CHAPTER VI

CONCERNING THE INTEGRATION OF OTHER SECOND ORDER DIFFERENTIAL EQUATIONS BY PUTTING IN PLACE SUITABLE MULTIPLIERS

PROBLEM 112

906. *With the element dx put constant if an equation of this kind is proposed*

$$ddy+\frac{Aydx^2}{(Byy+C+2Dx+Exx)^2}=0,$$

to find the multiplier by which it is rendered integrable.

SOLUTION

The form $2Pdy+2Qydx$ of such a multiplier may be tried, where P and Q are functions of x, and the integral to be extended :

$$2ddy(Pdy+Qydx)+\frac{2Aydx^2(Pdy+Qydx)}{(Byy+C+2Dx+Exx)^2}=0$$

with the [trial] integral put in place

$$Pdy^2+2Qydxdy+Vdx^2=\text{Const}.dx^2,$$

where V shall be a function of the two variables x and y. Hence there will be made from the equality [by differentiation of the latter equation, recalling that dx is constant, and subtracting from this equation the former equation:]

$$dPdy^2+2ydxdQdy+2Qdxdy^2+dx^2dV-\frac{2Aydx^2(Pdy+Qydx)}{(Byy+C+2Dx+Exx)^2}=0,$$

for which the value of V is unable to be put in place by integration, unless there shall be $dP+2Qdx=0$; and then there will be

$$dV=\frac{Ay(2Pdy-ydP)}{(Byy+C+2Dx+Exx)^2}-2ydy\frac{dQ}{dx},$$

[note that dP is a function of x and hence meanwhile is held constant; if the integral found is subsequently differentiated w.r.t. x with y held constant, then the term in dP must remain, while terms involving y only can be ignored; and likewise with the integration and differentiation performed in the order x and y.]

the integral of which formula, if indeed it should admit to integration, from the variability of y there will be

$$\frac{-AP}{B(Byy+C+2Dx+Exx)}-yy\frac{dQ}{dx}=V.$$

Hence with y assumed constant, it is necessary that

$$-\frac{AdP(Byy+C+2Dx+Exx)+2APdx(D+Ex)}{B(Byy+C+2Dx+Exx)^2}-yy\frac{ddQ}{dx}=\frac{-AyydP}{(Byy+C+2Dx+Exx)^2}$$

to which there is satisfaction, if

$$\frac{ddQ}{dx^2}=0 \text{ and} -dP(C+2Dx+Exx)+2Pdx(D+Ex)=0,$$

it is evident that these two conditions are consistent to each other. Moreover the latter gives

$$\frac{dP}{P}=\frac{2Ddx+2Exdx}{C+2Dx+Exx} \text{ and thus } P=C+2Dx+Exx,$$

from which there becomes

$$Q=\frac{dP}{2dx}=-D-Ex, \text{ hence } \frac{dQ}{dx}=-E \text{ and } \frac{ddQ}{dx^2}=0.$$

Hence the multiplier sought is $2dy(C+2Dx+Exx)-2ydx(D+Ex)$ and hence there is obtained the integral

$$\frac{dy^2}{dx^2}(C+2Dx+Exx)-\frac{2ydy}{dx}(D+Ex)-\frac{A(C+2Dx+Exx)}{B(Byy+C+2Dx+Exx)}+Eyy=\text{Const.},$$

[From :

$$2ddy\big(2dy(C+2Dx+Exx)-2ydx(D+Ex)\big)$$
$$+\frac{2Aydx^2\big(2dy(C+2Dx+Exx)-2ydx(D+Ex)\big)}{(Byy+C+2Dx+Exx)^2}=0]$$

or by adding $\frac{A}{B}$ to both sides :

$$\frac{dy^2}{dx^2}(C+2Dx+Exx)-\frac{2ydy}{dx}(D+Ex)-\frac{Ayy}{Byy+C+2Dx+Exx}+Eyy=\text{Const.}$$

COROLLARY 1

907. Hence this equation $ddy+\frac{aaydx^2}{(yy+xx)^2}=0,$ where
$A=aa,\ B=1,\ C=0, D=\ 0$ and $E=1,$ is rendered integrable by the multiplier $2xxdy-2yxdx$ and the integral of this will be

$$\frac{xxdy^2}{dx^2}-\frac{2xydy}{dx}+yy+\frac{aayy}{yy+xx}=bb\ .$$

COROLLARY 2

908. If here there is put $y = ux$, on account of $dy = udx + xdu$ we will have

$$\left.\begin{array}{l} xxuu+\frac{2ux^3du}{dx}+\frac{x^4du^2}{dx^2}+\frac{aauu}{1+uu} \\ -2xxuu-\frac{2ux^3du}{dx} \\ +\ xxuu \end{array}\right\}=bb$$

or

$$\frac{x^4du^2}{dx^2}=\frac{bb+(bb-aa)uu}{1+uu},\quad \text{hence}\quad \frac{dx}{xx}=\frac{du\sqrt{(1+uu)}}{\sqrt{(bb+(bb-aa)uu)}}$$

from which both x and y can be determined by u.

COROLLARY 3

909. In a similar manner also the integration can be completed in general. For the sake of brevity let

$$C+2Dx+Exx=Bzz;$$

there will be $D+Ex=\frac{Bzdz}{dx}$ and our equation becomes

$$\frac{Bzzdy^2}{dx^2}-\frac{2Byzdydz}{dx^2}+Eyy+\frac{Ayy}{B(yy+zz)}=\frac{K}{B},$$

which on putting $y=uz$ changes into

$$\frac{Bz^4du^2}{dx^2}-\frac{Buuzzdz^2}{dx^2}+Euuzz+\frac{Auu}{B(1+uu)}=\frac{K}{B}.$$

But $\frac{zzdz^2}{dx^2}=\frac{(D+Ex)^2}{BB}$, from which there arises

$$\frac{Bz^4du^2}{dx^2}+\frac{CE-DD}{B}uu=\frac{K+(K-A)uu}{B(1+uu)}$$

or

$$\frac{BBz^4du^2}{dx^2}=\frac{K+(K-A+DD-CE)uu+(DD-CE)u^4}{1+uu},$$

thus so that with the value of z restored :

$$\frac{dx}{C+2Dx+Exx}=\frac{du\sqrt{(1+uu)}}{\sqrt{\left(K+(K-A+DD-CE)uu+(DD-CE)u^4\right)}},$$

and thus x is defined by u and thence also

$$y=uz=u\sqrt{\frac{C+2Dx+Exx}{B}}.$$

SCHOLIUM

910. Hence a substitution is apparent, by which both the proposed second order differential equation as well as the multiplier must be reduced to a more convenient form. For by putting as an abbreviation $C+2Dx+Exx=Bzz$ our equation

$$ddy+\frac{Aydx^2}{B^2(yy+zz)^2}=0$$

[Note : the *O. O.* edition incorrectly has B^3 in this equation, although it is correct in the first edition.] with the aid of the substitution $y=uz$ is transformed into

$$zddu+2dzdu+uddz+\frac{Audx^2}{BBz^3(1+uu)^2}=0,$$

the multiplier of which is $2B(zzdy-yzdz)$ or $2Bz^3du$ or simply z^3du.

But since there shall be $dz=\frac{dx(D+Ex)}{Bz}$ there will be

$$ddz=\frac{Edx^2}{Bz}-\frac{dxdz(D+Ex)}{Bzz}=\frac{Edx^2}{Bz}-\frac{dx^2(D+Ex)^2}{BBz^3}=\frac{(CE-DD)dx^2}{BBz^3},$$

thus so that there shall be $z^3ddz=\frac{CE-DD}{BB}dx^2$, from which our equation multiplied by z^3du adopts this form

$$z^4duddu+2z^3dzdu^2+\frac{CE-DD}{BB}ududx^2+\frac{Audx^2}{BB(1+uu)^2}=0$$

evidently integrable with the integral arising [, the first term by parts] :

$$\frac{1}{2}z^4du^2+\frac{CE-DD}{2BB}uudx^2-\frac{Adx^2}{2BB(1+uu)}=\frac{1}{2}\text{Const.}dx^2,$$

of which so it is that a new integration soon appears on account of the function *z* of *x*, since there shall be

$$zzdu = dx\sqrt{\left(\text{Const.}+\tfrac{DD-CE}{BB}uu+\tfrac{A}{BB(1+uu)}\right)},$$

where the variables *u* and *x* can be separated at once.

Otherwise here it may be noted that the function assumed for *z* satisfies the equation $z^3ddz=\alpha dx^2$, since an account of this still shall not be evident. But on multiplying this equation by $\frac{2dz}{z^2}$ there appears $2dzddz=\frac{2\alpha dx^2dz}{z^3}$, the integral of which shall be $dz^2=\beta dx^2-\frac{\alpha dx^2}{zz}$ or $dx=\frac{zdz}{\sqrt{(\beta zz-\alpha)}}$, from which again there becomes $\beta x+\gamma=\sqrt{(\beta zz-\alpha)}$ and thus $\beta zz=\alpha+\gamma\gamma+2\beta\gamma x+\beta\beta xx$, which is itself our form.
[β and γ are of course constants of the integration.]

PROBLEM 113

911. *With the element dx assumed constant to find a more general form of second order equations, which are rendered integrable with the aid of a multiplier of this kind* $Mydx+Ndy$.

SOLUTION

Because the multiplier with the aid of the substitution $y=Ru$ can be changed into the most simple form *Sdu*, by this substitution the second order differential equation itself adopts this form

$$ddu+Pdxdu+\frac{Udx^2}{S}=0,$$

the latter part of which multiplied by *Sdu* is integrable at once, if indeed *U* should denote some function of *u*, while both *R*, *S* and *P* are functions of *x*. Therefore since the equation

$$Sduddu+PSdxdu^2+Udx^2du=0$$

must be integrable, on putting in place the integral

$$\tfrac{1}{2}Sdu^2+dx^2\int Udu=\tfrac{1}{2}Cdx^2$$

it is necessary that

$$\tfrac{1}{2}dSdu^2=PSdxdu^2 \quad \text{or} \quad Pdx=\frac{dS}{2S}.$$

[Note the device used by Euler on many occasions, whereby two integrals are made to cancel, the first coming from an integration by parts of the first term of the proposed diff. eq., as in the above case $\int Sduddu=\frac{1}{2}du^2S-\frac{1}{2}\int du^2dS$, while the latter term is present

already, here $\int PSdxdu^2$, thus effecting a great economy by replacing the term in P with an integrable term in S.]
On account of which this general form

$$ddu + \frac{dSdu}{2S} + \frac{Udx^2}{S} = 0$$

multiplied by Sdu gives the integral

$$Sdu^2 = dx^2\left(C - 2\int Udu\right),$$

which on integrating anew presents

$$\int \frac{dx}{\sqrt{S}} = \int \frac{du}{\left(C - 2\int Udu\right)}.$$

Therefore since these are evident, we can return to the more involved forms on putting $u = \frac{y}{R}$, since thus now there shall be $U = \text{funct.}\frac{y}{R}$. Now indeed there is

$$du = \frac{dy}{R} - \frac{ydR}{RR} \text{ and } ddu = \frac{ddy}{R} - \frac{2dRdy}{RR} - \frac{yddR}{RR} + \frac{2ydR^2}{R^3},$$

from which our equation [$ddu + \frac{dSdu}{2S} + \frac{Udx^2}{S} = 0$] becomes

$$\frac{ddy}{R} - \frac{2dRdy}{RR} - \frac{yddR}{RR} + \frac{2ydR^2}{R^3} + \frac{Udx^2}{S} + \frac{dSdy}{2RS} - \frac{ydRdS}{2RRS} = 0,$$

which multiplied by [$Sdu =$] $\frac{S}{RR}\left(Rdy - ydR\right)$ is rendered integrable.

Therefore in order that we may agree with the above proposed form, we may put in place $S = \alpha R^4$, and the equation

$$\frac{ddy}{R} - \frac{yddR}{RR} + \frac{Udx^2}{\alpha R^4} = 0$$

multiplied by $\alpha RR\left(Rdy - ydR\right)$ is rendered integrable or this equation

$$Rddy - yddR + \frac{dx^2}{RR} f\!:\!\frac{y}{R} = 0$$

multiplied by $Rdy - ydR$ becomes integrable.
[Note that Euler has used two explicit ways of representing functions in this section : $U = \text{funct.}\frac{y}{R}$ and $f\!:\!\frac{y}{R}$; in both cases for the argument $\frac{y}{R}$, while usually he just states the function property implicitly. The reader may find the concluding remarks in the next Scholium instructive concerning the method introduced here.]

In order that a way can be come upon to an obscure integration, there is put $f:\frac{y}{R}=\frac{\alpha y}{R}+V$, so that V is a homogeneous function of no dimensions of y and R, and there becomes $yddR=\frac{\alpha ydx^2}{R^3}$, so that there becomes, [substituting in $\frac{ddy}{R}-\frac{yddR}{RR}+\frac{Udx^2}{\alpha R^4}=0$ above ;]

$$Rddy+\frac{Vdx^2}{RR}=0 \text{ or } ddy+\frac{Vdx^2}{R^3}=0,$$

which is rendered integrable by the multiplier $R(Rdy-ydR)$. But since there shall be $ddR=\frac{\alpha dx^2}{R^3}$, there will be, as we have seen above [§ 910], $R=\sqrt{(\alpha+2\beta x+\gamma xx)}$, from which, while V is a homogeneous function of no dimensions of y and $R=\sqrt{(\alpha+2\beta x+\gamma xx)}$, the equation

$$ddy+\frac{Vdx^2}{(\alpha+2\beta x+\gamma xx)^{\frac{3}{2}}}=0$$

emerges integrable, with the aid of the multiplier $(\alpha+2\beta x+\gamma xx)dy-(\beta+\gamma x)ydx$.

