CHAPTER IX

CONCERNING THE TRANSFORMATION OF THE SECOND ORDER DIFFERENTIAL EQUATION $Lddy + Mdxdy + Nydx^2 = 0$

PROBLEM 125

993. *To transform the second order differential equation*

$$Lddy + Mdxdy + Nydx^2 = 0$$

into another form, in which L, M, N are any functions of x, with the element dx assumed constant, with the help of the substitution $y = e^{\int Pdx}z$.

SOLUTION

Since hence there shall be [§ 940] $\frac{dy}{y} = Pdx + \frac{dz}{z}$, there will be on differentiation

$$\frac{ddy}{y} - \frac{dy^2}{y^2} = dxdP + \frac{ddz}{z} - \frac{dz^2}{zz},$$

hence

$$\frac{ddy}{y} = \frac{ddz}{z} + \frac{2Pdxdz}{z} + dxdP + PPdx^2.$$

Whereby since our equation shall be

$$\frac{Lddy}{y} + \frac{Mdxdy}{y} + Ndx^2 = 0,$$

there will be with the substitution made

$$\frac{Lddz}{z} + \frac{2LPdxdz}{z} + LdxdP + LPPdx^2 + \frac{Mdxdz}{z} + MPdx^2 + Ndx^2 = 0$$

or on multiplying by z

$$Lddz + (2LP + M)dxdz + zdx(LdP + LPPdx + MPdx + Ndx) = 0,$$

where it is allowed to take some function of x for P, from which innumerable equations may be obtained between the two variables x and z.

COROLLARY 1

994. But if hence this transformed equation is permitted to be integrated or to be resolved by a series, from the value found of *z* there will be had $y = e^{\int Pdx} z$.

COROLLARY 2

995. The transformed equation is of the kind proposed, since therefore in that the variable *z* with its differentials *dz* and *ddz* occupies a single dimension everywhere and likewise *y* in the equation proposed.

COROLLARY 3

996. If it arises that both equations, the proposed and the transformed, can be resolved by series conveniently, in this way more resolutions of same equation can be shown.

SCHOLIUM 1

997. Since equations in this form can be resolved in series conveniently [§ 967, 985]

$$xx\left(a+bx^n\right)ddy + x\left(c+ex^n\right)dxdy + \left(f+gx^n\right)ydx^2 = 0,$$

where

$$L = xx\left(a+bx^n\right),\ M = x\left(c+ex^n\right),\ N = f+gx^n,$$

in order that the transformed shall have a similar form, there is required to become

$$LP = x\left(\mu+\nu x^n\right) \text{ and thus } P = \frac{\mu+\nu x^n}{x\left(a+bx^n\right)}$$

Hence there shall be

$$dP = \frac{-\mu a - \mu(n+1)bx^n + \nu(n-1)ax^n - \nu bx^{2n}}{xx\left(a+bx^n\right)^2}dx$$

and thus

$$LdP + LPPdx + MPdx = \frac{\left\{\begin{matrix} -\mu a - & (n+1)\mu bx^n + & (n-1)\nu ax^n - \nu bx^{2n} \\ +\mu\mu + & 2\mu\nu x^n + & \nu\nu x^{2n} \\ +\mu c + & \mu ex^n + & \nu cx^n + \nu ex^{2n} \end{matrix}\right\}}{a+bx^n}dx,$$

[both the first edition and the *O. O.* edition have the final *dx* missing here,] where division by $a+bx^n$ must succeed. The quotient $= \mu h + \nu kx^n$ is put in place and there becomes [on equating coefficients]

$$\mu = a - c + ah,\ \nu = b - e + bk$$

and in addition

$$2\mu\nu - (n+1)\mu b + (n\ -1)\nu a + \mu e + \nu c = \mu bh + \nu ak\ ,$$

where the above values substituted give

$$(h-k+n)(bc-ae) = nab(h-k) + ab(h-k)^2 \left[= ab(h-k)(n+h-k)\right],$$

from which there becomes

either $h-k = \frac{bc-ae}{ab}$ or $h-k=-n$.

Hence either of the letters h and k is left to our choice, and the transformed equation becomes

$$xx\left(a+bx^n\right)ddz + x\left(2\mu + c + (2\nu+e)x^n\right)dxdz + \left(f + \mu h + (g+\nu k)x^n\right)zdx^2 = 0\,.$$

But the resolution of this both by ascending as well as descending series requires like powers of x. But with the substitution itself there becomes

$$y = x^{\frac{a-c}{a}+h}\left(a+bx^n\right)^{\frac{bc-ae}{nab}-\frac{h-k}{n}} z\ ,$$

[as $y = e^{\int Pdx}z$, and $P = \frac{\mu+\nu x^n}{x\left(a+bx^n\right)}$, where $\mu = a-c+ah,\ \nu = b-e+bk$ and where $h-k = \frac{bc-ae}{ab}$ or $h-k=-n$. We have

$\int Pdx = \int \frac{\mu+\nu x^n}{x\left(a+bx^n\right)}dx = \frac{\mu}{a}\ln x - \frac{\mu b}{a}\ln\left(a+bx^n\right) + \frac{\nu}{nb}\ln\left(a+bx^n\right),$

$= \frac{\mu}{a}\ln x + \left(\frac{\nu}{nb} - \frac{\mu}{na}\right)\ln\left(a+bx^n\right) = \frac{a-c+ah}{a}\ln x + \left(\frac{b-e+bk}{nb} - \frac{a-c+ah}{na}\right)\ln\left(a+bx^n\right),$

leading to the required results.]

where, in order that only the power of x advances, there must be taken $h-k=-n$. There is no concern in what manner here h may be taken; hence on assuming $h=0$ there becomes $k=n$ and on substitution

$$y = x^{\frac{a-c}{a}}\left(a+bx^n\right)^{\frac{c-ae}{nab}+1} z\ ,$$

which leads to this equation

$$xx\left(a+bx^{n}\right)ddz+x\left(2a-c+\left(2\left(n+1\right)b-e\right)x^{n}\right)dxdz$$
$$+\left(f+\left(n\left(n+1\right)b-ne+g\right)x^{n}\right)zdx^{2}=0.$$

SCHOLIUM 2

998. We have seen above [§ 970] that the proposed equation between *x* and *y* admits an algebraic solution, if there should be [*i.e.* from a zero coefficient at some stage in the expansion]

$$\tfrac{c}{2a}-\tfrac{e}{2b}\cdot\cdot\frac{\sqrt{\left((a-c)^{2}-4af\right)}}{2a}\cdot\cdot\ \frac{\sqrt{\left((b-e)^{2}-4bg\right)}}{2b}=in,$$

which if the transformation should be treated in a similar way, will be able to be assigned an algebraic integral, if there should be

$$-\tfrac{c}{2a}+\tfrac{e}{2b}\cdot\cdot\frac{\sqrt{\left((a-c)^{2}-4af\right)}}{2a}\cdot\cdot\ \frac{\sqrt{\left((b-e)^{2}-4bg\right)}}{2b}-n=in,$$

from which conditions taken together it is possible to conclude that the conditions for an algebraic integral be satisfied, provided that this formula

$$\tfrac{bc-\mathrm{ae}}{2ab}\cdot\cdot\frac{\sqrt{\left((a-c)^{2}-4af\right)}}{2a}\cdot\cdot\ \frac{\sqrt{\left((b-e)^{2}-4bg\right)}}{2b}$$

has stood out as being divisible by the exponent *n.* Here I have used the sign .. to designate the ambiguity of positive and negative. Whereby if we put

$$f=\tfrac{(a-c)^{2}-hh}{4a}\quad\text{and}\quad g=\tfrac{(b-e)^{2}-kk}{4b}$$

integrability can be considered, as long as this expression

$$\tfrac{bc-ae\pm bh\pm ak}{2nab}$$

should be either a positive or negative whole number.

EXAMPLE

999. *For the proposed equation*

$$xx(1-xx)ddy + x(1+2mxx)dxdy - m(m+1)xxydx^2 = 0$$

to find the cases in which at least a particular integral can be assigned.

Here there is $a = 1,\ b = -1,\ c = 1,\ e = 2m,\ f = 0,\ g = -m(m+1)$ and $n = 2$.

