THE FOUNDATIONS OF INTEGRAL CALCULUS

VOLUME THREE,

IN WHICH A METHOD IS DISCUSSED FOR FINDING FUNCTIONS OF TWO AND MORE VARIABLES, FROM A GIVEN RELATION BETWEEN THE DIFFERENTIALS OF ANY ORDER .

TOGETHER WITH AN APPENDIX ON THE CALCULUS OF VARIATIONS AND WITH A RELATED SUPPLEMENT, ESTABLISHING THE DIRECT INTEGRATON OF PARTICULAR CASES OF DIFFERENTIAL EQUATIONS.

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FINAL BOOK OF THE INTEGRAL CALCULUS

FIRST PART

THE INVESTIGATION OF FUNCTIONS OF TWO VARIABLES FROM A GIVEN RELATION OF THE VARIABLES OF ANY ORDER.

SECTION ONE

THE INVESTIGATION OF FUNCTIONS OF TWO VARIABLES FROM A RELATION OF THE DIFFERENTIALS OF THE FIRST ORDER.

CHAPTER I

CONCERNING THE NATURE OF DIFFERENTIAL EQUATIONS IN GENERAL FROM WHICH FUNCTIONS OF TWO VARIABLES ARE DETERMINED

PROBLEM 1

1. *If z shall be a function of any two variables x and y, to define the nature of the differential equations, from which a relation of the differentials dx, dy and dz may be expressed.*

SOLUTION

Let

$$Pdx+Qdy+Rdz=0$$

be the equation expressing the relation of the variables *dx, dy* and *dz*, in which *P, Q* and *R* shall be any functions of *x, y* and *z.* And indeed in the first place it is necessary, that this equation shall have arisen from the differentialtion of some finite equation, after the differential should be divided by some finite quantity. Therefore a certain multiplier will be given, for example *M,* multiplied by which the formula $Pdx+Qdy+Rdz$ becomes integrable; for unless such a multiplier should arise, the proposed differential equation may become absurd and nothing at all can be said. Hence the whole business here is reduced to this, that some letter may be assigned, with the help of which differential equations of this absurd kind signifying nothing are able to be distinguished from real [*i.e.* genuine or valid] ones.

In the end we shall contemplate this proposed differential equation $Pdx+Qdy+Rdz=0$ as real. Let *M* be the multiplier rendering that integrable, thus so that this formula

$$MPdx+MQdy+MRdz$$

shall be a true differential of some function of the three variables *x, y* and *z;* which function if it shall be put $=V$, this equation $V=\text{Const.}$ shall become the complete integral of the proposed equation. Therefore if either *x* or *y* or *z* should be taken constant, it is required that the individual formulas

$$MQdy+MRdz,\ MRdz+MPdx,\ MPdx+MQdy$$

separately are integrable ; from which from the nature of the differentials there will be

$$\left(\frac{d.MQ}{dz}\right)-\left(\frac{d.MR}{dy}\right)=0,\ \left(\frac{d.MR}{dx}\right)-\left(\frac{d.MP}{dz}\right)=0,\ \left(\frac{d.MP}{dy}\right)-\left(\frac{d.MQ}{dx}\right)=0,$$

from which on expansion these three equations arise:

$$\text{I. } M\left(\frac{dQ}{dz}\right)+Q\left(\frac{dM}{dz}\right)-M\left(\frac{dR}{dy}\right)-R\left(\frac{dM}{dy}\right)=0,$$

$$\text{II. } M\left(\frac{dR}{dx}\right)+R\left(\frac{dM}{dz}\right)-M\left(\frac{dR}{dy}\right)-P\left(\frac{dM}{dy}\right)=0,$$

$$\text{III. } M\left(\frac{dP}{dy}\right)+P\left(\frac{dM}{dy}\right)-M\left(\frac{dQ}{dx}\right)-Q\left(\frac{dM}{dx}\right)=0;$$

of which if the first may be multiplied by P, the second by Q and the third by R, then in the sum all the differentials of M cancel out, and the remaining equation divided by M will be

$$P\left(\frac{dQ}{dz}\right)-P\left(\frac{dR}{dy}\right)+Q\left(\frac{dR}{dx}\right)-Q\left(\frac{dP}{dz}\right)+R\left(\frac{dP}{dy}\right)-R\left(\frac{dQ}{dx}\right)=0,$$

which contains a characteristic [differential equation] distinguishing between true [or valid] and absurd differential equations, and as often as this condition is in place between the quantities P, Q and R so also the proposed differential equation

$$Pdx+Qdy+Rdz=0$$

is a real equation. Otherwise here it is required to remember that a formula of this kind in brackets $\left(\frac{dQ}{dz}\right)$ indicates the value $\frac{dQ}{dz}$, if in the differentiation Q is to be treated as a function of the variable z only ; which likewise is to be understood for the others, which hence always are reduced to finite functions.

[The reader has noted perhaps the close resemblance of Euler's formulas above to those used in modern applications, for example, in the electrostatic version of Maxwell's Equations. Thus, if we consider an electrostatic potential function $\Phi(x,y,z)$, then the electric field associated with this potential in m.k.s. units is given by

$$\vec{E}=-\vec{\nabla}\Phi(x,y,z)=-\frac{\partial\Phi}{\partial x}\hat{i}-\frac{\partial\Phi}{\partial y}\hat{j}-\frac{\partial\Phi}{\partial z}\hat{k}=P\hat{i}+Q\hat{j}+R\hat{k}\ ;$$

from which it follows that with $\Phi(x,y,z)$ constant on an equipotential curve or surface, we have

$$\vec{E}.d\vec{r} = \left(P\hat{i}+Q\hat{j}+R\hat{k}\right).\left(dx\hat{i}+dy\hat{j}+dz\hat{k}\right)=Pdx+Qdy+Rdz=0\ .$$

Likewise, we may note that $\vec{\nabla}\times\vec{E}=\hat{i}\left(\frac{\partial R}{\partial y}-\frac{\partial Q}{\partial z}\right)+\hat{j}\left(\frac{\partial P}{\partial z}-\frac{\partial R}{\partial x}\right)+\hat{k}\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)$;

This is zero in any case in the electrostatic case, but also in general on forming the scalar or dot product

$$\vec{\nabla}\Phi.\left(\vec{\nabla}\times\vec{E}\right)=P\left(\frac{\partial R}{\partial y}-\frac{\partial Q}{\partial z}\right)+Q\left(\frac{\partial P}{\partial z}-\frac{\partial R}{\partial x}\right)+R\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)=0.$$

This vector identity necessarily is zero of course for any well-behaved function, and is the basis for Euler's characteristic equation, or the criterion for the existence of a valid solution. On setting the components of the curl to be *L*, *M*, and *N*, the same equation can be written below as $PL+QM+RN=0$, a useful form of the criterion.
Thus, in his usual unassuming manner, 100 years or so before it was required, Euler was quietly establishing the mathematics for the physics to come in the following century.]

COROLLARY 1

2. Hence for the proposed differential equation between the three variables

$$Pdx+Qdy+Rdz=0\,,$$

before all it is to be discerned whether or not each should have the characteric [equation] in place or not. In the former case the equation will be a real one, and in the latter truly absurd and planely signifying nothing, and neither at any time does such an equation prevail to produce the solution of any problem.

COROLLARY 2

3. The character found can also be expressed in this form

$$\left(\frac{PdQ-QdP}{dz}\right)+\left(\frac{QdR\text{-}RdQ}{dx}\right)+\left(\frac{RdP-PdR}{dy}\right)=0,$$

just as the brackets do not affect finite quantities, but are restricted to the differentiation of a certain variable only.

COROLLARY 3

4. In the same manner if an equation containing this characteristic is divided by *PQR*, it adopts this form

$$\left(\frac{d\,.l\frac{Q}{P}}{Rdz}\right)+\left(\frac{d\,.l\frac{R}{Q}}{Pdx}\right)+\left(\frac{d\,.l\frac{P}{R}}{Qdy}\right)=0,$$

which can be expressed thus :

$$\left(\frac{\frac{dQ}{Q}-\frac{dP}{P}}{Rdz}\right)+\left(\frac{\frac{dR}{R}-\frac{dQ}{Q}}{Pdx}\right)+\left(\frac{\frac{dP}{P}-\frac{dR}{R}}{Qdy}\right)=0.$$

SCHOLIUM 1

5. Just as all differential equations between two variables are real [genuine] always and by these a certain relation between the variables is defined always, thus hence we will learn that things which involve three variables are otherwise, and equations of this kind

$$Pdx+Qdy+Rdz=0$$

cannot be declared with certainty to have a finite relation between the three finite quantities *x, y* and *z*, unless the quantities *P, Q,* R should be prepared thus, so that the characteric equation is required to be found. From which it is understood that an infinite number of differential equations of this kind between three variables can be proposed, for which no straightforwards relation can be agreed upon and which hence therefore clearly may define nothing. Evidently by choice, equations of this kind can be formed with no proposed aim to which they may be applied ; and indeed immediately some certain problem should give rise to a differential equation between the three variables, it is always necessary to agree to assign a characteric quantity to that, since otherwise it may signify nothing. Such an equation signifying nothing for example is $zdx+xdy+ydz=0$ and indeed no function *z* of *x* and *y* can be become known, which satisfies that equation ; also the characteric for our example gives $-x-y-z,$ which quantity, since it does not vanish, declares the absurdity of that equation.

SCHOLIUM 2

6. So that the characteric quantity found is able to be applied more easily in whatever cases offered, first from the equation

$$Pdx+Qdy+Rdz=0$$

the following values are set out :

$$\left(\frac{dQ}{dz}\right)-\left(\frac{dR}{dy}\right)=L,\quad \left(\frac{dR}{dx}\right)-\left(\frac{dP}{dz}\right)=M,\quad \left(\frac{dP}{dy}\right)-\left(\frac{dQ}{dx}\right)=N$$

and our character will be found by this expression

$$LP+MQ+NR,$$

which if it vanishes, the proposed equation is a real [*i.e.* genuine] equation and a certain finite equation is acknowledged to exist; but if this is not reduced to zero, then the proposed equation is absurd and concerning the integration of this, it will indeed not be known.

