CHAPTER II

CONCERNING SECOND ORDER DIFFERENTIAL EQUATIONS IN WHICH ONE OF THE VARIABLES IS ABSENT

PROBLEM 95

750. *On putting dy = pdx and dp = qdx if some equation is given between the three quantities x, p and q, in which the other variable y is not present, to investigate the relation between the variables x and y.*

SOLUTION

Since the proposed equation contains these three quantities x, p and q, in place of q there may be written the value of this $\frac{dp}{dx}$, and a differential equation of the first order involving the two variables only x and p is obtained, as the integration of this is required to be treated following the precepts of the first part. Moreover from the integral found, since, if it should be complete, should be including an arbitrary constant, and from that either p in terms of x or x in terms of p can be determined. In the first case, in which p is permitted to be determined in terms of x, so that p is equal to a certain function of x, which shall be equal to X, on account of p = X becomes $pdx = dy = Xdx$, from which there is found $y = \int Xdx + \text{Const.}$, which equation defines the relation desired between x and y. In the second case, in which x is given in terms of p and is equal to some function P of p, so that there shall be $x = P$, then $y = \int pdx = \int pdP$ or $y = Pp - \int Pdp$. But if neither x by p nor p by x is able to be defined, it is required to be seen whether each can be expressed by a new variable u, from which there becomes $x = V$ and $p = U$; for then there will be had $y = \int UdV$.

COROLLARY 1

751. Therefore the resolution of second order differential equations thus can be put in place, so that it is recalled to a first order differential equation between the two variables x and p; which if it can be integrated, likewise the integration of that equation will be obtained with a certain new integration put in place.

COROLLARY 2

752. If the equation proposed between x, p and q should be prepared thus, so that q does not exceed a single dimension, or if it is allowed to be reduced to such a form, a simple equation arises involving only one dimension, where the precepts treated before are to be called into use.

COROLLARY 3

753. But if the quantity q has several dimensions or thus enters transcendently, these are to be treated by that trick, which have been treated at the end of the above section about the resolution of equations of this kind.

SCHOLIUM

754. When the letter q in the equation between x, p and q has only a single dimension and thus putting $q = \frac{dp}{dx}$, a simple differential equation arises, and the particular cases in which the integration succeeds are :

1) if this equation allows the separation of variables,

2) if either of the variables p and x, also does not surpass a single dimension in the ratio of the differentials had, and

3) if both the variables x and p everywhere constitute a number of the same dimensions, in which case the equation is called homogeneous.

The cases appearing less widely known, of the kind we have set out above, we shall not mention here.
Then if the quantity q either shall be involved with more dimensions or thus enters transcendently, allowing the resolution of particular cases, just as we have instructed above, are :

1) if some equation is proposed between x and q with p missing,

2) if the equation contains only p and q, which two cases indeed we have now treated in the previous chapter,

3) if in the proposed equation the two variables p and x everywhere make a number of the same dimensions,

4) if in the equation between x, p and q either of the two letters x or p should maintain a single dimension,

5) if an equation were prepared thus, so that on putting $x = v^{\mu}, p = z^{\mu+\nu}$ and $q = t^{\nu}$ a homogeneous equation should arise between v, z and t, which clearly constitute a number of the same dimensions everywhere.

Therefore we advance examples according to these cases.

EXAMPLE 1

755. *To investigate the equation between x and y, so that on putting dx constant this formula*

$$\frac{\left(dx^2+dy^2\right)^{\frac{3}{2}}}{dxddy}$$

is equal to a given function of x, which shall be equal to X.

On putting $dy = pdx$ and $dp = qdx$ there shall be $\frac{(1+pp)^{\frac{3}{2}}}{q} = X = \frac{(1+pp)^{\frac{3}{2}}dx}{dp}$ and thus $\frac{dx}{X} = \frac{dp}{(1+pp)^{\frac{3}{2}}}$; where since the variables x and p are separated from each other in turn, the integration gives

$$\frac{p}{\sqrt{(1+pp)}} = \int\frac{dx}{X}.$$

There is put $\int\frac{dx}{X} = V$ for the complete integral to be taken ; V is a function of x, hence

$$p = V\sqrt{(1+pp)} \quad \text{and} \quad p = \frac{V}{\sqrt{(1-VV)}}.$$

Whereby $dy = pdx = \frac{Vdx}{\sqrt{(1-VV)}}$, from which there is obtained

$$y = \int\frac{Vdx}{\sqrt{(1-VV)}}.$$

Then in addition the element is elicited

$$\sqrt{\left(dx^2+dy^2\right)} = dx\sqrt{(1+pp)} = \frac{dx}{\sqrt{(1-VV)}},$$

the integral of which gives

$$\int dx\sqrt{(1+pp)} = \int\frac{dx}{\sqrt{(1-VV)}}$$

COROLLARY 1

756. If x and y are orthogonal coordinates of the curve, the formula $\frac{(1+pp)^{\frac{3}{2}}}{q}$ will be the radius of curvature of this curve, from which hence the curve is defined, the radius of curvature is equal to some function of the abscissa x.

COROLLARY 2

757. Therefore if the radius of curvature should be inversely proportional to the abscissa x, there is assumed $X=\frac{aa}{2x}$ and there becomes $V=\int\frac{2xdx}{aa}=\frac{xx+ab}{aa}$, hence

$$y=\int\frac{(xx+ab)dx}{\sqrt{\left(a^4-(xx+ab)^2\right)}},$$

which condition gives the shapes of curves formed from an elastic plane.

COROLLARY 3

758. If there shall be $V=x^n$ or $X=\frac{1}{x^{n-1}}$, with the constant to be added ignored, there arises

$$y=\int\frac{x^ndx}{\sqrt{\left(1-x^{2n}\right)}},$$

since the integral can be shown algebraically in the cases in which either $n=\frac{1}{2i+1}$ or $n=\frac{-1}{2i}$, with i denoting a positive whole number.

EXAMPLE 2

759. *If on putting dx constant there is required to be*

$$dx\left(dx^2+dy^2\right)+xdyddy=addy\sqrt{\left(dx^2+dy^2\right)},$$

to find the equation between x and y.

On putting $dy = pdx$ our equation, on account of $ddy=dpdx$, adopts this form

$$dx(1+pp)+xpdp=adp\sqrt{(1+pp)},$$

which on division by $\sqrt{(1+pp)}$ is made integrable ; for there arises

$$x\sqrt{(1+pp)}=ap+b \quad \text{or} \quad x=\frac{ap+b}{\sqrt{(1+pp)}}.$$

Now since there shall be $y=\int pdx=px-\int xdp$, then there shall be

$$y=\frac{app+bp}{\sqrt{(1+pp)}}-\int\frac{dp(ap+b)}{\sqrt{(1+pp)}}$$

and with the integration expanded out

$$y=\frac{app+bp}{\sqrt{(1+pp)}}-a\sqrt{(1+pp)}\;-bl\frac{p+\sqrt{(1+pp)}}{n}$$

or

$$y=\frac{bp-a}{\sqrt{(1+pp)}}-bl\frac{p+\sqrt{(1+pp)}}{n}$$

thus so that both the variables *x* and *y* may be defined in terms of p. Therefore since from the first there is elicited

$$p=\frac{ab+x\sqrt{(aa+bb-xx)}}{xx-aa}\quad\text{and}\quad\sqrt{(1+pp)}=\frac{bx+a\sqrt{(aa+bb-xx)}}{xx-aa}$$

with these variables substituted there shall be

$$y=\frac{a(aa+bb-xx)+bx\sqrt{(aa+bb-xx)}}{bx+a\sqrt{(aa+bb-xx)}}-bl\frac{b+\sqrt{(aa+bb-xx)}}{n(x-a)}$$

or

$$y=\sqrt{(aa+bb-xx)}-bl\frac{b+\sqrt{(aa+bb-xx)}}{n(x-a)}.$$

COROLLARY

760. If the constant *b* entering into the first integration is assumed to vanish, the equation between *x* and *y* shall be come algebraic ; for there shall be $y=\sqrt{(aa-xx)}$. But if *b* does not vanish, the equation of the integral is transcending and involves logarithms.

EXEMPLUM 3

761. *On putting dx constant if there should be*

$$aaddy\sqrt{(aa+xx)}+aadxdy=xxdx^2,$$

to find the equation between x and y.

On putting $dy=pdx$ we shall have this equation

$$aadp\sqrt{(aa+xx)}+aapdx=xxdx$$

or

$$dp+\frac{pdx}{\sqrt{(aa+xx)}}=\frac{xxdx}{aa\sqrt{(aa+xx)}}$$

in which the variable *p* does not exceed one dimension. Therefore since there becomes

$$\int \frac{dx}{\sqrt{(aa+xx)}} = l\left(x+\sqrt{(aa+xx)}\right) \ ,$$

this equation [§ 469] is rendered integrable, if it is multiplied by $x+\sqrt{(aa+xx)}$;
for then there is produced

$$p\left(x+\sqrt{(aa+xx)}\right) = \int \frac{xxdx\left(x+\sqrt{(aa+xx)}\right)}{aa\sqrt{(aa+xx)}}$$

or

$$p\left(x+\sqrt{(aa+xx)}\right) = \frac{1}{aa}\int \frac{x^3dx}{\sqrt{(aa+xx)}} + \frac{x^3}{3aa} \ ;$$

but

$$\int \frac{x^3dx}{\sqrt{(aa+xx)}} = \frac{1}{3}(xx-2aa)\sqrt{(aa+xx)} + C$$

hence

$$p\left(x+\sqrt{(aa+xx)}\right) = \frac{(xx-2aa)\sqrt{(aa+xx)}+x^3}{3aa} + C \ .$$

This can be multiplied by $\sqrt{(aa+xx)}-x$, so that there is produced

$$aap = \frac{-xx-2aa+2x\sqrt{(aa+xx)}}{3} + C\sqrt{(aa+xx)} - Cx \ ,$$

and because $dy = pdx$, on integrating :

$$aay = -\tfrac{1}{9}x^3 - \tfrac{2}{3}aax + \tfrac{2}{9}(aa+xx)\sqrt{(aa+xx)} - \tfrac{1}{2}Cxx + C\int dx\sqrt{(aa+xx)} \ .$$

But if hence the constant C disappears, the equation between x and y will be algebraic, and clearly

$$9aay + 6aax + x^3 = 2(aa+xx)\sqrt{(aa+xx)}$$

EXAMPLE 4

762. *On putting dx constant to find the integral of this second order differential equation*

$$\left(aady^2+xxdx^2\right)ddy=nxdx^3dy.$$

There is put $dy=pdx$ and on account of $ddy=dpdx$ we shall have :

$$\left(aapp+xx\right)dp=npxdx;$$

which equation since it shall be homogeneous, we may put $x=pu$ and there shall be

$$pp\left(aa+uu\right)dp=nppu\left(pdu+udp\right)$$

or

$$\frac{dp}{p}=\frac{nudu}{aa+(1-n)uu}$$

which integrated gives

$$lp=\frac{n}{2(1-n)}l\left(aa+(1-n)uu\right)+\text{Const.}$$

Hence there is deduced

$$p=C\left(aa+(1-n)uu\right)^{\frac{n}{2(1-n)}}\ \text{and}\ x=Cu\left(aa+(1-n)uu\right)^{\frac{n}{2(1-n)}}.$$

Now since there shall be

$$y=px-\int xdp\quad\text{and}\quad dp=Cnudu\left(aa+(1-n)uu\right)^{\frac{3n-2}{2(1-n)}},$$

there becomes

$$y=CCu\left(aa+(1-n)uu\right)^{\frac{n}{1-n}}-nCC\int uudu\left(aa+(1-n)uu\right)^{\frac{2n-1}{1-n}}.$$

But in the case $n=1$ there shall be

$$lp=\frac{uu}{2aa}+C\quad\text{and}\quad u=a\sqrt{2l\frac{p}{c}},$$

hence

$$x=ap\sqrt{2l\frac{p}{c}}\quad\text{and}\quad y=app\sqrt{2l\frac{p}{c}}-a\int pdp\sqrt{2l\frac{p}{c}}.$$

COROLLARY

763. If there should be $n=\frac{1}{2}$, then there becomes :

$$x=Cu\sqrt{\left(aa+\tfrac{1}{2}uu\right)}$$

and

$$y=CCu\left(aa+\tfrac{1}{2}uu\right)-\tfrac{CCu^3}{6}+D=CCu\left(aa+\tfrac{1}{3}uu\right)+D$$

and thus the relation between *x* and *y* is expressed algebraically, as also happens, if $n=\frac{2}{3}$ or $n=\frac{3}{4}$ or $n=\frac{4}{5}$ etc.