COROLLARIUM 1

912. Moreover on putting $R=\sqrt{(\alpha+\beta x+\gamma xx)}$ our equation multiplied by $RRdy-RydR$ becomes

$$RRdyddy-RydRddy+\frac{Vdx^2(Rdy-ydR)}{RR}=0,$$

the integral of which will be

$$\tfrac{1}{2}RRdy^2-RydRdy+\int ydy\left(RddR+dR^2\right)+dx^2\int Vd.\tfrac{y}{R}=\text{Const}.dx^2$$

where there is

$$RddR+dR^2=d.RdR=d.(\beta+\gamma x)dx=\gamma dx^2,$$

and thus the integral is

$$RRdy^2-2RydRdy+\gamma yydx^2+2dx^2\int Vd.\tfrac{y}{R}=\text{Const}.dx^2.$$

COROLLARY 2

913. Because V is a function of $\frac{y}{R}$, the integral of the formula $\int Vd.\frac{y}{R}$ may be considered. Now for the integration of the final part on putting $y = Ru$ and $\int Vdu = U$ there will be had

$$R^4du^2 - RRuudR^2 + \gamma RRuudx^2 + 2Udx^2 = Gdx^2$$

or

$$R^4du^2 = dx^2\left(G - 2U + (\beta\beta - \alpha\gamma)uu\right)$$

and hence

$$dx\frac{dx}{\alpha + 2\beta x + \gamma xx} = \frac{du}{\sqrt{\left(G - 2U + (\beta\beta - \alpha\gamma)uu\right)}}$$

and again $y = u\sqrt{(\alpha + 2\beta x + \gamma xx)}$.

SCHOLIUM

914. Hence this equation $ddy + \frac{Vdx^2}{R^3} = 0$, with $R = \sqrt{(\alpha + 2\beta x + \gamma xx)}$ arising, extends much more widely than that which we have treated in the previous problem, since here it is permitted therefore to take for V any homogeneous function of zero dimensions of y and R. If indeed there is taken $V = \frac{AR^3y}{(myy+RR)^2}$, the first equation itself arises. Otherwise from the method, from which we have elicited that equation, it may appear that it is induced by the restriction to this hidden form, since that equation from which it has arisen,

$$Rddy - yddR + \frac{dx^2}{RR}f:\frac{y}{R} = 0$$

clearly admits to integration, if it is multiplied by $Rdy - ydR$. For there is

$$Rddy - yddR = d.(Rdy - ydR) \text{ and } \frac{Rdy - ydR}{RR} = d.\frac{y}{R},$$

with which multiplication performed we shall have

$$(Rdy - ydR)d.(Rdy - ydR) + dx^2f:\frac{y}{R}d.\frac{y}{R} = 0,$$

and each member of which equation is integrable by itself. But in the equation thence elicited the integrability is seen less clearly ; the integration is much more obscure in the following equations.

PROBLEMA 114

915. *With the element dx constant the integration of this equation*

$$yyddy + ydy^2 + Axdx^2 = 0$$

that with the aid of a rendering multiplier becomes completely integrable.

SOLUTION

Here a multiplier of this form $Ldy + Mdx$ is tried in vain; hence this form may be tried

$$3Ldy^2 + 2Mdxdy + Ndx^2$$

and an extended integral is put in place [Here the word *integral* means the integration to be performed, rather than the narrower sense we now use for the word. Need it be mentioned that this integral is thus assumed beforehand.]

$$Lyydy^3 + Myydxdy^2 + Nyydx^2dy + Vdx^3 = Cdx^3,$$

the differentiation of which leads to this equation :

$$\begin{aligned} dx^3dV = 3Lydy^4 &+ 2Mydxdy^3 + Nydx^2dy^2 + 2AMxdx^3dy + ANxdx^4 \\ -2Lydy^4 &- yydxdy^3\left(\tfrac{dL}{dx}\right) + 3ALxdx^2dy^2 - yydx^3dy\left(\tfrac{dN}{dx}\right) \\ - yydy^4\left(\tfrac{dL}{dy}\right) &- 2Mydxdy^3 - yydx^2dy^2\left(\tfrac{dM}{dx}\right) \\ &- yydxdy^3\left(\tfrac{dM}{dy}\right) - 2Nydx^2dy^2 \\ &- yydx^2dy^2\left(\tfrac{dN}{dy}\right) \end{aligned}$$

which formula in order that it allows integration, the members must vanish which contain dy^4, dy^3 and dy^2 ; from the first it is deduced that

$$L - y\left(\tfrac{dL}{dy}\right) = 0,$$

where $\left(\frac{dL}{dy}\right)$ is produced from the differentiation of L with x put constant. Therefore x may be considered as a constant quantity and there will be $\frac{dL}{L}=\frac{dy}{y}$ and thus $L=yf:x$; but we ignore this function of x in place of this we take unity, so that there shall be $L=y$ and $\left(\frac{dL}{dx}\right)=0$. Therefore with the second there must be $\left(\frac{dM}{dy}\right)=0$. Hence we take $M=0$, even if M is able to denote some function of x, since we will see that the task can be completed in this way. And thus for the third we will consider

$$-Ny+3Axy-yy\left(\frac{dN}{dy}\right)=0;$$

hence with x assumed constant there will be $3Axdy=Ndy+ydN$ and thus $Ny=3Axy$ or $N=3Ax$, where in turn we disregard the function of x, which may be introduced in place of a constant. Hence since at this stage we have found $L=y$, $M=0$ and $N=3Ax$, there will be $dV=-3Ayydy+3AAxxdx$; which formula since it is integrable by itself, evidently $V=-Ay^3+AAx^3$, the multiplier rendering our equation integrable will be

$$3ydy^2+3Axdx^2$$

and the extended integral will be considered

$$y^3dy^3+3Axyydx^2dy-Ay^3dx^3+AAx^3dx^3=Cdx^3,$$

which is a complete integral on account of the constant C.

COROLLARY 1

916. The first member of this integral can be resolved conveniently into three factors. If the factors $(z-\alpha)(z-\beta)(z-\gamma)$ of the formula z^3-A are put in place, so that there shall be

$$\alpha=\sqrt[3]{A},\quad \beta=\tfrac{-1+\sqrt{-3}}{2}\sqrt[3]{A}\quad\text{and}\quad \gamma=\tfrac{-1-\sqrt{-3}}{2}\sqrt[3]{A},$$

the integral found will be

$$\left(\frac{ydy}{dx}-\alpha y+\alpha\alpha x\right)\left(\frac{ydy}{dx}-\beta y+\beta\beta x\right)\left(\frac{ydy}{dx}-\gamma y+\gamma\gamma x\right)=C$$

with there arising

$$\alpha+\beta+\gamma=0,\alpha\beta+\alpha\gamma+\beta\gamma=0,\text{and } \alpha\beta\gamma=1;$$

for on putting $\frac{ydy}{dx} = z$ this form may be considered :

$$z^3 + 3Axyz - Ay^3 + AAx^3;$$

for which if there is put the factor $z - p - q$, it becomes $z^3 - 3pqz - p^3 - q^3 = 0$ and thus

$$p = y\sqrt[3]{A} \text{ et } q = -x\sqrt[3]{A^2}.$$

COROLLARY 2

917. Therefore on taking the constant $C = 0$ three particular integrals may be obtained :

$$ydy - \alpha ydx + \alpha\alpha xdx = 0$$

and on writing β and γ in place of α :

$$ydy - \beta ydx + \beta\beta xdx = 0 \text{ and } ydy - \gamma ydx + \gamma\gamma xdx = 0,$$

which on putting $y = ux$ give $\frac{dx}{x} = \frac{-udu}{uu - \alpha u + \alpha\alpha}$ and on integrating again :

$$lx = l\frac{\alpha}{\sqrt{(\alpha\alpha - \alpha u + uu)}} - \frac{1}{\sqrt{3}}\text{Ang. tang.}\frac{u\sqrt{3}}{2\alpha - u} + \text{Const.}$$

SCHOLIUM 1

918. But the differential equation of the first order found is integrated anew with difficulty. Indeed it is possible to be set free from the powers of the differentials on putting $dy = pdx$ and $y = ux$, from which there is made $\frac{dx}{x} = \frac{du}{p-u}$; for there is produced

$$x^3\left(u^3p^3 + 3Auup - Au^3 + AA\right) = C,$$

which with the logarithms taken differentiated gives

$$\frac{dx}{x} + \frac{uudp(upp + A) + udu\left(up^3 + 2Ap - Au\right)}{u^3p^3 + 3Auup - Au^3 + AA} = 0,$$

which by writing $\frac{du}{p-u}$ in place of $\frac{dx}{x}$ turns into

$$du\left(upp + A\right)^2 + uu\left(p - u\right)dp\left(upp + A\right) = 0,$$

and on dividing by $upp + A$ there arises

$$Adu + uppdu + puudp - u^3dp = 0,$$

which on putting $p=\frac{q}{u}$ is made a little simpler, clearly

$$Adu+qdq+qudu-uudq=0;$$

but for that on putting $A=m^3$ although it satisfies particularly $q=mu-mm$, yet thence the complete integral can scarcely be seen to be elicited. Otherwise this same equation between *p* and *u* can be immediately elicited from the second order differential equation proposed, since in that the two variables *x* and *y* everywhere constitute a number of the same dimensions. For on putting $dy=pdx$ and $y=ux$ that becomes

$$uuxdp+uppdx+Adx=0 \quad \text{or} \quad \frac{dx}{x}=\frac{-uudp}{A+upp}=\frac{du}{p-u},$$

which is the preceding equation itself.

SCHOLIUM 2

919. Yet meanwhile the proposed equation can be integrated completely and thus also these, which we have elicited from that. But this in short comes about from the singular reason as by that being raised to a differential equation of the third order.
Indeed since there shall be

$$yd.\frac{ydy}{dx}+Axdx=0,$$

there is put in place $\frac{dx}{y}=dv$, so that there becomes

$$yd.\frac{dy}{dv}+Axdx=0 \text{ or } d.\frac{dy}{dv}+Axdv=0,$$

which on taking the element *dv* constant on differentiating anew gives

$$\frac{d^3y}{dv}+Adxdv=0 \text{ or } d^3y+Aydv^3=0$$

which form thus has been prepared [§1117], so that if $y=P,\ y=Q,\ y=R$ satisfy this particularly, also there shall be satisfied $y=DP+EQ+FR$. Now truly $y=e^{-\alpha v}$ satisfies that equation, if there should be $\alpha^3=A$; therefore since in Corollary I the three letters $\alpha,\ \beta,\ \gamma$ shall be provided with the same condition, the complete integral will be considered

$$y=De^{-\alpha v}+Ee^{-\beta v}+Fe^{-\gamma v},$$

from which on account of $Ax=-\frac{ddy}{dv^2}$ there will be

$$x=\frac{-D\alpha\alpha e^{-\alpha v}-E\beta\beta e^{-\beta v}-F\gamma\gamma e^{-\gamma v}}{A}$$

or with the constants changed on account of $A=\alpha^3=\beta^3=\gamma^3$

$$x=\mathfrak{A}e^{-\alpha v}+\mathfrak{B}e^{-\beta v}+\mathfrak{C}e^{-\gamma v}, \quad y=-\mathfrak{A}\alpha e^{-\alpha v}-\mathfrak{B}\beta e^{-\beta v}-\mathfrak{C}\gamma e^{-\gamma v}.$$

Hence therefore the complete integral of the equation

$$Adu+qdq+qudu-uudq=0$$

is contained in this formula :

$$u=\frac{-\mathfrak{A}\alpha e^{-\alpha v}-\mathfrak{B}\beta e^{-\beta v}-\mathfrak{C}\gamma e^{-\gamma v}}{\mathfrak{A}e^{-\alpha v}+\mathfrak{B}e^{-\beta v}+\mathfrak{C}e^{-\gamma v}}, \quad \text{and} \quad q=\frac{\mathfrak{A}\alpha\alpha e^{-\alpha v}+\mathfrak{B}\beta\beta e^{-\beta v}+\mathfrak{C}\gamma\gamma e^{-\gamma v}}{\mathfrak{A}e^{-\alpha v}+\mathfrak{B}e^{-\beta v}+\mathfrak{C}e^{-\gamma v}}$$

on account of $q=pu=\frac{ydy}{xdx}=\frac{dy}{xdv}$ which is a conspicuous example of an integration by the direct method scarcely able to be perfected.