Thus in the example placed above there will be

$$P = z,\ Q = x,\ R = y,$$

hence

$$L = -1,\ M = -1 \ \text{ and } \ N = -1,$$

from which the characteric $-x - y - z$ indicates absurdity. Also we may offer an example of a genuine equation

$$dx(yy + nyz + zz) - x(y + nz)dy - xzdz = 0,$$

In which on account of

$$P = yy + nyz + zz,\ Q = -xy - nxz \ \text{ and } \ R = -xz$$

there will be

$$L = -nx,\ M = -3z - ny \ \text{ and } N = 3y + 2nz,$$

from which

$$\begin{aligned} LP + MQ + NR &= -nx(yy + nyz + zz) + x(y + nz)(3z + ny) - xz(3y + 2nz) \\ &= x(-nyy - nnyz - nzz + 3yz + 3nzz + nyy + nnyz - 3yz - 2nzz) = 0, \end{aligned}$$

whereby, since here the characteric quantity vanishes, this differential equation is to be taken as a genuine equation. In a similar manner this proposed equation

$$2dx(y + z) + dy(x + 3y + 2z) + dz(x + y) = 0$$

on account of

$$P = 2y + 2z,\ Q = x + 3y + 2z,\ R = x + y$$

makes

$$L = 2 - 1 = 1,\ M = 1 - 2 = -1 \ \text{ and } \ N = 2 - 1 = 1$$

and hence

$$LP + MQ + NR = 2y + 2z - x - 3y - 2z + x + y = 0,$$

from which this differential equation will be genuine.

PROBLEM 2

7. *With the proposed differential equation between the three variables x, y, z, which shall be a real equation, to find the integral of this, from which it may be apparent what function one of these shall be of the remaining two.*

SOLUTION

Let the proposed differential equation be

$$Pdx + Qdy + Rdz = 0,$$

in which *P, Q, R* shall be functions of the this kind of *x, y* et *z,* as should satisfy the reality characteristic as found before. For unless this equation should be real, the integration of this is held to ridicule. Hence we may assume this equation to be a real one, and a relation will be given between the quantities themselves *x, y* and *z* satisfying the proposed equation ; according to which being found by careful deliberation, if one of the variables in the equation of the integral, for example *z*, may be considered constant, from the differential of this put equal to zero the equation must arise

$$Pdx + Qdy = 0.$$

Therefore in turn with one of the variables, e.g. *z,* treated as constant the integration of the differential equation $Pdx + Qdy = 0$, which contains only two variables, will give rise to the integral equation sought, but only if that quantity *z* duly advanced is involved in the constant quantity. From which we deduce this rule for the integration of the proposed equation.

It may be considered with one of the variables, for example *z,* as a constant, so that this equation $Pdx + Qdy = 0$ implying only two variables *x* and *y* may be considered ; then the complete integral equation of this may be investigated, which hence includes an arbitrary constant *C*. Then this constant *C* can be considered as some function of *z* and now also this *z* is a variable , the equation of the integral found may be differentiated again, so that all three *x, y* and *z* may be treated as variable, and the differential equation resulting may be compared with the proposed $Pdx + Qdy + Rdz = 0$, where the functions *P* and *Q* will appear at once, but the function *R* taken with that quantity affected by that element *dz* , will be determined by the reasoning, by which the quantity *z* is introduced into that letter *C*, and thus the equation of the integral sought will be obtained, which likewise will be complete, since always in that a certain arbitrary constant part is left for the letter *C*, since this determination will be desired from the differential of *C*.

COROLLARY 1

8. Hence the integration of this kind of differential equation containing three variables is reduced to the integration of a differential equation between only two variables, which hence as often as it is allowed, can be put in place by the methods treated in the above book.

COROLLARY 2

9. Therefore this integraton can be put in place in three ways, thus as in the first place either z or y or x is considered as a constant. But it is necessary always, that the same equation of the integral should arise, if indeed the differential equation should be valid. realis.

COROLLARY 3

10. But if this method is attempted with an impossible differential equation, the determination of that constant C thus will not succeed, as that variable that has been taken as constant, is involved on its own ; and also can be sought from this criterion of validity.

SCHOLIUM

11. So that this operation can be understood easier, we will first make a trial with that impossible equation

$$zdx + xdy + ydz = 0 .$$

This on taking z as constant will be

$$zdx + xdy = 0 \text{ or } \tfrac{zdx}{x} + dy = 0 ,$$

the integral of this is $zlx + y = C$ with C arising as a function of z. Hence this equation may be differentiated on taking z variable also and on putting $dC = Ddz$, so that D will also be a function of z only, there will be

$$\tfrac{zdx}{x} + dy + dzlx = Ddz \text{ or } zdx + xdy + dz\left(xlx - Dx\right) = 0 ;$$

hence there must be $xlx - Dx = y$ or $D = lx - \tfrac{y}{x}$, which is absurd.

Then in the valid equation

$$2dx\left(y + z\right) + dy\left(x + 3y + 2z\right) + dz\left(x + y\right) = 0$$

the operation set out is thus put in place. There is assumed y constant, so that there shall be

$$2dx(y+z)+dz(x+y)=0 \text{ or } \frac{2dx}{x+y}+\frac{dz}{y+z}=0$$

of which the integral is

$$2l(x+y)+l(y+z)=C,$$

where C also may involve y. Therefore if $dC=Ddy$ and on taking also the differential of the variable y there arises

$$\frac{2dx+2dy}{x+y}+\frac{dy+dz}{y+z}=Ddy$$

or

$$2dx(y+z)+2dy(y+z)+dy(x+y)+dz(x+y)=Ddy(x+y)(y+z),$$

which expression since on taking with the proposed form gives $D=0$, and thus $dC=0$ and C becomes a true constant, thus so that the integral will be

$$(x+y)^2(y+z)=\text{Const}.$$

Therefore we will set out some examples of this kind.

EXAMPLE 1

12. *To investigate the integral of this valid differential equation*

$$dx(y+z)+dy(x+z)+dz(x+y)=0.$$

Indeed in the first place it is apparent that this equation is valid, since there shall be

$$P=y+z,\ Q=x+z,\ R=x+y,$$
$$L=1-1=0,\ M=1-1=0,\ N=1-1=0.$$

Therefore z is assumed constant and the equation will be produced

$$dx(y+z)+dy(x+z)=0 \text{ or } \frac{dx}{x+z}+\frac{dy}{y+z}=0,$$

the integral of which is

$$l(x+z)+l(y+z)=f:z;$$

hence there is established:

$$(x+z)(y+z)=Z,$$

where the nature of the function Z must be elicited from differentiation. Moreover this becomes

$$dx(y+z)+dy(x+z)+dz(x+y+2z)=dZ,$$

from which if the proposed equation is taken away, there is left $2zdz=dZ$, hence $Z=zz+C$, thus so that the complete equation of the integral shall be

$$(x+z)(y+z)=zz+C \text{ or } xy+xz+yz=C,$$

which indeed from the proposed equation itself

$$ydx+zdx+xdy+zdy+xdz+ydz=0$$

is readily elicited, since both members taken together will be integrable.

EXAMPLE 2

13. *To find the complete integral of this valid differential equation*

$$dx(ay-bz)+dy(cz-ax)+dz(bx-cy)=0.$$

The validity of this equation can be shown thus. Since there shall be

$$P=ay-bz,\ Q=cz-ax,\ R=bx-cy,$$

there will be

$$L=2c,\ M=2b,\ N=2a$$

and hence clearly $LP+MQ+NR=0$.

Now z is assumed constant, so that there may be considered

$$\frac{dx}{cz-ax}+\frac{dy}{ay-bz}=0, \text{ hence } \frac{1}{a}l\frac{ay-bz}{cz-ax}=f:z;$$

hence there is put in place

$$\frac{ay-bz}{cz-ax}=Z$$

and on differentiation there is produced

$$\frac{adx(ay-bz)+ady(cz-ax)+adz(bx-cy)}{(cz-ax)^2}=dZ\,,$$

and from the comparison of this with the proposed equation there becomes
$dZ=0$ and $Z=C$, thus so that the equation of the complete integral shall be

$$\frac{ay-bz}{cz-ax}=n \text{ or } ay+nax=(b+nc)z.$$

But if the equation of the integral is put [in the form]

$$Ax+By+Cz=0,$$

these constants must be prepared thus, so that there shall be

$$Ac+Bb+Ca=0\,,$$

[Since $a=B$; $na = A$; and $-b-nc = C$;
thus $Ac+Bb+Ca=nac+ab-ab-nac=0$]

and thus the arbitrary constant C is found more neatly.

COROLLARY

14. Therefore this equation is rendered integrable, if it is divided by $(cz-ax)^2$, and by the same reasoning also these divisors $(ay-bz)^2$ and $(bx-cy)^2$ likewise are available. For the strength of the integral keeps a constant ratio between these divisors.
In as much as if $\frac{ay-bz}{cz-ax}=n$, there will be

$$\frac{bx-cy}{cz-ax}=\frac{-b-nc}{a} \text{ and } \frac{bx-cy}{ay-bz}=\frac{-b-nc}{na}.$$

EXAMPLE 3

15. To investigate the complete integral of this valid equation

$$dx(yy+yz+zz)+dy(zz+xz+xx)+dz(xx+xy+yy)=0\,.$$

The validity of this equation thus is apparent, since there shall be

$$P=yy+yz+zz,\ \ Q=zz+xz+xx,\ R=xx+xy+yy$$

and hence

$$L = 2z + x - x - 2y = 2(z - y),\ M = 2x + y - y - 2z = 2(x - z),$$
$$N = 2y + z - z - 2x = 2(y - x),$$

from which there comes about

$$LP + MQ + NR = 2\left(z^3 - y^3\right) + 2\left(x^3 - z^3\right) + 2\left(y^3 - x^3\right) = 0 .$$

Therefore towards investigating the integral z is assumed constant and there will be

$$\frac{dx}{xx+xz+zz} + \frac{dy}{yy+yz+zz} = 0 ,$$

the integral of which is

$$\frac{2}{z\sqrt{3}}\text{Ang. tang.}\frac{x\sqrt{3}}{2z+x} + \frac{2}{z\sqrt{3}}\text{Ang. tang.}\frac{y\sqrt{3}}{2z+y} = f:z,$$

[For $xx + xz + zz = \left(x + \frac{z}{2}\right)^2 + \left(\frac{z\sqrt{3}}{2}\right)^2$; also from elementary calculus, $tan\left(arctan\left(\frac{x}{a}\right)\right) = \frac{x}{a}$; hence on diff., $\frac{1}{a} = sec^2\left(arctan\left(\frac{x}{a}\right)\right)D.arctan\left(\frac{x}{a}\right) = \left(1 + \left(\frac{x}{a}\right)^2\right)D.arctan\left(\frac{x}{a}\right)$ and hence $\int \frac{adx}{x^2+a^2} = arctan\left(\frac{x}{a}\right)$, and in this case $x \to x + \frac{z}{2}, a \to \frac{z\sqrt{3}}{2}$; an inversion of the argument can be made by adding a factor of $\frac{\pi}{2}$ to the constant z. This gives the symmetric denominator needed for adding the angles, accomplished by the familiar formula for the tangent of the sum of two angles, of which the inverse tangent is then taken.]

which by the collecting together of these angles changes to

$$\frac{2}{z\sqrt{3}}\text{Ang. tang.}\frac{(xz+yz+xy)\sqrt{3}}{2zz+xz+yz-xy} = f:z$$

