CHAPTER II

CONCERNING THE RESOLUTION OF EQUATIONS OF THIS FORM

$$Ay+\frac{Bdy}{dx}+\frac{Cddy}{dx^2}+\frac{Dd^3y}{dx^3}+\frac{Ed^4y}{dx^4}+\text{etc.}=0$$

ON ASSUMING THE ELEMENT *dx* CONSTANT

PROBLEM 144

1117. *To find the complete integral of this third order differential equation*

$$Ay+\frac{Bdy}{dx}+\frac{Cddy}{dx^2}+\frac{Dd^3y}{dx^3}=0$$

on assuming the element dx constant.

SOLUTION

Since *A, B, C, D* shall be constant quantities, it is apparent from the most trivial attention that the equation of the form $y=e^{\lambda x}$ satisfies this equation ; since indeed from this there shall be

$$\frac{dy}{dx}=\lambda e^{\lambda x},\quad \frac{ddy}{dx^2}=\lambda^2 e^{\lambda x},\quad \frac{d^3y}{dx^3}=\lambda^3 e^{\lambda x},$$

with these in place and the equation divided by $e^{\lambda x}$ there becomes

$$A+\lambda B+\lambda^2 C+\lambda^3 D=0,$$

from which the exponent λ is determined; which since three values may be chosen, which shall be α,β,γ, we will have three satisfying formulas

$$y=e^{\alpha x},\quad y=e^{\beta x},\quad y=e^{\gamma x}.$$

Now it is evident from the nature of the proposed equation, if the values $y=P,\quad y=Q,\quad y=R$ satisfy the equation, then $y=\mathfrak{A}P+\mathfrak{B}Q+\mathfrak{C}R$ will also satisfies the equation, with these taken together in some manner. Whereby from the three formulas found we arrive at this wider equally satisfying extended form :

$$y = \mathfrak{A}e^{\alpha x} + \mathfrak{B}e^{\beta x} + \mathfrak{C}e^{\gamma x};$$

which form, since it includes three arbitrary constants $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}$, actually will be the complete integral of our proposed equation.

COROLLARY 1

1118. Therefore the complete integral contains just as many parts, as the equation has roots or factors

$$A + \lambda B + \lambda^2 C + \lambda^3 D = 0;$$

if a factor of this equation were $\alpha + \lambda$, then the part of the integral will be $e^{-\alpha x}$.

COROLLARIUM 2

1119. Evidently this part will be the integral of this equation $ay + \frac{dy}{dx} = 0$.
from which if there shall be

$$A + B\lambda + C\lambda^2 + D\lambda^3 = (a+\lambda)(b+\lambda)(c+\lambda),$$

the values of *y* may be sought from these simple equations

$$ay + \frac{dy}{dx} = 0, \quad by + \frac{dy}{dx} = 0, \quad cy + \frac{dy}{dx} = 0;$$

which if they shall be $y = P, y = Q, y = R$, then the integral of the proposed equation will be

$$y = \mathfrak{A}P + \mathfrak{B}Q + \mathfrak{C}R.$$

COROLLARY 3

1120. If two roots shall be equal, for example $\beta = \alpha$, the difference $\beta = \alpha + \omega$ may be considered as vanishing, and since there shall be $e^{\beta x} = e^{\alpha x} \cdot e^{\omega x} = e^{\alpha x}(1 + \omega x)$, clearly in place of $\mathfrak{A}e^{\alpha x} + \mathfrak{B}e^{\beta x}$ there must be written

$$\mathfrak{A}e^{\alpha x} + \mathfrak{B}e^{\beta x}x = e^{\alpha x}(\mathfrak{A} + \mathfrak{B}x);$$

[Perhaps here Euler had in mind making
$\omega \to 0$ and $\mathfrak{B} \to \infty$ in such a manner that $\omega\mathfrak{B} = \mathfrak{B}'$, which he wrote as $\mathfrak{B}$.]
and if all three roots were equal $\alpha = \beta = \gamma$, so that the equation shall be

$$y + \frac{3dy}{adx} + \frac{3ddy}{a^2dx^2} + \frac{d^3y}{a^3dx^3} = 0$$

on account of $\alpha=\beta=\gamma=-a$ the complete integral will be

$$e^{-ax}\left(\mathfrak{A}+\mathfrak{B}x+\mathfrak{C}x^2\right).$$

COROLLARY 4

1121. If two roots are made imaginary, so that there shall be $\alpha=\mu+\nu\sqrt{-1}$ and $\beta=\mu-\nu\sqrt{-1}$, in place of $\mathfrak{A}e^{\alpha x}+\mathfrak{B}e^{\beta x}$ there must be written

$$e^{\mu x}\left(\mathfrak{A}e^{\nu x\sqrt{-1}}+\mathfrak{B}e^{-\nu x\sqrt{-1}}\right),$$

which is reduced to this form

$$e^{\mu x}\left(\mathfrak{A}\cos.\nu x+\mathfrak{B}\sin.\nu x\right).$$

SCHOLIUM 1

1122. Though the proposed equation demands a threefold integration, before the finished relation between x and y may be obtained, yet here we have reached that by a single operation which is not related to integration. Clearly by conjecture, we have assembled a particularly satisfying form of the equation together with three forms of this kind which we have taken to follow. From the nature of that equation we have understood, if the singular values $y=P,\ y=Q,\ y=R$ satisfy the equation, then also the form composed from these $y=\mathfrak{A}P+\mathfrak{B}Q+\mathfrak{C}R$ must satisfy the equation, because unless it should happen conveniently, from these three values nothing further can be concluded. Therefore from the same principle in general differential equations of this kind, whatever the orders should be, thus as if by a single action, are able to be resolved, so that the complete integral may thus be assigned.

SCHOLIUM 2

1123. Because the differential equation of the third order

$$Ay+B\frac{dy}{dx}+C\frac{ddy}{dx^2}+D\frac{d^3y}{dx^3}=0$$

in general is allowed to be resolved, to that the complete integral shall be

$$y=\mathfrak{A}e^{\alpha x}+\mathfrak{B}e^{\beta x}+\mathfrak{C}e^{\gamma x}$$

with the roots $\alpha,\ \beta,\ \gamma$ of this cubic equation arising

$$A+B\lambda+C\lambda^2+D\lambda^3=0,$$

hence the use is not to be disregarded for other equations, in which we will perceive that is allowed to be transformed.

But in the first place that equation is allowed to be recalled as a differential equation of the second order with the aid of the substitution $y = e^{\int udx}$; from which there arises

$$\frac{dy}{dx} = e^{\int udx}u, \quad \frac{ddy}{dx^2} = e^{\int udx}\left(\frac{du}{dx} + uu\right) \quad \text{and} \quad \frac{d^3y}{dx^3} = e^{\int udx}\left(\frac{ddu}{dx^2} + \frac{3udu}{dx} + u^3\right),$$

thus so that the transformed equation on division by $e^{\int udx}$ becomes

$$A + Bu + Cuu + Du^3 + C\frac{du}{dx} + \frac{3Dudu}{dx} + \frac{Dddu}{dx^2} = 0,$$

of which therefore on account of $u = \frac{dy}{ydx}$, the integral is

$$u = \frac{\alpha\mathfrak{A}e^{\alpha x} + \beta\mathfrak{B}e^{\beta x} + \gamma\mathfrak{C}e^{\gamma x}}{\mathfrak{A}e^{\alpha x} + \mathfrak{B}e^{\beta x} + \mathfrak{C}e^{\gamma x}}.$$

But this equation is reduced further to the first order on putting $dx = \frac{du}{t}$;
since indeed the element dx shall be taken constant, there will be $tddu - dtdu = 0$ or $ddu = \frac{dtdu}{t}$
from which there arises $\frac{du}{dx} = t$ and $\frac{ddu}{dx^2} = \frac{tdt}{du}$, thus so that this differential equation of the first order appears

$$A + Bu + Cuu + Du^3 + t\left(C + 3Du\right) + \frac{Dtdt}{du} = 0,$$

the resolution of this also therefore is in our power ; and clearly the value of each of the variables u and t can be expressed by the same variable x. For since y is given by x, in the first place there shall be $u = \frac{dy}{ydx}$, then indeed [as $t = \frac{du}{dx} = \frac{d}{dx}\left(\frac{dy}{ydx}\right) = \frac{ddy}{ydx^2} - \frac{dy^2}{y^2dx^2}$]
$t + uu = \frac{ddy}{ydx^2}$ on account of $\frac{du}{dx} = t$. Therefore with the above value found substituted in place of y there will be

$$u = \frac{\alpha\mathfrak{A}e^{\alpha x} + \beta\mathfrak{B}e^{\beta x} + \gamma\mathfrak{C}e^{\gamma x}}{\mathfrak{A}e^{\alpha x} + \mathfrak{B}e^{\beta x} + \mathfrak{C}e^{\gamma x}} \quad \text{and} \quad t + uu = \frac{\alpha\alpha\mathfrak{A}e^{\alpha x} + \beta\beta\mathfrak{B}e^{\beta x} + \gamma\gamma\mathfrak{C}e^{\gamma x}}{\mathfrak{A}e^{\alpha x} + \mathfrak{B}e^{\beta x} + \mathfrak{C}e^{\gamma x}},$$

while α, β, γ are the roots from this equation

$$A + B\lambda + C\lambda^2 + D\lambda^3 = 0.$$

But is appropriate to observe, that equation on putting $t + uu = z$, to change into this form

$$A + Bu + z(C + Du) + \tfrac{Dz}{du}(z - uu) = 0,$$

which is seen to appear wider than these equations of the same kind which we have examined above [Vol. I, § 433, § 488] ; because the method of integration of which by known methods cannot be agreed upon, the resolution is easily established on putting

$$u = \tfrac{dy}{ydx} \text{ and } z = \tfrac{ddy}{ydx^2},$$

from which there is made

$$du = \tfrac{ddy}{ydx} - \tfrac{dy^2}{yydx} \text{ and } dz = \tfrac{d^3y}{ydx^2} - \tfrac{dyddy}{yydx^2}$$

and thus

$$\tfrac{dz}{du} = \tfrac{yd^3y - dyddy}{dx(yddy - dy^2)} \quad \text{and} \quad z - uu = \tfrac{yddy - dy^2}{yydx^2},$$

and thus this equation results

$$A + \tfrac{Bdy}{ydx} + \tfrac{Cddy}{ydx^2} + \tfrac{Ddyddy}{yydx^3} + \tfrac{Dyd^3y - Ddyddy}{yydx^3} = 0$$

or

$$Ay + B\tfrac{dy}{dx} + C\tfrac{ddy}{dx^2} + D\tfrac{d^3y}{dx^3} = 0,$$

the resolution of which has been shown.

SCHOLIUM 3

1124. That differential equation of the first order

$$Dtdt + tdu(C + 3Du) + du\left(A + Bu + Cu^2 + Du^3\right) = 0,$$

of which we have found the integral, is worth setting out with more care. And indeed in the first place I note that it can be rendered integrable, if it is divided by this form

$$DDt^3 + Dtt(B + 2Cu + 3Duu) + t(C + 3Du)\left(A + Bu + Cu^2 + Du^3\right) + \left(A + Bu + Cu^2 + Du^3\right)^2,$$

from which also we conclude that this equation

$$Dzdz - Duudz + zdu(C + Du) + du(A + Bu) = 0$$

becomes integrable, if it is divided by the following form

$$DDz^3 + Dzz(B+2Cu) + z(AC+(3AD+BC)u+(BD+CC)uu)$$
$$+AA+2ABu+(AC+BB)uu+(BC-AD)u^3.$$

[There is a mistaken sign in the last term in the following expression in the first edition, corrected in the *O.O.* by the editor of that edition.]
But each such divisor equated to zero gives rise to a particular integral, from which, since *t* or *z* obtain three values each, [from which] the particular individual integrals will be shown [Vol. I, § 574].