PROBLEM 115

920. *With the element dx assumed constant if this equation is proposed*

$$2y^3ddy+yydy^2+Xdx^2=0$$

with $X=\alpha+\beta x+\gamma xx$ *arising, to find the multiplier, which renders that integrable.*

SOLUTION

Here multipliers of the form

$$Ldy+Mdx \quad \text{et} \quad Ldy^2+Mdxdy+Ndx^2;$$

are tried in vain, and hence we take a multiplier of this form :

$$2Ldy^3+Mdx^2dy+Ndx^3$$

and the integral is put in place

$$Ly^3dy^4+My^3dx^2dy^2+2Ny^3dx^3dy+Sdx^4=0,$$

from which by differentiation there is deduced

$$\begin{aligned} dx^4 dS = 2Lyydy^5 &+ Myydx^2dy^3 + Nyydx^3dy^2 + MXdx^4dy + NXdx^5 \\ -3Lyydy^5 &+ 2LXdx^2dy^3 - y^3dx^3dy^2\left(\tfrac{dM}{dx}\right) - 2y^3dx^4dy\left(\tfrac{dN}{dx}\right) \\ -y^3dy^5\left(\tfrac{dL}{dy}\right) &- 3Myydx^2dy^3 - 6Nyydx^3dy^2 \\ &- y^3dx^2dy^3\left(\tfrac{dM}{dy}\right) - 2y^3dx^3dy^2\left(\tfrac{dN}{dy}\right) \end{aligned}$$

where we have taken L to be a function of y only, so that hence the terms containing dy^5 are cancelled, and there will be

$$-L - \frac{ydL}{dy} = 0 \text{ and } L = \frac{1}{y}.$$

Then for the destruction of the terms affecting dy^3 there will be

$$-2Myy + \frac{2X}{y} - y^3\left(\frac{dM}{dy}\right) = 0$$

and on taking x constant

$$dM + \frac{2Mdy}{y} = \frac{2Xdy}{y^4},$$

which multiplied by yy and integrated gives

$$Myy = P - \frac{2X}{y} \text{ and } M = \frac{P}{yy} - \frac{2X}{y^3}$$

with P denoting some function of x. Now towards removing the terms dy^2 there will be

$$-5Nyy - y\frac{dP}{dx} + \frac{2dX}{dx} - 2y^3\left(\frac{dN}{dy}\right) = 0$$

and on taking x constant

$$2y^3dN + 5Nyydy = \frac{2dX}{dx}dy - \frac{dP}{dx}ydy,$$

which divided by $\sqrt{y}$ and integrated gives

$$2Ny^{\frac{5}{2}} = \frac{4dX}{dx}\sqrt{y} - \frac{2dP}{3dx}y\sqrt{y}$$

with a function of x to be added on ignored, because the irrational $\sqrt{y}$ does not enter into the calculation. Hence there will be

$$N = \frac{2dX}{yydx} - \frac{dP}{3ydx}$$

and therefore

$$dS = dy\left(\frac{PX}{yy} - \frac{2XX}{y^3} - \frac{4yddX}{dx^2} + \frac{2yyddP}{3dx^2}\right) + \frac{2XdX}{yy} - \frac{XdP}{3y},$$

from which there becomes on integrating [, noting that the two last terms cancel with others produced from the first bracket on integrating those terms by parts]

$$S = \frac{XX}{yy} - \frac{PX}{y} - \frac{2yyddX}{dx^2} + \frac{2y^3ddP}{9dx^2} + \int\left(\frac{PdX}{y} + \frac{2XdP}{3y} + \frac{2yyd^3X}{dx^2} - \frac{2y^3d^3P}{9dx^2}\right),$$

which may be expressed finitely, if $P = 0$, since on account of $X = \alpha + \beta x + \gamma xx$ there shall be $d^3X = 0$.
On which account we may consider

$$L = \frac{1}{y}M = -\frac{2X}{y^3},\ N = \frac{2dX}{yydx} \quad \text{and } S = \frac{XX}{yy} - \frac{2yyddX}{dx^2} + \text{Const.},$$

from which the equation of the integral is

$$y^2dy^4 - 2Xdx^2dy^2 + 4yXdx^2dy + \frac{XXdx^4}{yy} - 2yydx^2ddX = Cdx^4.$$

Hence the proposed equation

$$2y^3ddy + yydy^2 + dx^2(\alpha + \beta x + \gamma xx) = 0$$

is rendered integrable multiplied by

$$\frac{2dy^3}{y} - \frac{2(\alpha+\beta x+\gamma xx)dx^2dy}{y^3} + \frac{2dx^3(\beta+2\beta x)}{yy};$$

then indeed there is the integral

$$y^2dy^4 - 2dx^2dy^2(\alpha + \beta x + \gamma xx) + 4ydx^3dy(\beta + 2\gamma x) - 4\gamma yydx^4 + \frac{(\alpha+\beta x+\gamma xx)^2dx^4}{yy} = Cdx^4$$

or

$$\left(yydy^2 - (\alpha + \beta x + \gamma xx)dx^2\right)^2 + 4y^3dx^3dy(\beta + 2\gamma x) - 4\gamma y^4dx^4 = Cyydx^4.$$

SCHOLIUM 1

921. This integral thus is complex, so that scarcely should it be considered possible for it to find by another method, also truly thus it has been prepared, so that no method is apparent by which it can be integrated again, from which the first integration may be judged to have been brought little of gain. But just as in the preceding problem we have drawn up the complete integral from another source, thus here in a similar manner it is allowed to elicit the integral, which there was more worthy of note, since the equation proposed would have a lot of difficulty in the solution itself considered. Clearly we can put likewise $dx = ydv$, and since there shall be

$$ddy = dxd.\tfrac{dy}{dx} = ydvd.\tfrac{dy}{ydv},$$

there will be on now assuming the element *dv* constant

$$ddy = ydv\left(\tfrac{ddy}{ydv} - \tfrac{dy^2}{yydv}\right) = ddy - \tfrac{dy^2}{y}.$$

Hence our equation [$2y^3ddy + yydy^2 + dx^2(\alpha + \beta x + \gamma xx) = 0$] adopts this form

$$2y^3ddy - yydy^2 + yydv^2(\alpha + \beta x + yxx) = 0$$

or

$$2yddy - dy^2 + dv^2(\alpha + \beta x + \gamma xx) = 0,$$

which differentiated anew gives

$$2yd^3y + ydv^3(\beta + 2\gamma x) = 0 \text{ or } 2d^3y + dv^3(\beta + 2\gamma x) = 0;$$

it is differentiated again and there is produced

$$2d^4y + 2\gamma ydv^4 = 0 \quad \text{or} \quad d^4y + y\gamma dv^4 = 0;$$

as the equation if resolved by some other method and the value of this *y* may be allowed to be expressed in terms of *v*, then there will be $x = \int ydv$ or without integration $x = -\frac{d^3y}{\gamma dv^2} - \frac{\beta}{2\gamma}$. But it is evident that $y = e^{\lambda v}$ satisfies this differential equation of the fourth order, if there should be $\lambda^4 + \gamma = 0$. Hence we may put $\gamma = -n^4$ and the four values of λ may be considered $\lambda = \pm n$ and $\lambda = \pm n\sqrt{-1}$, from which [§ 1125, §1128] the complete integral is $y = Ae^{nv} + Be^{-nv} + C\sin.(nv + \zeta)$ and hence

$$x = \tfrac{A}{n}e^{nv} - \tfrac{B}{n}e^{-nv} - \tfrac{C}{n}\cos.(nv + \zeta) + \tfrac{\beta}{2n^4},$$

which hence also satisfy the values of the equation proposed between *x* and *y*, provided the constants A, B, C and ζ thus are taken depending on these, so that they also agree with the quantity α. Certainly with these values substituted there must become

$$\alpha + \beta x - n^4 xx + \tfrac{2yddy - dy^2}{dv^2} = 0,$$

[from $2yddy - dy^2 + dv^2(\alpha + \beta x + \gamma xx) = 0$, as $\gamma = -n^4$]

where it is sufficient to examine the constant terms only [*i. e.* independent of *v*] , to which these must be added those which contain the square of the sine or cosine of the angle $nv + \zeta$, obviously from the combination of which a constant quantity emerges. Hence since there shall be

$$2y = 2Ae^{nv} + 2Be^{-nv} + 2C\sin.(nv + \zeta),$$
$$\tfrac{ddy}{dv^2} = nnAe^{nv} + nnBe^{-nv} - nnC\sin.(nv + \zeta),$$
$$\tfrac{dy}{dv} = nAe^{nv} - nBe^{-nv} + nC\cos.(nv + \zeta),$$
$$x = \tfrac{A}{n}e^{nv} - \tfrac{B}{n}e^{-nv} - \tfrac{C}{n}\cos.(nv + \zeta) + \tfrac{\beta}{2n^4},$$

there will be from the mentioned terms taken:

$$\beta x = \tfrac{\beta\beta}{2n^4},$$
$$-n^4 xx = 2ABnn - nnCCcos.^2(nv + \zeta) - \tfrac{\beta\beta}{4n^4},$$
$$\tfrac{2yddy}{dv^2} = 4ABnn - 2nnCCsin.^2(nv + \zeta),$$
$$-\tfrac{dy^2}{dv^2} = 2ABnn - nnCC\cos.^2(nv + \zeta),$$

hence [from $\alpha + \beta x - n^4 xx + \frac{2yddy - dy^2}{dv^2} = 0,$]

$$\alpha + 8nnAB - 2nnCC + \tfrac{\beta\beta}{4n^4} = 0$$

and thus

$$C = \sqrt{\left(\tfrac{\alpha}{2nn} + \tfrac{\beta\beta}{8n^6} + 4AB\right)}$$

or

$$\alpha = 2nn(CC-4AB)-\tfrac{\beta\beta}{4n^4}$$

and

$$a+\beta x+\gamma xx = 2nn(CC-4AB)-\left(\tfrac{\beta}{2nn}-nnx\right)^2.$$

Hence there remains three indeterminate constants A, B and ζ, thus so that there shall be no doubt, whether or not the formulas given for x and y show the complete integral.

SCHOLIUM 2

922. The second order differential equations, which we have treated in these two problems, can be reduced to a similar form. For in the first place

$$y\left(yddy+dy^2\right)+Xdx^2=0$$

with $X = Ax$ or $X = \alpha+\beta x$ arising, if there is put $ydy=\frac{1}{2}dz$ or $yy=z$, adopts this form

$$\tfrac{1}{2}ddz\sqrt{z}+Xdx^2=0,$$

which with the aid of the multiplier $\frac{3dz^2}{4\sqrt{z}}+3Xdx^2$ is rendered integrable. Now the other equation

$$yy\left(2yddy+dy^2\right)+Xdx^2=0$$

with $X=\alpha+\beta x+\gamma xx$ arising, on putting $y=z^{\frac{2}{3}}$ shall make

$$dy=\tfrac{2}{3}z^{-\frac{1}{3}}dz \text{ and } ddy=\tfrac{2}{3}z^{-\frac{1}{3}}ddz-\tfrac{2}{9}z^{-\frac{4}{3}}dz^2,$$

hence $2yddy+dy^2=\frac{4}{3}z^{\frac{1}{3}}ddz$ and thus the equation adopts this form

$$\tfrac{4}{3}z^{\frac{5}{3}}ddz+Xdx^2=0,$$

which is rendered integrable with the aid of the multiplier $\frac{16dz^3}{27z^{\frac{5}{3}}}-\frac{4Xdx^2dz}{3z^{\frac{7}{3}}}+\frac{2dXdx^2}{z^{\frac{4}{3}}}$. Hence we deduce $dz^2+4Xdx^2\sqrt{z}$ to be the multiplier for the equation $ddz+\frac{2Xdx^2}{\sqrt{z}}=0$, but for the equation $ddz+\frac{3Xdx^2}{4z\sqrt[3]{zz}}=0$ the multiplier shall be $dz^3-\frac{9Xdx^2dz}{4\sqrt[3]{zz}}+\frac{27}{8}dXdx^2\sqrt[3]{z}$, or under the one view :

for the equation	the multiplier will be

$ddz+\frac{2Xdx^2}{\sqrt{z}}=0$	$\frac{16dz^3}{27z^{\frac{5}{3}}}-\frac{4Xdx^2dz}{3z^{\frac{7}{3}}}+\frac{2dXdx^2}{z^{\frac{4}{3}}}$
$ddz+\frac{3Xdx^2}{4z\sqrt[3]{zz}}=0$	$dz^3-\frac{9Xdx^2}{4\sqrt[3]{zz}}+\frac{27}{8}dXdx^2\sqrt[3]{z}$

Otherwise these equations are most noteworthy, since they are able to be completed from higher order differential equations. Thus since from that equation [§ 1117], where dv is constant,

$$d^3y+Advddy+Bdv^2dy+Cydv^3=0$$

there shall be

$$y=\mathfrak{A}e^{\alpha v}+\mathfrak{B}e^{\beta v}+\mathfrak{C}e^{\gamma v},$$

if α, β, γ should be the roots of this equation

$$r^3+Ar^2+Br+C=0,$$

we may put $dv=\frac{dx}{y}$, and since there shall be

$$ddy=dvd.\frac{dy}{dx}=\frac{dx}{y}d.\frac{ydy}{dx}$$

and

$$d^3y=dv^2d.\frac{ddy}{dv^2}=dv^2d.\left(\frac{1}{dv}d.\frac{dy}{dv}\right)=\frac{dx^2}{yy}d.\left(\frac{y}{dx}d.\frac{ydy}{dx}\right),$$

if now we take dx constant, there will be

$$ddy=ddy+\frac{dy^2}{y}\ \text{[i. e. in the sense } ddy\rightarrow ddy+\frac{dy^2}{y}\text{ ,]}$$

and

$$d^3y=\frac{1}{yy}d.y\left(yddy+dy^2\right)=d^3y+\frac{4dyddy}{y}+\frac{dy^3}{yy}\ \text{[as above,]}$$

and hence on being multiplied by yy
[and inserting into $d^3y+Advddy+Bdv^2dy+Cydv^3=0$, with $dv=\frac{dx}{y}$ and noting $ddy\rightarrow ddy+\frac{dy^2}{y}$,]

$$yyd^3y+4ydyddy+dy^3+Adx\left(yddy+dy^2\right)+Bdx^2dy+Cdx^3=0,$$

which integrated gives

$$yyddy+ydy^2+Aydxdy+Bydx^2+\left(Cx+D\right)dx^2=0,$$

which hence can be integrated by the above.

