CHAPTER X11

CONCERNING THE INTEGRATION OF SECOND ORDER DIFFERENTIAL EQUATIONS BY APPROXIMATIONS

PROBLEM 137

1082. *To explain the principles for any second order differential equation, by which the integration is required to be sought by approximations.*

SOLUTION

The proposed equation involves the two variables *x* and *y* and on putting $dy = pdx$ and $dp = qdx$ an equation will be given between the four quantities *x, y, p* and *q,* from which *q* is allowed to be defined, so that *q* can be equal to some function of the three quantities *x, y* and *p,* calling which equal to *V* there shall be $q = V$ or $dp = Vdx$. Here it is to be noted in the first place that the integration, so that it may be evaluated, requires a twofold determination or as if two conditions are to be prescribed as it pleases, from which the integration is satisfied. Evidently it does not suffice, that on putting $x = a$ there becomes $y = b$, just as in we have come to see used in differential equations of the first order, but also there is allowed to be another condition added on, which shall be, that on putting $x = a$ there becomes also $p = \frac{dy}{dx} = c$ for the given quantity. Hence with these conclusions in place, so that on putting $x = a$ there becomes $y = b$ and $p = c$, all the work of the integration is reduced here, so that by giving some other value to *x* the corresponding values of *y* and of *p* are to be investigated; for if we can do this properly, we will be able to define the integral of the proposed equation perfectly, so that nothing in addition is desired. Because in general since this cannot happen, an account of the approximation involved in this, it can be established so that a value of *x* must be given differing minimally with *a*, which shall be $x = a + \omega$, and it is asked, by how much the values of the quantities *y* and *p* are to differ from the first quantities *b* and *c*. Here we assume for the initial values, that while *x* increases from *a* to $a + \omega$, also the values of the quantities *y* and *p* are going to change so little, that hence the function *V* suffers no notable variation. Whereby if we put in place $x = a$, $y = b$ and $p = c$, there arises $V = F$, the quantity *F* will be thought of as retaining the same value *V*, while *x* increases from *a* as far as to $a + \omega$. Therefore since for this minimum interval we may consider $dp = Fdx$, there will be from the integration $p = Fx + \text{Const.}$; now because on putting $x = a$ there must come about $p = c$,then there will be

$$p = c + Fx - Fa.$$

Now there shall be $x=a+\omega$ and we will have $p=c+F\omega$, which is the value of p corresponding to $x=a+\omega$. And hence for the minimum interval there will be $dy=cdx$ and thus $y=b+cx-ac$ and for the value $x=a+\omega$ there becomes $y=b+c\omega$, which is the value of y corresponding to the value $x=a+\omega$. On which account if the initial values shall be $x=a$, $y=b$ and $p=\frac{dy}{dx}=c$, and from these there becomes $V=F$, the following values in the interval minimally distant from these will be

$$x=a+\omega,\ y=b+c\omega,\ p=c+F\omega;$$

which again if these are considered as initial values, it is permitted to progress from these in a like manner through an interval as small as possible, and thus finally the progress through an interval of any size will become known.

COROLLARY 1

1083. Where the smaller these intervals are taken, there the less they may stray from the true values, as long as the quantities c and F do not become exceedingly large ; but if in that interval they thus increase to infinity, evidently there is an error in the quantities y and p undertaken to be assigned.

COROLLARY 2

1084. If the quantities c or F become exceedingly great, the interval in which y or p increases can be taken as given ; thus on putting $c\omega=\psi$ the following values will be $x=a+\frac{\psi}{c}, y=b+\psi$ and $p=c+\frac{F\psi}{c}$. But if F should emerge to be a very great quantity, the value of p in the smallest interval φ is taken to increase, so that there shall be $F\omega=\varphi$, and the following values will be $x=a+\frac{\varphi}{F}$, $y=b+\frac{c\varphi}{F}$ and $p=c+\varphi$.

COROLLARY 3

1085. If b should be an infinite quantity, for the nearest value of y to be extracted to be defined

$$\frac{1}{y}=\frac{1}{b+c\omega}=\frac{1}{b}-\frac{c\omega}{bb}=\frac{-c\omega}{bb}$$

in order that which expression shall be finite, it is required that the quantity c also shall be infinite ; otherwise the corresponding value of y remains infinite not only at $x=a$, but also at $x=a+\omega$.

SCHOLIUM 1

1086. Just as the solution of any problem depends on the integration of some differential equation of the second order, so also the conditions of the problem are accustomed to be satisfied by two determinations ; the first of which, while indeed a certain value $x = a$ is attributed to x , so requires that y be given value $y = b$, and truly the other, so that also the ratio $\frac{dy}{dx} = p$ follows the given value $p = c$. But if hence in general we wish to integrate a certain second order differential equation, the integration thus is allowed to be put in place so that on putting $x = a$ there comes about $y = b$ and $p = c$ with the quantities *a, b, c* depending on our choice.

Yet meanwhile whenever it arises in use that on putting $x = a$, the values of *y* and *p* do not depend completely on our choice, but may be chosen from the nature of the equation now given, in which cases the weakness of the determination from the other conditions is compensated. Just as if this equation is proposed

$$xx(a-bx)ddy - 2x(2a-bx)dxdy + 2(3a-bx)ydx^2 = 6aadx^2,$$

in whatever manner that is to be determined by integration, on putting $x = 0$ by necessity there shall be $y = a$ and $\frac{dy}{dx} = p = b$, thus so that for the case $x = 0$ the values of the quantities *y* and *p* are left minimally from our choice. But the complete integral is found

$$y = a + bx + \frac{(A+Bx)xx}{a-bx},$$

where even if the constants A and B are assumed as it pleases, yet always on putting $x = 0$ there emerges $y = a$ and $p = b$. Hence in cases of this kind it is no wonder, if for the given value of the quantity x , arbitrary values are not permitted for the values *y* and *p*.

SCHOLIUM 2

1087. The account explaining the integrating of second order differential equations by approximations, while we progress through the smallest intervals, just as we have done for first order differential equations, in certain cases involves difficulties, so that unless a remedy is brought forwards, it is unable to be called upon to be used. This arises in the first place, when $c = \infty$; for then, however short the interval ω is taken, neither the value of *y* or of *p* is allowed to be known. Likewise too it is inconveniently disturbed if on putting $x = a$, $y = b$ and $p = c$ the function *V* becomes infinite and thus there arises $F = \infty$, in which case the value of p is not defined. Then also the case, in which either c or F vanishes, it is convenient to treat separately; and if indeed then the values of *y* and *p* can be shown to be satisfied well enough, yet, because they experience no change, while the change is expressed in a higher power of ω, it is useful to investigate this change itself, from which there is less departure from the true values. But if the quantity b emerges infinite, now we note that in place of y the reciprocal of this, $\frac{1}{y}$, must be examined. Therefore just as it may occur with the difficulties mentioned, we examine the case with more care.