Therefore there is taken

$$\frac{xz+yz+xy}{2zz+xz+yz-xy} = Z$$

and this equation may be differentiated with all three x, y and z assumed variable and there is produced

$$\frac{2zdx(yy+yz+zz)+2zdy(zz+xz+xx)-2xdz(zz+yz+yy)-2ydz(zz+xz+xx)}{(2zz+xz+yz-xy)^2} = dZ ;$$

since therefore from the equation proposed there shall be

$$dx\left(yy+yz+zz\right)+dy\left(zz+xz+xx\right)=-dz\left(xx+xy+yy\right),$$

with the substitution made there shall be

$$\frac{-\,2zdz\left(xx+xy+yy\right)-2xdz\left(zz+yz+yy\right)-2ydz\left(zz+xz+xx\right)}{\left(2zz+xz+yz-xy\right)^2}=dZ\ ,$$

or

$$\frac{-2dz\left(xxz+xzz+yyz+yzz+xxy+xyy+3xyz\right)}{\left(2zz+xz+yz-xy\right)^2}=dZ\ ,$$

which is reduced to this form

$$\frac{-2dz\left(x+y+z\right)\left(xy+xz+yz\right)}{\left(2zz+xz+yz-xy\right)^2}=dZ$$

But on account of $Z=\frac{xy+xz+yz}{2zz+xz+yz-xy}$ there will be

$$\frac{-2ZZdz\left(x+y+z\right)}{xy+xz+yz}=dZ \text{ or } \frac{-dZ}{ZZ}=\frac{2dz\left(x+y+z\right)}{xy+xz+yz}\ .$$

Hence it is necessary that also $\frac{xy+xz+yz}{x+y+z}$ shall be a function of *z* only, which may be called $\varSigma$, so that there shall be

$$\frac{-dZ}{ZZ}=\frac{2dz}{\varSigma}\ .$$

Truly from the form of the function *Z* alone it is required to finish the task, which can be brought about thus. Since there shall be $Z=\frac{xz+yz+xy}{2zz+xz+yz-xy}$, then there will be

$1+Z=\frac{2zz+2xz+2yz}{2zz+xz+yz-xy}$, and hence $\frac{1+Z}{Z}=\frac{2z\left(x+y+z\right)}{xy+xz+yz}$,

with the help of which value the quantities *x* and *y* are extricated from the differential equation, and there becomes

$$-\frac{dZ}{ZZ}=dz\frac{2\left(x+y+z\right)}{xy+xz+yz}=dz\frac{1+Z}{Zz}\ ,$$

from which

$$\frac{-dZ}{Z(1+Z)} = \frac{dz}{z} = \frac{-dZ}{Z} + \frac{dZ}{1+Z}$$

and on integrating $lz = l\frac{1+Z}{Z} + la.$ Hence

$$\frac{1+Z}{Z} = \frac{z}{a} \quad \text{and} \quad Z = \frac{a}{z-a},$$

thus so that the equation of the integral of the equation shall be

$$\frac{a}{z-a} = \frac{xy+xz+yz}{2zz+xz+yz-xy} \text{ or } xy+xz+yz = a(x+y+z),$$

from which the simplest form can be deduced at once from the equation

$$\frac{2z(x+y+z)}{xy+xz+yz} = \frac{1+Z}{Z} = \frac{z}{a}.$$

COROLLARY

15[a]. Since the complete integral of the proposed equation shall be

$$xy+xz+yz = a(x+y+z) \text{ or } \frac{xy+xz+yz}{x+y+z} = \text{Const.},$$

also the proposed equation is taken to result from the differentiation of this.
From which it is apparent that the equation returns the proposed integral, if it should be divided by $(x+y+z)^2$, or also by $(xy+xz+yz)^2$.
[Thus, Euler is aware that the proposed equation is integrable, since it passes the validity test, in which the original multiplier disappears in the proof, and so that is not an essential ingredient. He has therefore applied a certain amount of dexterity to obtain the solution; he now points out that he could get the integral from an integrating factor much more easily.]

SCHOLIUM

16. From this example it is realised that the determination of the function by integration sometimes cannot be arrived at without being liable to a certain amount of difficulties, if indeed here we may elicit the function Z by a circuitous route. And now here the investigation is able to be put in place much easier; for we may find immediately that this expression

$$\frac{xy+xz+yz}{2zz+xz+yz-xy} = Z = f:z,$$

may be returned much more neatly. Certainly since there shall be

$$\frac{1}{Z} = \frac{2zz+xz+ys-xy}{xy+xz+yz},$$

there will be

$$1+\frac{1}{Z}=\frac{2z(x+y+z)}{xy+xz+yz}$$

and thus

$$\frac{xy+xz+yz}{x+y+z}=\frac{2Zz}{1+Z}=f:z.$$

Therefore with the function remaining Z there is put at once

$$\frac{xy+xz+y}{x+y+z}=\Sigma=f:z$$

and with the differentials taken it becomes clear by itself to become $d\Sigma=0$ and thus $\Sigma=$ Const.

At this point this problem is easier to resolve, if the integral is sought with y also assumed constant ; for then in a similar manner we arrive at an equation of this kind

$$\frac{xy+xz+yz}{x+y+z}=Y=f:y\,;$$

whereby since this expression must be equal to a function of z and of y,
it is necessary, as that shall be constant, and therefore the complete equation of the integral shall be

$$xy+xz+yz=a(x+y+z).$$

EXAMPLE 4

17. *To investigate the complete integral equation of this valid differential equation*

$$dx(xx-yy+zz)-zzdy+zdz(y-x)+\frac{xdz}{z}(yy-xx)=0\,.$$

The validity of this equation can be shown thus. On account of

$$P=xx-yy+zz,\ Q=-zz,\ R=z(y-x)+\frac{x}{z}(yy-xx)$$

there will be

$$L=-3z-\frac{2xy}{z},\ M=-3z+\frac{yy}{z}-\frac{3xx}{z},\ N=-2y,$$

with which calculation undertaken, $LP+MQ+NR$ vanishes.

Now we may assume z constant and we will consider this equation

$$dx(xx-yy+zz)-zzdy=0,$$

of which indeed the integral may not exist, unless we might observe $y = x$ to satisfy that particularly. Moreover hence on putting $y = x + \frac{zz}{v}$ we are able to elicit the complete integral ; indeed it becomes

$$dx\left(zz - \frac{2xzz}{v} - \frac{z^4}{vv} \right) - zzdx + \frac{z^4 dv}{vv} = 0$$

and hence

$$dv - \frac{2xvdx}{zz} = dx$$

which multiplied by $e^{\frac{-xx}{zz}}$ gives this integral

$$e^{\frac{-xx}{zz}} v = \int e^{\frac{-xx}{zz}} dx + f:z,$$

where indeed it is to be noted in the integration of the formula $\int e^{\frac{-xx}{zz}} dx$ the quantity z is treated as a constant and there is to be considered $v = \frac{zz}{y-x}$, thus so that there shall be

$$\int e^{\frac{-xx}{zz}} dx = \frac{e^{\frac{-xx}{zz}} zz}{y-x} + Z.$$

But if now we wish to differentiate this equation on taking z to be variable also, here a difficulty occurs, just as the the differential of the quantity $\int e^{\frac{-xx}{zz}} dx$ arising from the variability of z must be defined. Here from first principles it must be repeated, if there should be $dV = Sdx + Tdz$, to be $\left(\frac{dT}{dx}\right) = \left(\frac{dS}{dz}\right)$ and thus, if z is assumed constant, $T = \int dx\left(\frac{dS}{dz}\right)$. Now in our case there is

$$S = e^{\frac{-xx}{zz}} \quad \text{and} \quad V = \int e^{\frac{-xx}{zz}} dx$$

on assuming z constant; whereby since there shall be $\left(\frac{dS}{dz}\right) = e^{\frac{-xx}{zz}} \frac{2xx}{z^3}$, hence

$$T = \frac{2}{z^3} \int e^{\frac{-xx}{zz}} xxdx .$$

On account of which the whole differential of the quantity $\int e^{\frac{-xx}{zz}} dx$ arises from the variablility of each of x and z :

$$e^{\frac{-xx}{zz}} dx + \frac{2dz}{z^3} \int e^{\frac{-xx}{zz}} xxdx ,$$

to which the differential of the other part $\frac{e^{\frac{-xx}{zz}}zz}{y-x}+Z$ must be equated, which is

$$e^{\frac{-xx}{zz}}\left(\frac{2zdz}{y-x}-\frac{2zzdy-zzdx}{(y-x)^2}+\frac{2xxdz-2zdx}{(y-x)}\right)+dZ.$$

Now at this point the integral $\int e^{\frac{-xx}{zz}}xxdx$ disturbs the formula in which z is considered as constant ; but it is able to be reduced to that previous integral $\int e^{\frac{-xx}{zz}}dx$, if there is put

$$\int e^{\frac{-xx}{zz}}xxdx=Ae^{\frac{-xx}{zz}}x+B\int e^{\frac{-xx}{zz}}dx\ ;$$

for with x alone considered to be variable on differentation

$$xxdx=Adx-\frac{2Axxdx}{zz}+Bdx,$$

hence

$$A=\tfrac{1}{2}zz \text{ and } B=-A=\tfrac{1}{2}zz,$$

thus so that there shall be

$$\int e^{\frac{-xx}{zz}}xxdx=-\tfrac{1}{2}e^{\frac{-xx}{zz}}xzz+\tfrac{1}{2}zz\int e^{\frac{-xx}{zz}}dx.$$

Whereby since there shall be

$$\int e^{\frac{-xx}{zz}}dx=\frac{e^{\frac{-xx}{zz}}zz}{y-x}+Z,$$

there will be

$$\int e^{\frac{-xx}{zz}}xxdx=-\tfrac{1}{2}e^{\frac{-xx}{zz}}xzz+\frac{e^{\frac{-xx}{zz}}z^4}{2(y-x)}+\tfrac{1}{2}Zzz.$$

Hence with the substitution made, this differential equation arises

$$e^{\frac{-xx}{zz}}\left(dx-\frac{xdz}{z}+\frac{zdz}{y-x}\right)+\frac{Zdz}{z}$$
$$=e^{\frac{-xx}{zz}}\left(\frac{2zdz}{y-x}-\frac{zzdy}{(y-x)^2}+\frac{zzdx}{(y-x)^2}-\frac{2xdx}{y-x}+\frac{2xxdz}{z(y-x)}\right)+dZ,$$

which changes into this form

$$e^{\frac{-xx}{zz}}\left(\frac{dx(y+x)}{y-x}-\frac{zzdx}{(y-x)^2}+\frac{zzdy}{(y-x)^2}-\frac{zdz}{y-x}-\frac{x(y+x)dz}{z(y-x)}\right)=\frac{zdZ-Zdz}{z}$$

$$=e^{\frac{-xx}{zz}}\left(\frac{2zdz}{y-x}-\frac{zzdy}{(y-x)^2}+\frac{zzdx}{(y-x)^2}-\frac{2xdx}{y-x}+\frac{2xxdz}{z(y-x)}\right)+dZ,$$

or

$$\frac{e^{\frac{-xx}{zz}}}{(y-x)^2}\left(dx(yy-xx-zz)+zzdy-zdz(y-x)-\frac{xdz}{z}(yy-xx)\right)=\frac{zdZ-Zdz}{z};$$

whereby since with the proposed equations gathered together it is evident that there must be

$$zdZ-Zdz=0 \text{ or } Z=nz,$$

thus so that the complete integral of the proposed equation shall be

$$\int e^{\frac{-xx}{zz}}dx=\frac{e^{\frac{-xx}{zz}}zz}{y-x}+nz,$$

if indeed in the integral $\int e^{\frac{-xx}{zz}}dx$ the quantity z may be considered constant.