PROBLEM 116

923. *To define the conditions of the functions P, Q, R and L, M, N, so that this second order differential equation*

$$ddy + Pdy^2 + Qdxdy + Rdx^2 = 0$$

is rendered integrable by the multiplier $3Ldy^2 + 2Mdxdy + Ndx^2$.

SOLUTION

With the multiplication performed the integration of the terms affected by *ddy* gives

$$Ldy^3 + Mdxdy^2 + Ndx^2dy,$$

whereby the integral is put [into the form]

$$Ldy^3 + Mdxdy^2 + Ndx^2dy + Vdx^3 = Cdx^3,$$

the differential of which must be equal to the proposed formula taken by the multiplier,

$$[i.\ e.\ \left(3Ldy^2 + 2Mdxdy + Ndx^2\right)\left(ddy + Pdy^2 + Qdxdy + Rdx^2\right) =$$
$$d.\left(Ldy^3 + Mdxdy^2 + Ndx^2dy + Vdx^3\right).]$$

from which there arises :

$$\begin{array}{llllll} dx^3dV = & 3LPdy^4 & +3LQdxdy^3 & +3LRdx^2dy^2 & & \\ & & +\ 2MP & +2MQ & +2MRdx^3dy & \\ & -\left(\frac{dL}{dy}\right) & -\left(\frac{dL}{dx}\right) & +\ NP & +\ NQ & +NRdx^4 \\ & & -\left(\frac{dM}{dy}\right) & -\left(\frac{dM}{dx}\right) & -\ \left(\frac{dN}{dx}\right) & \\ & & & -\left(\frac{dN}{dy}\right) & & \end{array}$$

[Note that the powers of the differentials are omitted.]
Hence here there is required to become :

$$3LP - \left(\tfrac{dL}{dy}\right) = 0, \quad 3LQ + 2MP - \left(\tfrac{dL}{dx}\right) - \left(\tfrac{dM}{dy}\right) = 0,$$
$$3LR + 2MQ + NP - \left(\tfrac{dM}{dx}\right) - \left(\tfrac{dN}{dy}\right) = 0.$$

Then the shall be

$$dV = \left(2MR + NQ - \left(\tfrac{dN}{dx}\right)\right)dy + NRdx\,,$$

which formula must be integrable. Moreover from these equations it is deduced

$$P = \tfrac{1}{3L}\left(\tfrac{dL}{dy}\right), \quad Q = \tfrac{1}{3L}\left(\tfrac{dL}{dx}\right) + \tfrac{1}{3L}\left(\tfrac{dM}{dy}\right) - \tfrac{2M}{9LL}\left(\tfrac{dL}{dy}\right),$$

and

$$R = \tfrac{1}{3L}\left(\tfrac{dM}{dx}\right) + \tfrac{1}{3L}\left(\tfrac{dN}{dy}\right) - \tfrac{N}{9LL}\left(\tfrac{dL}{dy}\right) - \tfrac{2M}{9LL}\left(\tfrac{dL}{dx}\right) - \tfrac{2M}{9LL}\left(\tfrac{dM}{dy}\right) + 4\tfrac{MM}{27L^3}\left(\tfrac{dL}{dy}\right).$$

COROLLARY 1

924. If L, *M* and *N* should be functions of *x* only, then there will be

$$P = 0, \;\; Q = \tfrac{dL}{3Ldx} \text{ and } R = \tfrac{dM}{3Ldx} - \tfrac{2MdL}{9LLdx},$$

hence

$$dV = \left(\tfrac{2MdM}{3Ldx} - \tfrac{4MMdL}{9LLdx} + \tfrac{NdL}{3Ldx} - \tfrac{dN}{dx}\right)dy + \tfrac{NdM}{3L} - \tfrac{2MNdL}{9LL}$$

and the coefficient of *dy* must be constant. Whereby on dividing by $L^{\frac{1}{3}}$ there will be considered

$$\tfrac{Cdx}{\sqrt[3]{L}} = \tfrac{2MdM}{3L\sqrt[3]{L}} - \tfrac{4MMdL}{9LL\sqrt[3]{L}} + \tfrac{NdL}{3L\sqrt[3]{L}} - \tfrac{dN}{\sqrt[3]{L}}$$

and on integrating

$$C\int \tfrac{dx}{\sqrt[3]{L}} = \tfrac{MM}{3L\sqrt[3]{L}} - \tfrac{N}{\sqrt[3]{L}} \quad \text{or} \quad N = \tfrac{MM}{3L} - CL^{\frac{1}{3}}\int \tfrac{dx}{\sqrt[3]{L}},$$

hence

$$V = Cy + \int\left(\tfrac{dM}{3L} - \tfrac{2MdL}{9LL}\right)\left(\tfrac{MM}{3L} - CL^{\frac{1}{3}}\int \tfrac{dx}{\sqrt[3]{L}}\right).$$

COROLLARY 2

925. Let $M = S\sqrt[3]{L^2}$; there will be $dM = dS\sqrt[3]{L^2} + \tfrac{2SdL}{3\sqrt[3]{L^2}}$ and

$$V = Cy + \tfrac{1}{3}\int \tfrac{dS}{\sqrt[3]{L}}\left(\tfrac{1}{3}SS\sqrt[3]{L} - CL^{\frac{1}{3}}\int \tfrac{dx}{\sqrt[3]{L}}\right)$$

or

$$V = Cy + \tfrac{1}{27}S^3 - \tfrac{1}{3}C\int dS\int \tfrac{dx}{\sqrt[3]{L}}\,,$$

then truly

$$N = \tfrac{1}{3}SS\sqrt[3]{L} - CL^{\frac{1}{3}}\int\tfrac{dx}{\sqrt[3]{L}} = \left(\tfrac{1}{3}SS - C\int\tfrac{dx}{\sqrt[3]{L}}\right)\sqrt[3]{L}$$

and

$$P = 0, \quad Q = \tfrac{dL}{3Ldx} \text{ and } R = \tfrac{dS}{3dx\sqrt[3]{L}}.$$

Whereby this equation

$$ddy + \tfrac{dLdy}{3L} + \tfrac{dSdx}{3\sqrt[3]{L}} = 0$$

is rendered integrable by the multiplier

$$3Ldy^2 + 2Sdxdy\sqrt[3]{L^2} + dx^2\left(\tfrac{1}{3}SS - C\int\tfrac{dx}{\sqrt[3]{L}}\right)\sqrt[3]{L}$$

[i. e. $3Ldy^2 + 2Sdxdy\sqrt[3]{L^2} + dx^2\left(\tfrac{1}{3}SS - C\int\tfrac{dx}{\sqrt[3]{L}}\right)\sqrt[3]{L}\left(ddy + \tfrac{dLdy}{3L} + \tfrac{dSdx}{3\sqrt[3]{L}}\right) = 0$]

and the integral is

$$Ldy^3 + Sdxdy^2\sqrt[3]{L^2} + dx^2dy\left(\tfrac{1}{3}SS - C\int\tfrac{dx}{\sqrt[3]{L}}\right)\sqrt[3]{L} + Cydx^3$$
$$+\tfrac{1}{27}S^3dx^3 - \tfrac{1}{3}Cdx^2\int dS\int\tfrac{dx}{\sqrt[3]{L}} = 0.$$

COROLLARY 3

926. Here whatsoever is assumed for the constant *C*, the same integral must appear. Hence if $C = 0$, the multiplier of the equation

$$ddy + \tfrac{dLdy}{3L} + \tfrac{dSdx}{3\sqrt[3]{L}} = 0$$

will be

$$3Ldy^2 + 2Sdxdy\sqrt[3]{L^2} + \tfrac{1}{3}SSdx^2\sqrt[3]{L}$$

and the integral will be

$$Ldy^3 + Sdxdy^2\sqrt[3]{L^2} + \tfrac{1}{3}SSdx^2dy\sqrt[3]{L} + \tfrac{1}{27}S^3dx^3 = Ddx^3$$

or

$$\left(dy\sqrt[3]{L} + \tfrac{1}{3}Sdx\right)^3 = Ddx^3.$$

SCHOLION 1

927. Also from the same conditions, if the functions *P, Q* and *R* should be given, the functions *L, M,* and *N* can be defined as far as the latter condition of integrability is allowed, just as if there shall be $P = \tfrac{n}{y}, Q = 0$ and *R* a function of *x* only, suppose $R = X$, so that this equation may be considered

$$ddy + \tfrac{ndy^2}{y} + Xdx^2 = 0;$$

the multiplier of which if there is taken $3Ldy^2 + 2Mdxdy + Ndx^2$, so that the integral shall be

$$Ldy^3 + Mdxdy^2 + Ndx^2dy + Vdx^3 = Cdx^3,$$

will be at first $\frac{3nL}{y} - \left(\frac{dL}{dy}\right) = 0$ and with x assumed constant $\frac{dL}{L} = \frac{3ndy}{y}$, hence $L = Sy^{3n}$ with S denoting some function of x. Then there is [in the second place],

$$\frac{2nM}{y} - y^{3n}\frac{dS}{dx} - \left(\frac{dM}{dy}\right) = 0$$

and with x assumed constant

$$dM - \frac{2nMdy}{y} + \frac{dS}{dx}y^{3n}dy = 0,$$

which multiplied by y^{-2n} and integrated gives

$$y^{-2n}M + \frac{dS}{(n+1)dx}y^{n+1} = T \text{, a function of } x.$$

Hence

$$M = Ty^{2n} - \frac{dS}{(n+1)dx}y^{3n+1}.$$

In the third place there must become

$$3SXy^{3n} + \frac{nN}{y} - \frac{dT}{dx}y^{2n} + \frac{ddS}{(n+1)dx^2}y^{3n+1} - \frac{dN}{dy} = 0,$$

from which with x assumed constant

$$dN - \frac{nNdy}{y} + \frac{dT}{dx}y^{2n}dy - \frac{ddS}{(n+1)dx^2}y^{3n+1}dy - 3SXy^{3n}dy = 0,$$

which multiplied by y^{-n} and integrated gives

$$y^{-n}N + \frac{dT}{(n+1)dx}y^{n+1} - \frac{ddS}{2(n+1)^2dx^2}y^{2n+2} - \frac{3SX}{2n+1}y^{2n+1} = U \text{, a function of } x,$$

or

$$N = Uy^n - \frac{dT}{(n+1)dx}y^{2n+1} + \frac{ddS}{2(n+1)^2dx^2}y^{3n+2} + \frac{3SX}{2n+1}y^{3n+1}.$$

Moreover from these there becomes

$$dV = dy\left\{\begin{array}{c} 2TXy^{2n} - \frac{2XdS}{(n+1)dx}y^{3n+1} - \frac{dU}{dx}y^{n} + \frac{ddT}{(n+1)dx^2}y^{2n+1} \\ - \frac{d^3S}{2(n+1)^2dx^3}y^{3n+2} - \frac{3d.SX}{(2n+1)dx}y^{3n+1} \end{array}\right\}$$
$$+Xdx\left(Uy^{n} - \frac{dT}{(n+1)dx}y^{2n+1} + \frac{ddS}{(n+1)^2dx^2}y^{3n+2} + \frac{3SX}{2n+1}y^{3n+1}\right);$$

which formula so that it is allowed to be integrated, is required to be

$$2y^{2n}d.TX - 2y^{3n+1}\frac{d.XdS}{(n+1)dx} - y^{n}\frac{ddU}{dx} + y^{2n+1}\frac{d^3T}{(n+1)dx^2}$$
$$-y^{3n+2}\frac{d^4S}{2(n+1)^2dx^3} - 3y^{3n+1}\frac{dd.SX}{(2n+1)dx} - nUXy^{n-1}dx + \frac{(2n+1)XdT}{(n+1)}y^{2n}$$
$$-\frac{(3n+2)XddS}{2(n+1)^2dx}y^{3n+1} - \frac{3(3n+1)SXXdx}{2n+1}y^{3n} = 0;$$

hence here the individual powers of y, as far as they are unequal, must cancel out separately. Whereby the power y^{n-1} gives $U = 0$, from which also the power y^{n} is reduced to zero. The power y^{2n} gives $(2n+2)TdX + (2n+2)XdT + (2n+1)XdT = 0$ or $X^{2n+2}T^{4n+3} = A$, but the power y^{2n+1} gives $d^3T = 0$ or $T = \alpha + \beta x + \gamma xx$. Now the power y^{3n} requires $S = 0$, unless there shall be $n = -\frac{1}{3}$, in which case also the powers y^{3n+1} and y^{3n+2} spontaneously vanish. Hence since there becomes $u = 0, S = 0$ and $T = \alpha + \beta x + \gamma xx$ and hence $X = B(\alpha + \beta x + \gamma xx)^{\frac{-4n-2}{2n+2}}$, this equation

$$ddy + \frac{ndy^2}{y} + B(\alpha + \beta x + \gamma xx)^{\frac{-4n-3}{2n+2}}dx^2 = 0$$

is rendered integrable with the help of the integrator

$$2(\alpha + \beta x + \gamma xx)y^{2n}dy - \frac{dx(\beta + 2\gamma x)}{n+1}y^{2n+1}.$$

SCHOLIUM 2

928. Though a great deal is absent, by which this method is fostered in a less satisfactory manner at this stage, yet the examples treated in this chapter abundantly indicate, how great an advance thence we are able to expect, by which the nourishment of this is considered to be recommended to the Geometers. Therefore because the methods, by which it is agreed to be used in the resolution of second order differential equations, have been set out clearly enough, we progress to the next chapter, where the integration of equations of this kind, as far as indeed that is conveniently possible, we will show by infinite series.