PROBLEM 138

1088. *If on establishing the integration through intervals, for the beginning of a certain interval it may happen on putting $x = a$, $y = b$ and $p = c$, that the quantity c shall be either vanishing or infinite, to resolve the integration through this interval.*

SOLUTION

The preceding approximation gave $y = b + c(x-a)$, from which if $c = 0$, the increment of y may be expressed by a higher power of $x-a$, evidently $y = b + A(x-a)^{\lambda}$ on taking $\lambda > 1$ and with the higher powers rejected , which besides, on account of the minimum interval $x-a$, are rightly disregarded here. But if there shall be $c = \infty$, the value of y can be represented in a similar manner, $y = b + A(x-a)^{\lambda}$ on taking $\lambda < 1$, thus yet so that it will be greater than zero ; therefore in either case the investigation is the same, in order that from the proposed equation $dp = Vdx$ both the coefficient A as well as the exponent λ may be defined. Now from that equation we may deduce that

$$\frac{dy}{dx} = p = \lambda A(x-a)^{\lambda-1} \text{ and } dp = \lambda(\lambda-1)A(x-a)^{\lambda-2}dx;$$

and by necessity the same expression results, if in the formula Vdx there is put in place

$$y = b + A(x-a)^{\lambda} \text{ and } p = \lambda A(x-a)^{\lambda-1};$$

from which it is apparent to be $c = 0$, if $\lambda > 1$; and $c = \infty$, if $\lambda < 1$.

Now since V shall be a function of x, y and p, and there is put everywhere $x = a$, unless as far as the formula $x = a$ is present which remains, then indeed $y = b$, unless this gives rise to $V = 0$ or $V = \infty$; for if this arises, there is written $b + A(x-a)^{\lambda}$ for the value y and in a like manner for p there is written $\lambda A(x-a)^{\lambda-1}$. But the higher powers of the formula $x = a$ are rejected before the lesser ones and thus there may arise this expression of the form $C(x-a)^{\mu}$ for V, which must be equal to the formula $\lambda(\lambda-1)A(x-a)^{\lambda-2}$, from which both the coefficient A assumed, as well as the exponent λ will be defined; and thus truly the approximate values $y = b + A(x-a)^{\lambda}$ and $p = \lambda A(x-a)^{\lambda-1}$ will become known, which therefore depart less from the true values, in which a small difference is put in place between a and x. But the case in which $\lambda = 1$ is by itself evident and we have treated it in the preceding problem, since it shall be the only case in which the quantity c is met with a finite value.

COROLLARY 1

1089. If it comes about that on putting $c = \infty$, in which case there must become $\lambda < 1$, the function V can obtain a finite value, to which the formula $\lambda(\lambda - 1)A(x-a)^{\lambda-2}$ cannot be equal, the case by itself presents no difficulty and also the value of y emerges for the minimum excess of x over a actually infinite .

COROLLARY 2

1090. This is easier to understand from a permitted example $dp = 6xdx$, from which there shall be,
$p = c + 3xx - 3aa = \frac{dy}{dx}$ and hence $y = b + (c - 3aa)(x-a) + x^3 - a^3$ or

$$y = b - ac + 2a^3 + (c - 3aa)x + x^3$$

If therefore the constant c is assumed infinite, the value of y will always be infinite except in the one case $x = a$.

COROLLARY 3

1091. But if on assuming $c = 0$, in which case there must be $\lambda > 1$, the function V should have finite value and that thus constant, on putting $x = a$ and $y = b$ the formula $\lambda(\lambda - 1)A(x-a)^{\lambda-2}$ will be equal to that on taking $\lambda = 2$ and $2A =$ to that value of the constant. Just as in the preceding example there becomes $V = 6a = 2A$, hence $\mathrm{A} = 3a$ and there will be approximately $y = b + 3a(x-a)^2$*;* which also agrees with that integral found, which on putting $c = 0$ is $y = b + 2a^3 - 3aax + x^3 = b + (x+2a)(x-a)^2$, which expression goes into that on making $x = a$.

SCHOLIUM 1

1092. On putting $c = 0$ the function V if in that there is written
$x = a$, $y = b$ and $p = c = 0$, the value arising is either infinitely large, finite, or thus vanishing. In the first case, in which $V = \infty$, so that for it to be equal to $\lambda(\lambda - 1)A(x-a)^{\lambda-2}$ on taking $x = a$, it is necessary that there shall be $\lambda < 2$ with $\lambda > 1$ arising. But in order that the quantities A and λ are defined from this, in the function V it is required to write

$$y = b + A(x-a)^{\lambda} \text{ and } p = \lambda A(x-a)^{\lambda-1}$$

likewise $x = a$, unless the formula $x - a$ occurs everywhere. In this way, because by hypothesis in the case $x = a$ there becomes $V = \infty$, this value will be produced

$=C(x-a)^{-\alpha}$, as A and λ may be found deduced from $\lambda(\lambda-1)A(x-a)^{\lambda-2}$, while lest there arises $\lambda<1$, which case cannot stand since $c=0$. In the second case, in which there is produced V equal to a finite quantity, there is required to be taken $\lambda=2$, but if in the third case there shall be $V=0$, there must be taken $\lambda>2$, so that the value of this may be contained in the formula $\lambda(\lambda-1)A(x-a)^{\lambda-2}$.

But if there must be $c=\infty$, it is unable to happen, as we have seen, that the function V be given a finite value, and much less vanishing, unless we wish to admit a certain incongruous case, in which y remains infinite always. Therefore then the function V by necessity adopts an infinite value since the formula $\lambda(\lambda-1)A(x-a)^{\lambda-2}$ being prepared thus, so that there shall be $\lambda<1$. Therefore from this it is apparent that the determination of the quantity c is not always left to our choice, but whenever from the nature of this equation to be prescribed for us. Just as if this equation is proposed $ddy=\frac{2dx^2}{(x-a)^3}$, there will be

$$\frac{dy}{dx}=p=A-\frac{1}{(x-a)^2} \text{ and } y=B+Ax+\frac{1}{x-a},$$

from which on putting $x=a$ there becomes $c=A-\frac{1}{0^2}$ and $b=B+Aa+\frac{1}{0}$, hence

$$A=c+\frac{1}{0^2} \text{ et } B=b-ac-\frac{a}{0^2}+\frac{1}{0};$$

whereby, lest the equation of the integral is rendered infinite everywhere, the letters b and c can only be infinite.

