CHAPTER VIII

CONCERNING THE RESOLUTION OF OTHER SECOND ORDER DIFFERENTIAL EQUATIONS BY INFINITE SERIES

PROBLEM 122

967. *To show the general form of second order differential equations that it is permitted conveniently to resolve by series, and to investigate the integrals of these.*

SOLUTION

In the first place other equations are not able to be resolved by series conveniently, other than those in which the other variable y with its differentials dy and ddy nowhere have dimensions greater than one, since on substituting the infinite series for y we come across great difficulties in the calculations, if greater dimensions should arise anywhere. Hence equations of this kind may be encountered in this form

$$ddy + Mdxdy + Nydx^2 = Xdx^2.$$

Then as any term of an assumed series of y can be determined by the preceding one only, which is the most noteworthy case of resolution, it is required for terms of two kinds only to be present in a ratio of the other variable x, since we consider the dimensions which x itself with it differential dx make. From which indeed initially by rejecting the term Xdx^2, the equations may be contained in this form in a resolvable way

$$xx\left(a+bx^n\right)ddy + x\left(c+ex^n\right)dxdy + \left(f+gx^n\right)ydx^2 = 0.$$

For the resolution of this, we put in place

$$y = Ax^{\lambda} + Bx^{\lambda+n} + Cx^{\lambda+2n} + Dx^{\lambda+3n} + \text{etc.}$$

and with the substitution made the following sum of the series must be reduced to zero :

$$\begin{array}{llll}
\lambda(\lambda-1)Aax^{\lambda} + & (\lambda+n)(\lambda+n-1)Bax^{\lambda+n} & +(\lambda+2n)(\lambda+2n-1)Cax^{\lambda+2n} & +\text{etc.}\\
+ & \lambda(\lambda-1)Ab & +(\lambda+n)(\lambda+n-1)Bb & \\
+\lambda Ac \quad + & (\lambda+n)Bc & +(\lambda+2n)Cc & \\
+ & \lambda Ae & +(\lambda+n)Be & \\
+Af \quad + & Bf & +Cf & \\
+ & Ag & +Bg &
\end{array}$$

Hence initially here the exponent λ must be taken thus, so that there shall be

$$\lambda(\lambda-1)a+\lambda c+f=0,$$

then it is necessary for the remaining terms to become :

$$\begin{gathered}
\big((\lambda+n)(\lambda+n-1)a+(\lambda+n)c+f\big)B=-\big(\lambda(\lambda-1)b+\lambda e+g\big)A,\\
((\lambda+2n)(\lambda+2n-1)a+(\lambda+2n)c+f)C=-((\lambda+n)(\lambda+n-1)b+(\lambda+n)e+g)B,\\
((\lambda+3n)(\lambda+3n-1)a+(\lambda+3n)c+f)D=-((\lambda+2n)(\lambda+2n-1)b+(\lambda+2n)e+g)C\\
\text{etc.}
\end{gathered}$$

Therefore since there shall be $\lambda(\lambda-1)a+\lambda c+f=0$, if for the case of brevity we put

$$\lambda(\lambda-1)b+\lambda e+g=h,$$

there will be

$$\begin{gathered}
(n(n+2\lambda-1)a+nc)B=-hA,\\
(2n(2n+2\lambda-1)a+2nc)C=-(n(n+2\lambda-1)b+ne+h)B,\\
(3n(3n+2\lambda-1)a+3nc)D=-(2n(2n+2\lambda-1)b+2ne+h)C\\
\text{etc.}
\end{gathered}$$

Hence unless $a=0$, because two values are found for λ,

$$\lambda=\frac{a-c\pm\sqrt{\left((a-c)^2-4af\right)}}{2a},$$

clearly two series are found for *y*, which combined in some manner present the complete integral of the equation.

OTHERWISE

For the proposed equation

$$xx\left(a+bx^{n}\right)ddy+x\left(c+ex^{n}\right)dxdy+\left(f+gx^{n}\right)ydx^{2}=0,$$

a series in reverse order also can be imagined :

$$y=Ax^{\lambda}+Bx^{\lambda-n}+Cx^{\lambda-2n}+Dx^{\lambda-3n}+\text{etc.},$$

from which there arises on reduction to zero :

$$\begin{array}{llll}
\lambda(\lambda-1)Abx^{\lambda+n} & +\ (\lambda-n)(\lambda-n-1)Bbx^{\lambda} & +(\lambda-2n)(\lambda-2n-1)Cbx^{\lambda-n} & +\text{etc.}\\
 & +\quad \lambda(\lambda-1)Aa & +\ (\lambda-n)(\lambda-n-1)Ba & \\
+\lambda Ae & +\quad (\lambda-n)Be & +\ (\lambda-2n)Ce & \\
 & +\quad \lambda Ac & +\ (\lambda-n)Bc & \\
+Ag & +\quad Bg & +\ Cg & \\
 & +\quad Af & +\ Bf &
\end{array}$$

Hence the exponent λ here must be taken thus, so that there is made

$$\lambda(\lambda-1)b+\lambda e+g=0.$$

Then if we put

$$\lambda(\lambda-1)a+\lambda c+f=h,$$

the determination of the coefficients may be considered thus :

$$\begin{aligned}
&(n(n-2\lambda+1)b-e)B=-hA,\\
&(2n(2n-2\lambda+1)b-e)C=-(n(n-2\lambda+1)a-nc+h)B,\\
&(3n(3n-2\lambda+1)b-e)D=-(2n(2n-2\lambda+1)a-2nc+h)C\\
&\qquad\text{etc.}
\end{aligned}$$

COROLLARIUM 1

968. From the first solution, if *i* should denote some positive integer, the assumed series may terminate somewhere, if there should be [with or without the abbreviation *h*]

$$in(in+2\lambda-1)b+ine+h=0 \text{ or } (\lambda+in)(\lambda+in-1)b+(\lambda+in)e+g=0,$$

that is

$$\left.\begin{matrix}\lambda\lambda b+\lambda(2in-1)b+in(in-1)b\\ +\lambda e \qquad\qquad +ine\ +g\end{matrix}\right\}=0.$$

COROLLARY 2

969. Hence our equation is allowed to be integrated [with a finite number of terms], if the letters *f* and *g* should be prepared thus, so that there shall be

$$f=-\lambda(\lambda-1)a-\lambda c \text{ and } g=-(\lambda+in)(\lambda+in-1)b-(\lambda+in)e,$$

or with two numbers μ and ν taken, so that $\nu-\mu$ shall be divisible by the exponent *n*, if there should be

$$f=-\mu(\mu-1)a-\mu c \text{ and } g=-\nu(\nu-1)b-\nu e.$$

COROLLARY 3

970. Hence since there shall be

$$\mu=\frac{a-c\pm\sqrt{\left((a-c)^2-4af\right)}}{2a} \text{ and } \nu=\frac{b-e\pm\sqrt{\left((b-e)^2-4bg\right)}}{2b}$$

the equation will be considered an algebraic integral, if there should be $\nu-\mu=in$ with *i* denoting a positive whole number, that is, if there should be

$$in=\tfrac{c}{2a}-\tfrac{e}{2b}\pm\frac{\sqrt{\left((b-e)^2-4bg\right)}}{2b}\mp\frac{\sqrt{\left((a-c)^2-4af\right)}}{2a}$$

COROLLARY 4

971. But if it should come about that the exponent λ for the series becomes imaginary, it is agreed to be noted that

$$x^{\alpha+\beta\sqrt{-1}}=x^{\alpha}e^{\beta\sqrt{-1}.lx}=x^{\alpha}(cos.\beta lx+\sqrt{-1}.sin.\beta lx),$$

from which the two series thus can be combined, so that the integral may follow a real form.

SCHOLIUM

972. Each solution generally considered supplies a twofold series for the variable *y* for the twin values of the exponent λ, the combination of which shows the complete integral. Clearly the first solution for the exponent λ gives these two values

$$\lambda = \frac{a-c\pm\sqrt{\left((a-c)^2-4af\right)}}{2a}$$

the latter solution

$$\lambda = \frac{b-e\pm\sqrt{\left((b-e)^2-4bg\right)}}{2b}$$

thus so that in this manner the complete integral can be expressed in two ways. Which two forms even if greatly different and thus meanwhile the one progressing by imaginary exponents, while the other has real exponents, yet they must be equivalent to each other [Recall that the first series is expressed in ascending powers of *x* while the second series is expressed in descending powers of *x*; Euler will treat mainly the ascending case in what follows, and he is most interested in the cases in which some 'inconvenience' arises between the values of λ found in the indicial equation, usually resulting in an infinite term in a series.]

Also since it may come about, that either solution or both may show a nonsensical complete integral, while a single series is sufficient [for a particular integral]. This inconvenience for each solution can happen in two ways; certainly for the first solution, where the exponent λ is required to be defined from this equation $\lambda(\lambda-1)a+\lambda c+f=0$, then it is elicited from a single value for λ, if there should be either $a=0$ or $4af=(a-c)^2$. Only in the first case there shall be $\lambda=-\frac{f}{c}$, in the other case the value of λ becoming as if infinite, now in the second case both values of λ become equal to each other, clearly $\lambda=\frac{a-c}{2a}$.

The same inconvenience has to be considered in the other solution, if there should be either $b = 0$ or $4bg=(b-e)^2$, from which it becomes apparent that one solution labours under this inconvenience, while the other is free from that, as also each indeed is corrupted by the same. On account of which that may be appropriate to be shown, as the complete integral must be investigated in these cases also; with which view also we refer to the case in which both the values of λ become imaginary, since it is necessary to remove the imaginary kind by a particular trick. Now finally also the two series for *y* shown are difficult to put into use, whenever the values of λ are divisible by a different exponent *n*, the setting out of which cases also is worth explaining.

PROBLEM 123

973. *For the proposed second order differential equation*

$$xx\left(a+bx^n\right)ddy+x\left(c+ex^n\right)dxdy+\left(f+gx^n\right)ydx^2=0,$$

if it comes about that the two ascending series assumed for y either merge together into one, or the other series becomes impossible, to express the complete integral by series.

SOLUTION

With the series assumed

$$y=Ax^{\lambda}+Bx^{\lambda+n}+Cx^{\lambda+2n}+Dx^{\lambda+3n}+\text{etc.}$$

if it should come about, that both the values of λ from the equation

$$\lambda\left(\lambda-1\right)a+\lambda c+f=0$$

either become equal or they maintain a difference divisible by *n*, [which is a degenerate case] then the value of *y* besides the powers of x also involves the logarithm of *x*. Whereby [§ 934] for the resolution of the equation we put at once $y=u+vlkx$, so that $y=u+vlx+\alpha v$ with α denoting some constant quantity. Hence there shall be

$$dy=du+\tfrac{vdx}{x}+dvlkx \text{ and } ddy=ddu+\tfrac{2dxdv}{x}-\tfrac{vdx^2}{xx}+ddvlkx,$$

with which values substituted our equation adopts this form

$$\left.\begin{aligned}&xx(a+bx^n)ddu+2x(a+bx^n)dxdv-(a+bx^n)vdx^2\\&\qquad+x(c+ex^n)dxdu+(c+ex^n)vdx^2\\&\qquad\qquad+(f+gx^n)udx^2\\&+(xx(a+bx^n)ddv+x(c+ex^n)dxdv+(f+gx^n)vdx^2)lkx\end{aligned}\right\}=0,$$

where the final part affected by the logarithm must separately be equal to zero. Whereby on putting

$$v=Ax^{\lambda}+Bx^{\lambda+n}+Cx^{\lambda+2n}+Dx^{\lambda+3n}+\text{etc.}$$

that value is attributed to the exponent λ from the equation $\lambda(\lambda-1)a+\lambda c+f=0$, which is in no manner inconvenient, and for the remainder of the coefficients there will be on putting

$$\lambda(\lambda-1)b+\lambda e+g=nh\,,$$

so that there follows,

$$\begin{aligned}
&((n+2\lambda-1)a+c)\;B+hA=0,\\
&2((2n+2\lambda-1)a+c)C+((n+2\lambda-1)b+e)B+hB=0,\\
&3((3n+2\lambda-1)a+e)E+2((2n+2\lambda-1)b+e)C+hC=0,\\
&4((4n+2\lambda-1)a+e)E+3((3n+2\lambda-1)b+e)D+hD=0
\end{aligned}$$

etc.

Thus with these coefficients defined, the first of which A is left to our choice, we may put

$$u=\mathit{\Delta}+\mathfrak{A}x^{\lambda}+\mathfrak{B}x^{\lambda+n}+\mathfrak{C}x^{\lambda+2n}+\mathfrak{D}x^{\lambda+3n}+\text{etc.}$$

which value if substituted into the first equation with the series found for v, the following series is required to be reduced to zero :

$$xx(a+bx^{n})\tfrac{dd\mathit{\Delta}}{dx^{2}}+x(c+ex^{n})\tfrac{d\mathit{\Delta}}{dx}+(f+gx^{n})\mathit{\Delta}$$

$$\begin{array}{llllll}
+\lambda(\lambda-1)\mathfrak{A}ax^{\lambda} & + & (\lambda+n)(\lambda+\mathrm{n}-1)\mathfrak{B}ax^{\lambda+n} & + & (\lambda+2n)(\lambda+2n-1)\mathfrak{C}ax^{\lambda+2n} & +\text{etc.}\\
 & + & \lambda(\lambda-1)\mathfrak{A}b & + & (\lambda+2n)(\lambda+n-1)\mathfrak{B}b & \\
+\quad\lambda\mathfrak{A}c & + & (\lambda+n)\mathfrak{B}c & + & (\lambda+2n)\mathfrak{C}c & \\
 & + & \lambda\mathfrak{A}e & + & (\lambda+n)\mathfrak{B}e & \\
+\quad\mathfrak{A}f & + & \mathfrak{B}f & + & \mathfrak{C}f & \\
 & + & \mathfrak{A}g & + & \mathfrak{B}g & \\
+\quad 2\lambda Aa & + & 2(\lambda+n)Ba & + & 2(\lambda+2n)Ca & \\
 & + & 2\lambda Ab & + & 2(\lambda+n)Bb & \\
+\quad A(c-a) & + & B(c-a) & + & C(c-a) & \\
 & + & A(e-b) & + & B(e-b) &
\end{array}$$

But since there shall be $\lambda(\lambda-1)a+\lambda c+f=0$ and $\lambda(\lambda-1)b+\lambda e+g=nh$, the expression will be transformed into this form

$$
\begin{aligned}
&xx\left(a+bx^{n}\right)\tfrac{dd\Delta}{dx^{2}}+x\left(c+ex^{n}\right)\tfrac{d\Delta}{dx}+\left(f+gx^{n}\right)\Delta+\left((2\lambda-1)a+c\right)Ax^{\lambda}\\
&+((2n+2\lambda-1)a+c)\,Bx^{\lambda+n}+((4n+2\lambda-1)a+c)\,Cx^{\lambda+2n}+\text{etc.}\\
&+\quad\left((2\lambda-1)b+e\right)A\qquad+\quad((2n+2\lambda-1)b+e)B\\
&+n((n+2\lambda-1)a+c)\mathfrak{B}\qquad+2n((2n+2\lambda-1)a+e)\mathfrak{C}\\
&+\qquad\qquad nh\mathfrak{A}\qquad+\quad n((n+2\lambda-1)b+e)\mathfrak{B}\\
&\qquad\qquad\qquad\qquad+\qquad\qquad nh\mathfrak{B}
\end{aligned}
$$

where Δ denotes certain terms of the series

$$\mathfrak{A}x^{\lambda}+\mathfrak{B}x^{\lambda+n}+\text{etc.}$$

to be put in place, thus so that in the reverse order there shall be

$$\Delta=\mathfrak{a}x^{\lambda-n}+\mathfrak{b}x^{\lambda-2n}+\mathfrak{c}x^{\lambda-3n}+\ldots\ldots+\mathfrak{i}x^{\lambda-in}.$$

Because a starting condition has to be put in place for any case, the following are to be observed.

