CHAPTER III

CONCERNING HOMOGENEOUS SECOND ORDER DIFFERENTIAL EQUATIONS AND THOSE WHICH CAN BE REDUCED TO THAT FORM

PROBLEM 97

790. *To set out the nature of homogeneous second order differential equations and to adapt those to a finite form by putting* $dy = pdx$ *and* $dp = pdx$.

SOLUTION

On assuming the element *dx* constant a second order differential equation is said to be expressed in the usual manner, if not only the variables themselves *x* and y, but also the differentials of these *dx* and *dy* and likewise *ddy* on being attributed a single dimension, all the terms of the equation shall contain a number of the same dimension, just as in this equation

$$xxddy + xdx^2 + ydy^2 = 0,$$

where in the individual terms there are three dimensions involved. But if hence we put $\frac{dy}{dx} = p$ and $\frac{dp}{dx} = \frac{ddy}{dx^2} = q$, the letter *p* will be considered to have no dimension, while the letter *q* will be considered to have a single negative dimension. Hence the second order differential equation is reduced to this received form, as it retains no quantities other than the finite quantities *x*, y, *p* and *q*, and it will be homogeneous, if the letters *x* and *y* are granted a single dimension, to the letter *p* truly no dimension, but a single negative dimension is given to the letter *q* , and in the individual terms of the equation there arises a number of the same dimensions. Hence in turn, as often as this property is seized upon in a proposed equation between the four quantities *x*, *y*,*p* and *q* , that equation will be homogeneous, expressed in the usual form and clearly presents itself as homogeneous.

COROLLARY 1

791. Hence if in such a homogeneous equation between *x*, *y*, *p* and *q* there is put in place $y = ux$ and $q = \frac{v}{x}$ all the terms contain the same power of *x*, which hence on being removed by division, an equation is produced involving the three variables only *u*, *v* et *p*.

COROLLARY 2

792. Therefore the criterion between the four proposed quantities *x*, *y*, *p* and *q* is consistent with this, that on putting $y = ux$ and $q = \frac{v}{x}$, the quantity *x* is removed completely from the calculation.

COROLLARY 3

793. And with this substitution made, from which an equation may be obtained between the three quantities *u*, *v* and *p*, from which as it pleases, it is possible to define either *p* in terms of *u* and *v*, *v* in terms of *u* and *p*, or *u* in terms of *v* and *p*.

SCHOLIUM

794. We have put in place the idea of homogeneity in second order differential equations in the same way as we have used in differential equations of the first order. Indeed in these since the differentials must at once constitute a number of the same dimension, the homogeneity is decided by the variables *x* and *y* themselves. But in second order differential equations besides the variables *x* and *y* also an account must be taken of the letter *q* in computing the dimension to be had, thus finally so that it has to be given a single negative dimension ; but the letter *p* plainly does not enter into this calculation, which hence, however it is involved in the equation, does not disturb the homogeneity But it is of the greatest concern to know properly the innate character of homogeneous second order differential equations, since the resolution of these can be reduced to the resolution of differential equations of the first order, thus so that, if this should succeed, the integration of the second order differential equation itself will be obtained, that we show more clearly in the following problem.

PROBLEM 98

795. *For the proposed second order homogeneous differential equation, to reduce the solution of this to the integration of a differential equation of the first order.*

SOLUTION

With the equation requiring to be reduced here to the form received, on putting $dy = pdx$ and $dp = qdx$, so that an equation may be obtained between the four finite quantities *x*, *y*, *p* and *q*, there is put $y = ux$ and $q = \frac{v}{x}$, and since the equation shall be homogeneous, in this manner the quantity *x* is completely removed from the calculation, thus so that there will be produced an equation between the three quantities *u*, *v* and *p*, from which it is possible to define one in terms of the other two remaining. Now therefore since there shall be $dy = pdx$, then $udx + xdu = pdx$ and hence $\frac{dx}{x} = \frac{du}{p-u}$. Then on account of $dp = qdx$ there will be $dp = \frac{vdx}{x}$ and thus $\frac{dx}{x} = \frac{dp}{p}$ from which from the two fold value of $\frac{dx}{x}$ there is deduced

$$\frac{du}{p-u}=\frac{dp}{p} \text{ or } vdu = pdp - udp.$$

But if hence from that equation the quantity v is defined in terms of the two quantities p and u, there will be had a differential equation of the first order between the two variables p et u, the integration of which if it should be in force, so that p is known in terms of u, the equation $\frac{dx}{x}=\frac{du}{p-u}$, in which the variables x and u are separated, can be integrated and thus x can be defined in terms of u, from which there becomes $y = ux$; or $\frac{y}{x}$ is written at once in this integral in place of u and the sought equation between x and y is obtained.

COROLLARY 1

796. Hence the whole undertaking is reduced to the integration of this simple differential equation $vdu = pdp - udp$; which if with the aid of the above rule of treatment this can be brought about, then likewise the integration of the second order differential equation will be obtained.

COROLLARY 2

797. But likewise it is apparent that the resolution of this kind of differential equation requires a double integration, from which two arbitrary constants will enter, by which the complete integral is established.

COROLLARY 3

798. But even if the integration $vdu = pdp - udp$ does not succeed, yet a great gain is advanced from that, since the above general method proposes how the integration of all differential equations of the first order can be assigned approximately.

SCHOLIUM

799. Hence it is worth the effort to consider carefully these cases, in which the equation $vdu = pdp - udp$ allows integration; on account of which we shall examine, what kind of function v must be of p and u, so that this eventuates. But first it is apparent this comes about, if v were a homogeneous function of one dimension of p and u, because then this equation shall be homogeneous and by the above rules established it is able to be integrated. Then also the integration will succeed, if v were some function of p, since then the other variable u does not succeed one dimension and the integral of the equation $du+\frac{udp}{v}=\frac{pdp}{v}$ is

$$e^{\int\frac{dp}{v}}u=\int e^{\int\frac{dp}{v}}\frac{pdp}{v}$$

In the third place the integration is allowed to be resolved, if v were some function of the quantity $p-u$. For on putting $p-u=s$, in order that v shall be a function of s, on account of $p=s+u$ our equation shall be $vdu = sds + sdu$ and thus

$du = \frac{sds}{(v-s)}$ and $u = \int \frac{sds}{(v-s)}$, which integration thus is to be referred to the simple formulas. In the fourth place on maintaining $s = p - u$ if *P, Q, R* denote some functions of *s*, our equation $vdu = sds + sdu$ is able to be treated, if there should be $v = s + \frac{Ps}{Qu + Ru^n}$; for then there becomes $Pdu = Quds + Ru^n ds$. In the fifth place also it is apparent, if on denoting some functions *V* and *U* of u there should be $v = s + Vss + Us^n$, the integration also shall be in force; for our equation becomes $Vsdu + Us^{n-1}du = ds$. And in general if the differential equation $ds = Zdu$ should be integrable with the function *Z* of the two variables *s* and *U*, since our equation shall be $sds = (v-s)du$, and we will have $v = s + Zs$ for all the cases allowing integration.

EXAMPLE 1

800. *On assuming the element dx constant if this equation is proposed*

$$xxddy = xdxdy + nydx^2,$$

to find the integral of this.

On putting $dy = pdx$ and $dp = qdx$ then there shall be $qxx = px + ny$, from which on making $y = ux$ there is produced $qx = p + nu = v$, thus so that *v* shall be a function of on dimension of *p* and *u* and our equation $(p + nu)du = pdp - udp$ is made homogeneous. Hence since there shall be $nudu + pdu + udp = pdp$, then on integrating

$$C + nuu + 2pu = pp \quad \text{and} \quad p = u + \sqrt{\left(C + (n+1)uu\right)}.$$

Hence we will have $\frac{dx}{x} = \frac{du}{\sqrt{\left(C + (n+1)uu\right)}}$, which on integrating again gives

$$lx = \frac{1}{\sqrt{(n+1)}} l \frac{u\sqrt{(n+1)} + \sqrt{\left(C + (n+1)uu\right)}}{D}$$

or

$$Dx^{\sqrt{(n+1)}} = u\sqrt{(n+1)} + \sqrt{\left(C + (n+1)uu\right)}$$

and hence

$$D^2 x^{2\sqrt{(n+1)}} - 2Dx^{\sqrt{(n+1)}} u\sqrt{(n+1)} = C.$$

Let $D = f\sqrt{(n+1)}$ and $C = g(n+1)$, so that there is obtained :

$$ffx^{2\sqrt{(n+1)}} - 2fx^{\sqrt{(n+1)}} u = g$$

with $u=\frac{y}{x}$ being present. For the case in which $n=-1$, on account of $\frac{dx}{x}=\frac{du}{u}$ there becomes $\alpha l\frac{x}{a}=u=\frac{y}{x}$ and thus $y=\alpha xl\frac{x}{a}$. But if $n+1$ should be a negative number, then the integration also involves angles.

COROLLARY 1

801. If there should be $n=0$, the complete integral of this equation $xxddy=xdxdy$ will be $ffx^2-2fy=g$, which case is seen by itself, since from $\frac{ddy}{dx}=\frac{dx}{x}$ there becomes $\frac{dy}{dx}=fx$ and $2y=fxx-\frac{g}{f}$.

COROLLARY 2

802. If there should be $n=3$, the complete integral of the equation $xxddy=xdxdy+3ydx^2$ is $ffx^4-2fxy=g$. The same comes about, if in place of $\sqrt{(n+1)}$ there is written –2; for there becomes $\frac{ff}{x^4}-\frac{2fy}{x^3}=g$ and $ff-2fxy=gx^4$; each is reduced to $y=\frac{\alpha}{x}+\beta x^3$.

EXAMPLE 2

803. *On assuming the element dx constant if this equation is proposed*

$$\frac{xxddy}{dx}=\sqrt{\left(mxxdy^2+nyydx^2\right)},$$

to find the complete integral of this.

On account of $dy=pdx$ and $dp=qdx$ we will have $qxx=\sqrt{(mppxx+nyy)}$, which on putting $y=ux$ will change into

$$qx=\sqrt{(mpp+nuu)}=v,$$

on account of $q=\frac{v}{x}$. Because there is the relation $\frac{dx}{x}=\frac{du}{p-u}=\frac{dp}{v}$ then there becomes

$$du\sqrt{(mpp+nuu)}=(p-u)dp,$$

which is a homogeneous equation. Hence there is put $p=su$ and there is produced

$$du\sqrt{(mss+n)}=(s-1)(sdu+uds)$$

and hence

$$\frac{du}{u}=\frac{(s-1)ds}{\sqrt{(mss+n)}-ss+s}$$

from which there becomes

$$\frac{dx}{x}=\frac{du}{(s-1)u}=\frac{ds}{\sqrt{(mss+n)}-ss+s}$$

from which both $u=\frac{y}{x}$ as well as x can be determined by the same variable s.

EXAMPLE 3

804. *On assuming the element dx constant, if this equation is proposed*

$$nx^3ddy=(ydx-xdy)^2$$

to find the integral of this equation..

On putting $dy=pdx$ and $dp=qdx$ there shall be

$$nx^3q=(y-px)^2=nxxv \text{ on account of } q=\frac{v}{x}.$$

Now if there is put in place $y=ux$, there shall become

$$nv=(u-p)^2 \text{ and } \frac{dx}{x}=\frac{du}{p-u}=\frac{dp}{v}=\frac{ndp}{(p-u)^2},$$

from which there is had $ndp=pdu-udu$, which on making $p-u=s$ will change into

$$ndu+nds=sdu \quad \text{or} \quad du=\frac{nds}{s-n}, \quad \text{hence} \quad u=nl\frac{s-n}{\alpha}.$$

Then on account of $p-u=s$ there shall be

$$\frac{dx}{x}=\frac{du}{s}=\frac{nas}{s(s-n)} \text{ and } lx=l\frac{s-n}{\beta s}$$

and thus

$$x=\frac{s-n}{\beta s} \text{ and } y=nxl\frac{s-n}{\alpha}.$$

Hence since there shall be $s=\frac{n}{1-\beta x}$, then there shall be

$$y=nxl\frac{n\beta x}{\alpha(1-\beta x)}.$$

COROLLARY

805. The equation $nx^3q=(y-px)^2$ is more easily resolved on putting $y-px=z$, from which there becomes $-xdp=ds$; whereby on account of $qdx=dp$ there shall be

$$nx^3dp=zzdx=-nxxdz$$

and thus

$$\frac{1}{x}=\frac{1}{a}-\frac{n}{z} \quad \text{or} \quad \frac{x-a}{ax}=\frac{n}{y-px},$$

hence

$$y-px=\frac{nax}{x-a}=\frac{ydx-xdy}{dx}.$$

Whereby

$$\frac{ydx-xdy}{xx}=\frac{nadx}{x(x-a)} \quad \text{and} \quad \frac{y}{x}=nl\frac{dx}{x-a}+C$$

as before.

EXAMPLE 4

806. *With dx constant, if this equation is proposed*

$$\left(dx^2+dy^2\right)\sqrt{\left(dx^2+dy^2\right)}=ndxddy\sqrt{(xx+yy)},$$

to find the integral of this.