SCHOLIUM 2

1093. But now we have observed that not all the orders of infinity and of vanishing are contained in the formula $(x-a)^{\lambda}$ in the $x=a$; evidently the expression $xxlx$ in the case $x=0$ surpasses the second power x^2 infinitely often, yet meanwhile the power $x^{2-\alpha}$ is less infinitely often, however small the fraction α is taken. Whereby if in the above solution we thus wish to construe the formulas, so that all orders both of infinitudes as well as of vanishing appear, it is convenient to put in place

$$y=b+A(x-a)^{\lambda}\left(l(x-a)\right)^{\mu},$$

from which there becomes

$$\frac{dy}{dx}=p=\lambda A(x-a)^{\lambda-1}\left(l(x-a)\right)^{\mu}+\mu A(x-a)^{\lambda-1}\left(l(x-a)\right)^{\mu-1};$$

but on putting $x=a$ the first part is to the last part as $l(x-a)$ to 1, that is as $\infty:1$, from which it suffices to have taken

$$p=\lambda A(x-a)^{\lambda-1}\left(l(x-a)\right)^{\mu},$$

from which there is deduced in a like manner

$$\tfrac{dp}{dx}=\lambda(\lambda-1)A(x-a)^{\lambda-2}\left(l(x-a)\right)^{\mu},$$

which expression since after in the function V we will have written in that $x=a$, $y=b$ and $p=c$ or rather

$$y=b+A(x-a)^{\lambda}\left(l(x-a)\right)^{\mu} \quad \text{and} \quad p=\lambda A(x-a)^{\lambda-1}\left(l(x-a)\right)^{\mu},$$

must be prepared, so that hence both the constant A as well as the exponents λ and μ are known. These therefore have been put forwards in this investigation, where that is extended to more cases.

PROBLEM 139

1094. *The approximation set out before is to be pursued with greater accuracy, so that with the intervals also taken a little larger there is less departure from the true values.*

SOLUTION

On putting $dy=pdx$ the second order differential equation shows this form $\tfrac{dp}{dx}=V$, from which before we have thus defined p, as if V should be a constant quantity, at least for an exceedingly small interval, from which we have obtained $p=c+V(x-a)$, evidently after we have put into V, $x=a$, $y=b$, and $p=c$, which are the initial values to be retained for the interval $x-a=\omega$. But since meanwhile the function V shall not be constant, because it involves x, y and p, actually it will be [on integrating Vdx by parts]

$$p=c+V(x-a)-\int(x-a)dV.$$

Therefore we put $dV=Pdx+Qdy+Rdp$, so that there shall be

$$dV=(P+Qp+RV)dx,$$

and now we may consider the quantity $P+Qp+RV$ as constant, the value of which arises on putting $x=a$, $y=b$ and $p=c$, with which done we have taken V above to change into F, and there shall be

$$p=c+F(x-a)-\tfrac{1}{2}(P+Qc+RF)(x-a)^2.$$

Again from this on account of $dy=pdx$ there becomes

$$y=b+c(x-a)+\tfrac{1}{2}F(x-a)^2-\tfrac{1}{6}(P+Qc+RF)(x-a)^3,$$

and in a like manner the approximation is allowed to be pursued further. But when the quantities P, Q, R and V include the formula $x-a$ or the powers of this, it cannot be regarded any further as constant, and an account of this in the integration has to be considered, from which there arises, that the powers of the formula $x-a$ ascent in a non ordered series in the approximation. Therefore then it is agreed that for p to assume initially a series of this kind :

$$p=c+A(x-a)^{\lambda},$$

from which there becomes

$$y=b+c(x-a)+\tfrac{A}{\lambda+1}(x-a)^{\lambda+1},$$

and because

$$\tfrac{dp}{dx}=\lambda A(x-a)^{\lambda-1},$$

the function V must be equal to this formula, afterwards in that we will have written the assumed values for y and p and a for x, unless the formula $x-a$ has increased; so in this way the exponent λ as well as the coefficient A will be found.

If c shall be $=0$ or $=\infty$, an account of this can be introduce into the calculation, as on putting

$$p=f(x-a)^{n}+A(x-a)^{\lambda},$$

from which there becomes

$$y=b+\tfrac{f}{n+1}(x-a)^{n+1}+\tfrac{A}{\lambda+1}(x-a)^{\lambda+1}$$

which values if they are substituted into the function V in place of x and p, must produce

$$nf(x-a)^{n-1}+\lambda A(x-a)^{\lambda-1}.$$

COROLLARY 1

1095. In this manner it is allowed to progress continually through further intervals , as long as no greater singularities are accepted, so that the errors as undertaken remain insignificant ; and by this correction these errors are diminished, so that larger intervals are able to be put in place.

COROLLARY 2

1096. Clearly for the first interval, the initial values $x=a$, $y=b$ and $p=c$ are assumed as it pleases, and the values found at the end of the interval give the values for the second interval, from which the calculation for this interval likewise is set out from the start; and thus it is to be continually advancing further.

SCHOLIUM

1097. We have given a twofold solution to the problem, the first of which if it is seen to be allowed the most widely, yet in certain cases it cannot be called into use ; therefore for these it is convenient to use another solution. But generally only very few intervals of this kind are present, which demand the latter method, while all the other intervals are allowed to be worked out with the aid of the first method. Here it comes about , when for a certain interval the quantities *V* and *c* are either vanishing or increase to infinity ; why not also can it happen, that however small the interval is taken, the quantities *y* and p shall be liable to infinite variations, the representation of which requires in short a singular determination. Just as if this equation is proposed

$$ddy + \frac{ydx^2}{xx} = 0\,,$$

and the interval from $x = 0$ as far as to $x = \omega$, even if ω is assume the smallest, shows an infinite change in the values of y and *p*, that which is evident from the complete integral of this ; because as there will be

$$y = Ax^{\frac{1}{2}}\sin.\left(\frac{\surd 3}{2}lx + \alpha\right)$$

and hence

$$p = \frac{A}{2\sqrt{x}}\sin.\left(\frac{\surd 3}{2}lx + \alpha\right) + \frac{A}{2\sqrt{x}}\cos.\left(\frac{\surd 3}{2}lx + \alpha\right)$$

or

$$p = \frac{A}{\sqrt{x}}\sin.\left(\frac{\surd 3}{2}lx + \alpha + 60^0\right),$$

it is clear, if $x = 0$, indeed to become $y = 0$, but the value of p becomes uncertain. But by attributing a minimum value of x, y will retain a certain minimum value, but which for the minimum interval now shall be positive, now vanishing, now negative, on account of the maximum change, as experienced by *lx*; but the quantity *p* meanwhile passes through all the possible changes. Likewise this is clearer from this example :

$$ddy + \frac{2dxdy}{x} - \frac{ffydx^2}{x^4} = 0\,,$$

the complete integral of which is

$$y = A\sin.\left(\frac{f}{x} + \alpha\right);$$

while *x* increases from 0 to ω, the angle $\frac{f}{x} + \alpha$ changes from being infinite to being finite and the sine meanwhile thus undergoes all the changes from $+1$ to -1 infinitely often. Therefore when intervals of this kind occur, it is no wonder if the accustomed methods of approximating fail, clearly which depend on this beginning, because the changes through the smallest intervals also shall be extremely small ; but with the exception of these intervals, the prescribed solution can always be used.