I. A starting term cannot be put in place, unless there should be

$$\left(\lambda-in\right)(\lambda-in-1)a+(\lambda-in)c+f=0;$$

therefore since $\lambda\left(\lambda-1\right)a+\lambda c+f=0$, then there shall be

$$\lambda=in+\frac{a-c-\sqrt{\left((a-c)^{2}-4af\right)}}{2a},$$

and hence

$$\lambda=\frac{a-c+\sqrt{\left((a-c)^{2}-4af\right)}}{2a},$$

since these two values cannot agree, unless the sign for the negative root in the former, is taken as positive in the latter. But with these values of the equation there shall be

$$in=\tfrac{1}{a}\sqrt{\left((a-c)^{2}-4af\right)}\quad\text{or}\quad iinnaa=(a-c)^{2}-4af$$

and hence $f=\frac{(a-c)^2}{4a}-\frac{1}{4}iinna$, from which there becomes

$$\lambda=\frac{a-c}{2a}+\frac{1}{2}in.$$

Whereby if the proposed equation should be prepared thus, so that there shall be

$$ina=\sqrt{((a-c)^2-4af)},$$

then on taking

$$\lambda=\frac{a-c}{2a}+\frac{1}{2}in$$

and for the series *v* taking

$$v=Ax^{\lambda}+Bx^{\lambda+n}+\text{etc}.$$

it is agreed thus to establish for the other series *u* :

$$u=\mathfrak{i}x^{\lambda-in}+\ldots+\mathfrak{a}x^{\lambda-n}+\mathfrak{A}x^{\lambda}+\mathfrak{B}x^{\lambda+n}+\mathfrak{C}x^{\lambda+2n}+\text{ etc}.$$

This is the case in which the two values of λ from the equation $\lambda(\lambda-1)a+\lambda c+f=0$ have a difference divisible by *n*, where it is to be observed that the series *v* must start from a greater value of λ, truly the series *u* must start from a lower value.

II. The starting term Δ cannot be omitted, unless there should be $(2\lambda-1)a+c=0$, in which case there becomes $\lambda=\frac{a-c}{2}$; and this is the case in which the two roots of the equation $\lambda(\lambda-1)a+\lambda c+f=0$ become equal to each other, and thus $f=\frac{(a-c)^2}{4a}$. Hence this case can be contained in the preceding on taking there $i=0$.
Whereby in this manner the cases may be resolved, in which the two values of λ either are equal to each other or they have a difference divisible by *n*. And thus the complete integral is found for the two ascending series *v* et *u* expressed, of which that one *v* is multiplied by *lx*.

COROLLARY 1

974. Hence when the coefficients *a, c* and *f* thus have been prepared in the proposed equation, so that the roots of the equation $\lambda(\lambda-1)a+\lambda c+f=0$ are $\lambda=\mu$ and $\lambda=\mu-in$, with *i* denoting some positive integer, the complete integral of this kind will have the form $y=u+av+vlx$.

COROLLARY 2

975. But here the two quantities v and u thus can be determined from these equations :

$$\text{I. } xx\left(a+bx^{n}\right)ddv+x\left(c+ex^{n}\right)dxdv+\left(f+gx^{n}\right)ydv^{2}=0$$

$$\text{II. } \left.\begin{aligned} xx\left(a+bx^{n}\right)ddu+x\left(c+ex^{n}\right)dxdu+\left(f+gx^{n}\right)udx^{2}\\ +2x\left(a+bx^{n}\right)dxdv-(a+bx^{n})vdx^{2}\\ +(c+ex^{n})vdx^{2}\end{aligned}\right\}=0,$$

as there is put

$$v=Ax^{\mu}+Bx^{\mu+n}+Cx^{\mu+2n}+Dx^{\mu+3n}+\text{etc.},$$
$$u=\mathfrak{A}x^{\mu-in}+\mathfrak{B}x^{\mu-in+n}+\mathfrak{C}x^{\mu-in+2n}+\mathfrak{D}x^{\mu-in+3n}+\text{etc.}$$

Evidently these series on being substituted all the coefficients are allowed to be defined except one.

SCHOLION

976. Hence with the logarithm of x called into help in these cases that we have mentioned, the complete integral of the proposed equation can be shown by an ascending series, while without this artifice only a particular integral can be found. For when the equation $\lambda(\lambda-1)a+\lambda c+f=0$ has two roots, the difference of which is divisible by the exponent n, for example $\lambda=\mu$ and $\lambda=\mu-in$, by the first method only the series, which starts with the power x^{μ}, is able to be determined ; for if the other starting from the power $x^{\mu-in}$ is assumed for y, the coefficient of a certain term is found to be infinite, from which all the following terms become infinite too, which inconvenience on introducing the logarithm of x is happily removed. Therefore it will be helpful to make clear the use of this resolution by a few examples.

EXAMPLE 1

977. *To show the complete integral of the second order differential equation*

$$xddy + dxdy + gx^{n-1}ydx^2 = 0$$

by an ascending series.

We will have this equation $xxddy + xdxdy + gx^n ydx^2 = 0$, on reducing our form, where hence there shall be $a = 1, b = 0, c = 1, e = 0$, and $f = 0$. Hence $\lambda(\lambda - 1) + \lambda = 0$ or $\lambda\lambda = 0$, thus so that the two values of λ are equal and $= 0$. Whereby on putting $y = u + \alpha v + vlx$ these equations must be resolved

$$\text{I. } xxddv + xdxdv + gx^n vdx^2 = 0$$

and

$$\text{II. } \left.\begin{array}{l} xxddu + xdxdu + gx^n udx^2 \\ \qquad + 2xdxdv \end{array}\right\} = 0$$

Hence we put in place

$$v = A + Bx^n + Cx^{2n} + Dx^{3n} + \text{etc.}$$

and

$$u = \mathfrak{A} + \mathfrak{B}x^n + \mathfrak{C}x^{2n} + \mathfrak{D}x^{3n} + \text{etc.}$$

and the first equation gives

$$\left.\begin{array}{llll} n(n-1)Bx^n & + 2n(2n-1)Cx^{2n} & + 3n(3n-1)Dx^{3n} & + \text{etc.} \\ + \; nB & + \quad 2nC & + \quad 3nD & \\ + \; Ag & + \quad Bg & + \quad Cg & \end{array}\right\} = 0,$$

from which there shall be

$$B = \tfrac{-Ag}{nn}, \quad C = \tfrac{-Bg}{4nn}, \quad D = \tfrac{-Cg}{9nn}, \quad E = \tfrac{-Dg}{16nn} \text{ etc.}$$

Then indeed the other equation gives

$$\left.\begin{array}{llll} nn\mathfrak{B}x^n & + 4nn\mathfrak{C}x^{2n} & + 9nn\mathfrak{D}x^{3n} & + \text{etc.} \\ + \; \mathfrak{A}g & + \; \mathfrak{B}g & + \; \mathfrak{C}g & \\ + 2nB & + 4nC & + 6nD & \end{array}\right\} = 0,$$

from which there is deduced

$$\mathfrak{B} = \tfrac{-\mathfrak{A}g}{nn} - \tfrac{2B}{n}, \quad \mathfrak{C} = \tfrac{-\mathfrak{B}g}{4nn} - \tfrac{2C}{2n}, \quad ,\mathfrak{D} = \tfrac{-\mathfrak{A}g}{9nn} - \tfrac{2D}{3n}, \quad \text{etc.}$$

But here it is allowed to assume without risk that $\mathfrak{A}=0$, because the terms arising from $\mathfrak{A}$ are contained in the part αv. Therefore since there shall be

$$B=\frac{-Ag}{nn},\quad C=\frac{+Agg}{1\cdot 4n^4},\quad D=\frac{-Ag^3}{1\cdot 4\cdot 9n^6},\quad E=\frac{+Ag^4}{1\cdot 4\cdot 9\cdot 16n^8}\quad \text{etc.},$$

there will be, as follows,

$$\mathfrak{B}=\frac{2Ag}{n^3},\quad \mathfrak{A}=\frac{-2Agg}{4n^5}-\frac{2Agg}{2\cdot 1\cdot 4n^5}=\frac{-6Agg}{2\cdot 1\cdot 4n^5},$$
$$\mathfrak{D}=\frac{6Ag^3}{2\cdot 1\cdot 4\cdot 9n^7}+\frac{2Ag^3}{3\cdot 1\cdot 4\cdot 9n^7}=\frac{22Ag^3}{2\cdot 3\cdot 1\cdot 4\cdot 9n^7},$$
$$\mathfrak{E}=\frac{-22Ag^4}{2\cdot 3\cdot 1\cdot 4\cdot 9\cdot 16n^9}-\frac{2Ag^4}{4\cdot 1\cdot 4\cdot 9\cdot 16n^9}=\frac{-100Ag^4}{2\cdot 3\cdot 1\cdot 4\cdot 9\cdot 16n^9},$$
$$\mathfrak{F}=\frac{100Ag^5}{2\cdot 3\cdot 4\cdot 1\cdot 4\cdot 9\cdot 16\cdot 25n^{11}}+\frac{2Ag^5}{5\cdot 1\cdot 4\cdot 9\cdot 16\cdot 25n^{11}}=\frac{548Ag^5}{2\cdot 3\cdot 4\cdot 5\cdot 1\cdot 4\cdot 9\cdot 16\cdot 25n^{11}}$$
$$\text{etc.}$$

and thus the following values may be obtained :

$$\mathfrak{B}=\frac{2Ag}{n^3},\quad \mathfrak{C}=\frac{-6Agg}{1\cdot 8n^5},\quad \mathfrak{D}=\frac{22Ag^3}{1\cdot 8\cdot 27n^7},\quad \mathfrak{E}=\frac{-100Ag^4}{1\cdot 8\cdot 27\cdot 64n^9},$$
$$\mathfrak{E}=\frac{548Ag^5}{1\cdot 8\cdot 27\cdot 64\cdot 125n^{11}},\quad \mathfrak{G}=\frac{-3528Ag^6}{1\cdot 8\cdot 27\cdot 64\cdot 125\cdot 216n^{13}}\quad \text{etc.},$$

where the individual numerators 2, 6, 22, 100, 548, 3528 etc. thus are defined by the two preceding

$$6=3\cdot 2-1\cdot 0,\ 22=5\cdot 6-4\cdot 2,\ 100=7\cdot 22-9\cdot 6,$$
$$548=9\cdot 100-16\cdot 22,\ 3528=11\cdot 548-25\cdot 100\quad \text{etc.}$$

Consequently the integral is expressed thus :

$$y=\frac{2Ag}{n^3}x^n-\frac{6Agg}{1\cdot 8n^5}x^{2n}+\frac{22Ag^3}{1\cdot 8\cdot 27n^7}x^{3n}-\frac{100Ag^4}{1\cdot 8\cdot 27\cdot 64n^9}x^{4n}+\text{etc.}$$
$$+A\left(1-\frac{g}{nn}x^n+\frac{gg}{1\cdot 4n^4}x^{2n}-\frac{g^3}{1\cdot 4\cdot 9n^6}x^{3n}+\frac{g^4}{1\cdot 4\cdot 9\cdot 16n^6}x^{4n}-\text{etc.}\right)lx$$
$$+\alpha-\frac{\alpha g}{nn}x^n+\frac{\alpha gg}{1\cdot 4n^4}x^{2n}-\frac{\alpha g^3}{1\cdot 4\cdot 9n^6}x^{3n}+\frac{\alpha g^4}{1\cdot 4\cdot 9\cdot 16n^8}x^{4n}-\text{etc.}$$

where A and *a* are two arbitrary constants.

EXAMPLE 2

978. *To assign the complete integral of the second order differential equation*

$$x(1-xx)ddy-(1+xx)dxdy+xydx^2=0$$

by an ascending series.

Our equation is reduced to the form

$$x(1-xx)ddy-(1+xx)dxdy+xydx^2=0\,,$$

thus, so that there shall be $n=2,\ a=1,\ b=-1,\ c=-1,\ e=-1, f=0$ and $g=1$, from which the roots of the equation $\lambda(\lambda-1)-\lambda=0$ shall be $\lambda=0$ et $\lambda=2$, the difference of which divided by $n=2$ gives 1. Hence on putting $y=u+\alpha v+vlx$ there must be put in place

$$v=Ax^2+Bx^4+Cx^6+Dx^8+\text{etc.}$$

and

$$u=\mathfrak{A}+\mathfrak{B}x^2+\mathfrak{C}x^4+\mathfrak{D}x^6+\mathfrak{E}x^8+\text{etc.}$$

which series must be determined from the following equations :

I. $xx(1-xx)ddv-x(1+xx)dxdv+xxvdx^2=0,$

II. $xx(1-xx)ddu-x(1+xx)dxdu+xxudx^2+2x(1-xx)dxdv-2vdx^2=0.$

Hence for the determination of the first there becomes

$$\left.\begin{array}{rrrrr}
2Ax^2 & +12Bx^4 & +30Cx^6 & +56Dx^8 & +\text{ etc.}\\
 & -2A & -12B & -30C & \\
-2A & -4B & -6C & -8D & \\
 & -2A & -4B & -6C & \\
 & +A & +B & +C &
\end{array}\right\}=0$$

and thus

$$2\cdot 4B=1\cdot 3A,\ 4\cdot 6C=3\cdot 5B,\ 6\cdot 8D=5\cdot 7C\ \text{ etc.}$$

or

$$B=\tfrac{1\cdot 3}{2\cdot 4}A,\ C=\tfrac{1\cdot 3\cdot 3\cdot 5}{2\cdot 4\cdot 4\cdot 6}A,\ D=\tfrac{1\cdot 3\cdot 3\cdot 5\cdot 5\cdot 7}{2\cdot 4\cdot 4\cdot 6\cdot 6\cdot 8}A\quad\text{etc.}$$

Now there is found from the other equation

$$\left.\begin{array}{llllll}
2\mathfrak{B}x^2 & +12\mathfrak{C}x^4 & +30\mathfrak{D}x^6 & +56\mathfrak{E}x^8 & + \text{ etc.}\\
 & -2\mathfrak{B} & -12\mathfrak{C} & -30\mathfrak{D} & \\
-2\mathfrak{B} & -4\mathfrak{C} & -\ 6\mathfrak{D} & -\ 8\mathfrak{E} & \\
 & -2\mathfrak{B} & -\ 4\mathfrak{C} & -\ 6\mathfrak{D} & \\
+\ \mathfrak{A} & +\ \mathfrak{B} & +\ \mathfrak{C} & +\ \mathfrak{D} & \\
+4A & +\ 8B & +12C & +16D & \\
 & -\ 4A & -\ 8B & -12C & \\
-2A & -\ 2B & -\ 2C & -2D &
\end{array}\right\} = 0,$$

from which there is required to become

$$\mathfrak{A}+2A=0,\quad 2\cdot4\mathfrak{C}-1\cdot3\mathfrak{B}+6B-4A=0,\ 4\cdot6\mathfrak{D}-3\cdot5\mathfrak{C}+10C-8B=0,$$
$$6\cdot8\mathfrak{E}-5\cdot7\mathfrak{D}+14D-12C=0\ \text{ etc.},$$

or since there shall be

$$B=\tfrac{1\cdot3}{2\cdot4}A,\quad C=\tfrac{3\cdot5}{4\cdot6}B,\quad D=\tfrac{5\cdot7}{6\cdot8}C\quad \text{etc.},$$

there shall be $\mathfrak{A}=-2A$, then indeed

$$2\cdot4\mathfrak{C}-\ 1\cdot3\mathfrak{B}-\tfrac{2\cdot7}{2\cdot4}A=0,\quad \mathfrak{C}=\tfrac{1\cdot3}{2\cdot4}\mathfrak{B}+\tfrac{2\cdot7}{2^2\cdot4^2}A,$$
$$4\cdot6\mathfrak{D}-\ 3\cdot5\mathfrak{C}-\tfrac{2\cdot21}{4\cdot6}B=0,\quad \mathfrak{D}=\tfrac{3\cdot5}{4\cdot6}\mathfrak{C}+\tfrac{2\cdot21}{4^2\cdot6^2}B,$$
$$6\cdot8\mathfrak{E}\ -5\cdot7\mathfrak{D}-\tfrac{2\cdot43}{6\cdot8}C=0,\quad \mathfrak{E}=\tfrac{5\cdot7}{6\cdot8}\mathfrak{D}+\tfrac{2\cdot43}{6^2\cdot8^2}C,$$
$$8\cdot10\mathfrak{F}-7\cdot9\mathfrak{D}-\tfrac{2\cdot73}{8\cdot10}D=0,\quad \mathfrak{F}=\tfrac{7\cdot9}{8\cdot10}\mathfrak{E}+\tfrac{2\cdot73}{8^2\cdot10^2}D$$
$$\text{etc.}$$

While therefore there may be taken $\mathfrak{A}=-2A$, nothing hinders the letter $\mathfrak{B}$ from taking any value it pleases, and it may be put equal to zero, if indeed the constant α above has been introduced.