Hence we deduce [from $f = \frac{(a-c)^2 - hh}{4a}$ and $g = \frac{(b-e)^2 - kk}{4b}$] that

$$h = \sqrt{\left((a-c)^2 - 4af\right)} = 0$$

and

$$k = \sqrt{\left((b-e)^2 - 4bg\right)} = \sqrt{\left((2m+1)^2 - 4m(m+1)\right)},$$

that is, $k = \pm 1$. Hence the formula for an integer number is [$i = \frac{bc-ae\pm bh\pm ak}{2nab} =$] $\frac{-1-2m\pm 1}{-4}$, from which we come upon twin cases for m

$$\text{either } 2m+2 = \pm 4i \text{ or } 2m = \pm 4i,$$

that is

$$\text{either } m = \pm 2i - 1 \text{ or } m = \pm 2i;$$

hence provided m shall be either a positive or negative integer, it is possible for an algebraic integral to be shown. Moreover the substitution giving the transformed equation [from $y = x^{\frac{a-c}{a}+h}\left(a+bx^n\right)^{\frac{bc-ae}{nab}-\frac{h-k}{n}} z$: here we are assuming that $h = 0$ and $h-k = -n$ as stated above]

$$y = (1-xx)^{\frac{-1-2m}{-2}+1} z = (1-xx)^{\frac{2m+3}{2}} z,$$

and indeed the transformed equation itself :

$$xx(1-xx)ddz + x\left(1-2(m+3)xx\right)dxdz - (m+2)(m+3)xxzdx^2 = 0,$$

as it is evident to arise from that former one, if in place of m there is written $-m-3$. Moreover this integration may be found from that on account of $\lambda\lambda = 0$ on putting

$$y = A + Bx^2 + Cx^4 + Dx^6 + Ex^8 + \text{etc.},$$

from which there becomes :

$$\left.\begin{array}{llll} 2Bxx & +\quad 12Cx^4 + & 30Dx^6 + & 56Ex^8 + \text{etc.} \\ & -\quad 2B & -\quad 12C & -\quad 30D \\ +2B & +\quad 4C & +\quad 6D & +\quad 8E \\ & +\quad 4mB & +\quad 8mC & +\quad 12mD \\ -m(m+1)A & -m(m+1)B & -m(m+1)C & -m(m+1)D \end{array}\right\} = 0.$$

Thus the determination of the coefficients may be put in place :

$$B = \tfrac{m(m+1)}{4}A,\ \ C = \tfrac{(m-1)(m-2)}{16}B,\ \ D = \tfrac{(m-3)(m-4)}{36}C \ \text{ etc.}$$

And if there is put

$$z = \mathfrak{A} + \mathfrak{B}x^2 + \mathfrak{C}x^4 + \mathfrak{D}x^6 + \mathfrak{E}x^8 + \text{etc.}$$

there will be [for the transformed equation, as a final check]

$$\mathfrak{B} = \tfrac{(m+2)(m+3)}{4}\mathfrak{A},\ \ \mathfrak{C} = \tfrac{(m+4)(m+5)}{16}\mathfrak{B},\ \ \mathfrak{D} = \tfrac{(m+6)(m+7)}{36}\mathfrak{C} \ \text{ etc.}$$

PROBLEM 126

1000. *To transform the second order differential equation*

$$Lddy + Mdxdy + Nydx^2 = 0$$

into another form of the same kind with the help of the substitution $\frac{dy}{y} = \frac{Pzdx^2}{dz}$.

SOLUTION

Here evidently there is sought, that such a function of x is required to be taken for P, in order that on making the substitution of the variable z with its differential dz and ddz, a single dimension shall be obtained everywhere. Therefore since there shall be $\frac{dy}{y} = \frac{Pzdx^2}{dz}$, there will be on differentiating,

$$\frac{ddy}{y} - \frac{dy^2}{y^2} = \frac{-Pzdx^2ddz}{dz^2} + \frac{zdx^2dP}{dz} + Pdx^2$$

and

$$\frac{ddy}{y} = \frac{-Pzdx^2ddz}{dz^2} + \frac{zdx^2dP}{dz} + Pdx^2 + \frac{PPzzdx^4}{dz^2},$$

with which values substituted there becomes

$$\frac{-LPzdx^2ddz}{dz^2} + \frac{Lzdx^2dP}{dz} + LPdx^2 + \frac{LPPzzdx^4}{dz^2} + \frac{MPzdx^3}{dz} + Ndx^2 = 0.$$

Hence we may take $LP+N=0$ or $P=\frac{-N}{L}$ and on multiplying by $\frac{-dz^2}{Pzdx^2}$ we find

$$Lddz-\tfrac{LdPdz}{P}-LPzdx^2-Mdxdz=0$$

or

$$Lddz-Mdxdz-\tfrac{LdNdz}{N}+dLdz+Nzdx^2=0.$$

Hence the proposed equation with the help of the substitution $\frac{dy}{y}=\frac{-Nzdx^2}{Ldz}$ is transformed into this :

$$Lddz+\left(\tfrac{dL}{dx}-M-\tfrac{LdN}{Ndx}\right)dxdz+Nzdx^2=0.$$

Hence if it should be possible to elicit the value of *z* also the value of *y* can be expressed by *x*.

COROLLARY 1

1001. If there is put in this transformed equation in turn $\frac{dz}{z}=\frac{-Nydx^2}{Ldy}$, the proposed equation itself emerges ; from which these two equations thus are consistent between themselves, so that the one from the other is produced by a similar substitution.

COROLLARY 2

1002. If in the transformed equation the following substitution is put in place first [§ 993] $\frac{dz}{z}=Qdx+\frac{dv}{v}$, this new transformation will be obtained :

$$\begin{aligned}&Lddv+\left(2LQ+\tfrac{dL}{dx}-M-\tfrac{LdN}{Ndx}\right)dxdv\\&+vdx\left(LdQ+LQQdx+QdL-MQdx-\tfrac{LQdN}{N}+Ndx\right)=0,\end{aligned}$$

which hence is deduced from that proposed on putting $\frac{dy}{y}=\frac{-Nvdx^2}{L(dv+Qvdx)}$.

SCHOLIUM 1

1003. Hence by combining both substitutions, which we have used in the two preceding problems, we arrive at a general substitution of this kind :

$$\frac{dy}{y}=\frac{Pdz+Qzdx}{Rdz+Szdx}dx,$$

which if it is substituted into the proposed equation

$$Lddy + Mdxdy + Nydx^2 = 0,$$

the functions *P, Q, R, S* thus must be defined, so that in the resulting equation of the variable *z* with its differentials nowhere is there held a dimension greater than one. But terms arise affecting the quadratic dz^2, towards removing which it is required to make

$$Ldx(PP + QR - PS) + L(RdP - PdR) + MPRdx + NRRdx = 0$$

or

$$Q = \frac{PS}{R} - \frac{PP}{R} - \frac{dP}{dx} + \frac{PdR}{Rdx} - \frac{MP}{L} - \frac{NR}{L};$$

then indeed this equation is come upon

$$\begin{aligned} &Lddz(PS - QR) + Ldz(RdQ - QdR + SdP - PdS) \\ &+ Lzdx(SdQ - QdS + QQdx) + dxdz(2LPQ + M(QR + PS) + 2NRS) \\ &+ Szdx^2(MQ + NS) = 0. \end{aligned}$$

Truly it is easier to arrive at this general equation, if both substitutions are called into use in turn.

SCHOLION 2

1004. But here the transformation is on that account more worthy of note, because, even if the transformed equation admits to resolution, since yet not without difficulty is that equation proposed resolved. Since indeed a function of *x* should be found, which substituted in place of *z* should satisfy the transformed equation, for the value of *y* to be found above it is necessary to investigate the integral of this equation $\frac{dy}{y} = \frac{-Nzdx^2}{Ldz}$, where, even if the variables *x* and *y* have been separated from each other in turn, yet significant difficulties are able to exert themselves in that integration. Hence it is required to come about, in order that by the aid of a substitution of this kind, the integrations of equations are able to be shown, which scarcely are able to be investigated in a straight forwards manner. Clearly if it comes about, that the integral of the transformed equation can be found, either with the help of a certain method established above, or able to be expressed by a terminating series, then also the complete integral of this equation can be considered. For even if in the latter case only a particular integral becomes known, yet from that it is possible always to elicit generally the complete integral. In as much as if the value $y = X$ particularly satisfies the equation

$$Lddy + Mdxdy + Nydx^2 = 0,$$

there may be put $y = Xv$ and there becomes

$$\left.\begin{aligned} LXddv + 2LdXdv \;\; + LvddX \\ + \;\; MXdxdv + MvdxdX \\ + \;\; NXvdx^2 \end{aligned}\right\} = 0.$$

But because $X = y$ by hypothesis satisfies the equation, there will be

$$LddX + MdxdX + NXdx^2 = \;0 \;\text{ and }\; LXddv + \left(2LdX + MXdx\right)dv = 0$$

or

$$\frac{ddv}{dv} + \frac{2dX}{X} + \frac{Mdx}{L} = 0,$$

from which on integrating there arises

$$XXdv = Ce^{-\int \frac{Mdx}{L}}dx$$

and again

$$v = \int \frac{Cdx}{XX} e^{-\int \frac{Mdx}{L}},$$

thus so that the complete integral will be

$$y = CX \int \frac{dx}{XX} e^{-\int \frac{Mdx}{L}},$$

which hence from some particular integral $y = X$ can be elicited.

EXAMPLE

1005. *To transform the second order differential equation*

$$xx\left(a + bx^n\right)ddy + x\left(c + ex^n\right)dxdy + \; fydx^2 = 0$$

and to integrate that by series.