Hence it is worth the effort to investigate in general the equation

$$ydy + yPdx + Qdx = 0,$$

which divided by the form

$$y^3 + Lyy + My + N$$

becomes integrable. Moreover by the above operation [Vol. I, § 517–§ 527] there is found set forth :
[Recall that the method of multipliers demands that the differential *dZ* of some function *Z* of the two variables *x* and *y* is complete, and thus in this case we have :

$$\frac{ydy+yPdx+Qdx}{y^3+Lyy+My+N} = 0, \quad \text{or} \quad \frac{ydy}{y^3+Lyy+My+N} + \frac{(yP+Q)dx}{y^3+Lyy+My+N} = 0; \quad \text{hence:}$$
$$\frac{d}{dx}\frac{y}{y^3+Lyy+My+N} = \frac{d}{dy}\frac{(yP+Q)}{y^3+Lyy+My+N},$$
$$-y(yydL+ydM+dN) = P(y^3+Lyy+My+N)dx - (yP+Q)(3y^2+2Ly+M)dx$$
$$dL = 2Pdx; \quad dM = -(PL-3Q-2LP)dx; \quad dN = -(PM-PM-2LQ)dx; \quad 0 = PN-MQ,$$

on equating coefficients of powers of y.]

$$dL = 2Pdx,\ dM = PLdx + 3Qdx,\ dN = 2QLdx \ \text{ and } \ PN - QM = 0,$$

from which there is deduced

$$Pdx = \tfrac{1}{2}dL,\ \ Qdx = \frac{dN}{2L},\ \ dM = \tfrac{1}{2}LdL + \tfrac{3dN}{2L}$$

and

$$NdL = \tfrac{MdN}{L} \ \text{ or } \ M = \tfrac{NLdL}{dN},$$

which value substituted everywhere on taking *dN* constant gives

$$3dN^2 = LLdLdN + 2NLLddL + 2NLdL^2 ,$$

which multiplied by *dL* changes into

$$3dLdN^2 = d.NL^2dL^2 .$$

Now more conveniently and indeed in a singular manner these equations can be resolved on putting

$$N = \alpha Z^2 \text{ and } L = \frac{dZ}{dz} ,$$

from which on taking the element *dz* constant there is deduced $M = \frac{ZddZ}{2dz^2}$ and hence

$$dM = \frac{Zd^3Z + dZddZ}{2dz^2} \text{ and } \frac{1}{2}LdL + \frac{3dN}{2L} = \frac{dZddZ}{2dz^2} + 3\alpha Zdz .$$

Therefore $d^3Z = 6\alpha dz^3$ and thus

$$Z = \alpha z^3 + \beta zz + \gamma z + \delta, \ Pdx = \frac{ddZ}{2dz} \text{ and } Qdx = \alpha Zdz .$$

On account of which on taking $Z = \alpha z^3 + \beta z^2 + \gamma z + \delta$ this equation

$$ydy + y\frac{ddZ}{2dz} + \alpha Zdz = 0$$

is returned integrable on division by this form

$$y^3 + y^2\frac{dZ}{dz} + y\frac{ZddZ}{2dz^2} + \alpha ZZ .$$

In addition if Z should have the factors, as this equation proposes

$$ydy + ydz(\alpha + \beta + \gamma + 3z) + dz(\alpha + z)(\beta + z)(\gamma + z) = 0 ,$$

the divisor returning that integrable will be

$$\big(y + (\alpha + z)(\beta + z)\big)\big(y + (\alpha + z)(\gamma + z)\big)\big(y + (\beta + z)(\gamma + z)\big),$$

the individual factors of which equated to zero give the particular integral [Vol. I, § 574].

But from each the complete integral thus can be elicited in the more accustomed manner.
There is put

$$y = v - (\alpha + z)(\beta + z)$$

and there is found

$$vdv + vdz(\gamma + z) - dv(\alpha + z)(\beta + z) = 0;$$

again let $dv = pdz$ and there will be $v = \frac{p(\alpha+z)(\beta+z)}{p+\gamma+z}$ and on differentiating, on putting pdz in place of dv, this equation arises

$$dp(\alpha + z)(\beta + z)(\gamma + z) = dz\Big(p^3 + (2\gamma - \alpha - \beta)p^2 + (\gamma - \alpha)(\gamma - \beta)p\Big),$$

which gives this separation

$$\frac{dz}{(\alpha+z)(\beta+z)(\gamma+z)} = \frac{dp}{p(p+\gamma-\alpha)(p+\gamma-\beta)}.$$

PROBLEM 145

1125. *To find the complete integral of the differential equation of any order*

$$Ay + \frac{Bdy}{dx} + \frac{Cddy}{dx^2} + \frac{Dd^3y}{dx^3} + \frac{Ed^4y}{dx^4} + \text{etc.} = 0$$

on taking the element dx.

SOLUTION

It is evident that the formula $y = e^{\lambda x}$ satisfies this equation ; since indeed hence there shall be $\frac{dy}{dx} = \lambda e^{\lambda x}$, $\frac{ddy}{dx^2} = \lambda^2 e^{\lambda x}$, $\frac{d^3y}{dx^3} = \lambda^3 e^{\lambda x}$, and in general $\frac{d^ny}{dx^n} = \lambda^n e^{\lambda x}$, on making the substitution this equation is come upon, clearly after we have divided those by $e^{\lambda x}$,

$$A + B\lambda + C\lambda^2 + D\lambda^3 + E\lambda^4 + \text{etc.} = 0,$$

and from which the value of λ is required to be found. Hence just as many values are obtained for the letter λ, as there should be for the order of the proposed differential equation, which the individual equations equally satisfy. Which values if they shall be $\alpha, \beta, \gamma, \delta$ etc., indeed will be the particular integrals

$$y = \mathfrak{A}e^{\alpha x}, \quad y = \mathfrak{B}e^{\beta x}, \quad y = \mathfrak{C}e^{\gamma x} \text{ etc.}$$

Now from the nature of the equation itself it is evident that the sum of whatever number of values of these and thus likewise of all satisfies the equation. Therefore since the sum of all

$$y = \mathfrak{A}e^{\alpha x} + \mathfrak{B}e^{\beta x} + \mathfrak{C}e^{\gamma x} + \mathfrak{D}e^{\delta x} + \text{etc.}.$$

includes as many arbitrary constants $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D}$ etc., as the number of orders of the proposed differential equation present, which is why there is no doubt that this form is the complete integral of this equation.

The differential equation ascends to the order *n*, so that there shall be

$$Ay + \frac{Bdy}{dx} + \frac{Cddy}{dx^2} + \frac{Dd^3y}{dx^3} + \ldots\ldots + N\frac{d^n y}{dx^n} = 0$$

and the complete integral is established from *n* parts, which must be defined from the resolution of this algebraic equation of order *n*, evidently

$$A + B\lambda + C\lambda^2 + D\lambda^3 + \ldots\ldots + N\lambda^n = 0 .$$

Without doubt the individual factors of this makes apparent these simple parts ; thus if $\alpha - \lambda$ shall be a factor, from that the part of the integral arising is $\mathfrak{A}e^{\alpha x}$, which as is obvious, arises from the integration of the simple differential equation

$$\alpha y - \frac{dy}{dx} = 0 .$$

In a like manner the two factors taken together

$$(\alpha - \lambda)(\beta - \lambda) = \alpha\beta - (\alpha + \beta)\lambda + \lambda\lambda$$

give rise to the part of the integral $\mathfrak{A}e^{\alpha x} + \mathfrak{B}e^{\beta x}$, which at the same time is the integral of this second order differential equation

$$\alpha\beta y - (\alpha + \beta)\frac{dy}{dx} + \frac{ddy}{dx^2} = 0.$$

And in general if the algebraic factor of that equation shall be

$$a + b\lambda + c\lambda^2 + f\lambda^3 + \text{etc.} = 0$$

from this in turn the differential equation may be formed :

$$ay + b\frac{dy}{dx} + c\frac{ddy}{dx^2} + f\,\frac{d^3y}{dx^3} + \text{etc.} = 0,$$

the complete integral of which, if it shall be $y = P$, that at the same time will be a part of the proposed integral of the equation. And in this manner from the individual factors of the algebraic equation

$$A + B\lambda + C\lambda^2 + D\lambda^3 + ... + N\lambda^n = 0$$

the individual parts of the integral sought are to be derived, which taken together constitute the complete integral, thus as the particular business of the resolution of this equation depends on.

COROLLARY 1

1126. Therefore if all the factors of this algebraic equation were real and likewise unequal, the integration may be had without any difficulty. For if a simple factor should be $f + g\lambda$, the part of the integral arising from that is $\mathfrak{A}e^{\frac{-fx}{g}}$.

COROLLARY 2

1127. If two factors should be equal or a factor of this algebraic equation were $(f + g\lambda)^2$, the part of the integral thus arising from this is $e^{\frac{-fx}{g}}(\mathfrak{A} + \mathfrak{B}x)$. If the factor shall be the cube $(f + g\lambda)^3$, the part of the integral arising thence is $e^{\frac{-fx}{g}}\left(\mathfrak{A} + \mathfrak{B}x + \mathfrak{C}x^2\right)$ and from the biquadratic factor $(f + g\lambda)^4$ the part of this $e^{\frac{-fx}{g}}\left(\mathfrak{A} + \mathfrak{B}x + \mathfrak{C}x^2 + \mathfrak{D}x^3\right)$ and thus again for however many equal factors, as is allowed to be deduced from §1120.

COROLLARY 3

1128. If imaginary factors occur, two taken together show a real factor of three terms, the form of which may be represented thus

$$ff + 2fg\lambda\cos.\zeta + gg\lambda\lambda$$

from which there is deduced

$$\lambda = -\frac{f}{g}\left(\cos.\zeta \pm \sqrt{-1}\cdot\sin.\zeta\right),,$$

from which taken with §1121 there becomes $\mu = \frac{-f\cos.\zeta}{g}$ and $\nu = \frac{f\sin.\zeta}{g}$. From which the part of the integral that will arise from such a factor will be

$$e^{\frac{-fx\cos.\zeta}{g}}\left(\mathfrak{A}\cos.\frac{fx\cos.\zeta}{g}+\mathfrak{B}\sin.\frac{fx\sin.\zeta}{g}\right).$$

COROLLARY 4

1129. If a quadratic factor of this kind of form occurs between the factors,

$$(ff+2fg\lambda\cos.\zeta+gg\lambda\lambda)^2$$

or two factors of this kind are equal, they may be considered as if differing by an infinitely small part, so that in the other in place of $\frac{f}{g}$ there shall be $\frac{f}{g}(1+\omega)$, and on account of

$$e^{\frac{-fx\cos.\zeta}{g}(1+\omega)}=e^{\frac{-fx\cos.\zeta}{g}}\left(1-\frac{\omega fx}{g}\cos.\zeta\right),$$

$$\cos.\frac{fx\sin.\zeta}{g}(1+\omega)=\cos.\frac{fx\sin.\zeta}{g}-\frac{\omega fx\sin.\zeta}{g}\sin.\frac{fx\sin.\zeta}{g}$$

and

$$\sin.\frac{fx\sin.\zeta}{g}(1+\omega)=\sin.\frac{fx\sin.\zeta}{g}+\frac{\omega fx\sin.\zeta}{g}\cos.\frac{fx\sin.\zeta}{g}$$

and from this the part of the integral is deduced :

$$e^{\frac{-fx\cos.\zeta}{g}}\left\{\begin{array}{l}\mathfrak{A}'\cos.\frac{fx\sin.\zeta}{g}-\mathfrak{A}'\frac{\omega fx\cos.\zeta}{g}\cos.\frac{fx\sin.\zeta}{g}-\mathfrak{A}'\frac{\omega fx\sin.\zeta}{g}\sin.\frac{fx\sin.\zeta}{g}\\+\mathfrak{B}'\sin.\frac{fx\sin.\zeta}{g}-\mathfrak{B}'\frac{\omega fx\cos.\zeta}{g}\sin.\frac{fx\sin.\zeta}{g}+\mathfrak{B}'\frac{\omega fx\sin.\zeta}{g}\cos.\frac{fx\sin.\zeta}{g}\end{array}\right\}$$

to which the former must be added. In the end we may assemble these constants thus on putting

$$\mathfrak{A}+\mathfrak{A}'=\mathfrak{E},\quad \frac{-\mathfrak{A}'\omega fx\cos.\zeta}{g}+\frac{\mathfrak{B}'\omega fx\sin.\zeta}{g}=\mathfrak{G},$$
$$\mathfrak{B}+\mathfrak{B}'=\mathfrak{F},\quad \frac{-\mathfrak{A}'\omega fx\sin.\zeta}{g}-\frac{\mathfrak{B}'\omega fx\cos.\zeta}{g}=\mathfrak{H},$$

from which these constants are determined everywhere, and the corresponding part of the integral