EXAMPLE 3

979. *To show the complete integral of the second-order differential equation*

$$xx(1+bxx)ddy + x(-5+exx)dxdy + (5+gxx)ydx^2 = 0$$

by an ascending integral.

Because here there is $a=1, c=-5$ and $f=5$, the equation $\lambda(\lambda-1)-5\lambda+5=0$ or $\lambda\lambda-6\lambda+5=0$ has the roots $\lambda=1$ and $\lambda=5$, the difference of which 4 can be divided by $n=2$. Hence on putting $y=u+\alpha v+vlx$ there is put in place

$$v = Ax^5 + Bx^7 + Cx^9 + Dx^{11} + Ex^{13} + \text{etc.}$$

and

$$u = \mathfrak{A}x + \mathfrak{B}x^3 + \mathfrak{C}x^5 + \mathfrak{D}x^7 + \text{etc.};$$

the equations to be resolved will be

$$\text{I. } xx(1+bxx)ddv + x(-5+exx)dxdv + (5+gxx)vdx^2 = 0$$

and

$$\left.\begin{array}{r} \text{II. } xx(1+bxx)ddu + x(-5+exx)dxdu + (5+gxx)udx^2 \\ +\ 2x(1+bxx)dxdv - (1+bxx)\,vax^2 \\ +(-5+exx)vdx^2 \end{array}\right\} = 0,$$

where the former leads to

$$\left.\begin{array}{lllll} 5\cdot 4Ax^5 & +7\cdot 6Bx^7 & +9\cdot 8Cx^9 & +11\cdot 10Dx^{11} & +\text{ etc.} \\ -5\cdot 5A & -5\cdot 7B & -5\cdot 9C & -5\cdot 11D & \\ +\ 5A & +\ 5B & +\ 5C & +\ 5D & \\ & +5\cdot 4Ab & +7\cdot 6Bb & +\ 9\cdot 8Cb & \\ & +\ 5Ae & +\ 7Be & +\ 9Ce & \\ & +\ Ag & +\ Bg & +\ Cg & \end{array}\right\} = 0,$$

the latter to

$$\left.\begin{array}{llllll}
 & +2\cdot 3\mathfrak{B}x^3 & +4\cdot 5\mathfrak{C}x^5 & +6\cdot 7\mathfrak{D}x^7 & +8\cdot 9\mathfrak{E}x^9 & +\text{etc.}\\
-5\mathfrak{A}x & -5\cdot 3\mathfrak{B} & -5\cdot 5\mathfrak{C} & -5\cdot 7\mathfrak{D} & -5\cdot 9\mathfrak{E} & \\
+5\mathfrak{A} & +5\mathfrak{B} & +5\mathfrak{C} & +5i\mathfrak{D} & +5\mathfrak{E} & \\
 & & +2\cdot 3\mathfrak{B}b & +4\cdot 5\mathfrak{C}b & +6\cdot 7\mathfrak{D}b & \\
 & +\mathfrak{A}e & +3\mathfrak{B}e & +5\mathfrak{C}e & +7\mathfrak{D}e & \\
 & +\mathfrak{A}g & +\mathfrak{B}g & +\mathfrak{C}g & +\mathfrak{D}g & \\
 & & +2\cdot 5A & +2\cdot 7B & +2\cdot 9C & \\
 & & -6A & -6B & -6C & \\
 & & & +2\cdot 5Ab & +2\cdot 7Bb & \\
 & & & -Ab & -Bb & \\
 & & & +Ae & +Be &
\end{array}\right\} = 0.$$

From thence there becomes

$$\begin{array}{lll}
12B+A(20b+5e+g)=0 & \text{or} & 2\cdot 6B+A(4\cdot 5b+5e+g)=0,\\
32C+B(42b+7e+g)=0 & & 4\cdot 8C+B(6\cdot 7b+7e+g)=0,\\
60D+C(72b+9e+g)=0 & & 6\cdot 10D+C(8\cdot 9b+9e+g)=0\\
\text{etc.} & & \text{etc.}
\end{array}$$

But hence

$$\begin{array}{l}
-4\mathfrak{B}+\mathfrak{A}(e+g)=0,\\
0\mathfrak{C}+\mathfrak{B}(2\cdot 3b+3e+g)+4A=0,\\
2\cdot 6\mathfrak{D}+\mathfrak{C}(4\cdot 5b+5e+g)+8B+A(9b+e)=0,\\
4\cdot 8\mathfrak{E}+\mathfrak{D}(6.7b+7e+g)+12C+B(13b+e)=0\\
\text{etc.}
\end{array}$$

From the former formulas the letters *B*, *C*, *D* etc. are determined by *A*, from the latter the second becomes $\mathfrak{B}=\frac{-4A}{2\cdot 3b+3e+g}$, but from the first $\mathfrak{A}=\frac{4\mathfrak{A}}{e+g}$, then truly $\mathfrak{C}$ can be for any it pleases to assume and then the remaining coefficients $\mathfrak{D}$, $\mathfrak{E}$, $\mathfrak{F}$ etc. are defined.

SCHOLIUM

980. This example supplies us with the opportunity to observe certain unusual phenomena. Evidently even if the complete integral in general involves lx, yet that produces certain cases free from logarithms.

Certainly in the first place if there should be $g=-e$, there becomes $\mathfrak{B}=0$ with $\mathfrak{A}$ remaining undefined, then truly on account of $\mathfrak{B}=0$ it is required to take $A=0,\ B=0$ etc. and thus $v=0$. Again indeed there shall be

$$2\cdot 6\mathfrak{D}+4\mathfrak{C}(5b+e)=0,$$
$$4\cdot 8\mathfrak{E}+6\mathfrak{D}(7b+e)=0,$$
$$6\cdot 10\mathfrak{F}+8\mathfrak{E}(9b+e)=0$$
$$\text{etc.},$$

where $\mathfrak{C}$ is an arbitrary constant, and the complete integral of the equation

$$xx(1+bxx)ddy+x(-5+exx)dxdy+(5-exx)ydx^2=0$$

will be

$$y=\mathfrak{A}x+\mathfrak{C}x^5+\mathfrak{D}x^7+\mathfrak{E}x^9+\mathfrak{F}x^{11}+\text{etc.},$$

which thus is expressed finitely if $e=-(2i+5)b$ by taking for i the numbers 0, 1, 2, 3, 4 etc.

In the second case if there shall be

$$2\cdot 3b+3e+g=0 \text{ or } g=-6b-3e, \text{there becomes } \mathfrak{B}=-\tfrac{1}{2}\mathfrak{A}(3\mathrm{b}+e),$$

then truly $A=0, B=0, C=0$ etc., hence $v=0$. Again indeed there is found

$$\mathfrak{D}=-\tfrac{1}{6}\mathfrak{C}(7b+e),\quad \mathfrak{E}=-\tfrac{1}{8}\mathfrak{D}(9b+e),\quad \mathfrak{F}=-\tfrac{1}{10}\mathfrak{E}(11b+e)\ \text{etc.}$$

and hence

$$y=\mathfrak{A}x-\tfrac{1}{2}\mathfrak{A}\left(3b+e\right)x^3+\mathfrak{C}x^5+\mathfrak{D}x^7+\mathfrak{E}x^9+\text{etc.},$$

where $\mathfrak{A}$ and $\mathfrak{C}$ remain for our arbitrary constants.

In the third place if there shall be $4\cdot 5b+5e+g=0$ or $g=-20b-5e$, in the first place there becomes $B=0, C=0, D=0$ etc. and thus $v=Ax^5$, then truly

$$\mathfrak{B}=-\mathfrak{A}(5b+e),\quad -\mathfrak{B}(14b+2e)+4A=0 \text{ or } \quad \mathfrak{B}=\tfrac{2A}{7b+e}$$

and hence

$$A=-\tfrac{1}{2}\mathfrak{A}(5b+e)(7b+e),$$

again

$$2\cdot 6\mathfrak{D} \;+\; A(9b+e)=0,$$
$$4\cdot 8\mathfrak{E} \;+2\mathfrak{D}(11b+e)=0,$$
$$6\cdot 10\mathfrak{F}+4\mathfrak{E}(13b+e)\;=0$$
etc.

Hence the coefficients $\mathfrak{B}, A, \mathfrak{D}, \mathfrak{E}, \mathfrak{F}$ etc. are defined by $\mathfrak{A}$, and $\mathfrak{C}$ too remains for our arbitrary constant, from which the complete integral in this case will be

$$y = Ax^5 lx + \mathfrak{C}x^5 + \mathfrak{A}x + \mathfrak{B}x^3 + * + \mathfrak{D}x^7 + \mathfrak{E}x^9 + \text{etc.},$$

which expression becomes finite, whenever $(2i+5)b+e=0$.

EXAMPLE 4

981. *If there shall be in the first example* $e=-7b$ *and* $g=15b$,

$$xx(1+bxx)ddy - x(5+7bxx)dxdy + 5(1+3bxx)ydx^2 = 0$$

to show the complete integral of the equation.

Hence there shall be $\mathfrak{B}=2\mathfrak{A}b,\ A=0,\ \mathfrak{D}=0,\ \mathfrak{E}=0$ and thus $v=0$ and

$$u = \mathfrak{A}x + 2\mathfrak{A}bx^3 + \mathfrak{C}x^5,$$

from which on taking some constants for $\mathfrak{A}$ and $\mathfrak{C}$ there will be the complete integral

$$y = \mathfrak{A}x(1+2bxx) + \mathfrak{C}x^5.$$

Hence there will be the particular integrals

$$y = ax(1+2bxx),\ y = ax^5,\ y = ax(1+bxx)^2.$$

COROLLARY 1

982. On putting $y = e^{\int zdx}$, so that there becomes $z = \frac{dy}{ydx}$, the complete integral of this first order differential equation

$$xx(1+bxx)dz + xx(1+bxx)zzdx - x(5+7bxx)zdx + 5(1+3bxx)dx = 0$$

will be

$$z = \frac{\mathfrak{A}(1+6bxx)+5\mathfrak{C}x^4}{\mathfrak{A}x(1+2bxx)+\mathfrak{C}x^5}.$$

COROLLARY 2

983. But the equation of the second order differential equation is reduced, if it is divided by $xx(1+bxx)^2$, and the integral will be

$$\frac{xdy-5ydx}{x(1+bxx)} = Cdx \quad \text{or} \quad dy - \frac{5ydx}{x} = Cdx(1+bxx),$$

which divided by x^5 gives the integral

$$\frac{y}{x^5} = \frac{-C}{4x^4} - \frac{bC}{2x^2} + D \quad \text{or} \quad y = -\tfrac{1}{4}Cx(1+2bxx) + Dx^5$$

as before.

SCHOLIUM

984. But at this stage the complete integration of our equation by ascending series fails in the case, in which $a = 0$ and thus $\lambda c + f = 0$, from which one value for the exponent λ is defined $\lambda = \frac{-f}{c}$, which supplies only a particular integral, and this also is removed, if there should be $c = 0$. Because moreover in these cases a is = 0, it is necessary that the coefficient b certainly be present, from which the complete integral can be shown by descending powers, since the equation $\lambda(\lambda-1)b + \lambda e + g = 0$ always contains two roots, from which the twofold series may be obtained. But likewise here an inconvenience in the use can come about, when the two roots of λ are either equal or they have a difference divisible by the exponent n. Now for this inconvenience on introducing a series multiplied by lx in a like manner a cure is brought about, which we have used in solving this problem, and it would be superfluous to repeat that derivation here. But if moreover the two roots of λ both for the ascending as well as the descending series become imaginary, that remains to be shown, how the complete integral is required to be expressed by infinite series.

PROBLEM 124

985. *For the proposed second order differential equation*

$$xx(a+bx^n)ddy + x(c+ex^n)dxdy + (f+gx^n)ydx^2 = 0$$

if it should come about, that the equation $\lambda(\lambda-1)a+\lambda c+f=0$ *should have imaginary roots, to show the complete integral by an ascending series.*

SOLUTION

From the solution produced above (§ 971), it is deduced in this case that there must be put in place

$$y = v\sin.\mu lx + u\cos.\mu lx,$$

from which there becomes

$$dy = (dv - \tfrac{\mu udx}{x})\sin.\mu lx + (\tfrac{\mu vdx}{x} + du)\cos.\mu lx$$

and

$$\begin{aligned} ddy = &(ddv - \tfrac{2\mu dxdu}{x} + \tfrac{\mu udx^2}{xx} - \tfrac{\mu\mu vdx^2}{xx})\sin.\mu lx \\ &+ (ddu + \tfrac{2\mu dxdv}{x} - \tfrac{\mu vdx^2}{xx} - \tfrac{\mu\mu udx^2}{xx})\cos.\mu lx; \end{aligned}$$

with which substitution made if we reduce separately both the terms affecting $\sin.\mu lx$ as well as $\cos.\mu lx$ to zero, we will obtain the following equations :

$$\text{I. } \left.\begin{aligned} xx(a+bx^n)ddv + x(c+ex^n)dxdv + (f+gx^n)vdx^2 \\ -2\mu x(a+bx^n)dxdu - \mu\mu(a+bx^n)vdx^2 \\ +\mu(a+bx^n)udx^2 \\ -\mu(c+ex^n)udx^2 \end{aligned}\right\} = 0,$$

$$\text{II. } \left.\begin{aligned} xx(a+bx^n)ddu + x(c+ex^n)dxdu + (f+gx^n)udx^2 \\ +2\mu x(a+bx^n)dxdv - \mu\mu(a+bx^n)udx^2 \\ -\mu(a+bx^n)vdx^2 \\ +\mu(c+ex^n)vdx^2 \end{aligned}\right\} = 0.$$

Now we assume these ascending series for *v* and *u*

$$v = Ax^{\lambda} + Bx^{\lambda+n} + Cx^{\lambda+2n} + Dx^{\lambda+3n} + \text{etc.},$$
$$u = \mathfrak{A}x^{\lambda} + \mathfrak{B}x^{\lambda+n} + \mathfrak{C}x^{\lambda+2n} + \mathfrak{D}x^{\lambda+3n} + \text{etc.}$$

and with these substituted the first equation will go into this :

$$\lambda(\lambda-1)Aax^{\lambda} + (\lambda+n)(\lambda+n-1)Bax^{\lambda+n} + (\lambda+2n)(\lambda+2n-1)Cax^{\lambda+2n} + \text{etc.} = 0$$
$$\begin{array}{lrr}
 & +\ \lambda(\lambda-1)Ab & +\ (\lambda+n)(\lambda+n-1)Bb \\
+\ \lambda Ac & +\ (\lambda+n)Bc & +\ (\lambda+2n)Cc \\
 & +\ \lambda Ae & +\ (\lambda+n)Be \\
+\ Af & +\ Bf & +\ Cf \\
 & +\ Ag & +\ Bg \\
-2\mu\lambda\mathfrak{A}a & -\ 2\mu(\lambda+n)\mathfrak{B}a & -\ 2\mu(\lambda+2n)\mathfrak{C}a \\
 & -\ 2\mu\lambda\mathfrak{A}b & -\ 2\mu(\lambda+n)\mathfrak{B}b \\
-\mu\mu Aa & -\ \mu\mu Ba & -\ \mu\mu Ca \\
 & -\ \mu\mu Ab & -\ \mu\mu Bb \\
+\mu\mathfrak{A}a & +\ \mu\mathfrak{B}a & +\ \mu\mathfrak{C}a \\
 & +\ \mu\mathfrak{A}b & +\ \mu\mathfrak{B}b \\
-\mu\mathfrak{A}c & -\ \mu\mathfrak{B}c & -\ \mu\mathfrak{C}c \\
 & -\ \mu\mathfrak{A}e & -\ \mu\mathfrak{B}e
\end{array}$$

Hence the other equation is easily formed by permuting the latin and gothic letters and by changing the sign of μ in addition.