Since here there shall be $L = xx\left(a + bx^n\right)$, $M = x\left(c + ex^n\right)$ and $N = f$, there is on using this substitution

$$\frac{dy}{y} = \frac{-\,fzdx^2}{xx\left(a+bx^n\right)dz},$$

by which our equation is reduced to this form

$$xx\left(a + bx^n\right)ddz + x\left(2a - c + \left(\left(n+2\right)b - e\right)x^n\right)dxdz + \; fzdx^2 = 0;$$

for the resolution of which if there is put in place

$$z = Ax^{\lambda} + Bx^{\lambda+n} + Cx^{\lambda+2n} + \text{ etc.},$$

there must become

$$\lambda(\lambda-1)a + \lambda(2a-c) + f = 0 \text{ or } \lambda\lambda a + \lambda(a-c) + f = 0,$$

hence

$$\lambda = \frac{-a+c\pm\sqrt{\left((a-c)^2-4af\right)}}{2a}$$

But the series may terminate by § 970, if this expression
[noting that for $\frac{b-e\pm\sqrt{\left((b-e)^2-4bg\right)}}{2b}$, with $g = 0$, we have a perfect square,]

$$-\frac{c}{2a} + \frac{e}{2b} - \frac{n}{2} \pm \left(\frac{e}{2b} - \frac{n}{2} - \frac{1}{2}\right) \pm \frac{\sqrt{\left((a-c)^2-4af\right)}}{2a} = in$$

with i denoting some positive whole number, that is either

$$-\frac{c}{2a} + \frac{e}{b} - \frac{1}{2} - n \pm \frac{\sqrt{\left((a-c)^2-4af\right)}}{2a} = in$$

or

$$-\frac{c}{2a} + \frac{1}{2} \pm \frac{\sqrt{\left((a-c)^2-4af\right)}}{2a} = in.$$

But if the proposed equation can be resolved in this way, this is terminated if there should be:

$$\frac{c}{2a} - \frac{e}{2b} \pm \frac{b-e}{2b} \pm \frac{\sqrt{\left((a-c)^2-4af\right)}}{2a} = in$$

that is either

$$\frac{c}{2a} - \frac{e}{b} + \frac{1}{2} \pm \frac{\sqrt{\left((a-c)^2-4af\right)}}{2a} = in$$

or

$$\frac{c}{2a} - \frac{1}{2} \pm \frac{\sqrt{\left((a-c)^2-4af\right)}}{2a} = in$$

From which it is understood that it is possible for a finite integral to be shown, if the whole number i should be either positive or negative. Now thus the above substitution

(§ 998) leads to a doubling, thus so that this new substitution supports no cases of integrability.

SCHOLION 1

1006. Yet in order that it may be apparent, how from a finite value of z a finite value of y is able to be elicited, we consider the case

$$xx\left(a+bx^2\right)ddy+x\left(3a+ex^2\right)dxdy-24aydx^2=0,$$

where $n=2,\ c=3a$ and $f=-24a$, which with the substitution made

$$\frac{dy}{y}=\frac{24azdx^2}{xx\left(a+bx^2\right)dz}$$

goes into this

$$xx\left(a+bx^2\right)ddz+x\left(-a+(4b-e)xx\right)dxdz-24azdx^2=0,$$

where for the ascending series

$$\lambda\lambda-2\lambda-24=0=(\lambda-6)(\lambda+4).$$

There is put in place

$$y=Ax^{-4}+Bx^{-2}+\mathrm{C}+Dx^2+\text{etc.};$$

$$\left.\begin{array}{lllll}
20Aax^{-4}+ & 6Bax^{-2} & * & + & 2Dax^2+\ \text{etc.}\\
 & +\ 20Ab & +\ 6Bb & & *\\
+\ 4Aa & +\ 2Ba & * & - & 2Da\\
 & -\ 4A(4b-e) & -\ 2B(4b-e) & & *\\
-\ 24Aa & -\ 24Ba & -\ 24Ca & - & 24Da
\end{array}\right\}=0.$$

Hence since there shall be $D=0$, all the following terms are removed. Then indeed there is

$$16Ba=4A(b+e),\quad 24Ca=-2Bb+2Be,$$

hence

$$B=\tfrac{b+e}{4a}A,\quad C=\tfrac{e-b}{12a}B=\tfrac{ee-bb}{48aa}A$$

and hence [as we take $z=Ax^{\lambda}+Bx^{\lambda+n}+Cx^{\lambda+2n}+$ etc.,]

$$z = A\left(\frac{1}{x^4} + \frac{b+e}{4ax^2} + \frac{ee-bb}{48aa}\right) = \frac{A\left(48aa+12a(b+e)xx+(ee-bb)x^4\right)}{48aax^4},$$

from which it follows

$$dz = Adx\left(\frac{-4}{x^5} - \frac{b+e}{2ax^3}\right) = \frac{-Adx}{2ax^5}\left(8a + (b+e)xx\right).$$

Hence

$$\frac{dy}{y} = \frac{-\left(48aa+12a(b+e)xx+(ee-bb)x^4\right)}{x\left(a+bx^2\right)\left(8a+(b+e)xx\right)}dx$$

or on resolving

$$\frac{dy}{y} = \frac{-6dx}{x} + \frac{(5b-e)xdx}{a+bxx} + \frac{2(b+e)xdx}{8a+(b+e)xx}$$

and hence on integrating

$$y = \frac{A}{x^6}\left(a+bxx\right)^{\frac{5b-e}{2b}}\left(8a + (b+e)xx\right).$$

SCHOLION 2

1007. Because here in this fortuitous case it is seen to come about, that from the value of z found it was possible conveniently to define the quantity y, it can be shown that the same always is required to come about in the following general way. For since the proposed equation

$$Lddy + Mdxdy + Nydx^2 = 0$$

with the aid of the substitution $\frac{dy}{y} = \frac{-Nzdx^2}{Ldz}$ shall be transformed into this :

$$Lddz - Mdxdz - \frac{LdNdz}{N} + dLdz + Nzdx^2 = 0,$$

if this is divided by Ldz, there is produced

$$\frac{ddz}{dz} - \frac{Mdx}{L} - \frac{dN}{N} + \frac{dL}{L} = -\frac{Nzdx^2}{Ldz} = \frac{dy}{y},$$

from which by integrating there is elicited

$$y = \frac{\alpha Ldz}{Ndx}e^{-\int\frac{Mdx}{L}},$$

which with the value of z found at once gives the value of y without further integration.

Since again there shall be $dy = \frac{-Nyzdx^2}{Ldz}$, there will be

$$dy = -\alpha z dx e^{-\int \frac{M}{L} dx} \quad \text{and hence} \quad y dy = \frac{-\alpha\alpha L z dz}{N} e^{-2\int \frac{M dx}{L}}$$

and these relations therefore are more noteworthy, because from these the proposed equation without further difficulties can be reduced according to the transformation. For the formula substituted for *y* leads to a differential equation of the third order, which moreover clearly admits to integration and the equation found here supplies the needs for this. Hence therefore we arrive at the opportunity to investigate substitutions of this kind, which ascend to certain differentials of the third order, yet by integration are allowed to be reduced to differentials of the second order.

PROBLEM 127

1008. *To transform the second order differential equation*

$$Lddy + Mdxdy + Nydx^2 = 0,$$

with the aid of the substitution of this kind $y = \frac{Pdz}{dx}$, *into another equally second order differential equation.*

SOLUTION

On account of $y = \frac{Pdz}{dx}$ there becomes

$$dy = \frac{Pddz + dPdz}{dx} \quad \text{and} \quad ddy = \frac{Pd^3z + 2dPddz + dzddP}{dx},$$

with which formulas substituted this differential equation of the third order arises

$$LPd^3z + 2LdPddz + LdzddP + MPdxddz + MdxdPdz + NPdx^2dz = 0,$$

in order that thus we assume it has been prepared, so that [multiplied by] a function of *x*, which shall be *Q*, it becomes integrable. The integrable equation hence shall have this form

$$LPQd^3z + 2LQdPddz + MPQdxddz + LQdzddP + MQdxdPdz + NPQdx^2dz = 0,$$

the integral of which shall be

$$LPQddz + Sdxdz + Tzdx^2 = Cdx^2,$$

from which there is deduced [Note : the integrations are all by parts: those products involving the integrals of *dddz*, *ddz*, and *dz* becoming simply *ddz*, *dz*, and *z*; the associated term involving *d.LPQ* becomes attached to the term involving *ddz*, etc.]

$$ddz(2LQdP + MPQdx) = ddz(d.LPQ + Sdx),$$
$$dz\left(LQddP + MQdxdP + NPQdx^2\right) = dz\left(dxdS + Tdx^2\right) \text{ and } zdx^2dT = 0$$

and thus T is a constant quantity.

But thence there becomes

$$Sdx = LQdP - LPdQ - PQdL + MPQdx,$$

from which, by the other condition, there is elicited

$$\begin{aligned}Tdx^2 &= LQddP + MQdxdP + NPQdx^2 - LQddP - LdPdQ - QdPdL\\ &+ LPddQ + LdPdQ + PdQdL + PQddL + PdQdL + QdPdL\\ &- MPdxdQ - MQdxdP - PQdxdM\end{aligned}$$

or

$$Tdx^2 = Pdd.LQ - Pdxd.MQ + PNQdx^2.$$

Whereby since T shall be a constant quantity, there is put $T = \alpha$ and hence the function P is defined conveniently, evidently

$$P = \frac{\alpha dx^2}{dd.LQ - dxd.MQ + NQdx^2},$$

and with this value assumed for P the proposed equation with the aid of the substitution $y = \frac{Pdz}{dx}$ is transformed into this :

$$LPQddz + dz(LQdP - LPdQ - PQdL + MPQdx) + \alpha zdx^2 = Cdx^2;$$

where since z is allowed to be increased by a constant quantity, the constant C can be omitted. Hence this equation may be divided by PQ and there will be produced :

$$Lddz + dz\left(\frac{LdP}{P} - \frac{LdQ}{Q} - dL + Mdx\right) + \frac{azdx^2}{PQ} = 0$$

or with the value of P substituted in the last term

$$Lddz + dz\left(\frac{LdP}{P} - \frac{d.LQ}{Q} + Mdx\right) + \frac{z}{Q}\left(dd.LQ - dxd.MQ + NQdx^2\right) = 0$$

and here for Q it is permitted to accept any function of x.