On putting $dy=pdx$ and $dp=qdx$ the proposed equation adopts this form

$$(1+pp)\sqrt{(1+pp)}=nq\sqrt{(xx+yy)},$$

which on putting $y=ux$ and $q=\frac{v}{x}$ is changed into this :

$$(1+pp)\sqrt{(1+pp)}=nv\sqrt{(1+uu)}.$$

Therefore since $\frac{dx}{x}=\frac{du}{p-u}=\frac{dp}{v}$,on account of $v=\frac{(1+pp)\sqrt{(1+pp)}}{n\sqrt{(1+uu)}}$ there becomes

$$(1+pp)^{\frac{3}{2}}du=n(p-u)dp\sqrt{(1+uu)},$$

the resolution of which is not at once apparent. But by reverting to the calculation of angles there shall be :

$p=\text{tang.}\varphi$ and $u=\text{tang.}\omega$; then there shall be

$$dp=\frac{d\varphi}{\cos.^2\varphi}, \quad du=\frac{d\omega}{\cos.^2\omega}, \quad \sqrt{(1+pp)}=\frac{1}{\cos.\varphi}, \quad \sqrt{(1+uu)}=\frac{1}{\cos.\omega}$$

and

$$p-u=\frac{\sin.(\varphi-\omega)}{\cos.\varphi\cos.\omega}$$

and hence

$$\frac{1}{cos^3.\varphi}\cdot\frac{d\omega}{cos^2.\omega}=\frac{n\,sin.(\varphi-\omega)}{cos.\varphi\,cos.\omega}\cdot\frac{1}{cos.\omega}\cdot\frac{d\varphi}{cos^2.\varphi}$$

or

$$d\omega=nd\varphi\,sin.(\varphi-\omega)=d\varphi-(d\varphi-d\omega),$$

hence

$$d\varphi=\frac{d\varphi-d\omega}{1-n\,sin.(\varphi-\omega)}.$$

Therefore on putting $\varphi-\omega=\psi$ there will be

$$\varphi=\int\frac{d\psi}{1-n\,sin.\psi}\quad\text{and}\quad\omega=\int\frac{d\psi}{1-n\,sin.\psi}-\psi,$$

hence on account of $p=\text{tang}.\varphi,\quad u=\text{tang}.\omega$ there is obtained

$$\frac{dx}{x}=\frac{du\,cos.\varphi\,cos.\omega}{sin.\omega}=\frac{d\omega\,cos.\varphi}{sin.\psi\,cos.\omega}=\frac{nd\psi\,cos.\varphi}{cos.\omega(1-n\,sin.\psi)}.$$

In the case $n=1$ there becomes

$$d\varphi=\frac{d\psi}{1-sin.\psi}=\frac{d\psi(1+sin.\psi)}{cos^2.\psi},$$

hence

$$\varphi=\text{tang}.\psi+\frac{1}{cos.\psi}+\alpha=\frac{1+sin.\psi}{cos.\psi}+\alpha,\quad\omega=\frac{1+sin.\psi}{cos.\psi}+\alpha-\psi$$

and

$$\frac{dx}{x}=\frac{d\varphi\,cos.\varphi}{cos.\omega}=\frac{d\varphi\,cos.\varphi}{cos.\varphi\,cos.\psi+sin.\varphi\,cos.\psi}.$$

But since there shall be $\varphi-\alpha=\sqrt{\frac{1+sin.\varphi}{1-sin.\varphi}}$, there becomes

$$sin.\psi=\frac{(\varphi-\alpha)^2-1}{(\varphi-\alpha)^2+1},\quad cos.\psi=\frac{2(\varphi-\alpha)}{(\varphi-\alpha)^2+1}.$$

Hence

$$\frac{dx}{x}=\frac{d\varphi\,cos.\varphi\left((\varphi-\alpha)^2+1\right)}{2(\varphi-\alpha)cos.\varphi+(\varphi-\alpha)^2\,sin.\varphi-sin.\varphi}.$$

COROLLARY 1

807. If in the case $n = 1$ the constant α is assumed infinite, then $sin.\psi = 1$, hence $\psi = 90^0$ and $\omega = \varphi - 90^0$; then indeed

$$\frac{dx}{x} = \frac{d\varphi \cos.\varphi}{\sin.\varphi}$$

and thus $x = a\sin\varphi$ and $u = -\cot.\varphi$, hence $y = -a\cos.\varphi$ and $xx + yy = aa$.

COROLLARY 2

808. But in the same case $n = 1$, in which the constant α is not assumed infinite, the numerator of the fraction, to which $\frac{dx}{x}$ is equal, conveniently is the differential of the denominator; from which there becomes

$$x = a\left((\varphi - \alpha)^2 \sin.\varphi - \sin.\varphi + 2(\varphi - \alpha)\cos.\varphi\right).$$

Then truly there shall be

$$\omega = \varphi - \text{Ang.tang.}\frac{(\varphi-\alpha)^2 - 1}{2(\varphi-\alpha)}$$

and thus

$$u = \frac{y}{x} = \text{tang.}\omega = \frac{\text{tang.}\varphi - \frac{(\varphi-\alpha)^2-1}{2(\varphi-\alpha)}}{1 + \frac{(\varphi-\alpha)^2-1}{2(\varphi-\alpha)}\text{tang.}\varphi}$$

or

$$\frac{y}{x} = \frac{2(\varphi-\alpha)\sin.\varphi - (\varphi-\alpha)^2\cos.\varphi + \cos.\varphi}{(\varphi-\alpha)^2\sin.\varphi - \sin.\varphi + 2(\varphi-\alpha)\cos.\varphi},$$

consequently

$$y = -a\left((\varphi-\alpha)^2\cos.\varphi - \cos.\varphi - 2(\varphi-\alpha)\sin.\varphi\right)$$

and

$$\sqrt{(xx + yy)} = a\left((\varphi-\alpha)^2 + 1\right).$$

SCHOLIUM 1

809. Also the integration can be carried out in general. For since there shall be

$$d\varphi = \frac{d\psi}{1-n\sin.\psi} \quad \text{and} \quad \frac{dx}{x} = \frac{nd\varphi\cos.\varphi}{\cos.\omega},$$

then there shall be

$$\varphi + \alpha = \frac{1}{\sqrt{(1-nn)}}\,\text{Ang.cos.}\frac{n-\sin.\psi}{1-n\sin.\psi},$$

[Note, though this does not show how Euler arrived at his result, we can at least prove its validity :

Since $\frac{n-\sin.\psi}{1-n\sin.\psi} = \cos\left(\cos^{-1}\frac{n-\sin.\psi}{1-n\sin.\psi}\right)$, we have

$$\frac{-\cos.\psi d\psi}{1-n\sin.\psi} + \frac{(n-\sin.\psi)n\cos.\psi d\psi}{(1-n\sin.\psi)^2} = -\sin\left(\cos^{-1}\frac{n-\sin.\psi}{1-n\sin.\psi}\right)\cdot d\left(\cos^{-1}\frac{n-\sin.\psi}{1-n\sin.\psi}\right);$$

hence,

$$d\left(\cos^{-1}\frac{n-\sin.\psi}{1-n\sin.\psi}\right) = \frac{-\cos.\psi d\psi(1-n\sin.\psi)+(n-\sin.\psi)n\cos.\psi d\psi}{-(1-n\sin.\psi)^2\sqrt{1-\left(\frac{n-\sin.\psi}{1-n\sin.\psi}\right)^2}}$$

$$= -\frac{(n^2-1)\cos.\psi d\psi}{(1-n\sin.\psi)\sqrt{(1-n\sin.\psi)^2-(n-\sin.\psi)^2}} = -\frac{(n^2-1)\cos.\psi d\psi}{(1-n\sin.\psi)\sqrt{(1-n+(1-n)\sin.\psi)(1+n-(1+n)\sin.\psi)}}$$

$$= \frac{(1-n^2)\cos.\psi d\psi}{(1-n\sin.\psi)\sqrt{(1-n)(1+n)}\sqrt{(1+\sin.\psi)(1-\sin.\psi)}} = \frac{(1-n^2)d\psi}{(1-n\sin.\psi)\sqrt{(1-n)(1+n)}} = \frac{\sqrt{(1-n^2)}d\psi}{(1-n\sin.\psi)}.$$

Note also one might presume that Euler has solved by his usual kind of trickery, an application of the general identity :
$\int\frac{f'(x)dx}{\sqrt{1-f^2(x)}} = -\cos^{-1}f(x)$, for a suitably defined function *f(x).* Thus the trick employed is to multiply the given function to be integrated , here $\frac{d\psi}{(1-n\sin.\psi)}$, by some function $g(\psi)/g(\psi)$, so that the integrand becomes $\frac{f'(\psi)d\psi}{\sqrt{1-f^2(\psi)}}$, the integral of which can be recognised as the negative of the inverse cosine of $f(\psi)$. The trick can be followed above by starting at the end and working backwards, where the multiplying function $g(\psi)/g(\psi)$ is $\frac{\cos.\psi}{\sqrt{(1+\sin.\psi)(1-\sin.\psi)}}$.]

from which on putting $(\varphi+\alpha)\sqrt{(1-nn)} = \theta$ there shall be

$$\cos.\theta = \frac{n-\sin.\psi}{1-n\sin.\psi}$$

and hence

$$sin.\psi = \frac{n-cos.\theta}{1-n\,cos.\theta} \text{ and } cos.\psi = \frac{sin.\theta\sqrt{(1-nn)}}{1-n\,cos.\theta}.$$

But on account of $\omega = \varphi - \psi$ there will be had

$$\frac{dx}{x} = \frac{nd\varphi\cos.\varphi(1-n\,cos.\theta)}{cos.\phi\,sin.\theta\sqrt{(1-nn)}+sin.\varphi(n-\cos.\theta)}.$$

Now since $d\theta = d\varphi\sqrt{(1-nn)}$, the differential of this denominator is

$$-\,d\varphi\,sin.\varphi\,sin.\theta\sqrt{(1-nn)} + d\varphi\,cos.\varphi\,cos.\theta(1-nn) + nd\varphi\,cos.\varphi$$
$$-\;dp\,cos.\varphi\,cos.\theta + d\varphi\,sin.\varphi\,sin.\theta\sqrt{(1-nn)},$$

which reduces to $nd\varphi\,cos.\varphi(1-n\,cos.\theta)$, clearly the numerator itself, thus so that there shall be

$$x = a\left(cos.\varphi\,sin.\theta\sqrt{(1-nn)} + sin.\varphi\left(n-cos.\theta\right)\right)$$

or

$$x = a\,cos.\omega(1-n\,cos.\theta) \quad \text{and thus} \quad y = ux = a\,sin.\omega(1-n\,cos.\theta).$$

Hence with the angle θ assumed the angle ψ is sought, so that there shall be

$$sin.\psi = \frac{n-cos.\theta}{1-n\,cos.\theta} \quad \text{and} \quad cos.\psi = \frac{sin.\theta}{1-n\,cos.\theta}\sqrt{(1-nn)} \quad ;$$

then indeed there becomes

$$\omega = \frac{\theta}{\sqrt{(1-nn)}} - \alpha - \psi$$

and there will be the complete integral

$$x = a\left(1-n\,cos.\theta\right)cos.\omega \text{ and } y = a\left(1-n\,cos.\theta\right)sin.\omega.$$

SCHOLIUM 2

810. But if the number *n* is greater than one, this integration becomes imaginary; so that this inconvenience may be removed, the equation $d\varphi = \frac{d\psi}{1-n\,sin.\psi}$ is to be noted, with the integral being

$$\varphi + \alpha = \frac{1}{\sqrt{(nn-1)}}\,l\,\frac{\sqrt{(n-1)}(1+sin.\psi)+\sqrt{(n+1)}(1-sin.\psi)}{\sqrt{(n-1)}(1+sin.\psi)-\sqrt{(n+1)}(1-sin.\psi)}.$$

Whereby if there is put $(\varphi+\alpha)\sqrt{(nn-1)}=\theta$, so that there becomes :

$$d\theta = d\varphi\sqrt{(nn-1)} \quad \text{et} \quad \omega=\varphi-\psi=\frac{\theta}{\sqrt{(nn-1)}}-\alpha-\psi,$$

then there shall be

$$\frac{e^{\theta}+1}{e^{\theta}-1}=\frac{\sqrt{(n-1)(1+\sin.\psi)}}{\sqrt{(n+1)(1-\sin.\psi)}}=\frac{(n-1)(1+\sin.\psi)}{cos.\psi\sqrt{(nn-1)}},$$

from which there is found

$$sin.\psi=\frac{e^{\theta}+2n+e^{-\theta}}{ne^{\theta}+2+ne^{-\theta}} \quad \text{and } cos.\psi=\frac{\left(e^{\theta}-e^{-\theta}\right)\sqrt{(nn-1)}}{ne^{\theta}\ +2\ +ne^{-\theta}}$$

thus so that from the angle θ the angles $\varphi,\ \psi$ and ω are defined. Since now there shall be

$$\frac{dx}{x}=\frac{nd\varphi\cos.\varphi}{cos.\omega}=\frac{nd\varphi\cos.\varphi}{cos.\phi cos.\psi+sin.\varphi sin.\psi},$$

there becomes

$$\frac{dx}{x}=\frac{nd\varphi\cos.\varphi\left(ne^{\theta}+2+ne^{-\theta}\right)}{cos.\phi\left(e^{\theta}-e^{-\theta}\right)\sqrt{(nn-1)}+sin.\varphi\left(e^{\theta}+2n\ +e^{-\theta}\right)},$$

were again it comes about conveniently, so that the numerator shall be the differential of the denominator, just as the differentiation soon to be put in place will show. Hence therefore there will be :

$$x=a\left(cos.\phi\left(e^{\theta}-e^{-\theta}\right)\sqrt{(nn-1)}+sin.\varphi\left(e^{\theta}+2n\ +e^{-\theta}\right)\right)$$

or

$$x=a\cos.\varphi\left(ne^{\theta}+2+ne^{-\theta}\right)$$

and on account of $u=\text{tang.}\omega$ there becomes

$$y=a\cos.\omega\left(ne^{\theta}+2+ne^{-\theta}\right)$$

On which account from the angle θ there is sought first the angle ψ, so that there shall be

$$sin.\psi=\frac{e^{\theta}+2n+e^{-\theta}}{ne^{\theta}+2+ne^{-\theta}} \quad \text{and} \quad cos.\psi=\frac{\left(e^{\theta}-e^{-\theta}\right)\sqrt{(nn-1)}}{ne^{\theta}+2+ne^{-\theta}};$$

from which found the angle $\omega=\frac{\theta}{\sqrt{(nn-1)}}-\alpha-\psi$ is taken and these formulas found for *x* and *y* will give the complete integral on account of the two constants *a* and α introduced.