EXAMPLE 1

1098. *With the equation* $ddy + \frac{ydx^2}{fx} = 0$ *proposed, to resolve the integration of this by an approximation.*

Hence since there shall be $dp = -\frac{ydx}{fx}$, then there will be $V = \frac{-y}{fx}$; whereby if for the initial interval there shall be put $x = a,\ y = b$ and $p = c$, from that on progressing a very small amount, by the first solution [§ 1094], on account of

$$P = \frac{y}{fxx} = \frac{b}{aaf},\quad Q = \frac{-1}{fx} = \frac{-1}{af} \text{ and } R = 0$$

we will have

$$p = c - \frac{b}{af}(x-a) - \frac{1}{2}\left(\frac{b}{aaf} - \frac{c}{af}\right)(x-a)^2$$

and

$$y = b + c(x-a) - \frac{b}{2af}(x-a)^2.$$

Hence on taking $x - a = \omega$ for the following interval the initial values will be

$$a' = a + \omega,\ b' = b + c\omega - \frac{b\omega^2}{2af} \text{ and } c' = c - \frac{b\omega}{2af} - \frac{(b-ac)\omega\omega}{2aaf},$$

from which in a similar manner the initial values for the following interval are deduced. Truly if for a certain interval there becomes $a = 0$, an operation by a singular method must be put in place. Clearly on putting for the start of this interval $x = 0,\ y = b$ and $p = c$, there is established

$$p = c + Ax^{\lambda} \text{ and } y = b + cx + \frac{Ax^{\lambda+1}}{\lambda+1}$$

there will be

$$\frac{dp}{dx} = \lambda Ax^{\lambda-1} = \frac{-y}{fx} = \frac{-b}{fx} - \frac{c}{f} - \frac{Ax^{\lambda}}{(\lambda+1)f}$$

for which, unless there shall be $b = 0$, is unable to be satisfied; for there arises $\lambda = 0$ and $A = \infty$, from which we conclude there must be put in place

$$y = b + Axlx,$$

so that there shall be

$$p = Alx + A \text{ and } \frac{dp}{dx} = \frac{A}{x} = \frac{-b-Axlx}{fx},$$

hence $A = -\frac{b}{f}$. Now from which hence p may be known more accurately, we may put in place

$$y = b + Axlx + Bx;$$

then there will be

$$p = Alx + A + B \text{ and } \frac{dp}{dx} = \frac{A}{x},$$

from which it is concluded as before $A = \frac{-b}{f}$, and B remains undetermined, thus so that there shall be

$$y = b - \frac{bx}{f}lx + Bx \text{ and } p = \frac{-b}{f}lx - \frac{b}{f} + B;$$

hence if there shall be $b = 0$ in the case $x = 0$, by necessity the quantity c is infinite. On account of which if the start of the interval shall be $x = 0, y = b$ and $p = \infty$, for the end of this and the beginning of the following there shall be

$$x = \omega,\ y = b - \frac{b\omega}{f}l\omega \text{ and } p = \frac{-b}{f}l\omega.$$

EXAMPLE 2

1099. *This equation shall be proposed*

$$xxddy - 2xdxdy + 2ydx^2 = \frac{x^2ydx^2}{ff},$$

which is required to be integrated by an approximation.

Since there shall be

$$\frac{dp}{dx} = \frac{2p}{x} - \frac{2y}{xx} + \frac{y}{ff} = V,$$

there will be

$$P = \frac{-2p}{xx} + \frac{4y}{x^3}, \quad Q = \frac{-2}{xx} + \frac{1}{ff}, \quad \text{and} \quad R = \frac{2}{x}.$$

From this, and if for the start of this interval there shall be $x = a,\ y = b,\ p = c$, on account of

$$F = \frac{2c}{a} - \frac{2b}{aa} + \frac{b}{ff}$$

there will be

$$p = c + \left(\frac{2c}{a} - \frac{2b}{aa} + \frac{b}{ff}\right)(x - a) - \frac{1}{2}\left(\frac{c}{ff} + \frac{2b}{aff}\right)(x - a)^2$$

and

$$y = b + c(x - a) + \frac{1}{2}\left(\frac{2c}{a} - \frac{2b}{aa} + \frac{b}{ff}\right)(x - a)^2$$

from which the calculation is continued easily through the intervals, while there shall not be $a = 0$. But in this case, when $a = 0$, the interval for the computation is defined with difficulty, because then it cannot happen that the quantities b and c be attributed given values, which is easily understood from this, as the complete integral of the proposed equation is

$$y = Ae^{\frac{x}{f}}x + Be^{\frac{-x}{f}}x;$$

for on putting $x = 0$ by necessity there becomes $y = 0$, unless we wish to take the coefficients A and B as infinite. But on taking $b = 0$ the approximation is apparent.

CAPUT X11
DE AEQUATIONUM DIFFERENTIO-DIFFERENTIALIUM INTEGRATIONE PER APPROXlMATIONES

PROBLEMA 137

1082. *Proposita aequatione differentio-differentiali quacunque principia explicare, ex quibus integrationem per approximationes peti oportet.*