COROLLARY 1

1009. Hence the preceding substitution may be derived [§ 1000] on putting

$$dd.LQ - dxd.MQ = 0 \quad \text{and thus} \quad d.LQ - MQdx = Cdx$$

or

$$e^{-\int \frac{Mdx}{L}} LQ = C\int e^{-\int \frac{Mdx}{L}} dx + D.$$

And if here there is taken $C = 0$, there will be

$$Q = \frac{D}{L} e^{\int \frac{Mdx}{L}} \text{ and } P = \frac{\alpha dx^2}{NQdx^2} = \frac{\alpha}{NQ} \quad \text{or} \quad P = \frac{\alpha L}{N} e^{-\int \frac{Mdx}{L}}$$

as before [§ 1001].

COROLLARY 2

1010. But if we put

$$dd.LQ - dxd.MQ = dXdx,$$

so that there shall be

$$P = \frac{\alpha dx}{dX + NQdx}$$

then there will be

$$d.LQ - MQdx = Xdx + Adx$$

and again on integrating

$$e^{-\int \frac{Mdx}{L}} LQ = \int e^{-\int \frac{Mdx}{L}} dx\left(X + A\right) + B$$

and

$$Q = \frac{1}{L} e^{\int \frac{Mdx}{L}} \int e^{-\int \frac{Mdx}{L}} dx\left(X + A\right) + \frac{B}{L} e^{\int \frac{Mdx}{L}}.$$

COROLLARY 3

1011. There is put $\int e^{-\int \frac{Mdx}{L}} Xdx = e^{-\int \frac{Mdx}{L}} V$ and $A = 0,\ B = 0;$ then there will be

$$X = \frac{dV}{dx} - \frac{MV}{L} \text{ and } Q = \frac{V}{L}$$

and thus

$$P = \frac{\alpha dx}{\frac{ddV}{dx} - \frac{M}{L} dV - Vd.\frac{M}{L} + \frac{N}{L} Vdx}$$

Therefore if there shall be $V = \alpha$, then

$$Q = \frac{\alpha}{L}, \quad P = \frac{LLdx}{LNdx - LdM + MdL}$$

and the resulting equation

$$Lddz + dz\left(\frac{LdP}{P} + MdX\right) + \frac{zdx(LNdx - LdM + MdL)}{L} = 0.$$

SCHOLIUM

1012. Moreover these are exceedingly general, as it will be possible to conclude from any common use whatsoever. But in whatever manner a transformation may be put in place and the transformed equation resolved in series, it is observed that these are not terminated in any cases other than these, in which the proposed equation itself , and from that transformed by the first substitution, is occasionally reduced into a terminating series. From which it is evident that with the help of transformations of this kind hardly any new integrable cases can be elicited.

Truly while at this stage we have introduced another variable z in place of the variable y by substitution with the same *x*, and from the powers of which series were formed, now with that kept in place, also we may investigate a little, how in place of this *x* by introducing another variable *t* it could bring about another transformation, where in the first place it may be noted, since before the element *dx* was assumed to be constant, now in the transformation the element *dt* must be taken as constant. Therefore here we write *t* in place of some certain function of *x*, but which ought to be prepared thus, so that the resulting equation is not made exceedingly complicated.

PROBLEM 128

1013. *With the proposed second order differential equation*

$$Lddy + Mdxdy + Nydx^2 = 0$$

to introduce another quantity t in place of x, which is equal to a certain function of x.

SOLUTION

With the equation divided by *dx* the equation may be shown thus :

$$Ld.\frac{dy}{dx} + Mdy + Nydx = 0,$$

in order that the consideration of the element *dx*, which was considered constant, shall now be excluded. Since *t* is equal to some function of *x*, from that there made $dt = Pdx$ or $dx = \frac{dt}{P}$, from which we obtain

$$Ld.\frac{Pdy}{dt} + Mdy + \frac{Nydt}{P} = 0$$

and assuming the element *dt* constant

$$LPddy + LdPdy + Mdtdy + \frac{Nydt^2}{P} = 0,$$

where it only remains, as with the finite quantities which at this stage include the variable *x*, that in place of this the other variable *t* may be introduced.

EXAMPLE

1014. *Let this equation be proposed*

$$xx\left(a+bx^{n}\right)ddy+x\left(c+ex^{n}\right)dxdy+\left(f+gx^{n}\right)ydx^{2}=0,$$

in which t may be introduced in place of the formula $h+kx^{n}$.

Hence since there shall be $t=h+kx^{n}$, then there will be $dt=nkx^{n-1}$
and thus

$P=nkx^{n-1}$ and $dP=n\left(n-1\right)kx^{n-2}dx=\frac{(n-1)dt}{x}$. [From which $dx=\frac{dt}{knx^{n-1}}$.]

Whereby we may consider

$$nkx^{n+1}\left(a+bx^{n}\right)ddy+\left(n-1\right)xdtdy\left(a+bx^{n}\right)+x\left(c+ex^{n}\right)dtdy+\frac{\left(f+gx^{n}\right)ydt^{2}}{nkx^{n-1}}=0$$

or

$$nk\left(a+bx^{n}\right)ddy+\frac{(n-1)dtdy\left(a+bx^{n}\right)+dtdy\left(c+ex^{n}\right)}{x^{n}}+\frac{\left(f+gx^{n}\right)ydt^{2}}{nkx^{2n}}=0.$$

Now indeed there shall be $x^{n}=\frac{t-h}{k}$, which value substituted gives

$$n\left(ak-bh+bt\right)ddy+\frac{(n-1)dtdy(ak-bh+bt)+dtdy(ck-eh+et)}{t-h}+\frac{(fk-gh+gt)ydt^{2}}{n(t-h)^{2}}=0.$$

Now thus here wherever $t-h$ occurs, in order that a simpler equation emerges, there may be written *u* in place of $t-h$, but then likewise this is the case, if in place of the power x^{n} we may write the quantity *u;* hence therefore here neither is there any gain elicited by the emergence of new series.

COROLLARY

1015. If in the general equation [§ 1013] we wish to write *t* in place of x^{m} , there will be

$$dt=mx^{m-1}dx \text{ and } P=mx^{m-1}$$

and the equation that will result on account of

$$dP=m\left(m-1\right)x^{m}dx=\frac{(m-1)dt}{x}$$

thus will be

$$mLx^{m-1}ddy+\frac{(m-1)Ldtdy}{x}+Mdtdy+\frac{Nydt^{2}}{mx^{m-1}}=0$$

or

$$mLddy+\frac{(m-1)Ldtdy}{t}+\frac{Mxdtdy}{t}+\frac{Nxxydt^2}{mtt}=0.$$

SCHOLIUM

1016. It is observed that it is not necessary to treat more of the transformed equations of this kind, since from these sources all the suitable transformations to be used can be derived without difficulty. But another straightforward method of an unusual kind is given for expressing the integration of second order differential equation, which is brought about by integral formulas involving two variables, while one is held as constant in the integration. Thus if P should be any function of the two variables x and u and there is $y=\int Pdx$, on considering u in the integration as constant, this integral $\int Pdx$ will be a function of x and u; which thus being determined, so that it vanishes on putting $x=0$, if there is then put $x=a$, a function will be obtained of u equal to y ; which if it should satisfy some differential equation proposed between u and y, and this equation can be resolved from the formula $y=\int Pdx$, which can be considered as the integral of this. And in this way innumerable second order differential equations can be shown to be integrable, which by other methods are clearly seen to be intractable.

But whenever the formula $\int Pdx$ regarding the quantity u as constant cannot actually be integrated, yet the integral of this in this business can be taken as known, because the value of this can be assigned at least by an approximation. While evidently on taking x for the abscissa if P denotes the applied line agreeing with that at right angles, the formula $\int Pdx$ expresses the area of the same curve standing on the abscissas x, and on putting $x=a$ the area is considered to be determined up to the value $y=\int Pdx$, just as we have defined it in that way equal, as it is customary to say, to that assigned by the quadrature of the curve, by which this account of integration is conveniently called the construction through quadratures.

But here in the first place it is agreed to consider that account, by which we have distinguished between particular and complete integrals ; from which one must be warned carefully, that the integrals found in this way are not to be considered as complete, unless they involve as many as two arbitrary constants. Therefore since the same differential equations may agree on an infinite number of particular integrals, it is not to be wondered at, if in this way for the same equation proposed we may come upon many different integrals. But this almost straight forward argument is new and has not been treated by anyone hitherto, if indeed several examples are to be excepted, which I have now given formerly [see E44, E45, E52, E70, & E274 in Series I, vol. 20 & 22 *Opera Omnia*.] ; from which without doubt, if this method is developed with care, finally perhaps an outstanding advance in analysis shall be produced.