Hence the first power x^{λ} removes these equations

$$A(\lambda(\lambda-1)a + \lambda c + f - \mu\mu a) - \mu\mathfrak{A}(2\lambda a - a + c) = 0,$$
$$\mathfrak{A}(\lambda(\lambda-1)a + \lambda c + f - \mu\mu a) + \mu A(2\lambda a - a + c) = 0,$$

for which it is necessary that there shall be both

$$2\lambda a - a + c = 0$$

as well as

$$\lambda(\lambda-1)a + \lambda c + f - \mu\mu a = 0\,.$$

Thence there becomes $\lambda = \frac{1}{2} - \frac{c}{2a}$, which value substituted here gives

$$-a(\tfrac{1}{4}-\tfrac{cc}{4aa})+\tfrac{c}{2}-\tfrac{cc}{2a}+f=\mu\mu a=-\tfrac{a}{4}+\tfrac{c}{2}-\tfrac{cc}{4a}+f$$

or

$$\mu\mu a=\tfrac{4af-(a-c)^2}{4a}\quad\text{and thus}\quad \mu=\tfrac{\sqrt{\left(4af-(a-c)^2\right)}}{2a}\quad\text{and}\quad \lambda=\tfrac{a-c}{2a}.$$

From which it is apparent that this solution has to be considered, if $4af>(a-c)^2$, in which case the preceding solution becomes imaginary [§ 967] . But here the quantities A and $\mathfrak{A}$ are left arbitrary in our equation.

Now the term $x^{\lambda+n}$ demands these equations on both sides :

$$B\big((\lambda+n)(\lambda+n-1)a+(\lambda+n)c+f-\mu\mu a\big)+A\big(\lambda(\lambda-1)b+\lambda e+g-\mu\mu b\big)$$
$$-\mu\mathfrak{B}\big(2(\lambda+n)a-a+c\big)-\mu\mathfrak{A}(2\lambda b-b+e)=0$$

and

$$\mathfrak{B}\big((\lambda+n)(\lambda+n-1)a+(\lambda+n)c+f-\mu\mu a\big)+\mathfrak{A}\big(\lambda(\lambda-1)b+\lambda e+g-\mu\mu b\big)$$
$$+\mu B\big(2(\lambda+n)a-a+c\big)+\mu A(2\lambda b-b+e)=0.$$

For the sake of brevity let

$$(\lambda+n)(\lambda+n-1)a+(\lambda+n)c+f-\mu\mu a=\alpha,$$
$$\lambda(\lambda-1)b+\lambda e+g-\mu\mu b=\beta,$$
$$2(\lambda+n)a-a+c=2na=\gamma,$$
$$2\lambda b-b+e=\delta,$$

so that we may have

$$B\alpha+A\beta-\mu\mathfrak{B}\gamma-\mu\mathfrak{A}\delta=0\quad\text{and}\quad\mathfrak{B}\alpha+\mathfrak{A}\beta+\mu B\gamma+\mu A\delta=0,$$

from which there is deduced

$$B=\tfrac{-A(\alpha\beta+\mu\mu\gamma\delta)+\mu\mathfrak{A}(\alpha\delta-\beta\gamma)}{\alpha\alpha+\mu\mu\gamma\gamma}\quad\text{and}\quad\mathfrak{B}=\tfrac{-\mathfrak{A}(\alpha\beta+\mu\mu\gamma\delta)-\mu A(\alpha\delta-\beta\gamma)}{\alpha\alpha+\mu\mu\gamma\gamma}.$$

But now from the assumed values there is

$$\alpha=nna,\quad\beta=\tfrac{(ae-bc)(a-c)}{2aa}-\tfrac{bf}{a}+g,\quad\gamma=2na,\quad\delta=\tfrac{ae-bc}{a},$$

from which the assumed values A and $\mathfrak{A}$ there are defined B and $\mathfrak{B}$ and hence again $C,\mathfrak{C},D,\mathfrak{D}$ etc.

EXAMPLE 1

986. *Let $c = a$ and $f = a$, so that there becomes $\mu = 1$, and the integral of this equation is investigated*

$$xx\left(a + bx^n\right)ddy + x\left(c + ex^n\right)dxdy + \left(f + gx^n\right)ydx^2 = 0.$$

Hence here there shall be $\lambda = 0$ and $\mu = 1$, from which on putting $y = v\sin.lx + u\cos.lx$ and on taking the series for *v* and *u*

$$v = A + Bx^n + Cx^{2n} + Dx^{3n} + \text{etc.},$$
$$u = \mathfrak{A} + \mathfrak{B}x^n + \mathfrak{C}x^{2n} + \mathfrak{D}x^{3n} + \text{etc.}$$

the coefficients *A* and $\mathfrak{A}$ are able to be taken as it pleases. From these in the first place on account of $\alpha = nna, \beta = g - b,\ \gamma = 2na$ and $\delta = e - b$ there shall be

$$B = \frac{-A(nna(g-b)+2na(e-b))+\mathfrak{A}(nna(e-b)-2na(g-b))}{n^4aa+4nnaa}$$

or

$$B = \frac{-A\left(n(g-b)+2(e-b)\right)+\mathfrak{A}\left(n(e-b)-2(g-b)\right)}{na(nn+4)}$$

and

$$\mathfrak{B} = \frac{-\mathfrak{A}\left(n(g-b)+2(e-b)\right)-A\left(n(e-b)-2(g-b)\right)}{na(nn+4)}$$

For the following coefficients we will have

$$C(2n(2n-1)a + 2na + a - a) + B\left(n(n-1)b + ne + g - b\right)$$
$$-\mathfrak{C}(4na - a + a) - \mathfrak{B}(2nb - b + e) = 0$$

or

$$4nnCa + B((nn - n - 1)b + ne + g) - 4n\mathfrak{C}a - \mathfrak{B}((2n-1)b + e) = 0$$

and

$$4nn\mathfrak{C}a + \mathfrak{B}((nn - n - 1)b + ne + g) + 4nCa + B((2n-1)b + e) = 0\ ,$$

of which that multiplied by *n* is added to this, so that there is produced

$$4n(nn+1)Ca + B((n^3 - nn + n - 1)b + (nn+1)e + ng)$$
$$+\mathfrak{B}(-(nn+1)b + g) = 0,$$

hence

$$C = \frac{-B((n-1)(nn+1)b+(nn+1)e+ng)+\mathfrak{B}((nn+1)b-g)}{4na(nn+1)}$$

and

$$\mathfrak{C} = \frac{-\mathfrak{B}((n-1)(nn+1)b+(nn+1)e+ng)-B((nn+1)b-g)}{4na(nn+1)}.$$

Again there shall be

$$9nnDa + C((4nn-2n-1)b+2ne+g) - 6n\mathfrak{D}a - \mathfrak{C}((4n-1)b+e) = 0,$$
$$9nn\mathfrak{D}a + \mathfrak{C}((4nn-2n-1)b+2ne+g) + 6nDa + C((4n-1)b+e) = 0,$$

of which that multiplied by *3n*, now this by 2 joined together give

$$3n(9nn+4)Da + C((12n^3-6nn+5n-2)b+2(3nn+1)e+3ng)$$
$$+\mathfrak{C}((-4nn-n-2)b+ne+2g) = 0,$$

from which it follows

$$D = \frac{-C((12n^3-6nn+5n-2)b+2(3nn+1)e+3ng)+\mathfrak{C}((4nn+n+2)b-ne-2g)}{3n(9nn+4)a},$$
$$\mathfrak{D} = \frac{-\mathfrak{C}((12n^3-6nn+5n-2)b+2(3nn+1)e+3ng)-C((4nn+n+2)b-ne-2g)}{3n(9nn+4)a}.$$

But in general from whatever coefficients M and $\mathfrak{M}$, the following N and $\mathfrak{N}$ are defined by these formulas

$$in(iinn+4)Na + M((i(i-1)^2n^3 - i(i-1)nn + (3i-4)n-2)b$$
$$+i(i-1)nne + 2e + ing) - \mathfrak{M}((2(i-1)nn + (i-2)n+2)b - (i-2)ne - 2g) = 0,$$
$$in(iinn+4)\mathfrak{N}a + \mathfrak{M}((i(i-1)^2n^3 - i(i-1)nn + (3i-4)n-2)b + i(i-1)nne + 2e + ing)$$
$$+M((2(i-1)nn + (i-2)n+2)b - (i-2)ne - 2g) = 0.$$

COROLLARY 1

987. If the quantities *b*, *e*, *g* are prepared thus, in order that the two corresponding letters N and $\mathfrak{N}$ vanish, all the following vanish and the complete integral is expressed in a finite form. Thus, in order that B and $\mathfrak{B}$ vanish, there must become

$$2(g-b) = n(e-b) \text{ and } n(g-b) = -2(e-b),$$

from which there becomes $g = e = b$, and the equation proposed itself has the factor $a + bx^n$.

COROLLARY 2

988. Moreover in general the integral will be expressed finitely, if with *i* denoting some positive whole number there shall be

$$g=\left(\left(i-1\right)nn+\tfrac{1}{2}\left(i-2\right)n+1\right)b-\tfrac{1}{2}\left(i-2\right)ne,$$

then indeed

$$e=-\left(2\left(i-1\right)n-1\right)b,$$

from which there becomes

$$g=\left(\left(i-1\right)^{2}nn+1\right)b.$$

EXAMPLE 2

989. *On taking* $n=1$ *if there shall be* $e=-b$ *and* $g=2b$, *to assign the complete integral of this equation*

$$xx(a+bx)ddy+x(a-bx)dxdy+(a+2bx)ydx^{2}=0.$$

From the formulas found in the manner we can deduce

$$B=\tfrac{-A(g+2e-3b)+\mathfrak{A}(e+b-2g)}{5a}=\tfrac{3Ab-4\mathfrak{A}b}{5a}\text{ and }\mathfrak{B}=\tfrac{3\mathfrak{A}b+4Ab}{5a},$$

then truly

$$C=\tfrac{-B\left(2e+g\right)+\mathfrak{B}\left(2b-g\right)}{8a}=0\text{ and }\mathfrak{C}=0.$$

On account of which we will have

$$v=A+\tfrac{(3A-4\mathfrak{A})b}{5a}x\text{ and }u=\mathfrak{A}+\tfrac{(3\mathfrak{A}+4A)b}{5a}x$$

and hence the complete integral is elicited

$$y=A\sin.lx+\mathfrak{A}\cos.lx+\tfrac{bx}{5a}((3A-4\mathfrak{A})\sin.lx+(3\mathfrak{A}+4A)\cos.lx).$$

COROLLARY 1

990. On taking $\mathfrak{A}=0$ the particular integral will be had :

$$y=A(\sin.lx+\tfrac{3bx}{5a}\sin.lx+\tfrac{4bx}{5a}\cos.lx)\ ;$$

but if there shall be $A=0$, another will be had :

$$y = \mathfrak{A}(\cos.lx - \tfrac{4bx}{5a}\sin.lx + \tfrac{3bx}{5a}\cos.lx).$$

COROLLARY 2

991. On putting $y = e^{\int sdx}$ our equation is reduced to this :

$$xx(a+bx)ds + xx(a+bx)ssdx + x(a-bx)sdx + (a+2bx)dx = 0,$$

the integral of which may be had $s = \frac{dy}{ydx}$ thence being defined ; which equation can be transformed into many other forms.

SCHOLIUM

992. In a similar manner the integration by descending series can be put in place, if the exponents of the individual terms become imaginary ; for which indeed there will be no need to be set out separately. And these are sufficient, as it may be apparent, for in the resolution of equations by infinite series they shall require to be used with caution. Moreover the greatest use of the development of these consists in this, that they are able to show second order differential equations, by which at least it is permitted to assign a particular algebraic integral, which cases we have indicated above § 969.

Again the integration by infinite series in a like manner can be extended to equations of the same kind

$$xx(a+bx^n+\beta x^{2n})ddy + x(c+ex^n+\varepsilon x^{2n})dxdy + (f+gx^n+\gamma x^{2n})ydx^2 = 0 \ ;$$

but then any term of the series sought is determined from the two preceding terms, thus so that, if two in contact vanish, all the following shall become zero. But if an empty term of *y* should appear, then the resolution in series becomes easier, to which therefore I agree not to tarry over. Just as if this equation is proposed :

$$xxddy - xdxdy + ax^n ydx^2 = bx^m dx^2$$

the series being considered starting from the power x^m

$$y = Ax^m + Bx^{m+n} + Cx^{m+2n} + Dx^{m+3n} + \text{etc.},$$

from which there becomes

$$\left.\begin{array}{llllll} m(m-1)Ax^m & + (m+n)(m+n-1)Bx^{m+n} & + (m+2n)(m+2n-1)Cx^{m+2n} & + \text{ etc.} \\ -mA & - \qquad (m+n)B & - \qquad (m+2n)C & \\ -b & + \qquad Aa & + \qquad Ba & \end{array}\right\} = 0$$

and hence

$$A = \frac{b}{m(m-2)},\ B = \frac{-Aa}{(m+n)(m+n-2)},\ C = \frac{-B}{(m+2n)(m+2n-2)} \ \text{ etc.},$$

where indeed many things to be heeded occur, which by the precepts given above can be set in order.

But in the first place in this affair it helps to transform the proposed equation into another form with the aid of a substitution, the resolution of which by series becomes simpler; because since this will be able to be done in many ways, it has been considered to treat this argument with more care in the following chapter and thus for the form of the equation

$$Lddy + Mdxdy + Nydx^2 = 0,$$

since rarely is a transformation to other forms of this kind of equation considered in other places.