SCHOLIUM 3

811. Since here a particular part of the integration appears to have succeeded fortuitously, and it will be worth the effort to inquire about the cause of this, for perhaps the account of the integration can be made clearer. Therefore since there shall be
$\varphi=\psi+\omega$ and $d\varphi=\frac{d\psi}{1-n\sin.\psi}$ and hence

$$d\omega=\frac{nd\psi\cos.\varphi}{1-n\sin.\psi}=nd\varphi\sin.\psi\,,$$

the equation $\frac{dx}{x}=\frac{nd\varphi\cos.\varphi}{\cos.\omega}$ on account of $\cos.\varphi=\cos.\psi\cos.\omega-\sin.\psi\sin.\omega$ is resolved into this :

$$\frac{dx}{x}=nd\varphi\sin.\psi-\frac{nd\varphi\sin.\psi\sin.\omega}{\cos.\omega},$$

which on account of $d\varphi=\frac{d\psi}{1-n\sin.\psi}$ and $nd\varphi\sin.\psi=d\omega$ adopts this integral form

$$\frac{dx}{x}=\frac{nd\psi\cos.\psi}{1-n\sin.\psi}-\frac{d\omega\sin.\omega}{\cos.\omega},$$

from which there is elicited

$$x=\frac{a\cos.\omega}{1-n\sin.\psi}\quad\text{and}\quad y=\frac{a\sin.\omega}{1-n\sin.\psi}$$

on account of $y=ux=x\,\mathrm{tang}.\omega$. Therefore behold in general this integration of our equation. The angles ω and ψ bear this relation between each other, so that there shall be :

$$d\omega=\frac{nd\psi\sin.\psi}{1-n\sin.\psi}$$

then truly there shall be

$$x=\frac{a\cos.\omega}{1-n\sin.\psi}\quad\text{and}\quad y=\frac{a\sin.\omega}{1-n\sin.\psi}$$

But if we put hence $\sqrt{(xx+yy)}=z$, so that $x=z\cos.\omega$ and $y=z\sin.\omega$, then

$$z=\frac{a}{1-n\sin.\psi}\quad\text{and}\quad \sin.\psi=\frac{z-a}{nz},\quad\text{hence}\quad d\omega=\frac{(z-a)d\psi}{a}.$$

But if there should be

$$d\psi=\frac{adz}{z\sqrt{\left(nnzz-(z-a)^2\right)}},\quad\text{hence}\quad d\omega=\frac{(z-a)dz}{z\sqrt{\left(nnzz-(z-a)^2\right)}}.$$

from which the angle ω is defined in terms of *z*. On removing the irrationality if we put

$$\sqrt{\left(nnzz-(z-a)^2\right)}=s\left(nz+z-a\right),$$

there becomes

$$z=\frac{a(ss+1)}{(n+1)ss-n+1}\quad\text{and}\quad d\omega=\frac{2nds(ss-1)}{(ss+1)((n+1)ss-(n-1))}$$

or

$$d\omega=\frac{2ds}{ss+1}-\frac{2ds}{(n+1)ss-n+1},$$

which integration is obvious.

PROBLEM 99

812. *If a second order differential equation then at last becomes homogeneous, if the others with the variable y are given n dimensions, the integration of this equation is reduced to a differential equation of the first order.*

SOLUTION

We may put $dy=pdx$ and $dp=qdx$, so that a finite equation arises between the four quantities *x, y, p* and *q;* which we may consider to be prepared just as in the manner of future homogeneous equations. Hence in the first place since a single dimension is to be enumerated for *x* the variable *y* shall have *n* dimensions, to the quantity $p=\frac{dy}{dx}$ are to be attributed $n-1$ dimensions and to the quantity $q=\frac{dp}{dx}$, $n-2$ dimensions. On account of which we can put

$$y=x^n u,\quad p=x^{n-1}t\quad\text{and}\quad q=x^{n-2}v$$

and on account of $dy=pdx$ and $dp=qdx$ we will have

$$xdu+nudx=tdx\text{ and }xdt+(n-1)tdx=vdx,$$

from which we deduce

$$\frac{dx}{x}=\frac{du}{t-nu}=\frac{dt}{v-(n-1)t}$$

and thus

$$du(v-(n-1)t)=dt(t-nu)\quad(*).$$

But with the above equations worked out substituted in the equation between *x, y, p* and *q* by hypothesis the variable *x* is excluded from the calculation, thus so that there is produced an equation only between the three variables *u, t* and *v,* from which the letter *v* is allowed to be defined in terms of the two variables *t* and *u*. With which value substituted there shall be had an differential equation of the first order between the two variables *u* and *t,* from which *t* is able to be determined in terms of *u* ; with the aid of the equation $\frac{dx}{x}=\frac{du}{t-nu}$ *x* can be determined in terms of *u* and hence on account of $u=\frac{y}{x^n}$ the equation of the integral will be obtained between *x* and *y* and that complete on account of the twofold integration.

COROLLARY 1

813. Hence this is the criterion for the treatment of the equation between *x, y, p* and *q* in this manner, so that on putting $y = x^n u$, $p = x^{n-1}t$ and $q = x^{n-2}v$ an exponent *n* may be allowed in a determination of this kind, in order that in short the variable *x* can be removed from the calculation by division.

COROLLARY 2

814. If there is put $n = 0$, then the equation is to be prepared thus, so that by granting this to *y* and no dimensions to the differentials it becomes homogeneous. Evidently in this case only the variable *x* with its differentials is thought to constitute dimensions.

COROLLARY 3

815. Now in the opposite case if the dimensions are considered from the variable *y* only, thus in order that that with its differentials *dy* and *ddy* everywhere constitute a number of the same dimensions, then the exponent *n* becomes infinite.

SCHOLIUM

816. If only the variable *x* with its differentials furnishes the number of the same dimensions everywhere, on account of $n = 0$ there shall be $u = y$ and in the equation between *x, y,p* and *q* it is convenient to put in place $p = \frac{t}{x}$ and $q = \frac{v}{xx}$, with which accomplished the variable *x* will be removed from the calculation and there will be produced an equation between *y, t* and *v*, with the help of which the differential equation $dy(v+t) = tdt$ will be reduced to two variables only ; with which resolved there will be $\frac{dx}{x} = \frac{dy}{t}$, where, since *t* may be given in terms of *y*, the integration presents no difficulty. Now in the other case, in which the variable *y* alone with its differentials has even dimensions, that must thus take an infinite exponent *n* everywhere, and the resolution must be put in place in another way, that we shall soon instruct [§ 822] ; unless perhaps with the variables *x* and *y* interchanged, it may please to reduce the case to the preceding one.

EXAMPLE 1

817. *To find the integral of this proposed equation :*

$$xxddy = \alpha y dx^2 + \beta xdxdy,$$

with the element dx assumed constant.

Here that condition may be considered carefully, in which only the variable *x* with its differential *dx* everywhere constitutes two dimensions, and there shall be $n = 0$. Since therefore on putting $dy = pdx$ and $dp = qdx$ we shall have $qxx = \alpha y + \beta px$, we can put $p=\frac{t}{x}$ and $q=\frac{v}{xx}$ and there is produced $v=\alpha y+\beta t$, from which we arrive at that differential equation

$$aydy+(\beta+1)tdy=tdt\,;$$

on account of which homogeneity we can make $t=yz$ and there shall be

$$ady+(\beta+1)zdy=yzdz+zzdy$$

or

$$\frac{dy}{\mathrm{y}}=\frac{zdz}{\alpha+(\beta+1)z-zz}$$

Let

$$\alpha+(\beta+1)z-zz=(f+z)(g-z),$$

so that there shall be $\alpha=fg$ and $\beta+1=g-f$, and there will be found

$$\frac{dy}{y}=-\frac{f}{f+g}\cdot\frac{dz}{f+z}+\frac{g}{f+g}\cdot\frac{dz}{g-z},$$

from which there is gathered on integration

$$ly=C-\frac{f}{f+g}l(f+z)-\frac{g}{f+g}l(g-z)$$

or

$$y(f+z)^{\frac{f}{f+g}}(g-z)^{\frac{f}{f+g}}=a.$$

Then there shall now be

$$\frac{dx}{x}=\frac{dy}{yz}=\frac{dz}{(f+z)(g-z)}\text{ or }\frac{dx}{x}=\frac{1}{f+g}\cdot\frac{dz}{f+z}+\frac{1}{f+g}\cdot\frac{dz}{g-z},$$

hence

$$x=b\left(\frac{f+z}{g-z}\right)^{\frac{1}{f+g}}\quad\text{or}\quad\frac{f+z}{g-z}=\left(\frac{x}{b}\right)^{f+g}.$$

From which since there shall be $z=\frac{gx^{f+g}-fb^{f+g}}{b^{f+g}+x^{f+g}}$, and with this value substituted there becomes :

$$(f+g)b^{g}x^{f}y=a\left(b^{f+g}+x^{f+g}\right)$$

or on putting $\frac{a}{f+g}=C$

$$y = c\left(\tfrac{b^f}{x^f} + \tfrac{x^g}{b^g}\right)$$

then truly :

$$g - f = \beta + 1 \text{ and } f + g = \sqrt{\left(\left(\beta+1\right)^2 + 4\alpha\right)}.$$

COROLLARY

818. Because in the proposed equation also both the variables x and y likewise everywhere have the same total dimensions, that also is allowed to be treated following the precepts of the proceeding problem.

EXAMPLE 2

819. *On putting dx constant if the second order differential equation contains only two terms, so that it shall be of this kind :*

$$ddy = cx^{\alpha} y^{\beta} dx^{2-\gamma} dy^{\gamma},$$

to investigate the integral of this.

On putting $dy = pdx$ and $dp = qdx$ this form will be taken : $q = cx^{\alpha} y^{\beta} p^{\gamma}$, where the exponent n thus is allowed to be defined, so that on putting $y = x^n u$, $p = x^{n-1} t$ and $q = x^{n-2} v$ the variable x can be removed by division; for it is necessary to take :

$$\alpha + \beta n + \gamma\left(n-1\right) - n + 2 = 0 \text{ or } n = \tfrac{-\alpha+\gamma-2}{\beta+\gamma-1}$$

and then there shall be $v = cu^{\beta} t^{\gamma}$. Therefore the differential equation of the first order to be resolved will be

$$cu^{\beta} t^{\gamma} du - \left(n-1\right) tdu = tdt - nudt ;$$

from which since the variable t can be determined in terms of u, it is necessary for the integral to assume this formula $\frac{dx}{x} = \frac{du}{t-nu}$ with which done on account of $u = \frac{y}{x^n}$ there will be obtained the equation of the integral sought between x and y.

Only in the case $\beta + \gamma = 1$, in which n becomes infinite, is the particular postulate to be set out below treated [§ 822], unless perhaps likewise there shall be $\gamma = \alpha + 2$; for then the exponent n is certainly removed from our choice, but the equation will be homogeneous.

EXAMPLE 3

820. *To find the integral of this proposed equation:*

$$x^4 ddy = x^3 dxdy + 2xydxdy - 4yydx^2,$$

with the element dx assumed constant.

Here it is evident, if for the differentials of y itself and for the differentials dy and ddy two dimensions are attributed, and truly for x and dx single dimensions are given, then in all the terms six dimensions are to be obtained. Whereby since on putting $dy = pdx$ and $dp = qdx$ we shall have this equation

$$x^4 q = x^3 p + 2xyp - 4yy,$$

we can make

$$y = x^2 u,\ p = xt \quad \text{and} \quad q = v$$

and there is produced

$$v = t + 2ut - 4uu.$$

But on account of $n = 2$ our differential equation [(*) in § 812] will be

$$du(v - t) = dt(t - 2u),$$

which becomes

$$2udu(t - 2u) = dt(t - 2u),$$

from which we deduce either $t = 2u$ or $t = uu + c$, we set out these two cases separately :

1) If $t = 2u$, on account of $\frac{dx}{x} = \frac{du}{t-2u}$ there becomes $du = 0$ and thus $u = C$ and therefore $y = Cxx$, which is the certainly satisfying the particular integral of the proposed equation.

2) Let $t = uu + c$; then there shall be $\frac{dx}{x} = \frac{du}{uu-2u+c}$, where three cases are to be considered. In the first case if $c = 1$, then

$$l\frac{x}{a} = \frac{1}{1-u} = \frac{xx}{xx-y} \text{ or } xx = (xx - y)l\frac{x}{a}.$$

In the second case if the constant $c = 1 - ff$ then $\frac{dx}{x} = \frac{du}{(u-1)^2 - ff}$ and hence

$$lx = \frac{-1}{2f} l\frac{f+u-1}{f-u+1} + C,$$

hence on account of $u=\frac{y}{xx}$ there shall be

$$x=a\left(\frac{(f+1)xx-y}{(f-1)xx+y}\right)^{\frac{1}{2f}}.$$

In the third case if the constant $c=1+ff$, then $\frac{dx}{x}=\frac{du}{(u-1)^2+ff}$, which integrated gives

$$l\frac{x}{a}=\frac{1}{f}\text{Ang. tang. }\frac{u-1}{f}\quad\text{or}\quad\frac{u-1}{f}=\frac{y-xx}{fxx}=\text{tang. }fl\frac{x}{a}.$$

Therefore on account of the arbitrary constant *c* the integration either succeeds algebraically, or depends on logarithms or angles, from which the general form cannot be expressed.