EXAMPLE 5

764. *On putting dx constant to integrate this second order differential equation*

$$adxdy^2+xxdxddy=nxdy\sqrt{\left(dx^4+aaddy^2\right)}.$$

There is put $dy=pdx$ and $dp=qdx$, so that there becomes $ddy=qdx^3$, and our equation adopts this form:

$$app+qxx=npx\sqrt{(1+aaqq)},$$

which is homogeneous between *p* and *x*. Hence there can be put in place $p=ux$ and there comes about

$$auu+q=nu\sqrt{(1+aaqq)}.$$

Now there is $dp=qdx=udx+xdu$, from which there becomes

$$\frac{dx}{x}=\frac{du}{q-u}.$$

But from that equation between *q* and *u* there is deduced

$$q=\frac{auu+nu\sqrt{\left(1-nnaauu+a^4u^4\right)}}{nnaauu-1}$$

and

$$q-u=\frac{u(1+au-nnaauu)+nu\sqrt{\left(1-nnaauu+a^4u^4\right)}}{nnaauu-1}$$

and thus

$$\frac{dx}{x}=\frac{du}{u}\cdot\frac{nnaauu-1}{1+au-nnaauu+n\sqrt{\left(1-nnaauu+a^4u^4\right)}}.$$

Hence x is given in terms of u and hence also $p = ux$ in terms of u, from which there is deduced

$$y=\int pdx=\int uxdx\,.$$

COROLLARY 1

765. That differential equation is transformed into this :

$$\frac{dx}{x}=\frac{du}{u}\cdot\frac{1+au-nnaauu-n\sqrt{\left(1-nnaauu+a^4u^4\right)}}{nn-1-2au+(nn-1)aauu},$$

from which an account of the integration is easier to be seen.

COROLLARY 2

766. Moreover the case $nn = 2$ is noteworthy, from which there is made

$$\frac{dx}{x}=\frac{du}{u}\cdot\frac{1+au-2aauu-(1-aauu)\sqrt{2}}{(1-au)^2}$$

or

$$\frac{dx}{x}=\frac{du}{u}\cdot\frac{1+2au-(1+au)\sqrt{2}}{1-au}=\frac{du\left(1-\sqrt{2}\right)}{u}+\frac{adu\left(3-\sqrt{2}\right)}{1-au}$$

from which there is deduced :

$$lx=\left(1-\sqrt{2}\right)lu-\left(3-2\sqrt{2}\right)l(1-au)+\text{Const.}$$

or

$$xu^{\sqrt{2}-1}(1-au)^{3-2\sqrt{2}}=C.$$

EXAMPLE 6

767. *On assuming the element* $ds=\sqrt{\left(dx^2+dy^2\right)}$ *constant, to find the integral of this equation* $dx^3dy-xds^2ddy=adxds\sqrt{\left(ddx^2+ddy^2\right)}$.

On putting $dy = pdx$ there becomes $ds=dx\sqrt{(1+pp)}$ and on account of $dds=0$ there will be made

$$ddx = -\frac{pdpdx}{1+pp} = \frac{-pqdx^2}{1+pp}$$

with $dp = qdx$ being present, then

$$ddy = pddx + dpdx = \frac{-ppqdx^2}{1+pp} + qdx^2 = \frac{qdx^2}{1+pp}$$

and thus

$$\sqrt{\left(ddx^2 + ddy^2\right)} = \frac{qdx^2}{\sqrt{(1+pp)}},$$

with which substituted our equation adopts this form

$$p - qx = aq,$$

which on differentiation gives $-xdq = adq$ and thus $dq = 0$ and $q = \frac{1}{c}$. Hence $p = \int qdx = \frac{x+a}{c}$, which is the same value obtained from the equation $p = (x+a)q$ without integration. Then indeed there shall be

$$y = \int pdx = \frac{xx+2ax}{2c} + b,$$

which is the equation of the complete integral involving the two constants *b* and *c*.

EXEMPLUM 7

768. *On taking the element* $ds = \sqrt{\left(dx^2 + dy^2\right)}$ *constant to find the integral of this higher order differential equation*

$$dx^3dy - xds^2ddy = \frac{bdx^4ds^2ddy}{\sqrt{\left(dx^8 + aads^4ddy^2\right)}}.$$

On putting $dy = pdx$ and $dp = qdx$ on account of $ds = dx\sqrt{(1+pp)}$ there shall be

$$ddx = \frac{-pqdx}{1+pp} \text{ and } ddy = \frac{-dxddx}{dy} = \frac{-ddx}{p} = \frac{qdx^2}{1+pp},$$

hence $ds^2ddy = qdx^4$ from which our equation shall become

$$p - qx = \frac{bq}{\sqrt{(1+aaqq)}},$$

which on differentiation gives

$$-xdq = \frac{bdq}{(1+aaqq)^{\frac{3}{2}}},$$

from which there is concluded

$$\text{either } dq = 0 \quad \text{or} \quad x = \frac{-b}{(1+aaqq)^{\frac{3}{2}}}.$$

In the former place there shall be

$$q = \tfrac{1}{c} \text{ and } p = \tfrac{x}{c} + \frac{b}{\sqrt{(cc+aa)}},$$

and hence

$$y = \int pdx = \tfrac{xx}{2c} + \frac{bx}{\sqrt{(cc+aa)}} + f.$$

In the latter case, in which $x = \frac{-b}{(1+aaqq)^{\frac{3}{2}}}$ there becomes

$$p = \frac{-bq}{(1+aaqq)^{\frac{3}{2}}} + \frac{bq}{\sqrt{(1+aaqq)}} = \frac{aabq^3}{(1+aaqq)^{\frac{3}{2}}}.$$

But there is the equation

$$dx = \frac{3aabqdq}{(1+aaqq)^{\frac{5}{2}}} \quad \text{and hence} \quad dy = pdx = \frac{3a^4bbq^4dq}{(1+aaqq)^4}$$

and with the aid of the reduction

$$y = \frac{-\frac{1}{2}bbq - aabbq^3}{(1+aaqq)^3} + \frac{1}{2}bb\int\frac{dq}{(1+aaqq)^3}.$$

Now there is

$$\int\frac{dq}{(1+aaqq)^{n+1}} = \frac{q}{2n(1+aaqq)^n} + \frac{2n-1}{2n}\int\frac{dq}{(1+aaqq)^n}.$$

Hence

$$\int\frac{dq}{(1+aaqq)^3} = \frac{q}{4(1+aaqq)^2} + \frac{3}{4}\int\frac{dq}{(1+aaqq)^2}$$

and

$$\int\frac{dq}{(1+aaqq)^2} = \frac{q}{2(1+aaqq)} + \frac{1}{2}\int\frac{dq}{1+aaqq} = \frac{q}{2(1+aaqq)} + \frac{1}{2a}\,\text{Ang.tang.}aq.$$

Hence

$$\int\frac{dq}{(1+aaqq)^3} = \frac{q}{4(1+aaqq)^2} + \frac{3q}{8(1+aaqq)} + \frac{3}{8a}\,\text{Ang.tang.}aq.$$

and thus

$$y = \frac{-bqq(1+2aaqq)}{2(1+aaqq)^3} + \frac{bbq}{8(1+aaqq)^2} + \frac{3bbq}{16(1+aaqq)} + \frac{3bb}{16a}\,\text{Ang.tang.}aq$$

with $x=\frac{-b}{(1+aaqq)^{\frac{3}{2}}}$ present from which there becomes $1+aaqq=\sqrt[3]{\frac{bb}{xx}}$, thus so that in this way it is possible to show the equation between *x* and *y*. But this integral, as we have seen above [§ 723] only is particular.

PROBLEM 96

769. *On putting $dy = pdx$ and $dp = qdx$ if there should be given some equation between y, p and q, thus so that the variable x should be missing from that, to investigate the integral equation between x and y.*

SOLUTION

Since there shall be $q=\frac{dp}{dx}$ and $dx=\frac{dy}{p}$, then there shall be $q=\frac{pdp}{dy}$; in the equation between *y*, *p* and *q* this value $\frac{pdp}{dy}$ is substituted in place of *q* and there is had an equation of the first order involving only the variables *p* and *y*, and it is required to attempt the resolution of which by the methods set out above. But with the equation of the integral found between *p* and *y* from which either *p* in terms of *y* or *y* in terms of *p* may be defined, from which the other integration may be easier to put in place. If *y* is able to be defined conveniently in terms of *p*, so that *y* is equal to some function of *p*, which shall be equal to *P*, to that there shall be $y = P$, then there shall be $dx=\frac{dP}{p}$ and hence $x=\int\frac{dP}{p}=\frac{P}{p}+\int\frac{Pdp}{pp}$. But if it is allowed to define *p* more conveniently in terms of *y*, so that $p = Y$ with *Y* denoting some function of *y*, on account of $dx=\frac{dy}{p}$ there will be obtained $x=\int\frac{dy}{Y}$. But if neither succeeds, on introducing a new variable *u* each quantity *p* and *y* can be defined, so that there becomes $p=U$ and $y=V$ with *U* and *V* being functions of *u*, and hence there shall be $dx=\frac{dV}{U}$ and $x=\int\frac{dV}{U}$, and in this way the complete integral can be obtained by a two–fold integration.

COROLLARIUM 1

770. Therefore the resolution of a higher order differential equation of this kind also is returned to a differential equation of the first order ; the resolution of which if it should be in a power, then likewise the integral of that can be shown.

COROLLARY 2

771. If the equation between *y*, *p* and *q* were prepared thus, so that from that the value of *q* can be elicited conveniently, and hence *q* is equal to some function of *y* and *p*, which shall be *T*, then $pdp = Tdy$, which is a simple differential equation of the first order.

COROLLARY 3

772. But if an expansion of this kind does not succeed, then the letter q either rises to higher powers or it becomes involved with the signs of roots or thus may enter transcendentally, the differential equation will be of the first grade, but complicated, and which has to be treated by the above methods put in place.

SCHOLIUM 1

773. Since in a few cases differential equations of the first order can be integrated, here too, it will be helpful to note the same and to make these clear by examples. Now meanwhile as it shall be convenient to consider the remaining cases as if they were solutions, since in differential equations of higher order, it is desired chiefly that the resolution of these is the reduction to a lower order. Indeed always in analysis, which proceed in the order of treatment, as the integrations are accustomed to be taken as completely finished, even if further solutions may be desired at this stage, so that in this manner the number of solutions sought is reduced. Thus however much may be lacking at this stage, at least we can prevail to resolve algebraic equations of all orders, as long as our efforts are not extended beyond this quarter, yet in higher analysis we will take the resolution of all these equations as known. Since also it is not without use, in practice it suffices that the resolution by approximation can be extended as far as it pleases. In a like manner also, since the method found by which we treated the integration of first order differential equations approximately, deservedly can be considered as clearly applicable to the whole calculation, if by that means we are able to reduce the resolution of differential equations of higher order. Whereby in this second part we may be led at once from a differential equation of the second order to one of the first order, and the whole calculation shall be had as completed.

SCHOLIUM 2

774. Hence higher order differential equations, which in this manner can be reduced to first order differential equations, thus are to be prepared, so that on putting $dy = pdx$ and $dp = qdx$ the variable x itself thus may be removed and an equation between only the three variables y, p et q arises. Hence the case, in which the resolution of such an equation is admitted, are of two kinds, to the first of which are to be referred these, in which q takes a single dimension, from which q can be taken equal to some function of y and p. Therefore since $q = \frac{pdp}{dy} = f:(y \text{ and } p)$, which we put equal to T, the resolution succeeds,

1) if T is a homogeneous function of one dimension of y of p,

2) if there should be $T = \frac{P}{y+Q}$ with P and Q some assigned functions of p only; hence indeed there becomes $Pdy = ypdp + Qpdp$, to which also the case is referred : $T = \frac{P}{y+Qy^n}$

3) if there should be $T = p(Yp + Z)$, if *Y* and *Z* are certain functions of some kind of *y*, because then the equation $dp = Ypdy + Zdy$ is integrable on account of the single dimension of *p*, to which also the case is to be referred : $T = p\left(Yp + Zp^n\right)$.