COROLLARY

18. Therefore the equation of the integral is returned, if it may be multiplied by

$$\frac{1}{(y-x)^2}e^{\frac{-xx}{zz}};$$

and then the integral is that equation that we have found.

SCHOLIUM 1

19. This example is especially noteworthy, since in the solution of this certain tricks have been called in to help, for which in the preceding there was no need. But by the formula the integral $\int e^{\frac{-xx}{zz}}dx$ may be considered not to be determined well enough. Since indeed in that z is considered constant, the constant being introduced by integration through nz is not defined, if indeed a rule is not prescribed, following which the integral is required to be taken $\int e^{\frac{-xx}{zz}}dx$, whither thus so that it vanishes on making $x = 0$, or by some other way. But this doubt may be refuted, if we divide the equation found by z, in order that the formula of the integral shall be $\int e^{\frac{-xx}{zz}}\frac{dx}{z}$ since everywhere $\frac{dx}{z}$ shall be $d.\frac{x}{z}$, that being evident to express a certain function of $\frac{x}{z}$, and if there is put $\frac{x}{z}=p$, to be our integral equation

$$\int e^{-pp}dp+\text{Const.}=e^{-pp}\frac{z}{y-x};$$

nor here shall that further condition be consided to have a place, by which in the formula of the integral the quantity z should be considered as constant, but the integral may be determined in the same way, if the equation should contain two variables. If we should consider carefull this circumstance, the full differential of the formula $\int e^{\frac{-xx}{zz}}dx$ from the variablilty of each of x and z should give rise to no difficulty. For after we have come upon the equation

$$\int e^{\frac{-xx}{zz}}dx=e^{\frac{-xx}{zz}}\frac{zz}{y-x}+f:z,$$

thus we may represent that :

$$\int e^{\frac{-xx}{zz}}\frac{dx}{z}=\int e^{\frac{-xx}{zz}}d.\frac{x}{z}=e^{\frac{-xx}{zz}}\frac{z}{y-x}+Z;$$

where since the variability of z shall also be introduced into the integral formula , if that should be differentiated with all variables x, y and z taken, there shall arise

$$e^{\frac{-xx}{zz}}\left(\frac{dx}{z}-\frac{xdz}{zz}\right)=e^{\frac{-xx}{zz}}\left(\frac{dz}{y-x}+\frac{zdx-zdy}{(y-x)^2}-\frac{2xdx}{z(y-x)}+\frac{2xxdz}{zz(y-x)}\right)+dZ,$$

or

$$e^{\frac{-xx}{zz}}\left(\frac{dx(y+x)}{z(y-x)}-\frac{zdx}{(y-x)^2}+\frac{zdy}{(y-x)^2}-\frac{xdz(y+x)}{zz(y-x)}-\frac{dz}{y-x}\right)=dZ,$$

which is reduced to this form

$$\frac{e^{\frac{-xx}{zz}}}{z(y-x)^2}\Big(dx\big(yy-xx-zz\big)+zzdy-zdz\big(y-x\big)-\frac{xdz}{z}\big(yy-xx\big)\Big)=dZ,$$

from which it is apparent that there must be $dZ=0$ and $Z=\text{Const.}$, and thus the equation of the integral is elicited as before.

SCHOLIUM 2

20. The same integral may be produced, if in place of z either of the remaining x or y should be assumed as constant ; where in general it is convenient to be noted, if the equation of this kind

$$Pdx+Qdy+Rdz=0$$

with z constant has been allowed to be treated, also the resolution, whichever of the three variables is assumed to be constant, must succeed, even if whenever that should be less evident. Thus in the proposed equation if y should be considered constant, on resolving there will be this equation

$$dx(xx+zz-yy)-zdz(x-y)-\tfrac{xdz}{z}(xx-yy)=0\,;$$

which multiplied by z changes into this form

$$(zdx-xdz)(xx+zz-yy)+yzzdz=0\,,$$

it is apparent that is easily rendered simpler on putting $x=pz;$ then indeed on account of

$$zdx-xdz=zzdp$$

there will be produced

$$dp(ppzz+zz-yy)+ydz=0\,.$$

Again on putting $z=qy$ there becomes

$$dp(ppqq+qq-1)+dq=0\,,$$

since that is satisfied by $q=\frac{1}{p}$, there is put in place $q=\frac{1}{p}+\frac{1}{r}$ and there will be had

$$dp\left(\tfrac{2p}{r}+\tfrac{pp}{rr}+\tfrac{1}{pp}+\tfrac{2}{pr}+\tfrac{1}{rr}\right)-\tfrac{dp}{pp}-\tfrac{dr}{rr}=0$$

or

$$dp\left(2ppr+p^3+2r+p\right)-pdr=0 \text{ or } dr-\tfrac{2rdp(pp+1)}{p}=dp(pp+1),$$

which multiplied by $\frac{1}{pp}e^{-pp}$ and integrated gives

$$e^{-pp}\tfrac{r}{pp}=\int e^{-pp}\tfrac{dp(pp+1)}{pp}.$$

But

$$\int e^{-pp}\frac{dp}{pp} = -e^{-pp}\frac{1}{p} - 2\int e^{-pp}dp$$

from which

$$e^{-pp}\left(\frac{r}{pp}+\frac{1}{p}\right) = -\int e^{-pp}dp.$$

Since now there shall be $p=\frac{x}{z}$ and $\frac{1}{r}=\frac{z}{y}-\frac{z}{x}=\frac{z(x-y)}{xy}$ there will be

$$r=\frac{xy}{z(x-y)},\quad \frac{r}{pp}=\frac{yz}{x(x-y)}\quad \text{and}\quad \frac{r}{pp}+\frac{1}{p}=\frac{z}{x-y}$$

From which our equation will be

$$\int e^{\frac{-xx}{zz}}d.\frac{x}{z} = e^{\frac{-xx}{zz}}\frac{z}{y-x} + f:y$$

the differential of which, if also y is considered to be variable, since the equation proposed prepared before will give as before $f:y=\text{Const.}$

Since otherwise in these examples the variables *x, y, z* everywhere satisfy a number of the same dimensions, I will set out a general method for treating equations of this kind.

PROBLEM 3

21. *If in the differential equation*

$$Pdx+Qdy+Rdz=0$$

P, Q, R should be homogeneous functions of x, y and z themselves of the same number of dimensions, to investigate the integral of this , if indeed it should be valid.

SOLUTION

Let n be the number of dimensions , which the three variables *x, y* and *z* establish in the functions *P, Q, R* , and on putting $x=pz$ and $y=qz$ there becomes

$$P=z^nS,\quad Q=z^nT\quad \text{et}\quad R=z^nV\,,$$

thus so that S, T, V shall become functions of the two variables only p and q. Since now there shall be

$$dx=pdz+zdp\quad \text{and}\quad dy=qdz+zdq\,,$$

our equation hence adopts this form

$$dz\left(pS+qT+V\right)+Szdp+Tzdq=0 \text{ or } \tfrac{dz}{z}+\tfrac{Sdp+Tdq}{pS+qT+V}=0,$$

which equation cannot be valid, unless the formula of the differentials $\frac{Sdp+Tdq}{pS+qT+V}$ involving the two variables p and q should be integrable by itself; which eventuates, if there should be :

$$\left(qT+V\right)\left(\tfrac{dS}{dq}\right)+PT\left(\tfrac{dS}{dp}\right)-\left(pS+V\right)\left(\tfrac{dT}{dp}\right)-qS\left(\tfrac{dT}{dq}\right)-S\left(\tfrac{dV}{dq}\right)+T\left(\tfrac{dV}{dp}\right)=0.$$

Therefore as long as this character has to be taken into account, our equation will be valid and the integral of this will be

$$lz+\int\tfrac{Sdp+Tdq}{pS+qT+V}=\text{Const.},$$

where there is a need only, that in place or the letters p and q the values assumed are put back in place.

COROLLARY 1

22. Thus in our first example (§ 12) since there shall be

$$P=y+z,\quad Q=x+z,\quad R=x+y,$$

then there will be

$$S=q+1,\quad T=p+1,\quad V=p+q$$

and

$$\tfrac{dz}{z}+\tfrac{(q+1)dp+(p+1)dq}{2pq+2p+2q}=0,$$

the integral of which is

$$lz+\tfrac{1}{2}l\left(pq+p+q\right)=\tfrac{1}{2}l\left(xy+xz+yz\right)=C$$

or

$$xy+xz+yz=C.$$

COROLLARY 2

23. In the second example (§ 13) there is

$$P=ay-bz,\ Q=cz-ax,\ R=bx-cy,$$

hence

$$S=aq-b,\ T=c-ap,\ V=bp-cq.$$

Therefore

$$\frac{dz}{z}+\frac{(aq-b)dp+(c-ap)dq}{0}=0$$

and hence

$$(aq-b)dp+(c-ap)dq=0$$

and on integrating,

$$l\frac{aq-b}{c-ap}=l\frac{ay-bz}{cz-ax}=C.$$

[One wonders at Euler not applying H's Rule here and below to such expressions.]