SCHOLIUM

821. Moreover with the integral found first on putting $y=Cxx$, it is unable to be taken in any of these forms in which the complete integral is established ; but by no means is it less satisfactory for the proposed second order differential equation. Hence in this example these [questions] are further illustrated, which we have commented on above concerning this paradox [see §546 & §564], because sometimes a finite equation satisfies a differential equation, which is barely present in the complete integral. Therefore we see this same paradox present also in differential equations of the second order. But either that equation $y=Cxx$ shall be admitted during the integration, while the other integral is sought, which may be considered to be not yet altogether completed ; here indeed the proposed equation itself is to be agreed on as if having factors, from which that other equation $y=Cxx$ arises; truly much is missing, in order that we could agree with this explanation. Whether or not that question should rather be investigated either by geometry or by some other discipline, that should lead to the solution of this kind of equation, it is seen that it must be considered with great care, generally where it can be judged without difficulty, whether or not, whatever differential equations it satisfies, that also it can be agreed not to question. Just as if the descent of a weight slipping from a height $=a$ has to be defined and the height, from which now it is away from the ground, shall be *x*, there the speed shall be as $\sqrt{(a-x)}$ and the element of time $dt=\frac{-dx}{\sqrt{(a-x)}}$. Here indeed it is clear thus for this differential equation to be satisfied on putting $x=a$, so that the time *t* remains indefinite, which still by no means is in agreement with the question, which for the integral to be resolved truly is $t=2\sqrt{(a-x)}$.

[We have met this kind of discussion before with Gregorius, where he contemplates the hare and the tortoise problem concerning motion arising from the ancient Greek paradox of Zeno.]

PROBLEM 100

822. *If in a differential equation of the second order of the variable y with its differentials dy and ddy , fulfilling everywhere a dimension of the same number, to reduce the integration to a differential equation of the first order.*

SOLUTION

On putting $dy = pdx$ and $dp = qdx$ an equation will thus be prepared, so that these three variables y, p, q everywhere maintain a number of the same dimension with the dimension of the other variable x of course not entering into calculation. Whereby if there is put in place $p = uy$ and $q = vy$, in all the terms there will be present the same power of y, with which removed by division there will be had an equation between only three variables x, u and v, from which it is allowed to define one by the two remaining, thus so that v may be equal to some function of x and u. Now on account of $p = uy$ there will be $dy = uydx$ but on account of

$$dp = qdx \text{ there becomes } udy + ydu = vydx,$$

from which it follows that

$$\frac{dy}{y} = udx \text{ and } \frac{dy}{y} = \frac{vdx - du}{u}$$

and thus

$$udu + uudx = vdx,$$

which differential equations embraces only the two variables x and u. As hence if the integration is permitted, in order that the relation between x and u thence becomes known, there remains that the integral of the formula udx be investigated, with which found there shall be $ly = \int udx$ and thus the equation of the integral arises between x and y, which on account of the twofold integration completed it involves two arbitrary constants and thus the complete integral will be shown.

COROLLARY 1

823. Hence the integration of equations of this kind is reduced to a differential equation of this kind

$$du + uudx = vdx,$$

the resolution of which if it succeeds, likewise the integration of this will be had, since the integration of the formula udx is free from difficulty.

COROLLARY 2

824. Since there shall be $\frac{dy}{y} = udx$, then there becomes $y = e^{\int udx}$, with which substituted the proposed second order differential equation is reduced at once to a differential equation of the first order ; for there shall be

$$dy = p = e^{\int udx}u \quad \text{and} \quad \frac{dp}{dx} = q = \frac{e^{\int udx}(du + uudx)}{dx}$$

and then the formula of the exponential is removed at once from the equation.

COROLLARY 3

825. In turn also the proposed differential equation of the first order $du + uudx = vdx$, in which *v* shall be some function of *x* and *u*, that on putting $u = \frac{dy}{ydx}$ is transformed into a second order differential equation of this kind, in which the variable y with its differentials *dy* and *ddy* everywhere constitute a number of the same dimensions.

SCHOLION 1

826. This reduction of differential equations of the first order to the second order may be considered contrary to the laws of analysis, yet meanwhile repeatedly this is not without use ; but if indeed by another method of this kind second order differential equations are enabled to be treated, then the integrals of these can be shown either by series or finitely, likewise the integrals of differential equations of the first order will become known, of which a general account is scarcely seen from elsewhere. Moreover in the following [Ch. VII & VIII] we will see second order differential equations of this kind, in which the other variables , *y* does not exceed a single dimension, can be integrated conveniently by a series and thus these series are cut off, thus so that an integral with a finite expression is shown. Otherwise for a proposed equation of this kind of the first grade $du + uudx = vdx$ by substituting $u = \frac{dy}{ydx}$ that is more noteworthy, since on taking the element *dx* constant there becomes

$$du = \frac{ddy}{ydx} - \frac{dy^2}{yydx} \text{ and thus } du + uudx = \frac{ddy}{ydx},$$

thus so that two terms coalesce into one in this manner.

SCHOLIUM 2

827. Here it is especially helpful for the case to be noted, in which the equation $du + uudx = vdx$ is allowed to be integrated. This in the end will be the form $du = Vdx$ of the general equations resolvable and *V* clearly a function of *x* and *u* and it is evident the integration succeeds, if there should be $v = uu + V$. Hence in the first place this comes about, if there shall be $V = \frac{X}{U}$ with *X* denoting some function of *x* and *U* of *u*. In the second place, if *V* shall be a homogeneous function of zero dimensions of *x* and *u*. In the third place, if with X and Ξ denoting some functions of x there should be $V = Xu + \Xi u^n$. In the fourth place, if with *P* and *Q* denoting some functions of *u* there should be $V = \frac{1}{Px + Qx^n}$. And in a like manner from the other integrable forms the other cases can be concluded.

EXAMPLE 1

828. *With the element dx assumed constant if this equation is put in place:*

$$\alpha yddy + \beta dy^2 = \frac{ydxdy}{\sqrt{(aa+xx)}},$$

to find the integral of this.

With $dy = pdx$ and $dp = qdx$ put in place there is produced :

$$\alpha yq + \beta pp = \frac{yp}{\sqrt{(aa+xx)}},$$

which by making $p = uy$ and $q = vy$ changes into this :

$$\alpha v + \beta uu = \frac{u}{\sqrt{(aa+xx)}} \quad \text{or} \quad v = \frac{u}{\alpha\sqrt{(aa+xx)}} - \frac{\beta uu}{\alpha},$$

from which this equation is required to be resolved :

$$du + uudx = \frac{udx}{\alpha\sqrt{(aa+xx)}} - \frac{\beta uudx}{\alpha}.$$

[Thus $u^2 + \frac{du}{dx} = \frac{1}{y^2}\left(\frac{dy}{dx}\right)^2 + \frac{d}{dx}\left(\frac{1}{y}\left(\frac{dy}{dx}\right)\right) = \frac{1}{y^2}\left(\frac{dy}{dx}\right)^2 - \frac{1}{y^2}\left(\frac{dy}{dx}\right)^2 + \frac{1}{y}\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{q}{y} = v.$]

There is put in place $u = \frac{1}{s}$ and there becomes :

$$ds + \frac{sdx}{\alpha\sqrt{(aa+xx)}} = \left(1 + \frac{\beta}{\alpha}\right)dx,$$

which multiplied by $\left(x + \sqrt{(aa+xx)}\right)^{\frac{1}{\alpha}}$ and integrated gives:

$$s\left(x + \sqrt{(aa+xx)}\right)^{\frac{1}{\alpha}} = \left(1 + \frac{\beta}{\alpha}\right)\int dx\left(x + \sqrt{(aa+xx)}\right)^{\frac{1}{\alpha}}.$$

There is made $x + \sqrt{(aa+xx)} = t^\alpha$; then there shall be $aa = t^{2\alpha} - 2t^\alpha x$, hence

$$x = \frac{t^{2\alpha} - aa}{2t^\alpha} = \frac{1}{2}t^\alpha - \frac{1}{2}aat^{-\alpha} \quad \text{and} \quad dx = \frac{\alpha}{2}dt\left(t^{\alpha-1} + aat^{-\alpha-1}\right),$$

thus so that there shall be

$$st=\left(1+\frac{\beta}{\alpha}\right)\int\frac{\alpha}{2}dt\left(t^{\alpha}+aat^{-\alpha}\right)\quad\text{or}\quad st=C+\left(\frac{\alpha+\beta}{2}\right)\left(\frac{t^{\alpha+1}}{\alpha+1}+\frac{aat^{1-\alpha}}{1-\alpha}\right).$$

Again there shall be $\frac{dy}{y}=udx=\frac{dx}{s}$; but from the differential equation there shall be

$$\left(1+\frac{\beta}{\alpha}\right)\frac{dx}{s}=\frac{ds}{s}+\frac{dx}{\alpha\sqrt{(aa+xx)}}$$

and hence

$$\left(1+\frac{\beta}{\alpha}\right)ly=ls+\frac{1}{\alpha}l\left(x+\sqrt{(aa+xx)}\right)=lst+D,$$

therefore $y=B(st)^{\frac{\alpha}{\alpha+\beta}}$. Whereby on assuming $C=\frac{\alpha+\beta}{2}A$ there is had

$$y=B\left(A+\frac{t^{\alpha+1}}{\alpha+1}+\frac{aat^{1-\alpha}}{1-\alpha}\right)^{\frac{\alpha}{\alpha+\beta}}$$

with

$$x=\tfrac{1}{2}\left(t^{\alpha}-aat^{-\alpha}\right)\quad\text{or}\quad t=\left(x+\sqrt{(aa+xx)}\right)^{\frac{1}{\alpha}}\quad\text{present},$$

thus in order that the equation between *x* and *y* shall be

$$Cy^{\frac{\alpha+\beta}{\alpha}}=A+\frac{1}{\alpha+1}\left(x+\sqrt{(aa+xx)}\right)^{\frac{\alpha+1}{\alpha}}+\frac{aa}{1-\alpha}\left(x+\sqrt{(aa+xx)}\right)^{\frac{1-\alpha}{\alpha}}.$$

SCHOLIUM

829. This same example thus has been prepared, so that it can be easily resolved by another method, for

$$\alpha yq+\beta pp=\frac{yp}{\sqrt{(aa+xx)}}$$

If it is multiplied by $\frac{dx}{yp}$, on account of $qdx=dp$ and $pdx=dy$ it becomes

$$\frac{\alpha dp}{p}+\frac{\beta dy}{y}=\frac{dx}{\sqrt{(aa+xx)}},$$

the individual terms of which are integrable. Hence on putting

$$p^{\alpha}y^{\beta}=C^{\alpha}\left(x+\sqrt{(aa+xx)}\right)$$

and hence

$$y^{\frac{\beta}{\alpha}}dy = Cdx\left(x+\sqrt{(aa+xx)}\right)^{\frac{1}{\alpha}},$$

which equation gives anew the integral found before.

Hence the general form of the equations resolved in this way is :

$$Pdp + Ydy + Xdx = 0$$

with P being a function of p, Y of y and X of x, which by us is usually represented by $Pq + Yp + X = 0$. Hence it is therefore seen, how also second order differential equations with the help of a suitable multiplier are able to be led through integration ; which method since it has an outstanding use with first order differential equations, from which it is required to be developed more, as also it can be extended to differential equations of higher order, and which argument we will try to set out further below [see Ch. V].

EXAMPLE 2

830. *With the element dx assumed constant, if this equation is put in place,*

$$xyddy = ydxdy + xdy^2 + \frac{bxdy^2}{\sqrt{(aa-xx)}},$$

to find the integral of this.

On putting $dy = pdx$ and $dp = qdx$ then there shall be

$$xyq = yp + xpp + \frac{bxpp}{\sqrt{(aa-xx)}},$$

which on making $p = uy$ and $q = vy$ changes into

$$xv = u + uux + \frac{buux}{\sqrt{(aa-xx)}},$$

from which this differential equation arises :

$$du + uudx = \frac{udx}{x} + uudx + \frac{buudx}{\sqrt{(aa-xx)}} \quad \text{or} \quad \frac{xdu-udx}{uu} = \frac{bxdx}{\sqrt{(aa-xx)}},$$

the integral of which is

$$C - \frac{x}{u} = -b\sqrt{(aa-xx)} \quad \text{or} \quad u = \frac{x}{C+b\sqrt{(aa-xx)}},$$

hence

$$\frac{dy}{y} = \frac{xdx}{C+b\sqrt{(aa-xx)}}.$$

There is put in place $\sqrt{(aa-xx)} = t$, so that there becomes $xdx = -tdt$; then

$$\frac{dy}{y}=\frac{-tdt}{C+bt}=-\frac{dt}{b}+\frac{Cdt}{b(C+bt)} \quad \text{and} \quad ly=-\frac{t}{b}+\frac{C}{bb}l(C+bt)+lc.$$

Let $C=nbb$; then there shall be

$$l\frac{y}{c}=-\frac{\sqrt{(aa-xx)}}{b}+nl\frac{nb+\sqrt{(aa-xx)}}{b},$$

where c and n are arbitrary constants.

