}{2}}(1+t)^{\frac{3}{2}}(1+xxt)^{\frac{3}{2}}}$$

from which there is accomplished

$$xx(1-xx)\frac{ddy}{dx^2}-x(1+xx)\frac{dy}{dx}+xxy=C\int\frac{xxdt(1-xxtt)}{t^{\frac{1}{2}}(1+t)^{\frac{3}{2}}(1+xxt)^{\frac{3}{2}}},$$

the integral of which formula is $\frac{2Cxx\sqrt{t}}{\sqrt{(1+t)(1+xxt)}}$; which since it vanishes both in the case $t=0$ as well as in the case $t=\infty$, the construction of our equation

$$y=C\int\frac{zdt}{(1+t)\sqrt{t(1+t)}}=C\int\frac{dt\sqrt{(1+xxt)}}{(1+t)\sqrt{t(1+t)}}$$

thus may be accomplished : on taking *x* as constant the integration is thus established, so that the integral vanishes on putting $t=0$, with which done there is put in place $t=\infty$ and the function of *x*, which will be produced for *y*, satisfies the proposed equation.

But if we select the other construction, with $\lambda=0$ there will be $m=-2,\ h=\pm1,\ n=2,\ k=-2,\ \mu=-2,\ \eta=\mp1,\ v=2,\ \theta=0$ and thus *z* must be defined from this equation

$$\frac{dz}{ds}=\frac{-2A+z(2\pm s)}{s(2\ +2s)}$$

with $s=xxt$, so that on putting $s=0$ there is made $z=A$. The upper sign may prevail, and since there shall be

$$dz-\frac{zds(2+s)}{2s(1+s)}=\frac{-Ads}{s(1+s)},$$

this may be multiplied by $\frac{\sqrt{(1+s)}}{s}$ and there will be the integral

$$\frac{z\sqrt{(1+s)}}{s}=\text{Const.}-A\int\frac{ds}{ss\sqrt{(1+s)}}$$

or

$$z=A-\frac{As}{\sqrt{(1+s)}}l\frac{1+\sqrt{(1+s)}}{\sqrt{s}}+\frac{Bs}{\sqrt{(1+s)}},$$

which on putting $s=0$ gives $z=A$, whatever *B* shall be. Following this there is

$$y=C\int t^{\frac{1}{2}-1}zdt(1+t)^{-\frac{1}{2}} \text{ or } y=C\int\frac{zdt}{\sqrt{t(1+t)}},$$

just as which value is not easily shown to be satisfactory ; and this method deserves to be developed more.

EXAMPLE

1075. *To show the construction of second order differential equation*

$$xx(1-xx)ddy - x(1+xx)dxdy + xxydx^2 = 0$$

arising from the preceding problem.

On account of $n=2,\ a=1,\ b=-1,\ c=-1,\ e=-1,\ f=0$ and $g=1$ for the first construction we have either $\lambda=0$ or $\lambda=2$; from which we may consider :

1. If $\lambda=0$, just as we have found,

$$m=-2,\quad h=\mp1,\quad n=2,\quad k=0,\quad \mu=-2,\quad \eta=\pm1,\quad \nu=2,\ \theta=-2,$$

from which there becomes

$$z=(1+xxt)^{\mp\frac{1}{2}} \quad \text{and} \quad y=C\int t^{\mp\frac{1}{2}-1}zdt(1+t)^{\pm\frac{1}{2}-1},$$

and thus a two–fold construction arises,

$$\text{either } z=\sqrt{(1+xxt)} \quad \text{and} \quad y=C\int\frac{zdt}{(1+t)\sqrt{t(1+t)}},$$

$$\text{or } z=\frac{1}{\sqrt{(1+xxt)}} \quad \text{and} \quad y=C\int\frac{zdt}{t\sqrt{t(1+t)}}.$$

2. If $\lambda=2$, there will be on account of $q=\pm2$

$$m=-2,\ h=-2\mp1,\ n=2,\ k=0,\ \mu=-2,\ \eta=-2\pm1,\ \nu=2,\ \theta=2,$$

from which there becomes

$$z=(1+xxt)^{\frac{-2\mp1}{2}} \quad \text{and} \quad y=Cx^2\int t^{1\mp\frac{1}{2}-1}zdt(1+t)^{\pm\frac{1}{2}}$$

and thus the two–fold construction may be considered,

$$\text{either } z=(1+xxt)^{\frac{-3}{2}} \quad \text{and} \quad y=Cx^2\int\frac{zdt\sqrt{(1+t)}}{\sqrt{t}},$$

$$\text{or } z=(1+xxt)^{\frac{-1}{2}} \quad \text{and} \quad y=Cx^2\int\frac{zdt\sqrt{t}}{\sqrt{(1+t)}}.$$

For the second general construction we may consider :

3. If $\lambda=0$, on interchanging these indices

$$m=-2,\ h=\pm1,\ n=2,\ k=-2,\ \mu=-2,\ \eta=\mp1,\ \nu=2,\ \theta=0,$$

from which on putting $xxt=s$ first z is sought from the equation $\frac{dz}{ds}=\frac{-2A+z(2\pm s)}{2s(1+s)}$, so that on putting $s=0$ there becomes $z=A$; then truly there will be

$$y=C\int t^{\pm\frac{1}{2}-1}zdt(1+t)^{\mp\frac{1}{2}}.$$

Hence therefore there arises the two–fold construction,

$$\text{either } \frac{dz}{ds}=\frac{-2A+z(2+s)}{2s(1+s)} \quad \text{and} \quad y=C\int\frac{zdt}{\sqrt{t}\sqrt{(1+t)}},$$

$$\text{or } \frac{dz}{ds}=\frac{-2A+z(2-s)}{2s(1+s)} \quad \text{and} \quad y=C\int\frac{zdt\sqrt{(1+t)}}{t\sqrt{t}}.$$

4. If $\lambda=2$, we will have

$$m=-2,\ h=-2\pm1,\ n=2,\ k=2,\ \mu=-2,\ \eta=-2\mp1,\ \nu=2,\ \theta=0$$

And on putting $xxt=s$ as before z is sought from the equation $\frac{dz}{ds}=\frac{2A-z(2+(2\mp1)s)}{2s(1+s)}$, and there will be

$$y=Cx^2\int t^{\pm\frac{1}{2}}zdt(1+t)^{\frac{-2\mp1}{2}}$$

and thus also a two–fold construction is elicited,

$$\text{either } \frac{dz}{ds}=\frac{2A-z(2+s)}{2s(1+s)} \quad \text{and} \quad y=Cx^2\int\frac{zdt\sqrt{t}}{(1+t)\sqrt{(1+t)}},$$

$$\text{or } \frac{dz}{ds}=\frac{2A-z(2+3s)}{2s(1+s)} \quad\text{and}\quad y=Cx^2\int\frac{zdt}{\sqrt{t(1+t)}}.$$

From the third solution we deduce at first that $-\lambda(\lambda-1)-\lambda+1=0$ or $\lambda\lambda=1$ and thus $\lambda=\pm1$ and $p=\sqrt{4}=\pm 2$, whereby:

5. If there is taken $\lambda=+1$, then there will be

$$m=2,\ h=\mp1,\ n=-2,\ k=0,\ \mu=2,\ \eta=\pm1,\ v=-2,\ \theta=2$$

and hence

$$z=\left(1+\frac{t}{xx}\right)^{\pm\frac{1}{2}} \quad\text{and}\quad y=Cx\int t^{\pm\frac{1}{2}-1}zdt(1+t)^{-1\mp\frac{1}{2}},$$

thus so that again there is a two–fold construction to be considered,

$$\text{either } z=\frac{1}{x}\sqrt{(xx+t)} \quad\text{and}\quad y=Cx\int\frac{zdt}{(1+t)\sqrt{t(1+t)}},$$

$$\text{or } z=\frac{x}{\sqrt{(xx+t)}} \quad\text{and}\quad y=Cx\int\frac{zdt}{t\sqrt{t(1+t)}}.$$

6. If $\lambda=-1$ is taken, then there will be

$$m=2,\ h=2\mp1,\ n=-2,\ k=0,\ \mu=2,\ \eta=2\pm1,\ v=-2,\ \theta=-2$$

and hence

$$z=\left(1+\frac{t}{xx}\right)^{-1\pm\frac{1}{2}} \quad\text{and}\quad y=\frac{C}{x}\int t^{\pm\frac{1}{2}}zdt(1+t)^{\mp\frac{1}{2}},$$

from which the two constructions emanate,

$$\text{either } z=\frac{x}{\sqrt{(xx+t)}} \quad\text{and}\quad y=\frac{C}{x}\int\frac{zdt\sqrt{t}}{\sqrt{(1+t)}},$$

$$\text{or } z=\frac{x^3}{(xx+t)^{\frac{3}{2}}} \quad\text{and}\quad y=\frac{C}{x}\int\frac{zdt\sqrt{(1+t)}}{\sqrt{t}}.$$

And hence from the fourth solution we may conclude:

7. If $\lambda=+1$,

$$m=2,\ h=\pm1,\ n=-2,\ k=2,\ \mu=2,\ \eta=\mp1,\ v=-2,\ \theta=0.$$

Now on putting $s=\frac{t}{xx}$, z is sought from this equation $\frac{dz}{ds}=\frac{-2A+z(2\mp s)}{2s(1+s)}$, so that on putting $s=0$ there becomes $z=A$, and then there will be

$$y = Cx\int t^{\mp\frac{1}{2}-1} zdt\left(1+t\right)^{\pm\frac{1}{2}},$$

from which the two–fold construction follows,

$$\text{either } \frac{dz}{ds} = \frac{-2A+z(2-s)}{2s(1+s)} \quad \text{and} \quad y = Cx\int \frac{zdt\sqrt{(1+t)}}{t\sqrt{t}},$$
$$\text{or } \frac{dz}{ds} = \frac{-2A+z(2+s)}{2s(1+s)} \quad \text{and} \quad y = Cx\int \frac{zdt}{\sqrt{t}(1+t)}.$$

8. If $\lambda = -1$, there will be considered

$$m = 2,\ h = 2\pm 1,\ n = -2,\ k = -2,\ \mu = 2,\ \eta = 2\mp 1,\ v = -2,\ \theta = 0,$$

and on putting $\frac{t}{xx} = s$, z must be sought from this equation $\frac{dz}{ds} = \frac{2A-z(2+(2\pm1)s)}{2s(1+s)}$, so that on putting $s = 0$ there becomes $z = A$; with which accomplished there will be

$$y = \frac{C}{x}\int t^{\mp\frac{1}{2}} zdt\left(1+t\right)^{-1\pm\frac{1}{2}}$$

and thus the two–fold construction,

$$\text{either } \frac{dz}{ds} = \frac{2A-z(2+3s)}{2s(1+s)} \quad \text{and} \quad y = \frac{C}{x}\int \frac{zdt}{\sqrt{t}(1+t)},$$
$$\text{or } \frac{dz}{ds} = \frac{2A-z(2+s)}{2s(1+s)} \quad \text{and} \quad y = \frac{C}{x}\int \frac{zdt\sqrt{t}}{(1+t)\sqrt{(1+t)}}.$$

Therefore we have the entire sixteen constructions to follow.