CAPUT VI

DE INTEGRATIONE ALIARUM AEQUATIONUM DIFFERENTIO-DIFFERENTIALIUM PER IDONEOS MULTIPLICATORES INSTITUENDA

PROBLEMA 112

906. *Posito elemento dx constante si proposita sit huiusmodi aequatio*

$$ddy+\frac{Aydx^2}{(Byy+C+2Dx+Exx)^2}=0,$$

invenire multiplicatorem, quo ea integrabilis reddatur.

SOLUTIO

Tentetur talis multiplicatoris forma $2Pdy+2Qydx$, ubi P et Q sint functiones ipsius x, et producti

$$2ddy(Pdy+Qydx)+\frac{2Aydx^2(Pdy+Qydx)}{(Byy+C+2Dx+Exx)^2}=0$$

integrale statuatur

$$Pdy^2+2Qydxdy+Vdx^2=\text{Const.}dx^2,$$

ubi V sit functio binas variabiles x et y complectens. Erit ergo facta aequalitate

$$dPdy^2+2ydxdQdy+2Qdxdy^2+dx^2dV-\frac{2Aydx^2(Pdy+Qydx)}{(Byy+C+2Dx+Exx)^2}=0,$$

quae per integrationem valorem ipsius V suppeditare nequit, nisi sit
$dP+2Qdx=0$; eritque tum

$$dV=\frac{Ay(2Pdy-ydP)}{(Byy+C+2Dx+Exx)^2}-2ydy\frac{dQ}{dx},$$

cuius formulae, siquidem integrationem admittat, integrale ex variabilitate ipsius y erit

$$\frac{-AP}{B(Byy+C+2Dx+Exx)}-yy\frac{dQ}{dx}=V \quad .$$

Sumto ergo y constante necesse est sit

$$-\frac{AdP(Byy+C+2Dx+Exx)+2APdx(D+Ex)}{B(Byy+C+2Dx+Exx)^2}-yy\frac{ddQ}{dx}=\frac{-AyydP}{(Byy+C+2Dx+Exx)^2}$$

cui satisfit, si

$$\frac{ddQ}{dx^2}=0 \text{ et } -dP(C+2Dx+Exx)+2Pdx(D+Ex)=0,$$

quae duae conditiones an simul consistere possint, videndum est. Posterior autem dat

$$\frac{dP}{P}=\frac{2Ddx+2Exdx}{C+2Dx+Exx} \text{ ideoque } P=C+2Dx+Exx,$$

unde fit

$$Q=\frac{dP}{2dx}=-D-Ex, \text{ hinc } \frac{dQ}{dx}=-E \text{ et } \frac{ddQ}{dx^2}=0.$$

Multiplicator ergo quaesitus est $2dy(C+2Dx+Exx)-2ydx(D+Ex)$ hincque obtinetur integrale

$$\frac{dy^2}{dx^2}(C+2Dx+Exx)-\frac{2ydy}{dx}(D+Ex)-\frac{A(C+2Dx+Exx)}{B(Byy+C+2Dx+Exx)}+Eyy=\text{Const.}.$$

seu utrinque addendo $\frac{A}{B}$

$$\frac{dy^2}{dx^2}(C+2Dx+Exx)-\frac{2ydy}{dx}(D+Ex)-\frac{Ayy}{Byy+C+2Dx+Exx}+Eyy=\text{Const.}$$

COROLLARIUM 1

907. Haec ergo aequation $ddy+\frac{aaydx^2}{(yy+xx)^2}=0$, ubi $A=aa$, $B=1$, $C=0$, $D=0$ et $E=1$, integrabilis redditur multiplicatore $2xxdy-2yxdx$ eiusque integrale erit

$$\frac{xxdy^2}{dx^2}-\frac{2xydy}{dx}+yy+\frac{aayy}{yy+xx}=bb.$$

COROLLARIUM 2

908. Si hic ponatur $y = ux$, ob $dy = udx + xdu$ habebimus

$$\left.\begin{aligned} &xxuu + \tfrac{2ux^3du}{dx} + \tfrac{x^4du^2}{dx^2} + \tfrac{aauu}{1+uu} \\ &-2xxuu - \tfrac{2ux^3du}{dx} \\ &+ xxuu \end{aligned}\right\} = bb$$

sive

$$\frac{x^4du^2}{dx^2} = \frac{bb+(bb-aa)uu}{1+uu}, \quad \text{ergo} \quad \frac{dx}{xx} = \frac{du\sqrt{(1+uu)}}{\sqrt{(bb+(bb-aa)uu)}}$$

unde tam *x* quam *y* per *u* determinatur.

COROLLARIUM 3

909. Simili etiam modo integratio in genere perfici potest. Sit enim brevitatis gratia

$$C + 2Dx + Exx = Bzz;$$

erit $D + Ex = \frac{Bzdz}{dx}$ et aequatio nostra fiet

$$\frac{Bzzdy^2}{dx^2} - \frac{2Byzdydz}{dx^2} + Eyy + \frac{Ayy}{B(yy+zz)} = \frac{K}{B},$$

quae posito $y = uz$ abit in

$$\frac{Bz^4du^2}{dx^2} - \frac{Buuzzdz^2}{dx^2} + Euuzz + \frac{Auu}{B(1+uu)} = \frac{K}{B}.$$

At $\frac{zzdz^2}{dx^2} = \frac{(D+Ex)^2}{BB}$, unde oritur

$$\frac{Bz^4du^2}{dx^2} + \frac{CE-DD}{B}uu = \frac{K+(K-A)uu}{B(1+uu)}$$

seu

$$\frac{BBz^4du^2}{dx^2} = \frac{K+(K-A+DD-CE)uu+(DD-CE)u^4}{1+uu},$$

ita ut sit restituto valore ipsius *z*

$$\frac{dx}{C+2Dx+Exx} = \frac{du\sqrt{(1+uu)}}{\sqrt{(K+(K-A+DD-CE)uu+(DD-CE)u^4)}},$$

sicque *x* definitur per *u* indeque etiam

$$y = uz = u\sqrt{\tfrac{C+2Dx+Exx}{B}}.$$

SCHOLION

910. Hinc patet substitutio, qua tam ipsa aequatio differentio-differentialis proposita quam multiplicator ad formam commodiorem reduci debet. Posito enim ad abbreviandum $C + 2Dx + Exx = Bzz$ aequatio nostra

$$ddy + \frac{Aydx^2}{B^2(yy+zz)^2} = 0$$

ope substitutionis $y = uz$ transformatur in

$$zddu + 2dzdu + uddz + \frac{Audx^2}{BBz^3(1+uu)^2} = 0,$$

cuius multiplicator est $2B(zzdy - yzdz)$ seu $2Bz^3du$ vel simpliciter z^3du.
Cum autem sit $dz = \frac{dx(D+Ex)}{Bz}$ erit

$$ddz = \frac{Edx^2}{Bz} - \frac{dxdz(D+Ex)}{Bzz} = Edx^2 - \frac{dx^2(D+Ex)^2}{BBz^3} = \frac{(CE-DD)dx^2}{BBz^3},$$

ita ut sit $z^3ddz = \frac{CE-DD}{BB}dx^2$, unde aequatio nostra per z^3du multiplicata induit hanc formam

$$z^4duddu + 2z^3dzdu^2 + \frac{CE-DD}{BB}ududx^2 + \frac{Aududx^2}{BB(1+uu)^2} = 0$$

manifesto integrabilem integrali existente

$$\frac{1}{2}z^4du^2 + \frac{CE-DD}{2BB}uudx^2 - \frac{Adx^2}{2BB(1+uu)} = \frac{1}{2}\text{Const.}dx^2,$$

cuius adeo nova integratio ob z functionem ipsius x mox in oculos incurrit, cum sit

$$zzdu = dx\sqrt{\left(\text{Const.} + \frac{DD-CE}{BB}uu + \frac{A}{BB(1+uu)}\right)},$$

ubi variabiles u et x sponte separantur.

Caeterum hic notetur functionem pro z assumtam satisfacere aequationi $z^3ddz = \alpha dx^2$, cum tamen eius ratio non sit manifesta. Multiplicando autem hanc aequationem per $\frac{2dz}{z^2}$ prodit $2dzddz = \frac{2\alpha dx^2dz}{z^3}$, cuius integrale est $dz^2 = \beta dx^2 - \frac{\alpha dx^2}{zz}$ seu $dx = \frac{zdz}{\sqrt{(\beta zz-\alpha)}}$, unde porro fit $\beta x + \gamma = \sqrt{(\beta zz - \alpha)}$

ideoque $\beta zz = \alpha + \gamma\gamma + 2\beta\gamma x + \beta\beta xx$, quae est ipsa nostra forma.

PROBLEMA 113

911. *Sumto elemento dx constante invenire formam generaliorem aequationum differentio-differentialium, quae ope huiusmodi multiplicatoris* $Mydx + Ndy$ *integrabiles reddantur.*

SOLUTIO

Quia multiplicator ope substitutionis $y = Ru$ in formam simplicissimam Sdu transmutari potest, hae substitutione ipsa aequatio differentio-differentialis induat hanc formam

$$ddu + Pdxdu + \tfrac{Udx^2}{S} = 0,$$

cuius postremum membrum per Sdu multiplicatum sponte est integrabile, siquidem U denotet functionem quamcunque ipsius u, dum R et S et P sint functiones ipsius x. Cum ergo aequatio

$$Sduddu + PSdxdu^2 + Udx^2du = 0$$

debeat esse integrabilis, posito integrali

$$\tfrac{1}{2}\, Sdu^2 + dx^2\int Udu = \tfrac{1}{2}Cdx^2$$

necesse est sit

$$\tfrac{1}{2}dSdu^2 = PSdxdu^2 \quad \text{seu} \quad Pdx = \tfrac{dS}{2S}.$$

Quocirca haec forma generalis

$$ddu + \tfrac{dSdu}{2S} + \tfrac{Udx^2}{S} = 0$$

per Sdu multiplicata dabit integrale

$$Sdu^2 = dx^2\left(C - 2\int Udu\right),$$

quod denuo integratum praebet

$$\int \tfrac{dx}{\sqrt{S}} = \int \tfrac{du}{\left(C - 2\int Udu\right)}.$$

Cum igitur haec sint manifesta, ponendo $u = \frac{y}{R}$ ad formas magis intricatas regrediamur, ita ut iam sit $U = \text{funct.}\frac{y}{R}$. Nunc vero est

$$du = \frac{dy}{R} - \frac{ydR}{RR} \text{ et } ddu = \frac{ddy}{R} - \frac{2dRdy}{RR} - \frac{yddR}{RR} + \frac{2ydR^2}{R^3},$$

unde aequatio nostra fit

$$\frac{ddy}{R} - \frac{2dRdy}{RR} - \frac{yddR}{RR} + \frac{2ydR^2}{R^3} + \frac{Udx^2}{S} + \frac{dSdy}{2RS} - \frac{ydRdS}{2RRS} = 0,$$

quae per $\frac{S}{RR}(Rdy - ydR)$ [multiplicata] integrabilis redditur.

Ut igitur ad formam supra propositam accedamus, statuamus $S = \alpha R^4$ et aequatio

$$\frac{ddy}{R} - \frac{yddR}{RR} + \frac{Udx^2}{\alpha R^4} = 0$$

per $\alpha RR(Rdy - ydR)$ multiplicata integrabilis redditur seu haec aequatio

$$Rddy - yddR + \frac{dx^2}{RR} f:\frac{y}{R} = 0$$

per $Rdy - ydR$ multiplicata fit integrabilis.

Ut via ad integrationem perveniendi magis occultetur, ponatur $f:\frac{y}{R} = \frac{\alpha y}{R} + V$ ut V sit functio homogenea nullius dimensionis ipsarum y et R, ae ponatur $yddR = \frac{\alpha ydx^2}{R^3}$, ut fiat

$$Rddy + \frac{Vdx^2}{RR} = 0 \text{ seu } ddy + \frac{Vdx^2}{R^3} = 0,$$

quae multiplicatore $R(Rdy - ydR)$ redditur integrabilis. At cum sit $ddR = \frac{\alpha dx^2}{R^3}$, erit, ut supra [§ 910] vidimus, $R = \sqrt{(\alpha + 2\beta x + \gamma xx)}$, unde, dum V sit functio homogenea nullius dimensionis ipsarum y et $R = \sqrt{(\alpha + 2\beta x + \gamma xx)}$, aequatio

$$ddy + \frac{Vdx^2}{(\alpha + 2\beta x + \gamma xx)^{\frac{3}{2}}} = 0$$

ope multiplicatoris $(\alpha + 2\beta x + \gamma xx)dy - (\beta + \gamma x)ydx$ integrabilis evadit.