COROLLARY 3

24. In the third example (§ 15) there becomes

$$S=qq+q+1,\ T=pp+p+1\ \text{ et }\ V=pp+pq+qq$$

and hence

$$\frac{dz}{z}+\frac{dp(qq+q+1)+dq(pp+p+1)}{pppq+pqq+pp+3pq+qq+p+q}=0,$$

which denominator is $=(p+q+1)(pq+p+q)$, from which this fraction is resolved into these two

$$\frac{-dp-dq}{p+q+1}+\frac{dp(q+1)+dq(p+1)}{pq+p+q},$$

from which the integral arises with the logarithms replaced by numbers

$$\frac{z(pq+p+q)}{p+q+1}=\frac{xy+xz+yz}{x+y+z}=C.$$

COROLLARY 4

25. In example four (§ 17) there becomes

$$S=pp-qq+1,\ T=-1,\ V=q-p+p(qq-pp)$$

and hence

$$\frac{dz}{z}+\frac{dp(pp-qq+1)-dq}{0}=0$$

and thus

$$dq=dp(pp-qq+1).$$

Therefore since it is satisfied by $q=p$, there is put $q=p+\frac{1}{r}$; there becomes $dr-2prdp=dp$ and on integrating

$$e^{-pp}r = \int e^{-pp}dp = e^{-pp}\,\tfrac{1}{q-p}\,,$$

thus so that the integral shall be

$$e^{\frac{-xx}{zz}}\,\tfrac{z}{y-x} = \int e^{\frac{-xx}{zz}}\,d.\tfrac{x}{z} + \text{Const.}$$

SCHOLION

26. Therefore since differential equations involving three variables themselves present no special difficulty, because the resolution of these, if indeed they should be valid, are possible always to be reduced to differential equations of two variables, I shall not pursue this argument further. Which indeed concerns differential equations of this kind of three variables, in which the differentials themselves rise to higher dimensions, just as there shall be

$$Pdx^2 + Qdy^2 + Rdz^2 + 2Sdxdy + 2Tdxdz + 2Vdydz = 0\,,$$

concerning these it is understood generally, unless they are able to be reduced by the extraction of roots to the form

$$Pdx + Qdy + Rdz = 0\,,$$

that they are invalid always. Indeed in whatever manner the equation of the integral should be prepared, from that the value of *z* thus should be able to be defined, so that *z* is equal to a function of the two variables *x* and *y,* from which there becomes $dz = pdx + qdy$, nor do these variables *x* and *y* depend on each other in any way. Therefore this value $pdx + qdy$ substituted in place of dz in the differential equation thus must satisfy the requirement that all the terms cancel each other out, but because that may not happen, if from the resolution of the equation *dz* may be defined thus, so that the differentials *dx* are *dy* may be involved with the signs of the roots. Hence the equation cannot be valid, to be put in place of that quoted in the example, as on resolution it shall give

$$dz = \frac{-Tdx - Vdy \pm \sqrt{\left((TT-PR)dx^2 + 2(TV-RS)dxdy + (VV-QR)\,dy^2\right)}}{R}\,,$$

unless the root is able to be extracted, that is, unless the equation can be resolved into factors of the form

$$Pdx + Qdy + Rdz\,.$$

And even if this comes about but these unequal factors are put in place, the equation still will not be valid, unless the criterion treated above shall be satisfied.

From these it is evident, truly that equations of this kind, which indeed involve four or more variables, shall be of greater difficulty.

PROBLEM 4

27. *If V shall be a function of any two variables x and y, but in the formula of the integral* $\int Vdx$ *the quantity y shall be considered constant, to define the the differential of this form* $\int Vdx$, *if in additon it is assumed that both x and y are variable.*

SOLUTION

This form of the integral may be put as

$$\int Vdx = Z$$

and everywhere Z will be a function of both variables x and y, even if in that integration y may be considered constant. But it is evident, if in turn in the differentiation y is assumed constant, to become $dZ = Vdx$. Whereby if also y is put in place to be variable, the differential of this $Z = \int Vdx$ will have a form of this kind

$$dZ = Vdx + Qdy$$

and the quaestion is reduced to this, that this quantity Q may be determined. But because the form $Vdx + Qdy$ is a true differential, it is necessary that there shall be $\left(\frac{dV}{dy}\right) = \left(\frac{dQ}{dx}\right)$ and hence $dx\left(\frac{dQ}{dx}\right) = dx\left(\frac{dV}{dy}\right)$. But $dx\left(\frac{dQ}{dx}\right)$ is the differential of Q, if y is considered constant ; from which Q may be found, if the formula $dx\left(\frac{dV}{dy}\right)$ thus may be integrated, as y is treated as a constant, or there shall be

$$Q = \int dx\left(\frac{dV}{dy}\right).$$

On account of which the differential of the formula $Z = \int Vdx$ arising from the variablility of each of x and y will be

$$dZ = Vdx + dy\int dx\left(\frac{dV}{dy}\right).$$

COROLLARY 1

28. Since V is a function of x and y, if there is put $dV = Rdx + Sdy$, then there shall be $S = \left(\frac{dV}{dy}\right)$, from which there becomes

$$dZ = d.\int Vdx = Vdx + dy\int Sdx,$$

evidently in the integration of the formula $\int Sdx$, and just as of the formula $\int Vdx$, only the quantity x is considered to be variable.

COROLLARIUM 2

29. If V should be a homogeneous function of the two variables x and y with the number of dimensions present $= n$, on putting $dV = Rdx + Sdy$ there will be $Rx + Sy = nV$ and thus $S = \frac{nV}{y} - \frac{Rx}{y}$ hence,

$$\int Sdx = \frac{n}{y}\int Vdx - \frac{1}{y}\int Rxdx.$$

But on account of y being constant there will be $Rdx = dV$, hence

$$\int Rxdx = \int xdV = Vx - \int Vdx \text{ and thus } \int Sdx = \frac{n+1}{y}\int Vdx - \frac{Vx}{y}$$

and

$$dZ = d.\int Vdx = Vdx - \frac{Vxdy}{y} + \frac{(n+1)dy}{y}\int Vdx.$$

COROLLARIUM 3

30. Likewise it is found easier by consideration, as the function $Z = \int Vdx$ shall become homogeneous of dimensions $n+1$, whereby on putting $dZ = Vdx + Qdy$ there will be $Vx + Qy = (n+1)Z$ and thus $Q = \frac{(n+1)Z}{y} - \frac{Vx}{y}$ as before.

SCHOLION

31. Now regarding the problem before, and indeed one that I have used in a certain preceding book, from what was to become from that, I have not considered treating it in that given work, since this book is occupied with functions of two or more variables.

Moreover the particular work did not depend on differential equations of this kind, such as I have shown how to integrate in this chapter, because indeed to be completely brief, as with the differentiation of functions of two variables x and y the duplicate formulas $\left(\frac{dV}{dx}\right)$ and $\left(\frac{dV}{dy}\right)$ depend on a function V of this kind being present, here we will consider mainly questions of this kind , in which such a function V is to be defined from some relation of these two formulas $\left(\frac{dV}{dx}\right)$ and $\left(\frac{dV}{dy}\right)$. But this relation is expressed by an equation between these formulas and the two variables x and $y,$ since also that function sought V can be introduced, from the nature of which equation a division of the treatment is to be desired. Clearly the general problem thus itself is considered, in which that division is occupied solving, so that the function V of the two variables x and y may be found, which satisfies some equation between the proposed quantities $x, y, V, \left(\frac{dV}{dx}\right)$ and $\left(\frac{dV}{dy}\right)$. But if in this equation only one of the two differential formulas $\left(\frac{dV}{dx}\right)$ or $\left(\frac{dV}{dy}\right)$ should be present, the resolution is not difficult and it is reduced to the case of the differential equation involving only two variables ; but when both these formulas are present in the proposed equation, the question is much harder and on many occasions equations indeed cannot be resolved, even if the resolution of the differential equation allowed includes only two variables ; for in this labour, just as often as the resolution is allowed to be reduced to the integration of a differential equation between two variables, so a problem will be considered for the resolution.

Therefores since from the equation the proposed formula $\left(\frac{dV}{dy}\right)$ is equal to some function put together from the quantities x, y, V and $\left(\frac{dV}{dx}\right)$, from the nature of this function we will assign the following treatment, according as it should be simpler or only the formula $\left(\frac{dV}{dx}\right)$, or in addition a single term from the rest, or even two or thus all the terms taken together. For with this order in place it will appear most easy to decide how many terms are allowed to be applied and how many at this stage should be desired. Besides several aids come to mind concerning the transformation of the two formulas of the differentials to other variables are to be explained.

THE DIVISION OF THIS SECTION

32. So that the parts which it is convenient to treat in this section may be set out to be understood more clearly, as these questions are concerned with functions of two variables, let x and y be the two variables and z a function of these to be defined from a certain differential relation, thus so that a finite equation between x, y and z may be required. Moreover we may put $dz = pdx + qdy$, thus so that it will be indicated only by the customery $p = \left(\frac{dz}{dx}\right)$ and $q = \left(\frac{dz}{dy}\right)$ and thus p and q shall be differential formulas, which enter into the proposed relation. Therefore in general there will be from that relation some equation between the proposed quantities p, q, x, y and z and this section may be perfectly resolved, if a method may be put in place from some given equation, and between these quantities p, q, x, y and z an equation can be elicited between x, y and $z;$ but since in general for functions of a single variable it is not indeed possible to excel, much less here is to be expected; from which it is only convenient to set out these cases which allow a resolution.

Moreover in the first place the resolution will succeed, if in the proposed equation either of the differential formulas p or q clearly is lacking, thus so that an equation either between p, x, y and z or between q, x, y and z is proposed.

Then the equations, which contain only the two differential formulas p and q, thus so that one must be some function of the other, is conveniently allowed to be resolved.

Then therefore there are the following equations, which besides p and q are completed by a single finite quantity of x, y or z, from which we may observe generally that cases of this kind are able to be resolved.

Again the order postulates, that we may progress to equations which besides the two differential formulas p and q in addition involve two of the finite quantities either x and y, x and z, or y and z; and finally we will proceed with the aid of transformations to be explained to the resolution of equations involving all the letters p, q, x, y and z.

INSTITUTIONUM CALCULI INTEGRALIS

VOLUMEN TERTIUM ,

IN QUO METHODUS INVENIENDI FUNCTIONES DUARUM ET PLURIUM VARIABILIUM, EX DATA RELATIONE DIFFERENTIALIUM CUIUSVIS GRADUS PERTRACTATUR.

UNA CUM APPENDICE
DE CALCULO VARIATIONUM ET SUPPLEMENTO,
EVOLUTIONEM CASUUM PRORSUS SINGULARIUM CIRCA INTEGRATIONEM AEQUATIONUM DIFFERENTIALUM CONTINENTE.

AUCTORE
LEONHARDO EULERO
ACAD. SCIENT. BORUSSIAE DIRECTORE VICENNALI ET SOCIO ACAD. PETROP. PARISIN. ET LONDIN.

PETROPOLI,
Impensis Academiae Imperialis Scientiarum
1770.

LIBER POSTERIOR
PARS PRIMA
INVESTIGATIO FUNCTIONUM DUARUM VARIABILIUM

CALCULI INTEGRALIS
LIBER POSTERIOR.
PARS PRIMA
SEU
INVESTIGATIO FUNCTIONUM DUARUM
VARIABILIUM EX DATA DIFFERENTIALIUM
CUIUSVIS GRADUS RELATIONE.