SCHOLION

1076. We may create a danger in showing how these constructions which are harder to be seen satisfy the proposed equation ; and to this end we select the latter construction *no.* 4, which has

$$dz + \frac{zds(2+3s)}{2s(1+s)} = \frac{Ads}{s(1+s)}.$$

This multiplied by $s\sqrt{(1+s)}$ gives the integral

$$sz\sqrt{(1+s)} = A\int \frac{ds}{\sqrt{(1+s)}} = 2A\sqrt{(1+s)} + B$$

or

$$z = \frac{2A}{s} + \frac{B}{s\sqrt{(1+s)}}.$$

Now on putting $s=0$, as there becomes $z=A$, there must be $B=-2A$,
[Thus, for small *s*, $z=\frac{2A}{s}+\frac{B}{s\sqrt{(1+s)}}=\frac{2A}{s}+\frac{B}{s}\left(1-\frac{s}{2}+..\right)=\frac{-B}{2}+...$]
so that there shall be

$$z=\frac{2A\left(\sqrt{(1+s)}-1\right)}{s\sqrt{(1+s)}}=\frac{2A}{txx}-\frac{2A}{txx\sqrt{(1+txx)}}.$$

Hence there becomes

$$\left(\frac{dz}{dx}\right)=\frac{-4A}{tx^3}+\frac{2A(2+3txx)}{tx^3(1+txx)^{\frac{3}{2}}}$$

and

$$\left(\frac{ddz}{dx^2}\right)=\frac{12A}{tx^4}-\frac{6A\left(2+5txx+4ttx^4\right)}{tx^4(1+txx)^{\frac{5}{2}}}.$$

Now since there shall be $y=C\int\frac{xxzdt}{\sqrt{t(1+t)}}$, then there will be

$$\left(\frac{dy}{dx}\right)=2C\int\frac{xzdt}{\sqrt{t(1+t)}}+C\int\frac{xxdt}{\sqrt{t(1+t)}}\left(\frac{dz}{dx}\right)$$

and

$$\left(\frac{ddy}{dx^2}\right)=2C\int\frac{zdt}{\sqrt{t(1+t)}}+4C\int\frac{xdt}{\sqrt{t(1+t)}}\left(\frac{dz}{dx}\right)+C\int\frac{xxdt}{\sqrt{t(1+t)}}\left(\frac{ddz}{dx^2}\right)$$

and hence

$$xx(1-xx)ddy-x(1+xx)dxdy+xxydx^2$$
$$=C\int\frac{dt}{\sqrt{t(1+t)}}\left(\frac{2Axx}{t}-\frac{2Axx(1+4txx+3ttxx)}{t(1+txx)^{\frac{5}{2}}}\right),$$

because the integral is

$$\frac{-4ACxx\sqrt{(1+t)}}{\sqrt{t}}+\frac{4ACxx\sqrt{(1+t)}}{(1+txx)^{\frac{3}{2}}\sqrt{t}}=\frac{4ACxx\sqrt{(1+t)}}{\sqrt{t}}\left(\frac{1}{(1+txx)^{\frac{3}{2}}}-1\right),$$

it can be expressed in this form

$$-2Cx^4\left(3z+x\left(\frac{dz}{dx}\right)\right)\sqrt{t(1+t)},$$

or also in this way

$$-2Cx^4\left(\tfrac{2A+z}{1+txx}\right)\sqrt{t(1+t)}.$$

But this expression itself becomes $=0$, in the first place if $t=-1$, and then also if $t=0$, from which the value found for y

$$y = D\int \frac{dt}{t\sqrt{t(1+t)}}\left(1-\frac{1}{\sqrt{(1+txx)}}\right)$$

thus must be defined by integration, so that it vanishes on putting $t=0$, then truly on putting $t=-1$. Or on putting $t=-v$ there will be

$$y = D\int \frac{dv}{v\sqrt{v(1-v)}}\left(1-\frac{1}{\sqrt{(1-vxx)}}\right)$$

with the integral thus taken, so that it vanished on putting $v=0$, and then on making $v=1$.

This example is sufficient to show how the constructions shown satisfy the second order differential equation ; now meanwhile if the quantity z is expressed transcendently evidently by logarithms, it is only possible to reach agreement by an exceedingly tedious calculation.

PROBLEM 136

1077. *On putting*

$$y = C\int (1+t)^{\nu-1}(a+tx)^{\lambda}\,dt\,,$$

in which integration the quantity x is considered as a constant, the definition of the integral henceforth must be investigated by the limits, so that y is equal to a certain function of x, and to find second order differential equations of the form

$$Lxx\frac{ddy}{dx^2} + Mx\frac{dy}{dx} + Ny = 0$$

that satisfy that function.

SOLUTION

Since there shall be from the principles established previously[§ 1017]

$$\frac{dy}{dx} = C\int \lambda t(1+t)^{\nu-1}(a+tx)^{\lambda-1}\,dt$$

and

$$\frac{ddy}{dx^2} = C\int \lambda(\lambda-1)tt(1+t)^{\nu-1}(a+tx)^{\lambda-2}\,dt,$$

there will be

$$Lxx\frac{ddy}{dx^2}+Mx\tfrac{dy}{dx}+Ny$$
$$=C\int(1+t)^{\nu-1}(a+tx)^{\lambda-2}\,dt\Big(\lambda(\lambda-1)Lttxx+\lambda Mtx(a+tx)+N(a+tx)^2\Big)$$
$$=C\int(1+t)^{\nu-1}(a+tx)^{\lambda-2}\,dt\begin{Bmatrix}Naa+ & Natx+ & Nttxx\\ & +\lambda Matx+ & \lambda Mttxx\\ & & +\lambda(\lambda-1)Lttxx\end{Bmatrix}$$

which formula with x constant must be completely integrable. Therefore there is put the integral

$$C(1+t)^{\nu}(a+tx)^{\lambda-1}(Paa+Qatx)$$

with P and Q denoting some functions of x; the differential of this will be

$$C(1+t)^{\nu-1}(a+tx)^{\lambda-2}\,dt$$
$$\times\Big(\nu(a+tx)(Paa+Qatx)+(\lambda-1)x(1+t)(Paa+Qatx)+Qax(1+t)(a+tx)\Big)$$

$$=C(1+t)^{\nu-1}(a+tx)^{\lambda-2}\,dt\begin{Bmatrix}\nu Pa^3 & + & \nu Qaatx+ & \nu Qattxx\\ +(\lambda-1)Paax+ & & \nu Paatx+ & (\lambda-1)Qattxx\\ + & Qaax+ & (\lambda-1)Paatx+ & Qattxx\\ & + & (\lambda-1)Qatxx & \\ & + & Qaatx & \\ & + & Qatxx & \end{Bmatrix}$$

with which form compared with that we come upon

$$N=\nu Pa+(\lambda-1)Px+Qx,$$
$$2N+\lambda M=(\nu+1)Qa+(\lambda+\nu-1)Pa+\lambda Qx,$$
$$N+\lambda M+\lambda(\lambda-1)L=(\lambda+\nu)Qa,$$

of which equations the first and third taken from the second give

$$\lambda(\lambda-1)L=-(\lambda-1)Pa+(\lambda-1)Px+(\lambda-1)Qa-(\lambda-1)Qx$$

and hence

$$\lambda L=(a-x)(Q-P) \text{ or } L=\tfrac{1}{\lambda}(a-x)(Q-P);$$

but twice the first taken from the second gives

$$\lambda M=(\lambda-v-1)Pa-2(\lambda-1)Px+(v+1)Qa+(\lambda-2)Qx$$

or

$$\lambda M=((v+1)a+(\lambda-2)x)(Q-P)+\lambda(a-x)P.$$

Whereby with some functions of *x* taken for *P* and *Q* if the functions *L, M*, N thus are defined, so that

$$L=\tfrac{1}{\lambda}(a-x)(Q-P),$$
$$M=\tfrac{1}{\lambda}((v+1)a+(\lambda-2)x)(Q-P)+(a-x)P,$$
$$N=x(Q-P)+(va+\lambda x)P,$$

with the second order differential equation

$$Lxxddy+Mxdxdy+Nydx^2=0$$

satisfied by the integral formula

$$y=C\int(1+t)^{v-1}(a+tx)^{\lambda}\,dt$$

with *x* treated as constant, provided the limits of the integration are thus put in place, so that in each the expression $(1+t)^{v}(a+tx)^{\lambda-1}(Pa+Qtx)$ vanishes. Moreover it is required to be noted that these limits do not depend on *x*. But first it is clear that this expression becomes $=0$ in the case $t=-1$, only if there shall be $v>0$. Hence on putting $t=\infty$ it also vanishes, only if $v+\lambda-1+1$ shall be a negative number or $v+\lambda<0$. On account of which if there shall be $v>0$ and $v+\lambda<0$, the integral $y=C\int(1+t)^{v-1}(a+tx)^{\lambda}\,dt$ thus must be taken , so that it vanishes on putting $t=-1$; then truly there is put in place $t=\infty$ and the function of *x* resulting for *y* should satisfy the proposed equation.

COROLLARY 1

1078. Because the functions *P* and *Q* do not enter into the integral formula assumed for *y*, it is evident that the same formula satisfies all the second order differential equations , whatever values may be attributed to the letters *P* and *Q*.

COROLLARY 2

1079. Therefore on taking $Q=P$ the same integral formula [*lit*. 'formula of the integral']

$$y=C\int(1+t)^{\nu-1}(a+tx)^{\lambda}\,dt$$

also satisfies this differential equation of the first order

$$(a-x)xdy+(\nu a+\lambda x)ydx=0.$$

Now the integral of this is

$$y=\frac{D(a-x)^{\lambda+\nu}}{x^{\nu}},$$

which value therefore also in general satisfies our second order differential equation ; which soon becomes apparent on being tested.