For the other kind if the quantity *q* should have more dimensions or implicated with the square root signs or thus entering transcendently, an equation between *y, p* and *q* will be allowed resolution,

1) if on putting $q = pu$,so that there becomes $u = \frac{dp}{dy}$ a homogeneous equation results between *y* and *p,* in which clearly *y* and *p* make a number of the same dimensions everywhere, in whatever manner the remaining *u* enters into that,

2) if in the equation between *y, p* and *u* after the substitution $q = pu$, with the other quantity *y* or *p* arising with a single dimension,

3) if on putting $y = v^{\mu}$, $p = z^{\mu+\nu}$ and $u = t^{\nu}$ a homogeneous equation arises between the three quantities *v, z* et *t;* for we have shown above how to resolve equations of this kind [§ 698].

EXAMPLE 1

775. *On putting the element dx constant if there found this second order differential equation* $ddy + Adxdy + Bydx^2 = 0$, *to find the exact integral of this equation.*

On putting $dy = pdx$ and $dp = qdx$ our equation shall be $q + Ap + By = 0$ or $pdp + Apdy + Bydy = 0$, which, since it shall be homogeneous, on putting $p = vy$ turns into

$$vvydy + vyydv + Avydy + Bydy = 0,$$

from which there becomes:

$$\frac{dy}{y} + \frac{vdv}{vv+Av+B} = 0.$$

Let $vv + Av + B = (v+\alpha)(v+\beta)$, so that $\alpha + \beta = A$ and $\alpha\beta = B$; then there becomes

$$\frac{dy}{y} + \frac{\alpha dv}{(\alpha-\beta)(v+\alpha)} - \frac{\beta dv}{(\alpha-\beta)(v+\beta)} = 0$$

and hence on integrating :

$$ly + \frac{\alpha}{(\alpha-\beta)} l(v+\alpha) - \frac{\beta}{(\alpha-\beta)} l(v+\beta) = C$$

or

$$y = a(v+\beta)^{\frac{\beta}{(\alpha-\beta)}}(v+\alpha)^{\frac{-\alpha}{(\alpha-\beta)}}$$

and thus

$$p = vy = av(v+\beta)^{\frac{\beta}{\alpha-\beta}}(v+\alpha)^{\frac{-\alpha}{\alpha-\beta}}.$$

Then there is certainly $dx = \frac{dy}{p} = \frac{dy}{vy}$, from which on account of $\frac{dy}{y} = \frac{-vdv}{vv+Av+B}$ there becomes

$$dx = \frac{-vdv}{vv+Av+B} = \frac{dv}{(\alpha-\beta)(v+\alpha)} - \frac{dv}{(\alpha-\beta)(v+\beta)} \quad \text{and} \quad x = \frac{1}{\alpha-\beta} l \frac{v+\alpha}{v+\beta} + \text{Const.}$$

Now this shall be resolved easier in the following manner. Since there shall be

$$\frac{dy}{y} = \frac{-vdv}{(v+\alpha)(v+\beta)} \quad \text{and} \quad dx = \frac{-vdv}{(v+\alpha)(v+\beta)},$$

then there becomes

$$\frac{dy}{y} + \alpha dx = \frac{-dv}{(v+\beta)} \quad \text{and} \quad \frac{dy}{y} + \beta dx = \frac{-dv}{(v+\alpha)},$$

hence

$$ly + \alpha x = la - l(v+\beta) \quad \text{and} \quad ly + \beta x = lb - l(v+\alpha).$$

Hence

$$v+\beta = \frac{a}{y}e^{-\alpha x} \quad \text{and} \quad v+\alpha = \frac{b}{y}e^{-\beta x},$$

from which there becomes

$$\alpha - \beta = \frac{1}{y}\left(be^{-\beta x} - ae^{-\alpha x}\right)$$

and thus with the constants changed :

$$y = \mathfrak{A}e^{-\alpha x} + \mathfrak{B}e^{-\beta x},$$

which integration can be put in place, if α and β are real unequal quantities.

Now since we have put $vv + Av + B = (v+\alpha)(v+\beta)$, then there becomes

$$\alpha = \tfrac{1}{2}A + \sqrt{\left(\tfrac{1}{4}AA - B\right)} \quad \text{and} \quad \beta = \tfrac{1}{2}A - \sqrt{\left(\tfrac{1}{4}AA - B\right)};$$

hence, since the expression $\frac{1}{4}AA - B$ should be either positive, negative, or vanishing,

we have three cases to be set out :

1) Let $\frac{1}{2}A = m$ and $\sqrt{\left(\frac{1}{4}AA - B\right)} = n$; then the complete integral of the proposed equation shall be

$$y = \mathfrak{A}e^{-(m+n)x} + \mathfrak{B}e^{-(m-n)x} = e^{-mx}\left(\mathfrak{A}e^{-nx} + \mathfrak{B}e^{nx}\right).$$

2) Let $\frac{1}{2}A = m$ and $\sqrt{\left(\frac{1}{4}AA - B\right)} = n\sqrt{-1}$; on account of

$$e^{nx\sqrt{-1}} = cos.nx + \sqrt{-1}.sin.nx \text{ et } e^{-nx\sqrt{-1}} = cos.nx - \sqrt{-1}.sin.nx$$

there will be with the constants changed :

$$y = e^{-mx}\left(\mathfrak{C}\, cos.nx + \mathfrak{D}\, sin.nx\right) = \mathfrak{E}e^{-mx}\, cos.\left(nx + \mathfrak{e}\right).$$

3) Let $\frac{1}{2}A = m$ and $\sqrt{\left(\frac{1}{4}AA - B\right)} = 0$ or in the first case $n = 0$; on account of which $e^{-nx} = 1 - nx$ and $e^{nx} = 1 + nx$ there becomes $y = e^{-mx}\left(\mathfrak{C} + \mathfrak{D}x\right)$.

COROLLARY 1

776. Hence it is required to investigated the roots of the equation arrived at for the proposed integral $vv + Av + B = 0$, with which found it is easy to assign the complete integral.

COROLLARY 2

777. But this quadratic equation $vv + Av + B = 0$ has a notable analogy with the proposed equation itself $ddy + Adydx + Bydx^2 = 0$, from which clearly it arises on writing $1, v, v^2$ in place of $y, \frac{dy}{dx}$ et $\frac{ddy}{dx^2}$.

COROLLARY 3

778. Moreover from that equation formed algebraically $vv + Av + B = 0$ if a factor of this shall be $v + \alpha$, from that a particular integral is deduced immediately $y = \mathfrak{B}e^{-\beta x}$, and likewise the other factor $v + \beta$ will give for the other particular integral $y = \mathfrak{B}e^{-\beta x}$, with which joined together there may be obtained the complete integral $y = \mathfrak{A}e^{-\alpha x} + \mathfrak{B}e^{-\beta x}$.

SCHOLIUM

779. Below [Ch. IV] an easier method of treating second order differential equations of this kind is described, which thus extend to such a form

$$ddy + Pdydx + Qydx^2 = 0,$$

where P and Q are some functions of x, which can also be extended to the form

$$ddy + Pdydx + Qydx^2 = Xdx^2$$

on assuming some function of x for X. Evidently the method hence may be drawn from that, because in equations of the variable y of this kind, since the differentials dy and ddy of this everywhere constitute a single dimension, or also even none, and with the help of this the resolution can be reduced to a differential equation of the first order, from which the calculation for the completion can be had. But when in this manner the second order differential equation is reduced to an equation of the first order, precautions must be taken properly, so that a reduction for the integration may be had, clearly that is arrived at with the help of a suitable substitution; for no less than two integrations are required at this stage to be resolved, from which just as many arbitrary constants are to be introduced, if indeed the complete integral is desired, just as we see clearly in this example and in the preceding.

EXAMPLE 2

780. *For the proposed higher order differential equation*

$$abddy = dx\sqrt{\left(yydx^2 + aady^2\right)},$$

to investigate the integral of this.

On putting $dy = pdx$ and $dp = qdx$, this equation changes into this :

$$abq = \sqrt{\left(yy + aapp\right)} = \frac{abpdp}{dy}$$

on account of $q = \frac{pdp}{dy}$; which since it shall be homogeneous, there is put $p = \frac{y}{u}$; then it becomes

$$ydy\sqrt{\left(1 + \frac{aa}{uu}\right)} = \frac{aby}{u^3}\left(udy - ydu\right)$$

or

$$uudy\sqrt{\left(aa + uu\right)} = abudy - abydu,$$

from which there is made

$$\frac{dy}{y}=\frac{abdu}{abu-uu\sqrt{(aa+uu)}}.$$

There is put $\sqrt{(aa+uu)}=su$; then there shall be

$$uu=\frac{aa}{ss-1},\quad \frac{du}{u}=\frac{-sds}{ss-1}\quad \text{and}\quad \frac{dy}{y}=\frac{-bsds}{bss-as-b}=\frac{-sds}{ss-2ns-1}$$

on putting $\frac{a}{b}=2n$. Hence

$$\frac{2dy\sqrt{(nn+1)}}{y}=\frac{-ds\left(n+\sqrt{(nn+1)}\right)}{s-n-\sqrt{(nn+1)}}+\frac{ds\left(n-\sqrt{(nn+1)}\right)}{s-n+\sqrt{(nn+1)}}$$

and thus

$$y^{2\sqrt{(nn+1)}}=\frac{C\left(s-n+\sqrt{(nn+1)}\right)^{n-\sqrt{(nn+1)}}}{\left(s-n-\sqrt{(nn+1)}\right)^{n+\sqrt{(nn+1)}}}.$$

Therefore y is given in terms of s, so that there shall be $y=S$ and hence

$$u=\frac{a}{\sqrt{(ss-1)}}\quad \text{and}\quad p=\frac{S\sqrt{(ss-1)}}{a}\quad \text{and}$$

$$dx=\frac{adS}{S\sqrt{(ss-1)}}\quad \text{or}\quad dx=\frac{-asds}{(ss-2ns-1)\sqrt{(ss-1)}},$$

which formulas lead to rationality and are possible to be integrated by logarithms or circular arcs.

EXAMPLE 3

781. *On putting $dy=pdx$ and $dp=qdx$, to find the integral of this equation*

$$\frac{(pp+yy)\sqrt{(pp+yy)}}{2pp+yy-qy}=ny.$$

Since there shall be $q=\frac{pdp}{dy}$, then there becomes

$$dy(pp+yy)\sqrt{(pp+yy)}=2nppydy+ny^3-nyypdp\ ;$$

on account of the homogeneity of this there is put $p=uy$ and there comes about

$$y^3dy(uu+1)^{\frac{3}{2}}=2nuuy^3dy+ny^3dy-nu^2y^3dy-nuy^4du,$$

from which there is deduced

$$\frac{dy}{y}=\frac{-nudu}{(uu+1)\sqrt{(1+uu)}-nuu-n}=\frac{nudu}{(uu+1)\left(n-\sqrt{(1+uu)}\right)}$$

and y is defined in terms of u ; from which there becomes $p=uy$ and

$$dx=\frac{dy}{uy}=\frac{ndu}{(uu+1)\left(n-\sqrt{(1+uu)}\right)}$$

In the case, in which $n=1$, there shall be

$$\frac{dy}{y}=\frac{-udu}{(uu+1)\left(\sqrt{(1+uu)}-1\right)}=\frac{-du\left(n+\sqrt{(1+uu)}\right)}{u(uu+1)},$$

and

$$dx=\frac{-du\left(1+\sqrt{(1+uu)}\right)}{uu(uu+1)}.$$

Now there is

$$\int\frac{du}{u(uu+1)}=l\frac{u}{\sqrt{(uu+1)}},\quad \int\frac{du}{uu(uu+1)}=-\frac{1}{u}-\text{Ang.tang.}u,$$

$$\int\frac{du}{u\sqrt{(uu+1)}}=l\frac{\sqrt{(uu+1)}-1}{u},\quad \int\frac{du}{uu\sqrt{(uu+1)}}=-\frac{\sqrt{(uu+1)}}{u},$$

from which there is deduced:

$$y=\frac{C\sqrt{(uu+1)}}{\sqrt{(uu+1)}-1}=C\frac{\sqrt{(uu+1)}}{uu}\left(\sqrt{(uu+1)}+1\right)$$

and

$$x=D+\frac{\sqrt{(uu+1)}+1}{u}+\text{Ang. tang.}u$$

From which there is $\frac{1}{\sqrt{(uu+1)}}=1-\frac{a}{y}$ and $u=\frac{\sqrt{(2ay-aa)}}{y-a}$ and thus

$$x=D+\sqrt{\frac{2y-a}{a}}+\text{Ang.cos.}\frac{y-a}{a},$$

which formulas on introducing the angle φ, the cosine of which is $\frac{y-a}{y}$, thus can be shown more conveniently :

$$y=\frac{a}{1-cos.\varphi}\quad\text{and}\quad x=\zeta+\varphi+\cot.\tfrac{1}{2}\varphi.$$

.