CAPUT VIII
DE ALIARUM AEQUATIONUM DIFFERENTIO-DIFFERENTIALUM RESOLUTIONE PER SERIES INFINITAS

PROBLEMA 122

967. *Formam generalem aequationum differentio-differentialium, quas commode per series resolvere licet, exhibere earumque integralia investigare.*

SOLUTIO

Primo alias aequationes commode per series resolvere non licet, nisi in quibus altera variabilis y cum suis differentialibus dy et ddy nusquam plus una dimensione obtinet, quoniam pro y seriem infinitam substituendo in calculos nimis molestos incideremus, si usquam plures dimensiones ingrederentur. Huiusmodi ergo aequationes in hac forma

$$ddy + Mdxdy + Nydx^2 = Xdx^2$$

continentur. Tum vero ut seriei pro y assumtae quilibet terminus per solum praecedentem determinetur, qui est casus resolutionis maxime notabilis, duplicis tantum generis terminos ratione alterius variabilis x inesse oportet, siquidem ad dimensiones, quas ipsa x cum suo differentiali dx constituit, respiciamus. Unde primo quidem reiecto termino Xdx^2 aequationes hoc modo resolubiles in hac forma continentur

$$xx\left(a+bx^n\right)ddy + x\left(c+ex^n\right)dxdy + \left(f+gx^n\right)ydx^2 = 0.$$

Pro cuius resolutione fingamus

$$y = Ax^{\lambda} + Bx^{\lambda+n} + Cx^{\lambda+2n} + Dx^{\lambda+3n} + \text{etc.}$$

et facta substitutione sequens serierum summa ad nihilum redigi debet

$$\begin{array}{llll}
\lambda(\lambda-1)Aax^{\lambda} & +\ (\lambda+n)(\lambda+n-1)Bax^{\lambda+n} & +(\lambda+2n)(\lambda+2n-1)Cax^{\lambda+2n} & +\text{etc.}\\
 & +\ \lambda(\lambda-1)Ab & +\ (\lambda+n)(\lambda+n-1)Bb & \\
+\lambda Ac & +\ (\lambda+n)Bc & +\ (\lambda+2n)Cc & \\
 & +\ \lambda Ae & +\ (\lambda+n)Be & \\
+Af & +\ Bf & +\ Cf & \\
 & +\ Ag & +\ Bg &
\end{array}$$

Hic ergo primo exponens λ ita accipi debet, ut sit

$$\lambda(\lambda-1)a+\lambda c+f=0,$$

tum vero pro reliquis fieri oportet

$$((\lambda+n)(\lambda+n-1)a+(\lambda+n)c+f)B=-(\lambda(\lambda-1)b+\lambda e+g)A,$$
$$((\lambda+2n)(\lambda+2n-1)a+(\lambda+2n)c+f)C=-((\lambda+n)(\lambda+n-1)b+(\lambda+n)e+g)B,$$
$$((\lambda+3n)(\lambda+3n-1)a+(\lambda+3n)c+f)D=-((\lambda+2n)(\lambda+2n-1)b+(\lambda+2n)e+g)C$$
etc.

Cum igitur sit $\lambda(\lambda-1)a+\lambda c+f=0$, si ponamus brevitatis causa

$$\lambda(\lambda-1)b+\lambda e+g=h,$$

erit

$$(n(n+2\lambda-1)a+nc)B=-hA,$$
$$(2n(2n+2\lambda-1)a+2nc)C=-(n(n+2\lambda-1)b+ne+h)B,$$
$$(3n(3n+2\lambda-1)a+3nc)D=-(2n(2n+2\lambda-1)b+2ne+h)C$$
etc.

Quia ergo, nisi $a=0$, pro λ gemini inveniuntur valores, scilicet

$$\lambda=\frac{a-c\pm\sqrt{((a-c)^2-4af)}}{2a},$$

binae series pro y inveniuntur, quae utcunque combinatae integrale completum aequationis propositae praebent.

ALITER

Proposita aequatione hac

$$xx(a+bx^n)ddy+x(c+ex^n)dxdy+(f+gx^n)ydx^2=0.$$

series quoque ordine retrogrado fingi potest

$$y=Ax^{\lambda}+Bx^{\lambda-n}+Cx^{\lambda-2n}+Dx^{\lambda-3n}+\text{etc.},$$

unde oritur ad nihilum reducendum

$$\begin{array}{llll}
\lambda(\lambda-1)Abx^{\lambda+n} + & (\lambda-n)(\lambda-n-1)Bbx^{\lambda} & +(\lambda-2n)(\lambda-2n-1)Cbx^{\lambda-n} & +\text{etc.} \\
+ & \lambda(\lambda-1)Aa & + (\lambda-n)(\lambda-n-1)Ba & \\
+\lambda Ae \quad + & (\lambda-n)Be & + (\lambda-2n)Ce & \\
+ & \lambda Ac & + (\lambda-n)Bc & \\
+Ag \quad + & Bg & + Cg & \\
+ & Af & + Bf &
\end{array}$$

Hic ergo exponentem λ ita accipi oportet, ut fiat

$$\lambda(\lambda-1)b+\lambda e+g=0 .$$

Tum vero si ponamus

$$\lambda(\lambda-1)a+\lambda c+f=h ,$$

determinatio coefficientium ita se habebit

$$\begin{aligned}
&(n(n-2\lambda+1)b-e)B=-hA,\\
&(2n(2n-2\lambda+1)b-e)C=-(n(n-2\lambda+1)a-nc+h)B,\\
&(3n(3n-2\lambda+1)b-e)D=-(2n(2n-2\lambda+1)a-2nc+h)C\\
&\qquad\qquad \text{etc.}
\end{aligned}$$

COROLLARIUM 1

968. Ex priore solutione, si *i* denotet numerum integrum positivum, series assumta alicubi abrumpetur, si fuerit

$$in(in+2\lambda-1)b+ine+h=0 \text{ vel } (\lambda+in)(\lambda+in-1)b+(\lambda+in)e+g=0 ,$$

hoc est

$$\left.\begin{array}{l}\lambda\lambda b+\lambda(2in-1)b+in(in-1)b\\ \quad+\lambda e \qquad +ine \ +g\end{array}\right\}=0.$$

COROLLARIUM 2

969. Aequatio ergo nostra integrationem admittit, si litterae *f* et *g* ita fuerint comparatae, ut sit

$$f=-\lambda(\lambda-1)a-\lambda c \text{ et } g=-(\lambda+in)(\lambda+in-1)b-(\lambda+in)e ,$$

vel sumtis duobus numeris μ et ν, ut sit $\nu-\mu$ divisibile per exponentem *n*, si fuerit

$$f=-\mu(\mu-1)a-\mu c \text{ et } g=-\nu(\nu-1)b-\nu e.$$

COROLLARIUM 3

970. Cum hinc sit

$$\mu=\frac{a-c\pm\sqrt{\left((a-c)^2-4af\right)}}{2a} \text{ et } \nu=\frac{b-e\pm\sqrt{\left((b-e)^2-4bg\right)}}{2b}$$

aequatio habebit integrale algebraicum, si fuerit $\nu-\mu=in$ denotante i numerum integrum positivum, hoc est, si sit

$$in=\frac{c}{2a}-\frac{e}{2b}\pm\frac{\sqrt{\left((b-e)^2-4bg\right)}}{2b}\mp\frac{\sqrt{\left((a-c)^2-4af\right)}}{2a}$$

COROLLARIUM 4

971. Pro serie autem invenienda si eveniat, ut exponens λ fiat imaginarius, notari convenit esse

$$x^{\alpha+\beta\sqrt{-1}}=x^{\alpha}e^{\beta\sqrt{-1}.lx}=x^{\alpha}(cos.\beta lx+\sqrt{-1}.sin.\beta lx),$$

unde binae series ita combinari poterunt, ut integrale consequatur formam realem.

SCHOLION

972. Utraque solutio generatim spectata duplicem seriem pro variabili y suppeditat pro gemino exponentis λ valore, quarum combinatio integrale completum exhibet. Solutio scilicet prior pro exponente λ hos duos praebet valores

$$\lambda=\frac{a-c\pm\sqrt{\left((a-c)^2-4af\right)}}{2a}$$

solutio vero posterior

$$\lambda=\frac{b-e\pm\sqrt{\left((b-e)^2-4bg\right)}}{2b}$$

ita ut hoc modo integrale completum duplici modo exprimi possit. Quae binae formae etiamsi maxime diversae atque adeo interdum altera per exponentes imaginarios progrediatur, dum altera habet reales, tamen sibi aequipollentes esse debent.

Quin etiam evenire potest, ut altera solutio vel etiam utraque ad integrale completum exhibendum sit inepta, dum unicam seriem suppeditet. Incommodum hoc pro utraque solutione duplici modo accidere potest; pro priori nempe solutione, ubi exponentem λ ex hac aequatione

$$\lambda(\lambda-1)a+\lambda c+f=0$$

definiri oportet, unicus inde pro λ eruitur valor, si fuerit vel $a=0$ vel $4af=(a-c)^2$. Priori casu tantum fit $\lambda=-\frac{f}{c}$ altero ipsius λ valore quasi in infinitum abeunte, posteriori casu vero ambo ipsius λ valores fiunt inter se aequales, scilicet $\lambda=\frac{a-c}{2a}$.

Idem incommodum in altera solutione locum habet, si fuerit vel $b=0$ vel $4bg=(b-e)^2$, unde patet fieri posse, ut altera solutio huiusmodi incommodo laboret, dum altera eo careat, quin etiam ut utraque eodem inquinetur. Quocirca ostendi conveniet, quemadmodum etiam his casibus integrale completum investigari debeat; quorsum etiam casum referamus, quo ambo ipsius λ valores fiunt imaginarii, quandoquidem ad imaginariam speciem tollendam singulari artificio est opus. Denique vero etiam binae series pro y exhibendae difficultate premuntur, quoties bini valores ipsius λ differentiam habent per exponentem n divisibilem, quorum casuum evolutio etiam explicari meretur.

PROBLEMA 123

973. *Proposita aequatione differentio-differentiali*

$$xx\left(a+bx^n\right)ddy+x\left(c+ex^n\right)dxdy+\left(f+gx^n\right)ydx^2=0$$

si eveniat, ut binae series ascendentes pro y assumtae vel in unam coalescant vel altera fiat impossibilis, integrale completum per series exprimere.

SOLUTIO

Assumta serie

$$y=Ax^{\lambda}+Bx^{\lambda+n}+Cx^{\lambda+2n}+Dx^{\lambda+3n}+\text{etc.}$$

si eveniat, ut bini valores ipsius λ ex aequatione $\lambda(\lambda-1)a+\lambda c+f=0$ vel fiant aequales vel differentiam per n divisibilem obtineant, valor ipsius y praeter potestates ipsius x etiam logarithmum ipsius x involvet. Quare [§ 934] pro aequationis resolutione statim ponamus $y=u+vlkx$, ut sit $y=u+vlx+\alpha v$ denotante α quantitatem constantem quamcunque. Hinc erit

$$dy=du+\frac{vdx}{x}+dvlkx \text{ atque } ddy=ddu+\frac{2dxdv}{x}-\frac{vdx^2}{xx}+ddvlkx,$$

quibus valoribus substitutis aequatio nostra hanc induet formam

$$\left.\begin{aligned} &xx(a+bx^n)ddu+2x(a+bx^n)dxdv-(a+bx^n)vdx^2\\ &\qquad\qquad\qquad + x(c+ex^n)dxdu+(c+ex^n)vdx^2\\ &\qquad\qquad\qquad\qquad\qquad\qquad +(f+gx^n)udx^2\\ &+(xx(a+bx^n)ddv+x(c+ex^n)dxdv+(f+gx^n)vdx^2)lkx \end{aligned}\right\}=0,$$

ubi partem postremam logarithmo affectam seorsim nihilo aequari oportet.
Quare posito

$$v = Ax^{\lambda}+Bx^{\lambda+n}+Cx^{\lambda+2n}+Dx^{\lambda+3n}+\text{etc.}$$

exponenti λ ex aequatione $\lambda(\lambda-1)a+\lambda c+f=0$ is valor tribuatur, qui nulli incommodo est obnoxius, eritque pro reliquis coefficientibus ponendo

$$\lambda(\lambda-1)b+\lambda e+g=nh,$$

ut sequitur,

$$\begin{aligned} &((n+2\lambda-1)a+c)\,B+hA=0,\\ &2((2n+2\lambda-1)a+c)C+((n+2\lambda-1)b+e)B+hB=0,\\ &3((3n+2\lambda-1)a+e)E+2((2n+2\lambda-1)b+e)C+hC=0,\\ &4((4n+2\lambda-1)a+e)E+3((3n+2\lambda-1)b+e)D+hD=0\\ &\qquad\qquad\qquad\text{etc.} \end{aligned}$$

His coefficientibus ita definitis, quorum primus *A* arbitrio nostro relinquitur, ponamus

$$u=\varDelta+\mathfrak{A}x^{\lambda}+\mathfrak{B}x^{\lambda+n}+\mathfrak{C}x^{\lambda+2n}+\mathfrak{D}x^{\lambda+3n}+\text{etc.}$$

qui valor si in priori aequatione cum serie pro *v* inventa substituatur, sequentes series ad nihilum reduci oportet

$$xx(a+bx^n)\frac{dd\varDelta}{dx^2}+x(c+ex^n)\frac{d\varDelta}{dx}+(f+gx^n)\varDelta$$

$$\begin{array}{llllll}
+\lambda(\lambda-1)\mathfrak{A}ax^{\lambda} & +(\lambda+n)(\lambda+\mathrm{n}-1)\mathfrak{B}ax^{\lambda+n} & +(\lambda+2n)(\lambda+2n-1)\mathfrak{C}ax^{\lambda+2n}+\text{etc.}\\
 & +\quad \lambda(\lambda-1)\mathfrak{A}b & +(\lambda+2n)(\lambda+n-1)\mathfrak{B}b\\
+\quad \lambda\mathfrak{A}c & +\quad (\lambda+n)\mathfrak{B}c & +\quad (\lambda+2n)\mathfrak{C}c\\
 & +\quad \lambda\mathfrak{A}e & +\quad (\lambda+n)\mathfrak{B}e\\
+\quad \mathfrak{A}f & +\quad \mathfrak{B}f & +\quad \mathfrak{C}f\\
 & +\quad \mathfrak{A}g & +\quad \mathfrak{B}g\\
+\quad 2\lambda Aa & +\quad 2(\lambda+n)Ba & +\quad 2(\lambda+2n)Ca\\
 & +\quad 2\lambda Ab & +\quad 2(\lambda+n)Bb\\
+\quad A(c-a) & +\quad B(c-a) & +\quad C(c-a)\\
 & +\quad A(e-b) & +\quad B(e-b)
\end{array}$$

Cum autem sit $\lambda(\lambda-1)a+\lambda c+f=0$ et $\lambda(\lambda-1)b+\lambda e+g=nh$, expressio haec transmutabitur in hanc formam

$$\begin{array}{ll}
\multicolumn{2}{l}{xx\left(a+bx^n\right)\frac{dd\varDelta}{dx^2}+x\left(c+ex^n\right)\frac{d\varDelta}{dx}+\left(f+gx^n\right)\varDelta+\left((2\lambda-1)a+c\right)Ax^{\lambda}}\\
+((2n+2\lambda-1)a+c)\,Bx^{\lambda+n} & +((4n+2\lambda-1)a+c)\,Cx^{\lambda+2n}+\text{etc.}\\
+\quad \left((2\lambda-1)b+e\right)A & +\quad ((2n+2\lambda-1)b+e)B\\
+\,n((n+2\lambda-1)a+c)\mathfrak{B} & +2n((2n+2\lambda-1)a+e)\mathfrak{C}\\
+\quad nh\mathfrak{A} & +\quad n((n+2\lambda-1)b+e)\mathfrak{B}\\
 & +\quad nh\mathfrak{B}
\end{array}$$

ubi $\varDelta$ denotat quosdam tenninos seriei

$$\mathfrak{A}x^{\lambda}+\mathfrak{B}x^{\lambda+n}+\text{etc.}$$

praemittendos, ita ut ordine retrogrado sit

$$\varDelta=\mathfrak{a}x^{\lambda-n}+\mathfrak{b}x^{\lambda-2n}+\mathfrak{c}x^{\lambda-3n}+\ldots\ldots+\mathfrak{i}x^{\lambda-in}.$$

Quod principium quomodo quovis casu sit constituendum, sequentia sunt observanda.