SOLUTIO

Versetur aequatio proposita inter binas variabiles x et y ac posito $dy = pdx$ et $dp = qdx$ dabitur aequatio inter quatuor quantitates x, y, p et q, ex qua q ita definire licebit, ut q aequetur functioni cuidam trium quantitatum x, y et p, qua vocata = V sit $q = V$ seu $dp = Vdx$. Hic primo observandum est integrationem, ut sit determinata, duplicem determinationem requirere seu duas conditiones quasi pro lubitu praescribi posse, quibus satisfaciat. Scilicet non sufficit, ut posito $x = a$ fiat $y = b$, quemadmodum in aequationibus differentialibus primi gradus usu venire vidimus, sed aliam insuper conditionem adiicere licet, quae sit, ut posito $x = a$ fiat etiam $p = \frac{dy}{dx} = c$ quantitati datae. His ergo determinationibus constitutis, ut posito $x = a$ fiat $y = b$ et $p = c$, totum integrationis negotium huc reducitur, ut ipsi x alium quemcunque valorem tribuendo investigentur valores respondentes ipsius y et ipsius p; hoc enim si praestiterimus, aequationis propositae integrale perfecte definiverimus, ut nihil praeterea desiderari possit. Quod cum in genere fieri nequeat, approximationis ratio in hoc consistit, ut ipsi x valor quam minimum ab a discrepans tribuatur, qui sit $x = a + \omega$, et inquiratur, quantum valores quantitatum y et p a primitivis b et c sint discrepaturi. Hic pro principio assumimus, dum x ab a ad $a + \omega$ increscit, etiam quantitatum y et p valores tam parum mutatum iri, ut inde functio V nullam variationem notabilem patiatur. Quare si ponamus statuendo $x = a$, $y = b$ et $p = c$ fieri $V = F$, eundem valorem F quantitas V retinere censebitur, dum x ab a usque ad $a + \omega$ augetur. Cum igitur pro hoc intervallo minimo habeamus $dp = Fdx$, erit integrando $p = Fx + \text{Const.}$; verum quia posito $x = a$ fieri debet $p = c$,erit

$$p = c + Fx - Fa.$$

Sit nunc $x = a + \omega$ atque habebimus $p = c + F\omega$, qui est valor ipsius p valori $x = a + \omega$ respondens. Denique pro hoc minimo intervallo erit $dy = cdx$ ideoque $y = b + cx - ac$ et pro valore $x = a + \omega$ fit $y = b + c\omega$, qui est valor ipsius y valori $x = a + \omega$ conveniens. Quocirca si valores primitivi sint $x = a$, $y = b$ et $p = \frac{dy}{dx} = c$ ex iisque fiat $V = F$, sequentes valores intervallo quam minimo ab illis remoti erunt

$$x = a + \omega,\ y = b + c\omega,\ p = c + F\omega;$$

qui si porro ut primitivi spectentur, ex iis simili modo per intervallum quam minimum progredi licet sicque tandem progressus per intervallum quantumvis magnum innotescet.

COROLLARIUM 1

1083. Quo minora capiantur haec intervalla, eo minus a vero aberrabitur, dummodo quantitates c et F non sint nimis magnae; sin autem ea adeo in infinitum excrescant, manifestum est errorem in quantitatibus y et p insignem commissum iri.

COROLLARIUM 2

1084. Si quantitas c vel F fiat vehementer magna, intervallum, quo y vel p crescit, pro dato accipi potest; ita posito $c\omega = \psi$ erunt sequentes valores

$x = a + \frac{\psi}{c}, y = b + \psi$ et $p = c + \frac{F\psi}{c}$. At si F prodeat quantitas permagna, valor ipsius p intervallo minimo φ augeri sumatur, ut sit $F\omega = \varphi$, eruntque valores sequentes

$x = a + \frac{\varphi}{F}$, $y = b + \frac{c\varphi}{F}$ et $p = c + \varphi$.

COROLLARIUM 3

1085. Si b sit quantitas infinita, pro valore proximo ipsius y expediet definiri

$$\frac{1}{y} = \frac{1}{b+c\omega} = \frac{1}{b} - \frac{c\omega}{bb} = \frac{-c\omega}{bb}$$

quae expressio ut sit finita, etiam quantitas c infinita sit oportet; alioquin valor ipsius y respondens non solum ipsi $x = a$, sed etiam ipsi $x = a + \omega$ maneret infinitus.

SCHOLION 1

1086. Quoties solutio alicuius problematis pendet ab integratione cuiuspiam aequationis differentialis secundi gradus, toties conditiones problematis binas determinationes suppeditare solent; quarum altera, dum ipsi x certus quidam valor $x = a$ tribuitur, exigit, ut y datum valorem $y = b$, altera vero, ut etiam ratio $\frac{dy}{dx} = p$ datum valorem $p = c$ consequatur. Quodsi ergo in genere aequationem quandam differentio-differentialem integrare velimus, integrationem ita instituere licet, ut posito $x = a$ fiat $y = b$ et $p = c$ quantitatibus a, b, c ab arbitrio nostro pendentibus.

Interim tamen quandoque usu venire potest, ut posito $x = a$ valores ipsarum y et p non penitus ab arbitrio nostro pendeant, sed ex natura aequationis iam datos valores sortiantur, quibus casibus defectus determinationis allis conditionibus compensatur. Veluti si proponatur haec aequatio

$$xx(a-bx)ddy - 2x(2a-bx)dxdy + 2(3a-bx)ydx^2 = 6aadx^2$$

,

quomodocunque ea per integrationem determinetur, posito $x = 0$ necessario fit $y = a$ et $\frac{dy}{dx} = p = b$, ita ut pro casu $x = 0$ valores quantitatum y et p minime arbitrio nostro relinquantur. Integrale autem completum reperitur

$$y = a + bx + \frac{(A+Bx)xx}{a-bx},$$

ubi etiamsi constantes .A et B pro lubitu assumantur, tamen semper posito $x = 0$ prodit $y = a$ et $p = b$. Huiusmodi ergo casibus mirum non est, si pro dato ipsius x valore quantitatum y et p valores arbitrio nostro haud permittantur.

SCHOLION 2

1087. Exposita ratio aequationes differentio- differentiales per approximationes integrandi, dum per intervalla minima progredimur, quemadmodum etiam in aequationibus differentialibus primi gradus fecimus, certis casibus difficultatibus involvitur, nt, nisi remedium afferatur, in usum vocari nequeat. Primum hoc evenit, quando $c = \infty$; tum enim, quantumvis exiguum accipiatur intervallum ω, neque ipsius y neque ipsius p valorem cognoscere licet. Simile quoque incommodum turbat, si positis $x = a$, $y = b$ et $p = c$ functio V fiat infinita ideoque prodeat $F = \infty$, quo casu valor ipsius p non definitur. Deinde etiam casus, quibus vel c vel F evanescit, seorsim tractari convenit; etsi enim tum valores ipsarum y et p satis accurate ostenduntur, tamen, quia nullam mutationem patiuntur, dum mutatio altiore ipsius ω potestate exprimitur, hanc ipsam mutationem investigare utile est, quo in progressu minus a veritate aberretur. Sin autem quantitas b evadat infinita, iam animadvertimus loco ipsius y eius reciprocum $\frac{1}{y}$ explorari debere. Quemadmodum ergo difficultatibus ante memoratis sit occurrendum, diligentius perpendamus.