SCHOLIUM

1130. Behold therefore the general method of finding the integrals of differential equations of this kind

$$Ay+\frac{Bdy}{dx}+\frac{Cddy}{dx^2}+\frac{Dd^3y}{dx^3}+\ldots..+N\frac{d^ny}{dx^n}=0$$

thus drawn together in a abridged form. There is written, as indicated in this little table,

in place of	y	$\frac{dy}{dx}$	$\frac{ddy}{dx^2}$	$\frac{d^3y}{dx^3}$	$\frac{d^4y}{dx^4}$	$\ldots$	$\frac{d^ny}{dx^n}$
there is written	1	z	z^2	z^3	z^4	$\ldots$	z^n

so that this algebraic equation arises

$$A+Bz+Cz^2+Dz^3+Ez^4+\ldots+Nz^n=0,$$

the individual factors of which, real, either simple or doubled, are to be noted, and in the above case, in which two or more are equal to each other, are to be observed properly. Then the parts of this kind arise for the integral sought from the individual factors, and are allowed to be understood in the following table:

Factors	Parts of the integral
$f+gz$	$\mathfrak{A}e^{\frac{-fx}{g}}$
$(f+gz)^2$	$(\mathfrak{A}+\mathfrak{B}x)e^{\frac{-fx}{g}}$
$(f+gz)^3$	$(\mathfrak{A}+\mathfrak{B}x+\mathfrak{C}x^2)e^{\frac{-fx}{g}}$
$(f+gz)^4$	$(\mathfrak{A}+\mathfrak{B}x+\mathfrak{C}x^2+\mathfrak{D}x^3)e^{\frac{-fx}{g}}$
etc.	etc.
$ff+2fgz\cos.\zeta+ggzz$	$e^{\frac{-fx\cos.\zeta}{g}}\left(\mathfrak{A}\cos.\frac{fx\cos.\zeta}{g}+\mathfrak{B}\sin.\frac{fx\sin.\zeta}{g}\right).$
$(ff+2fgz\cos.\zeta+ggzz)^2$	$e^{\frac{-fx\cos.\zeta}{g}}\left\{\begin{matrix}(\mathfrak{A}+\mathfrak{B}x)\cos.\frac{fx\cos.\zeta}{g}\\+(\mathfrak{a}+\mathfrak{b}x)\sin.\frac{fx\cos.\zeta}{g}\end{matrix}\right\}$
$(ff+2fgz\cos.\zeta+ggzz)^3$	$e^{\frac{-fx\cos.\zeta}{g}}\left\{\begin{matrix}(\mathfrak{A}+\mathfrak{B}x+\mathfrak{C}xx)\cos.\frac{fx\cos.\zeta}{g}\\+(\mathfrak{a}+\mathfrak{b}x+\mathfrak{c}xx)\sin.\frac{fx\cos.\zeta}{g}\end{matrix}\right\}$

etc.	etc.

But for the individual factors different constant letters must be written, so that the integral is obtained complete with all the numbers.

EXAMPLE 1

1131. *To assign the complete integral of this fourth order differential equation*

$$y-\frac{2dy}{dx}+\frac{2ddy}{dx^2}-\frac{2d^3y}{dx^3}+\frac{d^4y}{dx^4}=0.$$

Hence there arises this algebraic equation

$$1-2z+2zz-2z^3+z^4=0,$$

which is resolved into these factors $(1-z)^2(1+zz)$, of which the first on account of $f=1$ and $g=-1$ give this part of the integral $(\mathfrak{A}+\mathfrak{B}x)e^x$, now the other factor on account of $f=1$, $\cos.\zeta=0$, $g=1$ and $\sin.\zeta=1$ gives $\mathfrak{A}\cos.x+\mathfrak{B}\sin.x$. Whereby the complete integral, which is sought, will be

$$y=(\mathfrak{A}+\mathfrak{B}x)e^x+\mathfrak{C}\cos.x+\mathfrak{D}\sin.x$$

containing four arbitrary constants.

But if we wish, so that on putting $x=0$ there becomes $y=0$, there is necessary to be made $\mathfrak{A}+\mathfrak{C}=0$; if also $\frac{dy}{dx}$ must vanish in the same case, on account of

$$\frac{dy}{dx}=(\mathfrak{A}+\mathfrak{B}+\mathfrak{B}x)e^x-\mathfrak{C}\sin.x+\mathfrak{D}\cos.x$$

there must become $\mathfrak{A}+\mathfrak{B}+\mathfrak{D}=0$. If in addition $\frac{d^2y}{dx^2}$ must vanish, on account of

$$\frac{d^2y}{dx^2}=(\mathfrak{A}+2\mathfrak{B}+\mathfrak{B}x)e^x-\mathfrak{C}\cos.x-\mathfrak{D}\sin.x$$

there must become $\mathfrak{A}+2\mathfrak{B}-\mathfrak{D}=0$. Whereby we may satisfy these three conditions on taking $\mathfrak{C}=-\mathfrak{A}$, $\mathfrak{B}=-\mathfrak{A}$ and $\mathfrak{D}=0$, thus so that there shall be the integral

$$y=\mathfrak{A}(1-x)e^x-\mathfrak{A}\cos.x.$$

EXAMPLE 2

1132. *To integrate the differential equation of the fourth order*

$$Ay+\frac{Cddy}{dx^2}+\frac{Ed^4y}{dx^4}=0$$

on assuming the element dx constant.

The algebraic equation leading to the integration is

$$A+Czz+Ez^4=0,$$

which always has two real double factors, of which the twofold form can be either

$$(aa+2maz+nzz)(aa-2maz+nzz) \text{ or } (aa+mzz)(aa+nzz).$$

From the former there is

$$A=a^4,\ C=2naa-4mmaa,\ E=nn,$$

and from the latter truly

$$A=a^4,\ C=(m+n)aa,\ E=mn;$$

but it is permitted always to represent the first term A by the biquadratic a^4 and in the first place the resolution can be done, if E shall be a positive number and $2naa-C$ or $2\sqrt{AE}-C$ also is positive and thus $4AE>CC$, and truly the latter, if $CC>4AE$. Therefore it is then to be considered, to which division the individual factors may be related, from which the following cases occur :

1. If all four simple factors are real, then there will be

$$A+Czz+Ez^4=(a+z)(a-z)(b+z)(b-z);$$

this equation will be considered

$$aabby-(aa+bb)\frac{ddy}{dx^2}+\frac{d^4y}{dx^4}=0 \ ,$$

the complete integral of which is

$$y=\mathfrak{A}e^{ax}+\mathfrak{B}e^{-ax}+\mathfrak{C}e^{bx}+\mathfrak{D}e^{-bx}.$$

And if there shall be $b = a$, the complete integral of this equation

$$a^4 y - \frac{2aaddy}{dx^2} + \frac{d^4 y}{dx^4} = 0$$

will be

$$y = (\mathfrak{A} + \mathfrak{B}x)e^{ax} + (\mathfrak{C} + \mathfrak{D}x)e^{-ax}.$$

II. If two simple factors are real, and indeed two imaginary, then there will be

$$A + Czz + E^4 = (a+z)(a-z)(bb+zz);$$

this equation will be considered

$$aabby + (aa-bb)\frac{ddy}{dx^2} - \frac{d^4 y}{dx^4} = 0$$

the complete integral of this is

$$y = \mathfrak{A}e^{ax} + \mathfrak{B}e^{-ax} + \mathfrak{C}\cos.bx + \mathfrak{D}\sin.bx.$$

III. If all the simple factors are imaginary, then there are two cases to be pursued :

1) If

$$A + Czz + Ez^4 = (aa+zz)(bb+zz),$$

from which the complete integral of this equation

$$aabby + (aa+bb)\frac{ddy}{dx^2} + \frac{d^4 y}{dx^4} = 0$$

will be

$$y = \mathfrak{A}\cos.ax + \mathfrak{B}\sin.ax + \mathfrak{C}\cos.bx + \mathfrak{D}\sin.bx.$$

2) If

$$A + Czz + Ez^4 = (aa + 2az\cos.\zeta + zz)(aa - 2as\cos.\zeta + zz),$$

from which the complete integral of this equation

$$a^4 y - \frac{2aaddy}{dx^2}\cos.2\zeta + \frac{d^4 y}{dx^4} = 0$$

will be

$$y=e^{+ax\cos.\zeta}\left(\mathfrak{A}\cos.\left(ax\sin.\zeta\right)+\mathfrak{B}\sin.\left(ax\sin.\zeta\right)\right)$$
$$+e^{-ax\cos.\zeta}\left(\mathfrak{C}\cos.\left(ax\sin.\zeta\right)+\mathfrak{D}\sin.\left(ax\sin.\zeta\right)\right)$$

But if there shall be in the former case $b=a$ or in the latter $\cos.\zeta=0$, then the complete integral of this equation

$$a^4y+\frac{2aaddy}{dx^2}+\frac{d^4y}{dx^4}=0$$

will be

$$y=\left(\mathfrak{A}+\mathfrak{B}x\right)\cos.ax+\left(\mathfrak{C}+\mathfrak{D}x\right)\sin.ax\,.$$

SCHOLIUM 1

1133. Therefore since it is possible to assign the complete integral of the equation

$$Ay+\frac{Cddy}{dx^2}+\frac{Ed^4y}{dx^4}=0,$$

all the equations, which are allowed to be derived from this, can be integrated. But this equation first multiplied by *2dy* is reduced by integration to a differential equation of the third order

$$Ayy+\frac{Cd^2y}{dx^2}+\frac{2Edyd^3y-Eddy^2}{dx^4}=\text{Const.}$$

[Note that $\int 2dyd^4y=2dyd^3y-\int 2ddyd^3y=2dyd^3y-ddy^2$.]

But in the integral found before , it is allowed to define the constants $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$ thus, so that the constant Const. vanishes, and thus the complete integral of the equation

$$Ayy+\frac{Cd^2y}{dx^2}+\frac{2Edyd^3y-Eddy^2}{dx^4}=0$$

will be in our power. Now there is put $y=e^{\int vdx}$, so that there shall be $v=\frac{dy}{ydx}$, and on account of

$$\frac{dy}{dx}=e^{\int vdx}v,\quad \frac{ddy}{dx^2}=e^{\int vdx}\left(\frac{dv}{dx}+vv\right)\quad\text{and}\quad \frac{d^3y}{dx^3}=e^{\int vdx}\left(\frac{ddv}{dx^2}+\frac{3vdv}{dx}+v^3\right)$$ our equation adopts

this form

$$A+Cvv+E\left(\frac{2vddv}{dx^2}+\frac{4vvdv}{dx}+v^4-\frac{dv^2}{dx^2}\right)=0\,.$$

If again there shall be $dx = \frac{dv}{s}$, so that there shall be

$$s = \frac{dv}{dx} = \frac{ddy}{ydx^2} - \frac{d^2y}{yydx^2}\text{; there will be } \frac{ddv}{dx} = ds \text{ and } \frac{ddv}{dx^2} = \frac{sds}{dv},$$

from which this equation of the first order results

$$A + Cvv + E\left(\frac{2vsds}{dv} - ss + 4vvs + v^4\right) = 0,$$

the relation of which between v and s thus is defined from the relation between x and y, so that there shall be

$$v = \frac{dy}{ydx} \text{ and } s = \frac{yddy - dy^2}{yydx^2}.$$

SCHOLIUM 2

1134. But on retaining that constant introduced by the integration, so that there may be considered

$$Ayy + \frac{Cd^2y}{dx^2} + \frac{2Edyd^3y - Eddy^2}{dx^4} = G$$

in the complete integral, in which y is expressed by x, the constants $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D}$ for this quantity G can be determined similarly. Therefore now there is put $dx = \frac{dy}{u}$, so that there shall be $\frac{dy}{dx} = u$; then there will be $\frac{ddy}{dx^2} = \frac{udu}{dy}$ and $\frac{d^3y}{dx^2} = d.\frac{udu}{dy}$ and thus $\frac{d^3y}{dx^3} = \frac{u}{dy}d.\frac{udu}{dy}$.
From which this second order differential equation may be obtained :

$$Ayy + Cuu + E\left(\frac{2uu}{dy}d.\frac{udu}{dy} - \frac{uudu^2}{dy^2}\right) = G,$$

where the consideration of the element assumed to be constant has been laid aside. Therefore nothing stands in the way why we may not take dy for the constant, and there becomes

$$Ayy + Cuu + E\left(\frac{2u^3ddu}{dy^2} + \frac{uudu^2}{dy^2}\right) = G$$

which equation therefore also can be integrated.