I. Principium hoc locum habere nequit, nisi fuerit

$$(\lambda - in)(\lambda - in - 1)a + (\lambda - in)c + f = 0;$$

cum igitur sit $\lambda(\lambda - 1)a + \lambda c + f = 0$, inde erit

$$\lambda = in + \frac{a - c - \sqrt{\left((a-c)^2 - 4af\right)}}{2a},$$

hinc vero

$$\lambda = \frac{a - c + \sqrt{\left((a-c)^2 - 4af\right)}}{2a},$$

quandoquidem hi duo valores convenire nequeunt, nisi ibi signum radicale negative, hic vero positive accipiatur. Aequatis autem his valoribus fit

$$in = \tfrac{1}{a}\sqrt{\left((a-c)^2 - 4af\right)} \quad \text{seu} \quad iinnaa = (a-c)^2 - 4af$$

hincque $f = \frac{(a-c)^2}{4a} - \frac{1}{4}iinna$, unde fit

$$\lambda = \tfrac{a-c}{2a} + \tfrac{1}{2}in.$$

Quare si aequatio proposita ita fuent comparata, ut sit

$$ina = \sqrt{((a-c)^2 - 4af)},$$

tum sumto

$$\lambda = \tfrac{a-c}{2a} + \tfrac{1}{2}in$$

ac pro v sumta serie

$$v = Ax^{\lambda} + Bx^{\lambda+n} + \text{etc}.$$

alteram seriem u ita constitui convenit

$$u = \mathfrak{i}x^{\lambda - in} + \ldots + \mathfrak{a}x^{\lambda - n} + \mathfrak{A}x^{\lambda} + \mathfrak{B}x^{\lambda+n} + \mathfrak{C}x^{\lambda+2n} + \text{ etc}.$$

Hic est casus, quo bini valores ipsius λ ex aequatione $\lambda(\lambda-1)a+\lambda c+f=0$ differentiam habent per *n* divisibilem, ubi notandum seriem *v* a maiore valore ipsius λ, seriem vero *u* a minore inchoari debere.

II. Principium Δ omitti nequit, nisi fuerit $(2\lambda-1)a+c=0$, quo casu fit $\lambda=\frac{a-c}{2}$; atque hic est casus, quo aequationis $\lambda(\lambda-1)a+\lambda c+f=0$ binae radices fiunt inter se aequales ideoque $f=\frac{(a-c)^2}{4a}$ .Continetur ergo hic casus in praecedente sumendo ibi $i=0$. Quare hoc modo resolventur casus, quibus bini valores ipsius λ vel sunt inter se aequales vel differentiam habent per exponentem *n* divisibilem. Sicque reperitur integrale completum per duas series ascendentes *v* et *u* expressum, quarum ilIa *v* per *lx* multiplicatur.

COROLLARIUM 1

974. Quando ergo in aequatione proposita coeffieientes *a*, *c* et *f* ita sunt comparati, ut aequationis $\lambda(\lambda-1)a+\lambda c+f=0$ radices sint $\lambda=\mu$ et $\lambda=\mu-in$ denotante *i* numerum integrum positivum, integrale completum huiusmodi habebit formam $y=u+\alpha v+vlx$.

COROLLARIUM 2

975. Hic autem binae quantitates *v* et *u* ex his aequationibus

$$\text{I. } xx\left(a+bx^n\right)ddv+x\left(c+ex^n\right)dxdv+\left(f+gx^n\right)ydv^2=0$$

$$\text{II. } \left.\begin{aligned} xx\left(a+bx^n\right)ddu+x\left(c+ex^n\right)dxdu+\left(f+gx^n\right)udx^2 \\ +2x\left(a+bx^n\right)dxdv-(a+bx^n)vdx^2 \\ +(c+ex^n)vdx^2 \end{aligned}\right\}=0$$

ita determinari poterunt, ut ponatur

$$\begin{aligned} v &= Ax^{\mu}+Bx^{\mu+n}+Cx^{\mu+2n}+Dx^{\mu+3n}+\text{etc.}, \\ u &= \mathfrak{A}x^{\mu-in}+\mathfrak{B}x^{\mu-in+n}+\mathfrak{C}x^{\mu-in+2n}+\mathfrak{D}x^{\mu-in+3n}+\text{etc.} \end{aligned}$$

Has scilicet series substituendo omnes coefficientes ex uno definire licebit.

SCHOLION

976. Logarithmo ergo ipsius *x* in subsidium vocato his casibus, quos commemoravimus, integrale completum aequationis propositae per series ascendentes exhiberi potest, dum sine hoc artificio integrale tantum particulare invenitur. Quando enim aequatio

$\lambda(\lambda-1)a+\lambda c+f=0$ duas radices habet, quarum differentia per exponentem n est divisibilis, puta $\lambda=\mu$ et $\lambda=\mu-in$, priore methodo sola series, quae incipit a potestate x^{μ}, determinari potest; si enim altera a potestate $x^{\mu-in}$ incipiens pro y assumeretur, coefficiens cuiusdam termini reperiretur infinitus, unde sequentes omnes forent quoque infiniti, quod incommodum introducendo logarithmum ipsius x feliciter tollitur. Hunc igitur usum istius resolutionis aliquot exemplis illustrasse iuvabit.

EXEMPLUM 1

977. *Aequationis differentio-differentialis* $xddy+dxdy+gx^{n-1}ydx^2=0$ *integrale completum per series ascendentes exhibere.*

Hanc aequationem ad nostram formam reducendo habebimus $xxddy+xdxdy+gx^{n}ydx^2=0$, ubi ergo est $a=1, b=0, c=1, e=0,$ et $f=0$. Hinc $\lambda(\lambda-1)+\lambda=0$ seu $\lambda\lambda=0$, ita ut bini valores ipsius λ sint aequales et = 0. Quare posito $y=u+\alpha v+vlx$ resolvi oportet has aequationes

$$\text{I. } xxddv+xdxdv+gx^{n}vdx^2=0$$

et

$$\text{II. } \left.\begin{matrix} xxddu+xdxdu+gx^{n}udx^2 \\ +2xdxdv \end{matrix}\right\}=0$$

Statuamus ergo

$$v=A+Bx^{n}+Cx^{2n}+Dx^{3n}+\text{etc.}$$

et

$$u=\mathfrak{A}+\mathfrak{B}x^{n}+\mathfrak{C}x^{2n}+\mathfrak{D}x^{3n}+\text{etc.}$$

ac prior aequatio praebet

$$\left.\begin{matrix} n(n-1)Bx^{n} & +2n(2n-1)Cx^{2n} & +3n(3n-1)Dx^{3n} & +\text{etc.} \\ +\ nB & +\ 2nC & +\ 3nD & \\ +\ Ag & +\ Bg & +\ Cg & \end{matrix}\right\}=0,$$

unde fit

$$B=\frac{-Ag}{nn},\quad C=\frac{-Bg}{4nn},\quad D=\frac{-Cg}{9nn},\quad E=\frac{-Dg}{16nn}\quad \text{etc.}$$

Tum vero altera aequatio dat

$$\left.\begin{array}{llll} nn\mathfrak{B}x^n & +4nn\mathfrak{C}x^{2n} & +9nn\mathfrak{D}x^{3n} & +\text{etc.} \\ +\ \mathfrak{A}g & +\ \mathfrak{B}g & +\ \mathfrak{C}g & \\ +2nB & +4nC & +6nD & \end{array}\right\} = 0,$$

unde colligitur

$$\mathfrak{B} = \frac{-\mathfrak{A}g}{nn} - \frac{2B}{n}, \quad \mathfrak{C} = \frac{-\mathfrak{B}g}{4nn} - \frac{2C}{2n}, \quad ,\mathfrak{D} = \frac{-\mathfrak{A}g}{9nn} - \frac{2D}{3n}, \quad \text{etc.}$$

Hic autem tuto assumere licet $\mathfrak{A} = 0$, quoniam termini ex $\mathfrak{A}$ oriundi continentur in membro αv. Cum igitur sit

$$B = \frac{-Ag}{nn}, \quad C = \frac{+Agg}{1\cdot 4n^4}, \quad D = \frac{-Ag^3}{1\cdot 4\cdot 9n^6}, \quad E = \frac{+Ag^4}{1\cdot 4\cdot 9\cdot 16n^8} \quad \text{etc.},$$

erit, ut sequitur,

$$\mathfrak{B} = \frac{2Ag}{n^3}, \quad \mathfrak{A} = \frac{-2Agg}{4n^5} - \frac{2Agg}{2\cdot 1\cdot 4n^5} = \frac{-6Agg}{2\cdot 1\cdot 4n^5},$$

$$\mathfrak{D} = \frac{6Ag^3}{2\cdot 1\cdot 4\cdot 9n^7} + \frac{2Ag^3}{3\cdot 1\cdot 4\cdot 9n^7} = \frac{22Ag^3}{2\cdot 3\cdot 1\cdot 4\cdot 9n^7},$$

$$\mathfrak{E} = \frac{-22Ag^4}{2\cdot 3\cdot 1\cdot 4\cdot 9\cdot 16n^9} - \frac{2Ag^4}{4\cdot 1\cdot 4\cdot 9\cdot 16n^9} = \frac{-100Ag^4}{2\cdot 3\cdot 1\cdot 4\cdot 9\cdot 16n^9},$$

$$\mathfrak{F} = \frac{100Ag^5}{2\cdot 3\cdot 4\cdot 1\cdot 4\cdot 9\cdot 16\cdot 25n^{11}} + \frac{2Ag^5}{5\cdot 1\cdot 4\cdot 9\cdot 16\cdot 25n^{11}} = \frac{548Ag^5}{2\cdot 3\cdot 4\cdot 5\cdot 1\cdot 4\cdot 9\cdot 16\cdot 25n^{11}}$$

etc.

sicque obtinentur sequentes valores

$$\mathfrak{B} = \frac{2Ag}{n^3}, \quad \mathfrak{C} = \frac{-6Agg}{1\cdot 8n^5}, \quad \mathfrak{D} = \frac{22Ag^3}{1\cdot 8\cdot 27n^7}, \quad \mathfrak{E} = \frac{-100Ag^4}{1\cdot 8\cdot 27\cdot 64n^9},$$

$$\mathfrak{E} = \frac{548Ag^5}{1\cdot 8\cdot 27\cdot 64\cdot 125n^{11}}, \quad \mathfrak{G} = \frac{-3528Ag^6}{1\cdot 8\cdot 27\cdot 64\cdot 125\cdot 216n^{13}} \quad \text{etc.},$$

ubi numeratores 2, 6, 22, 100, 548, 3528 etc. singuli ita per binos praecedentes definiuntur

$$6 = 3\cdot 2 - 1\cdot 0,\ 22 = 5\cdot 6 - 4\cdot 2,\ 100 = 7\cdot 22 - 9\cdot 6,$$
$$548 = 9\cdot 100 - 16\cdot 22,\ 3528 = 11\cdot 548 - 25\cdot 100 \quad \text{etc.}$$

Consequenter integrale ita exprimetur

$$y = \frac{2Ag}{n^3}x^n - \frac{6Agg}{1\cdot 8n^5}x^{2n} + \frac{22Ag^3}{1\cdot 8\cdot 27n^7}x^{3n} - \frac{100Ag^4}{1\cdot 8\cdot 27\cdot 64n^9}x^{4n} + \text{etc.}$$
$$+A\left(1 - \frac{g}{nn}x^n + \frac{gg}{1\cdot 4n^4}x^{2n} - \frac{g^3}{1\cdot 4\cdot 9n^6}\ x^{3n} + \frac{g^4}{1\cdot 4\cdot 9\cdot 16n^6}\ x^{4n} - \text{ etc.}\right)lx$$
$$+\alpha - \frac{\alpha g}{nn}x^n + \frac{\alpha gg}{1\cdot 4n^4}x^{2n} - \frac{\alpha g^3}{1\cdot 4\cdot 9n^6}\ x^{3n} + \frac{\alpha g^4}{1\cdot 4\cdot 9\cdot 16n^8}\ x^{4n} - \text{etc.}$$

ubi A et *a* sunt binae constantes arbitrariae.

EXEMPLUM 2

978. *Aequationis differentio-differentialis*

$$x(1-xx)ddy-(1+xx)dxdy+xydx^2=0$$

integrale completum per series ascendentes assignare.

Ad formam nostram reducta est

$$x(1-xx)ddy-(1+xx)dxdy+xydx^2=0,$$

ita ut sit $n=2,\ a=1,\ b=-1,\ c=-1,\ e=-1,\ f=0$ et $g=1$, unde aequationis $\lambda(\lambda-1)-\lambda=0$ radices sint $\lambda=0$ et $\lambda=2$, quarum differentia per $n=2$ divisa dat 1. Posito ergo $y=u+\alpha v+vlx$ statui debet

$$v=Ax^2+Bx^4+Cx^6+Dx^8+\text{etc.}$$

et

$$u=\mathfrak{A}+\mathfrak{B}x^2+\mathfrak{C}x^4+\mathfrak{D}x^6+\mathfrak{E}x^8+\text{etc.}$$

quae series ex sequentibus aequationibus determinari debent

I. $xx(1-xx)ddv-x(1+xx)dxdv+xxvdx^2=0,$

II. $xx(1-xx)ddu-x(1+xx)dxdu+xxudx^2+2x(1-xx)dxdv-2vdx^2=0.$

Hinc pro prioris determinatione fit

$$\left.\begin{array}{lllll}
2Ax^2 & +12Bx^4 & +30Cx^6 & +56Dx^8 & +\text{ etc.}\\
 & -2A & -12B & -30C & \\
-2A & -4B & -6C & -8D & \\
 & -2A & -4B & -6C & \\
 & +A & +B & +C &
\end{array}\right\}=0$$

ideoque

$$2\cdot 4B=1\cdot 3A,\ 4\cdot 6C=3\cdot 5B,\ 6\cdot 8D=5\cdot 7C \quad \text{etc.}$$

seu

$$B=\tfrac{1\cdot 3}{2\cdot 4}A,\ C=\tfrac{1\cdot 3\cdot 3\cdot 5}{2\cdot 4\cdot 4\cdot 6}A,\ D=\tfrac{1\cdot 3\cdot 3\cdot 5\cdot 5\cdot 7}{2\cdot 4\cdot 4\cdot 6\cdot 6\cdot 8}A \quad \text{etc.}$$

Ex altera vero aequatione reperitur

$$\left.\begin{array}{lllll}
2\mathfrak{B}x^2 & +12\mathfrak{C}x^4 & +30\mathfrak{D}x^6 & +56\mathfrak{E}x^8 & +\text{ etc.}\\
 & -2\mathfrak{B} & -12\mathfrak{C} & -30\mathfrak{D} & \\
-2\mathfrak{B} & -4\mathfrak{C} & -6\mathfrak{D} & -8\mathfrak{E} & \\
 & -2\mathfrak{B} & -4\mathfrak{C} & -6\mathfrak{D} & \\
+\mathfrak{A} & +\mathfrak{B} & +\mathfrak{C} & +\mathfrak{D} & \\
+4A & +8B & +12C & +16D & \\
 & -4A & -8B & -12C & \\
-2A & -2B & -2C & -2D &
\end{array}\right\} = 0,$$

unde fieri oportet

$$\mathfrak{A}+2A=0,\quad 2\cdot 4\mathfrak{C}-1\cdot 3\mathfrak{B}+6B-4A=0,\ 4\cdot 6\mathfrak{D}-3\cdot 5\mathfrak{C}+10C-8B=0,$$
$$6\cdot 8\mathfrak{E}-5\cdot 7\mathfrak{D}+14D-12C=0\ \text{ etc.},$$

seu cum sit

$$B=\tfrac{1\cdot 3}{2\cdot 4}A,\quad C=\tfrac{3\cdot 5}{4\cdot 6}B,\quad D=\tfrac{5\cdot 7}{6\cdot 8}C\quad \text{etc.},$$

erit $\mathfrak{A}=-2A$, tum vero

$$2\cdot 4\mathfrak{C}-1\cdot 3\mathfrak{B}-\tfrac{2\cdot 7}{2\cdot 4}A=0,\quad \mathfrak{C}=\tfrac{1\cdot 3}{2\cdot 4}\mathfrak{B}+\tfrac{2\cdot 7}{2^2\cdot 4^2}A,$$
$$4\cdot 6\mathfrak{D}-3\cdot 5\mathfrak{C}-\tfrac{2\cdot 21}{4\cdot 6}B=0,\quad \mathfrak{D}=\tfrac{3\cdot 5}{4\cdot 6}\mathfrak{C}+\tfrac{2\cdot 21}{4^2\cdot 6^2}B,$$
$$6\cdot 8\mathfrak{E}-5\cdot 7\mathfrak{D}-\tfrac{2\cdot 43}{6\cdot 8}C=0,\quad \mathfrak{E}=\tfrac{5\cdot 7}{6\cdot 8}\mathfrak{D}+\tfrac{2\cdot 43}{6^2\cdot 8^2}C,$$
$$8\cdot 10\mathfrak{F}-7\cdot 9\mathfrak{D}-\tfrac{2\cdot 73}{8\cdot 10}D=0,\quad \mathfrak{F}=\tfrac{7\cdot 9}{8\cdot 10}\mathfrak{E}+\tfrac{2\cdot 73}{8^2\cdot 10^2}D$$
etc.