PROBLEMA 138

1088. *Si integrationem per intervalla instituendo pro initio cuiuspiam intervalli posito $x = a$, $y = b$ et $p = c$ eveniat, ut quantitas c sit vel evanescens vel infinita, integrationem per hoc intervallum absolvere.*

SOLUTIO

Praecedens approximatio dederat $y = b + c(x-a)$, unde, si $c = 0$, incrementum ipsius y altiore potestate ipsius $x - a$ exprimetur, scilicet $y = b + A(x-a)^{\lambda}$ existente $\lambda > 1$ reiectisque altioribus potestatibus, quae prae hac ob intervallum $x - a$ minimum recte contemnuntur. Sin autem sit $c = \infty$, valor ipsius y simili modo repraesentari potest, $y = b + A(x-a)^{\lambda}$ existente $\lambda < 1$, ita tamen, ut nihilo sit major; utroque ergo casu eadem investigatio est, ut ex aequatione proposita $dp = Vdx$ tam coefficiens A quam exponens λ definiatur. Iam ex illa aequatione deducimus

$$\tfrac{dy}{dx} = p = \lambda A(x-a)^{\lambda-1} \text{ et } dp = \lambda(\lambda-1)A(x-a)^{\lambda-2}dx\,;$$

ac necesse est eandem expressionem resultare, si in formula Vdx statuatur

$$y = b + A(x-a)^{\lambda} \text{ et } p = \lambda A(x-a)^{\lambda-1};$$

unde evidens est fore $c = 0$, si $\lambda > 1$, et $c = \infty$, si $\lambda < 1$.

Cum iam V sit functio ipsarum x, y et p, ponatur ubique $x = a$, nisi quatenus formula $x = a$ inest, quae relinquatur, tum vero $y = b$, nisi hinc prodeat $V = 0$ vel $V = \infty$; hoc enim si eveniat, pro y valor $b + A(x-a)^{\lambda}$ scribatur similique modo pro p scribatur $\lambda A(x-a)^{\lambda-1}$. Reiiciantur autem formulae $x = a$ potestates altiores prae inferioribus sicque pro V orietur expressio huius formae $C(x-a)^{\mu}$, quae formulae $\lambda(\lambda-1)A(x-a)^{\lambda-2}$ aequari debet, unde tam coefficiens assumtus A quam exponens λ definietur; ideoque vero proximi valores $y = b + A(x-a)^{\lambda}$ et $p = \lambda A(x-a)^{\lambda-1}$ innotescent, qui eo minus a veritate recedent, quo minor differentia inter a et x constituatur. Casus autem, quo $\lambda = 1$, per se est perspicuus atque in praecedente problemate pertractatus, cum is sit solus, quo quantitas c finitum nanciscitur valorem.

COROLLARIUM 1

1089. Si eveniat, ut posito $c = \infty$, quo casu esse debebat $\lambda < 1$, functio V finitum obtineat valorem, cui formula $\lambda(\lambda-1)A(x-a)^{\lambda-2}$ aequari nequit, casus per se nihil habet difficultatis et valor ipsius y etiam pro minimo excessu ipsius x super a revera infinitus evadet.

COROLLARIUM 2

1090. Facilius hoc perspicere licet ex exemplo $dp = 6xdx$, unde fit $p = c + 3xx - 3aa = \tfrac{dy}{dx}$ hincque $y = b + (c-3aa)(x-a) + x^3 - a^3$ seu

$$y = b - ac + 2a^3 + (c-3aa)x + x^3$$

Si ergo constans c sumatur infinita, valor ipsius y semper erit infinitus solo excepto casu $x = a$.

COROLLARIUM 3

1091. Sin autem sumto $c = 0$, quo casu esse debet $\lambda > 1$, functio V finitum habeat valorem eumque adeo constantem, posito $x = a$ et $y = b$ ei formula $\lambda(\lambda-1)A(x-a)^{\lambda-2}$ aequabitur sumendo $\lambda = 2$ et $2A =$ illi valori constanti. Veluti in praecedente exemplo fit $V = 6a = 2A$, hinc $A = 3a$ eritque proxime $y = b + 3a(x-a)^2$; quod etiam congruit

cum integrali invento, quod posito $c=0$ est
$y=b+2a^3-3aax+x^3=b+(x+2a)(x-a)^2$, quae expressio facto $x=a$ in illam abit.

SCHOLION 1

1092. Posito $c=0$ functio V si in ea scribatur $x=a$, $y=b$ et $p=c=0$, valorem nanciscetur vel infinite magnum vel finitum vel adeo evanescentem. Primo casu, quo fit $V=\infty$, ut ei aequari possit $\lambda(\lambda-1)A(x-a)^{\lambda-2}$ sumto $x=a$, necesse est sit $\lambda<2$ existente $\lambda>1$. Ut autem hinc quantitates A et λ definiantur, in functione V scribi oportet

$$y=b+A(x-a)^{\lambda} \text{ et } p=\lambda A(x-a)^{\lambda-1}$$

itemque $x=a$, nisi ubi formula $x-a$ occurrit. Hoc modo, quia per hypothesin casu $x = a$ fit $V=\infty$, iste valor prodibit $=C(x-a)^{-\alpha}$, quo collato cum $\lambda(\lambda-1)A(x-a)^{\lambda-2}$ reperientur A et λ, dum ne prodeat $\lambda<1$, qui casus cum $c=0$ subsistere nequit. Secundo casu, quo prodit V = quantitati finitae, capi oportet $\lambda=2$, sin autem tertio casu sit $V=0$, sumi debet $\lambda>2$, ut eius valor in formula $\lambda(\lambda-1)A(x-a)^{\lambda-2}$ contineatur.

At si debeat esse $c=\infty$, fieri nequit, ut vidimus, ut functio V finitum obtineat valorem, multo minus evanescentem, nisi quidem casus incongruos, quibus y perpetuo maneat infinita, admittere velimus. Tum igitur functio V necessario valorem infinitum induit cum formula $\lambda(\lambda-1)A(x-a)^{\lambda-2}$ comparandum ita, ut sit $\lambda<1$. Hinc igitur patet determinationem quantitatis c non semper arbitrio nostro relinqui, sed quandoque ex indole ipsius aequationis nobis praescribi. Veluti si proponatur haec aequatio $ddy=\frac{2dx^2}{(x-a)^3}$ erit

$$\frac{dy}{dx}=p=A-\frac{1}{(x-a)^2} \text{ et } y=B+Ax+\frac{1}{x-a},$$

unde posito $x=a$ fit $c=A-\frac{1}{0^2}$ et $b=B+Aa+\frac{1}{0}$, ergo

$$A=c+\frac{1}{0^2} \text{ et } B=b-ac-\frac{a}{0^2}+\frac{1}{0};$$

quare, ne aequatio integralis omnino in infinitis versetur, litterae b et c non possunt non esse infinitae.