COROLLARY 3

1080. Therefore this value of the integral $y=C\int(1+t)^{\nu-1}(a+tx)^{\lambda}\,dt$ taken following the defined limits must agree with the algebraic formula $y=\frac{D(a-x)^{\lambda+\nu}}{x^{\nu}}$, but only if there shall be $\nu>0$ and $\lambda+\nu<0$.

SCHOLIUM

1081. Therefore the integration in this problem may be considered less hidden. Truly the reduction of the formula of the integral $y=C\int(1+t)^{\nu-1}(a+tx)^{\lambda}\,dt$ to $y=\frac{D(a-x)^{\lambda+\nu}}{x^{\nu}}$, to which it is reduced, therefore is thus more noteworthy if the integral is taken so that it vanishes on putting $t=-1$ and $t=\infty$. Hence on putting $\lambda+\nu=-\mu$, so that μ and ν are positive numbers, there will be $C\int\frac{(1+t)^{\nu-1}dt}{(a+tx)^{\mu+\nu}}=\frac{D}{x^{\nu}(a-x)^{\mu}}$.

Or there is put $1+t=z$; then there will be

$$C\int\frac{z^{\nu-1}dz}{(a-x+xz)^{\mu+\nu}}=\frac{D}{x^{\nu}(a-x)^{\mu}}$$

with the limits of this integration $z=0$ and $z=\infty$ present. Now also this observation is not of great note; for on putting $a-x=ux$ there is produced

$$\frac{C}{x^{\mu+\nu}}\int\frac{z^{\nu-1}dz}{(u+z)^{\mu+\nu}}=\frac{D}{x^{\mu+\nu}u^{\mu}}$$

and this formula $\int\frac{z^{\nu-1}dz}{(u+z)^{\mu+\nu}}$ thus integrated, so that it vanishes on putting $z=0$, if then there is put $z=\infty$, adopts this form $\frac{A}{u^{\mu}}$ in which A denotes a constant quantity not depending on *u*. But it may depend on the exponents μ and *v* by a rule easily observed from the cases. Evidently on putting

$$\int\frac{z^{v-1}dz}{(u+z)^{\mu+v}}=\frac{A}{u^{\mu}}$$

if there shall be $v=1$, then that integral becomes $-\frac{1}{\mu(u+z)^{\mu}}+\frac{1}{\mu u^{\mu}}$ and on putting $z=\infty$ there is produced $\frac{1}{\mu u^{\mu}}$, thus so that in this case there shall be $A=\frac{1}{\mu}$. If there shall be $v=2$, then the integration is also successful and there is found $A=\frac{1}{\mu(\mu+1)}$, if $v=3$, there becomes $A=\frac{1\cdot 2}{\mu(\mu+1)(\mu+2)}$, and if $v=4$ there becomes $A=\frac{1\cdot 2\cdot 3}{\mu(\mu+1)(\mu+2)(\mu+3)}$, from which in general we may conclude that there is

$$A=\frac{1\cdot 2\cdot 3\cdots(v-1)}{\mu(\mu+1)(\mu+2)\cdots(\mu+v-1)}.$$

Whereby the integration according to the prescribed rule established will be

$$\frac{1}{\mu}\cdot\frac{2}{\mu+1}\cdot\frac{3}{\mu+2}\cdots\frac{v-1}{\mu+v-1}=\mu u^{\mu}\int\frac{z^{v-1}dz}{(u+z)^{\mu+v}}$$

But if the exponent *v* should not be a whole number, the value of A with the aid of the interpolation of this formula may be defined by the preceding factors. Evidently the quadratures of the circle may be present, if the exponent *v* involves the fraction $\frac{1}{2}$, but concerning interpolations of this kind, we are urged further elsewhere as this is neither this place nor this the argument to be fruitfully pursued.

There remains the final chapter of this section, in which there will be taught the integration of second order differential equations by approximations.

CAPUT XI

DE CONSTRUCTIONE AEQUATIONUM DIFFERENTIO–DIFFERENTIALUM EX EARUM RESOLUTIONE PER SERIES INFINITAS PETITA

PROBLEMA 133

1059. *Proposita serie infinita*

$$A+Bs+Cs^2+\ldots+Ms^{i-1}+Ns^i+\text{etc.},$$

in qua sit

$$B=\tfrac{0m+h}{1n+k}A,\quad C=\tfrac{1m+h}{2n+k}B,\quad D=\tfrac{2m+h}{3n+k}C$$

et in genere

$$N=\tfrac{(i-1)m+h}{in+k}M,$$

eius summam per formulam integralem exprimere.

SOLUTIO

Ponatur summa quaesita = z, ita ut sit

$$z=A+Bs+Cs^2+\ldots+Ms^{i-1}+Ns^i+\text{etc.},$$

eritque differentiando

$$\tfrac{sdz}{ds}=0A+1Bs+2Cs^2+3Ds^3+\ldots+(i-1)Ms^{i-1}+iNs^i+\text{etc.},$$

ex cuius combinatione cum praecedente oritur

$$\tfrac{msdz}{ds}+hz=hA+(m+h)Bs+(2m+h)Cs^2+\ldots+((i-1)m+h)Ms^{i-1}+(im+h)Ns^i+\text{etc.}$$

Deinde vero etiam simili modo est

$$\tfrac{nsdz}{ds}+kz=kA+(n+k)Bs+(2n+k)Cs^2+\ldots+((i-1)n+k)Ms^{i-1}+(in+k)Ns^i+\text{etc.};$$

ergo ob

$$(n+k)B=hA,\quad (2n+k)C=(m+h)B,\quad \text{etc.}$$

erit

$$\tfrac{nsdz}{ds}+kz=kA+hAs+(m+h)Bs^2+(2m+h)Cs^3+\text{etc.},$$

unde manifesto conficitur

$$\frac{nsdz}{ds}+kz=kA+\frac{mssdz}{ds}+hsz \quad \text{or} \quad sdz(n-ms)+zds(k-hs)=kAds,$$

hoc est

$$dz+\frac{zds(k-hs)}{s(n-ms)}=\frac{kAds}{s(n-ms)}.$$

Cum nunc sit

$$\frac{ds(k-hs)}{s(n-ms)}=\frac{kds}{ns}+\frac{(mk-nh)ds}{n(n-ms)},$$

aequatio ista integrabilis fit multiplicata per $s^{\frac{k}{n}}(n-ms)^{\frac{nh-mk}{mn}}$ proditque

$$(n-ms)^{\frac{nh-mk}{mn}}s^{\frac{k}{n}}z=Ak\int s^{\frac{k}{n}-1}ds(n-ms)^{\frac{nh-mk}{mn}-1}$$

quod integrale ita capi oportet, ut posito $s=0$ fiat $z=A$, quo observato habebimus

$$z=Aks^{-\frac{k}{n}}(n-ms)^{\frac{mk-nh}{mn}}\int s^{\frac{k}{n}-1}ds(n-ms)^{\frac{nh-mk}{mn}-1}.$$

COROLLARIUM 1

1060. Peculiari solutione eget casus $m=0$, quo fit

$$dz+\frac{zds(k-hs)}{ns}=\frac{Akds}{ns},$$

quae per $s^{\frac{k}{n}}e^{-\frac{hs}{n}}$ multiplicata praebet

$$e^{-\frac{hs}{n}}s^{\frac{k}{n}}z=\frac{Ak}{n}\int e^{-\frac{hs}{n}}s^{\frac{k}{n}-1}ds \text{ ideoque } z=\frac{Ak}{n}e^{\frac{hs}{n}}s^{\frac{-k}{n}}\int e^{-\frac{hs}{n}}s^{\frac{k}{n}-1}ds$$

integrali ita sumto, ut fiat $z=A$ posito $s=0$.

COROLLARIUM 2

1061. Casus etiam $n=0$ seorsim resolvi debet; aequatio enim

$$dz+zds\frac{hs-k}{mss}=-\frac{Akds}{mss}$$

multiplicari debet per $s^{\frac{h}{m}}e^{\frac{k}{ms}}$, et invenitur integrale

$$e^{\frac{k}{ms}}s^{\frac{h}{n}}z=-\frac{Ak}{n}\int e^{\frac{k}{ms}}s^{\frac{k}{m}-2}ds \text{ ideoque } z=-\frac{Ak}{n}e^{\frac{-k}{ms}}s^{\frac{-h}{m}}\int e^{\frac{k}{ms}}s^{\frac{h}{m}-2}ds.$$

COROLLARIUM 3

1062. Si fuerit et $m=0$ et $n=0$, ob $N=\frac{h}{k}M$ series nostra erit geometrica, aequatio vero nostra erit

$$zds(k-hs)=Akds \quad \text{seu} \quad z=\frac{Ak}{k-hs},$$

uti natura rei manifesto postulate

SCHOLION

1063. Imprimis hic casus notari meretur, quo est $k=0$ et summa z sine signo integrali exprimi potest; erit namque

$$(n-ms)^{\frac{h}{m}}z=\text{Const.},$$

et quia, si $s=0$, fieri debet $z=A$, erit $\text{Const.}=A^{\frac{h}{m}}$ ideoque

$$z=A^{\frac{h}{m}}(n-ms)^{\frac{-h}{m}} \text{ seu } z=A\left(\frac{n}{n-ms}\right)^{\frac{h}{m}} \text{ vel etiam } z=A\left(1-\frac{ms}{n}\right)^{\frac{-h}{m}}.$$

At vero integratio etiam succedit casu, quo $k=n$; erit enim

$$(n-ms)^{\frac{h}{m}-1}sz=An\int ds(n-ms)^{\frac{h}{m}-2},$$

quod integrale est

$$\text{Const.}-\frac{An(n-ms)^{\frac{h}{m}-1}}{h-m},$$

et quia posito $s=0$ fit $z=A$, erit

$$0=\text{Const.}-\frac{A}{h-m}n^{\frac{h}{m}}$$

hincque

$$z=\frac{An}{(h-m)s}\left(\left(\frac{n}{n-ms}\right)^{\frac{h}{m}-1}-1\right)=\frac{An}{(h-m)s}\left(\left(1-\frac{ms}{n}\right)^{1-\frac{h}{m}}-1\right).$$

Porro perspicuum est integrationem expediri posse casu $k=2n$; quo cum sit

$$(n-ms)^{\frac{h}{m}-2}ssz=2An\int sds(n-ms)^{\frac{h}{m}-3},$$

erit hoc integrale

$$=\text{Const.}-\frac{2Ans}{h-2m}(n-ms)^{\frac{h}{m}-2}+\frac{2An}{h-2m}\int ds(n-ms)^{\frac{h}{m}-2}$$

$$=\text{Const.}-\frac{2Ans}{h-2m}(n-ms)^{\frac{h}{m}-2}-\frac{2An(n-ms)^{\frac{h}{m}-1}}{(h-m)(h-2m)},$$

ubi

$$\text{Const.} = \frac{2An^{\frac{h}{m}}}{(h-m)(h-2m)}$$

ideoque

$$z = \frac{2Ann}{(h-m)(h-2m)ss}\left(\left(\frac{n}{n-ms}\right)^{\frac{h}{m}-2} - 1 - \frac{(h-2m)s}{n}\right).$$

Similique modo etiam integratio casibus $k = 3n,\ k = 4n$ etc. absolvetur.