If either we put $yy = p$ and $uu = q$, on taking the element dp constant there arises this equation

$$Ap + Cq + E\frac{4pqddq + 2qdpdq - pdq^2}{dp^2} = G.$$

Or id in that equation we put $u = r^{\frac{2}{3}}$, then there will be

$$Ayy + Cr^{\frac{4}{3}} + \frac{4}{3}Er^{\frac{5}{3}}\frac{ddr}{dy^2} = G.$$

The integration of which forms without this aid is seen to be very hard.

PROBLEM 146

1135. *For the proposed differential equation of any order* $a^n y \pm \frac{d^n y}{dx^n} = 0$, *where the element dx has been taken constant, to find the complete integral of this.*

SOLUTION

The algebraic equation attending the solution is $a^n \pm z^n = 0$, for the resolution of which it is agreed upon to examine two cases, according as either the upper or lower sign prevails.

I. The upper prevails, so that this equation shall be proposed :

$$a^n y + \frac{d^n y}{dx^n} = 0$$

and the real factors of the formula $a^n + z^n$ are

$$aa - 2azcos.\tfrac{\pi}{n} + zz,\ \ aa - 2azcos.\tfrac{3\pi}{n} + zz,\ aa - 2azcos.\tfrac{5\pi}{n} + zz \quad \text{etc.},$$

the last of which is either either $aa - 2az\cos.\frac{n\pi}{n} + zz$ or $aa - 2az\cos.\frac{(n-1)\pi}{n} + zz$,

as either n or $n-1$ should be an odd number, and indeed in that case in place of the quadratic factor $aa + 2az + zz$, the root of this $a + z$ must be taken.

Hence the complete integral of this equation is

$$\begin{aligned} y = \ & e^{ax\cos.\frac{\pi}{n}}\left(\mathfrak{A}\cos.\left(ax\sin.\tfrac{\pi}{n}\right) + \mathfrak{B}\sin.\left(ax\sin.\tfrac{\pi}{n}\right)\right) \\ & + e^{ax\cos.\frac{3\pi}{n}}\left(\mathfrak{C}\cos.\left(ax\sin.\tfrac{3\pi}{n}\right) + \mathfrak{D}\sin.\left(ax\sin.\tfrac{3\pi}{n}\right)\right) \\ & + e^{ax\cos.\frac{5\pi}{n}}\left(\mathfrak{E}\cos.\left(ax\sin.\tfrac{5\pi}{n}\right) + \mathfrak{F}\sin.\left(ax\sin.\tfrac{5\pi}{n}\right)\right) \\ & \text{etc.,} \end{aligned}$$

of which expression, if n shall be odd, the final part shall be $\mathfrak{N}e^{-ax}$. Because the integral thus can also be shown :

$$y = \mathfrak{A}e^{ax\cos.\frac{\pi}{n}}\cos.\left(ax\sin.\frac{\pi}{n}+\mathfrak{a}\right)+\mathfrak{B}e^{ax\cos.\frac{3\pi}{n}}\cos.\left(ax\sin.\frac{3\pi}{n}+\mathfrak{b}\right)$$
$$+\mathfrak{C}e^{ax\cos.\frac{5\pi}{n}}\cos.\left(ax\sin.\frac{5\pi}{n}+\mathfrak{c}\right)+\mathfrak{D}e^{ax\cos.\frac{7\pi}{n}}\cos.\left(ax\sin.\frac{7\pi}{n}+\mathfrak{d}\right)$$
etc.,

which form must be continued so far, as long as similar terms recur.

II. If the lower sign prevails and this equation is proposed :

$$a^n y-\frac{d^n y}{dx^n}=0$$

then the factors of the formula $a^n - z^n$ are real :

$$a-z,\quad aa-2az\cos.\frac{2\pi}{n}+zz,\quad aa-2az\cos.\frac{4\pi}{n}+zz,\quad aa-2az\cos.\frac{6\pi}{n}+zz,\ \text{etc.},$$

of which, if n is an even number, the final is $a+z$, but if it is odd, then

$$aa-2az\cos.\frac{(n-1)\pi}{n}+zz$$

Whereby the complete integral of this equation is :

$$y=\mathfrak{A}e^{ax}+e^{ax\cos.\frac{2\pi}{n}}\left(\mathfrak{B}\cos.\left(ax\sin.\frac{2\pi}{n}\right)+\mathfrak{C}\sin.\left(ax\sin.\frac{2\pi}{n}\right)\right)$$
$$+e^{ax\cos.\frac{4\pi}{n}}\left(\mathfrak{D}\cos.\left(ax\sin.\frac{4\pi}{n}\right)+\mathfrak{E}\sin.\left(ax\sin.\frac{4\pi}{n}\right)\right)$$
$$+e^{ax\cos.\frac{6\pi}{n}}\left(\mathfrak{F}\cos.\left(ax\sin.\frac{6\pi}{n}\right)+\mathfrak{G}\sin.\left(ax\sin.\frac{6\pi}{n}\right)\right)$$
etc.,

because the integral can also be expressed thus :

$$y = \mathfrak{A}e^{ax} + \mathfrak{B}e^{ax\cos.\frac{2\pi}{n}}\cos.\left(ax\sin.\frac{2\pi}{n} + \mathfrak{b}\right)$$
$$+ \mathfrak{C}e^{ax\cos.\frac{4\pi}{n}}\cos.\left(ax\sin.\frac{4\pi}{n} + \mathfrak{c}\right)$$
$$+ \mathfrak{D}e^{ax\cos.\frac{6\pi}{n}}\cos.\left(ax\sin.\frac{6\pi}{n} + \mathfrak{d}\right)$$
etc.,

which form is to be continued so far, as long as different terms from the beginning shall be produced.

SCHOLIUM 1

1136. Therefore for the various values of the exponent *n* the following integrals themselves will be considered, and in the first place indeed from the equation

$$a^n y + \frac{d^n y}{dx^n} = 0.$$

I. The integral of the equation $ay + \frac{dy}{dx} = 0$ is $y = \mathfrak{A}e^{-ax}$.

II. The integral of the equation $a^2 y + \frac{ddy}{dx^2} = 0$ is $\mathfrak{A}\cos.(ax + \mathfrak{a})$

Ill. The integral of the equation $a^3 y + \frac{d^3 y}{dx^3} = 0$ is $y = \mathfrak{A}e^{\frac{1}{2}ax}\cos.\left(\frac{ax\sqrt{3}}{2} + \mathfrak{a}\right) + \mathfrak{B}e^{-ax}$.

IV. The integral of the equation $a^4 y + \frac{d^4 y}{dx^4} = 0$ is
$y = \mathfrak{A}e^{\frac{ax}{\sqrt{2}}}\cos.\left(\frac{ax}{\sqrt{2}} + \mathfrak{a}\right) + \mathfrak{B}e^{\frac{-ax}{\sqrt{2}}}\cos.\left(\frac{ax}{\sqrt{2}} + \mathfrak{b}\right)$.

V. The integral of the equation $a^5 y + \frac{d^5 y}{dx^5} = 0$ is

$$y = \mathfrak{A}e^{ax\cos.36^0}\cos.\left(ax\sin.36^0 + \mathfrak{a}\right) + \mathfrak{B}e^{-ax\cos.72^0}\cos.\left(ax\sin.72^0 + \mathfrak{b}\right) + \mathfrak{C}e^{-ax}.$$

VI. The integral of the equation $a^6 y + \frac{d^6 y}{dx^6} = 0$ is

$$y = \mathfrak{A}e^{\frac{ax\sqrt{3}}{2}}\cos.\left(\tfrac{1}{2}ax + \mathfrak{a}\right) + \mathfrak{B}\cos.\left(ax + \mathfrak{b}\right) + \mathfrak{C}e^{\frac{-ax\sqrt{3}}{2}}\cos.\left(\tfrac{1}{2}ax + \mathfrak{c}\right)$$
etc.

Moreover in a similar manner for the other form

$$a^n y - \frac{d^n y}{dx^n} = 0$$

the integrations themselves thus will be considered according to the simpler values of the exponent *n*.

1. The integral of the equation $ay - \frac{dy}{dx} = 0$ is $y = \mathfrak{A}e^{ax}$.

II. The integral of the equation $a^2 y - \frac{ddy}{dx^2} = 0$ is $y = \mathfrak{A}e^{ax} + \mathfrak{B}e^{-ax}$.

Ill. The integral of the equation $a^3 y - \frac{d^3 y}{dx^3} = 0$ is $y = \mathfrak{A}e^{ax} + \mathfrak{B}e^{-\frac{1}{2}ax}\cos.\left(\frac{ax\sqrt{3}}{2} + \mathfrak{b}\right)$.

IV. The integral of the equation $a^4 y - \frac{d^4 y}{dx^4} = 0$ is $y = \mathfrak{A}e^{ax} + \mathfrak{B}\cos.\left(ax + \mathfrak{b}\right) + \mathfrak{C}e^{-ax}$.

V. The integral of the equation $a^5 y - \frac{d^5 y}{dx^5} = 0$ is

$$y = \mathfrak{A}e^{ax} + \mathfrak{B}e^{ax\cos.72^0}\cos.\left(ax\sin.72^0 + \mathfrak{b}\right) + \mathfrak{C}e^{-ax\cos.36^0}\cos.\left(ax\sin.36^0 + \mathfrak{c}\right).$$

VI. The integral of the equation $a^6 y - \frac{d^6 y}{dx^6} = 0$ is

$$y = \mathfrak{A}e^{ax} + \mathfrak{B}e^{\frac{1}{2}ax}\cos.\left(ax\frac{\sqrt{3}}{2} + \mathfrak{b}\right) + \mathfrak{C}e^{-\frac{1}{2}ax}\cos.\left(ax\sin.ax\frac{\sqrt{3}}{2} + \mathfrak{c}\right) + \mathfrak{D}e^{-ax}$$

and thus, as far as it is pleases, it is allowed to progress.

SCHOLIUM 2

1137. Although the method which I have used here, may make available readily the integral of equations contained in the proposed form, yet it differs entirely from the principles of integration. For when a differential equation is of a higher order, the rules of integration require that it may be integrated separately so often, before a finite relation between the variables is arrived at, and meanwhile the individual integrations receive an arbitrary constant, and at last in this manner the complete integral is found. But at this point as if by a single operation we have elicited the last integral with all the constants, by which that is returned complete ; actually of course by using a single conjecture I have arrived at all the particular integrals and the nature of the equation has permitted conveniently, that from these the complete integral can be formed. Now if we wish to observe strictly the rules of integration, for argument's sake for a fourth order differential equation there is need of a fourfold integration, of which by the first that may be reduced to a differential equation of the third order, then indeed that itself by a new integration is reduced to a differential equation of the second order, which integrated anew leads to a first order differential equation, and this finally integrated again makes apparent the sought relation between the two variables. And in this way here also a form of treating equations to resolution is allowed, so that by continual integrations the form is reduced to a simpler form, in which finally the same integrals that we have elicited here are come upon.

But since this method extends wider than to the forms considered here and with the aid of this method, the more general equation may be able to be integrated

$$X = A + \frac{Bdy}{dx} + \frac{Cddy}{dx^2} + \frac{Dd^3y}{dx^3} + \text{etc.}$$

with X denoting some function of x, for the resolution of which the preceding method barely suffices, I will adapt a new method at once to this more general form.

CAPUT II

DE RESOLUTIONE AEQUATIONUM HUIUS FORMAE

$$Ay + \frac{Bdy}{dx} + \frac{Cddy}{dx^2} + \frac{Dd^3y}{dx^3} + \frac{Ed^4y}{dx^4} + \text{etc.} = 0$$

SUMTO ELEMENTO *dx* CONSTANTE

PROBLEMA 144

1117. *Aequationis differentialis tertii gradus*

$$Ay + \frac{Bdy}{dx} + \frac{Cddy}{dx^2} + \frac{Dd^3y}{dx^3} = 0$$

sumto elemento dx constante integrale completum invenire.