COROLLARIUM 1

912. Posito autem $R = \sqrt{(\alpha + \beta x + \gamma xx)}$ aequatio nostra per $RRdy - RydR$ multiplicata fit

$$RRdyddy - RydRddy + \frac{Vdx^2(Rdy - ydR)}{RR} = 0,$$

cuius integrale est

$$\tfrac{1}{2}RRdy^2 - RydRdy + \int ydy\left(RddR + dR^2\right) + dx^2\int Vd.\tfrac{y}{R} = \text{Const.}dx^2$$

ubi est

$$RddR + dR^2 = d.RdR = d.(\beta + \gamma x)dx = \gamma dx^2,$$

sicque integrale est

$$RRdy^2 - 2RydRdy + \gamma yydx^2 + 2dx^2\int Vd.\tfrac{y}{R} = \text{Const.}dx^2.$$

COROLLARIUM 2

913. Quia V est functio ipsius $\frac{y}{R}$, formulae $\int Vd.\frac{y}{R}$ integrale habetur. Pro ulteriore vero integratione posito $y = Ru$ et $\int Vdu = U$ habebitur

$$R^4du^2 - RRuudR^2 + \gamma RRuudx^2 + 2Udx^2 = Gdx^2$$

seu

$$R^4du^2 = dx^2\left(G - 2U + (\beta\beta - \alpha\gamma)uu\right)$$

hincque

$$dx\frac{dx}{\alpha + 2\beta x + \gamma xx} = \frac{du}{\sqrt{\left(G - 2U + (\beta\beta - \alpha\gamma)uu\right)}}$$

ac porro $y = u\sqrt{(\alpha + 2\beta x + \gamma xx)}$.

SCHOLION

914. Haec ergo aequatio $ddy + \frac{Vdx^2}{R^3} = 0$ existente $R = \sqrt{(\alpha + 2\beta x + \gamma xx)}$ multo latius patet ea, quam in praecedente problemate tractavimus, propterea quod hic pro *V* accipere licet functionem quamcunque homogeneam nullius dimensionis ipsarum *y* et *R.* Si enim sumatur $V = \frac{AR^3y}{(myy+RR)^2}$, ipsa aequatio primum tractata oritur. Caeterum ex methodo, qua illam aequationem elicuimus, apparet eam per restrictionem ad hanc formam occultam esse perductam, cum ea aequatio, unde est nata,

$$Rddy - yddR + \frac{dx^2}{RR}f : \frac{y}{R} = 0$$

perspicue integrationem admittat, si per $Rdy - ydR$ multiplicetur. Est enim

$$Rddy - yddR = d.(Rdy - ydR) \text{ et } \frac{Rdy - ydR}{RR} = d.\frac{y}{R},$$

unde facta multiplicatione habebimus

$$(Rdy - ydR)d.(Rdy - ydR) + dx^2 f : \tfrac{y}{R} d.\tfrac{y}{R} = 0,$$

euius aequationis utrumque membrum per se est integrabile. In aequatione autem inde eruta integrabilitas minus perspicitur; multo magis integratio est abscondita in aequationibus sequentibus.

PROBLEMA 114

915. *Sumto elemento dx constante integrationem huius aequationis*

$$yyddy + ydy^2 + Axdx^2 = 0$$

ope multiplicatoris eam integrabilem reddentis perficere.

SOLUTIO

Hic frustra tentatur multiplicator huius formae *Ldy +Mdx;* tentetur ergo haec forma

$$3Ldy^2 + 2Mdxdy + Ndx^2$$

ac ponatur producti integrale

$$Lyydy^3 + Myydxdy^2 + Nyydx^2dy + Vdx^3 = Cdx^3,$$

euius differentiatio perducit ad hanc aequationem

$$\begin{aligned} dx^3dV = 3Lydy^4 &+ 2Mydxdy^3 + Nydx^2dy^2 + 2AMxdx^3dy + ANxdx^4 \\ -2Lydy^4 &- yydxdy^3\left(\tfrac{dL}{dx}\right) + 3ALxdx^2dy^2 - yydx^3dy\left(\tfrac{dN}{dx}\right) \\ -\ yydy^4\left(\tfrac{dL}{dy}\right) &- 2Mydxdy^3 - yydx^2dy^2\left(\tfrac{dM}{dx}\right) \\ &- yydxdy^3\left(\tfrac{dM}{dy}\right) - 2Nydx^2dy^2 \\ &- yydx^2dy^2\left(\tfrac{dN}{dy}\right) \end{aligned}$$

quae formula ut integrationem admittat, membra, quae dy^4, dy^3 et dy^2 continent, evanescere debent; unde primo colligitur

$$L - y\left(\tfrac{dL}{dy}\right) = 0,$$

ubi $\left(\frac{dL}{dy}\right)$ nascitur ex differentiatione ipsius *L* posito *x* constante. Consideretur ergo *x* ut quantitas constans eritque $\frac{dL}{L}=\frac{dy}{y}$ ideoque $L=yf:x$; negligamus autem hanc functionem ipsius *x* seu eius loco unitatem sumamus, ut sit $L=y$ et $\left(\frac{dL}{dx}\right)=0$. Secundo ergo esse debet $\left(\frac{dM}{dy}\right)=0$. Sumamus igitur $M=0$, etiamsi *M* denotare possit functionem quamcunque ipsius *x*, quandoquidem videbimus hoc modo negotium confici posse. Tertio itaque habebimus

$$-Ny+3Axy-yy\left(\frac{dN}{dy}\right)=0;$$

sumto ergo *x* constante erit $3Axdy=Ndy+ydN$ ideoque $Ny=3Axy$ seu $N=3Ax$, ubi iterum functionem ipsius *x*, quae loco constantis ingrederetur, negligimus. Cum igitur hactenus invenimus $L=y$, $M=0$ et $N=3Ax$, erit $dV=-3Ayydy+3AAxxdx$; quae formula cum sponte sit integrabilis, scilicet $V=-Ay^3+AAx^3$, multiplicator nostram aequationem integrabilem reddens erit

$$3ydy^2+3Axdx^2$$

et producti integrale habebitur

$$y^3dy^3+3Axyydx^2dy-Ay^3dx^3+AAx^3dx^3=Cdx^3,$$

quod ob constantem *C* est integrale completum.

COROLLARIUM 1

916. Huius integralis membrum primum commode in tres factores resolvi potesta. Si ponantur formulae z^3-A factores $(z-\alpha)(z-\beta)(z-\gamma)$, ut sit

$$\alpha=\sqrt[3]{A},\quad \beta=\frac{-1+\sqrt{-3}}{2}\sqrt[3]{A}\quad \text{et}\quad \gamma=\frac{-1-\sqrt{-3}}{2}\sqrt[3]{A},$$

erit integrale inventum

$$\left(\frac{ydy}{dx}-\alpha y+\alpha\alpha x\right)\left(\frac{ydy}{dx}-\beta y+\beta\beta x\right)\left(\frac{ydy}{dx}-\gamma y+\gamma\gamma x\right)=C$$

existente

$$\alpha+\beta+\gamma=0,\quad \alpha\beta+\alpha\gamma+\beta\gamma=0\ \text{ et }\ \alpha\beta\gamma=1;$$

posito enim $\frac{ydy}{dx}=z$ habetur haec forma

$$z^3 + 3Axyz - Ay^3 + AAx^3;$$

cuius factor si ponatur $z - p - q$, fit $z^3 - 3pqz - p^3 - q^3 = 0$ ideoque

$$p = y\sqrt[3]{A} \text{ et } q = -x\sqrt[3]{A^2}$$

COROLLARIUM 2

917. Sumto ergo constante $C = 0$ tria obtinentur integralia particularia

$$ydy - \alpha ydx + \alpha\alpha xdx = 0$$

et loco α scribendo β et γ

$$ydy - \beta ydx + \beta\beta xdx = 0 \text{ et } ydy - \gamma ydx + \gamma\gamma xdx = 0,$$

quae posito $y = ux$ dant $\frac{dx}{x} = \frac{-udu}{uu-\alpha u+\alpha\alpha}$ et porro integrando

$$lx = l\frac{\alpha}{\sqrt{(\alpha\alpha-\alpha u+uu)}} - \frac{1}{\sqrt{3}}\text{Ang. tang.}\frac{u\sqrt{3}}{2\alpha-u} + \text{Const.}$$

SCHOLION 1

918. Aequationem autem differentialem primi ordinis inventam difficile est denuo integrare. A potestatibus quidem differentialium ponendo $dy = pdx$ et $y = ux$, unde fit $\frac{dx}{x} = \frac{du}{p-u}$, liberari potest; prodit enim

$$x^3\left(u^3p^3 + 3Auup - Au^3 + AA\right) = C,$$

quae sumtis logarithmis differentiata dat

$$\frac{dx}{x} + \frac{uudp(upp+A)+udu\left(up^3+2Ap-Au\right)}{u^3p^3+3Auup-Au^3+AA} = 0,$$

quae loco $\frac{dx}{x}$ scripto $\frac{du}{p-u}$ abit in

$$du(upp + A)^2 + uu(p-u)dp(upp + A) = 0,$$

ac per $upp + A$ dividendo oritur

$$Adu + uppdu + puudp - u^3dp = 0,$$

quae ponendo $p = \frac{q}{u}$ aliquanto fit simplicior, scilicet

$$Adu + qdq + qudu - uudq = 0;$$

cui autem posito $A = m^3$ etsi particulariter satisfacit $q = mu - mm$, tamen inde integratio completa erui vix posse videtur. Caeterum eadem haec aequatio inter *p* et *u* immediate elicitur ex aequatione differentio-differentiali proposita, quoniam in ea binae variabiles *x* et *y* ubique eundem dimensionum numerum constituunt. Posito enim $dy = pdx$ et $y = ux$ abit ea in

$$uuxdp + uppdx + Adx = 0 \quad \text{seu} \quad \frac{dx}{x} = \frac{-uudp}{A+upp} = \frac{du}{p-u},$$

quae est ipsa praecedens aequatio.

SCHOLION 2

919. Interim tamen aequatio proposita complete integrari potest indeque etiam eae, quas ex ea elicuimus. Hoc autem prorsus singulari ratione praestatur aequationem illam adeo ad differentialia tertii ordinis evehendo.
Cum enim sit

$$yd.\frac{ydy}{dx} + Axdx = 0,$$

statuatur $\frac{dx}{y} = dv$, ut fiat

$$yd.\frac{dy}{dv} + Axdx = 0 \ \text{ seu } \ d.\frac{dy}{dv} + Axdv = 0,$$

quae sumto elemento *dv* constante denuo differentiata praebet

$$\frac{d^3y}{dv} + Adxdv = 0 \ \text{ seu } \ d^3y + Aydv^3 = 0$$

quae forma [§1117] ita est comparata, ut, si ei particulariter satisfaciant $y = P,\ y = Q,\ y = R$, etiam satisfaciat $y = DP + EQ + FR$. Iam vero illi satisfacit $y = e^{-\alpha v}$, si fuerit $\alpha^3 = A$; cum igitur in Corollario 1 ternae litterae $\alpha,\ \beta,\ \gamma$ eadem conditione sint praeditae, habebitur integrale completum

$$y = De^{-\alpha v} + Ee^{-\beta v} + Fe^{-\gamma v},$$

unde ob $Ax = -\frac{ddy}{dv^2}$ erit

$$x = \frac{-D\alpha\alpha e^{-\alpha v} - E\beta\beta e^{-\beta v} - F\gamma\gamma e^{-\gamma v}}{A}$$

seu mutatis constantibus ob $A = \alpha^3 = \beta^3 = \gamma^3$

$$x = \mathfrak{A}e^{-\alpha v} + \mathfrak{B}e^{-\beta v} + \mathfrak{C}e^{-\gamma v}, \quad y = -\mathfrak{A}\alpha e^{-\alpha v} - \mathfrak{B}\beta e^{-\beta v} - \mathfrak{C}\gamma e^{-\gamma v}.$$

Hinc ergo aequationis

$$Adu + qdq + qudu - uudq = 0$$

integrale completum his formulis continetur

$$u=\frac{-\mathfrak{A}\alpha e^{-\alpha v}-\mathfrak{B}\beta e^{-\beta v}-\mathfrak{C}\gamma e^{-\gamma v}}{\mathfrak{A}e^{-\alpha v}+\mathfrak{B}e^{-\beta v}+\mathfrak{C}e^{-\gamma v}}, \quad \text{et} \quad q=\frac{\mathfrak{A}\alpha\alpha e^{-\alpha v}+\mathfrak{B}\beta\beta e^{-\beta v}+\mathfrak{C}\gamma\gamma e^{-\gamma v}}{\mathfrak{A}e^{-\alpha v}+\mathfrak{B}e^{-\beta v}+\mathfrak{C}e^{-\gamma v}}$$

ob $q=pu=\frac{ydy}{xdx}=\frac{dy}{xdv}$ quod insigne est specimen integrationis methodo directa vix perficiendae.