CAPUT IX

DE TRANSFORMATIONE AEQUATIONUM DIFFERENTIO-DIFFERENTIALIUM

$$Lddy + Mdxdy + Nydx^2 = 0$$

PROBLEMA 125

993. *Aequationem differentio-differentialem*

$$Lddy + Mdxdy + Nydx^2 = 0,$$

in qua L, M, N sunt functiones quaecunque ipsius x, sumto elemento dx constante ope substitutionis $y = e^{\int Pdx}z$ *in aliam formam transmutare.*

SOLUTIO

Cum [§ 940] hinc sit $\frac{dy}{y} = Pdx + \frac{dz}{z}$, erit differentiando

$$\frac{ddy}{y} - \frac{dy^2}{y^2} = dxdP + \frac{ddz}{z} - \frac{dz^2}{zz},$$

ergo

$$\frac{ddy}{y} = \frac{ddz}{z} + \frac{2Pdxdz}{z} + dxdP + PPdx^2.$$

Quare cum aequatio nostra sit

$$\frac{Lddy}{y} + \frac{Mdxdy}{y} + Ndx^2 = 0,$$

erit facta substitutione

$$\frac{Lddz}{z} + \frac{2LPdxdz}{z} + LdxdP + LPPdx^2 + \frac{Mdxdz}{z} + MPdx^2 + Ndx^2 = 0$$

seu per z multiplicando

$$Lddz + (2LP + M)dxdz + zdx(LdP + LPPdx + MPdx + Ndx) = 0,$$

ubi pro P functionem quamcunque ipsius x accipere licet, unde innumerabiles aequationes inter binas variabiles x et z obtinentur.

COROLLARIUM 1

994. Quodsi ergo hanc aequationem transformatam integrare vel per seriem resolvere liceat, ex invento valore ipsius s habebitur $y = e^{\int Pdx}z$.

COROLLARIUM 2

995. Aequatio transformata similis est propositae, propterea quod in ea variabilis *z* cum suis differentialibus *dz* et *ddz* ubique unicam dimensionem occupat perinde ac *y* in aequatione proposita.

COROLLARIUM 3

996. Si eveniat, ut ambae aequationes, proposita ac transformata, aeque commode per series resolvi possint, hoc modo plures resolutiones eiusdem aequationis exhiberi possunt.

SCHOLION 1

997. Cum aequationes commode per series resolubiles [§ 967, 985] in hac forma contineantur

$$xx\left(a+bx^{n}\right)ddy+x\left(c+ex^{n}\right)dxdy+\left(f+gx^{n}\right)ydx^{2}=0,$$

ubi est

$$L=xx\left(a+bx^{n}\right),\ M=x\left(c+ex^{n}\right),\ N=f+gx^{n},$$

ut transformata similem obtineat formam, fieri oportet

$$LP=x\left(\mu+\nu x^{n}\right)\ \text{ ideoque }\ P=\frac{\mu+\nu x^{n}}{x\left(a+bx^{n}\right)}$$

Hinc erit

$$dP=\frac{-\mu a-\mu(n+1)bx^{n}+\nu(n-1)ax^{n}-\nu bx^{2n}}{xx\left(a+bx^{n}\right)^{2}}dx$$

ideoque

$$LdP+LPPdx+MPdx=\frac{\left\{\begin{matrix}-\mu a & -(n+1)\mu bx^{n} & +(n-1)\nu ax^{n}-\nu bx^{2n}\\ +\mu\mu & +2\mu\nu x^{n} & +\nu\nu x^{2n}\\ +\mu c & +\mu ex^{n} & +\nu cx^{n}+\nu ex^{2n}\end{matrix}\right\}}{a+bx^{n}}dx,$$

ubi divisio per $a+bx^{n}$ succedere debet. Statuatur quotus $=\mu h+\nu kx^{n}$ fietque

$$\mu=a-c+ah,\ \nu=b-e+bk$$

ac praeterea

$$2\mu\nu-(n+1)\mu b+(n-1)\nu a+\mu e+\nu c=\mu bh+\nu ak,$$

ubi priores valores substituti praebent

$$(h-k+n)(bc-ae)=nab(h-k)+ab(h-k)^2,$$

unde fit

$$\text{vel } h-k=\tfrac{bc-ae}{ab} \text{ vel } h-k=-n.$$

Litterarum ergo *h* et *k* altera arbitrio nostro relinquitur fitque aequatio transformata

$$xx\left(a+bx^n\right)ddz+x\left(2\mu+c+(2\nu+e)x^n\right)dxdz+\left(f+\mu h+(g+\nu k)x^n\right)zdx^2=0.$$

Huius autem resolutio tam per series ascendentes quam descendentes similes
ipsius *x* postulat potestates. Substitutio autem ipsa fit

$$y=x^{\frac{a-c}{c}+h}\left(a+bx^n\right)^{\frac{bc-ae}{nab}-\frac{h-k}{n}}z,$$

ubi, ne sola potestas ipsius *x* ingrediatur, sumi debet $h-k=-n$. Nihil interest, quomodo hic *h* accipiatur; sumto ergo $h=0$ fit $k=n$ et substitutio

$$y=x^{\frac{a-c}{c}}\left(a+bx^n\right)^{\frac{c-ae}{nab}+1}z,$$

quae ducit ad hanc aequationem

$$xx\left(a+bx^n\right)ddz+x\left(2a-c+\left(2(n+1)b-e\right)x^n\right)dxdz$$
$$+\left(f+\left(n(n+1)b-ne+g\right)x^n\right)zdx^2=0.$$

SCHOLION 2

998. Supra § 970 vidimus aequationem propositam inter *x* et *y* algebraicum admittere integrale, si fuerit

$$\tfrac{c}{2a}-\tfrac{e}{2b}\cdot\cdot\frac{\sqrt{\left((a-c)^2-4af\right)}}{2a}\cdot\cdot\frac{\sqrt{\left((b-e)^2-4bg\right)}}{2b}=in,$$

quae si transformata simili modo tractetur, integrale algebraicum assignari poterit, si fuerit

$$-\frac{c}{2a}+\frac{e}{2b}\cdot\cdot\frac{\sqrt{\left((a-c)^2-4af\right)}}{2a}\cdot\cdot\frac{\sqrt{\left((b-e)^2-4bg\right)}}{2b}-n=in,$$

quibus conditionibus coniunctis concludere licet integrale algebraicum satisfacere, dummodo haec formula

$$\frac{bc-ae}{2ab}\cdot\cdot\frac{\sqrt{\left((a-c)^2-4af\right)}}{2a}\cdot\cdot\frac{\sqrt{\left((b-e)^2-4bg\right)}}{2b}$$

divisibilis extiterit per exponentem *n.* Hic signum .. ad ambiguitatem positivi ac negativi designandam adhibui. Quare si ponamus

$$f=\frac{(a-c)^2-hh}{4a}\quad\text{et}\quad g=\frac{(b-e)^2-kk}{4b}$$

integrabilitas locum habet, quoties haec expressio

$$\frac{bc-ae\pm bh\pm ak}{2nab}$$

fuerit numerus integer sive positivus sive negativus.

EXEMPLUM

999. *Proposita aequatione*

$$xx(1-xx)ddy+x(1+2mxx)dxdy-m(m+1)xxydx^2=0$$

invenire casus, quibus integrale algebraicum saltem particulare assignari potest.

Hic est $a=1,\ b=-1,\ c=1,\ e=2m,\ f=0,\ g=-m(m+1)$ et $n=2$. Hinc deducimus

$$h=\sqrt{\left((a-c)^2-4af\right)}=0$$

et

$$k=\sqrt{\left((b-e)^2-4bg\right)}=\sqrt{\left((2m+1)^2-4m(m+1)\right)},$$

hoc est $k=\pm 1$. Formula ergo numero integro aequalis est $\frac{-1-2m\pm 1}{-4}$, unde geminos pro *m* casus nanciscimur

$$\text{vel}\ \ 2m+2=\pm 4i\ \ \text{vel}\ 2m=\pm 4i,$$

hoc est

$$\text{vel } m = \pm 2i - 1 \text{ vel } m = \pm 2i\,;$$

dummodo ergo *m* sit numerus integer sive positivus sive negativus, integrale particulare algebraicum exhiberi potest. Substitutio autem aequationem transformatam praebens est

$$y = (1-xx)^{\frac{-1-2m}{-2}+1} z = (1-xx)^{\frac{2m+3}{2}} z\,,$$

ipsa vero aequatio transformata

$$xx(1-xx)ddz + x(1-2(m+3)xx)dxdz - (m+2)(m+3)xxzdx^2 = 0\,,$$

quam ex illa oriri manifestum est, si loco *m* scribatur $-m-3$. Ipsa autem haec integralia reperiuntur ob $\lambda\lambda = 0$ ponendo

$$y = A + Bx^2 + \mathrm{C}x^4 + Dx^6 + Ex^8 + \text{etc.},$$

unde fit

$$\left.\begin{array}{lllllllll} 2Bxx & + & 12Cx^4 & + & 30Dx^6 & + & 56Ex^8 + \text{etc.} \\ & - & 2B & - & 12C & - & 30D \\ +2B & + & 4C & + & 6D & + & 8E \\ & + & 4mB & + & 8mC & + & 12mD \\ -m(m+1)A & & -m(m+1)B & & -m(m+1)C & & -m(m+1)D \end{array}\right\} = 0.$$