SCHOLION 2

1093. Non autem omnes ordines infinitorum et evanescentium in formula $(x-a)^{\lambda}$ casu $x=a$ contineri iam observavimus; expressio scilicet $xxlx$ casu $x=0$ infinities superat potestatem secundam x^2, interim tamen infinities minor est potestate $x^{2-\alpha}$, quantumvis etiam exigua fractio α accipiatur. Quare si in superiori solutione formulas ita instruere velimus, ut ad omnes ordines tam infinitorum quam evanescentium pateant, statui conveniet

$$y=b+A(x-a)^{\lambda}\left(l(x-a)\right)^{\mu},$$

unde fit

$$\tfrac{dy}{dx}=p=\lambda A(x-a)^{\lambda-1}\left(l(x-a)\right)^{\mu}+\mu A(x-a)^{\lambda-1}\left(l(x-a)\right)^{\mu-1};$$

at posito $x=a$ pars prior est ad posteriorem ut $l(x-a)$ ad 1, hoc est ut $\infty:1$, ex quo sufiicit sumsisse

$$p=\lambda A(x-a)^{\lambda-1}\left(l(x-a)\right)^{\mu},$$

ex quo simili modo colligitur

$$\tfrac{dp}{dx}=\lambda(\lambda-1)A(x-a)^{\lambda-2}\left(l(x-a)\right)^{\mu},$$

quae expressio cum functione V postquam in ea scripserimus $x=a$, $y=b$ et $p=c$ seu potius

$$y=b+A(x-a)^{\lambda}\left(l(x-a)\right)^{\mu} \quad \text{et} \quad p=\lambda A(x-a)^{\lambda-1}\left(l(x-a)\right)^{\mu},$$

comparari debet, ut inde tam constans A quam exponentes λ et μ innotescant. Haec ergo tenenda sunt in ista investigatione, quo ea ad plures casus extendatur.

PROBLEMA 139

1094. *Approximationem ante expositam accuratius persequi, ut sumtis intervallis etiam paulo maioribus minus a vero aberretur.*

SOLUTIO

Posito $dy=pdx$ aequatio differentio-differentialis hac forma $\frac{dp}{dx}=V$ exhibeatur, ex qua ante p ita definivimus, quasi V esset quantitas constans, pro intervallo saltem vehementer parvo, unde obtinuimus $p=c+V(x-a)$, postquam scilicet in V

posuerimus $x=a$ et $y=b$ et $p=c$, qui sunt valores primitivi per intervallum $x-a=\omega$ retinendi. Cum autem functio *V* interea non sit constans, quia *x*, *y* et *p* involvit, revera erit

$$p=c+V(x-a)-\int(x-a)dV.$$

Ponamus igitur $dV=Pdx+Qdy+Rdp$, ut sit

$$dV=(P+Qp+RV)dx,$$

et nunc quantitatem $P+Qp+RV$ ut constantem spectemus, cuius valor prodeat ponendo $x=a$, $y=b$ et $p=c$, quo facto *V* in *F* abire supra sumsimus, eritque

$$p=c+F(x-a)-\tfrac{1}{2}(P+Qc+RF)(x-a)^2.$$

Hinc porro ob $dy=pdx$ fit

$$y=b+c(x-a)+\tfrac{1}{2}F(x-a)^2-\tfrac{1}{6}(P+Qc+RF)(x-a)^3$$

similique modo approximationem ulterius prosequi licet. Quando autem quantitates *P*, *Q*, *R* et *V* formulam $x-a$ eiusve potestates complectuntur, quam non amplius ut constantem spectare licet, eius ratio in integratione est habenda, qua fit, ut in seriebus approximantibus formulae $x-a$ potestates non ordine ascendant. Tum igitur conveniet pro *p* eiusmodi seriei initium assumi

$$p=c+A(x-a)^{\lambda},$$

unde fit

$$y=b+c(x-a)+\tfrac{A}{\lambda+1}(x-a)^{\lambda+1},$$

et quia

$$\tfrac{dp}{dx}=\lambda A(x-a)^{\lambda-1},$$

huic formulae aequari debet functio *V*, postquam in ea pro *y* et *p* valores assumtos et *a* pro *x* scripserimus, nisi formula $x-a$ ingrediatur; hoc modo tam exponens λ quam coefficiens *A* determinabitur.

Si *c* sit = 0 vel $=\infty$, eius ratio potest in calculum introduci, ut ponatur

$$p=f(x-a)^n+A(x-a)^{\lambda},$$

unde fit

$$y=b+\tfrac{f}{n+1}(x-a)^{n+1}+\tfrac{A}{\lambda+1}(x-a)^{\lambda+1}$$

qui valores si loco *x* et *p* substituantur in functione *V* prodire debet

$$nf(x-a)^{n-1}+\lambda A(x-a)^{\lambda-1}.$$

COROLLARIUM 1

1095. Hoc modo per intervalla continuo ulterius progredi licet, dummodo singula non maiora accipiantur, quam ut errores commissi maneant insensibiles; atque hac quidem correctione errores illi diminuuntur, ut intervalla etiam maiora statui queant.

COROLLARIUM 2

1096. Pro primo scilicet intervallo valores primitivi $x=a$, $y=b$ et $p=c$ pro lubitu assumuntur et valores in fine intervalli inventi praebent valores initiales pro secundo intervallo, ex quibus calculus pro hoc intervallo perinde expeditur ac pro primo; sicque continuo ulterius est progrediendum.