COROLLARY 1

782. From the separated equation found in the first place a particular solution is elicited being attributed to a constant value of u of this kind, so that the denominator vanishes, which is $u=\sqrt{(nn-1)}$ *;* hence $p=y\sqrt{(nn-1)}$ and $dx\sqrt{(nn-1)}=\frac{dy}{y}$; from which there becomes $ly=la+x\sqrt{(nn-1)}$.

COROLLARY 2

783. In the case, in which $n = 1$, this particular case gives $y = a$ for some value of the other variable ; for let it be $u = 0$ and thus as $p = 0$, thus from the equation $dy = pdx$ the quantity X cannot be determined.

SCHOLION

784. If y designates the radius drawn [It is interesting to note that Euler uses the word *vectorem* here, indicating the notion of a point being carried from one place to another.] from a fixed point to some point curve and x the angle, which this radius makes with a certain straight line in a given position, the formula $\frac{(pp+yy)\sqrt{(pp+yy)}}{2pp+yy-qy}$ expresses the radius of curvature of this curve. Hence in the example proposed a curve of this kind is sought, the radius of curvature of which is equal to ny, to which question in the case $n = 1$ the value $y = a$ satisfies everywhere, which gives a circle, which also is deduced from the equation of the integral [corrected in the O. O.]

$$\frac{1}{y}=C\left(1-\frac{1}{\sqrt{(uu+1)}}\right)$$

on assuming the constant a infinite; for then it is necessary that there shall be $u = 0$ and $p = 0$ and thus the angle x is not determined. But besides a circle an infinite number of other curves lines satisfy the condition. But if $n>1$, a particular solution gives the logarithmic spiral $ly=la+x\sqrt{(nn-1)}$, but besides which also an infinite number of other curves are satisfactory; but in the cases $n < 1$ no particular solution of this kind has a place, but the formulas found for $\frac{dy}{y}$ and dx actually must be integrated.

EXAMPLE 4

785. *On putting* $dy = pdx$ *and* $dp = qdx$ *to find the relation between x and y, so that there becomes*

$$\frac{(pp+yy)\sqrt{(pp+yy)}}{2pp+yy-qy} = a$$

Since there shall be $q = \frac{pdp}{y}$, on putting $pp + yy = zz$; on account of $pdp = qdy$ there shall be $qdy + ydy = zdz$ or $q + y = \frac{zdz}{dy}$. But the proposed equation adopts this form :

$$z^3 = a(2zz - yy - qy) = a\left(2zz - \frac{yzdz}{dy}\right) \text{ or } zzdy = 2azdy - aydz,$$

from which there becomes :

$$\frac{dy}{y} = \frac{adz}{2az-zz} \text{ or } \frac{2dy}{y} = \frac{dz}{z} + \frac{dz}{2a-z},$$

whereby on integrating there is deduced :

$$yy = \frac{Cz}{2a-z} \text{ and } pp = zz - \frac{Cz}{2a-z} = \frac{-Cz+2azz-z^3}{2a-z}.$$

But there shall be, $z = \frac{2ayy}{C+yy}$ and hence

$$pp = \frac{4aay^4}{(C+yy)^2} - yy = \frac{yy\left(4aayy-(C+yy)^2\right)}{(C+yy)^2}.$$

Hence there arises therefore

$$dx = \frac{(C+yy)dy}{y\sqrt{\left(4aayy-(C+yy)^2\right)}}$$

let $yy = u$; then

$$dx = \frac{(C+u)du}{2u\sqrt{\left(4aau-(C+u)^2\right)}}$$

This equation is rendered more manageable on putting

$$u = 2aa - C + 2a\cos.\varphi\sqrt{(aa-C)},$$

for there becomes :

$$dx = \frac{-ad\varphi\left(a+\cos.\varphi\sqrt{(aa-C)}\right)}{2aa-C+2a\cos.\varphi\sqrt{(aa-C)}}$$

or

$$2dx = -d\varphi - \frac{Cd\varphi}{2aa - C + 2a\cos.\varphi\sqrt{(aa-C)}},$$

which integrated gives [§ 261]

$$2x = \zeta - \varphi - \text{Ang.cos.}\frac{m+\cos.\varphi}{1+m\cos.\varphi}$$

on putting $m = \frac{2a\sqrt{(aa-C)}}{2aa-C}$ so that $C = \frac{2aa\sqrt{(1-mm)}}{1+\sqrt{(1-mm)}}$ and $\sqrt{(aa-C)} = \frac{ma}{1+\sqrt{(1-mm)}}$,
and hence

$$yy = \frac{2aa(1+m\cos.\varphi)}{1+\sqrt{(1-mm)}},$$

from which there becomes

$$\cos.\varphi = \frac{yy\left(1+\sqrt{(1-mm)}\right)-2aa}{2maa}$$

and

$$\frac{m+\cos.\varphi}{1+m\cos.\varphi} = \frac{yy\left(1+\sqrt{(1-mm)}\right)-2aa(1-mm)}{myy\left(1+\sqrt{(1-mm)}\right)}.$$

COROLLARY 1

786. Since there shall be $yy = \frac{2aa(1+m\cos.\varphi)}{1+\sqrt{(1-mm)}}$, then there will be

$$yy = aa + bb + 2ab\cos.\varphi,$$

if there is put $b = \frac{a\left(1-\sqrt{(1-mm)}\right)}{m}$, from which there becomes

$$m = \frac{2ab}{aa+bb} \text{ and } \sqrt{(1-mm)} = \frac{aa-bb}{aa+bb},$$

and hence

$$2x = \zeta - \varphi - \text{Ang.cos.}\frac{2ab+(aa+bb)\cos.\varphi}{aa+bb+2ab\cos.\varphi}$$

or

$$2x = \zeta - \varphi - \text{Ang.sin.}\frac{(aa-bb)\sin.\varphi}{yy}.$$

COROLLARY 2

787. If as above the radius vector *y* with the angle *x* is referred to a curved line, this curve is required to be a circle described with radius = *a*. Moreover there shall become
$dx = \frac{d\varphi(aa-ab\cos.\varphi)}{aa+bb-2ab\cos.\varphi}$ on taking $yy = aa+bb-2ab\cos.\varphi$ and hence

$$x = \zeta + \text{Ang.tang.}\frac{a\sin.\varphi}{a\cos.\varphi - b}$$

the application of which to geometry is made evident.
[A peculiarity of this book is a complete lack of diagrams.]

EXAMPLE 5

788. *On assuming the element dx constant, if this equation is proposed :*

$$ddy\left(ydy+adx\right)=dy\left(dx^2+dy^2\right),$$

to find the integral of this.

On putting $dy=pdx$ and $dp=qdx$ we will have $q\left(py+a\right)=p\left(1+pp\right)$ and on account of

$$q=\tfrac{pdp}{dy}$$

$$dp\left(py+a\right)=dy\left(1+pp\right) \text{ or } dy-\tfrac{pydp}{1+pp}=\tfrac{adp}{1+pp},$$

which integrated gives

$$\frac{y}{\sqrt{\left(1+pp\right)}}=\frac{ap}{\sqrt{\left(1+pp\right)}}+b$$

and thus

$$y=ap+b\sqrt{\left(1+pp\right)} \quad \text{and} \quad x=\int\tfrac{dy}{p}=alp+bl\left(p+\left(1+pp\right)\right)+C,$$

thus as x and y can be expressed by the same variable p. If the constant b is taken $=0$, the particular integral may be obtained $y=ap$ and $x=alp+C=al\tfrac{y}{a}+C$

or by exponentials : $y=Ce^{x:a}$. But if there is taken $b=a$, on account of

$$p+\sqrt{\left(1+pp\right)}=\tfrac{y}{a} \quad \text{and} \quad p=\tfrac{yy-aa}{2ay}$$

there will be

$$x=al\frac{yy-aa}{2aa}+\mathrm{C} \text{ or } yy=aa+Ce^{x:a}.$$

EXAMPLE 6

789. *On assuming dx constant, to find the integral of this differential equation :*

$$dy^2-yddy=n\sqrt{\left(dx^2dy^2+aaddy^2\right)}.$$

On putting $dy=pdx$ and $dp=qdx$ there shall be

$$pp - qy = n\sqrt{(pp + aaqq)},$$

which on making $q = pu$, so that there shall be $\frac{pdp}{dy} = pu$ and thus $dp = udy$, will become

$$pp - puy = np\sqrt{(1 + aauu)} \text{ or } p - uy = n\sqrt{(1 + aauu)}.$$

Now because $dp = udy$, this equation can be differentiated and there is produced :

$$-ydu = \frac{naaudu}{\sqrt{(1+aauu)}},$$

hence either $du = 0$ or $y = \frac{-naau}{\sqrt{(1+aauu)}}$.

and hence

1) In case $du = 0$ there becomes $u = \alpha$, $p = \alpha y + \beta$ and $dx = \frac{dy}{\alpha y+\beta}$ and hence $\alpha x = l(\alpha y + \beta) + C$.

2) If $y = \frac{-naau}{\sqrt{(1+aauu)}}$, then there will be

$$p = uy + n\sqrt{(1 + aauu)} = \frac{n}{\sqrt{(1+aauu)}}$$

$$dx = \frac{dy}{p} = \frac{-aadu}{1+aauu} \text{ et } x = -a\text{Ang. tang.}au + C$$

or on account of $u = \frac{y}{a\sqrt{(nnaa-yy)}}$ the equation sought between x and y will be

$$\frac{b-x}{a} = \text{Ang.tang.}\frac{y}{\sqrt{(nnaa-yy)}} = \text{Ang.sin.}\frac{y}{na},$$

from which there becomes $y = na\sin.\frac{b-x}{a}$. But this relation is to be had only for a particular integral.

CAPUT II

DE AEQUATIONIBUS DIFFERENTIO-DIFFERENTIALIBUS IN QUIBUS ALTERA VARIABILIUM IPSA DEEST

PROBLEMA 95

750. *Posito $dy = pdx$ et $dp = qdx$ si detur aequatio quaecunque inter tres quantitates x, p et q, in quam altera variabilis y non ingrediatur, investigare relationem inter ipsas variabiles x et y.*

SOLUTIO

Cum aequatio proposita has tres quantitates x, p et q contineat, loco q scribatur eius valor $\frac{dp}{dx}$ atque habebitur aequatio differentialis primi gradus duas tantum quantitates variabiles x et p involvens, quam secundum praecepta prioris partis tractari eiusque integrale investigari oportet. Integrali autem invento, quod, si fuerit completum, constantem arbitrariam complectetur, inde vel p per x vel x per p determinari poterit. Priori casu, quo p per x definire licet, ut p aequetur functioni cuidam ipsius x, quae sit = X, ob $p = X$ fiet $pdx = dy = Xdx$, unde reperitur $y = \int Xdx + \text{Const.}$, quae aequatio relationem desideratam inter x et y, definit. Posteriori casu, quo x per p datur et functioni cuidam P ipsius p aequatur, ut sit $x = P$, erit $y = \int pdx = \int pdP$ seu $y = Pp - \int Pdp$. Sin autem neque x per p neque p per x definiri queat, videndum est, num utramque per novam variabilem u exprimere liceat, unde fiat $x = V$ et $p = U$; tum enim habebitur $y = \int UdV$.