Ergo determinatio coefficientium ita se habet

$$B = \frac{m(m+1)}{4}A,\ \ C = \frac{(m-1)(m-2)}{16}B,\ \ D = \frac{(m-3)(m-4)}{36}C \text{ etc.}$$

Ac si ponatur

$$z = \mathfrak{A} + \mathfrak{B}x^2 + \mathfrak{C}x^4 + \mathfrak{D}x^6 + \mathfrak{E}x^8 + \text{etc.}$$

erit

$$\mathfrak{B} = \frac{(m+2)(m+3)}{4}\mathfrak{A},\ \ \mathfrak{C} = \frac{(m+4)(m+5)}{16}\mathfrak{B},\ \ \mathfrak{D} = \frac{(m+6)(m+7)}{36}\mathfrak{C} \text{ etc.}$$

PROBLEMA 126

1000. *Aequationem differentio-differentialem*

$$Lddy + Mdxdy + Nydx^2 = 0$$

ope substitutionis $\frac{dy}{y} = \frac{Pzdx^2}{dz}$ *in aliam eiusdem formae transmutare.*

SOLUTIO

Hic scilicet quaeritur, qualem functionem ipsius x pro P accipi oporteat, ut facta substitutione variabilis z cum suis differentialibus dz et ddz, ubique unicam dimensionem obtineat. Cum igitur sit $\frac{dy}{y} = \frac{Pzdx^2}{dz}$, erit differentiando

$$\frac{ddy}{y} - \frac{dy^2}{y^2} = \frac{-Pzdx^2ddz}{dz^2} + \frac{zdx^2dP}{dz} + Pdx^2$$

et

$$\frac{ddy}{y} = \frac{-Pzdx^2ddz}{dz^2} + \frac{zdx^2dP}{dz} + Pdx^2 + \frac{PPzzdx^4}{dz^2},$$

quibus valoribus substitutis fit

$$\frac{-LPzdx^2ddz}{dz^2} + \frac{Lzdx^2dP}{dz} + LPdx^2 + \frac{LPPzzdx^4}{dz^2} + \frac{MPzdx^3}{dz} + Ndx^2 = 0.$$

Sumamus ergo $LP + N = 0$ seu $P = \frac{-N}{L}$ et multiplicando per $\frac{-dz^2}{Pzdx^2}$ nanciscemur

$$Lddz - \frac{LdPdz}{P} - LPzdx^2 - Mdxdz = 0$$

seu

$$Lddz - Mdxdz - \frac{LdNdz}{N} + dLdz + Nzdx^2 = 0.$$

Aequatio ergo proposita ope substitutionis $\frac{dy}{y} = \frac{-Nzdx^2}{Ldz}$ transformatur in hanc

$$Lddz + \left(\frac{dL}{dx} - M - \frac{LdN}{Ndx}\right)dxdz + Nzdx^2 = 0.$$

Quodsi ergo hinc valor ipsius z erui possit, habebitur etiam valor ipsius y per x expressus.

COROLLARIUM 1

1001. Si in hac aequatione transformata vicissim ponatur $\frac{dz}{z} = \frac{-Nydx^2}{Ldy}$, ipsa aequatio proposita exoritur; unde hae duae aequationes ita inter se cohaerent, ut altera ex altera per similem substitutionenl producatur.

COROLLARIUM 2

1002. Si in aequatione transformata secundum substitutionem priorem [§ 993] ponatur $\frac{dz}{z} = Qdx + \frac{dv}{v}$, obtinebitur haec nova transformata

$$Lddv+\left(2LQ+\tfrac{dL}{dx}-M-\tfrac{LdN}{Ndx}\right)dxdv$$
$$+vdx\left(LdQ+LQQdx+QdL-MQdx-\tfrac{LQdN}{N}+Ndx\right)=0,$$

quae ergo ex ipsa proposita deducitur ponendo $\frac{dy}{y}=\frac{-Nvdx^2}{L(dv+Qvdx)}$.

SCHOLION 1

1003. Hinc combinando ambas substitutiones, quibus in binis praecedentibus problematibus sumus usi, substitutionem huiusmodi generalem adipiscimur

$$\frac{dy}{y}=\frac{Pdz+Qzdx}{Rdz+Szdx}dx,$$

quae si in aequatione proposita

$$Lddy+Mdxdy+Nydx^2=0$$

substituatur, functiones *P*, *Q*, *R*, *S* ita definiri debent, ut in aequatione resultante variabilis *z* cum suis differentialibus nusquam plus una dimensione teneat. Oriuntur autem termini quadrato dz^2 affecti, ad quos destruendos fieri oportet

$$Ldx(PP+QR-PS)+L(RdP-PdR)+MPRdx+NRRdx=0$$

seu

$$Q=\frac{PS}{R}-\frac{PP}{R}-\frac{dP}{dx}+\frac{PdR}{Rdx}-\frac{MP}{L}-\frac{NR}{L};$$

tum vero pervenitur ad hanc aequationem

$$Lddz(PS-QR)+Ldz(RdQ-QdR+SdP-PdS)$$
$$+Lzdx(SdQ-QdS+QQdx)+dxdz(2LPQ+M(QR+PS)+2NRS)$$
$$+Szdx^2(MQ+NS)=0.$$

Verum facilius ad hanc aequationem generalem pervenitur, si ambae substitutiones alternatim in usum vocentur.

SCHOLION 2

1004. Transformatio autem hic exposita eo magis est notatu digna, quod, etiamsi aequatio transformata resolutionem admittat, inde tamen nonnisi difficulter ipsa proposita resolvatur. Cum enim reperta fuerit functio ipsius *x*, quae loco *z* substituta aequationi transformatae satisfaciat, pro valore ipsius *y* inveniendo insuper integrale huius aequationis $\frac{dy}{y} = \frac{-Nzdx^2}{Ldz}$ investigari oportet, ubi, etsi variabiles x et y a se invicem sunt separatae, tamen difficultates insignes in ipsa integratione se exerere possunt. Fieri ergo poterit, ut ope huius substitutionis eiusmodi aequationum integralia exhiberi queant, quae directa via vix investigare liceat. Scilicet si eveniat, ut integrale aequationis transformatae vel ope methodi cuiusdam supra expositae inveniri vel per seriem abruptam exprimi possit, tum etiam ipsius aequationis propositae integrale habebitur. Etsi enim casu posteriori integrale tantum particulare innotescit, tamen ex eo semper in hoc aequationum genere integrale completum elici potest. Namque si aequationi

$$Lddy + Mdxdy + Nydx^2 = 0$$

particulariter satisfaciat valor $y = X$, ponatur $y = Xv$ fietque

$$\left.\begin{aligned} LXddv + 2LdXdv &+ LvddX \\ + MXdxdv &+ MvdxdX \\ &+ NXvdx^2 \end{aligned}\right\} = 0.$$

At quia $X = y$ per hypothesin aequationi satisfacit, erit

$$LddX + MdxdX + NXdx^2 = 0 \text{ et } LXddv + (2LdX + MXdx)dv = 0$$

seu

$$\frac{ddv}{dv} + \frac{2dX}{X} + \frac{Mdx}{L} = 0,$$

unde integrando oritur

$$XXdv = Ce^{-\int \frac{Mdx}{L}} dx$$

porroque

$$v = \int \frac{Cdx}{XX} e^{-\int \frac{Mdx}{L}},$$

ita ut integrale completum sit

$$y = CX \int \frac{dx}{XX} e^{-\int \frac{Mdx}{L}},$$

quod ergo ex quolibet integrali particulari $y = X$ elici potest.

EXEMPLUM

1005. *Aequationem differentio-differentialem*

$$xx(a + bx^n)ddy + x(c + ex^n)dxdy + fydx^2 = 0$$

transformare ac per seriem integrare.