PROBLEMA 134

1064. *Proposita huiusmodi serie infinita*

$$A\mathfrak{A} + B\mathfrak{B}u + C\mathfrak{C}u^2 + D\mathfrak{D}u^3 + \ldots.. + M\mathfrak{M}u^{i-1} + N\mathfrak{N}u^i + \text{etc.}$$

coefficientium lege existente

$$B = \frac{0m+h}{1n+k}A,\quad C = \frac{1m+h}{2n+k}B,\quad D = \frac{2m+h}{3n+k}C,\quad \ldots\ldots\quad N = \frac{(i-1)m+h}{in+k}M,$$
$$\mathfrak{B} = \frac{0\mu+\eta}{1\nu+\theta}\mathfrak{A},\quad \mathfrak{C} = \frac{1\mu+\eta}{2\nu+\theta}\mathfrak{B},\quad \mathfrak{D} = \frac{2\mu+\eta}{3\nu+\theta}\mathfrak{C},\quad \ldots\ldots\quad \mathfrak{N} = \frac{(i-1)\mu+\eta}{i\nu+\theta}\mathfrak{M}$$

eius summam per formulam integralem exprimere.

SOLUTIO

Posita summa

$$y = A\mathfrak{A} + B\mathfrak{B}u + C\mathfrak{C}u^2 + D\mathfrak{D}u^3 + E\mathfrak{E}u^4 + \text{etc.}$$

consideretur series hoc modo formata

$$z = A + Bux + Cu^2x^2 + Du^3x^3 + Eu^4x^4 + \text{etc.}$$

cuius summa posito $ux = s$ est, ut modo [§ 1059] invenimus,

$$z = Aks^{-\frac{k}{n}}(n-ms)^{\frac{mk-nh}{mn}}\int s^{\frac{k}{n}-1}ds(n-ms)^{\frac{nh-mk}{mn}-1}$$

integrali ita determinato, ut posito $s = 0$ fiat $z = A$.

Formetur huiusmodi formula integralis

$$V = \int Pzdx = \int Pdx\left(A + Bux + Cu^2x^2 + Du^3x^3 + Eu^4x^4 + \text{etc.}\right),$$

in qua u spectetur ut constans, pro P autem eiusmodi functio ipsius x accipiatur, ut fiat

$$\int Pxdx = \tfrac{\mathfrak{B}}{\mathfrak{A}}\int Pdx, \quad \int Px^2dx = \tfrac{\mathfrak{C}}{\mathfrak{B}}\int Pxdx, \quad \int Px^3dx = \tfrac{\mathfrak{D}}{\mathfrak{C}}\int Px^2dx, \quad \text{etc.},$$

erit

$$V = \left(A + \tfrac{B\mathfrak{B}}{\mathfrak{A}}u + \tfrac{C\mathfrak{C}}{\mathfrak{A}}u^2 + \tfrac{D\mathfrak{D}}{\mathfrak{A}}u^3 + \text{etc.}\right)\int Pdx,$$

unde patet fore

$$y = \tfrac{\mathfrak{A}V}{\int Pdx} = \tfrac{\mathfrak{A}\int Pzdx}{\int Pdx}.$$

Quare cum valor ipsius z sit cognitus, tantum superest, ut functio P ipsius x conditionibus memoratis praedita investigetur. In genere autem esse oportet

$$\int Px^idx = \tfrac{\mathfrak{N}}{\mathfrak{M}}\int Px^{i-1}dx = \tfrac{(i-1)\mu+\eta}{iv+\theta}\int Px^{i-1}dx;$$

quae aequalitas, cum sufficiat, ut tantum certo quodam casu, quo ipsi x datus tribuitur valor, subsistat, ponamus in genere esse

$$(iv+\theta)\int Px^idx = \left((i-1)\mu+\eta\right)\int Px^{i-1}dx + x^iQ,$$

ita ut pro terminis integralibus sit $Q = 0$. Differentiando ergo habebimus

$$(iv+\theta)Px^idx = (i\mu-\mu+\eta)Px^{i-1}dx + x^idQ + ix^{i-1}Qdx$$

seu per x^{i-1} dividendo

$$(iv+\theta)Pxdx = (i\mu-\mu+\eta)Pdx + xdQ + iQdx;$$

quae aequalitas cum pro omnibus valoribus ipsius i aeque subsistere debeat, hinc duas adipiscimur aequationes

$$vPxdx = \mu Pdx + Qdx \quad \text{et} \quad \theta Pxdx = (\eta-\mu)Pdx + xdQ,$$

unde duplici modo colligimus

$$Pdx = \tfrac{Qdx}{vx-\mu} \quad \text{et} \quad Pdx = \tfrac{xdQ}{\theta x-(\eta-\mu)}$$

sicque alterum valorem per alterum dividendo

$$\frac{xdQ}{Qdx} = \frac{\theta x + \mu - \eta}{\nu x - \mu} \quad \text{seu} \quad \frac{dQ}{Q} = \frac{dx(\theta x + \mu - \eta)}{x(\nu x - \mu)},$$

quae evolvitur in

$$\frac{dQ}{Q} = \frac{\eta - \mu}{\mu x}dx + \frac{\mu\theta + \mu\nu - \eta\nu}{\mu(\nu x - \mu)}dx.$$

Hinc integrando elicitur

$$Q = x^{\frac{\eta}{\mu}-1}(\nu x - \mu)^{\frac{\theta\mu - \eta\nu}{\mu\nu}+1} \quad \text{in Const.}$$

seu

$$Q = -x^{\frac{\eta}{\mu}-1}(\mu - \nu x)^{\frac{\theta\mu - \eta\nu}{\mu\nu}+1},$$

unde fit

$$Pdx = x^{\frac{\eta}{\mu}-1}dx(\mu - \nu x)^{\frac{\theta\mu - \eta\nu}{\mu\nu}}.$$

Cum igitur sit

$$(i\nu + \theta)\int Px^i dx = ((i-1)\mu + \eta)\int Px^{i-1}dx - x^{i+\frac{\eta}{\mu}-1}(\mu - \nu x)^{\frac{\theta\mu - \eta\nu}{\mu\nu}+1},$$

si haec integralia ita capiantur, ut evanescant posito $x = 0$, tum vero statuatur $x = \frac{\mu}{\nu}$, fiet, uti hypothesis nostra postulat,

$$\int Px^i dx = \frac{(i-1)\mu + \eta}{i\nu + \theta}\int Px^{i-1}dx\ ;$$

at vero in hunc finem necesse est, ut sit

$$i + \frac{\eta}{\mu} - 1 > 0 \quad \text{et} \quad \frac{\theta\mu - \eta\nu}{\mu\nu} + 1 > 0.$$

Quae conditio si locum habeat, seriei propositae summa ita exprimetur, ut sit

$$y\int x^{\frac{\eta}{\mu}-1}dx(\mu - \nu x)^{\frac{\theta\mu - \eta\nu}{\mu\nu}} = \mathfrak{A}\int x^{\frac{\eta}{\mu}-1}zdx(\mu - \nu x)^{\frac{\theta\mu - \eta\nu}{\mu\nu}}$$

existente

$$z = Aks^{\frac{-k}{n}}(n - ms)^{\frac{mk - nh}{mn}}\int s^{\frac{k}{n}-1}ds(n - ms)^{\frac{nh - mk}{mn}-1}$$

integrali hoc ita sumto, ut fiat $z = A$ posito $s = 0$. Hoc autem integrali invento pro s scribatur ux et hoc valore ipsius z in illa formula substituto quantitatem u tanquam constantem tractari oportet, quoad illae integrationes lege praescripta fuerint absolutae; tum enim pro *y* prodibit functio ipsius *u* summam seriei propositae exprimens.

COROLLARIUM 1

1065. Quia in geminatis coefficientibus nostrae seriei similis lex progressionis assumitur, singulas series *A, B, C, D* etc.. et $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$ etc . inter se permutare licet, unde hac methodo duplex formula summam seriei exprimens obtinetur.

COROLLARIUM 2

1066. Etsi functio *Q* non in calculum ingreditur, eam tamen nosse oportet, quoniam ex eius indole termini integrationis constitui debent, ita ut pro utroque fiat $Q=0$. Hi scilicet termini sunt $x=0$ et $x=\frac{\mu}{\nu}$, dum fuerit $i+\frac{\eta}{\nu}-1>0$ et $\frac{\theta\mu-\eta\nu}{\mu\nu}+1>0$, ubi *i* est numerus integer positivus.

COROLLARIUM 3

1067. Pro functione *Q* casus, quo vel $\mu=0$ vel $\nu=0$, seorsim sunt evolvendi. Priori, quo $\mu=0$, est

$$\frac{dQ}{Q}=\frac{dx(\theta x-\eta)}{\nu xx}=\frac{\theta dx}{\nu x}-\frac{\eta dx}{\nu xx},$$

unde fit

$$Q=e^{\frac{\eta}{\nu x}}x^{\frac{\theta}{\nu}}.$$

Posteriori, quo $\nu=0$, est

$$\frac{dQ}{Q}=\frac{dx(\theta x+\mu-\eta)}{-\mu x}=-\frac{\theta dx}{\mu}+\frac{\eta-\mu}{\mu}\cdot\frac{dx}{x}$$

ideoque

$$Q=e^{-\frac{\theta x}{\mu}}x^{\frac{\eta}{\mu}-1}.$$

SCHOLION

1068. Constructiones hoc modo adornandae prorsus similes sunt iis, quas capite praecedente tradidimus, cum res etiam ad formulam integralem huiusmodi $\int Vdx$ reducatur, in qua *V* est functio binarum variabilium *u* et *x*, quarum illa autem *u* in ipsa integratione constans reputatur, post integrationem vero ipsi *x* datus quidam valor assignatur. Verum tamen haec constructio ad casus in superiori methodo non contentos extenditur, quandoquidem fieri potest, ut quantitas *z* functiones maxime transcendentes involvat. Vicissim autem vidimus methodum praecedentem ad eiusmodi aequationes applicari posse, quae per series, quales hic tractamus, evolvi nequeant, unde in Analysin ex hoc fonte haud contemnenda incrementa hauriri posse videntur.