Dum ergo capiatur $\mathfrak{A}=-2A$, littera $\mathfrak{B}$ pro lubitu accipi potest nihilque impedit, quominus nihilo aequalis statuatur, siquidem constans α supra est inducta.

EXEMPLUM 3

979. *Aequationis differentio–differentialis*
$$xx(1+bxx)ddy+x(-5+exx)dxdy+(5+gxx)ydx^2=0$$
integrale completum per series ascendentes exhibere.

Quia hic est $a=1, c=-5$ et $f=5$, aequatio $\lambda(\lambda-1)-5\lambda+5=0$ seu $\lambda\lambda-6\lambda+5=0$ radices habet $\lambda=1$ et $\lambda=5$, quarum differentia 4 per $n=2$ dividi potest. Posito ergo $y=u+\alpha v+vlx$ statuatur

$$v=Ax^5+Bx^7+Cx^9+Dx^{11}+Ex^{13}+\text{etc.}$$

et

$$u=\mathfrak{A}x+\mathfrak{B}x^3+\mathfrak{C}x^5+\mathfrak{D}x^7+\text{etc.};$$

aequationes resolvendae erunt

$$\text{I. } xx(1+bxx)ddv + x(-5+exx)dxdv + (5+gxx)vdx^2 = 0$$

et

$$\text{II. } \left.\begin{array}{r} xx(1+bxx)ddu + x(-5+exx)dxdu + (5+gxx)udx^2 \\ +\ 2x(1+bxx)dxdv - (1+bxx)\ vax^2 \\ +(-5+exx)vdx^2 \end{array}\right\} = 0,$$

ubi prior ducit ad

$$\left.\begin{array}{llll} 5\cdot 4Ax^5 & +7\cdot 6Bx^7 & +9\cdot 8Cx^9 & +11\cdot 10Dx^{11} + \text{ etc.} \\ -5\cdot 5A & -5\cdot 7B & -5\cdot 9C & -\ 5\cdot 11D \\ +\ 5A & +\ 5B & +\ 5C & +\ 5D \\ & +5\cdot 4Ab & +7\cdot 6Bb+ & 9\cdot 8Cb \\ & +\ 5Ae & +\ 7Be+ & 9Ce \\ & +\ Ag & +\ Bg+ & Cg \end{array}\right\} = 0,$$

posterior vero ad

$$\left.\begin{array}{llllll} & +2\cdot 3\mathfrak{B}x^3 + & 4\cdot 5\mathfrak{C}x^5 & +\ 6\cdot 7\mathfrak{D}x^7 + & 8\cdot 9\mathfrak{E}x^9 + \text{ etc.} \\ -5\mathfrak{A}x & -5\cdot 3\mathfrak{B} & -\ 5\cdot 5\mathfrak{C} & -\ 5\cdot 7\mathfrak{D} & -\ 5\cdot 9\mathfrak{E} \\ +5\mathfrak{A} & +\ 5\mathfrak{B} & +\ 5\mathfrak{C} & +\ 5i\mathfrak{D} & +\ 5\mathfrak{E} \\ & & +2\cdot 3\mathfrak{B}b & +\ 4\cdot 5\mathfrak{C}b & +6\cdot 7\mathfrak{D}b \\ & +\ \mathfrak{A}e & +\ 3\mathfrak{B}e & +\ 5\mathfrak{C}e & +\ 7\mathfrak{D}e \\ & +\ \mathfrak{A}g & +\ \mathfrak{B}g & +\ \mathfrak{C}g & +\ \mathfrak{D}g \\ & & +\ 2\cdot 5A & +\ 2\cdot 7B & +\ 2\cdot 9C \\ & & -\ 6A & -\ 6B & -\ 6C \\ & & & +\ 2\cdot 5Ab & +2\cdot 7Bb \\ & & & -\ Ab & -\ Bb \\ & & & +\ Ae & +\ Be \end{array}\right\} = 0.$$

Inde fit

$$\begin{array}{lll} 12B + A(20b+5e+g) = 0 & \text{or} & 2\cdot 6B + A(4\cdot 5b+5e+g) = 0, \\ 32C + B(42b+7e+g) = 0 & & 4\cdot 8C + B(6\cdot 7b+7e+g) = 0, \\ 60D + C(72b+9e+g) = 0 & & 6\cdot 10D + C(8\cdot 9b+9e+g) = 0 \\ \text{etc.} & & \text{etc.} \end{array}$$

Hinc autem

$$-4\mathfrak{B}+ \mathfrak{A}\left(e+g\right)=0,$$
$$0\mathfrak{C} \quad + \mathfrak{B}(2\cdot 3b+3e+g)+4A=0,$$
$$2\cdot 6\mathfrak{D}+ \mathfrak{C}(4\cdot 5b+5e+g)+8B + A(9b+e)=0,$$
$$4\cdot 8\mathfrak{E}+ \mathfrak{D}(6.7b+7e+g)+12C+B(13b+e)=0$$
etc.

Ex prioribus formulis litterae *B*, *C*, *D* etc. per *A* determinantur, ex posteriorum vero secunda fit $\mathfrak{B}=\frac{-4A}{2\cdot 3b+3e+g}$, ex prima autem $\mathfrak{A}=\frac{4\mathfrak{A}}{e+g}$, tum vero $\mathfrak{C}$ pro lubitu assumi potest indeque reliqui coefficientes $\mathfrak{D}$, $\mathfrak{E}$, $\mathfrak{F}$ etc. definiuntur.

SCHOLION

980. Exemplum hoc occasionem nobis suppeditat phaenomena quaedam singularia observandi. Scilicet etiamsi integrale completum in genere *lx* involvat, tamen id a logarithmo liberum prodit certis casibus.

Primo nempe si sit $g=-e$, fit $\mathfrak{B}=0$ manente $\mathfrak{A}$ indefinito, tum vero ob $\mathfrak{B}=0$ capi oportet $A=0, B=0$ etc. ideoque $v=0$. Porro vero erit

$$2\cdot 6\mathfrak{D}+4\mathfrak{C}(5b+e)=0,$$
$$4\cdot 8\mathfrak{E}+6\mathfrak{D}(7b+e)=0,$$
$$6\cdot 10\mathfrak{F}+8\mathfrak{E}(9b+e)=0$$
etc.,

ubi $\mathfrak{C}$ altera est constans arbitraria, eritque aequationis

$$xx(1+bxx)ddy+x(-5+exx)dxdy+(5-exx)ydx^2=0$$

integrale completum

$$y=\mathfrak{A}x+\mathfrak{C}x^5+\mathfrak{D}x^7+\mathfrak{E}x^9+\mathfrak{F}x^{11}+\text{etc.},$$

quod adeo finite exprimitur, si $e=-(2i+5)b$ pro *i* sumendo numeros 0, 1, 2, 3, 4 etc.

Secundo si sit $2\cdot 3b+3e+g=0$ seu $g=-6b-3e$, fit $\mathfrak{B}=-\frac{1}{2}\mathfrak{A}(3b+e)$, tum vero $A=0, B=0, C=0$ etc., ergo $v=0$. Porro vero reperitur

$$\mathfrak{D}=-\tfrac{1}{6}\mathfrak{C}(7b+e),\quad \mathfrak{E}=-\tfrac{1}{8}\mathfrak{D}(9b+e),\quad \mathfrak{F}=-\tfrac{1}{10}\mathfrak{E}(11b+e)\ \text{etc.}$$

hincque

$$y=\mathfrak{A}x-\tfrac{1}{2}\mathfrak{A}\left(3b+e\right)x^3+\mathfrak{C}x^5+\mathfrak{D}x^7+\mathfrak{E}x^9+\text{etc.},$$

ubi $\mathfrak{A}$ et $\mathfrak{C}$ arbitrio nostro relinquuntur.
Tertio si sit $4\cdot 5b+5e+g=0$ seu $g=-20b-5e$, primo fit $B=0, C=0, D=0$ etc. ideoque $v=Ax^5$, tum vero

$$\mathfrak{B}=-\mathfrak{A}(5b+e),\quad -\mathfrak{B}(14b+2e)+4A=0 \text{ seu } \quad \mathfrak{B}=\tfrac{2A}{7b+e}$$

hincque

$$A=-\tfrac{1}{2}\mathfrak{A}(5b+e)(7b+e),$$

porro

$$2\cdot 6\mathfrak{D} + A(9b+e)=0,$$
$$4\cdot 8\mathfrak{E} +2\mathfrak{D}(11b+e)=0,$$
$$6\cdot 10\mathfrak{F}+4\mathfrak{E}(13b+e) =0$$
etc.

Per $\mathfrak{A}$ ergo definiuntur coefficientes $\mathfrak{B}, A, \mathfrak{D}, \mathfrak{E}, \mathfrak{F}$ etc. ac $\mathfrak{C}$ quoque arbitrio nostro relinquitur, unde integrale completum hoc casu erit

$$y=Ax^5lx+\mathfrak{C}x^5+\mathfrak{A}x+\mathfrak{B}x^3+*+\mathfrak{D}x^7+\mathfrak{E}x^9+\text{etc.},$$

quae expressio fit finita, quoties $(2i+5)b+e=0$.

EXEMPLUM 4

981. *Si in priori exemplo sit $e=-7b$ et $g=15b$, aequationis*

$$xx(1+bxx)ddy-x(5+7bxx)dxdy+5(1+3bxx)ydx^2=0$$

integrale completum algebraice exhibere.

Erit ergo $\mathfrak{B}=2\mathfrak{A}b, A=0, \mathfrak{D}=0, \mathfrak{E}=0$ ideoque $v=0$ et

$$u=\mathfrak{A}x+2\mathfrak{A}bx^3+\mathfrak{C}x^5,$$

unde pro $\mathfrak{A}$ et $\mathfrak{C}$ sumendo constantes quascunque erit integrale completum

$$y=\mathfrak{A}x(1+2bxx)+\mathfrak{C}x^5.$$

Integralia ergo particularia erunt

$$y=ax(1+2bxx),\ y=ax^5,\ y=ax(1+bxx)^2.$$

COROLLARIUM 1

982. Posito $y=e^{\int zdx}$, ut sit $z=\frac{dy}{ydx}$, aequationis huius differentialis primi gradus

$$xx(1+bxx)dz+xx(1+bxx)zzdx-x(5+7bxx)zdx+5(1+3bxx)dx=0$$

integrale completum est

$$z=\frac{\mathfrak{A}(1+6bxx)+5\mathfrak{C}x^4}{\mathfrak{A}x(1+2bxx)+\mathfrak{C}x^5}.$$

COROLLARIUM 2

983. Aequatio autem differentio-differentialis integrabilis redditur, si dividatur per $xx(1+bxx)^2$, eritque integrale

$$\frac{xdy-5ydx}{x(1+bxx)}=Cdx \quad\text{seu}\quad dy-\frac{5ydx}{x}=Cdx(1+bxx),$$

quae per x^5 divisa integrale praebet

$$\frac{y}{x^5}=\frac{-C}{4x^4}-\frac{bC}{2x^2}+D \quad\text{seu}\quad y=-\tfrac{1}{4}Cx(1+2bxx)+Dx^5$$

ut ante.

SCHOLION

984. Deficit autem adhuc integratio completa nostrae aequationis generalis per series ascendentes casu, quo $a=0$ ideoque $\lambda c+f=0$, unde unicus pro exponente λ valor definitur $\lambda=\frac{-f}{c}$, qui tantum integrale particulare suppeditat, atque hoc etiam tollitur, si fuerit $c=0$. Quia autem his casibus *a* est = 0, coefficiens *b* certo adsit necesse est, ex quo integrale completum per series descendentes exhiberi poterit, cum aequatio $\lambda(\lambda-1)b+\lambda e+g=0$ duas semper contineat radices, ex quibus duplex series obtinetur. Simile autem hic incommodum usu venire potest, quando binae radices ipsius λ vel prodeunt aequales vel differentiam habent per exponentem *n* divisibilem. Verum huic incommodo seriem per *lx* multiplicatam introducendo simili methodo medela affertur, qua in hoc problemate sumus usi, ac superfluum foret istam evolutionem hic repetere. Quodsi autem binae radices ipsius λ tam pro seriebus ascendentibus quam descendentibus fiant imaginariae, ostendendum restat, quomodo integrale completum per series infinitas exprimi oporteat.

PROBLEMA 124

985. *Proposita aequatione differentio-differentiali*

$$xx(a+bx^n)ddy+x(c+ex^n)dxdy+(f+gx^n)ydx^2=0$$

si eveniat, ut aequatio $\lambda(\lambda-1)a+\lambda c+f=0$ *radices habeat imagiarias, eius integrale completum per series ascendentes exhibere.*

SOLUTIO

Ex supra allatis (§ 971) colligitur hoc casu statui debere

$$y = v\sin.\mu lx + u\cos.\mu lx\,,$$

unde fit

$$dy = (dv - \tfrac{\mu udx}{x})\sin.\mu lx + (\tfrac{\mu vdx}{x} + du)\cos.\mu lx$$

et

$$\begin{aligned} ddy = &(ddv - \tfrac{2\mu dxdu}{x} + \tfrac{\mu udx^2}{xx} - \tfrac{\mu\mu vdx^2}{xx})\sin.\mu lx \\ &+ (ddu + \tfrac{2\mu dxdv}{x} - \tfrac{\mu vdx^2}{xx} - \tfrac{\mu\mu udx^2}{xx})\cos.\mu lx; \end{aligned}$$

qua facta substitutione si terminos tam $\sin.\mu lx$ quam $\cos.\mu lx$ affectos seorsim ad nihilum redigamus, obtinebimus duas sequentes aequationes

$$\text{I. } \left.\begin{aligned} xx(a+bx^n)ddv + x(c+ex^n)dxdv + (f+gx^n)vdx^2 \\ -2\mu x(a+bx^n)dxdu - \mu\mu(a+bx^n)vdx^2 \\ + \mu(a+bx^n)udx^2 \\ - \mu(c+ex^n)udx^2 \end{aligned}\right\} = 0,$$

$$\text{II. } \left.\begin{aligned} xx(a+bx^n)ddu + x(c+ex^n)dxdu + (f+gx^n)udx^2 \\ + 2\mu x(a+bx^n)dxdv - \mu\mu(a+bx^n)udx^2 \\ - \mu(a+bx^n)vdx^2 \\ + \mu(c+ex^n)vdx^2 \end{aligned}\right\} = 0.$$

Iam pro v et u assumamus has series ascendentes

$$\begin{aligned} v &= Ax^{\lambda} + Bx^{\lambda+n} + Cx^{\lambda+2n} + Dx^{\lambda+3n} + \text{etc.}, \\ u &= \mathfrak{A}x^{\lambda} + \mathfrak{B}x^{\lambda+n} + \mathfrak{C}x^{\lambda+2n} + \mathfrak{D}x^{\lambda+3n} + \text{etc.} \end{aligned}$$

iisque substitutis prior aequatio abit in hanc

$$\begin{array}{llllll}
\lambda(\lambda-1)Aax^{\lambda} & +(\lambda+n)(\lambda+n-1)Bax^{\lambda+n} & & +(\lambda+2n)(\lambda+2n-1)Cax^{\lambda+2n} & +\text{etc.}=0 \\
 & + & \lambda(\lambda-1)Ab & + & (\lambda+n)(\lambda+n-1)Bb \\
+\ \lambda Ac & + & (\lambda+n)Bc & + & (\lambda+2n)Cc \\
 & + & \lambda Ae & + & (\lambda+n)Be \\
+\ Af & + & Bf & + & Cf \\
 & + & Ag & + & Bg \\
-2\mu\lambda\mathfrak{A}a & - & 2\mu(\lambda+n)\mathfrak{B}a & - & 2\mu(\lambda+2n)\mathfrak{C}a \\
 & - & 2\mu\lambda\mathfrak{A}b & - & 2\mu(\lambda+n)\mathfrak{B}b \\
-\mu\mu Aa & - & \mu\mu Ba & - & \mu\mu Ca \\
 & - & \mu\mu Ab & - & \mu\mu Bb \\
+\mu\mathfrak{A}a & + & \mu\mathfrak{B}a & + & \mu\mathfrak{C}a \\
 & + & \mu\mathfrak{A}b & + & \mu\mathfrak{B}b \\
-\mu\mathfrak{A}c & - & \mu\mathfrak{B}c & - & \mu\mathfrak{C}c \\
 & - & \mu\mathfrak{A}e & - & \mu\mathfrak{B}e
\end{array}$$

Hinc altera aequatio facile formatur permutandis litteris latinis et germanicis atque insuper signum numeri μ mutando.