Cum hic sit $L = xx(a + bx^n)$, $M = x(c + ex^n)$ et $N = f$, utendum est hac substitutione

$$\frac{dy}{y}=\frac{-fzdx^2}{xx\left(a+bx^n\right)dz},$$

qua nostra aequatio reducitur ad hanc formam

$$xx\left(a+bx^n\right)ddz+x\left(2a-c+\left((n+2)b-e\right)x^n\right)dxdz+fzdx^2=0;$$

pro cuius resolutione si ponatur

$$z=Ax^{\lambda}+Bx^{\lambda+n}+Cx^{\lambda+2n}+\text{ etc.},$$

fieri debet

$$\lambda(\lambda-1)a+\lambda(2a-c)+f=0 \text{ seu } \lambda\lambda a+\lambda(a-c)+f=0,$$

ergo

$$\lambda=\frac{-a+c\pm\sqrt{\left((a-c)^2-4af\right)}}{2a}$$

Series autem abrumpetur per § 970, si haec expressio

$$-\frac{c}{2a}+\frac{e}{2b}-\frac{n}{2}\pm\left(\frac{e}{2b}-\frac{n}{2}-\frac{1}{2}\right)\pm\frac{\sqrt{\left((a-c)^2-4af\right)}}{2a}=in$$

denotante *i* numerum integrum positivum, hoc est vel

$$-\frac{c}{2a}+\frac{e}{b}-\frac{1}{2}-n\pm\frac{\sqrt{\left((a-c)^2-4af\right)}}{2a}=in$$

vel

$$-\frac{c}{2a}+\frac{1}{2}\pm\frac{\sqrt{\left((a-c)^2-4af\right)}}{2a}=in$$

Sin autem ipsa aequatio proposita hoc modo in seriem resolvatur, haec abrumpetur, si fuerit

$$\frac{c}{2a}-\frac{e}{2b}\pm\frac{b-e}{2b}\pm\frac{\sqrt{\left((a-c)^2-4af\right)}}{2a}=in$$

hoc est vel

$$\frac{c}{2a}-\frac{e}{b}+\frac{1}{2}\pm\frac{\sqrt{\left((a-c)^2-4af\right)}}{2a}=in$$

vel

$$\frac{c}{2a}-\frac{1}{2}\pm\frac{\sqrt{\left((a-c)^2-4af\right)}}{2a}=in$$

Unde intelligitur integrale finitum exhiberi posse, sive numerus integer i sit positivus sive negativus. Ad hanc vero duplicitatem iam prior substitutio perduxerat (§ 998), ita ut haec nova substitutio nullos novos casus integrabiles suppeditet.

SCHOLION 1

1006. Ut tamen pateat, quomodo ex valore finito ipsius z valor finitus ipsius y elici queat, contemplemur casum

$$xx\left(a+bx^2\right)ddy+x\left(3a+ex^2\right)dxdy-24aydx^2=0,$$

ubi $n=2$, $c=3a$ et $f=-24a$, quae facta substitutione

$$\frac{dy}{y}=\frac{24azdx^2}{xx\left(a+bx^2\right)dz}$$

abit in hanc

$$xx\left(a+bx^2\right)ddz+x\left(-a+(4b-e)xx\right)dxdz-24azdx^2=0,$$

ubi pro serie ascendente fit

$$\lambda\lambda-2\lambda-24=0=(\lambda-6)(\lambda+4).$$

Statuatur

$$y=Ax^{-4}+Bx^{-2}+\mathrm{C}+Dx^2+\text{etc.};$$

erit

$$\left.\begin{array}{lllll}
ax^{-4}+ & 6Bax^{-2} & * & + & 2Dax^2+\text{ etc.}\\
 & +\ 20Ab & +\ 6Bb & & *\\
20A+\ 4Aa & +\ 2Ba & * & - & 2Da\\
 & -\ 4A(4b-e) & -\ 2B(4b-e) & & *\\
-\ 24Aa & -\ 24Ba & -\ 24Ca & - & 24Da
\end{array}\right\}=0.$$

Cum ergo sit $D=0$, sequentes termini omnes tolluntur. Tum vero est

$$16Ba=4A(b+e),\quad 24Ca=-2Bb+2Be,$$

ergo

$$B=\frac{b+e}{4a}A,\quad C=\frac{e-b}{12a}B=\frac{ee-bb}{48aa}A$$

hincque

$$z = A\left(\frac{1}{x^4} + \frac{b+e}{4ax^2} + \frac{ee-bb}{48aa}\right) = \frac{A\left(48aa+12a(b+e)xx+(ee-bb)x^4\right)}{48aax^4},$$

unde sequitur

$$dz = Adx\left(\frac{-4}{x^5} - \frac{b+e}{2ax^3}\right) = \frac{-Adx}{2ax^5}\left(8a+(b+e)xx\right).$$

Ergo

$$\frac{dy}{y} = \frac{-\left(48aa+12a(b+e)xx+(ee-bb)x^4\right)}{x\left(a+bx^2\right)\left(8a+(b+e)xx\right)}dx$$

seu resolvendo

$$\frac{dy}{y} = \frac{-6dx}{x} + \frac{(5b-e)xdx}{a+bxx} + \frac{2(b+e)xdx}{8a+(b+e)xx}$$

hincque integrando

$$y = \frac{A}{x^6}\left(a+bxx\right)^{\frac{5b-e}{2b}}\left(8a+(b+e)xx\right).$$

SCHOLION 2

1007. Quod hic casu fortuito evenisse videtur, ut ex valore ipsius *z* invento quantitas *y* commode definiri potuerit, idem perpetuo evenire oportere sequenti modo in genere ostendi potest. Cum enim aequatio proposita

$$Lddy + Mdxdy + Nydx^2 = 0$$

ope substitutionis $\frac{dy}{y} = \frac{-Nzdx^2}{Ldz}$ in hanc sit transformata

$$Lddz - Mdxdz - \frac{LdNdz}{N} + dLdz + Nzdx^2 = 0,$$

si haec per *Ldz* dividatur, prodit

$$\frac{ddz}{dz} - \frac{Mdx}{L} - \frac{dN}{N} + dL = -\frac{Nzdx^2}{Ldz} = \frac{dy}{y},$$

ex qua integrando elicitur

$$y = \frac{\alpha Ldz}{Ndx}e^{-\int \frac{M}{L}dx},$$

quae invento valore ipsius *z* statim sine ulteriori integratione praebet valorem ipsius *y*.

Cum porro sit $dy = \frac{-Nyzdx^2}{Ldz}$, erit

$$dy = -\alpha z dx e^{-\int \frac{M}{L} dx} \quad \text{hincque} \quad y dy = \frac{-\alpha\alpha L z dz}{N} e^{-2\int \frac{M dx}{L}}$$

atque hae relationes eo magis sunt notatu dignae, quod ex iis aequatio proposita nonnisi per plures ambages ad transformatam reduci possit. Ipsa enim formula pro y substituta perducit ad aequationem differentialem tertii gradus, quae autem manifesto integrationem admittit ipsamque aequationem hic inventam suppeditat. Hinc igitur occasionem adipiscimur eiusmodi substitutiones investigandi, quae quidem ad differentialia tertii gradus ascendant, verum tamen per integrationem ad differentialia secunda redigi se patiantur.

PROBLEMA 127

1008 *Aequationem differentio-differentialem*

$$Lddy + Mdxdy + Nydx^2 = 0$$

ope huiusmodi substitutionis $y = \frac{Pdz}{dx}$ *in aliam aequationem pariter differentio-differentialem transformare.*

SOLUTIO

Ob $y = \frac{Pdz}{dx}$ fit

$$dy = \frac{Pddz + dPdz}{dx} \quad \text{et} \quad ddy = \frac{Pd^3z + 2dPddz + dzddP}{dx},$$

quibus formulis substitutis oritur haec aequatio differentialis tertii gradus

$$LPd^3z + 2LdPddz + LdzddP + MPdxddz + MdxdPdz + NPdx^2dz = 0,$$

quam ita comparatam assumamus, ut per functionem ipsius x, quae sit Q, integrabilis evadat. Integrabilis ergo sit haec forma

$$LPQd^3z + 2LQdPddz + MPQdxddz + LQdzddP + MQdxdPdz + NPQdx^2dz = 0,$$

cuius integrale sit

$$LPQddz + Sdxdz + Tzdx^2 = Cdx^2,$$

unde colligitur

$$ddz\left(2LQdP + MPQdx\right) = ddz\left(d.LPQ + Sdx\right),$$

$$dz\left(LQddP + MQdxdP + NPQdx^2\right) = dz\left(dxdS + Tdx^2\right) \text{ et } zdx^2dT = 0$$

ideoque T quantitas constans.
Inde autem fit

$$Sdx = LQdP - LPdQ - PQdL + MPQdx,$$

ex quo per alteram conditionem elicitur

$$\begin{aligned}Tdx^2 &= LQddP + MQdxdP + NPQdx^2 - LQddP - LdPdQ - QdPdL\\ &+ LPddQ + LdPdQ + PdQdL + PQddL + PdQdL + QdPdL\\ &- MPdxdQ - MQdxdP - PQdxdM\end{aligned}$$

sive

$$Tdx^2 = Pdd.LQ - Pdxd.MQ + PNQdx^2.$$

Quare cum T sit quantitas constans, ponatur $T = \alpha$ atque hinc commode functio P definitur, scilicet

$$P = \frac{\alpha dx^2}{dd.LQ - dxd.MQ + NQdx^2},$$

hocque valore pro P assumto aequatio proposita ope substitutionis $y = \frac{Pdz}{dx}$ transformatur in hanc

$$LPQddz + dz\left(LQdP - LPdQ - PQdL + MPQdx\right) + \alpha zdx^2 = Cdx^2;$$

ubi cum z constante quantitate augere liceat, constans C omitti potest. Dividatur ergo haec aequatio per PQ et prodibit

$$Lddz + dz\left(\frac{LdP}{P} - \frac{LdQ}{Q} - dL + Mdx\right) + \frac{\alpha zdx^2}{PQ} = 0$$

seu in postremo termino valorem ipsius P substituendo

$$Lddz + dz\left(\frac{LdP}{P} - \frac{d.LQ}{Q} + Mdx\right) + \frac{z}{Q}\left(dd.LQ - dxd.MQ + NQdx^2\right) = 0$$

atque hic pro Q functionem quamcunque ipsius x accipere licet.