CAPUT III

DE AEQUATIONIBUS DIFFERENTIO-DIFFERENTIALIBUS HOMOGENEIS ET QUAE AD EAM FORMAM REDUCI POSSUNT

PROBLEMA 97

790. *Aequationum differentio-differentialium homogenearum naturam explicare atque ad formam finitam ponendo* $dy = pdx$ *et* $dp = pdx$ *accommodare.*

SOLUTIO

Sumto elemento *dx* constante aequatio differentio-differentialis vulgari modo expressa dicitur homogenea, si non solum ipsis variabilibus *x* et y, sed etiam earum differentialibus *dx* et *dy* itemque ipsi *ddy* singulis unam dimensionem tribuendo omnes aequationis termini eundem dimensionum numerum contineant, veluti in hac aequatione

$$xxddy + xdx^2 + ydy^2 = 0,$$

ubi in singulis terminis ternae insunt dimensiones. Quodsi ergo ponamus $\frac{dy}{dx} = p$ ac $\frac{dp}{dx} = \frac{ddy}{dx^2} = q$, littera *p* nullam dimensionem, littera vero *q* unam dimensionem negativam continere erit censenda. Hinc aequatio differentio–differentialis ad formam hic receptam reducta, ut nonnisi quantitates finitas *x*, y, *p* et *q* contineat, erit homogenea, si litteris *x* et *y* unam dimensionem tribuendo, litterae *p* vero nullam, at litterae *q* unam dimensionem negativam in singulis aequationis terminis idem oriatur dimensionum numerus. Vicissim ergo, quoties haec proprietas in aequatione inter quaternas quantitates *x, y,p* et *q* proposita deprehenditur, ea aequatio erit homogenea et forma vulgari expressa manifesto homogeneitatem prae se feret.

COROLLARIUM 1

791. Si ergo in aequatione tali homogenea inter *x, y, p* et *q* statuatur $y = ux$ et $q = \frac{v}{x}$ omnes termini eandem potestatem ipsius *x* continebunt, qua ergo per divisionem sublata aequatio prodibit tres tantum variabiles *u*, *v* et *p* involvens.

COROLLARIUM 2

792. Criterium igitur aequationis homogeneae inter quatuor quantitates *x*, *y*, *p* et *q* propositae in hoc consistit, ut posito $y = ux$ et $q = \frac{v}{x}$ quantitas *x* prorsus ex calculo exterminetur.

COROLLARIUM 3

793. Facta itaque hac substitutione, qua obtinetur aequatio inter ternas quantitates u, v et p, ex ea pro lubitu vel p per u et v, vel v per u et p,vel u per v et p definiri poterit.

SCHOLION

794. Simili modo ideam homogeneitatis in aequationibus differentio–differentialibus constituimus, quo in aequationibus differentialibus primi gradus sumus usi. In his quidem cum differentialia sponte eundem dimensionum numerum constituere debeant, homogeneitas ex solis ipsis variabilibus x et y diiudicatur. At in aequationibus differentio–differentialibus praeter ipsas variabiles x et y etiam litterae q ratio in computo dimensionum haberi debet, ita tamen, ut ipsi una dimensio negativa sit tribuenda; littera autem p in, hunc computum plane non ingreditur, quae ergo, utcunque aequationi implicetur, homogeneitatem non turbat. Plurimum autem interest probe nosse indolem differentio–differentialium aequationum homogenearum, cum earum resolutio ad resolutionem aequationum differentialium primi gradus reduci possit, ita ut, si haec successerit, etiam ipsarum aequationum differentio-differentialium integratio habeatur, id quod in sequenti problemate luculentius ostendemus.

PROBLEMA 98

795. *Proposita aequatione differentio–differentiali homogenea eius resolutionem ad integrationem aequationis differentialis primi gradus reducere.*

SOLUTIO

Reducta aequatione ponendo $dy = pdx$ et $dp = qdx$ ad formam hic receptam, ut habeatur aequatio inter quatuor quantitates finitas x, y, p et q, ponatur $y = ux$ et $q = \frac{v}{x}$, ac cum aequatio sit homogenea, hoc modo quantitas x penitus ex calculo elidetur, ita ut proditura sit aequatio inter ternas quantitates u, v et p, ex qua unam per binas reliquas definire liceat. Nunc igitur cum sit $dy = pdx$, erit $udx + xdu = pdx$ hincque $\frac{dx}{x} = \frac{du}{p-u}$. Deinde ob $dp = qdx$ erit $dp = \frac{vdx}{x}$ ideoque $\frac{dx}{x} = \frac{dp}{p}$ ex quo duplici ipsius $\frac{dx}{x}$ va1ore colligitur

$$\frac{du}{p-u} = \frac{dp}{p} \text{ seu } vdu = pdp - udp.$$

Quodsi ergo ex ilIa aequatione quantitas v definiatur per binas p et u, habebitur aequatio differentialis primi gradus inter binas variabiles p et u, cuius integratio si fuerit in potestate, ut p per u innotescat, aequatio $\frac{dx}{x} = \frac{du}{p-u}$, in qua variabiles x et u sunt separatae, integretur sicque x per u definietur, unde fit $y = ux$; seu statim in hoc integrali loco u scribatur $\frac{y}{x}$ et habebitur aequatio inter x et y quaesita.

COROLLARIUM 1

796. Totum ergo negotium reducitur ad integrationem huius aequationis differentialis simplicis $vdu = pdp - udp$; quae si ope regularum supra traditarum expediri queat, simul aeqnationis differentio–differentialis integratio habetur.

COROLLARIUM 2

797. Simul autem patet resolutionem huiusmodi aequationum duplicem integrationem requirere, unde duae quantitates arbitrariae constantes ingredientur, quibus integrale completum constituitur.

COROLLARIUM 3

798. Etiamsi autem integratio $vdu = pdp - udp$ non succedat, tamen ingens lucrum est rem eo perduxisse, cum supra methodus generalis sit tradita integralia omnium aequationum differentialium primi gradus proxime assignandi.

SCHOLION

799. Operae igitur pretium erit eos casus perpendere, quibus aequatio $vdu = pdp - udp$ integrationem admittit; quamobrem examinemus, qualis functio *v* debeat esse ipsarum *p* et *u*, ut hoc eveniat. Primum autem patet hoc fieri, si *v* fuerit functio homogenea unius dimensionis ipsarum *p* et *u*, quoniam tum ipsa haec aequatio fit homogenea ac per regulas supra expositas ad integrationem perduci potest. Deinde etiam integratio succedit, si fuerit *v* functio quaecunque ipsius *p*, quoniam tum altera variabiIis *u* unam dimensionem non superat et aequationis $du + \frac{udp}{v} = \frac{pdp}{v}$ integrale est

$$e^{\int \frac{dp}{v}} u = \int e^{\int \frac{dp}{v}} \frac{pdp}{v}$$

Tertio integrationem absolvere licebit, si *v* fuerit functio quaecunque quantitatis $p - u$. Posito enim $p - u = s$, ut sit *v* functio ipsius *s*, ob $p = s + u$ nostra aequatio erit $vdu = sds + sdu$ ideoque $du = \frac{sds}{(v-s)}$ et $u = \int \frac{sds}{(v-s)}$, quae integratio adeo ad formulas simplices est referenda. Quarto manente $s = p - u$ si *P, Q, R* denotent functiones quascunque ipsius *s,* aequatio nostra $vdu = sds + sdu$ tractari poterit, si fuerit $v = s + \frac{Ps}{Qu + Ru^n}$; tum emm fit $Pdu = Quds + Ru^n ds$. Quinto etiam patet, si denotantibus *V* et *U* functiones quascunque ipsius *u* fuerit $v = s + Vss + Us^n$, integrationem quoque fore in potestate; fit enim aequatio nostra $Vsdu + Us^{n-1}du = ds$. Atque in genere si aequatio differentialis $ds = Zdu$ fuerit integrabilis existente *Z* functione binarum variabilium *s* et *U,* cum nostra aequatio sit $sds = (v - s)du$, habebimus $v = s + Zs$ pro omnibus casibus integrationem admittentibus.

EXEMPLUM 1

800. *Sumto elemento dx constante si proponatur haec aequatio* $xxddy = xdxdy + nydx^2$ *,eius integrale invenire.*

Posito $dy = pdx$ et $dp = qdx$ erit $qxx = px + ny$, unde facto $y = ux$ prodit $qx = p + nu = v$, ita ut v sit functio unius dimensionis ipsarum p et u et aequatio nostra $(p + nu)du = pdp - udp$ fiat homogenea. Cum ergo sit $nudu + pdu + udp = pdp$, erit integrando

$$C + nuu + 2pu = pp \quad \text{et} \quad p = u + \sqrt{\left(C + (n+1)uu\right)}.$$

Habebimus ergo $\frac{dx}{x} = \frac{du}{\sqrt{\left(C+(n+1)uu\right)}}$, quae denuo integrata dat

$$lx = \frac{1}{\sqrt{(n+1)}} l \frac{u\sqrt{(n+1)} + \sqrt{\left(C+(n+1)uu\right)}}{D}$$

seu

$$Dx^{\sqrt{(n+1)}} = u\sqrt{(n+1)} + \sqrt{\left(C + (n+1)uu\right)}$$

hincque

$$D^2 x^{2\sqrt{(n+1)}} - 2Dx^{\sqrt{(n+1)}} u\sqrt{(n+1)} = C.$$

Sit $D = f\sqrt{(n+1)}$ et $C = g(n+1)$, ut habeatur

$$ffx^{2\sqrt{(n+1)}} - 2fx^{\sqrt{(n+1)}} u = g$$

existente $u = \frac{y}{x}$. Casu, quo $n = -1$, ob $\frac{dx}{x} = \frac{du}{u}$ erit $\alpha l\frac{x}{a} = u = \frac{y}{x}$ ideoque $y = \alpha x l\frac{x}{a}$. At si $n+1$ sit numerus negativus, integratio etiam angulos implicabit.

COROLLARlUM 1

801. Si sit $n = 0$, huius aequationis $xxddy = xdxdy$ integrale completum erit $ffx^2 - 2fy = g$, qui casus per se est perspicuus, cum ex $\frac{ddy}{dx} = \frac{dx}{x}$ fluat $\frac{dy}{dx} = fx$ et $2y = fxx - \frac{g}{f}$.

COROLLARIUM 2

802. Si sit $n = 3$, aequationis $xxddy = xdxdy + 3ydx^2 \cdot$ integrale completum est $ffx^4 - 2fxy = g$. Idem evenit, si loco $\sqrt{(n+1)}$ scribatur –2; fit enim $\frac{ff}{x^4} - \frac{2fy}{x^3} = g$ et $ff - 2fxy = gx^4$; utraque redit ad $y = \frac{\alpha}{x} + \beta x^3$.

EXEMPLUM 2

803. *Sumto elemento dx constante si proponatur haec aequatio*

$$\frac{xxddy}{dx} = \sqrt{\left(mxxdy^2 + nyydx^2\right)},$$

eius integrale completum invenire.

Ob $dy = pdx$ et $dp = qdx$ habebimus $qxx = \sqrt{(mppxx + nyy)}$, quae posito $y = ux$ abit in

$$qx = \sqrt{(mpp + nuu)} = v$$

ob $q = \frac{v}{x}$. Quia ergo est $\frac{dx}{x} = \frac{du}{p-u} = \frac{dp}{v}$ erit

$$du\sqrt{(mpp + nuu)} = (p - u)dp,$$

quae est aequatio homogenea. Ponatur ergo $p = su$ et prodit

$$du\sqrt{(mss + n)} = (s-1)(sdu + uds)$$

hincque

$$\frac{du}{u} = \frac{(s-1)ds}{\sqrt{(mss+n)} - ss + s}$$

ex qua fit

$$\frac{dx}{x} = \frac{du}{(s-1)u} = \frac{ds}{\sqrt{(mss+n)} - ss + s}$$

unde tam $u = \frac{y}{x}$ quam x per eandem variabilem s determinatur.

EXEMPLUM 3

804. *Sumto elemento dx constante si proponatur haec aequatio*

$$nx^3ddy = (ydx - xdy)^2$$

eius integrale invenire.

Posito $dy = pdx$ et $dp = qdx$ erit $nx^3q = (y - px)^2 = nxxv$ ob $q = \frac{v}{x}$.

Si iam statuatur $y = ux$, fiet

$$nv = (u-p)^2 \text{ et } \frac{dx}{x} = \frac{du}{p-u} = \frac{dp}{v} = \frac{ndp}{(p-u)^2},$$

unde habetur $ndp = pdu - udu$, quae facto $p - u = s$ abit in

$$ndu + nds = sdu \quad \text{seu} \quad du = \frac{nds}{s-n}, \quad \text{hinc} \quad u = nl\frac{s-n}{\alpha}.$$

Tum vero ob $p - u = s$ erit

$$\frac{dx}{x} = \frac{du}{s} = \frac{nas}{s(s-n)} \text{ et } lx = l\frac{s-n}{\beta s}$$

ideoque

$$x = \frac{s-n}{\beta s} \text{ et } y = nxl\frac{s-n}{\alpha}.$$

Cum ergo sit $s = \frac{n}{1-\beta x}$, erit

$$y = nxl\frac{n\beta x}{\alpha(1-\beta x)}.$$

COROLLARIUM

805. Aequatio $nx^3q = (y - px)^2$ facilius resolvitur ponendo $y - px = z$, unde fit $-xdp = ds$; quare ob $qdx = dp$ erit

$$nx^3dp = zzdx = -nxxdz$$

ideoque

$$\frac{1}{x} = \frac{1}{a} - \frac{n}{z} \quad \text{seu} \quad \frac{x-a}{ax} = \frac{n}{y-px},$$

ergo

$$y - px = \frac{nax}{x-a} = \frac{ydx - xdy}{dx}.$$

Quare

$$\frac{ydx - xdy}{xx} = \frac{nadx}{x(x-a)} \quad \text{et} \quad \frac{y}{x} = nl\frac{dx}{x-a} + C$$

ut ante.