PROBLEMA 135

1069. *Proposita aequatione differentio-differentiali*

$$xx\left(a+bx^n\right)ddy+x\left(c+ex^n\right)dxdy+\left(f+gx^n\right)ydx^2 = 0$$

valorem ipsius y per formulam integralem construere.

SOLUTIO I

Hanc aequationem supra (§ 967) ita in seriem evolvimus, ut posito

$$y=x^\lambda\left(A+Bx^n+Cx^{2n}+Dx^{3n}+Ex^{4n}+\text{etc.}\right)$$

primo exponenti λ tribui debeat radix huius aequationis $\lambda(\lambda-1)a+\lambda c+f=0$; tum vero posito brevitatis gratia $\lambda(\lambda-1)b+\lambda e+g=h$ fit

$$B=\frac{-h}{n\left(na+(2\lambda-1)a+c\right)}A,\ \ C=\frac{-nnb-(2\lambda-1)nb-ne-h}{2n\left(2na+(2\lambda-1)a+c\right)}B,\ \ D=\frac{-4nnb-2(2\lambda-1)nb-2ne-h}{3n\left(3na+(2\lambda-1)a+c\right)}C$$

ideoque illius seriei positis binis terminis contiguis indefinite $Mx^{(i-1)n}+Nx^{in}$ generaliter

$$N=\frac{-(i-1)^2nnb-(2\lambda-1)(i-1)nb-(i-1)ne-h}{in\left(ina+(2\lambda-1)a+c\right)}M$$

ubi cum denominator iam habeat factores, quales ante [§ 1064] assumsimus, numeratorem quoque in factores resolvamus, quo nihilo aequali posito invenitur

$$(i-1)n=-\tfrac{1}{2}(2\lambda-1)-\tfrac{e}{2b}\pm\sqrt{\left(\tfrac{1}{4}(2\lambda-1)^2+\tfrac{(2\lambda-1)e}{2b}+\tfrac{ee}{4bb}-\tfrac{h}{b}\right)}$$

seu

$$(i-1)n=-\tfrac{1}{2}(2\lambda-1)-\tfrac{e}{2b}\pm\tfrac{1}{2b}\sqrt{\left((b-e)^2-4bg\right)}.$$

Ponatur brevitatis gratia

$$\sqrt{\left((b-e)^2-4bg\right)}=q,$$

ut sit

$$(i-1)n=\frac{-(2\lambda-1)b-e\pm q}{2b},$$

et nostra relatio fiet

$$N=\frac{-\left((i-1)^2nb+\frac{1}{2}(2\lambda-1)b+\frac{1}{2}e-\frac{1}{2}q\right)\left((i-1)nb+\frac{1}{2}(2\lambda-1)b+\frac{1}{2}e+\frac{1}{2}q\right)}{inb\left(ina+(2\lambda-1)a+c\right)}M\ .$$

Ponamus iam $x^n=u$ et seriem inventam ita repraesentemus

$$\frac{y}{x^\lambda}=A\mathfrak{A}+B\mathfrak{B}u+C\mathfrak{C}u^2+....+M\mathfrak{M}u^{i-1}+N\mathfrak{N}u^i+\text{etc.}$$

horumque duplicium coefficientium lex ita se habebit

$$N=\frac{(i-1)nb+\frac{1}{2}(2\lambda-1)b+\frac{1}{2}e-\frac{1}{2}q}{-inb}M$$

et

$$\mathfrak{N}=\frac{(i-1)nb+\frac{1}{2}(2\lambda-1)b+\frac{1}{2}e+\frac{1}{2}q}{ina+(2\lambda-1)a+c}\mathfrak{M}.$$

Cum igitur haec series similis sit ei, quam ante construximus, comparationem instituamus et habebimus

$$m=nb,\quad h=\tfrac{1}{2}(2\lambda-1)b+\tfrac{1}{2}e-\tfrac{1}{2}q,$$
$$n=-nb\quad\text{et}\quad k=0,$$
$$\mu=nb,\quad \eta=\tfrac{1}{2}(2\lambda-1)b+\tfrac{1}{2}e+\tfrac{1}{2}q,$$
$$\nu=na,\quad\text{et}\quad \theta=(2\lambda-1)a+c.$$

Primum ergo quaeramus quantitatem *z*, et ne littera *x* ambiguitatem creet, loco litterae *x* in praecedente problemate usurpatae utamur littera *t* sitque $ut=s$, et quia est $k=0$, erit per § 1063

$$z=A(1+s)^{\frac{-(2\lambda-1)b-e+q}{2nb}}=A(1+ut)^{\frac{-(2\lambda-1)b-e+q}{2nb}}.$$

Hoc valore invento tractetur in sequentibus integrationibus quantitas *u* ut constans, et cum, quod supra [§ 1064] erat *y*, hic sit $\frac{y}{x^\lambda}$ et *t*, quod supra erat *x*, erit

$$\frac{y}{x^\lambda}\int t^{\frac{\eta}{\mu}-1}dt(\mu-\nu t)^{\frac{\theta\mu-\eta\nu}{\mu\nu}}=\mathfrak{A}\int t^{\frac{\eta}{\mu}-1}zdt(\mu-\nu t)^{\frac{\theta\mu-\eta\nu}{\mu\nu}};$$

ubi cum sit $u=x^n$, hic valor statim pro *u* scribi potest, ut sit

$$z=A(1+x^nt)^{\frac{-(2\lambda-1)b-e+q}{2nb}},$$

et in his integrationibus littera *x* ut constans spectari debet. Quodsi autem fuerit $i+\frac{\eta}{\mu}-1>0$ et $\frac{\theta\mu-\eta\nu}{\mu\nu}+1>0$, haec integralia ita capi debent, ut evanescant posito $t=0$, quo facto ipsi *t* tribui debet valor $t=\frac{\mu}{\nu}=\frac{b}{a}$.
Cum unitas sit minimus valor ipsius *i*, sufficit, ut sit

$$\frac{(2\lambda-1)b+e+q}{2nb}>0 \text{ , tum vero } \frac{(2\lambda-1)ab+2bc-ae-aq}{2nab}+1>0.$$

Nunc vero cum $\int t^{\frac{\eta}{\mu}-1}dt(\mu-\nu t)^{\frac{\theta\mu-\eta\nu}{\mu\nu}}$ fiat quantitas constans, pro *y* autem eius multiplum quodvis nostrae aequationi aeque satisfaciat, eius integrale ita exprimetur

$$y=Cx^{\lambda}\int t^{\frac{\eta}{\mu}-1}zdt(\mu-\nu t)^{\frac{\theta\mu-\eta\nu}{\mu\nu}}$$

existente $z=(1+x^{n}t)^{\frac{-(2\lambda-1)b-e+q}{2nb}}$.

SOLUTIO 2

1071. Si coefficientes geminatos inter se permutemus, ut sit

$$\begin{aligned}
&m=nb, \quad h=\tfrac{1}{2}(2\lambda-1)b+\tfrac{1}{2}e+\tfrac{1}{2}q,\\
&n=na \quad k=(2\lambda-1)a+c,\\
&\mu=nb, \quad \eta=\tfrac{1}{2}(2\lambda-1)b+\tfrac{1}{2}e-\tfrac{1}{2}q,\\
&\nu=-nb, \quad \theta=0,
\end{aligned}$$

sumaturque λ ex aequatione $\lambda(\lambda-1)a+\lambda c+f=0$, primo ponatur $x^{n}t=s$ et quaeratur *z*, ut sit

$$z=Aks^{\frac{-k}{n}}(n-ms)^{\frac{mk-nh}{mn}}\int s^{\frac{k}{n}-1}ds(n-ms)^{\frac{nh-mk}{mn}-1}$$

integrali ita definito, ut posito $s=0$ fiat $z=A$, qui quidem valor *A* est arbitrarius; tum spectata *x* ut constante erit

$$y=Cx^{\lambda}\int t^{\frac{\eta}{\mu}-1}zdt(\mu-\nu t)^{\frac{\theta\mu-\eta\nu}{\mu\nu}}$$

integrali ita sumto, ut evanescat posito $t=0$, tum vero facto $t=\frac{\mu}{\nu}$, si modo fuerit $\frac{\eta}{\mu}>0$ et $1-\frac{\eta}{\mu}>0$ ob $\theta=0$; ubi notetur *z* per hanc aequationem differentialem definiri

$$\frac{dz}{ds}=\frac{Ak-z(k-hs)}{s(n-ms)}.$$

SOLUTIO 3

1072. Per seriem descendentem aequationem propositam [§ 967] ita resolvimus, ut posito

$$y = x^{\lambda}\left(A + Bx^{-n} + Cx^{-2n} + Dx^{-3n} + Ex^{-4n} + \text{etc.}\right)$$

exponens λ definiri debeat ex hac aeqnatione $\lambda(\lambda-1)b + \lambda e + g = 0$, tum vero posito $\lambda(\lambda-1)a + \lambda c + f = h$ sit

$$B = \frac{-h}{n(nb-(2\lambda-1)b-e)}A, \quad C = \frac{-nna+(2\lambda-1)na+nc-h}{n(nb-(2\lambda-1)b-e)}B$$

et generatim

$$N = \frac{-(i-1)^2nna+(2\lambda-1)(i-1)na+(i-1)nc-h}{in(inb-(2\lambda-1)b-e)}M \ ,$$

quae aeqnalitas posito $\sqrt{\left((a-c)^2 - 4af\right)} = p$ ita per factores exhibetur

$$N = \frac{-\left((i-1)na-\frac{1}{2}(2\lambda-1)a-\frac{1}{2}c-\frac{1}{2}p\right)\left((i-1)na-\frac{1}{2}(2\lambda-1)a-\frac{1}{2}c+\frac{1}{2}p\right)}{ina(inb+(2\lambda-1)b-e)}M$$