SCHOLION

1097. Huius problematis duplicem solutionem dedimus, quarum prior etsi latissime patere videtur, certis tamen casibus in usum vocari nequit; iis ergo altera solutione uti conveniet. Existunt autem tantum plerumque paucissima eiusmodi intervalla, quae posteriorem methodum postulant, dum reliqua omnia ope prioris expedire licet. Evenit hoc, quando pro quopiam intervallo quantitates V et c vel evanescunt vel in infinitum excrescunt; quin etiam fieri potest, ut, quantumvis exiguum intervallum accipiatur, quantitates y et p variationibus infinitis sint obnoxiae, quarum repraesentatio determinationem prorsus singularem requirit. Veluti si proponatur haec aequatio

$$ddy+\frac{ydx^2}{xx}=0,$$

intervallum ab $x=0$ usque ad $x=\omega$, etiamsi ω quam minimum assumatur, infinitam mutationem in valoribus y et p indicat, id quod ex eius integrali completo perspicitur; quod cum sit

$$y=Ax^{\frac{1}{2}}\sin.\left(\frac{\sqrt{3}}{2}lx+\alpha\right)$$

hincque

$$p=\frac{A}{2\sqrt{x}}\sin.\left(\frac{\sqrt{3}}{2}lx+\alpha\right)+\frac{A}{2\sqrt{x}}\cos.\left(\frac{\sqrt{3}}{2}lx+\alpha\right)$$

seu

$$p=\frac{A}{\sqrt{x}}\sin.\left(\frac{\sqrt{3}}{2}lx+\alpha+60^0\right),$$

evidens est, si $x=0$, fore quidem $y=0$, sed ipsius p valorem esse incertum. At ipsi x valorem quam minimum tribuendo y quidem minimum retinebit valorem, sed qui pro minimo intervallo modo sit positivus modo evanescens modo negativus, ob maximam mutationem, quam lx patitur; quantitas autem p interea transit per omnes mutationes possibiles. Idem luculentius perspicitur ex hoc exemplo

$$ddy + \frac{2dxdy}{x} - \frac{ffydx^2}{x^4} = 0,$$

cuius integrale completum est

$$y = A\sin.\left(\frac{f}{x} + \alpha\right);$$

dum enim x a 0 ad ω crescit, angulus $\frac{f}{x} + \alpha$ ab infinito ad finitum transibit eiusque sinus interea omnes mutationes ab +1 ad –1 infinities adeo subiit. Quando ergo eiusmodi intervalla occurrunt, mirum non est, si consuetae methodi approximandi deficiant, quippe quae hoc principio innituntur, quod mutationes per intervalla minima sint etiam valde parvae; his autem intervallis exceptis solutio praescripta semper cum usu adhiberi potest.

EXEMPLUM 1

1098. *Proposita aequatione* $ddy + \frac{ydx^2}{fx} = 0$ *eius integrationem per approximationem absolvere.*

Cum ergo sit $dp = -\frac{ydx}{fx}$, erit $V = \frac{-y}{fx}$; quare si pro initio intervalli sit $x = a$, $y = b$ et $p = c$, inde tantillum progrediendo per solutionem priorem [§ 1094] ob $P = \frac{y}{fxx} = \frac{b}{aaf}$, $Q = \frac{-1}{fx} = \frac{-1}{af}$ et $R = 0$ habebimus

$$p = c - \frac{b}{af}(x-a) - \frac{1}{2}\left(\frac{b}{aaf} - \frac{c}{af}\right)(x-a)^2$$

et

$$y = b + c(x-a) - \frac{b}{2af}(x-a)^2.$$

Sumto ergo $x - a = \omega$ pro intervallo sequente erunt valores initiales

$$a' = a + \omega,\ b' = b + c\omega - \frac{b\omega^2}{2af} \text{ et } c' = c - \frac{b\omega}{2af} - \frac{(b-ac)\omega\omega}{2aaf},$$

unde simili modo valores initiales pro intervallo sequente colliguntur. Verum si pro quopiam intervallo fiat $a = 0$, operatio peculiari modo institui debet. Posito scilicet pro initio huius intervalli $x = 0$, $y = b$ et $p = c$ statuatur

$$p = c + Ax^{\lambda} \text{ et } y = b + cx + \frac{Ax^{\lambda+1}}{\lambda+1}$$

erit

$$\frac{dp}{dx} = \lambda Ax^{\lambda-1} = \frac{-y}{fx} = \frac{-b}{fx} - \frac{c}{f} - \frac{Ax^{\lambda}}{(\lambda+1)f}$$

cui, nisi sit $b = 0$, satisfieri nequit; prodiret enim $\lambda = 0$ et $A = \infty$, unde concludimus poni debere

$$y = b + Axlx,$$

ut sit

$$p = Alx + A \text{ et } \frac{dp}{dx} = \frac{A}{x} = \frac{-b-Axlx}{fx},$$

hinc $A = -\frac{b}{f}$. Verum quo hinc p accuratius cognoscere liceat, statuamus

$$y = b + Axlx + Bx;$$

erit

$$p = Alx + A + B \text{ et } \frac{dp}{dx} = \frac{A}{x},$$

unde concluditur ut ante $A = \frac{-b}{f}$, et B manet indeterminatum, ita ut sit

$$y = b - \frac{bx}{f}lx + Bx \text{ et } p = \frac{-b}{f}lx - \frac{b}{f} + B;$$

nisi ergo sit $b = 0$ casu $x = 0$, quantitas c necessario est infinita. Quamobrem si intervalli initio sit $x = 0, y = b$ et $p = \infty$, pro eius fine et initio sequentis erit
$x = \omega,\ y = b - \frac{b\omega}{f}l\omega$ et $p = \frac{-b}{f}l\omega.$

EXEMPLUM 2

1099. *Proposita sit haec aequatio*

$$xxddy - 2xdxdy + 2ydx^2 = \frac{x^2ydx^2}{ff},$$

quam per approximationem integrari oporteat.

Cum sit

$$\frac{dp}{dx} = \frac{2p}{x} - \frac{2y}{xx} + \frac{y}{ff} = V,$$

erit

$$P = \frac{-2p}{xx} + \frac{4y}{x^3}, \quad Q = \frac{-2}{xx} + \frac{1}{ff}, \quad \text{et} \quad R = \frac{2}{x}.$$

Hinc, si pro cuiusque intervalli initio sit $x = a,\ y = b,\ p = c$, ob

$$F = \frac{2c}{a} - \frac{2b}{aa} + \frac{b}{ff}$$

erit

$$p = c + \left(\frac{2c}{a} - \frac{2b}{aa} + \frac{b}{ff}\right)(x-a) - \frac{1}{2}\left(\frac{c}{ff} + \frac{2b}{aff}\right)(x-a)^2$$

et

$$y = b + c(x-a) + \frac{1}{2}\left(\frac{2c}{a} - \frac{2b}{aa} + \frac{b}{ff}\right)(x-a)^2$$

unde calculus per intervalla facile continuatur, dum ne sit $a = 0$. Hoc autem casu, quo $a = 0$, difficulter intervallum computo definitur, quia tum fieri nequit, ut quantitatibus b

et c dati valores tribuantur, id quod inde facillime intelligitur, quod aequationis propositae integrale completum est

$$y = Ae^{\frac{x}{f}}x + Be^{\frac{-x}{f}}x\,;$$

posito enim $x = 0$ necessario fit $y = 0$, nisi coefficientes A et B infinitos capere velimus. Sumto autem $b = 0$ approximatio est in promtu.