SECTIO PRIMA
INVESTIGATIO DUARUM VARIABILIUM
FUNCTIONUM EX DATA DIFFERENTIALIUM
PRIMI GRADVS RELATIONE.

CAPUT I
DE NATURA AEQUATIONUM DIFFERENTIALIUM QUIBUS FUNCTIONES DUARUM VARIABILIUM DETERMINANTUR IN GENERE

PROBLEMA 1

1. *Si z sit functio quaecunque duarum variabilium x et y, definire indolem aequationis differentialis, qua relatio differentialium dx, dy et dz exprimitur.*

SOLUTIO

Sit

$$Pdx+Qdy+Rdz=0$$

aequatio relationem differentialium *dx, dy* et *dz* exprimens, in qua *P, Q* et *R* sint functiones quaecunque ipsarum *x, y* et *z.* Ac primo quidem necesse est, ut haec aequatio nata sit ex differentiatione aequationis cuiuspiam finitae, postquam differentiale per quampiam quantitatem fuerit divisum. Dabitur ergo quidam multiplicator, puta *M,* per quem formula $Pdx+Qdy+Rdz$ multiplicata fiat integrabilis; nisi enim talis multiplicator existeret, aequatio differentialis proposita foret absurda nihilque omnino declararet. Totum ergo negotium huc redit, ut character assignetur, cuius ope huiusmodi aequationes differentiales absurdae nihilque significantes a realibus dignosci queant.

Hunc in finem contemplemur aequationem propositam $Pdx+Qdy+Rdz=0$ tanquam realem. Sit *M* multiplicator eam reddens integrabilem, ita ut haec formula

$$MPdx+MQdy+MRdz$$

sit verum differentiale cuiuspiam functionis trium variabilium *x, y* et *z;* quae functio si ponatur $=V$ haec aequatio $V=\text{Const.}$ futura sit integrale completum aequationis propositae. Sive igitur *x* sive *y* sive *z* accipiatur constans, singulas has formulas

$$MQdy+MRdz,\ MRdz+MPdx,\ MPdx+MQdy$$

seorsum integrabiles esse oportet; unde ex natura differentialium erit

$$\left(\frac{d.MQ}{dz}\right)-\left(\frac{d.MR}{dy}\right)=0,\ \left(\frac{d.MR}{dx}\right)-\left(\frac{d.MP}{dz}\right)=0,\ \left(\frac{d.MP}{dy}\right)-\left(\frac{d.MQ}{dx}\right)=0,$$

unde per evolutionem hae tres oriuntur aequationes

$$\text{I. } M\left(\frac{dQ}{dz}\right)+Q\left(\frac{dM}{dz}\right)-M\left(\frac{dR}{dy}\right)-R\left(\frac{dM}{dy}\right)=0,$$
$$\text{II. } M\left(\frac{dR}{dx}\right)+R\left(\frac{dM}{dz}\right)-M\left(\frac{dR}{dy}\right)-P\left(\frac{dM}{dy}\right)=0,$$
$$\text{III. } M\left(\frac{dP}{dy}\right)+P\left(\frac{dM}{dy}\right)-M\left(\frac{dQ}{dx}\right)-Q\left(\frac{dM}{dx}\right)=0;$$

quarum si prima per P, secunda per Q et tertia per R multiplicetur, in summa omnia differentialia ipsius M se tollent et reliqua aequatio per M divisa erit

$$P\left(\frac{dQ}{dz}\right)-P\left(\frac{dR}{dy}\right)+Q\left(\frac{dR}{dx}\right)-Q\left(\frac{dP}{dz}\right)+R\left(\frac{dP}{dy}\right)-R\left(\frac{dQ}{dx}\right)=0,$$

quae continet characterem aequationes differentiales reales ab absurdis discernentem, et quoties inter quantitates P, Q et R haec conditio locum habet, toties aequatio differentialis proposita

$$Pdx+Qdy+Rdz=0$$

est realis. Caeterum hic meminisse oportet huiusmodi formulam uncinulis inclusam $\left(\frac{dQ}{dz}\right)$ significare, valorem $\frac{dQ}{dz}$ si in differentiatione ipsius Q , sola quantitas z ut variabilis tractetur; quod idem de caeteris est tenendum, quae ergo semper ad functiones finitas reducuntur.

COROLLARIUM 1

2. Proposita ergo aequatione differentiali inter tres variabiles

$$Pdx+Qdy+Rdz=0$$

ante omnia dispiciendum est, utrum character inventus locum habeat necne. Priori casu aequatio erit realis, posteriori vero absurda et nihil plane significans neque unquam ad talem aequationem ullius problematis solutio perducere valet.

COROLLARIUM 2

3. Character inventus etiam hoc modo exprimi potest

$$\left(\frac{PdQ-QdP}{dz}\right)+\left(\frac{QdR\text{-}RdQ}{dx}\right)+\left(\frac{RdP-PdR}{dy}\right)=0,$$

quandoquidem uncinulae non quantitates finitas afficiunt, sed solam differentiationem ad certam variabilem restringunt.

COROLLARIUM 3

4. Simili modo si aequatio haec characterem continens per *PQR* dividatur, ea hanc formam induet

$$\left(\frac{d.l\frac{Q}{P}}{Rdz}\right)+\left(\frac{d.l\frac{R}{Q}}{Pdx}\right)+\left(\frac{d.l\frac{P}{R}}{Qdy}\right)=0,$$

quae etiam ita exprimi potest

$$\left(\frac{\frac{dQ}{Q}-\frac{dP}{P}}{Rdz}\right)+\left(\frac{\frac{dR}{R}-\frac{dQ}{Q}}{Pdx}\right)+\left(\frac{\frac{dP}{P}-\frac{dR}{R}}{Qdy}\right)=0.$$

SCHOLION 1

5. Quemadmodum omnes aequationes differentiales inter binas variabiles semper sunt reales semperque per eas relatio certa inter ipsas variabiles definitur, ita hinc discimus rem secus se habere in aequationibus differentialibus, quae tres variabiles involvant, atque huiusmodi aequationes

$$Pdx+Qdy+Rdz=0$$

non certam relationem inter ipsas quantitates finitas *x, y* et *z* declarare, nisi quantitates *P, Q,* R ita fuerint comparatae, ut character inventus locum habeat. Ex quo intelligitur infinitas huiusmodi aequationes differentiales inter ternas variabiles proponi posse, quibus nulla prorsus relatio finita conveniat et quae propterea nihil plane definiant. Pro arbitrio scilicet huiusmodi aequationes formari possunt nullo scopo proposito, ad quem sint accommodatae; statim enim ac certum quoddam problema ad aequationem differentialem inter ternas variabiles perducit, semper necesse est characterem assignatum ei convenire, cum alioquin nihil omnino significaret. Talis aequatio nihil significans est exempli gratia $zdx+xdy+ydz=0$ neqne pro *z* uIla quidem functio ipsarum *x* et *y* cogitari potest, quae isti aequationi satisfaciat; quin etiam character noster pro hoc exemplo dat $-x-y-z,$ quae quantitas, cum non evanescat, absurditatem illius aequationis declarat.

SCHOLION 2

6. Quo character inventus facilius ad quosvis casus oblatos accommodari queat, ex aequatione

$$Pdx+Qdy+Rdz=0$$

primo evolvantur sequentes valores

$$\left(\frac{dQ}{dz}\right)-\left(\frac{dR}{dy}\right)=L,\quad \left(\frac{dR}{dx}\right)-\left(\frac{dP}{dz}\right)=M,\quad \left(\frac{dP}{dy}\right)-\left(\frac{dQ}{dx}\right)=N$$

et character noster hac continebitur expressione

$$LP+MQ+NR,$$

quae si evanescat, aequatio proposita erit realis et aequationem quandam finitam agnoscet; sin autem ea ad nihilum non redigatur, aequatio proposita erit absurda atque de eius integratione ne cogitandum quidem erit.

Ita in exemplo supra posito erit

$$P=z,\ Q=x,\ R=y,$$

hinc

$$L=-1,\ M=-1 \text{ et } N=-1,$$

unde character $-x-y-z$ absurditatem indicat. Proferamus vero etiam exemplum aequationis realis

$$dx(yy+nyz+zz)-x(y+nz)dy-xzdz=0,$$

In qua ob

$$P=yy+nyz+zz,\ Q=-xy-nxz \text{ et } R=-xz$$

erit

$$L=-nx,\ M=-3z-ny \text{ et } N=3y+2nz,$$

unde

$$\begin{aligned}LP+MQ+NR&=-nx(yy+nyz+zz)+x(y+nz)(3z+ny)-xz(3y+2nz)\\&=x(-nyy-nnyz-nzz+3yz+3nzz+nyy+nnyz-3yz-2nzz)=0,\end{aligned}$$

quare, cum hic character evanescat, aequatio haec differentialis pro reali est habenda. Simili modo proposita hac aequatione

$$2dx(y+z)+dy(x+3y+2z)+dz(x+y)=0$$

ob

$$P=2y+2z,\ Q=x+3y+2z,\ R=x+y$$

fit

$$L=2-1=1,\ M=1-2=-1 \text{ et } N=2-1=1$$

hincque

$$LP+MQ+NR=2y+2z-x-3y-2z+x+y=0,$$

unde ista aequatio differentialis erit realis.

PROBLEMA 2

7. *Proposita aequatione differentiali inter ternas variabiles x, y, z, quae sit realis, eius integrale investigare, unde pateat, qualis functio una earum sit binarum reliquarum.*

SOLUTIO

Sit aequatio differentialis proposita

$$Pdx + Qdy + Rdz = 0,$$

in qua P, Q, R eiusmodi sint functiones ipsarum x, y et z, ut character realitatis ante inventus satisfaciat. Nisi enim ista aequatio esset realis, ridiculum foret eius integrationem tentare. Sumamus ergo hanc aequationem esse realem atque dabitur relatio inter ipsas quantitates x, y et z aequationi propositae satisfaciens; ad quam inveniendam perpendatur, si in aequatione integrali una variabilium, puta z, constans spectetur, ex eius differentiali nihilo aequali posito nasci debere aequationem

$$Pdx + Qdy = 0.$$

Vicissim ergo una variabili, puta z, ut constante tractata integratio aequationis differentialis $Pdx + Qdy = 0$, quae duas tantum variabiles continet, perducet ad aequationem integralem quaesitam, si modo in quantitatem constantem per integrationem ingressam illa quantitas z rite involvatur. Ex quo hanc regulam pro integratione aequationis propositae colligimus.