Quodsi iam ponamus $x^{-n} = u$ et talem seriem constituamus

$$\frac{y}{x^{\lambda}} = A\mathfrak{A} + B\mathfrak{B}u + C\mathfrak{C}u^2 + ... + M\mathfrak{M}u^{i-1} + N\mathfrak{N}u^i + \text{etc.},$$

erit

$$N = \frac{(i-1)na-\frac{1}{2}(2\lambda-1)a-\frac{1}{2}c-\frac{1}{2}p}{-ina}M$$

et

$$\mathfrak{N} = \frac{(i-1)na-\frac{1}{2}(2\lambda-1)a-\frac{1}{2}c+\frac{1}{2}p}{inb-(2\lambda-1)b-e}\mathfrak{M};$$

habebimus comparatione instituta cum constructione generali

$$m = na, \quad h = -\tfrac{1}{2}(2\lambda - 1)a - \tfrac{1}{2}c - \tfrac{1}{2}p,$$
$$n = -na \quad et \quad k = 0,$$
$$\mu = nb, \quad \eta = -\tfrac{1}{2}(2\lambda - 1)a - \tfrac{1}{2}c + \tfrac{1}{2}p,$$
$$v = nb, \quad \theta = -(2\lambda - 1)b - e.$$

Hinc posito $s = ut = x^{-n}t$ erit

$$z = A(1+s)^{\frac{-h}{m}} = A\left(1 + x^{-n}t\right)^{\frac{-h}{m}};$$

quo valore invento si iam sola quantitas t pro variabili habeatur, orietur haec constructio

$$y = Cx^{\lambda}\int t^{\frac{\eta}{\mu}-1}zdt(\mu - vt)^{\frac{\theta\mu - \eta v}{\mu v}}$$

ubi termini integrationis ita sunt constituendi, ut utroque fiat

$$t^{\frac{\eta}{\mu}}(\mu - vt)^{\frac{\theta\mu - \eta v}{\mu v}+1} = 0.$$

SOLUTIO 4

1073. Hic etiam coefficientes geminati permutari possunt, ut sit

$$m = na, \quad h = -\tfrac{1}{2}(2\lambda - 1)a - \tfrac{1}{2}c + \tfrac{1}{2}p,$$
$$n = nb \quad k = -(2\lambda - 1)b - e,$$
$$\mu = na, \quad \eta = -\tfrac{1}{2}(2\lambda - 1)a - \tfrac{1}{2}c - \tfrac{1}{2}p,$$
$$v = -na, \quad \theta = 0,$$

sumtoque ut ante [§ 1072] λ ex aequatione $\lambda(\lambda - 1)b + \lambda e + g = 0$ et posito $\sqrt{\left((a-c)^2 - 4af\right)} = p$ sit $s = x^{-n}t$ et quaeratur z ex hac aequatione

$$\frac{dz}{ds} = \frac{Ak - z(k - hs)}{s(n - ms)},$$

ita ut posito $s = 0$ fiat $z = A$, unde fit

$$z = Aks^{\frac{-k}{n}}(n - ms)^{\frac{mk - nh}{mn}}\int s^{\frac{k}{n}-1}ds(n - ms)^{\frac{nh - mk}{mn}-1};$$

tum vero spectata x ut constante erit

$$y = Cx^{\lambda}\int t^{\frac{\eta}{\mu}-1} zdt\left(\mu - \nu t\right)^{\frac{\theta\mu-\eta\nu}{\mu\nu}},$$

ubi bini termini integrationis ita sumi debent, ut utroque fiat

$$t^{\frac{\eta}{\mu}}\left(\mu - \nu t\right)^{\frac{\theta\mu-\eta\nu}{\mu\nu}+1} = 0.$$

SCHOLION

1074. Singulae hae constructiones plerumque pluribus modis exhiberi possunt, cum non solum λ duplicem valorem habere queat, sed etiam formulae radicales *p* et *q* signo ambiguo gaudeant. At hae constructiones alio modo negotium conficiunt atque praecedentes; quod quo clarius appareat, consideremus aequationem [§ 978]

$$xx(1-xx)ddy - x(1+xx)dxdy + xxydx^2 = 0,$$

ita ut sit $a=1,\ b=-1,\ c=-1,\ e=-1,\ f=0$ et $g=1$ atque $n=2$, unde pro duabus prioribus constructionibus habetur $\lambda(\lambda-1)-\lambda=0$, ergo vel $\lambda=0$ vel $\lambda=2,$ tum vero $q=\pm 2.$ Constructio ergo prima dat

$$\lambda=0,\ m=-2,\ h=\mp 1,\ n=2,\ k=0,\ \mu=-2,\ \eta=\pm 1, \nu=2,\ \theta=-2,$$

unde colligitur

$$z = A(1+xxt)^{\mp\frac{1}{2}} \quad \text{et} \quad y = C\int t^{\mp\frac{1}{2}-1} zdt(1+t)^{\pm\frac{1}{2}-1}.$$

Valeant signa inferiora, ut sit

$$z = \sqrt{(1+xxt)} \quad \text{et} \quad y = C\int \frac{zdt}{(1+t)\sqrt{t(1+t)}};$$

quae formulae quomodo satisfaciant, videamus. Sumta nempe sola *x* variabili fit

$$\frac{dz}{dx} = \frac{xt}{\sqrt{(1+xxt)}} \quad \text{et} \quad \frac{ddz}{dx^2} = \frac{t}{(1+xxt)^{\frac{3}{2}}},$$

hinc

$$\frac{dy}{dx} = C\int \frac{xtdt}{t^{\frac{1}{2}}(1+t)^{\frac{3}{2}}(1+xxt)^{\frac{1}{2}}} \quad \text{et} \quad \frac{ddy}{dx^2} = C\int \frac{tdt}{t^{\frac{1}{2}}(1+t)^{\frac{3}{2}}(1+xxt)^{\frac{3}{2}}}$$

unde conficitur

$$xx(1-xx)\frac{ddy}{dx^2} - x(1+xx)\frac{dy}{dx} + xxy = C\int \frac{xxdt(1-xxtt)}{t^{\frac{1}{2}}(1+t)^{\frac{3}{2}}(1+xxt)^{\frac{3}{2}}},$$

cuius formulae integrale est $\frac{2Cxx\sqrt{t}}{\sqrt{(1+t)(1+xxt)}}$; quod cum evanescat tam casu $t=0$ quam casu $t=\infty$, constructio nostrae aequationis

$$y = C\int \frac{zdt}{(1+t)\sqrt{t(1+t)}} = C\int \frac{dt\sqrt{(1+xxt)}}{(1+t)\sqrt{t(1+t)}}$$

ita confici debet: sumto *x* pro constante integratio ita instituatur, ut integrale evanescat posito $t = 0$, quo facto statuatur $t = \infty$ et functio ipsius *x*, quae pro *y* prodibit, satisfaciet aequationi propositae.

Sin autem secundam constructionem eligamus, sumta $\lambda = 0$ erit $m = -2,\ h = \pm 1,\ n = 2,\ k = -2,\ \mu = -2,\ \eta = \mp 1,\ v = 2,\ \theta = 0$ atque *z* ita definiri debet ex hac aequatione

$$\frac{dz}{ds} = \frac{-2A + z(2 \pm s)}{s(2\ \ + 2s)}$$

existente $s = xxt$, ut posito $s = 0$ fiat $z = A$. Valeat signum superius, et cum sit

$$dz - \frac{zds(2+s)}{2s(1+s)} = \frac{-Ads}{s(1+s)},$$

multiplicetur per $\frac{\sqrt{(1+s)}}{s}$ eritque integrale

$$\frac{z\sqrt{(1+s)}}{s} = \text{Const.} - A\int \frac{ds}{ss\sqrt{(1+s)}}$$

seu

$$z = A - \frac{As}{\sqrt{(1+s)}}\, l\, \frac{1+\sqrt{(1+s)}}{\sqrt{s}} + \frac{Bs}{\sqrt{(1+s)}},$$

quae posito $s = 0$ dat $z = A$, quicquid sit *B.* Deinde est

$$y = C\int t^{\frac{1}{2}-1} zdt(1+t)^{-\frac{1}{2}} \text{ seu } y = C\int \frac{zdt}{\sqrt{t(1+t)}},$$

qui valor quomodo satisfaciat, haud facile ostendi potest; hocque magis ista methodus excoli meretur.

EXEMPLUM

1075. *Constructiones aequationis differentio-differentialis*

$$xx(1-xx)ddy - x(1+xx)dxdy + xxydx^2 = 0$$

ex praecedente problemate oriundas exhibere.

Ob $n = 2,\ a = 1,\ b = -1,\ c = -1,\ e = -1,\ f = 0$ et $g = 1$ pro prima constructione habemus vel $\lambda = 0$ vel $\lambda = 2$; unde habemus:

1. Si $\lambda = 0$, ut modo invenimus,

$$m=-2,\quad h=\mp 1,\quad n=2,\quad k=0,\quad \mu=-2,\quad \eta=\pm 1,\quad v=2,\ \theta=-2,$$

unde fit

$$z=\left(1+xxt\right)^{\mp\frac{1}{2}}\quad \text{et}\quad y=C\int t^{\mp\frac{1}{2}-1}zdt\left(1+t\right)^{\pm\frac{1}{2}-1},$$

sicque duplex oritur constructio,

$$\text{altera } z=\sqrt{\left(1+xxt\right)}\quad \text{et}\quad y=C\int\frac{zdt}{(1+t)\sqrt{t(1+t)}},$$

$$\text{altera } z=\frac{1}{\sqrt{(1+xxt)}}\quad \text{et}\quad y=C\int\frac{zdt}{t\sqrt{t(1+t)}}.$$

2. Si $\lambda=2$, erit ob $q=\pm 2$

$$m=-2,\ h=-2\mp 1,\ n=2,\ k=0,\ \mu=-2,\ \eta=-2\pm 1,\ v=2,\ \theta=2,$$

unde fit

$$z=\left(1+xxt\right)^{\frac{-2\mp 1}{2}}\quad \text{et}\quad y=Cx^2\int t^{1\mp\frac{1}{2}-1}zdt\left(1+t\right)^{\pm\frac{1}{2}}$$

sicque duplex habetur constructio,

$$\text{altera } z=\left(1+xxt\right)^{\frac{-3}{2}}\quad \text{et}\quad y=Cx^2\int\frac{zdt\sqrt{(1+t)}}{\sqrt{t}},$$

$$\text{altera } z=\left(1+xxt\right)^{\frac{-1}{2}}\quad \text{et}\quad y=Cx^2\int\frac{zdt\sqrt{t}}{\sqrt{(1+t)}}.$$