EXEMPLUM 4

806. *Sumto dx constante si proponatur haec aequatio*

$$\left(dx^2 + dy^2\right)\sqrt{\left(dx^2 + dy^2\right)} = ndxddy\sqrt{(xx + yy)},$$

eius integrale invenire.

Posito $dy = pdx$ et $dp = qdx$ aequatio proposita hanc induit formam

$$(1+pp)\sqrt{(1+pp)}=nq\sqrt{(xx+yy)},$$

quae facto $y=ux$ et $q=\frac{v}{x}$ transit in hanc

$$(1+pp)\sqrt{(1+pp)}=nv\sqrt{(1+uu)}.$$

Cum igitur sit $\frac{dx}{x}=\frac{du}{p-u}=\frac{dp}{v}$,ob $v=\frac{(1+pp)\sqrt{(1+pp)}}{n\sqrt{(1+uu)}}$ erit

$$(1+pp)^{\frac{3}{2}}du=n(p-u)dp\sqrt{(1+uu)},$$

cuius resolutio non sponte patet. Calculum autem ad angulos revocando sit $p=\text{tang}.\varphi$ et $u=\text{tang}.\omega$; erit

$$dp=\frac{d\varphi}{cos.^2\varphi},\quad du=\frac{d\omega}{cos.^2\omega},\quad \sqrt{(1+pp)}=\frac{1}{cos.\varphi},\quad \sqrt{(1+uu)}=\frac{1}{cos.\omega}$$

et

$$p-u=\frac{sin.(\varphi-\omega)}{cos.\varphi cos.\omega}$$

hincque

$$\frac{1}{cos^3.\varphi}\cdot\frac{d\omega}{cos^2.\omega}=\frac{n\,sin.(\varphi-\omega)}{cos.\varphi cos.\omega}\cdot\frac{1}{cos.\omega}\cdot\frac{d\varphi}{cos^2.\varphi}$$

sive

$$d\omega=nd\varphi\,sin.(\varphi-\omega)=d\varphi-(d\varphi-d\omega),$$

ergo

$$d\varphi=\frac{d\varphi-d\omega}{1-n\,sin.(\varphi-\omega)}.$$

Posito ergo $\varphi-\omega=\psi$ erit

$$\varphi=\int\frac{d\psi}{1-n\,sin.\psi}\quad\text{et}\quad\omega=\int\frac{d\psi}{1-n\,sin.\psi}-\psi,$$

hinc ob $p=\text{tang}.\varphi$, $u=\text{tang}.\omega$ obtinetur

$$\frac{dx}{x}=\frac{du\,cos.\varphi cos.\omega}{sin.\omega}=\frac{d\omega\,cos.\varphi}{sin.\psi\,cos.\omega}=\frac{nd\psi\,cos.\varphi}{cos.\omega(1-n\,sin.\psi)}.$$

Casu $n=1$ fit

$$d\varphi=\frac{d\psi}{1-sin.\psi}=\frac{d\psi(1+sin.\psi)}{cos^2.\psi},$$

ergo

$$\varphi=\text{tang}.\psi+\frac{1}{cos.\psi}+\alpha=\frac{1+sin.\psi}{cos.\psi}+\alpha,\quad \omega=\frac{1+sin.\psi}{cos.\psi}+\alpha-\psi$$

et

$$\frac{dx}{x} = \frac{d\varphi \cos.\varphi}{\cos.\omega} = \frac{d\varphi \cos.\varphi}{\cos.\varphi \cos.\psi + \sin.\varphi \cos.\psi}.$$

Cum autem sit $\varphi - \alpha = \sqrt{\frac{1+\sin.\varphi}{1-\sin.\varphi}}$, erit

$$\sin.\psi = \frac{(\varphi-\alpha)^2-1}{(\varphi-\alpha)^2+1}, \quad \cos.\psi = \frac{2(\varphi-\alpha)}{(\varphi-\alpha)^2+1}.$$

Ergo

$$\frac{dx}{x} = \frac{d\varphi \cos.\varphi\left((\varphi-\alpha)^2+1\right)}{2(\varphi-\alpha)\cos.\varphi + (\varphi-\alpha)^2 \sin.\varphi - \sin.\varphi}.$$

COROLLARIUM 1

807. Si casu $n = 1$ sumatur constans α infinita, erit $\sin.\psi = 1$, hinc $\psi = 90^0$ et $\omega = \varphi - 90^0$; tum vero

$$\frac{dx}{x} = \frac{d\varphi \cos.\varphi}{\sin.\varphi}$$

ideoque $x = a\sin\varphi$ et $u = -\cot.\varphi$, ergo $y = -a\cos.\varphi$ et $xx + yy = aa$.

COROLLARIUM 2

808. Eodem autem casu $n = 1$, quo constans α non sumitur infinita, fractionis, cui $\frac{dx}{x}$ aequatur, numerator commode est differentiale denominatoris;
unde fit

$$x = a\left((\varphi-\alpha)^2 \sin.\varphi - \sin.\varphi + 2(\varphi-\alpha)\cos.\varphi\right).$$

Tum vero est

$$\omega = \varphi - \text{Ang.tang.}\frac{(\varphi-\alpha)^2-1}{2(\varphi-\alpha)}$$

ideoque

$$u = \frac{y}{x} = \text{tang.}\omega = \frac{\text{tang.}\varphi - \frac{(\varphi-\alpha)^2-1}{2(\varphi-\alpha)}}{1+\frac{(\varphi-\alpha)^2-1}{2(\varphi-\alpha)}\text{tang.}\varphi}$$

seu

$$\frac{y}{x} = \frac{2(\varphi-\alpha)\sin.\varphi - (\varphi-\alpha)^2 \cos.\varphi + \cos.\varphi}{(\varphi-\alpha)^2 \sin.\varphi - \sin.\varphi + 2(\varphi-\alpha)\cos.\varphi},$$

consequenter

$$y = -a\left((\varphi-\alpha)^2 \cos.\varphi - \cos.\varphi - 2(\varphi-\alpha)\sin.\varphi\right)$$

et

$$\sqrt{(xx+yy)} = a\left((\varphi-\alpha)^2 + 1\right).$$

SCHOLION 1

809. In genere etiam integrationem absolvere licet. Cum enim sit

$$d\varphi = \frac{d\psi}{1-n\sin.\psi} \quad \text{and} \quad \frac{dx}{x} = \frac{nd\varphi\cos.\varphi}{\cos.\omega},$$

erit

$$\varphi+\alpha = \frac{1}{\sqrt{(1-nn)}}\text{Ang.cos.}\frac{n-\sin.\psi}{1-n\sin.\psi},$$

unde posito $(\varphi+\alpha)\sqrt{(1-nn)} = \theta$ erit

$$\cos.\theta = \frac{n-\sin.\psi}{1-n\sin.\psi}$$

hincque

$$\sin.\psi = \frac{n-\cos.\theta}{1-n\cos.\theta} \quad \text{et} \quad \cos.\psi = \frac{\sin.\theta\sqrt{(1-nn)}}{1-n\cos.\theta}.$$

At ob $\omega = \varphi - \psi$ habebitur

$$\frac{dx}{x} = \frac{nd\varphi\cos.\varphi(1-n\cos.\theta)}{\cos.\phi\sin.\theta\sqrt{(1-nn)}+\sin.\varphi(n-\cos.\theta)}.$$

Iam cum sit $d\theta = d\varphi\sqrt{(1-nn)}$, differentiale huius denominatoris est

$$\begin{aligned} &- d\varphi\sin.\varphi\sin.\theta\sqrt{(1-nn)} + d\varphi\cos.\varphi\cos.\theta(1-nn) + nd\varphi\cos.\varphi \\ &- dp\cos.\varphi\cos.\theta + d\varphi\sin.\varphi\sin.\theta\sqrt{(1-nn)}, \end{aligned}$$

quod redit ad $nd\varphi\cos.\varphi(1-n\cos.\theta)$, ipsum scilicet numeratorem, ita ut sit

$$x = a\left(\cos.\varphi\sin.\theta\sqrt{(1-nn)} + \sin.\varphi(n-\cos.\theta)\right)$$

seu

$$x = a\cos.\omega(1-n\cos.\theta) \quad \text{ideoque} \quad y = ux = a\sin.\omega(1-n\cos.\theta).$$

Assumto ergo angulo θ quaeratur angulus ψ, ut sit

$$sin.\psi = \tfrac{n-cos.\theta}{1-n\,cos.\theta} \quad \text{et} \quad cos.\psi = \tfrac{sin.\theta}{1-n\,cos.\theta}\sqrt{(1-nn)} \quad ;$$

tum vero fiat

$$\omega = \tfrac{\theta}{\sqrt{(1-nn)}} - \alpha - \psi$$

eritque integrale completum

$$x = a(1-n\,cos.\theta)cos.\omega \ \text{ et } \ y = a(1-n\,cos.\theta)sin.\omega.$$

SCHOLION 2

810. At si numerus n sit unitate maior, haec integratio fit imaginaria; quod incommodum ut tollatur, notandum est aequationis $d\varphi = \frac{d\psi}{1-n\,sin.\psi}$,

integrale esse

$$\varphi + \alpha = \tfrac{1}{\sqrt{(nn-1)}}\, l\, \tfrac{\sqrt{(n-1)}(1+sin.\psi)+\sqrt{(n+1)}(1-sin.\psi)}{\sqrt{(n-1)}(1+sin.\psi)-\sqrt{(n+1)}(1-sin.\psi)}.$$

Quare si ponatur $(\varphi+\alpha)\sqrt{(nn-1)} = \theta$, ut sit

$$d\theta = d\varphi\sqrt{(nn-1)} \quad \text{et} \quad \omega = \varphi - \psi = \tfrac{\theta}{\sqrt{(nn-1)}} - \alpha - \psi,$$

erit

$$\tfrac{e^{\theta}+1}{e^{\theta}-1} = \tfrac{\sqrt{(n-1)(1+\sin.\psi)}}{\sqrt{(n+1)(1-\sin.\psi)}} = \tfrac{(n-1)(1+\sin.\psi)}{cos.\psi\sqrt{(nn-1)}},$$

unde reperitur

$$sin.\psi = \tfrac{e^{\theta}+2n+e^{-\theta}}{ne^{\theta}+2+ne^{-\theta}} \quad \text{et} \quad cos.\psi = \tfrac{(e^{\theta}-e^{-\theta})\sqrt{(nn-1)}}{ne^{\theta}+2+ne^{-\theta}}$$

ita ut ex angulo θ definiantur anguli $\varphi,\ \psi$ et ω. Cum iam sit

$$\tfrac{dx}{x} = \tfrac{nd\varphi\cos.\varphi}{cos.\omega} = \tfrac{nd\varphi\cos.\varphi}{cos.\phi\, cos.\psi + sin.\varphi\, sin.\psi},$$

erit

$$\tfrac{dx}{x} = \tfrac{nd\varphi\cos.\varphi\left(ne^{\theta}+2+ne^{-\theta}\right)}{cos.\phi\left(e^{\theta}-e^{-\theta}\right)\sqrt{(nn-1)}+sin.\varphi\left(e^{\theta}+2n+e^{-\theta}\right)},$$

ubi iterum commode evenit, ut numerator sit ipsum differentiale denominatoris, quemadmodum differentiationem instituenti mox patebit. Hinc ergo erit

$$x = a\left(cos.\phi\left(e^{\theta}-e^{-\theta}\right)\sqrt{(nn-1)} + sin.\varphi\left(e^{\theta}+2n+e^{-\theta}\right)\right)$$

seu

$$x = a\cos.\varphi\left(ne^{\theta} + 2 + ne^{-\theta}\right)$$

et ob $u = \text{tang}.\omega$ fit

$$y = a\cos.\omega\left(ne^{\theta} + 2 + ne^{-\theta}\right)$$

Quocirca ex angulo θ primo quaeratur angulus ψ, ut sit

$$sin.\psi = \frac{e^{\theta}+2n+e^{-\theta}}{ne^{\theta}+2+ne^{-\theta}} \quad \text{et} \quad cos.\psi = \frac{\left(e^{\theta}-e^{-\theta}\right)\sqrt{(nn-1)}}{ne^{\theta}+2+ne^{-\theta}};$$

quo invento capiatur angulus $\omega = \frac{\theta}{\sqrt{(nn-1)}} - \alpha - \psi$ ac formulae illae pro x et y inventae dabunt integrale eompletum ob duas constantes a et α introductas.