PROBLEMA 115

920. *Sumto elemento dx constante si proponatur haec aequatio*

$$2y^3ddy+yydy^2+Xdx^2=0$$

existente $X=\alpha+\beta x+\gamma xx$ *, invenire multiplicatorem, qui eam integrabilem reddat.*

SOLUTIO

Hic frustra tentantur multiplicatores formae

$$Ldy+Mdx \quad \text{et} \quad Ldy^2+Mdxdy+Ndx^2\,;$$

sumamus ergo multiplicatorem huius formae

$$2Ldy^3+Mdx^2dy+Ndx^3$$

et integrale statuatur

$$Ly^3dy^4+My^3dx^2dy^2+2Ny^3dx^3dy+Sdx^4=0,$$

unde per differentiationem colligitur

$$\begin{aligned} dx^4dS = {} & 2Lyydy^5+Myydx^2dy^3+Nyydx^3dy^2+MXdx^4dy+NXdx^5 \\ & -3Lyydy^5+2LXdx^2dy^3-y^3dx^3dy^2\left(\tfrac{dM}{dx}\right)-2y^3dx^4dy\left(\tfrac{dN}{dx}\right) \\ & -y^3dy^5\left(\tfrac{dL}{dy}\right)-3Myydx^2dy^3-6Nyydx^3dy^2 \\ & -y^3dx^2dy^3\left(\tfrac{dM}{dy}\right)-2y^3dx^3dy^2\left(\tfrac{dN}{dy}\right) \end{aligned}$$

ubi sumimus L esse functionem ipsius y tantum. Ut ergo termini dy^5 continentes destruantur, erit

$$-L-\frac{ydL}{dy}=0 \;\text{ et }\; L=\frac{1}{y}\cdot$$

Deinde pro destructione terminorum per dy^3 affectorum erit

$$-2Myy+\frac{2X}{y}-y^3\left(\frac{dM}{dy}\right)=0$$

et sumto *x* constante

$$dM+\frac{2Mdy}{y}=\frac{2Xdy}{y^4},$$

quae per *yy* multiplicata et integrata praebet

$$Myy=P-\frac{2X}{y} \quad \text{et} \quad M=\frac{P}{yy}-\frac{2X}{y^3}$$

denotante *P* functionem quamcunque ipsius *x*. Iam ad terminos dy^2 tollendos erit

$$-5Nyy-y\frac{dP}{dx}+\frac{2dX}{dx}-2y^3\left(\frac{dN}{dy}\right)=0$$

et sumto *x* constante

$$2y^3dN+5Nyydy=\frac{2dX}{dx}dy-\frac{dP}{dx}ydy,$$

quae per $\sqrt{y}$ divisa et integrata dat

$$2Ny^{\frac{5}{2}}=\frac{4dX}{dx}\sqrt{y}-\frac{2dP}{3dx}y\sqrt{y}$$

neglecta functione ipsius *x* addenda, quoniam irrationalitas $\sqrt{y}$ in calculum non ingreditur. Erit ergo

$$N=\frac{2dX}{yydx}-\frac{dP}{3ydx}$$

ac propterea

$$dS=dy\left(\frac{PX}{yy}-\frac{2XX}{y^3}-\frac{4yddX}{dx^2}+\frac{2yyddP}{3dx^2}\right)+\frac{2XdX}{yy}-\frac{XdP}{3y},$$

unde fit integrando

$$S=\frac{XX}{yy}-\frac{PX}{y}-\frac{2yyddX}{dx^2}+\frac{2y^3ddP}{9dx^2}+\int\left(\frac{PdX}{y}+\frac{2XdP}{3y}+\frac{2yyd^3X}{dx^2}-\frac{2y^3d^3P}{9dx^2}\right),$$

quae finite exprimetur, si $P=0$, cum ob $X=\alpha+\beta x+\gamma xx$ sit $d^3X=0$. Quocirca habemus

$$L=\tfrac{1}{y}M=-\tfrac{2X}{y^3}\quad \text{et}\quad N=\tfrac{2dX}{yydx}\quad \text{atque}\quad S=\tfrac{XX}{yy}-\tfrac{2yyddX}{dx^2}+\text{Const.},$$

unde aequatio integralis est

$$y^2dy^4-2Xdx^2dy^2\ +4yXdx^2dy+\tfrac{XXdx^4}{yy}-2yydx^2ddX=Cdx^4.$$

Aequatio ergo proposita

$$2y^3ddy+yydy^2+dx^2\left(\alpha+\beta x+\gamma xx\right)=0$$

integrabilis redditur multiplicata per

$$\tfrac{2dy^3}{y}-\tfrac{2(\alpha+\beta ix+\gamma xx)dx^2dy}{y^3}+\ \tfrac{2dx^3(\beta+2\beta x)}{yy};$$

tum vero est integrale

$$y^2dy^4-2dx^2dy^2\left(\alpha+\beta x+\gamma xx\right)+4ydx^3dy\left(\beta+2\gamma x\right)-4\gamma yydx^4+\tfrac{(\alpha+\beta x+\gamma xx)^2dx^4}{yy}=Cdx^4$$

seu

$$\left(yydy^2-\left(\alpha+\beta x+\gamma xx\right)dx^2\right)^2+4y^3dx^3dy\left(\beta+2\gamma x\right)-4\gamma y^4dx^4=Cyydx^4.$$

SCHOLION 1

921. Integrale hoc ita est intricatum, ut alia methodo vix inveniri potuisse videatur, verum etiam ita est comparatum, ut nulla pateat methodus id porro integrandi, unde prima integratio parum lucri attulisse est iudicanda. Quemadmodum autem in praecedente problemate integrale completum ex alio fonte hausimus, ita hic simili modo integrale eruere licet, quod eo magis est notatu dignum, cum aequatio proposita in se spectata solutu sit difficillima. Ponamus scilicet itidem $dx=ydv$, et cum sit

$$ddy=dxd.\tfrac{dy}{dx}=ydvd.\tfrac{dy}{ydv},$$

erit sumendo iam elementum dv constans

$$ddy=ydv\left(\tfrac{ddy}{ydv}-\tfrac{dy^2}{yydv}\right)=ddy-\tfrac{dy^2}{y}.$$

Hinc nostra aequatio induit hanc formam

$$2y^3ddy-yydy^2+yydv^2\left(\alpha+\beta x+yxx\right)=0$$

seu

$$2yddy - dy^2 + dv^2(\alpha + \beta x + \gamma xx) = 0,$$

quae denuo differentiata praebet

$$2yd^3y + ydv^3(\beta + 2\gamma x) = 0 \text{ seu } 2d^3y + dv^3(\beta + 2\gamma x) = 0;$$

differentietur iterum prodibitque

$$2d^4y + 2\gamma ydv^4 = 0 \text{ seu } d^4y + y\gamma dv^4 = 0;$$

quam aequationem si aliunde resolvere valoremque ipsius y per v exprimere liceat, erit $x = \int ydv$ seu sine integratione $x = -\frac{d^3y}{\gamma dv^2} - \frac{\beta}{2\gamma}$. At manifestum est isti aequationi differentiali quarti ordinis satisfacere $y = e^{\lambda v}$, si sit $\lambda^4 + \gamma = 0$. Ponamus ergo $\gamma = -n^4$ et quatuor ipsius λ habebuntur valores $\lambda = \pm n$ et $\lambda = \pm n\sqrt{-1}$, unde [§ 1125, §1128] eius integrale completum est
$y = Ae^{nv} + Be^{-nv} + C\sin.(nv + \zeta)$
hincque

$$x = \tfrac{A}{n}e^{nv} - \tfrac{B}{n}e^{-nv} - \tfrac{C}{n}\cos.(nv + \zeta) + \tfrac{\beta}{2n^4},$$

qui ergo valores quoque satisfacient aequationi inter x et y propositae, dummodo constantes A, B, C et ζ ita a se pendentes capiantur, ut quantitati quoque α conveniant. His nempe valoribus substitutis fieri debet

$$\alpha + \beta x - n^4xx + \frac{2yddy - dy^2}{dv^2} = 0,$$

ubi tantum terminos constantes considerasse sufficit, quibus accenseri debent ii, qui quadratum sinus cosinusve anguli $nv + \zeta$ continent, quippe ex quorum combinatione quantitas constans exsurgit. Cum ergo sit

$$2y = 2Ae^{nv} + 2Be^{-nv} + 2C\sin.(nv + \zeta),$$
$$\frac{ddy}{dv^2} = nnAe^{nv} + nnBe^{-nv} - nnC\sin.(nv + \zeta),$$
$$\frac{dy}{dv} = nAe^{nv} - nBe^{-nv} + nC\cos.(nv + \zeta),$$
$$x = \tfrac{A}{n}e^{nv} - \tfrac{B}{n}e^{-nv} - \tfrac{C}{n}\cos.(nv + \zeta) + \tfrac{\beta}{2n^4},$$

erit sumtis terminis memoratis

$$\beta x = \frac{\beta\beta}{2n^4},$$
$$-n^4 xx = 2ABnn - nnCCcos.^2(nv+\zeta) - \frac{\beta\beta}{4n^4},$$
$$\frac{2yddy}{dv^2} = 4ABnn - 2nnCCsin.^2(nv+\zeta),$$
$$-\frac{dy^2}{dv^2} = 2ABnn - nnCC\cos.^2(nv+\zeta),$$

ergo

$$\alpha + 8nnAB - 2nnCC + \frac{\beta\beta}{4n^4} = 0$$

ideoque

$$C = \sqrt{\left(\frac{\alpha}{2nn} + \frac{\beta\beta}{8n^6} + 4AB\right)}$$

vel

$$\alpha = 2nn(CC - 4AB) - \frac{\beta\beta}{4n^4}$$

et

$$a + \beta x + \gamma xx = 2nn(CC - 4AB) - \left(\frac{\beta}{2nn} - nnx\right)^2.$$

Manent ergo tres constantes A, B et ζ indeterminatae, ita ut nullum sit dubium, quin formulae pro x et y datae integrale completum exhibeant.

SCHOLION 2

922. Aequationes differentio-differentiales, quas in his duobus problematibus tractavimus, ad similem formam reduci possunt. Prior enim

$$y(yddy + dy^2) + Xdx^2 = 0$$

existente $X = Ax$ vel $X = \alpha + \beta x$, si ponatur $ydy = \frac{1}{2}dz$ seu $yy = z$, induit hanc formam

$$\tfrac{1}{2}ddz\sqrt{z} + Xdx^2 = 0,$$

quae ope multiplicatoris $\frac{3dz^2}{4\sqrt{z}} + 3Xdx^2$ integrabilis redditur. Altera vero aequatio

$$yy(2yddy + dy^2) + Xdx^2 = 0$$

existente $X = \alpha + \beta x + \gamma xx$ posito $y = z^{\frac{2}{3}}$ fit

$$dy = \tfrac{2}{3}z^{-\frac{1}{3}}dz \quad \text{et} \quad ddy = \tfrac{2}{3}z^{-\frac{1}{3}}ddz - \tfrac{2}{9}z^{-\frac{4}{3}}dz^2,$$

hinc $2yddy+dy^2=\frac{4}{3}z^{\frac{1}{3}}ddz$ sicque aequatio hanc induit formam

$$\frac{4}{3}z^{\frac{5}{3}}ddz+Xdx^2=0,$$

quae integrabilis redditur ope multiplicatoris $\frac{16dz^3}{27z^{\frac{5}{3}}}-\frac{4Xdx^2dz}{3z^{\frac{7}{3}}}+\frac{2dXdx^2}{z^{\frac{4}{3}}}$. Hinc colligimus

$ddz+\frac{2Xdx^2}{\sqrt{z}}=0$

$dz^2+4Xdx^2\sqrt{z}$, pro aequatione autem $ddz+\frac{3Xdx^2}{4z\sqrt[3]{zz}}=0$ multiplicatorem fore

$dz^3-\frac{9Xdx^2}{4\sqrt[3]{zz}}+\frac{27}{8}dXdx^2\sqrt[3]{z}$ seu sub uno conspectu

pro aequatione	multiplicator erit
$ddz+\frac{2Xdx^2}{\sqrt{z}}=0$	$\frac{16dz^3}{27z^{\frac{5}{3}}}-\frac{4Xdx^2dz}{3z^{\frac{7}{3}}}+\frac{2dXdx^2}{z^{\frac{4}{3}}}$
$ddz+\frac{3Xdx^2}{4z\sqrt[3]{zz}}=0$	$dz^3-\frac{9Xdx^2}{4\sqrt[3]{zz}}+\frac{27}{8}dXdx^2\sqrt[3]{z}$

Caeterum hae integrationes maxime sunt notatu dignae, cum ex aequationibus differentialibus altioribus perfici queant. Ita cum [§ 1117] ex hac aequatione, ubi *dv* constans,

$$d^3y+Advddy+Bdv^2dy+Cydv^3=0$$

sit

$$y=\mathfrak{A}e^{\alpha v}+\mathfrak{B}e^{\beta v}+\mathfrak{C}e^{\gamma v},$$

si fuerint α, β, γ radices huius aequationis

$$r^3+Ar^2+Br+C=0,$$

ponamus $dv=\frac{dx}{y}$, et cum sit

$$ddy=dvd.\frac{dy}{dx}=\frac{dx}{y}d.\frac{ydy}{dx}$$

et

$$d^3y=dv^2d.\frac{ddy}{dv^2}=dv^2d.\left(\frac{1}{dv}d.\frac{dy}{dv}\right)=\frac{dx^2}{yy}d.\left(\frac{y}{dx}d.\frac{ydy}{dx}\right),$$

si iam *dx* constans sumamus, erit

$$ddy=ddy+\frac{dy^2}{y}$$

et

$$d^3y=\frac{1}{yy}d.y\left(yddy+dy^2\right)=d^3y+\frac{4dyddy}{y}+\frac{dy^3}{yy}$$

hincque per *yy* multiplicando

$$yyd^3y+4ydyddy+dy^3+Adx\left(yddy+dy^2\right)+Bdx^2dy+Cdx^3=0\,,$$

quae integrata dat

$$yyddy+ydy^2+Aydxdy+Bydx^2+\left(Cx+D\right)dx^2=0,$$

quae ergo per superiora integrari potest.