Consideretur una variabilium, puta z, ut constans, ut habeatur haec aequatio $Pdx + Qdy = 0$ duas tantum variabiles x et y implicans; tum eius investigetur aequatio integralis completa, quae ergo constantem arbitrariam Ccomplectetur. Deinde haec constans C consideretur ut functio quaecunque ipsius z atque hac z nunc etiam pro variabili habita aequatio integralis inventa denuo differentietur, ut omnes tres x, y et z tanquam variabile tractentur, et aequatio differentialis resultans comparetur cum proposita $Pdx + Qdy + Rdz = 0$, ubi quidem functiones P et Q sponte prodibunt, at functio R cum ea quantitate, qua elementum dz afficitur, collata determinabit rationem, qua quantitas z in illam litteram C ingreditur, sicque obtinebitur aequatio integralis quaesita, quae simul erit completa, cum semper in illa litterae C pars quaedam constans vere arbitraria relinquatur, cum haec determinatio ex differentiali ipsius C sit petenda.

COROLLARIUM 1

8. Reducitur ergo integratio huiusmodi aequationum differentialium tres variabiles continentium ad integrationem aequationum differentialium inter duas tantum variabiles, quae ergo, quoties licet, per methodos in superiori libro traditas est instituenda.

COROLLARIUM 2

9. Haec ergo integratio tribus modis institui potest, prout primo vel z vel y vel x tanquam constans spectatur. Semper autem necesse est, ut eadem aequatio integralis resultet, siquidem aequatio differentialis fuerit realis.

COROLLARIUM 3

10. Quodsi, haec methodus tentetur in aequatione differentiali impossibili, determinatio illius constantis C non ita succedet, ut eam variabilem, quae pro constante est habita, solam involvat; atque etiam ex hoc criterium realitatis peti poterit.

SCHOLION

11. Quo haec operatio facilius intelligatur, periculum faciamus primo in aequatione impossibili hac

$$zdx + xdy + ydz = 0 .$$

Hic sumta z pro constante erit

$$zdx + xdy = 0 \quad \text{seu} \quad \tfrac{zdx}{x} + dy = 0 ,$$

cuius integrale est $zlx + y = C$ existente C functione ipsius z. Differentietur ergo haec aequatio sumendo etiam z variabile positoque $dC = Ddz$, ut D sit etiam functio ipsius z tantum, erit

$$\tfrac{zdx}{x} + dy + dzlx = Ddz \quad \text{seu} \quad zdx + xdy + dz\left(xlx - Dx\right) = 0 ;$$

deberet ergo esse $xlx - Dx = y$ seu $D = lx - \frac{y}{x}$, quod est absurdum.

Deinde in aequatione reali

$$2dx\left(y+z\right) + dy\left(x+3y+2z\right) + dz\left(x+y\right) = 0$$

operatio exposita ita instituatur. Sumatur y constans, ut sit

$$2dx\left(y+z\right) + dz\left(x+y\right) = 0 \quad \text{seu} \quad \tfrac{2dx}{x+y} + \tfrac{dz}{y+z} = 0$$

cuius integrale est

$$2l\left(x+y\right) + l\left(y+z\right) = C ,$$

ubi C etiam y involvat. Sit ergo $dC = Ddy$ et sumto etiam y variabili differentiatio praebet

$$\tfrac{2dx+2dy}{x+y} + \tfrac{dy+dz}{y+z} = Ddy$$

seu

$$2dx(y+z)+2dy(y+z)+dy(x+y)+dz(x+y)=Ddy(x+y)(y+z),$$

quae expressio cum forma proposita collata praebet $D=0$, ideoque $dC=0$ et C fit constans vera, ita ut integrale sit

$$(x+y)^2(y+z)=\text{Const}.$$

Huiusmodi igitur exempla aliquot evolvamus.

EXEMPLUM 1

12. *Huius aequationis differentialis realis*

$$dx(y+z)+dy(x+z)+dz(x+y)=0$$

integrale investigare.

Primo quidem patet hanc aequationem esse realem, cum sit

$$P=y+z,\ Q=x+z,\ R=x+y,$$
$$L=1-1=0,\ M=1-1=0,\ N=1-1=0.$$

Sumatur igitur z constans et aequatio prodibit

$$dx(y+z)+dy(x+z)=0 \text{ seu } \tfrac{dx}{x+z}+\tfrac{dy}{y+z}=0,$$

cuius integrale est

$$l(x+z)+l(y+z)=f:z;$$

statuatur ergo

$$(x+z)(y+z)=Z,$$

ubi natura functionis Z ex differentiatione debet erui. Fit autem

$$dx(y+z)+dy(x+z)+dz(x+y+2z)=dZ,$$

a qua si proposita auferatur, relinquitur $2zdz=dZ$, hinc $Z=zz+C$, ita ut aequatio integralis completa sit

$(x+z)(y+z)=zz+C$ seu $xy+xz+yz=C$,

quae quidem ex ipsa proposita

$$ydx + zdx + xdy + zdy + xdz + ydz = 0$$

facile elicitur, cum bina membra iuncta sint integrabilia.

EXEMPLUM 2

13. *Huius differentialis aequationis realis*

$$dx(ay - bz) + dy(cz - ax) + dz(bx - cy) = 0$$

aequationem integralem completam invenire.

Realitas huius aequationis ita ostenditur. Cum sit

$$P = ay - bz,\ Q = cz - ax,\ R = bx - cy,$$

erit

$$L = 2c,\ M = 2b,\ N = 2a$$

hincque manifesto $LP + MQ + NR = 0$.

Iam sumatur z constans, ut habeatur

$$\frac{dx}{cz-ax} + \frac{dy}{ay-bz} = 0, \text{ ergo } \frac{1}{a} l \frac{ay-bz}{cz-ax} = f:z;$$

statuatur ergo

$$\frac{ay-bz}{cz-ax} = Z$$

et differentiatio praebet

$$\frac{adx(ay-bz)+ady(cz-ax)+adz(bx-cy)}{(cz-ax)^2} = dZ,$$

ex cuius comparatione cum proposita fit $dZ = 0$ et $Z = C$, ita ut aequatio integralis completa sit

$$\frac{ay-bz}{cz-ax} = n \text{ seu } ay + nax = (b + nc)z.$$

Quodsi aequatio integralis ponatur

$$Ax + By + Cz = 0,$$

hae constantes ita debent esse comparatae, ut sit

$$Ac + Bb + Ca = 0,$$

sicque constans arbitraria concinnius inducitur.

COROLLARIUM

14. Haec ergo aequatio integrabilis redditur, si dividatur per $(cz-ax)^2$, atque ob eandem rationem etiam hi divisores $(ay-bz)^2$ et $(bx-cy)^2$ idem praestant. Vi enim integralis hi divisores constantem inter se tenent rationem.
Namque si $\frac{ay-bz}{cz-ax}=n,$ erit

$$\frac{bx-cy}{cz-ax}=\frac{-b-nc}{a} \text{ et } \frac{bx-cy}{ay-bz}=\frac{-b-nc}{na}.$$

EXEMPLUM 3

15. *Huius aequationis differentialis realis*

$$dx(yy+yz+zz)+dy(zz+xz+xx)+dz(xx+xy+yy)=0$$

aequationem integralem completam investigare.

Realitas huius aequationis inde patet, quod sit

$$P=yy+yz+zz,\ Q=zz+xz+xx,\ R=xx+xy+yy$$

hincque

$$L=2z+x-x-2y=2(z-y),\ M=2x+y-y-2z=2(x-z),$$
$$N=2y+z-z-2x=2(y-x),$$

unde fit

$$LP+MQ+NR=2\left(z^3-y^3\right)+2\left(x^3-z^3\right)+2\left(y^3-x^3\right)=0\,.$$

Ad integrale ergo investigandum sumatur z constans eritque

$$\frac{dx}{xx+xz+zz}+\frac{dy}{yy+yz+zz}=0\,,$$

cuius integrale est

$$\frac{2}{z\sqrt{3}}\text{Ang. tang.}\frac{x\sqrt{3}}{2z+x}+\frac{2}{z\sqrt{3}}\text{Ang. tang.}\frac{y\sqrt{3}}{2z+y}=f:z,$$

quae per collectionem horum angulorum abit in

$$\frac{2}{z\sqrt{3}}\text{Ang. tang.}\frac{(xz+yz+xy)\sqrt{3}}{2zz+xz+yz-xy}=f:z$$

Statuatur ergo

$$\frac{xz+yz+xy}{2zz+xz+yz-xy}=Z$$

haecque aequatio differentietur sumtis omnibus tribus *x*, *y* et *z* variabilibus ac prodibit

$$\frac{2zdx(yy+yz+zz)+2zdy(zz+xz+xx)-2xdz(zz+yz+yy)-2ydz(zz+xz+xx)}{(2zz+xz+yz-xy)^2}=dZ\ ;$$

cum igitur ex aequatione proposita sit

$$dx(yy+yz+zz)+dy(zz+xz+xx)=-dz(xx+xy+yy),$$

erit facta substitutione

$$\frac{-2zdz(xx+xy+yy)-2xdz(zz+yz+yy)-2ydz(zz+xz+xx)}{(2zz+xz+yz-xy)^2}=dZ\ ,$$

seu

$$\frac{-2dz(xxz+xzz+yyz+yzz+xxy+xyy+3xyz)}{(2zz+xz+yz-xy)^2}=dZ\ ,$$

quae in hanc formam reducitur

$$\frac{-2dz(x+y+z)(xy+xz+yz)}{(2zz+xz+yz-xy)^2}=dZ$$

At ob $Z=\frac{xy+xz+yz}{2zz+xz+yz-xy}$ erit

$$\frac{-2ZZdz(x+y+z)}{xy+xz+yz}=dZ \text{ seu } \frac{-dZ}{ZZ}=\frac{2dz(x+y+z)}{xy+xz+yz}.$$

Necesse ergo est, ut etiam $\frac{xy+xz+yz}{x+y+z}$ sit functio ipsius *z* tantum, quae vocetur Σ, ut sit

$$\frac{-dZ}{ZZ}=\frac{2dz}{\Sigma}.$$

Verum ex sola forma functionis *Z* negotium confici oportet, quod ita expediri pot est. Cum sit $Z=\frac{xz+yz+xy}{2zz+xz+yz-xy}$, erit

$$1+Z=\frac{2zz+2xz+2yz}{2zz+xz+yz-xy}, \text{ hinc } \frac{1+Z}{Z}=\frac{2z(x+y+z)}{xy+xz+yz},$$

cuius valoris ope quantitates *x* et *y* ex aequatione differentiali eliduntur, fitque

$$-\frac{dZ}{ZZ}=dz\frac{2(x+y+z)}{xy+xz+yz}=dz\frac{1+Z}{Zz},$$

unde

$$\frac{-dZ}{Z(1+Z)}=\frac{dz}{z}=\frac{-dZ}{Z}+\frac{dZ}{1+Z}$$

et integrando $lz=l\frac{1+Z}{Z}+la.$ Ergo

$$\frac{1+Z}{Z}=\frac{z}{a}\quad \text{et}\quad Z=\frac{a}{z-a},$$

ita ut aequatio integralis quaesita sit

$$\frac{a}{z-a}=\frac{xy+xz+yz}{2zz+xz+yz-xy}\ \text{ seu }\ xy+xz+yz=a\left(x+y+z\right),$$

quae simplicissima forma statim colligitur ex aequatione

$$\frac{2z\left(x+y+z\right)}{xy+xz+yz}=\frac{1+Z}{Z}=\frac{z}{a}.$$

COROLLARIUM

15[a]. Cum aequationis propositae integrale completum sit

$$xy+xz+yz=a\left(x+y+z\right)\ \text{ seu }\ \frac{xy+xz+yz}{x+y+z}=\text{Const.},$$

ex huius differentiatione etiam ipsa aequatio proposita resultare deprehenditur. Unde patet aequationem propositam integrabilem reddi, si dividatur per $\left(x+y+z\right)^2$, vel etiam per $\left(xy+xz+yz\right)^2$.