SOLUTIO

Cum *A, B, C, D* sint quantitates constantes, levi attentione adhibita patet isti aequationi huiusmodi formam $y = e^{\lambda x}$ satisfacere; cum enim hinc sit

$$\frac{dy}{dx} = \lambda e^{\lambda x}, \quad \frac{ddy}{dx^2} = \lambda^2 e^{\lambda x}, \quad \frac{d^3y}{dx^3} = \lambda^3 e^{\lambda x},$$

his substitutis et aequatione per $e^{\lambda x}$ divisa fit

$$A + \lambda B + \lambda^2 C + \lambda^3 D = 0,$$

unde exponens λ determinatur; qui cum tres valores sortiatur, qui sint α, β, γ, habebimus tres formulas satisfacientes

$$y = e^{\alpha x}, \quad y = e^{\beta x}, \quad y = e^{\gamma x}.$$

Verum ex natura aequationis propositae perspicuum est, si ei satisfaciant valores $y = P, \quad y = Q, \quad y = R$, etiam his utcunque coniungendis satisfacturum $y = \mathfrak{A}P + \mathfrak{B}Q + \mathfrak{C}R$. Quare ex ternis formulis inventis nanciscimur hanc latissime patentem aeque satisfacientem

$$y = \mathfrak{A}e^{\alpha x} + \mathfrak{B}e^{\beta x} + \mathfrak{C}e^{\gamma x};$$

quae forma cum tres constantes arbitrarias $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}$ complectatur, revera erit integrale completum aequationis nostrae propositae.

COROLLARIUM 1

1118. Integrale ergo completum tot constat partibus, quot radices habeat seu factores aequatio

$$A+\lambda B+\lambda^2 C+\lambda^3 D=0\,;$$

cuius si factor fuerit $\alpha+\lambda$, pars integralis erit $e^{-\alpha x}$.

COROLLARIUM 2

1119. Haec scilicet pars erit integrale huius aequationis $ay+\frac{dy}{dx}=0$.
Unde si sit

$$A+B\lambda+C\lambda^2+D\lambda^3=(a+\lambda)(b+\lambda)(c+\lambda)\,,$$

quaerantur valores ipsius y ex his aequationibus simplicibus

$$ay+\frac{dy}{dx}=0,\quad by+\frac{dy}{dx}=0,\quad cy+\frac{dy}{dx}=0\,;$$

qui si sint $y=P,\, y=Q,\, y=R$, integrale aequationis propositae erit

$$y=\mathfrak{A}P+\mathfrak{B}Q+\mathfrak{C}R\,.$$

COROLLARIUM 3

1120. Si binae radices sint aequales, puta $\beta=\alpha$, consideretur differentia ut evanescens $\beta=\alpha+\omega$, et cum sit $e^{\beta x}=e^{\alpha x}\cdot e^{\omega x}=e^{\alpha x}(1+\omega x)$, evidens est loco $\mathfrak{A}e^{\alpha x}+\mathfrak{B}e^{\beta x}$ scribi debere

$$\mathfrak{A}e^{\alpha x}+\mathfrak{B}e^{\beta x}x=e^{\alpha x}(\mathfrak{A}+\mathfrak{B}x)\,;$$

ac si omnes tres radices fuerint aequales $\alpha=\beta=\gamma$, ut aequatio sit

$$y+\frac{3dy}{adx}+\frac{3ddy}{a^2dx^2}+\frac{d^3y}{a^3dx^3}=0$$

ob $\alpha=\beta=\gamma=-a$ integrale completum erit

$$e^{-ax}\left(\mathfrak{A}+\mathfrak{B}x+\mathfrak{C}x^2\right).$$

COROLLARIUM 4

1121. Si binae radices fiant imaginariae, ut sit $\alpha = \mu + \nu\sqrt{-1}$ et $\beta = \mu - \nu\sqrt{-1}$, loco $\mathfrak{A}e^{\alpha x} + \mathfrak{B}e^{\beta x}$ scribi debet

$$e^{\mu x}\left(\mathfrak{A}e^{\nu x\sqrt{-1}} + \mathfrak{B}e^{-\nu x\sqrt{-1}}\right),$$

quae reducitur ad hanc formam

$$e^{\mu x}\left(\mathfrak{A}\cos.\nu x + \mathfrak{B}\sin.\nu x\right).$$

SCHOLION 1

1122. Quanquam aequatio proposita triplicem integrationem postulat, antequam ad relationem finitam inter *x* et *y* perveniatur, hic tamen una operatione, quae ne integrationi quidem est affinis, eo pertigimus. Coniectura scilicet formam collegimus aequationi particulariter satisfacientem simulque tres huiusmodi formas sumus consecuti. Deinde ex ipsa aequationis indole intelleximus, si valores singuli $y = P$, $y = Q$, $y = R$ satisfaciant, etiam formam ex iis compositam $y = \mathfrak{A}P + \mathfrak{B}Q + \mathfrak{C}R$ satisfacere debere, quod nisi commode evenisset, ex illis ternis valoribus nihil amplius concludi potuisset. Ex eodem ergo principio in genere huiusmodi aequationes differentiales, quoticunque fuerint gradus, uno quasi actu ita resolvi poterunt, ut adeo integrale completum assignetur.

SCHOLION 2

1123. Quoniam aequationem differentialem tertii gradus

$$Ay + B\frac{dy}{dx} + C\frac{ddy}{dx^2} + D\frac{d^3y}{dx^3} = 0$$

in genere resolvere licuit, ut integrale completum esset

$$y = \mathfrak{A}e^{\alpha x} + \mathfrak{B}e^{\beta x} + \mathfrak{C}e^{\gamma x}$$

existentibus α, β, γ radicibus huius aequationis cubicae

$$A + B\lambda + C\lambda^2 + D\lambda^3 = 0,$$

hinc usum non contemnendum pro allis aequationibus, in quas illam transformare licet, percipiemus.

Primo autem illam aequationem ad differentialem secundi gradus revocare licet ope substitutionis $y = e^{\int udx}$; unde fit

$$\frac{dy}{dx} = e^{\int udx}u, \quad \frac{ddy}{dx^2} = e^{\int udx}\left(\frac{du}{dx} + uu\right) \quad \text{et} \quad \frac{d^3y}{dx^3} = e^{\int udx}\left(\frac{ddu}{dx^2} + \frac{3udu}{dx} + u^3\right),$$

ita ut aequatio transformata sit divisione per $e^{\int udx}$ facta

$$A+Bu+Cuu+Du^3+C\frac{du}{dx}+\frac{3Dudu}{dx}+\frac{Dddu}{dx^2}=0,$$

cuius ergo ob $u=\frac{dy}{ydx}$ integrale est

$$u=\frac{\alpha\mathfrak{A}e^{\alpha x}+\beta\mathfrak{B}e^{\beta x}+\gamma\mathfrak{C}e^{\gamma x}}{\mathfrak{A}e^{\alpha x}+\mathfrak{B}e^{\beta x}+\mathfrak{C}e^{\gamma x}}.$$

Haec autem aequatio ulterius ad gradum primum reducitur ponendo $dx=\frac{du}{t}$;
cum enim elementum dx sumtum sit constans, erit $tddu-dtdu=0$ seu $ddu=\frac{dtdu}{t}$
unde fit $\frac{du}{dx}=t$ et $\frac{ddu}{dx^2}=\frac{tdt}{du}$. Ita ut prodeat haec aequatio
differentialis primi gradus

$$A+Bu+Cuu+Du^3+t(C+3Du)+\frac{Dtdt}{du}=0,$$

cuius ergo resolutio quoque est in potestate; utriusque scilicet variabilis u et t valor per eandem variabilem x exprimi potest. Cum enim y per x detur, erit primo $u=\frac{dy}{ydx}$, tum vero $t+uu=\frac{ddy}{ydx^2}$ ob $\frac{du}{dx}=t$. Loco y ergo substituto valore supra invento erit

$$u=\frac{\alpha\mathfrak{A}e^{\alpha x}+\beta\mathfrak{B}e^{\beta x}+\gamma\mathfrak{C}e^{\gamma x}}{\mathfrak{A}e^{\alpha x}+\mathfrak{B}e^{\beta x}+\mathfrak{C}e^{\gamma x}} \quad \text{et} \quad t+uu=\frac{\alpha\alpha\mathfrak{A}e^{\alpha x}+\beta\beta\mathfrak{B}e^{\beta x}+\gamma\gamma\mathfrak{C}e^{\gamma x}}{\mathfrak{A}e^{\alpha x}+\mathfrak{B}e^{\beta x}+\mathfrak{C}e^{\gamma x}},$$

dummodo α,β,γ sint radices ex hac aequatione

$$A+B\lambda+C\lambda^2+D\lambda^3=0.$$

Observari autem convenit illam aequationem posito $t+uu=z$ abire in hanc formam

$$A+Bu+z(C+Du)+\frac{Dz}{du}(z-uu)=0,$$

quae latius patere videtur quam illae eiusdem generis aequationes, quas supra [Vol. I, § 433, 488] tractavimus; cuius quia ratio per methodos cognitas integrandi non constat, resolutio facillime instituitur ponendo

$$u=\frac{dy}{ydx} \text{ et } z=\frac{ddy}{ydx^2},$$

unde fit

$$du = \frac{ddy}{ydx} - \frac{dy^2}{yydx} \quad \text{et} \quad dz = \frac{d^3y}{ydx^2} - \frac{dyddy}{yydx^2}$$

ideoque

$$\frac{dz}{du} = \frac{yd^3y - dyddy}{dx\left(yddy - dy^2\right)} \quad \text{et} \quad z - uu = \frac{yddy - dy^2}{yydx^2},$$

sicque resultat haec aequatio

$$A + \frac{Bdy}{ydx} + \frac{Cddy}{ydx^2} + \frac{Ddyddy}{yydx^3} + \frac{Dyd^3y - Ddyddy}{yydx^3} = 0$$

seu

$$Ay + B\frac{dy}{dx} + C\frac{ddy}{dx^2} + D\frac{d^3y}{dx^3} = 0 \ ,$$

euius resolutio est ostensa.

SCHOLION 3

1124. Aequatio ilIa differentialis primi gradus

$$Dtdt + tdu\left(C + 3Du\right) + du\left(A + Bu + Cu^2 + Du^3\right) = 0\,,$$

cuius integrale invenimus, diligentiore evolutione est digna. Ac primo quidem observo eam integrabilem reddi, si dividatur per hanc formam

$$DDt^3 + Dtt\left(B + 2Cu + 3Duu\right) + t\left(C + 3Du\right)\left(A + Bu + Cu^2 + Du^3\right) + \left(A + Bu + Cu^2 + Du^3\right)^2,$$

unde concludimus et hanc aequationem

$$Dzdz - Duudz + zdu\left(C + Du\right) + du\left(A + Bu\right) = 0$$

integrabilem fieri, si dividatur per hanc formam

$$\begin{aligned} &DDz^3 + Dzz\left(B + 2Cu\right) + z\left(AC + \left(3AD + BC\right)u + \left(BD + CC\right)uu\right) \\ &+ AA + 2ABu + \left(AC + BB\right)uu + \left(BC - AD\right)u^3 \end{aligned}$$