PROBLEMA 116

923. *Definire conditiones functionum P, Q, R et L, M, N, ut haec aequatio differentio-differentialis*

$$ddy+Pdy^2+Qdxdy+Rdx^2=0$$

integrabilis reddatur multiplicatore $3Ldy^2+2Mdxdy+Ndx^2$.

SOLUTIO

Facta multiplicatione integratio terminorum per *ddy* affectorum dat

$$Ldy^3+Mdxdy^2+Ndx^2dy\,,$$

quare ponatur integrale

$$Ldy^3+Mdxdy^2+Ndx^2dy+Vdx^3=Cdx^3\,,$$

cuius differentiale aequari debet formulae propositae in multiplicatorem ductae, unde oritur

$$\begin{aligned}dx^3dV=3LPdy^4&+3LQdxdy^3&&+3LRdx^2dy^2\\&+\;2MP&&+2MQ&&+2MRdx^3dy\\-\left(\tfrac{dL}{dy}\right)&-\left(\tfrac{dL}{dx}\right)&&+\;NP&&+\;NQ&&+NRdx^4\\&-\left(\tfrac{dM}{dy}\right)&&-\left(\tfrac{dM}{dx}\right)&&-\left(\tfrac{dN}{dx}\right)\\&&&-\left(\tfrac{dN}{dy}\right)\end{aligned}$$

Hic ergo fieri oportet

$$3LP-\left(\tfrac{dL}{dy}\right)=0,\quad 3LQ+2MP-\left(\tfrac{dL}{dx}\right)-\left(\tfrac{dM}{dy}\right)=0,$$

$$3LR+2MQ+NP-\left(\tfrac{dM}{dx}\right)-\left(\tfrac{dN}{dy}\right)=0.$$

Tum vero erit

$$dV = \left(2MR + NQ - \left(\tfrac{dN}{dx}\right)\right)dy + NRdx\,,$$

quae formula integrabilis esse debet. Ex illis autem aequationibus colligitur

$$P = \tfrac{1}{3L}\left(\tfrac{dL}{dy}\right), \quad Q = \tfrac{1}{3L}\left(\tfrac{dL}{dx}\right) + \tfrac{1}{3L}\left(\tfrac{dM}{dy}\right) - \tfrac{2M}{9LL}\left(\tfrac{dL}{dy}\right),$$

et

$$R = \tfrac{1}{3L}\left(\tfrac{dM}{dx}\right) + \tfrac{1}{3L}\left(\tfrac{dN}{dy}\right) - \tfrac{N}{9LL}\left(\tfrac{dL}{dy}\right) - \tfrac{2M}{9LL}\left(\tfrac{dL}{dx}\right) - \tfrac{2M}{9LL}\left(\tfrac{dM}{dy}\right) + 4\tfrac{MM}{27L^3}\left(\tfrac{dL}{dy}\right).$$

COROLLARIUM 1

924. Si L, *M* et *N* fuerint functiones ipsius *x* tantum, erit

$$P = 0, \;\; Q = \tfrac{dL}{3Ldx} \;\text{ et }\; R = \tfrac{dM}{3Ldx} - \tfrac{2MdL}{9LLdx}\,,$$

hinc

$$dV = \left(\tfrac{2MdM}{3Ldx} - \tfrac{4MMdL}{9LLdx} + \tfrac{NdL}{3Ldx} - \tfrac{dN}{dx}\right)dy + \tfrac{NdM}{3L} - \tfrac{2MNdL}{9LL}$$

ac coefficiens ipsius *dy* debet esse constans. Quare per $L^{\frac{1}{3}}$ dividendo habebitur

$$\tfrac{Cdx}{\sqrt[3]{L}} = \tfrac{2MdM}{3L\sqrt[3]{L}} - \tfrac{4MMdL}{9LL\sqrt[3]{L}} + \tfrac{NdL}{3L\sqrt[3]{L}} - \tfrac{dN}{\sqrt[3]{L}}$$

et integrando

$$C\int \tfrac{dx}{\sqrt[3]{L}} = \tfrac{MM}{3L\sqrt[3]{L}} - \tfrac{N}{\sqrt[3]{L}} \quad \text{seu} \;\; N = \tfrac{MM}{3L} - CL^{\frac{1}{3}}\int \tfrac{dx}{\sqrt[3]{L}},$$

ergo

$$V = Cy + \int\left(\tfrac{dM}{3L} - \tfrac{2MdL}{9LL}\right)\left(\tfrac{MM}{3L} - CL^{\frac{1}{3}}\int \tfrac{dx}{\sqrt[3]{L}}\right).$$

COROLLARIUM 2

925. Sit $M = S\sqrt[3]{L^2}$; erit $dM = dS\sqrt[3]{L^2} + \frac{2SdL}{3\sqrt[3]{L^2}}$ et

$$V = Cy + \tfrac{1}{3}\int \tfrac{dS}{\sqrt[3]{L}}\left(\tfrac{1}{3}SS\sqrt[3]{L} - CL^{\frac{1}{3}}\int \tfrac{dx}{\sqrt[3]{L}}\right)$$

seu

$$V = Cy + \tfrac{1}{27}S^3 - \tfrac{1}{3}C\int dS\int \tfrac{dx}{\sqrt[3]{L}}\,,$$

tum vero

$$N = \tfrac{1}{3}SS\sqrt[3]{L} - CL^{\frac{1}{3}}\int \tfrac{dx}{\sqrt[3]{L}} = \left(\tfrac{1}{3}SS - C\int \tfrac{dx}{\sqrt[3]{L}}\right)\sqrt[3]{L}$$

atque

$$P = 0, \quad Q = \frac{dL}{3Ldx} \text{ et } R = \frac{dS}{3dx\sqrt[3]{L}}.$$

Quare haec aequatio

$$ddy + \frac{dLdy}{3L} + \frac{dSdx}{3\sqrt[3]{L}} = 0$$

integrabilis redditur multiplicatore

$$3Ldy^2 + 2Sdxdy\sqrt[3]{L^2} + dx^2\left(\tfrac{1}{3}SS - C\int\frac{dx}{\sqrt[3]{L}}\right)\sqrt[3]{L}$$

et integrale est

$$Ldy^3 + Sdxdy^2\sqrt[3]{L^2} + dx^2dy\left(\tfrac{1}{3}SS - C\int\frac{dx}{\sqrt[3]{L}}\right)\sqrt[3]{L} + Cydx^3$$
$$+\tfrac{1}{27}S^3dx^3 - \tfrac{1}{3}Cdx^2\int dS\int\frac{dx}{\sqrt[3]{L}} = 0.$$

COROLLARIUM 3

926. Hic quidquid pro constante C assumatur, idem integrale prodire debet. Hinc si $C = 0$, aequationis

$$ddy + \frac{dLdy}{3L} + \frac{dSdx}{3\sqrt[3]{L}} = 0$$

multiplicator erit

$$3Ldy^2 + 2Sdxdy\sqrt[3]{L^2} + \tfrac{1}{3}SSdx^2\sqrt[3]{L}$$

et integrale

$$Ldy^3 + Sdxdy^2\sqrt[3]{L^2} + \tfrac{1}{3}SSdx^2dy\sqrt[3]{L} + \tfrac{1}{27}S^3dx^3 = Ddx^3$$

seu

$$\left(dy\sqrt[3]{L} + \tfrac{1}{3}Sdx\right)^3 = Ddx^3.$$

SCHOLION 1

927. Ex iisdem quoque conditionibus, si dentur functiones P, Q et R, definiri poterunt functiones L, M, N, quatenus quidem postrema conditio integrabilitatis patitur, veluti si sit $P = \frac{n}{y}$, $Q = 0$ et R functio ipsius x tantum, puta $R = X$, ut habeatur haec aequatio

$$ddy + \frac{ndy^2}{y} + Xdx^2 = 0;$$

cuius multiplicator si sumatur $3Ldy^2 + 2Mdxdy + Ndx^2$, ut integrale sit

$$Ldy^3 + Mdxdy^2 + Ndx^2dy + Vdx^3 = Cdx^3,$$

erit primo $\frac{3nL}{y}-\left(\frac{dL}{dy}\right)=0$ et sumta x constante $\frac{dL}{L}=\frac{3ndy}{y}$, hinc $L=Sy^{3n}$ denotante S functionem ipsius x. Deinde est

$$\frac{2nM}{y}-y^{3n}\frac{dS}{dx}-\left(\frac{dM}{dy}\right)=0,$$

et sumta x constante

$$dM-\frac{2nMdy}{y}+\frac{dS}{dx}y^{3n}dy=0,$$

quae per y^{-2n} multiplicata et integrata dat

$$y^{-2n}M+\frac{dS}{(n+1)dx}y^{n+1}=T \text{, funct. ipsius } x.$$

Ergo

$$M=Ty^{2n}-\frac{dS}{(n+1)dx}y^{3n+1}.$$

Tertio fieri debet

$$3SXy^{3n}+\frac{nN}{y}-\frac{dT}{dx}y^{2n}+\frac{ddS}{(n+1)dx^2}y^{3n+1}-\frac{dN}{dy}=0$$

unde sumta x constante

$$dN-\frac{nNdy}{y}+\frac{dT}{dx}y^{2n}dy-\frac{ddS}{(n+1)dx^2}y^{3n+1}dy-3SXy^{3n}dy=0,$$

quae per y^{-n} multiplicata et integrata dat

$$y^{-n}N+\frac{dT}{(n+1)dx}y^{n+1}-\frac{ddS}{2(n+1)^2dx^2}y^{2n+2}-\frac{3SX}{2n+1}y^{2n+1}=U \text{ func. ipius } x,$$

seu

$$N=Uy^{n}-\frac{dT}{(n+1)dx}y^{2n+1}+\frac{ddS}{2(n+1)^2dx^2}y^{3n+2}+\frac{3SX}{2n+1}y^{3n+1}.$$

Ex his autem fit

$$dV=dy\left\{\begin{array}{l}2TXy^{2n}-\frac{2XdS}{(n+1)dx}y^{3n+1}-\frac{dU}{dx}y^{n}+\frac{ddT}{(n+1)dx^2}y^{2n+1}\\ \qquad -\frac{d^3S}{2(n+1)^2dx^3}y^{3n+2}-\frac{3d.SX}{(2n+1)dx}y^{3n+1}\end{array}\right\}$$
$$+Xdx\left(Uy^{n}-\frac{dT}{(n+1)dx}y^{2n+1}+\frac{ddS}{(n+1)^2dx^2}y^{3n+2}+\frac{3SX}{2n+1}y^{3n+1}\right);$$

quae formula ut integrationem admittat, esse oportet

$$2y^{2n}d.TX-2y^{3n+1}\frac{d.XdS}{(n+1)dx}-y^{n}\frac{ddU}{dx}+y^{2n+1}\frac{d^3T}{(n+1)dx^2}$$
$$-y^{3n+2}\frac{d^4S}{2(n+1)^2dx^3}-3y^{3n+1}\frac{dd.SX}{(2n+1)dx}-nUXy^{n-1}dx+\frac{(2n+1)XdT}{(n+1)}y^{2n}$$
$$-\frac{(3n+2)XddS}{2(n+1)^2dx}y^{3n+1}-\frac{3(3n+1)SXXdx}{2n+1}y^{3n}=0;$$

hic ergo singulae potestates ipsius y, quatenus sunt inaequales, seorsim destrui debent. Quare potestas y^{n-1} dat $U=0$, unde etiam potestas y^{n} ad nihilum redigitur. Potestas y^{2n} dat $(2n+2)TdX+(2n+2)XdT+(2n+1)XdT=0$ seu $X^{2n+2}T^{4n+3}=A$, at potestas y^{2n+1} praebet $d^3T=0$ seu $T=\alpha+\beta x+\gamma xx$. Potestas vero y^{3n} postulat $S=0$, nisi sit $n=-\frac{1}{3}$, quo casu etiam potestates y^{3n+1} et y^{3n+2} sponte evanescunt. Cum ergo sit $u=0$, $S=0$ et $T=\alpha+\beta x+\gamma xx$ hincque $X=B(\alpha+\beta x+\gamma xx)^{\frac{-4n-2}{2n+2}}$, haec aequatio

$$ddy+\frac{ndy^2}{y}+B(\alpha+\beta x+\gamma xx)^{\frac{-4n-3}{2n+2}}dx^2=0$$

integrabilis redditur ope multiplicatoris

$$2(\alpha+\beta x+\gamma xx)y^{2n}dy-\frac{dx(\beta+2\gamma x)}{n+1}y^{2n+1}.$$

SCHOLION 2

928. Quanquam plurimum abest, quominus haec methodus satis adhuc sit culta, tamen specimina in hoc capite tradita abunde declarant, quanta incrementa inde expectare queamus, unde eius cultura maxime Geometris commendanda videtur. Quoniam igitur methodi, quibus in resolutione aequationum differentio-differentialium uti convenit, satis luculenter sunt expositae, ad sequens caput progrediamur, ubi integrationem huiusmodi aequationum, quatenus quidem id commode fieri potest, per series infinitas ostendemus.