COROLLARIUM 1

751. Huiusmodi ergo aequationum differentio-differentialium resolutio ita instituitur, ut revocetur ad aequationem differentialem primi gradus inter binas variabiles x et p; quae si integrari queat, simul illius aequationis integratio habebitur accedente quadam nova integratione.

COROLLARIUM 2

752. Si aequatio inter x, p et q proposita ita fuerit comparata, ut q unicam dimensionem non excedat, vel si ad talem formam reduci patiatur, orietur aequatio differentialis simplex differentialia unius tantum dimensionis involvens, ubi praecepta ante tradita in usum sunt vocanda.

COROLLARIUM 3

753. Sin autem quantitas *q* plures obtineat dimensiones vel adeo transcendenter ingrediatur, tentanda sunt ea artificia, quae in fine superioris partis circa resolutionem huiusmodi aequationum sunt tradita.

SCHOLION

754. Quando in aequatione inter *x*, *p* et *q* littera *q* unicam habet dimensionem indeque posito $q=\frac{dp}{dx}$ aequatio differentialis simplex nascitur, praecipui casus, quibus integratio succedit, sunt:
1) si aequatio haec differentialis separationem admittat,
2) si alterutra variabilium *p* et *x*, differentialium quoque ratione habita, unam dimensionem non superet, ac
3) si ambae variabiles *x* et *p* ubique eundem dimensionum numerum constituant, quo casu aequatio homogenea appellatur.

Casus minus late patentes, cuiusmodi supra evolvimus, hic non commemoramus. Deinde si quantitas *q* vel pluribus dimensionibus sit implicata vel adeo transcendenter ingrediatur, casus praecipui resolutionem admittentes, quemadmodum supra docuimus, sunt:
1) si proponatur aequatio quaecunque inter *x* et *q* deficiente *p*,
2) si aequatio tantum *p* et *q* contineat, quos binos quidem casus iam capite praecedente tractavimus,
3) si in aequatione proposita binae variabiles *p* et *x* ubique eundem dimensionum numerum constituant,
4) si in aequatione inter *x*, *p* et *q* altera binarum litterarum *x* vel *p* unicam dimensionem obtineat,
5) si aequatio ita fuerit comparata, ut posito $x=v^{\mu}, p=z^{\mu+\nu}$ et $q=t^{\nu}$ aequatio oriatur homogenea inter *v*, *z* et *t*, quae scilicet ubique eundem dimensionum numerum constituant.

Secundum hos ergo casus exempla proferamus.

EXEMPLUM 1

755. *Investigare aequationem inter x et y, ut posito dx constante haec formula* $\frac{\left(dx^2+dy^2\right)^{\frac{3}{2}}}{dxddy}$ *aequetur datae functioni ipsius x, quae sit* = X.

Posito $dy = pdx$ et $dp = qdx$ erit $\frac{(1+pp)^{\frac{3}{2}}}{q}=X=\frac{(1+pp)^{\frac{3}{2}}dx}{dp}$ ideoque $\frac{dx}{X}=\frac{dp}{(1+pp)^{\frac{3}{2}}}$;

ubi cum variabiles *x* et *p* sint a se invicem separatae, integratio dat
$$\frac{p}{\sqrt{(1+pp)}}=\int\frac{dx}{X}.$$

Ponatur $\int \frac{dx}{X} = V$ integrali completo sumto; erit *V* functio ipsius *x*, hinc

$$p = V\sqrt{(1+pp)} \quad \text{et} \quad p = \frac{V}{\sqrt{(1-VV)}}.$$

Quare $dy = pdx = \frac{Vdx}{\sqrt{(1-VV)}}$, unde obtinetur

$$y = \int \frac{Vdx}{\sqrt{(1-VV)}}.$$

Tum vero praeterea elicitur elementum

$$\sqrt{(dx^2 + dy^2)} = dx\sqrt{(1+pp)} = \frac{dx}{\sqrt{(1-VV)}},$$

cuius integrale praebet

$$\int dx\sqrt{(1+pp)} = \int \frac{dx}{\sqrt{(1-VV)}}$$

COROLLARIUM 1

756. Si *x* et *y* sint coordinatae orthogonales curvae, erit formula $\frac{(1+pp)^{\frac{3}{2}}}{q}$ eius radius curvedinis, unde hinc curva definitur, cuius radius curvedinis aequetur functioni cuicunque abscissae *x*.

COROLLARIUM 2

757. Si ergo radius curvedinis debeat esse reciproce proportionalis abscissae *x*, sumatur $X = \frac{aa}{2x}$ eritque $V = \int \frac{2xdx}{aa} = \frac{xx+ab}{aa}$, hince

$$y = \int \frac{(xx+ab)dx}{\sqrt{\left(a^4-(xx+ab)^2\right)}},$$

quae conditio praebet curvas a lamina elastica formatas.

COROLLARIUM 3

758. Si sit $V = x^n$ seu $X = \frac{1}{x^{n-1}}$, neglecta constante addenda oritur

$$y = \int \frac{x^n dx}{\sqrt{(1-x^{2n})}},$$

quod integrale algebraice exhiberi potest casibus, quibus est vel $n=\frac{1}{2i+1}$ vel $n=\frac{-1}{2i}$ denotante *i* numerum integrum positivum.

EXEMPLUM 2

759. *Si posito dx constante oporteat esse* $dx\left(dx^2+dy^2\right)+xdyddy=addy\sqrt{\left(dx^2+dy^2\right)}$, *invenire aequationem inter x et y.*

Posito $dy=pdx$ nostra aequatio ob $ddy=dpdx$ induit hanc formam

$$dx(1+pp)+xpdp=adp\sqrt{(1+pp)},$$

quae per $\sqrt{(1+pp)}$ divisa fit integrabilis; oritur enim

$$x\sqrt{(1+pp)}=ap+b \quad \text{seu} \quad x=\frac{ap+b}{\sqrt{(1+pp)}}.$$

Cum nunc sit $y=\int pdx=px-\int xdp$, erit

$$y=\frac{app+bp}{\sqrt{(1+pp)}}-\int\frac{dp(ap+b)}{\sqrt{(1+pp)}}$$

et integratione evoluta

$$y=\frac{app+bp}{\sqrt{(1+pp)}}-a\sqrt{(1+pp)}\ -bl\,\frac{p+\sqrt{(1+pp)}}{n}$$

seu

$$y=\frac{bp-a}{\sqrt{(1+pp)}}-bl\,\frac{p+\sqrt{(1+pp)}}{n}$$

ita ut ambae variabiles *x* et *y* per *p* definiantur. Cum igitur ex priori eliciatur

$$p=\frac{ab+x\sqrt{(aa+bb-xx)}}{xx-aa} \quad \text{et} \quad \sqrt{(1+pp)}=\frac{bx+a\sqrt{(aa+bb-xx)}}{xx-aa}$$

erit his valoribus substitutis

$$y=\frac{a(aa+bb-xx)+bx\sqrt{(aa+bb-xx)}}{bx+a\sqrt{(aa+bb-xx)}}-bl\,\frac{b+\sqrt{(aa+bb-xx)}}{n(x-a)}$$

seu

$$y=\sqrt{(aa+bb-xx)}-bl\,\frac{b+\sqrt{(aa+bb-xx)}}{n(x-a)}$$

COROLLARIUM

760. Si constans priori integratione ingressa b evanescens sumatur, aequatio inter x et y fit algebraica; erit enim $y=\sqrt{(aa-xx)}$. Sin autem b non evanescat, aequatio integralis est transcendens et logarithmos involvit.

EXEMPLUM 3

761. *Posito dx constante si debeat esse*

$$aaddy\sqrt{(aa+xx)}+aadxdy=xxdx^2,$$

invenire aequationem inter x et y.

Posito $dy=pdx$ habebimus hanc aequationem

$$aadp\sqrt{(aa+xx)}+aapdx=xxdx$$

seu

$$dp+\frac{pdx}{\sqrt{(aa+xx)}}=\frac{xxdx}{aa\sqrt{(aa+xx)}}$$

in qua variabilis p unam dimensionem non superat. Cum ergo sit

$$\int\frac{dx}{\sqrt{(aa+xx)}}=l\left(x+\sqrt{(aa+xx)}\right),$$

haec aequatio [§ 469] integrabilis redditur, si multiplicetur per $x+\sqrt{(aa+xx)}$; tum enim prodit

$$p\left(x+\sqrt{(aa+xx)}\right)=\int\frac{xxdx\left(x+\sqrt{(aa+xx)}\right)}{aa\sqrt{(aa+xx)}}$$

seu

$$p\left(x+\sqrt{(aa+xx)}\right)=\frac{1}{aa}\int\frac{x^3dx}{\sqrt{(aa+xx)}}+\frac{x^3}{3aa};$$

at

$$\int\frac{x^3dx}{\sqrt{(aa+xx)}}=\frac{1}{3}(xx-2aa)\sqrt{(aa+xx)}+C$$

hinc

$$p\left(x+\sqrt{(aa+xx)}\right)=\frac{(xx-2aa)\sqrt{(aa+xx)}+x^3}{3aa}+C.$$

Haec multiplicetur per $\sqrt{(aa+xx)}-x$, ut prodeat

$$aap=\frac{-xx-2aa+2x\sqrt{(aa+xx)}}{3}+C\sqrt{(aa+xx)}-Cx,$$

et quia $dy=pdx$, erit integrando

$$aay=-\tfrac{1}{9}x^3-\tfrac{2}{3}aax+\tfrac{2}{9}(aa+xx)\sqrt{(aa+xx)}-\tfrac{1}{2}Cxx+C\int dx\sqrt{(aa+xx)}.$$

Quodsi ergo constans C evanescat, aequatio inter x et y erit algebraica, scilicet

$$9aay+6aax+x^3=2(aa+xx)\sqrt{(aa+xx)}$$

EXEMPLUM 4

762. *Posito dx constante invenire integrale huius aequationis differentio–differentialis*

$$\left(aady^2+xxdx^2\right)ddy=nxdx^3dy.$$

Fiat $dy=pdx$ et ob $ddy=dpdx$ habebimus

$$(aapp+xx)dp=npxdx;$$

quae aequatio cum sit homogenea, statuamus $x=pu$ eritque

$$pp(aa+uu)dp=nppu(pdu+udp)$$

seu

$$\frac{dp}{p}=\frac{nudu}{aa+(1-n)uu}$$

quae integrata dat

$$lp=\frac{n}{2(1-n)}l\left(aa+(1-n)uu\right)+\text{Const}.$$

Hinc colligitur

$$p=C\left(aa+(1-n)uu\right)^{\frac{n}{2(1-n)}}\quad\text{atque}\quad x=Cu\left(aa+(1-n)uu\right)^{\frac{n}{2(1-n)}}.$$

Cum nunc sit

$$y=px-\int xdp\quad\text{et}\quad dp=Cnudu\left(aa+(1-n)uu\right)^{\frac{3n-2}{2(1-n)}},$$

erit

$$y = CCu\left(aa+(1-n)uu\right)^{\frac{n}{1-n}} - nCC\int uudu\left(aa+(1-n)uu\right)^{\frac{2n-1}{1-n}}.$$

Casu autem $n = 1$ erit

$$lp = \tfrac{uu}{2aa} + C \quad \text{et} \quad u = a\sqrt{2}l\,\tfrac{p}{c},$$

hinc

$$x = ap\sqrt{2}l\,\tfrac{p}{c} \quad \text{et} \quad y = app\sqrt{2}l\,\tfrac{p}{c} - a\int pdp\sqrt{2}l\,\tfrac{p}{c}.$$

COROLLARIUM

763. Si fuerit $n = \frac{1}{2}$, erit

$$x = Cu\sqrt{\left(aa+\tfrac{1}{2}uu\right)}$$

et

$$y = CCu\left(aa+\tfrac{1}{2}uu\right) - \tfrac{CCu^3}{6} + D = CCu\left(aa+\tfrac{1}{3}uu\right) + D$$

sicque relatio inter x et y algebraice exprimitur, quod etiam fit, si $n = \frac{2}{3}$ vel $n = \frac{3}{4}$ vel $n = \frac{4}{5}$ etc.