Utrinque autem divisor iste nihilo aequatus praebet integrale particulare, unde, cum t vel z ternos obtineant valores, singuli exhibebunt integralia particularia [Vol. I, § 574].
Hinc operae pretium erit in genere aequationem

$$ydy + yPdx + Qdx = 0$$

investigare, quae per formam

$$y^3 + Lyy + My + N$$

divisa integrabilis evadat. Per operationem autem supra [Vol. I, § 517-527] explicatam invenitur

$$dL = 2Pdx,\ dM = PLdx + 3Qdx,\ dN = 2QLdx \ \text{ et } \ PN - QM = 0\,,$$

unde colligitur

$$Pdx = \tfrac{1}{2}dL,\ \ Qdx = \frac{dN}{2L},\ \ dM = \tfrac{1}{2}LdL + \tfrac{3dN}{2L}$$

et

$$NdL = \tfrac{MdN}{L} \ \text{ seu } \ M = \tfrac{NLdL}{dN}\,,$$

qui valor ibi substitutus sumto dN constante dat

$$3dN^2 = LLdLdN + 2NLLddL + 2NLdL^2\,,$$

quae per dL multiplicata transit in

$$3dLdN^2 = d.NL^2dL^2\,.$$

Verum commodius ac singulari quidem modo illae aequationes resolvuntur statuendo

$$N = \alpha Z^2 \ \text{ et } \ L = \tfrac{dZ}{dz}\,,$$

unde sumto elemento dz constante deducitur $M = \frac{ZddZ}{2dz^2}$ hincque

$$dM = \tfrac{Zd^3Z + dZddZ}{2dz^2} \ \text{ et } \ \tfrac{1}{2}LdL + \tfrac{3dN}{2L} = \tfrac{dZddZ}{2dz^2} + 3\alpha Zdz\,.$$

Ergo $d^3Z = 6\alpha dz^3$ ideoque

$$Z = \alpha z^3 + \beta zz + \gamma z + \delta,\ Pdx = \tfrac{ddZ}{2dz} \ \text{ et } \ Qdx = \alpha Zdz\,.$$

Quocirca sumto $Z = \alpha z^3 + \beta z^2 + \gamma z + \delta$ haec aequatio

$$ydy + y\tfrac{ddZ}{2dz} + \alpha Zdz = 0$$

integrabilis redditur divisa per hanc formam

$$y^3 + y^2\tfrac{dZ}{dz} + y\tfrac{ZddZ}{2dz^2} + \alpha ZZ\,.$$

Praeterea si Z habeat factores, ut proposita sit haec aequatio

$$ydy + ydz(\alpha + \beta + \gamma + 3z) + dz(\alpha + z)(\beta + z)(\gamma + z) = 0,$$

divisor eam integrabilem reddens erit

$$\big(y + (\alpha + z)(\beta + z)\big)\big(y + (\alpha + z)(\gamma + z)\big)\big(y + (\beta + z)(\gamma + z)\big),$$

cuius singuli factores nihilo aequati praebent integrale particulare [Vol. I, § 574].
Ex unoquoque autem more magis consueto integrale completum ita elicitur. Ponatur

$$y = v - (\alpha + z)(\beta + z)$$

ac reperitur

$$vdv + vdz(\gamma + z) - dv(\alpha + z)(\beta + z) = 0;$$

sit porro $dv = pdz$ eritque $v = \frac{p(\alpha+z)(\beta+z)}{p+\gamma+z}$ et differentiando locoque dv ponendo pdz orietur haee aequatio

$$dp(\alpha + z)(\beta + z)(\gamma + z) = dz\Big(p^3 + (2\gamma - \alpha - \beta)\,p^2 + (\gamma - \alpha)(\gamma - \beta)\,p\Big),$$

quae dat hanc separatam

$$\frac{dz}{(\alpha+z)(\beta+z)(\gamma+z)} = \frac{dp}{p(p+\gamma-\alpha)(p+\gamma-\beta)}.$$

PROBLEMA 145

1125. *Aequationis differentialis cuiuscunque gradus*

$$Ay + \frac{Bdy}{dx} + \frac{Cddy}{dx^2} + \frac{Dd^3y}{dx^3} + \frac{Ed^4y}{dx^4} + \text{etc.} = 0$$

sumto elemento dx constante integrale completum invenire.

SOLUTIO

Et huic aequationi evidens est satisfacere formulam $y = e^{\lambda x}$; cum enim hinc sit $\frac{dy}{dx} = \lambda e^{\lambda x}$, $\frac{ddy}{dx^2} = \lambda^2 e^{\lambda x}$, $\frac{d^3y}{dx^3} = \lambda^3 e^{\lambda x}$, et in genere $\frac{d^ny}{dx^n} = \lambda^n e^{\lambda x}$, facta substitutione pervenietur ad hanc aequationem, postquam scilicet per ei $e^{\lambda x}$ diviserimus,

$$A + B\lambda + C\lambda^2 + D\lambda^3 + E\lambda^4 + \text{etc.} = 0,$$

ex qua valorem ipsius λ definiri oportet. Hinc littera λ totidem valores obtinebit, quoti fuerit ordinis aequatio differentialis proposita, quorum singuli aequationi aeque satisfacient. Qui valores si sint $\alpha, \beta, \gamma, \delta$ etc., integralia quidem particularla erunt

$$y = \mathfrak{A}e^{\alpha x}, \quad y = \mathfrak{B}e^{\beta x}, \quad y = \mathfrak{C}e^{\gamma x} \text{ etc.}$$

Verum ex ipsa aequationis natura perspicuum est aggregata quotcunque horum valorum ideoque etiam omnium perinde satisfacere. Cum igitur aggregatum omnium

$$y = \mathfrak{A}e^{\alpha x} + \mathfrak{B}e^{\beta x} + \mathfrak{C}e^{\gamma x} + \mathfrak{D}e^{\delta x} + \text{etc.}.$$

tot complectatur constantes arbitrarias $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D}$ etc., quoti ordinis differentialis est aequatio proposita, quin haec forma eius sit integrale completum, dubitari nequit. Ascendat aequatio differentialis ad gradum *n*, ut sit

$$Ay + \frac{Bdy}{dx} + \frac{Cddy}{dx^2} + \frac{Dd^3y}{dx^3} + \ldots.. + N\frac{d^ny}{dx^n} = 0$$

atque integrale completum ex *n* partibus constabit, quas ex resolutione huius aequationis algebraicae ordinis *n*, scilicet

$$A + B\lambda + C\lambda^2 + D\lambda^3 + \ldots. + N\lambda^n = 0,$$

definiri oportet. Singuli nimirum eius factores simplices partes illas patefacient; ita si factor sit $\alpha - \lambda$, ex eo integralis pars nascitur $\mathfrak{A}e^{\alpha x}$, quae, uti manifestum est, ex integratione aequationis differentialis simplicis

$$\alpha y - \frac{dy}{dx} = 0$$

nascitur. Simili modo duo factores coniunctim

$$(\alpha - \lambda)(\beta - \lambda) = \alpha\beta - (\alpha + \beta)\lambda + \lambda\lambda$$

integralis portionem $\mathfrak{A}e^{\alpha x} + \mathfrak{B}e^{\beta x}$ suppeditant, quae simul est integrale huius aequationis differentialis secundi gradus

$$\alpha\beta y - (\alpha + \beta)\frac{dy}{dx} + \frac{ddy}{dx^2} = 0.$$

Atque in genere si aequationis illius algebraicae factor sit

$$a + b\lambda + c\lambda^2 + f\lambda^3 + \text{etc.} = 0$$

ex hoc vicissim formetur aequatio differentialis

$$ay+b\frac{dy}{dx}+c\frac{ddy}{dx^2}+f\frac{d^3y}{dx^3}+\text{etc.}=0,$$

cuius integrale completum si sit $y=P$, id simul erit pars integralis aequationis propositae. Atque hoc modo ex singulis factoribus aequationis algebraicae

$$A+B\lambda+C\lambda^2+D\lambda^3+....+N\lambda^n=0$$

derivabuntur singulae partes integralis quaesiti, quae iunctae eius integrale completum constituent, ita ut praecipuum negotium resolutioni huius aequationis innitatur.

COROLLARIUM 1

1126. Si igitur istius aequationis algebraicae omnes factores simplices fuerint reales simulque inaequales, integratio nullam habet difficultatem. Si enim factor simplex sit $f+g\lambda$, integralis pars inde oriunda est $\mathfrak{A}e^{\frac{-fx}{g}}$.

COROLLARIUM 2

1127. Si bini factores simplices sint aequales seu factor fuerit $(f+g\lambda)^2$, pars integralis inde oriunda est $e^{\frac{-fx}{g}}(\mathfrak{A}+\mathfrak{B}x)$. Si factor sit cubus $(f+g\lambda)^3$, inde oritur pars integralis $e^{\frac{-fx}{g}}(\mathfrak{A}+\mathfrak{B}x+\mathfrak{C}x^2)$ et ex factore biquadrato $(f+g\lambda)^4$ huiusmodi pars $e^{\frac{-fx}{g}}(\mathfrak{A}+\mathfrak{B}x+\mathfrak{C}x^2+\mathfrak{D}x^3)$ et ita porro pro quotcunque factoribus aequalibus, uti ex §1120 colligere licet.

COROLLARIUM 3

1128. Si factores occurrant imaginarii, bini coniuncti exhibent factorem trinomium realem, cuius forma ita repraesentatur

$$ff+2fg\lambda\cos.\zeta+gg\lambda\lambda$$

unde deducitur

$$\lambda=-\frac{f}{g}\left(\cos.\zeta\pm\sqrt{-1}\cdot\sin.\zeta\right),,$$

quo cum § 1121 collato fit $\mu=\frac{-f\cos.\zeta}{g}$ et $\nu=\frac{f\sin.\zeta}{g}$. Ex quo pars integralis ex tali factore oriunda erit

$$e^{\frac{-fx\cos.\zeta}{g}}\left(\mathfrak{A}\cos.\frac{fx\cos.\zeta}{g}+\mathfrak{B}\sin.\frac{fx\sin.\zeta}{g}\right).$$

COROLLARIUM 4

1129. Si huiusmodi formae quadratum inter factores occurrat

$$(ff + 2fg\lambda\cos.\zeta + gg\lambda\lambda)^2$$

seu duo huiusmodi factores sint aequales, considerentur quasi infinite parum discrepantes, ut in altero loco $\frac{f}{g}$ sit $\frac{f}{g}(1+\omega)$, et ob

$$e^{\frac{-fx\cos.\zeta}{g}(1+\omega)} = e^{\frac{-fx\cos.\zeta}{g}}\left(1 - \frac{\omega fx}{g}\cos.\zeta\right),$$

$$\cos.\frac{fx\sin.\zeta}{g}(1+\omega) = \cos.\frac{fx\sin.\zeta}{g} - \frac{\omega fx\sin.\zeta}{g}\sin.\frac{fx\sin.\zeta}{g}$$

et

$$\sin.\frac{fx\sin.\zeta}{g}(1+\omega) = \sin.\frac{fx\sin.\zeta}{g} + \frac{\omega fx\sin.\zeta}{g}\cos.\frac{fx\sin.\zeta}{g}$$

ex hoc factore colligitur pars integralis

$$e^{\frac{-fx\cos.\zeta}{g}}\left\{\begin{array}{l}\mathfrak{A}'\cos.\frac{fx\sin.\zeta}{g} - \mathfrak{A}'\frac{\omega fx\cos.\zeta}{g}\cos.\frac{fx\sin.\zeta}{g} - \mathfrak{A}'\frac{\omega fx\sin.\zeta}{g}\sin.\frac{fx\sin.\zeta}{g} \\ +\mathfrak{B}'\sin.\frac{fx\sin.\zeta}{g} - \mathfrak{B}'\frac{\omega fx\cos.\zeta}{g}\sin.\frac{fx\sin.\zeta}{g} + \mathfrak{B}'\frac{\omega fx\sin.\zeta}{g}\cos.\frac{fx\sin.\zeta}{g}\end{array}\right\}$$

cui prior addi debet. Hunc in finem constantes ita contrahamus ponendo

$$\mathfrak{A} + \mathfrak{A}' = \mathfrak{E}, \quad \frac{-\mathfrak{A}'\omega fx\cos.\zeta}{g} + \frac{\mathfrak{B}'\omega fx\sin.\zeta}{g} = \mathfrak{G},$$

$$\mathfrak{B} + \mathfrak{B}' = \mathfrak{F}, \quad \frac{-\mathfrak{A}'\omega fx\sin.\zeta}{g} - \frac{\mathfrak{B}'\omega fx\cos.\zeta}{g} = \mathfrak{H},$$

unde illae constantes utique determinantur, eritque pars integralis respondens

SCHOLION

1130. En ergo universam methodum huiusmodi aequationum differentialium integralia inveniendi

$$Ay + \frac{Bdy}{dx} + \frac{Cddy}{dx^2} + \frac{Dd^3y}{dx^3} + \ldots\ldots + N\frac{d^ny}{dx^n} = 0$$

ita in compendium contractam. Scribatur, ut iste laterculus indicat,

loco	y	$\frac{dy}{dx}$	$\frac{ddy}{dx^2}$	$\frac{d^3y}{dx^3}$	$\frac{d^4y}{dx^4}$	$\cdots$	$\frac{d^ny}{dx^n}$
scribatur	1	z	z^2	z^3	z^4	$\cdots$	z^n

ut oriatur haec aequatio algebraica

$$A + Bz + Cz^2 + Dz^3 + Ez^4 + \ldots + Nz^n = 0,$$

cuius singuli factores reales, sive simplices sive duplicati, notentur, atque insuper casus, quibus duo pluresve sunt inter se aequales, probe observentur. Tum cuiusmodi partes pro integrali quaesito ex singulis factoribus oriantur, ex sequente tabella intelligere licit:

Factores	Partes integralis
$f + gz$	$\mathfrak{A}e^{\frac{-fx}{g}}$
$(f + gz)^2$	$(\mathfrak{A} + \mathfrak{B}x)e^{\frac{-fx}{g}}$
$(f + gz)^3$	$(\mathfrak{A} + \mathfrak{B}x + \mathfrak{C}x^2)e^{\frac{-fx}{g}}$
$(f + gz)^4$	$(\mathfrak{A} + \mathfrak{B}x + \mathfrak{C}x^2 + \mathfrak{D}x^3)e^{\frac{-fx}{g}}$
etc.	etc.
$ff + 2fgz\cos.\zeta + ggzz$	$e^{\frac{-fx\cos.\zeta}{g}}\left(\mathfrak{A}\cos.\frac{fx\cos.\zeta}{g} + \mathfrak{B}\sin.\frac{fx\sin.\zeta}{g}\right).$
$(ff + 2fgz\cos.\zeta + ggzz)^2$	$e^{\frac{-fx\cos.\zeta}{g}}\left\{\begin{array}{l}(\mathfrak{A} + \mathfrak{B}x)\cos.\frac{fx\cos.\zeta}{g} \\ +(\mathfrak{a} + \mathfrak{b}x)\sin.\frac{fx\cos.\zeta}{g}\end{array}\right\}$
$(ff + 2fgz\cos.\zeta + ggzz)^3$	$e^{\frac{-fx\cos.\zeta}{g}}\left\{\begin{array}{l}(\mathfrak{A} + \mathfrak{B}x + \mathfrak{C}xx)\cos.\frac{fx\cos.\zeta}{g} \\ +(\mathfrak{a} + \mathfrak{b}x + \mathfrak{c}xx)\sin.\frac{fx\cos.\zeta}{g}\end{array}\right\}$
etc.	etc.

Pro singulis autem factoribus diversae litterae constantes scribi debent, ut integrale omnibus numeris completum obtineatur.

EXEMPLUM 1

1131. *Aequationis differentialis quarti gradus*

$$y-\frac{2dy}{dx}+\frac{2ddy}{dx^2}-\frac{2d^3y}{dx^3}+\frac{d^4y}{dx^4}=0$$

integrale completum assignare.

Hinc oritur aequatio algebraica

$$1-2z+2zz-2z^3+z^4=0,$$

quae in hos factores resolvitur $(1-z)^2(1+zz)$, quorum prior ob $f=1$ et $g=-1$ praebet hanc partem integralis $(\mathfrak{A}+\mathfrak{B}x)e^x$, alter vero factor ob $f=1,\ \cos.\zeta=0,\ g=1$ et $\sin.\zeta=1$ dat $\mathfrak{A}\cos.x+\mathfrak{B}\sin.x$. Quare integrale completum, quod quaeritur, erit

$$y=(\mathfrak{A}+\mathfrak{B}x)e^x+\mathfrak{C}\cos.x+\mathfrak{D}\sin.x$$

continens quatuor constantes arbitrarias.

Quodsi velimus, ut posito $x=0$ fiat $y=0$, fieri oportet $\mathfrak{A}+\mathfrak{C}=0$; si etiam $\frac{dy}{dx}$ eodem casu evanescere debeat, ob

$$\frac{dy}{dx}=(\mathfrak{A}+\mathfrak{B}+\mathfrak{B}x)e^x-\mathfrak{C}\sin.x+\mathfrak{D}\cos.x$$

fieri debet $\mathfrak{A}+\mathfrak{B}+\mathfrak{D}=0$. Si praeterea $\frac{d^2y}{dx^2}$ evanescere debeat, ob

$$\frac{d^2y}{dx^2}=(\mathfrak{A}+2\mathfrak{B}+\mathfrak{B}x)e^x-\mathfrak{C}\cos.x-\mathfrak{D}\sin.x$$

fieri debet $\mathfrak{A}+2\mathfrak{B}-\mathfrak{D}=0$. Quare his tribus conditionibus satisfaciemus sumendo $\mathfrak{C}=-\mathfrak{A},\ \mathfrak{B}=-\mathfrak{A}$ et $\mathfrak{D}=0$, ita ut sit integrale

$$y=\mathfrak{A}(1-x)e^x-\mathfrak{A}\cos.x.$$

EXAMPLE 2

1132. *Aequationem differentialent quarti ordinis*

$$Ay+\frac{Cddy}{dx^2}+\frac{Ed^4y}{dx^4}=0$$

sumto elemento dx constante integrare.

Aequatio algebraica ad integrationem perducens est

$$A + Czz + Ez^4 = 0,$$

quae semper duos factores duplicatos reales habet, quorum forma duplex esse potest

vel $(aa + 2maz + nzz)(aa - 2maz + nzz)$ vel $(aa + mzz)(aa + nzz)$.
Ex priori est

$$A = a^4,\ C = 2naa - 4mmaa,\ E = nn,$$

ex posteriori vero

$$A = a^4,\ C = (m+n)aa,\ E = mn;$$

semper autem terminum primum A biquadrato a^4 repraesentare licet et prior resolutio locum habet, si E sit numerus positivus et $2naa - C$ seu $2\sqrt{AE} - C$ quoque positivus ideoque $4AE > CC$, posterior vero, si $CC > 4AE$. Tum igitur videndum est, ad quamnam classem singuli factores pertineant, unde sequentes casus occurrent:

1. Si omnes quatuor factores simplices sunt reales, erit

$$A + Czz + Ez^4 = (a+z)(a-z)(b+z)(b-z);$$

haec habebitur aequatio

$$aabby - (aa + bb)\frac{ddy}{dx^2} + \frac{d^4y}{dx^4} = 0 ,$$

cuius integrale completum est

$$y = \mathfrak{A}e^{ax} + \mathfrak{B}e^{-ax} + \mathfrak{C}e^{bx} + \mathfrak{D}e^{-bx}.$$

Ac si sit $b = a$, huius aequationis

$$a^4y - \frac{2aaddy}{dx^2} + \frac{d^4y}{dx^4} = 0$$

integrale completum erit

$$y = (\mathfrak{A} + \mathfrak{B}x)e^{ax} + (\mathfrak{C} + \mathfrak{D}x)e^{-ax}.$$

II. Si duo factores simplices sint reales, duo vero imaginarii, erit

$$A+Czz+E^4=(a+z)(a-z)(bb+zz);$$

haec habebitur aequatio

$$aabby+(aa-bb)\frac{ddy}{dx^2}-\frac{d^4y}{dx^4}=0$$

cuius integrale completum est

$$y=\mathfrak{A}e^{ax}+\mathfrak{B}e^{-ax}+\mathfrak{C}\cos.bx+\mathfrak{D}\sin.bx.$$

III. Si omnes factores simplices sint imaginarii, duo casus sunt evolvendi.
1) Si

$$A+Czz+Ez^4=(aa+zz)(bb+zz),$$

unde huius aequationis

$$aabby+(aa+bb)\frac{ddy}{dx^2}+\frac{d^4y}{dx^4}=0$$

integrale completum erit

$$y=\mathfrak{A}\cos.ax+\mathfrak{B}\sin.ax+\mathfrak{C}\cos.bx+\mathfrak{D}\sin.bx.$$

2) Si

$$A+Czz+Ez^4=(aa+2az\cos.\zeta+zz)(aa-2as\cos.\zeta+zz),$$

unde huius aequationis

$$a^4y-\frac{2aaddy}{dx^2}\cos.2\zeta+\frac{d^4y}{dx^4}=0$$

integrale completum est

$$y=e^{+ax\cos.\zeta}\left(\mathfrak{A}\cos.(ax\sin.\zeta)+\mathfrak{B}\sin.(ax\sin.\zeta)\right)$$
$$+e^{-ax\cos.\zeta}\left(\mathfrak{C}\cos.(ax\sin.\zeta)+\mathfrak{D}\sin.(ax\sin.\zeta)\right)$$

At si sit priori casu $b=a$ seu posteriori $\cos.\zeta=0$, huius aequationis

$$a^4y+\frac{2aaddy}{dx^2}+\frac{d^4y}{dx^4}=0$$

integrale completum est

$$y=(\mathfrak{A}+\mathfrak{B}x)\cos.ax+(\mathfrak{C}+\mathfrak{D}x)\sin.ax.$$

SCHOLION 1

1133. Cum igitur aequationis

$$Ay+\frac{Cddy}{dx^2}+\frac{Ed^4y}{dx^4}=0$$

integrale assignari possit, omnes aequationes, quas inde derivare licet, integrari poterunt. At haec aequatio per *2dy* multiplicata primo per integrationem ad differentialem tertii ordinis reducitur

$$Ayy+\frac{Cd^2y}{dx^2}+\frac{2Edyd^3y-Eddy^2}{dx^4}=\text{Const.}$$

In integrali autem ante invento constantes $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$ ita definire licet, ut haec constans Const. evanescat, ideoque huius aequationis

$$Ayy+\frac{Cd^2y}{dx^2}+\frac{2Edyd^3y-Eddy^2}{dx^4}=0$$

integrale completum erit in nostra potestate. Nunc ponatur $y=e^{\int vdx}$, ut sit $v=\frac{dy}{ydx}$, et ob

$\frac{dy}{dx}=e^{\int vdx}v,\quad \frac{ddy}{dx^2}=e^{\int vdx}\left(\frac{dv}{dx}+vv\right)$ atque $\frac{d^3y}{dx^3}=e^{\int vdx}\left(\frac{ddv}{dx^2}+\frac{3vdv}{dx}+v^3\right)$ aequatio nostra

hanc induit formam

$$A+Cvv+E\left(\frac{2vddv}{dx^2}+\frac{4vvdv}{dx}+v^4-\frac{dv^2}{dx^2}\right)=0\,.$$

Sit porro $dx=\frac{dv}{s}$, ut sit $s=\frac{dv}{dx}=\frac{ddy}{ydx^2}-\frac{d^2y}{yydx^2}$; erit $\frac{ddv}{dx}=ds$ et $\frac{ddv}{dx^2}=\frac{sds}{dv}$,

unde resultat haec aequatio differentialis primi gradus

$$A+Cvv+E\left(\frac{2vsds}{dv}-ss+4vvs+v^4\right)=0\,,$$

cuius relatio inter v et s ita ex relatione inter x et y inventa definitur, ut sit

$$v=\frac{dy}{ydx}\quad \text{et}\quad s=\frac{yddy-dy^2}{yydx^2}.$$

SCHOLION 2

1134. Retenta autem illa constanta per integrationem ingressa, ut habeatur

$$Ayy+\frac{Cd^2y}{dx^2}+\frac{2Edyd^3y-Eddy^2}{dx^4}=G$$

in integrali completo, quo *y* per *x* exprimitur, constantes $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$ quantitati huic *G* conformiter determinari poterunt. Nunc igitur ponatur $dx=\frac{dy}{u}$, ut sit $\frac{dy}{dx}=u$; erit $\frac{ddy}{dx^2}=\frac{udu}{dy}$ et $\frac{d^3y}{dx^2}=d.\frac{udu}{dy}$ ideoque $\frac{d^3y}{dx^3}=\frac{u}{dy}d.\frac{udu}{dy}$. Unde obtinetur haec aequatio differentialis secundi gradus

$$Ayy+Cuu+E\left(\frac{2uu}{dy}d.\frac{udu}{dy}-\frac{uudu^2}{dy^2}\right)=G\,,$$

ubi consideratio elementi pro constante assumti est exuta. Nihil ergo impedit, quominus sumamus *dy* pro constante, fietque

$$Ayy+Cuu+E\left(\frac{2u^3ddu}{dy^2}+\frac{uudu^2}{dy^2}\right)=G$$

quae ergo aequatio etiam integrari potest.