COROLLARIUM 1

1009. Hinc praecedens [§ 1000] substitutio derivatur ponendo

$$dd.LQ - dxd.MQ = 0 \quad \text{ideoque} \quad d.LQ - MQdx = Cdx$$

seu

$$e^{-\int \frac{Mdx}{L}} LQ = C\int e^{-\int \frac{Mdx}{L}} dx + D.$$

Namque si hic capiatur $C=0$, erit

$$Q=\frac{D}{L}e^{\int\frac{Mdx}{L}} \text{ et } P=\frac{\alpha dx^2}{NQdx^2}=\frac{\alpha}{NQ} \quad \text{seu} \quad P=\frac{\alpha L}{N}e^{-\int\frac{Mdx}{L}}$$

ut ante [§ 1001].

COROLLARIUM 2

1010. Sin autem ponamus

$$dd.LQ-dxd.MQ=dXdx,$$

ut sit

$$P=\frac{\alpha dx}{dX+NQdx}$$

erit

$$d.LQ-MQdx=Xdx+Adx$$

porroque integrando

$$e^{-\int\frac{Mdx}{L}}LQ=\int e^{-\int\frac{Mdx}{L}}dx\left(X+A\right)+B$$

et

$$Q=\frac{1}{L}e^{\int\frac{Mdx}{L}}\int e^{-\int\frac{Mdx}{L}}dx\left(X+A\right)+\frac{B}{L}e^{\int\frac{Mdx}{L}}.$$

COROLLARIUM 3

1011. Ponatur $\int e^{-\int\frac{Mdx}{L}}Xdx=e^{-\int\frac{Mdx}{L}}V$ and $A=0,\ B=0;$ erit

$$X=\frac{dV}{dx}-\frac{MV}{L} \text{ et } Q=\frac{V}{L}$$

ideoque

$$P=\frac{\alpha dx}{\frac{ddV}{dx}-\frac{M}{L}dV-Vd.\frac{M}{L}+\frac{N}{L}Vdx}$$

Si igitur sit $V=\alpha$, erit

$$Q=\frac{\alpha}{L},\quad P=\frac{LLdx}{LNdx-LdM+MdL}$$

aequatioque resultans

$$Lddz+dz\left(\frac{LdP}{P}+MdX\right)+\frac{zdx\left(LNdx-LdM+MdL\right)}{L}=0.$$

SCHOLION

1012. Haec autem nimis sunt generalia, quam ut inde quicquam ad usum communem concludi possit. Utcunque autem transformatio instituatur et aequatio transformata in seriem resolvatur, haec nullis aliis casibus abrumpi videtur nisi iis, quibus ipsa aequatio proposita et inde per primam substitutionem transformata ad seriem alicubi terminatam reducitur. Ex quo perspicuum est ope huiusmodi transformationum vix unquam novos casus integrabiles erui posse.

Verum dum hactenus loco variabilis y aliam z per substitutionem introduximus altera x, ex cuius potestatibus series formabantur, retenta, nunc etiam paucis exploremus, quomodo loco ipsius x aliam variabilem t introducendo transformationem perfici oporteat, ubi imprimis notetur necesse est, cum ante elementum dx assumtum fuerit constans, iam in transformata elementum dt constans accipi debere. Hic igitur t scribetur loco certae cuiuspiam functionis ipsius x, quam autem ita comparatam esse decet, ut aequatio resultans ne nimis fiat complicata.

PROBLEMA 128

1013. *Proposita aequatione differentio-differentiali*

$$Lddy + Mdxdy + Nydx^2 = 0$$

loco quantitatis x aliam t introducere, quae functioni cuipiam ipsius x aequetur

SOLUTIO

Divisa aequatione per dx repraesentetur aequatio ita

$$Ld.\frac{dy}{dx} + Mdy + Nydx = 0,$$

ut iam consideratio elementi dx, quod constans erat assumtum, sit exclusa.

Cum t aequetur functioni cuipiam ipsius x, fiat inde $dt = Pdx$ seu $dx = \frac{dt}{P}$,

unde nanciscimur

$$Ld.\frac{Pdy}{dt} + Mdy + \frac{Nydt}{P} = 0$$

ac sumto elemento dt constante

$$LPddy + LdPdy + Mdtdy + \frac{Nydt^2}{P} = 0,$$

ubi tantum superest, ut in quantitatibus finitis, quae adhuc variabilem x complectuntnr, eius loco altera t introducatur.

EXEMPLUM

1014. *Proposita sit haec aequatio*

$$xx\left(a + bx^n\right)ddy + x\left(c + ex^n\right)dxdy + \left(f + gx^n\right)ydx^2 = 0,$$

in quam loco formulae $h + kx^n$ *introducatur t.*

Cum ergo sit $t = h + kx^n$, erit $dt = nkx^{n-1}$
ideoque
$$P = nkx^{n-1} \text{ et } dP = n(n-1)kx^{n-2}dx = \frac{(n-1)dt}{x}.$$
Quare habebimus
$$nkx^{n+1}\left(a+bx^n\right)ddy + (n-1)xdtdy\left(a+bx^n\right) + x\left(c+ex^n\right)dtdy + \frac{\left(f+gx^n\right)ydt^2}{nkx^{n-1}} = 0$$
sive
$$nk\left(a+bx^n\right)ddy + \frac{(n-1)dtdy\left(a+bx^n\right)+dtdy\left(c+ex^n\right)}{x^n} + \frac{\left(f+gx^n\right)ydt^2}{nkx^{2n}} = 0.$$

Nunc vero est $x^n = \frac{t-h}{k}$, qui valor substitutus praebet
$$n(ak-bh+bt)ddy + \frac{(n-1)dtdy(ak-bh+bt)+dtdy(ck-eh+et)}{t-h} + \frac{(fk-gh+gt)ydt^2}{n(t-h)^2} = 0.$$

Verum hic ita ubique $t-h$ occurrit, ut aequatio simplicior evadat loco $t-h$ scribendo *u*, tum autem perinde est, ac si loco potestatis x^n scripsissemus quantitatem *u;* neque ergo hinc quicquam lucri pro novis seriebus eruendis redundat.

COROLLARIUM

1015. Si in aequatione generali [§ 1013] loco x^m scribere velimus *t*, erit
$$dt = mx^{m-1}dx \text{ et } P = mx^{m-1}$$
et aequatio resultabit ob
$$dP = m(m-1)x^m dx = \frac{(m-1)dt}{x}$$
ista
$$mLx^{m-1}ddy + \frac{(m-1)Ldtdy}{x} + Mdtdy + \frac{Nydt^2}{mx^{m-1}} = 0$$
seu
$$mLddy + \frac{(m-1)Ldtdy}{t} + \frac{Mxdtdy}{t} + \frac{Nxxydt^2}{mtt} = 0\,.$$

SCHOLION

1016. Plura de huiusmodi aequationum transformationibus tradere haud necesse videtur, cum ex his fontibus haud difficulter omnes transformationes ad usum idoneae derivari queant. Datur autem alia methodus prorsus singularis huiusmodi aequationum differentio-differentialium integralia exprimendi, quae per formulas integrales binas variabiles involventes expeditur, dum altera in integratione ut constans tractatur. Ita si P fuerit functio quaecunque binarum variabilium x et u ac ponatur $y=\int Pdx$ considerando u in integratione ut constantem, integrale hoc $\int Pdx$ erit functio ipsarum x et u; quod ita determinatum, ut evanescat posito $x=0$, si deinceps statuatur $x=a$, obtinebitur functio ipsius u ipsi y aequalis; quae si satisfaciat aequationi cuipiam differentiali inter u et y propositae, haec aequatio resolvetur formula $y=\int Pdx$, quae ut eius integrale spectari poterit. Atque hoc modo innumerabilium aequationum differentio-differentialium integralia exhiberi possunt, quae aliis methodis prorsus intractabiles videntur.

Quanquam autem formula $\int Pdx$ spectata quantitate u ut constante actu integrari nequit, tamen eius integrale in hoc negotio pro cognito accipi potest, quia eius valor saltem per approximationes assignari potest. Scilicet dum sumta x pro abscissa si P denotet applicatam orthogonalem ei convenientem, formula $\int Pdx$ exprimet aream eiusdem curvae abscissae x insistentem ac posito $x=a$ area habetur determinata valori $y\,y=\int Pdx$, prouti eum modo definivimus, aequalis, quae ergo, uti loqui solent, per quadraturas curvarum assignari potest, ex quo haec integrandi ratio commode appellatur constructio per quadraturas.

Hic autem imprimis ad eam rationem, qua integralia in particularia et completa distinximus, attendi conveniet; unde sollicite est cavendum, ne integralia hoc modo inventa pro completis habeantur, nisi quatenus binas constantes arbitrarias involvant. Cum igitur eidem aequationi differentiali infinita integralia particularia conveniant, mirandum non est, si hoc modo pro eadem aequatione proposita plura integralia diversa inveniamus. Hoc autem argumentum fere prorsus est novum neque a quoquam adhuc pertractatum, siquidem nonnulla specimina, quae equidem iam dudum dedi, excipiantur; ex quo dubitare non licet, quin ista methodus, si diligentius excolatur, aliquando forte praeclara incrementa in Analysin sit allatura.