EXEMPLUM 5

764. *Posito dx constante integrare hanc aequationem differentio-differentialem*

$$adxdy^2 + xxdxddy = nxdy\sqrt{\left(dx^4 + aaddy^2\right)}.$$

Fiat $dy = pdx$ et $dp = qdx$, ut sit $ddy = qdx^3$, et nostra aequatio induet hanc formam

$$app + qxx = npx\sqrt{(1+aaqq)},$$

quae est homogenea inter p et x. Statuatur ergo $p = ux$ fietque

$$auu + q = nu\sqrt{(1+aaqq)}.$$

Iam vero est $dp = qdx = udx + xdu$, unde fit

$$\tfrac{dx}{x} = \tfrac{du}{q-u}.$$

At ex illa aequatione inter q et u colligitur

$$q=\frac{auu+nu\sqrt{(1-nnaauu+a^4u^4)}}{nnaauu-1}$$

et

$$q-u=\frac{u(1+au-nnaauu)+nu\sqrt{(1-nnaauu+a^4u^4)}}{nnaauu-1}$$

sicque

$$\frac{dx}{x}=\frac{du}{u}\cdot\frac{nnaauu-1}{1+au-nnaauu+n\sqrt{(1-nnaauu+a^4u^4)}}.$$

Dabitur ergo x per u hincque etiam $p = ux$ per u, unde deducitur

$$y=\int pdx=\int uxdx.$$

COROLLARIUM 1

765. lIla aequatio differentialis transformatur in hanc

$$\frac{dx}{x}=\frac{du}{u}\cdot\frac{1+au-nnaauu-n\sqrt{(1-nnaauu+a^4u^4)}}{nn-1-2au+(nn-1)aauu},$$

unde ratio integrationis facilius perspicitur.

COROLLARIUM 2

766. Notatu dignus autem est casus $nn = 2$, quo fit

$$\frac{dx}{x}=\frac{du}{u}\cdot\frac{1+au-2aauu-(1-aauu)\sqrt{2}}{(1-au)^2}$$

seu

$$\frac{dx}{x}=\frac{du}{u}\cdot\frac{1+2au-(1+au)\sqrt{2}}{1-au}=\frac{du(1-\sqrt{2})}{u}+\frac{adu(3-\sqrt{2})}{1-au}$$

unde colligitur

$$lx=(1-\sqrt{2})lu-(3-2\sqrt{2})l(1-au)+\text{Const.}$$

seu

$$xu^{\sqrt{2}-1}(1-au)^{3-2\sqrt{2}}=C.$$

EXEMPLUM 6

767. *Sumto elemento* $ds=\sqrt{(dx^2+dy^2)}$ *constante invenire integrale huius* *aequationis* $dx^3dy - xds^2ddy = adxds\sqrt{\left(ddx^2 + ddy^2\right)}$.

Posito $dy = pdx$ erit $ds = dx\sqrt{(1+pp)}$ et ob $dds = 0$ fit

$$ddx = -\tfrac{pdpdx}{1+pp} = \tfrac{-pqdx^2}{1+pp}$$

existente $dp = qdx$, tum vero

$$ddy = pddx + dpdx = \tfrac{-ppqdx^2}{1+pp} + qdx^2 = \tfrac{qdx^2}{1+pp}$$

ideoque

$$\sqrt{\left(ddx^2 + ddy^2\right)} = \tfrac{qdx^2}{\sqrt{(1+pp)}},$$

quibus substitutis aequatio nostra induit hanc formam

$$p - qx = aq,$$

quae differentiata praebet $-xdq = adq$ ideoque $dq = 0$ et $q = \frac{1}{c}$. Hinc $p = \int qdx = \frac{x+a}{c}$, qui idem valor ex aequatione $p = (x+a)q$ sine integratione obtinetur. Tum vero est

$$y = \int pdx = \tfrac{xx+2ax}{2c} + b,$$

quae est aequatio integralis completa binas eonstantes b et c involvens.

EXEMPLUM 7

768. *Sumto elemento* $ds = \sqrt{\left(dx^2 + dy^2\right)}$ *constante invenire integrale huius aequationis differentio-differentialis*

$$dx^3dy - xds^2ddy = \tfrac{bdx^4ds^2ddy}{\sqrt{\left(dx^8+aads^4ddy^2\right)}}.$$

Posito $dy = pdx$ et $dp = qdx$ ob $ds = dx\sqrt{(1+pp)}$ erit

$$ddx = \tfrac{-pqdx}{1+pp} \quad \text{et} \quad ddy = \tfrac{-dxddx}{dy} = \tfrac{-ddx}{p} = \tfrac{qdx^2}{1+pp},$$

ergo $ds^2ddy = qdx^4$ unde aequatio nostra fit

$$p - qx = \tfrac{bq}{\sqrt{(1+aaqq)}},$$

quae differentiata praebet

$$-xdq = \frac{bdq}{(1+aaqq)^{\frac{3}{2}}},$$

unde concluditur

$$\text{vel } dq = 0 \quad \text{vel } x = \frac{-b}{(1+aaqq)^{\frac{3}{2}}}.$$

Priori casu est

$$q = \tfrac{1}{c} \text{ et } p = \tfrac{x}{c} + \frac{b}{\sqrt{(cc+aa)}},$$

hincque

$$y = \int pdx = \tfrac{xx}{2c} + \frac{bx}{\sqrt{(cc+aa)}} + f.$$

Posteriori casu, quo $x = \frac{-b}{(1+aaqq)^{\frac{3}{2}}}$ fit

$$p = \frac{-bq}{(1+aaqq)^{\frac{3}{2}}} + \frac{bq}{\sqrt{(1+aaqq)}} = \frac{aabq^3}{(1+aaqq)^{\frac{3}{2}}}.$$

At est

$$dx = \frac{3aabqdq}{(1+aaqq)^{\frac{5}{2}}} \quad \text{hincque} \quad dy = pdx = \frac{3a^4bbq^4dq}{(1+aaqq)^4}$$

et ope reductionum

$$y = \frac{-\frac{1}{2}bbq - aabbq^3}{(1+aaqq)^3} + \frac{1}{2}bb\int \frac{dq}{(1+aaqq)^3}.$$

Est vero

$$\int \frac{dq}{(1+aaqq)^{n+1}} = \frac{q}{2n(1+aaqq)^n} + \frac{2n-1}{2n}\int \frac{dq}{(1+aaqq)^n}.$$

Ergo

$$\int \frac{dq}{(1+aaqq)^3} = \frac{q}{4(1+aaqq)^2} + \frac{3}{4}\int \frac{dq}{(1+aaqq)^2}$$

et

$$\int \frac{dq}{(1+aaqq)^2} = \frac{q}{2(1+aaqq)} + \frac{1}{2}\int \frac{dq}{1+aaqq} = \frac{q}{2(1+aaqq)} + \frac{1}{2a}\,\text{Ang.tang.}aq.$$

Hinc

$$\int \frac{dq}{(1+aaqq)^3} = \frac{q}{4(1+aaqq)^2} + \frac{3q}{8(1+aaqq)} + \frac{3}{8a}\,\text{Ang.tang.}aq.$$

ideoque

$$y=\frac{-bqq(1+2aaqq)}{2(1+aaqq)^3}+\frac{bbq}{8(1+aaqq)^2}+\frac{3bbq}{16(1+aaqq)}+\frac{3bb}{16a}\text{Ang.tang.}aq$$

existente $x=\frac{-b}{(1+aaqq)^{\frac{3}{2}}}$ unde fit $1+aaqq=\sqrt[3]{\frac{bb}{xx}}$, ita ut hoc modo aequatio inter *x* et *y* exhiberi possit. Hoc autem integrale, ut supra [§ 723] vidimus, tantum est particulare.

PROBLEMA 96

769. *Posito $dy=pdx$ et $dp=qdx$ si detur aequatio quaecunque inter y,p et q, ita ut variabilis x ipsa in ea desit, investigare aequationem integralem inter x et y.*

SOLUTIO

Cum sit $q=\frac{dp}{dx}$ et $dx=\frac{dy}{p}$, erit $q=\frac{pdp}{dy}$; in aequatione ergo inter *y, p* et *q* ubique loco *q* substituatur iste valor $\frac{pdp}{dy}$ atque habebitur aequatio differentialis primi gradus binas tantum variabiles *p* et *y* involvens, cuius resolutionem per methodos supra expositas tentari oportet. Inventa autem aequatione integrali inter *p* et *y* inde vel *p* per *y* vel *y* per *p* definiatur, quo facilius altera integratio institui possit. Si *y* per *p* commode definiri queat, ut *y* aequetur functioni cuipiam ipsius *p*, quae sit = *P*, ut sit $y=P$, erit $dx=\frac{dP}{p}$ hincque $x=\int\frac{dP}{p}=\frac{P}{p}+\int\frac{Pdp}{pp}$. Sin autem commodius *p* per *y* definire liceat, ut sit *p* = *Y* denotante *Y* functionem quampiam ipsius *y*, ob $dx=\frac{dy}{p}$ habebitur $x=\int\frac{dy}{Y}$. At si neutrum succedat, novam variabilem *u* introducendo per eam utraque quantitas *p* et *y* definiatur, ut fiat $p=U$ et $y=V$ existentibus *U* et *V* functionibus ipsius *u*, atque hinc erit $dx=\frac{dV}{U}$ et $x=\int\frac{dV}{U}$ hocque modo per duplicem integrationem integrale completum obtinebitur.

COROLLARIUM 1

770. Huiusmodi ergo aequationum differentio–differentialium resolutio quoque revocatur ad aequationem differentialem primi gradus; cuius resolutio si fuerit in potestate, simul illius integrale exhiberi poterit.

COROLLARIUM 2

771. Si aequatio inter *y, p* et *q* ita fuerit comparata, ut ex ea commode valor ipsius *q* elici queat hincque *q* aequetur functioni ipsarum *y* et *p*, quae sit *T*, erit $pdp=Tdy$, quae est aequatio differentialis primi gradus simplex.

COROLLARIUM 3

772. Sin autem huiusmodi evolutio non succedat, dum littera *q* vel ad altiores potestates exsurgit vel signis radicalibus involvitur vel adeo transcendenter ingreditur, aequatio differentialis quidem erit primi gradus, sed complicata, quae methodis supra expositis erit tractanda.

SCHOLION 1

773. Cum paucis casibus aequationes differentiales primi gradus integrari queant, eosdem etiam hic notasse et per exempla illustrasse iuvabit. Interim vero et reliquos casus quasi solutos spectari convenit, quandoquidem in aequationibus differentialibus altiorum ordinum id potissimum desideratur, ut earum resolutio ad ordinem inferiorem reducatur. Perpetuo enim in Analysi, quae ordine tractationis praecedunt, tanquam penitus confecta spectari solent, etiamsi plurima adhuc desiderentur, ut hoc modo multitudo desideratorum diminuatur. Ita quamvis longe adhuc absit, quominus aequationes algebraicas omnium ordinum resolvere valeamus, dum adeo vires nostrae non ultra quartum extenduntur, tamen in Analysi sublimiori omnium istarum aequationum resolutionem pro cognita habemus. Quod etiam usu non caret, cum in praxi resolutio per approximationem, quam, quousque lubuerit, extendere licet, sufficere possit. Simili modo etiam, quoniam methodum tradidimus aequationum differentialium primi gradus integralia proxime inveniendi, merito totum negotium ut plane confectum est censendum, si eo resolutionem aequationum differentialium altiorum graduum reducere potuerimus. Quare in hac secunda parte statim atque aequationem differentialem secundi gradus ad primum gradum perduxerimus, totum negotium pro confecto erit habendum.