Vel si ponamus $yy=p$ et $uu=q$, sumto elemento *dp* constante prodibit haec aequatio

$$Ap+Cq+E\,\frac{4pqddq+2qdpdq-pdq^2}{dp^2}=G.$$

Vel si in illa aequatione ponatur $u=r^{\frac{2}{3}}$, erit

$$Ayy+Cr^{\frac{4}{3}}+\frac{4}{3}Er^{\frac{5}{3}}\frac{ddr}{dy^2}=G.$$

Quarum formarum integratio sine hoc subsidio maxime ardua videtur.

PROBLEMA 146

1135. *Proposita aequatione differentiali ordinis cuiuscunque* $a^ny\pm\frac{d^ny}{dx^n}=0$, *ubi elementum dx constans est assumtum, eius integrale completum investigare.*

SOLUTIO

Aequatio algebraica solutioni inserviens est $a^n\pm z^n=0$, pro cuius resolutione duos casus considerari convenit, prout signum vel superius vel inferius valeat.

I. Valeat superius, ut haec proposita sit aequatio

$$a^n y + \frac{d^n y}{dx^n} = 0$$

et formulae $a^n + z^n$ factores reales sunt

$$aa - 2az\cos.\tfrac{\pi}{n} + zz,\quad aa - 2az\cos.\tfrac{3\pi}{n} + zz,\ aa - 2az\cos.\tfrac{5\pi}{n} + zz \quad \text{etc.},$$

quorum ultimus est vel $aa - 2az\cos.\frac{n\pi}{n} + zz$ vel $aa - 2az\cos.\frac{(n-1)\pi}{n} + zz$,

prout vel n vel $n-1$ fuerit numerus impar, atque illo quidem casu loco factoris quadrati $aa + 2az + zz$ eius radix $a + z$ sumi debet.

Hinc istius aequationis integrale completum est

$$\begin{aligned} y = \ & e^{ax\cos.\frac{\pi}{n}}\left(\mathfrak{A}\cos.\left(ax\sin.\tfrac{\pi}{n}\right) + \mathfrak{B}\sin.\left(ax\sin.\tfrac{\pi}{n}\right)\right) \\ & + e^{ax\cos.\frac{3\pi}{n}}\left(\mathfrak{C}\cos.\left(ax\sin.\tfrac{3\pi}{n}\right) + \mathfrak{D}\sin.\left(ax\sin.\tfrac{3\pi}{n}\right)\right) \\ & + e^{ax\cos.\frac{5\pi}{n}}\left(\mathfrak{E}\cos.\left(ax\sin.\tfrac{5\pi}{n}\right) + \mathfrak{F}\sin.\left(ax\sin.\tfrac{5\pi}{n}\right)\right) \\ & \text{etc.}, \end{aligned}$$

cuius expressionis, si n sit numerus impar, ultima pars fit $\mathfrak{N}e^{-ax}$. Quod integrale etiam ita potest exhiberi

$$\begin{aligned} y = \ & \mathfrak{A}e^{ax\cos.\frac{\pi}{n}}\cos.\left(ax\sin.\tfrac{\pi}{n} + \mathfrak{a}\right) + \mathfrak{B}e^{ax\cos.\frac{3\pi}{n}}\cos.\left(ax\sin.\tfrac{3\pi}{n} + \mathfrak{b}\right) \\ & + \mathfrak{C}e^{ax\cos.\frac{5\pi}{n}}\cos.\left(ax\sin.\tfrac{5\pi}{n} + \mathfrak{c}\right) + \mathfrak{D}e^{ax\cos.\frac{7\pi}{n}}\cos.\left(ax\sin.\tfrac{7\pi}{n} + \mathfrak{d}\right) \\ & \text{etc.}, \end{aligned}$$

quae forma eousque continuari debet, quoad similes termini recurrant.

II. Si valeat signum inferius propositaque sit haec aequatio

$$a^n y - \frac{d^n y}{dx^n} = 0$$

formulae $a^n - z^n$ factores reales sunt

$$a - z, aa - 2az\cos.\tfrac{2\pi}{n} + zz,\quad aa - 2az\cos.\tfrac{4\pi}{n} + zz,\quad aa - 2az\cos.\tfrac{6\pi}{n} + zz, \text{ etc.},$$

quorum, si n numerus par, ultimus est $a + z$, sin autem impar,

$$aa-2az\cos.\frac{(n-1)\pi}{n}+zz$$

Quare aequationis huius integrale completum est

$$\begin{aligned} y=\mathfrak{A}e^{ax}&+e^{ax\cos.\frac{2\pi}{n}}\left(\mathfrak{B}\cos.\left(ax\sin.\frac{2\pi}{n}\right)+\mathfrak{C}\sin.\left(ax\sin.\frac{2\pi}{n}\right)\right)\\ &+e^{ax\cos.\frac{4\pi}{n}}\left(\mathfrak{D}\cos.\left(ax\sin.\frac{4\pi}{n}\right)+\mathfrak{E}\sin.\left(ax\sin.\frac{4\pi}{n}\right)\right)\\ &+e^{ax\cos.\frac{6\pi}{n}}\left(\mathfrak{F}\cos.\left(ax\sin.\frac{6\pi}{n}\right)+\mathfrak{G}\sin.\left(ax\sin.\frac{6\pi}{n}\right)\right)\\ &\text{etc.,}\end{aligned}$$

quod integrale etiam ita exprimi potest

$$\begin{aligned} y=\mathfrak{A}e^{ax}&+\mathfrak{B}e^{ax\cos.\frac{2\pi}{n}}\cos.\left(ax\sin.\frac{2\pi}{n}+\mathfrak{b}\right)\\ &+\mathfrak{C}e^{ax\cos.\frac{4\pi}{n}}\cos.\left(ax\sin.\frac{4\pi}{n}+\mathfrak{c}\right)\\ &+\mathfrak{D}e^{ax\cos.\frac{6\pi}{n}}\cos.\left(ax\sin.\frac{6\pi}{n}+\mathfrak{d}\right)\\ &\text{etc.,}\end{aligned}$$

quae forma eousque est continuanda, quamdiu termini a prioribus diversi prodeunt.

SCHOLION 1

1136. Pro variis ergo exponentis n valoribus integralia sequenti modo se habebunt ac primo quidem pro aequatione

$$a^n y+\frac{d^n y}{dx^n}=0.$$

I. Aequationis $ay+\frac{dy}{dx}=0$ integrale est $y=\mathfrak{A}e^{-ax}$.

II. Aequationis $a^2y+\frac{ddy}{dx^2}=0$ integrale est $\mathfrak{A}\cos.(ax+\mathfrak{a})$

Ill. Aequationis $a^3y+\frac{d^3y}{dx^3}=0$ integrale est $y=\mathfrak{A}e^{\frac{1}{2}ax}\cos.\left(\frac{ax\sqrt{3}}{2}+\mathfrak{a}\right)+\mathfrak{B}e^{-ax}$.

IV. Aequationis $a^4y+\frac{d^4y}{dx^4}=0$ integrale est $y=\mathfrak{A}e^{\frac{ax}{\sqrt{2}}}\cos.\left(\frac{ax}{\sqrt{2}}+\mathfrak{a}\right)+\mathfrak{B}e^{\frac{-ax}{\sqrt{2}}}\cos.\left(\frac{ax}{\sqrt{2}}+\mathfrak{b}\right)$.

V. Aequationis $a^5y+\frac{d^5y}{dx^5}=0$ integrale est

$$y=\mathfrak{A}e^{ax\cos.36^0}\cos.\left(ax\sin.36^0+\mathfrak{a}\right)+\mathfrak{B}e^{-ax\cos.72^0}\cos.\left(ax\sin.72^0+\mathfrak{b}\right)+\mathfrak{C}e^{-ax}.$$

VI. Aequationis $a^6y+\frac{d^6y}{dx^6}=0$ integrale est

$$y=\mathfrak{A}e^{\frac{ax\sqrt{3}}{2}}\cos.\left(\tfrac{1}{2}ax+\mathfrak{a}\right)+\mathfrak{B}\cos.\left(ax+\mathfrak{b}\right)+\mathfrak{C}e^{\frac{-ax\sqrt{3}}{2}}\cos.\left(\tfrac{1}{2}ax+\mathfrak{c}\right)$$

etc.

Simili autem modo pro altera forma

$$a^ny-\frac{d^ny}{dx^n}=0$$

integrationes ad valores simpliciores exponentis n accommodatae ita se habebunt.

1. Aequationis $ay-\frac{dy}{dx}=0$ integrale est $y=\mathfrak{A}e^{ax}$.

II. Aequationis $a^2y-\frac{ddy}{dx^2}=0$ integrale est $y=\mathfrak{A}e^{ax}+\mathfrak{B}e^{-ax}$.

Ill. Aequationis $a^3y-\frac{d^3y}{dx^3}=0$ integrale est $y=\mathfrak{A}e^{ax}+\mathfrak{B}e^{-\frac{1}{2}ax}\cos.\left(\frac{ax\sqrt{3}}{2}+\mathfrak{b}\right)$.

IV. Aequationis $a^4y-\frac{d^4y}{dx^4}=0$ integrale est $y=\mathfrak{A}e^{ax}+\mathfrak{B}\cos.\left(ax+\mathfrak{b}\right)+\mathfrak{C}e^{-ax}$.

V. Aequationis $a^5y-\frac{d^5y}{dx^5}=0$ integrale est

$$y=\mathfrak{A}e^{ax}+\mathfrak{B}e^{ax\cos.72^0}\cos.\left(ax\sin.72^0+\mathfrak{b}\right)+\mathfrak{C}e^{-ax\cos.36^0}\cos.\left(ax\sin.36^0+\mathfrak{c}\right).$$

VI. Aequationis $a^6y-\frac{d^6y}{dx^6}=0$ integrale est

$$y=\mathfrak{A}e^{ax}+\mathfrak{B}e^{\frac{1}{2}ax}\cos.\left(ax\frac{\sqrt{3}}{2}+\mathfrak{b}\right)+\mathfrak{C}e^{-\frac{1}{2}ax}\cos.\left(ax\sin.ax\frac{\sqrt{3}}{2}+\mathfrak{c}\right)+\mathfrak{D}e^{-ax}$$

sicque, quousque libuerit, progredi licet.

SCHOLION 2

1137. Quamvis methodus, qua hic sum usus, expedite integralia aequationum in proposita forma contentarum suppeditet, a principiis tamen integrationis omnino abhorret. Cum enim aequatio differentialis est altioris gradus, leges integrationis postulant, ut toties seorsim integretur, antequam ad relationem finitam inter binas variabiles perveniatur, et dum singulae integrationes constantem arbitrariam recipiunt, hoc demum modo integrale completum obtinetur. Hactenus autem una quasi operatione integrale postremum eruimus cum omnibus constantibus, quibus id completum redditur; revera scilicet sola coniectura utentes plura integralia particularia sumus adepti atque natura aequationis commode permisit, ut ex iis integrale completum formare liceret. Verum si leges integrandi stricte observare velimus, proposita verbi gratia aequatione differentiali quarti gradus quadruplici integratione opus erit, quarum prima ea reducatur ad aequationem differentialem tertii gradus, tum vero ista per novam integrationem ad aequationem differentialem secundi gradus, quae denuo integrata ad gradum primum perducatur, haecque tandem iterum integrata relationem quaesitam inter binas variabiles patefaciat. Atque hoc modo etiam aequationum hic tractatarum formam resolvere licet, ut per continuas integrationes ad gradus simpliciores redigatur, quibus tandem eadem integralia, quae hic elicuimus, inveniantur.

Cum autem haec methodus latius pateat quam ad formas hic consideratas eiusque ope haec aequatio generalior integrari queat

$$X = A + \frac{Bdy}{dx} + \frac{Cddy}{dx^2} + \frac{Dd^3y}{dx^3} + \text{etc.}$$

denotante X functionem quamcunque ipsius x, cui resolvendae praecedens methodus minime sufficit, novam methodum statim ad hanc formam generaliorem accommodabo.