SCHOLION 3

811. Cum hic praecipua pars integrationis fortuito successisse videatur, operae pretium erit in eius causam inquirere, num forte ratio integrandi clarius perspici queat. Cum igitur sit $\varphi = \psi + \omega$ et $d\varphi = \frac{d\psi}{1-n\,sin.\psi}$ hincque

$$d\omega = \frac{nd\psi\,cos.\varphi}{1-n\,sin.\psi} = nd\varphi\,sin.\psi\ ,$$

aequatio $\frac{dx}{x} = \frac{nd\varphi\,cos.\varphi}{cos.\omega}$ ob $cos.\varphi = cos.\psi\,cos.\omega - sin.\psi\,sin.\omega$ in hanc resolvitur

$$\frac{dx}{x} = nd\varphi\,sin.\psi - \frac{nd\varphi\,sin.\psi\,sin.\omega}{cos.\omega},$$

quae ob $d\varphi = \frac{d\psi}{1-n\,sin.\psi}$ et $nd\varphi\,sin.\psi = d\omega$ induit hanc formam integrabilem

$$\frac{dx}{x} = \frac{nd\psi\,cos.\psi}{1-n\,sin.\psi} - \frac{d\omega\,sin.\omega}{cos.\omega},$$

ex qua elicitur

$$x = \frac{a\,cos.\omega}{1-n\,sin.\psi} \quad \text{et} \quad y = \frac{a\,sin.\omega}{1-n\,sin.\psi}$$

ob $y = ux = x$ tang. w. En ergo in genere aequationis nostrae hanc integrationem. Anguli ω et ψ hanc inter se teneant relationem, ut sit

$$d\omega = \frac{nd\psi\,sin.\psi}{1-n\,sin.\psi}$$

tum vero erit

$$x = \frac{a\,cos.\omega}{1-n\,sin.\psi} \quad \text{et} \quad y = \frac{a\,sin.\omega}{1-n\,sin.\psi}$$

Quodsi ergo ponamus $\sqrt{(xx+yy)} = z$, ut sit $x = z\,cos.\omega$ et $y = z\,sin.\omega$, erit

$$z=\frac{a}{1-n\sin.\psi} \quad \text{et} \quad \sin.\psi=\frac{z-a}{nz}, \quad \text{hinc} \quad d\omega=\frac{(z-a)d\psi}{a}.$$

At fit

$$d\psi=\frac{adz}{z\sqrt{\left(nnzz-(z-a)^2\right)}}, \quad \text{ergo} \quad d\omega=\frac{(z-a)dz}{z\sqrt{\left(nnzz-(z-a)^2\right)}}.$$

unde angulus ω per z definitur. Ad irrationalitatem tollendam si ponamus

$$\sqrt{\left(nnzz-(z-a)^2\right)}=s(nz+z-a),$$

fit

$$z=\frac{a(ss+1)}{(n+1)ss-n+1} \quad \text{et} \quad d\omega=\frac{2nds(ss-1)}{(ss+1)\left((n+1)ss-(n-1)\right)}$$

seu

$$d\omega=\frac{2ds}{ss+1}-\frac{2ds}{(n+1)ss-n+1}$$

quae integratio est manifesta.

PROBLEMA 99

812. *Si aequatio differentio–differentialis tum demum fiat homogenea, si alteri variabili y tribuantur n dimensiones, eius integrationem ad aequationem differentialem primi gradus reducere.*

SOLUTIO

Ponamus $dy=pdx$ et $dp=qdx$, ut oriatur aequatio inter quaternas quantitates finitas *x, y, p* et *q;* quae quomodo ratione homogeneitatis futura sit comparata, videamus. Primo ergo cum pro *x* unam dimensionem numerando variabilis *y* habeat *n* dimensiones, quantitati $p=\frac{dy}{dx}$ tribuendae sunt $n-1$ dimensiones et quantitati $q=\frac{dp}{dx}$ $n-2$ dimensiones. Quocirca ponamus

$$y=x^n u, \quad p=x^{n-1}t \quad \text{et} \quad q=x^{n-2}v$$

et ob $dy=pdx$ et $dp=qdx$ habebimus

$$xdu+nudx=tdx \quad \text{et} \quad xdt+(n-1)tdx=vdx,,$$

unde colligimus

$$\frac{dx}{x}=\frac{du}{t-nu}=\frac{dt}{v-(n-1)t}$$

ideoque

$$du\left(v-(n-1)t\right)=dt\left(t-nu\right).$$

At factis superioribus substitutionibus in aequatione inter *x, y, p* et *q* per hypothesin variabilis *x* ex calculo extruditur, ita ut prodeat aequatio inter tres tantum variabiles *u, t* et *v,* ex qua litteram *v* per binas *t* et *u* definire licebit. Quo valore substituto habebitur aequatio differentialis primi gradus inter binas variabiles *u* et *t,* ex qua *t* per *u* determinari queat; ope aequationis $\frac{dx}{x}=\frac{du}{t-nu}$ definietur *x* per *u* hincque ob $u=\frac{y}{x^n}$ obtinebitur aequatio integralis inter *x* et *y* eaque ob duplicem integrationem completa.

COROLLARIUM 1

813. Aequationum ergo inter *x, y, p* et *q* hoc modo tractabilium hoc est criterium, ut posito $y=x^n u,\quad p=x^{n-1}t$ et $q=x^{n-2}v$ exponens *n* eiusmodi determinationem patiatur, ut variabilis *x* prorsus ex calculo per divisionem egrediatur.

COROLLARIUM 2

814. Si sit $n=0$, aequatio ita est comparata, ut tribuendo ipsi *y* eiusque differentialibus nullam dimensionem fiat homogenea. Hoc scilicet casu sola variabilis *x* cum suis differentialibus dimensiones constituere censetur.

COROLLARIUM 3

815. Contra vero si dimensiones ex sola variabili *y* aestimentur, ita ut ea cum suis differentialibus *dy* et *ddy* ubique eundem dimensionum numerum constituat, exponens *n* fiet infinitus.

SCHOLION

816. Si sola variabilis *x* cum suis differentialibus ubique eundem dimensionum numerum complet, ob $n=0$ fit $u=y$ et in aequatione inter *x, y,p* et *q* statui conveniet $p=\frac{t}{x}$ et $q=\frac{v}{xx}$, quo facto variabilis *x* ex calculo deturbabitur prodibitque aequatio inter *y, t* et *v,* cuius ope aequatio differentialis $dy\left(v+t\right)=tdt$ ad duas tantum variabiles reducetur; qua resoluta erit $\frac{dx}{x}=\frac{dy}{t}$, ubi, cum *t* detur per *y,* integratio nullam habet difficultatem. Verum altero casu, quo variabilis *y* sola cum suis differentialibus pares ubique dimensiones habet ideoque exponens *n* infinitus capi deberet, resolutio alio modo institui debet, quem mox [§ 822] docebimus; nisi forte permutando variabiles *x* et *y* casum ad praecedentem reducere lubuerit.

EXEMPLUM 1

817. *Sumto elemento dx constante si proponatur haec aequatio*

$$xxddy = \alpha ydx^2 + \beta xdxdy,$$

eius integrale invenire.

Perpendatur hic ista conditio, qua sola variabilis x cum suo differentiali dx ubique duas constituit dimensiones, eritque $n = 0$. Cum ergo posito $dy = pdx$ et $dp = qdx$ habeamus $qxx = \alpha y + \beta px$, statuamus $p = \frac{t}{x}$ et $q = \frac{v}{xx}$ fietque $v = \alpha y + \beta t$, unde adipiscimur istam aequationem differentialem

$$aydy + (\beta + 1)tdy = tdt\ ;$$

ob cuius homogeneitatem faciamus $t = yz$ eritque

$$ady + (\beta + 1)zdy = yzdz + zzdy$$

seu

$$\frac{dy}{\mathrm{y}} = \frac{zdz}{\alpha + (\beta + 1)z - zz}$$

Sit

$$\alpha + (\beta + 1)z - zz = (f + z)(g - z),$$

ut sit $\alpha = fg$ et $\beta + 1 = g - f$, reperieturque

$$\frac{dy}{y} = -\frac{f}{f+g} \cdot \frac{dz}{f+z} + \frac{g}{f+g} \cdot \frac{dz}{g-z},$$

unde colligitur integrando

$$ly = C - \frac{f}{f+g} l(f + z) - \frac{g}{f+g} l(g - z)$$

seu

$$y(f + z)^{\frac{f}{f+g}} (g - z)^{\frac{f}{f+g}} = a.$$

Tum vero est

$$\frac{dx}{x} = \frac{dy}{yz} = \frac{dz}{(f+z)(g-z)} \text{ seu } \frac{dx}{x} = \frac{1}{f+g} \cdot \frac{dz}{f+z} + \frac{1}{f+g} \cdot \frac{dz}{g-z},$$

hinc

$$x = b\left(\frac{f+z}{g-z}\right)^{\frac{1}{f+g}} \text{ seu } \frac{f+z}{g-z} = \left(\frac{x}{b}\right)^{f+g}.$$

Unde cum sit $z = \frac{gx^{f+g} - fb^{f+g}}{b^{f+g} + x^{f+g}}$, erit hoc valore ibi substituto

$$(f + g)b^g x^f y = a\left(b^{f+g} + x^{f+g}\right)$$

seu posito $\frac{a}{f+g}=C$

$$y=c\left(\frac{b^f}{x^f}+\frac{x^g}{b^g}\right)$$

est vero

$$g-f=\beta+1 \text{ et } f+g=\sqrt{\left((\beta+1)^2+4\alpha\right)}.$$

COROLLARIUM

818. Quoniam in aequatione proposita etiam ambae variabiles x et y simul ubique totidem dimensiones habent, earn etiam secundum praecepta praecedentis problematis tractare licet.

EXEMPLUM 2

819. *Posito dx constante si aequatio differentio-differentialis duobus tantum terminis constet, ut sit huiusmodi*

$$ddy=cx^\alpha y^\beta dx^{2-\gamma}dy^\gamma,$$

eius integrale investigare.

Posito $dy=pdx$ et $dp=qdx$ habebitur haec forma $q=cx^\alpha y^\beta p^\gamma$, ubi exponentem n ita definire licet, ut posito $y=x^n u$, $p=x^{n-1}t$ et $q=x^{n-2}v$ variabilis x per divisionem tolli possit; capi enim oportet

$$\alpha+\beta n+\gamma(n-1)-n+2=0 \text{ seu } n=\frac{-\alpha+\gamma-2}{\beta+\gamma-1}$$

tumque erit $v=cu^\beta t^\gamma$. Aequatio ergo differentialis primi gradus resolvenda erit

$$cu^\beta t^\gamma du-(n-1)tdu=tdt-nudt;$$

ex qua cum variabilis t per u fuerit determinata, integrari oportet hanc formulam $\frac{dx}{x}=\frac{du}{t-nu}$ quo facto ob $u=\frac{y}{x^n}$ obtinebitur aequatio integralis quaesita inter x at y.

Casus tantum $\beta+\gamma=1$, quo n fit infinitus, peculiarem postulat tractationem infra [§ 822] exponendam, nisi forte simul sit $\gamma=\alpha+2$; tum enim exponens n prorsus arbitrio nostro relinquitur, at aequatio erit homogenea.

EXEMPLUM 3

820. *Sumto elemento dx constante si proponatur haec aequatio*

$$x^4 ddy = x^3 dxdy + 2xydxdy - 4yydx^2,$$

eius integrale invenire.

Hic evidens est, si ipsi y eiusque differentialibus dy et ddy binae dimensiones, ipsi x vero et dx singulae tribuantur, in omnibus terminis obtineri sex dimensiones. Quare cum posito $dy = pdx$ et $dp = qdx$ habeamus hanc aequationem

$$x^4 q = x^3 p + 2xyp - 4yy,$$

faciamus

$$y = x^2 u,\ p = xt \quad \text{et} \quad q = v$$

prodibitque

$$v = t + 2ut - 4uu.$$

At ob $n = 2$ aequatio differentialis nostra erit

$$du(v - t) = dt(t - 2u),$$

quae abit in

$$2udu(t - 2u) = dt(t - 2u),$$

unde deducimus vel $t = 2u$ vel $t = uu + c$, quos binos casus seorsim evolvamus.

1) Si $t = 2u$, ob $\frac{dx}{x} = \frac{du}{t-2u}$ fit $du = 0$ ideoque $u = C$ ac propterea $y = Cxx$, quod est integrale particulare aequationi propositae utique satisfaciens.

2) Sit $t = uu + c$; erit $\frac{dx}{x} = \frac{du}{uu-2u+c}$, ubi tres casus sunt considerandi.
Primo si constans $c = 1$, erit

$$l\frac{x}{a} = \frac{1}{1-u} = \frac{xx}{xx-y} \text{ seu } xx = (xx - y)l\frac{x}{a}.$$

Secundo si constans $c = 1 - ff$ erit $\frac{dx}{x} = \frac{du}{(u-1)^2 - ff}$ hincque

$$lx = \frac{-1}{2f} l\frac{f+u-1}{f-u+1} + C,$$

ergo ob $u = \frac{y}{xx}$ erit

$$x = a\left(\frac{(f+1)xx-y}{(f-1)xx+y}\right)^{\frac{1}{2f}}.$$

Tertio si constans $c = 1 + ff$, erit $\frac{dx}{x} = \frac{du}{(u-1)^2 + ff}$, quae integrata dat

$$l\frac{x}{a} = \frac{1}{f}\text{Ang. tang. }\frac{u-1}{f} \quad \text{seu} \quad \frac{u-1}{f} = \frac{y-xx}{fxx} = \text{tang.}\, fl\frac{x}{a}.$$

Pro ratione ergo constantis arbitrariae *c* integratio vel algebraice succedit vel a logarithmis vel ab angulis pendet, unde forma generali exprimi nequit.