SCHOLION 2

774. Aequationes ergo differentio–differentiales, quae hoc modo ad differentiales primi gradus reducuntur, ita sunt comparatae, ut posito $dy = pdx$ et $dp = qdx$ variabilis *x* ipsa inde tollatur et aequatio inter solas tres variabiles *y, p* et *q* oriatur. Casus ergo, quibus talis aequatio resolutionem admittit, duplicis sunt generis, ad quorum prius referendi sunt ii, quibus *q* unicam obtinet dimensionem, unde *q* functioni cuipiam ipsarum *y* et *p* aequari potest. Cum igitur sit $q = \frac{pdp}{dy} = f:(y \text{ et } p)$, quam ponamus = *T*, resolutio succedet,

1) si *T* sit functio homogenea unius dimensionis ipsarum *y* et *p*,

2) si fuerit $T = \frac{P}{y+Q}$ designantibus *P* et *Q* functiones quascunque ipsius *p* tantum; hinc enim fit $Pdy = ypdp + Qpdp$, quorsum etiam refertur casus $T = \frac{P}{y+Qy^n}$

3) si fuerit $T = p(Yp + Z)$, siquidem *Y* et *Z* sint functiones quaecunque ipsius *y*, quia tum aequatio $dp = Ypdy + Zdy$ ob unicam dimensionem ipsius *p* est integrabilis, quorsum etiam referendus est casus $T = p\left(Yp + Zp^n\right)$.

Pro altero genere si quantitas *q* plures habeat dimensiones vel signis radicalibus sit implicata vel adeo transcendenter ingrediatur, aequatio inter *y, p* et *q* resolutionem admittet,

1) si posito $q = pu$, ut sit $u = \frac{dp}{dy}$ aequatio resultet homogenea inter *y* et *p*, in qua scilicet *y* et *p* ubique eundem dimensionum numerum compleant, utcunque caeterum *u* in eam ingrediatur,
2) si in aequatione post substitutionem $q = pu$ inter *y*, *p* et *u* orta altera quantitas *y* vel *p* unicam obtineat dimensionem,
3) si posito $y = v^{\mu}$, $p = z^{\mu+\nu}$ et $u = t^{\nu}$ aequatio oriatur homogenea inter ternas quantitates *v*, *z* et *t;* huiusmodi enim aequationes supra [§ 698] resolvere docuimus.

EXEMPLUM 1

775. *Posito elemento dx constante si habeatur haec aequatio differentio-differentialis* $ddy + Adxdy + Bydx^2 = 0$, *eius integrale completum invenire.*

Posito $dy = pdx$ et $dp = qdx$ aequatio nostra erit, $q + Ap + By = 0$ seu $pdp + Apdy + Bydy = 0$, quae, cum sit homogenea, posito $p = vy$ abit in

$$vvydy + vyydv + Avydy + Bydy = 0,$$

unde fit

$$\frac{dy}{y} + \frac{vdv}{vv+Av+B} = 0.$$

Sit $vv + Av + B = (v+\alpha)(v+\beta)$, ut sit $\alpha + \beta = A$ et $\alpha\beta = B$; erit

$$\frac{dy}{y} + \frac{\alpha dv}{(\alpha-\beta)(v+\alpha)} - \frac{\beta dv}{(\alpha-\beta)(v+\beta)} = 0$$

hincque integrando

$$ly + \frac{\alpha}{(\alpha-\beta)} l(v+\alpha) - \frac{\beta}{(\alpha-\beta)} l(v+\beta) = C$$

seu

$$y = a(v+\beta)^{\frac{\beta}{(\alpha-\beta)}} (v+\alpha)^{\frac{-\alpha}{(\alpha-\beta)}}$$

ideoque

$$p = vy = av(v+\beta)^{\frac{\beta}{\alpha-\beta}} (v+\alpha)^{\frac{-\alpha}{\alpha-\beta}}.$$

Tum vero est $dx = \frac{dy}{p} = \frac{dy}{vy}$, unde ob $\frac{dy}{y} = \frac{-vdv}{vv+Av+B}$ erit

$$dx = \frac{-vdv}{vv+Av+B} = \frac{dv}{(\alpha-\beta)(v+\alpha)} - \frac{dv}{(\alpha-\beta)(v+\beta)} \quad \text{et} \quad x = \frac{1}{\alpha-\beta} l \frac{v+\alpha}{v+\beta} + \text{Const.}$$

Verum haec resolutio fit facilior sequenti modo. Cum sit

$$\frac{dy}{y}=\frac{-vdv}{(v+\alpha)(v+\beta)} \quad \text{et} \quad dx=\frac{-vdv}{(v+\alpha)(v+\beta)},$$

erit

$$\frac{dy}{y}+\alpha dx=\frac{-dv}{(v+\beta)} \quad \text{et} \quad \frac{dy}{y}+\beta dx=\frac{-dv}{(v+\alpha)},$$

hinc

$$ly+\alpha x=la-l(v+\beta) \quad \text{et} \quad ly+\beta x=lb-l(v+\alpha).$$

Ergo

$$v+\beta=\tfrac{a}{y}e^{-\alpha x} \quad \text{et} \quad v+\alpha=\tfrac{b}{y}e^{-\beta x},$$

unde fit

$$\alpha-\beta=\tfrac{1}{y}\left(be^{-\beta x}-ae^{-\alpha x}\right)$$

ideoque mutatis constantibus

$$y=\mathfrak{A}e^{-\alpha x}+\mathfrak{B}e^{-\beta x},$$

quae integratio locum habet, si α et β sint quantitates reales et inaequales.

Cum igitur posuerimus $vv+Av+B=(v+\alpha)(v+\beta)$, erit

$$\alpha=\tfrac{1}{2}A+\sqrt{\left(\tfrac{1}{4}AA-B\right)} \quad \text{et} \quad \beta=\tfrac{1}{2}A-\sqrt{\left(\tfrac{1}{4}AA-B\right)};$$

hinc, prout expressio $\frac{1}{4}AA-B$ fuerit vel positiva vel negativa vel evanescens, tres habebimus casus evolvendos:

1) Sit $\frac{1}{2}A=m$ et $\sqrt{\left(\frac{1}{4}AA-B\right)}=n$; erit aequationis propositae integrale completum

$$y=\mathfrak{A}e^{-(m+n)x}+\mathfrak{B}e^{-(m-n)x}=e^{-mx}\left(\mathfrak{A}e^{-nx}+\mathfrak{B}e^{nx}\right).$$

2) Sit $\frac{1}{2}A=m$ et $\sqrt{\left(\frac{1}{4}AA-B\right)}=n\sqrt{-1}$; ob

$$e^{nx\sqrt{-1}}=\cos.nx+\sqrt{-1}.\sin.nx \quad \text{et} \quad e^{-nx\sqrt{-1}}=\cos.nx-\sqrt{-1}.\sin.nx$$

erit constantibus mutandis

$$y=e^{-mx}\left(\mathfrak{C}\cos.nx+\mathfrak{D}\sin.nx\right)=\mathfrak{E}e^{-mx}\cos.(nx+\mathfrak{e}).$$

3) Sit $\frac{1}{2}A = m$ et $\sqrt{\left(\frac{1}{4}AA - B\right)} = 0$ seu in casu primo $n = 0$; ob $e^{-nx} = 1 - nx$ et $e^{nx} = 1 + nx$ fiet $y = e^{-mx}\left(\mathfrak{C} + \mathfrak{D}x\right)$

COROLLARY 1

776. Ad aequationis ergo propositae integrale inveniendum aequationis $vv + Av + B = 0$ radices investigari oportet, quibus inventis facile erit integrale completum assignare.

COROLLARIUM 2

777. Haec autem aequatio quadratica $vv + Av + B = 0$ insignem habet analogiam cum ipsa aequatione proposita $ddy + Adydx + Bydx^2 = 0$, ex qua quippe oritur scribendo $1, v, v^2$ loco $y, \frac{dy}{dx}$ et $\frac{ddy}{dx^2}$.

COROLLARIUM 3

778. Formata autem aequatione hac algebraica $vv + Av + B = 0$ si eius factor sit $v + \alpha$, ex eo statim integrale particulare deducitur $y = \mathfrak{B}e^{-\beta x}$ similiterque alter factor $v + \beta$ integrale particulare dabit quibus coniunctis $y = \mathfrak{B}e^{-\beta x}$, obtinetur integrale completum $y = \mathfrak{A}e^{-\alpha x} + \mathfrak{B}e^{-\beta x}$.

SCHOLION

779. Infra [cap. IV] methodus facilior tradetur huiusmodi aequationes differentio–differentiales tractandi, quae adeo ad talem formam

$$ddy + Pdydx + Qydx^2 = 0$$

patet, ubi P et Q sint functiones quaecunque ipsius x, quae etiam extendetur ad formam

$$ddy + Pdydx + Qydx^2 = Xdx^2$$

sumendo pro X functionem quamcunque ipsius x. Methodus scilicet ea inde haurietur, quod in huiusmodi aequationibus variabilis y cum suis differentialibus dy et ddy ubique unicam dimensionem constituat vel etiam nullam eiusque ope resolutio ad aequationem differentialem primi gradus reducetur, quo ipso negotium pro confecto erit habendum. Quando autem hoc modo aequatio differentio–differentialis ad aequationem differentialem primi gradus reducitur, probe cavendum est, ne haec reductio pro integratione habeatur, quippe ad quam tantum ope idoneae substitutionis est perventum; nihilo enim minus duae adhuc integrationes supersunt absolvendae, quibus totidem constantes arbitrariae introducantur, siquidem integrale completum desideretur, quemadmodum in hoc exemplo et praecedentibus clare videmus.

EXEMPLUM 2

780. *Proposita aequatione differentio–differentiali*

$$abddy = dx\sqrt{\left(yydx^2 + aady^2\right)}$$

eius integrale investigare.

Posito $dy = pdx$ et $dp = qdx$ haec aequatio abit in hanc

$$abq = \sqrt{\left(yy + aapp\right)} = \tfrac{abpdp}{dy}$$

ob $q = \frac{pdp}{dy}$; quae cum sit homogenea, ponatur $p = \frac{y}{u}$; erit

$$ydy\sqrt{\left(1 + \tfrac{aa}{uu}\right)} = \tfrac{aby}{u^3}\left(udy - ydu\right)$$

seu

$$uudy\sqrt{\left(aa + uu\right)} = abudy - abydu,$$

unde fit

$$\frac{dy}{y} = \frac{abdu}{abu - uu\sqrt{(aa+uu)}}.$$

Ponatur $\sqrt{\left(aa + uu\right)} = su$; erit

$$uu = \tfrac{aa}{ss-1}, \quad \tfrac{du}{u} = \tfrac{-sds}{ss-1} \quad \text{et} \quad \tfrac{dy}{y} = \tfrac{-bsds}{bss-as-b} = \tfrac{-sds}{ss-2ns-1}$$

posito $\frac{a}{b} = 2n$. Ergo

$$\frac{2dy\sqrt{(nn+1)}}{y} = \frac{-ds\left(n+\sqrt{(nn+1)}\right)}{s-n-\sqrt{(nn+1)}} + \frac{ds\left(n-\sqrt{(nn+1)}\right)}{s-n+\sqrt{(nn+1)}}$$

ideoque

$$y^{2\sqrt{(nn+1)}} = \frac{C\left(s-n+\sqrt{(nn+1)}\right)^{n-\sqrt{(nn+1)}}}{\left(s-n-\sqrt{(nn+1)}\right)^{n+\sqrt{(nn+1)}}}.$$

Datur igitur y per s, ut sit $y = S$ hincque $u = \frac{a}{\sqrt{(ss-1)}}$ et $p = \frac{S\sqrt{(ss-1)}}{a}$ atque

$$dx = \frac{adS}{S\sqrt{(ss-1)}} \quad \text{seu} \quad dx = \frac{-asds}{(ss-2ns-1)\sqrt{(ss-1)}},$$

quae formula ad rationalitatem perduci et per logarithmos seu arcus circulares integrari poteat.