Pro secunda constructione generali habemus:

3. Si $\lambda=0$, permutando illos indices

$$m=-2,\ h=\pm 1,\ n=2,\ k=-2,\ \mu=-2,\ \eta=\mp 1,\ v=2,\ \theta=0,$$

unde posito $xxt=s$ primo quaeratur z ex hac aequatione $\frac{dz}{ds}=\frac{-2A+z(2\pm s)}{2s(1+s)}$,
ut posito $s=0$ fiat $z=A$; tum vero erit

$$y=C\int t^{\pm\frac{1}{2}-1}zdt\left(1+t\right)^{\mp\frac{1}{2}}.$$

Hinc ergo nascitur duplex constructio,

$$\text{altera } \frac{dz}{ds}=\frac{-2A+z(2+s)}{2s(1+s)} \quad \text{et} \quad y=C\int\frac{zdt}{\sqrt{t}\sqrt{(1+t)}},$$

$$\text{altera } \frac{dz}{ds}=\frac{-2A+z(2-s)}{2s(1+s)} \quad \text{et} \quad y=C\int\frac{zdt\sqrt{(1+t)}}{t\sqrt{t}}.$$

4. Si $\lambda=2$, habebimus

$$m=-2,\ h=-2\pm1,\ n=2,\ k=2,\ \mu=-2,\ \eta=-2\mp1,\ v=2,\ \theta=0$$

Positoque $xxt=s$ ut ante quaeratur z ex aequatione $\frac{dz}{ds}=\frac{2A-z(2+(2\mp1)s)}{2s(1+s)}$,
eritque

$$y=Cx^2\int t^{\pm\frac{1}{2}}zdt(1+t)^{\frac{-2\mp1}{2}}$$

ideoque etiam duplex constructio elicitur,

$$\text{altera } \frac{dz}{ds}=\frac{2A-z(2+s)}{2s(1+s)} \quad \text{et} \quad y=Cx^2\int\frac{zdt\sqrt{t}}{(1+t)\sqrt{(1+t)}},$$

$$\text{altera } \frac{dz}{ds}=\frac{2A-z(2+3s)}{2s(1+s)} \quad \text{et} \quad y=Cx^2\int\frac{zdt}{\sqrt{t(1+t)}}.$$

Ex solutione tertia colligimus primo $-\lambda(\lambda-1)-\lambda+1=0$ seu $\lambda\lambda=1$
ideoque $\lambda=\pm1$ et $p=\sqrt{4}=\pm2$, quare:
5. Si capiatur $\lambda=+1$, erit

$$m=2,\ h=\mp1,\ n=-2,\ k=0,\ \mu=2,\ \eta=\pm1,\ v=-2,\ \theta=2$$

hincque

$$z=\left(1+\tfrac{t}{xx}\right)^{\pm\frac{1}{2}} \quad \text{et} \quad y=Cx\int t^{\pm\frac{1}{2}-1}zdt(1+t)^{-1\mp\frac{1}{2}},$$

ita ut iterum duplex habeatur constructio,

$$\text{altera } z=\tfrac{1}{x}\sqrt{(xx+t)} \quad \text{et} \quad y=Cx\int\frac{zdt}{(1+t)\sqrt{t(1+t)}},$$

$$\text{altera } z=\frac{x}{\sqrt{(xx+t)}} \quad \text{et} \quad y=Cx\int\frac{zdt}{t\sqrt{t(1+t)}}.$$

6. Si capiatur $\lambda=-1$, erit

$$m=2,\ h=2\mp1,\ n=-2,\ k=0,\ \mu=2,\ \eta=2\pm1,\ v=-2,\ \theta=-2$$

hincque

$$z=\left(1+\tfrac{t}{xx}\right)^{-1\pm\frac{1}{2}} \quad \text{et} \quad y=\tfrac{C}{x}\int t^{\pm\frac{1}{2}}zdt\left(1+t\right)^{\mp\frac{1}{2}},$$

unde binae constructiones fluunt,

$$\text{altera } z=\frac{x}{\sqrt{(xx+t)}} \quad \text{et} \quad y=\tfrac{C}{x}\int\frac{zdt\sqrt{t}}{\sqrt{(1+t)}},$$

$$\text{altera } z=\frac{x^3}{(xx+t)^{\frac{3}{2}}} \quad \text{et} \quad y=\tfrac{C}{x}\int\frac{zdt\sqrt{(1+t)}}{\sqrt{t}}.$$

Ex solutione quarta denique concludimus:

7. Si $\lambda=+1$,

$m=2,\ h=\pm1,\ n=-2,\ k=2,\ \mu=2,\ \eta=\mp1,\ v=-2,\ \theta=0.$

Posito nunc $s=\frac{t}{xx}$ quaeratur z ex hac aequatione $\frac{dz}{ds}=\frac{-2A+z(2\mp s)}{2s(1+s)}$, ut posito $s=0$ fiat $z=A$, tumque erit

$$y=Cx\int t^{\mp\frac{1}{2}-1}zdt\left(1+t\right)^{\pm\frac{1}{2}},$$

unde duplex constructio,

$$\text{altera } \frac{dz}{ds}=\frac{-2A+z(2-s)}{2s(1+s)} \quad \text{et} \quad y=Cx\int\frac{zdt\sqrt{(1+t)}}{t\sqrt{t}},$$

$$\text{altera } \frac{dz}{ds}=\frac{-2A+z(2+s)}{2s(1+s)} \quad \text{et} \quad y=Cx\int\frac{zdt}{\sqrt{t}(1+t)}.$$

8. Si $\lambda=-1$, habebitur

$$m=2,\ h=2\pm1,\ n=-2,\ k=-2,\ \mu=2,\ \eta=2\mp1,\ v=-2,\ \theta=0,$$

et posito $\frac{t}{xx}=s$ quari debet z ex hac aequatione $\frac{dz}{ds}=\frac{2A-z(2+(2\pm1)s)}{2s(1+s)}$, ut posito $s=0$ fiat $z=A$; quo facto erit

$$y=\tfrac{C}{x}\int t^{\mp\frac{1}{2}}zdt\left(1+t\right)^{-1\pm\frac{1}{2}}$$

sicque duplex constructio,

$$\text{altera } \frac{dz}{ds}=\frac{2A-z(2+3s)}{2s(1+s)} \quad \text{et} \quad y=\tfrac{C}{x}\int\frac{zdt}{\sqrt{t}(1+t)},$$

$$\text{altera } \frac{dz}{ds}=\frac{2A-z(2+s)}{2s(1+s)} \quad \text{et} \quad y=\tfrac{C}{x}\int\frac{zdt\sqrt{t}}{(1+t)\sqrt{(1+t)}}.$$

Omnino ergo sedecim constructiones sumus consecuti.

SCHOLION

1076. Periculum faciamus ostendendi, quomodo hae constructiones, quae magis arduae videntur, aequationi propositae satisfaciant; atque in hunc finem eligamus eonstructionem posteriorem *n*° 4, quae habet

$$dz+\frac{zds(2+3s)}{2s(1+s)}=\frac{Ads}{s(1+s)}.$$

Haec per $s\sqrt{(1+s)}$ multiplicata praebet integrale

$$sz\sqrt{(1+s)}=A\int\frac{ds}{\sqrt{(1+s)}}=2A\sqrt{(1+s)}+B$$

seu

$$z=\frac{2A}{s}+\frac{B}{s\sqrt{(1+s)}}.$$

lam ut posito $s=0$ fiat $z=A$, debet esse $B=-2A$, ut sit

$$z=\frac{2A\left(\sqrt{(1+s)}-1\right)}{s\sqrt{(1+s)}}=\frac{2A}{txx}-\frac{2A}{txx\sqrt{(1+txx)}}.$$

Hinc fit

$$\left(\frac{dz}{dx}\right)=\frac{-4A}{tx^3}+\frac{2A(2+3txx)}{tx^3(1+txx)^{\frac{3}{2}}}$$

et

$$\left(\frac{ddz}{dx^2}\right)=\frac{12A}{tx^4}-\frac{6A\left(2+5txx+4ttx^4\right)}{tx^4(1+txx)^{\frac{5}{2}}}.$$

Cum nunc sit $y=C\int\frac{xxzdt}{\sqrt{t(1+t)}}$, erit

$$\left(\frac{dy}{dx}\right)=2C\int\frac{xzdt}{\sqrt{t(1+t)}}+C\int\frac{xxdt}{\sqrt{t(1+t)}}\left(\frac{dz}{dx}\right)$$

et

$$\left(\frac{ddy}{dx^2}\right)=2C\int\frac{zdt}{\sqrt{t(1+t)}}+4C\int\frac{xdt}{\sqrt{t(1+t)}}\left(\frac{dz}{dx}\right)+C\int\frac{xxdt}{\sqrt{t(1+t)}}\left(\frac{ddz}{dx^2}\right)$$

hincque

$$xx(1-xx)ddy - x(1+xx)dxdy + xxydx^2$$
$$= C\int \frac{dt}{\sqrt{t(1+t)}}\left(\frac{2Axx}{t} - \frac{2Axx(1+4txx+3ttxx)}{t(1+txx)^{\frac{5}{2}}}\right),$$

quod integrale est

$$\frac{-4ACxx\sqrt{(1+t)}}{\sqrt{t}} + \frac{4ACxx\sqrt{(1+t)}}{(1+txx)^{\frac{3}{2}}\sqrt{t}} = \frac{4ACxx\sqrt{(1+t)}}{\sqrt{t}}\left(\frac{1}{(1+txx)^{\frac{3}{2}}} - 1\right)$$

et hac forma exprimi potest

$$-2Cx^4\left(3z + x\left(\frac{dz}{dx}\right)\right)\sqrt{t(1+t)}$$

vel etiam hoc modo

$$-2Cx^4\left(\frac{2A+z}{1+txx}\right)\sqrt{t(1+t)}.$$

Expressio autem ista fit = 0, primo si $t=-1$, deinde etiam si $t=0$, unde valor pro *y* inventus

$$y = D\int \frac{dt}{t\sqrt{t(1+t)}}\left(1 - \frac{1}{\sqrt{(1+txx)}}\right)$$

ita per integrationem definiri debet, ut evanescat posito $t=0$, tum vero ponatur $t=-1$. Vel posito $t=-v$ erit

$$y = D\int \frac{dv}{v\sqrt{v(1-v)}}\left(1 - \frac{1}{\sqrt{(1-vxx)}}\right)$$

integrali ita sumto, ut evanescat posito $v=0$, tum vero facto $v=1$.