Utrinque ergo potestas prima x^{λ} exigit has aequationes

$$A(\lambda(\lambda-1)a+\lambda c+f-\mu\mu a)-\mu\mathfrak{A}(2\lambda a-a+c)=0,$$
$$\mathfrak{A}(\lambda(\lambda-1)a+\lambda c+f-\mu\mu a)+\mu A(2\lambda a-a+c)=0,$$

unde necesse est, ut sit tam

$$2\lambda a-a+c=0$$

quam

$$\lambda(\lambda-1)a+\lambda c+f-\mu\mu a=0\,.$$

Inde fit $\lambda=\frac{1}{2}-\frac{c}{2a}$, qui valor hic substitutus dat

$$-a(\tfrac{1}{4}-\tfrac{cc}{4aa})+\tfrac{c}{2}-\tfrac{cc}{2a}+f=\mu\mu a=-\tfrac{a}{4}+\tfrac{c}{2}-\tfrac{cc}{4a}+f$$

seu

$$\mu\mu a=\frac{4af-(a-c)^2}{4a}\quad\text{ideoque}\quad\mu=\frac{\sqrt{\left(4af-(a-c)^2\right)}}{2a}\quad\text{et}\quad\lambda=\frac{a-c}{2a}.$$

Unde patet hanc solutionem locum habere, si $4af > (a-c)^2$, quo ipso casu praecedens solutio [§ 967] fiebat imaginaria. Hic autem quantitates A et $\mathfrak{A}$ arbitrio nostro relinquuntur.

Terminus vero $x^{\lambda+n}$ utrinque postulat has aequationes

$$B\big((\lambda+n)(\lambda+n-1)a+(\lambda+n)c+f-\mu\mu a\big)+A\big(\lambda(\lambda-1)b+\lambda e+g-\mu\mu b\big)$$
$$-\mu\mathfrak{B}\big(2(\lambda+n)a-a+c\big)-\mu\mathfrak{A}(2\lambda b-b+e)=0$$

et

$$\mathfrak{B}\big((\lambda+n)(\lambda+n-1)a+(\lambda+n)c+f-\mu\mu a\big)+\mathfrak{A}\big(\lambda(\lambda-1)b+\lambda e+g-\mu\mu b\big)$$
$$+\mu B\big(2(\lambda+n)a-a+c\big)+\mu A(2\lambda b-b+e)=0.$$

Sit brevitatis gratia

$$(\lambda+n)(\lambda+n-1)a+(\lambda+n)c+f-\mu\mu a=\alpha,$$
$$\lambda(\lambda-1)b+\lambda e+g-\mu\mu b=\beta,$$
$$2(\lambda+n)a-a+c=2na=\gamma,$$
$$2\lambda b-b+e=\delta,$$

ut habeamus

$$B\alpha+A\beta-\mu\mathfrak{B}\gamma-\mu\mathfrak{A}\delta=0 \quad \text{et} \quad \mathfrak{B}\alpha+\mathfrak{A}\beta+\mu B\gamma+\mu A\delta=0,$$

unde colligitur

$$B=\frac{-A(\alpha\beta+\mu\mu\gamma\delta)+\mu\mathfrak{A}(\alpha\delta-\beta\gamma)}{\alpha\alpha+\mu\mu\gamma\gamma} \quad \text{et} \quad \mathfrak{B}=\frac{-\mathfrak{A}(\alpha\beta+\mu\mu\gamma\delta)-\mu A(\alpha\delta-\beta\gamma)}{\alpha\alpha+\mu\mu\gamma\gamma}.$$

At vero ex valoribus assumtis est

$$\alpha=nna,\quad \beta=\frac{(ae-bc)(a-c)}{2aa}-\frac{bf}{a}+g,\quad \gamma=2na,\quad \delta=\frac{ae-bc}{a},$$

unde ex assumtis A et $\mathfrak{A}$ definiuntur B et $\mathfrak{B}$ hincque porro $C, \mathfrak{C}, D, \mathfrak{D}$ etc.

EXEMPLUM 1

986. *Sit $c=a$ et $f=a$, ut fiat $\mu=1$, et investigetur integrale huius aequationis*

$$xx\left(a+bx^n\right)ddy+x\left(c+ex^n\right)dxdy+\left(f+gx^n\right)ydx^2=0.$$

Hic ergo erit $\lambda=0$ et $\mu=1$, unde posito $y=v\sin.lx+u\cos.lx$ ac pro v et u sumtis seriebus

$$v = A + Bx^n + Cx^{2n} + Dx^{3n} + \text{etc.},$$
$$u = \mathfrak{A} + \mathfrak{B}x^n + \mathfrak{C}x^{2n} + \mathfrak{D}x^{3n} + \text{etc.}$$

coefficientes A et $\mathfrak{A}$ pro lubitu accipi possunt. Ex iis primo ob $\alpha = nna, \beta = g - b,\ \gamma = 2na$ et $\delta = e - b$ erit

$$B = \frac{-A(nna(g-b)+2na(e-b))+\mathfrak{A}(nna(e-b)-2na(g-b))}{n^4aa+4nnaa}$$

seu

$$B = \frac{-A\big(n(g-b)+2(e-b)\big)+\mathfrak{A}\big(n(e-b)-2(g-b)\big)}{na(nn+4)}$$

et

$$\mathfrak{B} = \frac{-\mathfrak{A}\big(n(g-b)+2(e-b)\big)-A\big(n(e-b)-2(g-b)\big)}{na(nn+4)}$$

Pro sequentibus coefficientibus habebimus

$$C(2n(2n-1)a+2na+a-a) + B\big(n(n-1)b+ne+g-b\big)$$
$$-\mathfrak{C}(4na-a+a)-\mathfrak{B}(2nb-b+e)=0$$

seu

$$4nnCa + B((nn-n-1)b+ne+g) - 4n\mathfrak{C}a - \mathfrak{B}((2n-1)b+e) = 0$$

et

$$4nn\mathfrak{C}a + \mathfrak{B}((nn-n-1)b+ne+g) + 4nCa + B((2n-1)b+e) = 0\ ,$$

quarum illa per n multiplicata huic addatur, ut prodeat

$$4n(nn+1)Ca + B((n^3-nn+n-1)b+(nn+1)e+ng)$$
$$+\mathfrak{B}(-(nn+1)b\ +g)=0,$$

hinc

$$C = \frac{-B((n-1)(nn+1)b+(nn+1)e+ng)+\mathfrak{B}((nn+1)b-g)}{4na(nn+1)}$$

et

$$\mathfrak{C} = \frac{-\mathfrak{B}((n-1)(nn+1)b+(nn+1)e+ng)-B((nn+1)b-g)}{4na(nn+1)}.$$

Porro erit

$$9nnDa + C((4nn-2n-1)b+2ne+g) - 6n\mathfrak{D}a - \mathfrak{C}((4n-1)b+e) = 0,$$
$$9nn\mathfrak{D}a + \mathfrak{C}((4nn-2n-1)b+2ne+g) + 6nDa + C((4n-1)b+e) = 0,$$

quarum illa per *3n*, haec vero per 2 multiplicata iunctim dant

$$3n(9nn+4)Da+C((12n^3-6nn+5n-2)b+2(3nn+1)e+3ng)$$
$$+\mathfrak{C}((-4nn-n-2)b+ne+2g)=0,$$

unde sequitur

$$D=\frac{-C((12n^3-6nn+5n-2)b+2(3nn+1)e+3ng)+\mathfrak{C}((4nn+n+2)b-ne-2g)}{3n(9nn+4)a},$$
$$\mathfrak{D}=\frac{-\mathfrak{C}((12n^3-6nn+5n-2)b+2(3nn+1)e+3ng)-C((4nn+n+2)b-ne-2g)}{3n(9nn+4)a}.$$

In genere autem ex coefficientibus quibuscunque M et $\mathfrak{M}$ sequentes N et $\mathfrak{N}$ definiuntur per has formulas

$$in(iinn+4)Na+M((i(i-1)^2n^3-i(i-1)nn+(3i-4)n-2)b$$
$$+i(i-1)nne+2e+ing)-\mathfrak{M}((2(i-1)nn+(i-2)n+2)b-(i-2)ne-2g)=0,$$
$$in(iinn+4)\mathfrak{N}a+\mathfrak{M}((i(i-1)^2n^3-i(i-1)nn+(3i-4)n-2)b+i(i-1)nne+2e+ing)$$
$$+M((2(i-1)nn+(i-2)n+2)b-(i-2)ne-2g)=0.$$

COROLLARIUM 1

987. Si quantitates *b, e, g* ita sint comparatae, ut binae litterae sibi respondentes *N* et in evanescant, sequentes omnes evanescent et integrale completum forma finita exprimetur. Ita, ut *B* et $\mathfrak{B}$ evanescant, fieri debet

$$2(g-b)=n(e-b) \text{ et } n(g-b)=-2(e-b),$$

unde fit $g=e=b$, et ipsa aequatio proposita factorem habebit $a+bx^n$.

COROLLARIUM 2

988. In genere autem integrale finite exprimetur, si denotante *i* numerum integrum quemcunque positivum sit

$$g=\left((i-1)nn+\tfrac{1}{2}(i-2)n+1\right)b-\tfrac{1}{2}(i-2)ne,$$

tum vero

$$e=-\left(2(i-1)n-1\right)b,$$

unde fit

$$g=\left((i-1)^2nn+1\right)b.$$

EXEMPLUM 2

989. *Sumto* $n = 1$ *si sit* $e = -b$ *et* $g = 2b$ *, huius aequationis*

$$xx(a+bx)ddy + x(a-bx)dxdy + (a+2bx)ydx^2 = 0$$

integrale completum assignare.

Ex formulis modo inventis colligimus

$$B = \frac{-A(g+2e-3b)+\mathfrak{A}(e+b-2g)}{5a} = \frac{3Ab-4\mathfrak{A}b}{5a} \text{ et } \mathfrak{B} = \frac{3\mathfrak{A}b+4Ab}{5a},$$

tum vero

$$C = \frac{-B(2e+g)+\mathfrak{B}(2b-g)}{8a} = 0 \text{ et } \mathfrak{C} = 0.$$

Quocirca habebimus

$$v = A + \frac{(3A-4\mathfrak{A})b}{5a}x \text{ et } u = \mathfrak{A} + \frac{(3\mathfrak{A}+4A)b}{5a}x$$

hincque integrale completum elicitur

$$y = A\sin.lx + \mathfrak{A}\cos.lx + \frac{bx}{5a}((3A-4\mathfrak{A})\sin.lx + (3\mathfrak{A}+4A)\cos.lx).$$

COROLLARIUM 1

990. Sumto $\mathfrak{A} = 0$ habebitur integrale particulare

$$y = A(\sin.lx + \frac{3bx}{5a}\sin.lx + \frac{4bx}{5a}\cos.lx) \;;$$

sin autem sit $A = 0$, aliud habebitur

$$y = \mathfrak{A}(\cos.lx - \frac{4bx}{5a}\sin.lx + \frac{3bx}{5a}\cos.lx).$$

COROLLARIUM 2

991. Posito $y = e^{\int sdx}$ aequatio nostra reducitur ad hanc

$$xx(a+bx)ds + xx(a+bx)ssdx + x(a-bx)sdx + (a+2bx)dx = 0,$$

cuius integrale habetur $s = \frac{dy}{ydx}$ inde definiendum; quae aequatio in plures alias formas transfundi potest.

SCHOLION

992. Simili modo integratio per series descendentes instituitur, si exponentes singulorum terminorum prodeant imaginarii; quod seorsim exposuisse ne opus quidem erit. Atque haec sufficiunt, ut pateat, quibusnam cautelis in resolutione aequationum per series infinitas sit utendum. Summus autem usus istarum evolutionum in hoc consistit, ut aequationes differentio-differentiales exhiberi queant, quarum saltem integrale particulare algebraicum assignare liceat, quos casus supra § 969 indicavimus.

Similis porro integratio per series infinitas pari modo extendi potest ad huiusmodi aequationes

$$xx(a+bx^n+\beta x^{2n})ddy+x(c+ex^n+\varepsilon x^{2n})dxdy+(f+gx^n+\gamma x^{2n})ydx^2=0\ ;$$

tum autem seriei quaesitae quilibet terminus per duos praecedentes determinatur, ita ut, si bini contigui evanescant, sequentes omnes in nihilum sint abituri. Quodsi autem terminus ab y vacuus affuerit, resolutio in series fit facilior, cui propterea non immorandum censeo. Veluti si proponatur haec aequatio

$$xxddy-xdxdy+ax^nydx^2=bx^mdx^2$$

series a potestate x^m est inchoanda ponendo

$$y=Ax^m+Bx^{m+n}+Cx^{m+2n}+Dx^{m+3n}+\text{etc.},$$

unde fit

$$\left.\begin{array}{llllll} m(m-1)Ax^m & + & (m+n)(m+n-1)Bx^{m+n} & + & (m+2n)(m+2n-1)Cx^{m+2n} & + \text{ etc.}\\ -mA & - & (m+n)B & - & (m+2n)C & \\ -b & + & Aa & + & Ba & \end{array}\right\}=0$$

hincque

$$A=\frac{b}{m(m-2)},\ B=\frac{-Aa}{(m+n)(m+n-2)},\ C=\frac{-B}{(m+2n)(m+2n-2)}\quad \text{etc.},$$

ubi quidem multa observanda occurrunt, quae per praecepta supra data expedire licet.

Imprimis autem in hoc negotio iuvat aequationem propositam ope substitutionis in alias transformasse, quarum resolutio per series fiat simplicior; quod cum pluribus modis fieri possit, hoc argumentum sequenti capite diligentius pertractare visum est idque pro forma aequationum

$$Lddy+Mdxdy+Nydx^2=0,$$

quandoquidem pro aliis formis huiusmodi transformatio raro locum invenit.