EXEMPLUM 3

781. *Posito* $dy = pdx$ *et* $dp = qdx$ *invenire integrale huius aequationis*

$$\frac{(pp+yy)\sqrt{(pp+yy)}}{2pp+yy-qy} = ny.$$

Cum sit $q = \frac{pdp}{dy}$, erit

$$dy(pp+yy)\sqrt{(pp+yy)} = 2nppydy + ny^3 - nyypdp\ ;$$

ob cuius homogeneitatem ponatur $p = uy$ fietque

$$y^3dy(uu+1)^{\frac{3}{2}} = 2nuuy^3dy + ny^3dy - nu^2y^3dy - nuy^4du,$$

unde colligitur

$$\frac{dy}{y} = \frac{-udu}{(uu+1)\left(\sqrt{(1+uu)}-1\right)} = \frac{-du\left(n+\sqrt{(1+uu)}\right)}{u(uu+1)}$$

et y per u definitur; ex quo erit $p = uy$ et

$$dx = \frac{dy}{uy} = \frac{ndu}{(uu+1)\left(n-\sqrt{(1+uu)}\right)}$$

Casu, quo $n = 1$, erit

$$\frac{dy}{y} = \frac{-udu}{(uu+1)\left(\sqrt{(1+uu)}-nuu-1\right)} = \frac{-du\left(n+\sqrt{(1+uu)}\right)}{u(uu+1)},$$

et

$$dx = \frac{-du\left(1+\sqrt{(1+uu)}\right)}{uu(uu+1)}.$$

Est vero

$$\int\frac{du}{u(uu+1)} = l\frac{u}{\sqrt{(uu+1)}}, \quad \int\frac{du}{uu(uu+1)} = -\frac{1}{u} - \text{Ang.tang.}u,$$

$$\int\frac{du}{u\sqrt{(uu+1)}} = l\frac{\sqrt{(uu+1)}-1}{u}, \quad \int\frac{du}{uu\sqrt{(uu+1)}} = -\frac{\sqrt{(uu+1)}}{u},$$

unde colligitur

$$y = \frac{C\sqrt{(uu+1)}}{\sqrt{(uu+1)}-1} = C\frac{\sqrt{(uu+1)}}{uu}\left(\sqrt{(uu+1)}+1\right)$$

et

$$x = D + \frac{\sqrt{(uu+1)}+1}{u} + \text{Ang. tang.}u$$

Inde est $\frac{1}{\sqrt{(uu+1)}} = 1 - \frac{a}{y}$ et $u = \frac{\sqrt{(2ay-aa)}}{y-a}$ ideoque

$$x = D + \sqrt{\frac{2y-a}{a}} + \text{Ang.cos.}\frac{y-a}{a},$$

quae formulae introducendo angulum φ, cuius cosinus est $\frac{y-a}{y}$, ita commodius exhibentur

$$y = \frac{a}{1-cos.\varphi} \quad \text{et } x = \zeta + \varphi + \cot.\tfrac{1}{2}\varphi.$$

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COROLLARIUM 1

782. Ex aequatione separata primum inventa solutio particularis eruitur tribuendo ipsi u eiusmodi valorem constantem, ut denominator evanescat, qui est $u = \sqrt{(nn-1)}$ *;* hinc $p = y\sqrt{(nn-1)}$ et $dx\sqrt{(nn-1)} = \frac{dy}{y}$; unde fit $ly = la + x\sqrt{(nn-1)}$.

COROLLARIUM 2

783. Casu, quo $n = 1$, hic casus particularis praebet $y = a$ pro valore quocunque alterius variabilis; fit enim $u = 0$ ideoque et $p = 0$, ita ut ex aequatione $dy = pdx$ quantitas X non determinetur.

SCHOLION

784. Si y designet radium vectorem ex puncto fixo ad curvam quampiam ductum et x angulum, quem iste radius cum recta quadam positione data constituit formula,
$\frac{(pp+yy)\sqrt{(pp+yy)}}{2pp+yy-qy}$ exprimit huius curvae radium curvedinis.
In exemplo ergo proposito eiusmodi quaeritur curva, cuius radius curvedinis aequetur ipsi ny, cui quaestioni casu $n = 1$ utique satisfacit valor $y = a$, qui praebet circulum, qui etiam ex aequatione integrali colligitur

$$\frac{1}{y} = C\left(1 - \frac{1}{\sqrt{(uu+1)}}\right)$$

sumendo constantem a infinitam; tum enim necesse est sit $u = 0$ et $p = 0$ sicque angulus x non determinatur. Praeter circulum autem infinitae aliae lineae curvae satisfaciunt. At si $n>1$, solutio particularis $ly = la + x\sqrt{(nn-1)}$ praebet spiralem logarithmicam, praeter quam autem etiam infinitae aliae curvae satisfaciunt; casibus autem $n < 1$ nulla huiusmodi solutio particularis locum habet, sed formulas pro $\frac{dy}{y}$ et dx inventas revera integrari oportet.

EXEMPLUM 4

785. *Posito* $dy = pdx$ *et* $dp = qdx$ *invenire relationem inter x et y, ut fiat*

$$\frac{(pp+yy)\sqrt{(pp+yy)}}{2pp+yy-qy} = a$$

Cum sit $q = \frac{pdp}{y}$, ponatur $pp + yy = zz$; ob $pdp = qdy$ erit $qdy + ydy = zdz$ seu $q + y = \frac{zdz}{dy}$. Aequatio autem proposita induit hanc formam

$$z^3 = a(2zz - yy - qy) = a\left(2zz - \frac{yzdz}{dy}\right) \text{ seu } zzdy = 2azdy - aydz,$$

unde fit

$$\frac{dy}{y} = \frac{adz}{2az-zz} \text{ seu } \frac{2dy}{y} = \frac{dz}{z} + \frac{dz}{2a-z},$$

quare integrando colligitur

$$yy = \frac{Cz}{2a-z} \text{ et } pp = zz - \frac{Cz}{2a-z} = \frac{-Cz+2azz-z^3}{2a-z}.$$

At est, $z = \frac{2ayy}{C+yy}$ ergo

$$pp = \frac{4aay^4}{(C+yy)^2} - yy = \frac{yy\left(4aayy-(C+yy)^2\right)}{(C+yy)^2}.$$

Hinc igitur oritur

$$dx = \frac{(C+yy)dy}{y\sqrt{\left(4aayy-(C+yy)^2\right)}}$$

sit $yy = u$; erit

$$dx = \frac{(C+u)du}{2u\sqrt{\left(4aau-(C+u)^2\right)}}$$

Haec aequatio tractabilior redditur ponendo

$$u = 2aa - C + 2a\cos.\varphi\sqrt{(aa - C)}$$

fit enim

$$dx = \frac{-ad\varphi\left(a + \cos.\varphi\sqrt{(aa-C)}\right)}{2aa - C + 2a\cos.\varphi\sqrt{(aa-C)}}$$

seu

$$2dx = -d\varphi - \frac{Cd\varphi}{2aa - C + 2a\cos.\varphi\sqrt{(aa-C)}},$$

quae integrata dat [§ 261]

$$2x = \zeta - \varphi - \text{Ang.cos.}\frac{m + \cos.\varphi}{1 + m\cos.\varphi}$$

Posito $m = \frac{2a\sqrt{(aa-C)}}{2aa-C}$ ut sit $C = \frac{2aa\sqrt{(1-mm)}}{1+\sqrt{(1-mm)}}$ et $\sqrt{(aa-C)} = \frac{ma}{1+\sqrt{(1-mm)}}$,
hincque

$$yy = \frac{2aa(1 + m\cos.\varphi)}{1 + \sqrt{(1-mm)}},$$

unde fit

$$\cos.\varphi = \frac{yy\left(1 + \sqrt{(1-mm)}\right) - 2aa}{2maa}$$

et

$$\frac{m + \cos.\varphi}{1 + m\cos.\varphi} = \frac{yy\left(1 + \sqrt{(1-mm)}\right) - 2aa(1-mm)}{myy\left(1 + \sqrt{(1-mm)}\right)}.$$

COROLLARIUM 1

786. Cum sit $yy = \frac{2aa(1+m\cos.\varphi)}{1+\sqrt{(1-mm)}}$, erit

$$yy = aa + bb + 2ab\cos.\varphi,$$

si ponatur $b = \frac{a\left(1-\sqrt{(1-mm)}\right)}{m}$, unde fit $m = \frac{2ab}{aa+bb}$ et $\sqrt{(1-mm)} = \frac{aa-bb}{aa+bb}$,
hincque

$$2x = \zeta - \varphi - \text{Ang.cos.}\frac{2ab + (aa+bb)\cos.\varphi}{aa + bb + 2ab\cos.\varphi}$$

seu

$$2x = \zeta - \varphi - \text{Ang.sin.}\frac{(aa-bb)\sin.\varphi}{yy}.$$

COROLLARIUM 2

787. Si ut supra radius vector *y* cum angulo *x* referatur ad lineam curvam, hanc curvam circulum esse oportet radio = *a* descriptum. Fit autem

$dx = \frac{d\varphi(aa - ab\cos.\varphi)}{aa + bb - 2ab\cos.\varphi}$ sumto $yy = aa + bb - 2ab\cos.\varphi$ hincque

$$x = \zeta + \text{Ang.tang.}\frac{a\sin.\varphi}{a\cos.\varphi - b}$$

cuius applicatio ad Geometriam rem facit perspicuam.

EXEMPLUM 5

788. *Sumto elemento dx constante si proponatur haec aequatio*

$$ddy(ydy + adx) = dy(dx^2 + dy^2),$$

eius integrale invenire.

Posito $dy = pdx$ et $dp = qdx$ habebimus $q(py + a) = p(1 + pp)$ et ob

$$q = \frac{pdp}{dy}$$

$$dp(py + a) = dy(1 + pp) \text{ sive } dy - \frac{pydp}{1+pp} = \frac{adp}{1+pp},$$

quae integrata dat

$$\frac{y}{\sqrt{(1+pp)}} = \frac{ap}{\sqrt{(1+pp)}} + b$$

ideoque

$$y = ap + b\sqrt{(1 + pp)} \quad \text{et} \quad x = \int \frac{dy}{p} = alp + bl\left(p + (1 + pp)\right) + C,$$

ita ut x et y per eandem variabilem p exprimantur. Si constans b sumatur = 0, obtinetur integrale particulare

$$y = ap \quad \text{et} \quad x = alp + C = al\frac{y}{a} + C$$

seu in exponentialibus $y = Ce^{x:a}$. Sin autem sumatur $b = a$, ob

$$p + \sqrt{(1 + pp)} = \frac{y}{a} \quad \text{et} \quad p = \frac{yy - aa}{2ay}$$

erit

$$x = al\frac{yy - aa}{2aa} + \text{C} \text{ seu } yy = aa + Ce^{x:a}.$$

EXEMPLUM 6

789. *Sumto dx constante huius aequationis differentio-differentialis*

$$dy^2 - yddy = n\sqrt{\left(dx^2dy^2 + aaddy^2\right)}$$

integrale invenire.

Posito $dy = pdx$ et $dp = qdx$ erit

$$pp - qy = n\sqrt{(pp + aaqq)},$$

quae facto $q = pu$, ut sit $\frac{pdp}{dy} = pu$ ideoque $dp = udy$, abit in

$$pp - puy = np\sqrt{(1 + aauu)} \text{ seu } p - uy = n\sqrt{(1 + aauu)}.$$

Iam quia $dp = udy$, differentietur haec aequatio prodibitque

$$-ydu = \frac{naaudu}{\sqrt{(1+aauu)}},$$

hinc vel $du = 0$ vel $y = \frac{-naau}{\sqrt{(1+aauu)}}$.
hincque
1) Casu $du = 0$ fit $u = \alpha$, $p = \alpha y + \beta$ et $dx = \frac{dy}{\alpha y+\beta}$ hincque $\alpha x = l(\alpha y + \beta) + C$.
2) Si $y = \frac{-naau}{\sqrt{(1+aauu)}}$, erit

$$p = uy + n\sqrt{(1 + aauu)} = \frac{n}{\sqrt{(1+aauu)}}$$

$$dx = \frac{dy}{p} = \frac{-aadu}{1+aauu} \text{ et } x = -a\text{Ang. tang.}au + C$$

vel ob $u = \frac{y}{a\sqrt{(nnaa-yy)}}$ aequatio inter x et y quaesita erit

$$\frac{b-x}{a} = \text{Ang.tang.}\frac{y}{\sqrt{(nnaa-yy)}} = \text{Ang.sin.}\frac{y}{na},$$

unde fit $y = na\sin.\frac{b-x}{a}$. Haec autem relatio tantum pro integrali particulari est habenda.