Exemplum hoc sufficit ad ostendendum, quomodo constructiones exhibitae aequationi differentio-differentiali satisfaciant; interim vero si quantitas *z* transcendenter, per logarithmos scilicet, exprimitur, consensum nonnisi per calculos nimium taediosos declarare licet.

PROBLEMA 136

1077. *Posito*

$$y = C\int (1+t)^{\nu-1}(a+tx)^{\lambda}\,dt\,,$$

in qua integratione quantitas x ut constans spectatur, integrali per terminos deinceps investigandos definito, ut y aequetur certae functioni ipsius x, invenire aequationes differentio– differentiales formae

$$Lxx\frac{ddy}{dx^2} + Mx\tfrac{dy}{dx} + Ny = 0\,,$$

cui ea functio satisfaciat.

SOLUTIO

Cum sit ex principiis ante [§ 1017] stabilitis

$$\tfrac{dy}{dx} = C\int \lambda t(1+t)^{\nu-1}(a+tx)^{\lambda-1}\,dt$$

et

$$\tfrac{ddy}{dx^2} = C\int \lambda(\lambda-1)tt(1+t)^{\nu-1}(a+tx)^{\lambda-2}\,dt,$$

erit

$$\begin{aligned}&Lxx\frac{ddy}{dx^2} + Mx\tfrac{dy}{dx} + Ny\\ &= C\int (1+t)^{\nu-1}(a+tx)^{\lambda-2}\,dt\Big(\lambda(\lambda-1)Lttxx + \lambda Mtx(a+tx) + N(a+tx)^2\Big)\\ &= C\int (1+t)^{\nu-1}(a+tx)^{\lambda-2}\,dt\left\{\begin{matrix} Naa+ & Natx+ & Nttxx\\ & +\lambda Matx+ & \lambda Mttxx\\ & & +\lambda(\lambda-1)Lttxx\end{matrix}\right\}\end{aligned}$$

quae formula sumta *x* constanta absolute integrabilis esse debet. Ponatur ergo integrale

$$C(1+t)^{\nu}(a+tx)^{\lambda-1}(Paa+Qatx)$$

denotantibus *P* et *Q* functionibus quibuscunque ipsius *x;* erit eius differentiale aequationi differentio–differentiali

$$\begin{gathered}C(1+t)^{\nu-1}(a+tx)^{\lambda-2}\,dt\\ \times\Big(\nu(a+tx)(Paa+Qatx) + (\lambda-1)x(1+t)(Paa+Qatx) + Qax(1+t)(a+tx)\Big)\end{gathered}$$

$$=C(1+t)^{\nu-1}(a+tx)^{\lambda-2}\,dt\left\{\begin{matrix} \nu Pa^3 & + & \nu Qaatx+ & \nu Qattxx \\ +(\lambda-1)Paax+ & & \nu Paatx+ & (\lambda-1)Qattxx \\ + & Qaax+(\lambda-1) & Paatx+ & Qattxx \\ & +(\lambda-1) & Qatxx & \\ & + & Qaatx & \\ & + & Qatxx & \end{matrix}\right\}$$

qua forma cum illa comparata adipiscimur

$$N=\nu Pa+(\lambda-1)Px+Qx,$$
$$2N+\lambda M=(\nu+1)Qa+(\lambda+\nu-1)Pa+\lambda Qx,$$
$$N+\lambda M+\lambda(\lambda-1)L=(\lambda+\nu)Qa,$$

quarum aequationum extremae demta media praebent

$$\lambda(\lambda-1)L=-(\lambda-1)Pa+(\lambda-1)Px+(\lambda-1)Qa-(\lambda-1)Qx$$

hincque

$$\lambda L=(a-x)(Q-P) \quad \text{seu} \quad L=\tfrac{1}{\lambda}(a-x)(Q-P);$$

secunda autem demto primae duplo dat

$$\lambda M=(\lambda-\nu-1)Pa-2(\lambda-1)Px+(\nu+1)Qa+(\lambda-2)Qx$$

seu

$$\lambda M=\big((\nu+1)a+(\lambda-2)x\big)(Q-P)+\lambda(a-x)P.$$

Quare sumtis pro *P* et *Q* functionibus quibuscunque ipsius *x* si functiones *L*, *M*, N ita definiantur, ut sit

$$L=\tfrac{1}{\lambda}(a-x)(Q-P),$$
$$M=\tfrac{1}{\lambda}\big((\nu+1)a+(\lambda-2)x\big)(Q-P)+(a-x)P,$$
$$N=x(Q-P)+(\nu a+\lambda x)P,$$

aequationi differentio–differentiali

$$Lxxddy+Mxdxdy+Nydx^2=0$$

satisfaciet formula integralis

$$y = C\int (1+t)^{\nu-1}(a+tx)^{\lambda}\, dt$$

tractata *x* ut constante, dummodo integrationis termini ita constituantur, ut utroque haec expressio $(1+t)^{\nu}(a+tx)^{\lambda-1}(Pa+Qtx)$ evanescat. Notari autem oportet hos terminos non ab *x* pendere debere. Primo autem patet hanc expressionem fieri = 0 casu $t=-1$, si modo sit $\nu>0$. Deinde posito $t=\infty$ etiam evanescet, si modo sit $\nu+\lambda-1+1$ numerus negativus seu $\nu+\lambda<0$. Quocirca si sit $\nu>0$ et $\nu+\lambda<0$, integrale $y = C\int (1+t)^{\nu-1}(a+tx)^{\lambda}\, dt$ ita capi debet, ut posito $t=-1$ evanescat; tum vero statuatur $t=\infty$ functioque ipsius *x* pro *y* resultans satisfaciet aequationi propositae.

COROLLARIUM 1

1078. Quoniam functiones *P* et *Q* in formulam integralem pro *y* assumtam non ingrediuntur, manifestum est eandem formulam satisfacere omnibus aequationibus differentio–differentialibus, quicunque valores litteris *P* et *Q* tribuantur.

COROLLARIUM 2

1079. Sumto ergo $Q=P$ eadem formula integralis

$$y = C\int (1+t)^{\nu-1}(a+tx)^{\lambda}\, dt$$

satisfacit etiam huic aequationi differentiali primi gradus

$$(a-x)xdy+(\nu a+\lambda x)ydx=0.$$

Huius vero integrale est

$$y=\frac{D(a-x)^{\lambda+\nu}}{x^{\nu}},$$

qui ergo valor quoque in genere nostrae aequationi differentio–differentiali satisfacit; id quod tentanti mox patebit.

COROLLARIUM 3

1080. Hic ergo valor integralis $y = C\int (1+t)^{\nu-1}(a+tx)^{\lambda}\, dt$ secundum terminos definitos sumtus congruere debet cum formula algebraica $y=\frac{D(a-x)^{\lambda+\nu}}{x^{\nu}}$, si modo sit $\nu>0$ et $\lambda+\nu<0$.

SCHOLION

1081. Parum ergo integratio hoc problemate exhibita habet in recessu. Verum reductio formulae integralis $y = C\int (1+t)^{\nu-1}(a+tx)^{\lambda}\, dt$ ad $y=\frac{D(a-x)^{\lambda+\nu}}{x^{\nu}}$, eo magis est notatu digna, ad quam illa reducitur, si integrali ita sumto, ut

evanescat posito $t=-1$, ponatur $t=\infty$. Posito ergo $\lambda+v=-\mu$, ut μ et v sint numeri positivi, erit $C\int\frac{(1+t)^{v-1}dt}{(a+tx)^{\mu+v}}=\frac{D}{x^{v}(a-x)^{\mu}}$.
Vel ponatur $1+t=z$ erit

$$C\int\frac{z^{v-1}dz}{(a-x+xz)^{\mu+v}}=\frac{D}{x^{v}(a-x)^{\mu}}$$

terminis huius integrationis existentibus $z=0$ et $z=\infty$. Verum etiam haec observatio non magni est momenti; nam posito $a-x=ux$ fit

$$\frac{C}{x^{\mu+v}}\int\frac{z^{v-1}dz}{(u+z)^{\mu+v}}=\frac{D}{x^{\mu+v}u^{\mu}}$$

ideoque haec formula $\int\frac{z^{v-1}dz}{(u+z)^{\mu+v}}$ ita integrata, ut evanescat posito $z=0$, si tum ponatur $z=\infty$, hanc induet formam $\frac{A}{u^{\mu}}$ in qua A quantitatem constantem denotat ab u non pendentem. Pendet autem ab exponentibus μ et v lege ex casibus facile observanda. Scilicet posito

$$\int\frac{z^{v-1}dz}{(u+z)^{\mu+v}}=\frac{A}{u^{\mu}}$$

si sit $v=1$, integrale illud praebet $-\frac{1}{\mu(u+z)^{\mu}}+\frac{1}{\mu u^{\mu}}$ et posito $z=\infty$ prodit $\frac{1}{\mu u^{\mu}}$, ita ut hoc casu sit $A=\frac{1}{\mu}$. Si sit $v=2$, integratio quoque succedit reperiturque $A=\frac{1}{\mu(\mu+1)}$, si $v=3$, fit $A=\frac{1\cdot 2}{\mu(\mu+1)(\mu+2)}$, et si $v=4$ fit $A=\frac{1\cdot 2\cdot 3}{\mu(\mu+1)(\mu+2)(\mu+3)}$, unde in genere concludimus fore

$$A=\frac{1\cdot 2\cdot 3\cdots(v-1)}{\mu(\mu+1)(\mu+2)\cdots(\mu+v-1)}.$$

Quare integratione secundum regulam praescriptam instituta erit

$$\frac{1}{\mu}\cdot\frac{2}{\mu+1}\cdot\frac{3}{\mu+2}\cdots\frac{v-1}{\mu+v-1}=\mu u^{\mu}\int\frac{z^{v-1}dz}{(u+z)^{\mu+v}}$$

Quodsi exponens v non fuerit integer, valor ipsius A ope interpolationis huius formulae per factores procedentis definietur. Quadratura scilicet circuli ingredietur, si exponens v fractionem $\frac{1}{2}$ involvat, de huiusmodi autem interpolationibus alibi fusius egimus neque hic locus est hoc argumentum uberius prosequendi.

Restat ultimum huius sectionis caput, quo aequationum differentio–differentialium integratio per approximationes docebitur.