SCHOLION

821. Integrale autem particulare primo inventum $y = Cxx$ in nulla harum formarum, quibus integrale completum constituitur, contineri deprehenditur; nihilo vero minus satisfacit aequationi differentio–differentiali propositae. Hoc ergo exemplo magis illustrantur ea, quae supra [§ 546, 564] circa hoc paradoxon sumus commentati, quod interdum aequationi differentiali satisfaciat aequatio finita, quae in integrali completo minime contineatur. Vidimus igitur hoc idem paradoxon etiam in aequationibus differentio–differentialibus locum habere. Utrum autem illa aequatio $y = Cxx$ inter integralia sit admittenda, alia est quaestio, quae nondum penitus videtur confecta; hic quidem ipsa aequatio proposita quasi factores habens est censenda, ex quorum altero illa aequatio $y = Cxx$ nascatur, verum multum abest, ut in hac explicatione acquiescere queamus. Quin potius ipsa illa quaestio, sive geometrica fuerit sive alius disciplinae, cuius solutio ad huiusmodi aequationem perduxerit, accurate perpendi debere videtur; ubi plerumque haud difficulter iudicari potest, utrum, quicquid aequationi differentiali satisfaciat, id etiam ipsi quaestioni conveniat necne. Veluti si descensus gravis \ex altitudine $= a$ labentis definiri debeat et altitudo, qua iam a terra distat, sit *x*, erit ibi celeritas ut $\sqrt{(a-x)}$ et elementum temporis $dt = \frac{-dx}{\sqrt{(a-x)}}$. Hic quidem evidens est isti aequationi ideoque differentiali satisfieri ponendo $x = a$, ita ut tempus *t* maneat indefinitum, quod tamen quaestioni neutiquam convenit, quae nonnisi vero integrali $t = 2\sqrt{(a-x)}$ resolvitur.

PROBLEMA 100

822. *Si in aequatione differentio–differentiali variabilis y cum suis differentialibus dy et ddy ubique eundem dimensionum numerum adimpleat, eius integrationem ad aequationem differentialem primi gradus reducere.*

SOLUTIO

Posito $dy = pdx$ et $dp = qdx$ aequatio ita erit comparata, ut in ea ternae variabiles *y*, *p*, *q* ubique eundem dimensionum numerum obtineant altera variabili *x* in computum dimensionum prorsus non ingrediente. Quare si statuatur $p = uy$ et $q = vy$, in omnibus terminis inerit eadem ipsius *y* potestas, qua per divisionem sublata habebitur aequatio inter ternas tantum variabiles *x*, *u* et *v*, ex qua unam per binas reliquas definire licebit, ita ut *v* aequetur functioni cuipiam ipsarum *x* et *u*. Iam ob $p = uy$ erit $dy = uydx$ at ob $dp = qdx$ fiet $udy + ydu = vydx$, unde sequitur

$$\frac{dy}{y} = udx \quad \text{et} \quad \frac{dy}{y} = \frac{vdx - du}{u}$$

ideoque

$$udu + uudx = vdx,$$

quae aequatio differentialis duas tantum variabiles *x* et *u* complectitur. Quam ergo si integrare liceat, ut relatio inter *x* et *u* inde innotescat, superest, ut formulae *udx* integrale investigetur, quo invento erit $ly = \int udx$ sicque aequatio orietur integralis inter *x* et *y*, quae ob duplicem integrationem peractam duas constantes arbitrarias involvet ideoque integrale completum exhibebit.

COROLLARIUM 1

823. Huiusmodi ergo aequationum integratio reducitur ad huiusmodi aequationem differentialem

$$du + uudx = vdx,$$

cuius resolutio si succedat, simul illarum integratio habetur, cum formulae *udx* integratio difficultate careat.

COROLLARIUM 2

824. Cum sit $\frac{dy}{y} = udx$, erit $y = e^{\int udx}$, qua substitutione aequatio differentio–differentialis proposita statim reducitur ad aequationem differentialem primi gradus; erit enim

$$dy = p = e^{\int udx}u \quad \text{et} \quad \frac{dp}{dx} = q = \frac{e^{\int udx}(du + uudx)}{dx}$$

ac tum formula exponentialis sponte ex aequatione egreditur.

COROLLARIUM 3

825. Vicissim etiam proposita aequatione differentiali primi gradus
$du + uudx = vdx$, in qua *v* sit functio quaecunque ipsarum *x* at *u*, ea posito $u = \frac{dy}{ydx}$ in eiusmodi aequationem differentio–differentialem transformatur, in qua variabilis *y* cum suis differentialibus *dy* et *ddy* ubique eundem dimensionum numerum constituat.

SCHOLION 1

826. Haec reductio aequationum differentialium primi gradus ad gradum secundum legibus analyseos adversari videtur, interim tamen subinde usu non caret; quodsi enim alia methodo huiusmodi aequationes differentio–differentiales tractare liceat, dum earum integralia vel per series exhibentur vel finite, simul integralia aequationum differentialium primi gradus innotescunt, quorum ratio plerumque aliunde vix perspicitur. In sequentibus [Cap. VII, VIII] autem videbimus eiusmodi aequationes differentio–differentiales, in quibus variabilis altera *y* unam dimensionem non superat, per series commode integrari posse atque adeo interdum has series abrumpi, ita ut integrale finita expressione exhibeatur. Caeterum proposita huiusmodi aequatione differentiali primi gradus $du + uudx = vdx$ substitutio $u = \frac{dy}{ydx}$ eo magis est notatu digna, quod sumto elemento dx constante fiat

$$du = \frac{ddy}{ydx} - \frac{dy^2}{yydx} \text{ ideoque } du + uudx = \frac{ddy}{ydx},$$

ita ut duo termini hoc modo in unum coalescant.

SCHOLION 2

827. Casus hic imprimis notasse iuvabit, quibus aequatio $du + uudx = vdx$ integrationem admittit. Hunc in finem sit $du = Vdx$ forma generalis aequationum resolubilium et V certa functio ipsarum x et u ac manifestum est, si fuerit $v = uu + V$ integrationem succedere. Primum ergo hoc eveniat, si sit $V = \frac{X}{U}$ denotante X functionem ipsius x et U ipsius u. Secundo, si V sit functio homogenea nullius dimensionis ipsarum x et u. Tertio, si denotantibus X at Ξ functiones quascunque ipsius x fuerit $V = Xu + \Xi u^n$. Quarto, si denotantibus P at Q functiones quascunque ipsius u fuerit $V = \frac{1}{Px + Qx^n}$. Similique modo ex alIis formis integrabilibus alii casus concludentur.

EXEMPLUM 1

828. *Sumto elemento dx constante si proponatur haec aequatio*

$$\alpha yddy + \beta dy^2 = \frac{ydxdy}{\sqrt{(aa+xx)}},$$

eius integrale invenire.

Posito $dy = pdx$ et $dp = qdx$ prodit

$$\alpha yq + \beta pp = \frac{yp}{\sqrt{(aa+xx)}},$$

quae facto $p = uy$ et $q = vy$ abit in hanc

$$\alpha v + \beta uu = \frac{u}{\sqrt{(aa+xx)}} \text{ seu } v = \frac{u}{\alpha\sqrt{(aa+xx)}} - \frac{\beta uu}{\alpha},$$

unde hanc aequationam resolvi oportet

$$du + uudx = \frac{udx}{\alpha\sqrt{(aa+xx)}} - \frac{\beta uudx}{\alpha}.$$

Statuatur $u = \frac{1}{s}$ fietque

$$ds+\frac{sdx}{\alpha\sqrt{(aa+xx)}}=\left(1+\frac{\beta}{\alpha}\right)dx,$$

quae per $\left(x+\sqrt{(aa+xx)}\right)^{\frac{1}{\alpha}}$ multiplicata et integrata dat

$$s\left(x+\sqrt{(aa+xx)}\right)^{\frac{1}{\alpha}}=\left(1+\frac{\beta}{\alpha}\right)\int dx\left(x+\sqrt{(aa+xx)}\right)^{\frac{1}{\alpha}}.$$

Fiat $x+\sqrt{(aa+xx)}=t^{\alpha}$; erit $aa=t^{2\alpha}-2t^{\alpha}x$, hinc

$$x=\frac{t^{2\alpha}-aa}{2t^{\alpha}}=\frac{1}{2}t^{\alpha}-\frac{1}{2}aat^{-\alpha}\quad\text{et}\quad dx=\frac{\alpha}{2}dt\left(t^{\alpha-1}+aat^{-\alpha-1}\right),$$

ita ut sit

$$st=\left(1+\frac{\beta}{\alpha}\right)\int\frac{\alpha}{2}dt\left(t^{\alpha}+aat^{-\alpha}\right)\quad\text{seu}\quad st=C+\left(\frac{\alpha+\beta}{2}\right)\left(\frac{t^{\alpha+1}}{\alpha+1}+\frac{aat^{1-\alpha}}{1-\alpha}\right).$$

Porro est $\frac{dy}{y}=udx=\frac{dx}{s}$; at ex aequatione differentiali est

$$\left(1+\frac{\beta}{\alpha}\right)\frac{dx}{s}=\frac{ds}{s}+\frac{dx}{\alpha\sqrt{(aa+xx)}}$$

hincque

$$\left(1+\frac{\beta}{\alpha}\right)ly=ls+\frac{1}{\alpha}l\left(x+\sqrt{(aa+xx)}\right)=lst+D,$$

ergo $y=B(st)^{\frac{\alpha}{\alpha+\beta}}$. Quare sumto $C=\frac{\alpha+\beta}{2}A$ habebitur

$$y=B\left(A+\frac{t^{\alpha+1}}{\alpha+1}+\frac{aat^{1-\alpha}}{1-\alpha}\right)^{\frac{\alpha}{\alpha+\beta}}$$

existente

$$x=\frac{1}{2}\left(t^{\alpha}-aat^{-\alpha}\right)\quad\text{vel}\quad t=\left(x+\sqrt{(aa+xx)}\right)^{\frac{1}{\alpha}},$$

ita ut aequatio inter *x* et *y* sit

$$Cy^{\frac{\alpha+\beta}{\alpha}}=A+\frac{1}{\alpha+1}\left(x+\sqrt{(aa+xx)}\right)^{\frac{\alpha+1}{\alpha}}+\frac{aa}{1-\alpha}\left(x+\sqrt{(aa+xx)}\right)^{\frac{1-\alpha}{\alpha}}.$$

SCHOLION

829. Hoc idem exemplum ita est comparatum, ut alia ratione facillime resolvi possit; aequatio enim

$$\alpha yq+\beta pp=\frac{yp}{\sqrt{(aa+xx)}}$$

Si per $\frac{dx}{yp}$ multiplicetur, ob $qdx=dp$ et $pdx=dy$ abit in

$$\frac{\alpha dp}{p}+\frac{\beta dy}{y}=\frac{dx}{\sqrt{(aa+xx)}},$$

cuius singuli termini sunt integrabiles. Prodit ergo

$$p^{\alpha}y^{\beta}=C^{\alpha}\left(x+\sqrt{(aa+xx)}\right)$$

hincque

$$y^{\frac{\beta}{\alpha}}dy=Cdx\left(x+\sqrt{(aa+xx)}\right)^{\frac{1}{\alpha}},$$

quae aequatio denuo integrata praebet integrale ante inventum.

Forma ergo generalis aequationum hoc modo resolubilium est

$$Pdp+Ydy+Xdx=0$$

existente P functione ipsius p, Y ipsius y et X ipsius x, quae nostro more repraesentatur per $Pq+Yp+X=0$. Hinc ergo perspicitur, quomodo etiam aequationes differentio–differentiales ope idonei multiplicatoris ad integrationem perduci queant; quae methodus cum in aequationibus differentialibus primi gradus insignem usum praestiterit, eo magis excolenda videtur, quod etiam ad aequationes differentiales altiorum graduum pateat, quod argumentum infra [Cap. V] fusius pertractare conabimur.

EXEMPLUM 2

830. *Sumto elemento dx constante si proponatur haec aequatio*

$$xyddy=ydxdy+xdy^2+\frac{bxdy^2}{\sqrt{(aa-xx)}},$$

eius integrale invenire.

Posito $dy=pdx$ et $dp=qdx$ erit

$$xyq=yp+xpp+\frac{bxpp}{\sqrt{(aa-xx)}},$$

quae facto $p=uy$ et $q=vy$ abit in

$$xv = u + uux + \frac{buux}{\sqrt{(aa-xx)}},$$

unde oritur haec aequatio differentialis

$$du + uudx = \frac{udx}{x} + uudx + \frac{buudx}{\sqrt{(aa-xx)}} \quad \text{seu} \quad \frac{xdu-udx}{uu} = \frac{bxdx}{\sqrt{(aa-xx)}},$$

cuius integrale

$$C - \frac{x}{u} = -b\sqrt{(aa-xx)} \quad \text{seu} \quad u = \frac{x}{C+b\sqrt{(aa-xx)}},$$

ergo

$$\frac{dy}{y} = \frac{xdx}{C+b\sqrt{(aa-xx)}}.$$

Statuatur $\sqrt{(aa-xx)} = t$, ut sit $xdx = -tdt$; erit

$$\frac{dy}{y} = \frac{-tdt}{C+bt} = -\frac{dt}{b} + \frac{Cdt}{b(C+bt)} \quad \text{et} \quad ly = -\frac{t}{b} + \frac{C}{bb} l(C+bt) + lc.$$

Sit $C = nbb$; erit

$$l\frac{y}{c} = -\frac{\sqrt{(aa-xx)}}{b} + nl\frac{nb+\sqrt{(aa-xx)}}{b},$$

ubi c et n sunt constantes arbitrariae.