SCHOLION

16. Ex hoc exemplo intelligitur determinationem functionis per integrationem illatae interdum haud exiguis diffieultatibus esse obnoxiam, siquidem hic functionem Z non sine ambagibus elicuimus. Verum et hie ista investigatio multo facilius institui potuisset; statim enim atque invenimus

$$\frac{xy+xz+yz}{2zz+xz+yz-xy}=Z=f:z,$$

hanc ipsam expressionem concinniorem reddere licuisset. Nempe cum sit

$$\frac{1}{Z}=\frac{2zz+xz+ys-xy}{xy\ +xz\ +yz},$$

erit

$$1+\frac{1}{Z}=\frac{2z\left(x+y+z\right)}{xy\ +xz+yz}$$

ideoque

$$\frac{xy+xz+yz}{x+y+z}=\frac{2Zz}{1+Z}=f:z.$$

Relicta ergo functione Z statim ponatur

$$\frac{xy+xz+y}{x+y+z}=\Sigma=f:z$$

et sumtis differentialibus per se liquebit fieri $d\Sigma=0$ ideoque $\Sigma=\text{Const.}$

Adhuc facilius hoc problema resolvitur, si etiam sumto y constante eius integrale quaeratur; tum enim simili modo pervenitur ad huiusmodi aequationem

$$\frac{xy+xz+yz}{x+y+z}=Y=f:y;$$

quare cum haec expressio aeque esse debeat functio ipsius z atque ipsius y, necesse est, ut ea sit constans, eritque propterea aequatio integralis completa

$$xy+xz+yz=a\left(x+y+z\right).$$

EXEMPLUM 4

17. *Huius aequationis differentialis realis*

$$dx\left(xx-yy+zz\right)-zzdy+zdz\left(y-x\right)+\frac{xdz}{z}\left(yy-xx\right)=0$$

aequationem integralem completam investigare.

Realitas huius aequationis ita ostenditur. Ob

$$P=xx-yy+zz,\ Q=-zz,\ R=z\left(y-x\right)+\frac{x}{z}\left(yy-xx\right)$$

erit

$$L=-3z-\frac{2xy}{z},\ M=-3z+\frac{yy}{z}-\frac{3xx}{z},\ N=-2y,$$

unde calculo subducto formula $LP+MQ+NR$ evanescit.

Sumamus iam z constans et habebimus hanc aequationem

$$dx\left(xx-yy+zz\right)-zzdy=0,$$

cuius quidem integratio non constaret, nisi perspiceremus ei satisfacere particulariter $y=x$. Hinc autem ponendo $y=x+\frac{zz}{v}$ integrale completum eruere poterimus; fit enim

$$dx\left(zz-\frac{2xzz}{v}-\frac{z^4}{vv}\right)-zzdx+\frac{z^4dv}{vv}=0$$

hincque

$$dv - \frac{2xvdx}{zz} = dx$$

quae per $e^{\frac{-xx}{zz}}$ multiplicata praebet integrale

$$e^{\frac{-xx}{zz}} v = \int e^{\frac{-xx}{zz}} dx + f : z,$$

ubi quidem notandum est in integratione formulae $\int e^{\frac{-xx}{zz}} dx$ quantitatem z ut constantem tractari esseque $v = \frac{zz}{y-x}$, ita ut sit

$$\int e^{\frac{-xx}{zz}} dx = \frac{e^{\frac{-xx}{zz}} zz}{y-x} + Z.$$

Quodsi iam hanc aequationem differentiare velimus sumta etiam z variabili, difficultas hic occurrit, quomodo quantitatis $\int e^{\frac{-xx}{zz}} dx$ differentiale ex variabilitate ipsius z oriundum definiri debeat. Hic ex principiis repeti debet, si fuerit $dV = Sdx + Tdz$, fore $\left(\frac{dT}{dx}\right) = \left(\frac{dS}{dz}\right)$ ideoque, si z constans sumatur,
$T = \int dx\left(\frac{dS}{dz}\right)$. Iam nostro casu est

$$S = e^{\frac{-xx}{zz}} \quad \text{et} \quad V = \int e^{\frac{-xx}{zz}} dx$$

sumta z constante; quare cum sit $\left(\frac{dS}{dz}\right) = e^{\frac{-xx}{zz}} \frac{2xx}{z^3}$, ergo

$$T = \frac{2}{z^3} \int e^{\frac{-xx}{zz}} xxdx.$$

Quocirca quantitatis $\int e^{\frac{-xx}{zz}} dx$ differentiale plenum ex variabilitate utriusque x et z oriundum est

$$e^{\frac{-xx}{zz}} dx + \frac{2dz}{z^3} \int e^{\frac{-xx}{zz}} xxdx,$$

cui aequari debet alterius partis $\frac{e^{\frac{-xx}{zz}} zz}{y-x} + Z$ differentiale, quod est

$$e^{\frac{-xx}{zz}} \left(\frac{2zdz}{y-x} - \frac{2zzdy - zzdx}{(y-x)^2} + \frac{2xxdz - 2zdx}{(y-x)} \right) + dZ.$$

Turbat vero adhuc formula integralis $\int e^{\frac{-xx}{zz}} xxdx$, in qua z pro constante habetur; reduci autem potest ad priorem $\int e^{\frac{-xx}{zz}} dx$, si ponatur

$$\int e^{\frac{-xx}{zz}} xxdx = Ae^{\frac{-xx}{zz}} x + B\int e^{\frac{-xx}{zz}} dx \ ;$$

prodit enim sola x pro variabili habita differentiando

$$xxdx = Adx - \tfrac{2Axxdx}{zz} + Bdx,$$

ergo

$$A = \tfrac{1}{2} zz \ \text{ et } \ B = -A = \tfrac{1}{2} zz,$$

ita ut sit

$$\int e^{\frac{-xx}{zz}} xxdx = -\tfrac{1}{2} e^{\frac{-xx}{zz}} xzz + \tfrac{1}{2} zz \int e^{\frac{-xx}{zz}} dx.$$

Quare cum sit

$$\int e^{\frac{-xx}{zz}} dx = \frac{e^{\frac{-xx}{zz}} zz}{y-x} + Z,$$

erit

$$\int e^{\frac{-xx}{zz}} xxdx = -\tfrac{1}{2} e^{\frac{-xx}{zz}} xzz + \frac{e^{\frac{-xx}{zz}} z^4}{2(y-x)} + \tfrac{1}{2} Zzz.$$

Facta ergo substitutione haec orietur aequatio differentialis

$$e^{\frac{-xx}{zz}} \left(dx - \tfrac{xdz}{z} + \tfrac{zdz}{y-x} \right) + \tfrac{Zdz}{z}$$
$$= e^{\frac{-xx}{zz}} \left(\frac{2zdz}{y-x} - \frac{zzdy}{(y-x)^2} + \frac{zzdx}{(y-x)^2} - \frac{2xdx}{y-x} + \frac{2xxdz}{z(y-x)} \right) + dZ,$$

quae transit in hanc formam

$$e^{\frac{-xx}{zz}} \left(\frac{dx(y+x)}{y-x} - \frac{zzdx}{(y-x)^2} + \frac{zzdy}{(y-x)^2} - \frac{zdz}{y-x} - \frac{x(y+x)dz}{z(y-x)} \right) = \frac{zdZ - Zdz}{z}$$
$$= e^{\frac{-xx}{zz}} \left(\frac{2zdz}{y-x} - \frac{zzdy}{(y-x)^2} + \frac{zzdx}{(y-x)^2} - \frac{2xdx}{y-x} + \frac{2xxdz}{z(y-x)} \right) + dZ,$$

seu

$$\frac{e^{\frac{-xx}{zz}}}{(y-x)^2}\left(dx\left(yy-xx-zz\right)+zzdy-zdz\left(y-x\right)-\tfrac{xdz}{z}\left(yy-xx\right)\right)=\frac{zdZ-Zdz}{z};$$

qua cum proposita collata evidens est esse debere

$$zdZ-Zdz=0 \text{ seu } Z=nz,$$

ita ut aequationis propositae integrale completum sit

$$\int e^{\frac{-xx}{zz}}dx=\frac{e^{\frac{-xx}{zz}}zz}{y-x}+nz,$$

siquidem in integrali $\int e^{\frac{-xx}{zz}}dx$ quantitas z pro constante habeatur.

COROLLARIUM

18. Aequatio ergo proposita integrabilis redditur, si multiplicetur per

$$\frac{1}{(y-x)^2}e^{\frac{-xx}{zz}};$$

ac tum integrale est ipsa aequatio, quam invenimus.

SCHOLION 1

19. Exemplum hoc imprimis est notatu dignum, quod in eius solutione quaedam artificia sunt in subsidium vocata, quibus in praecedentibus non erat opus. Per formulam autem $\int e^{\frac{-xx}{zz}}dx$ integrale non satis determinatum videtur. Cum enim in ea z constans ponatur, constans per integrationem introducenda per nz non definitur, siquidem lex non praescribitur, secundum quam integrale $\int e^{\frac{-xx}{zz}}dx$ capi oporteat, utrum ita, ut evanescat facto $x = 0$, an alio quocunque modo. Dubium autem hoc diluetur, si aequationem inventam per z dividamus, ut formula integralis sit $\int e^{\frac{-xx}{zz}}\frac{dx}{z}$ ubi cum $\frac{dx}{z}$ sit $d.\frac{x}{z}$, evidens est ea exprimi functionem quandam ipsius $\frac{x}{z}$, ac si ponatur $\frac{x}{z}=p$, fore aequationem nostram integralem

$$\int e^{-pp}dp+\text{Const.}=e^{-pp}\frac{z}{y-x};$$

neque hic amplius conditio illa, qua in formula integrali quantitas z pro constante sit habenda, locum habet, sed integrale perinde determinatur, ac si aequatio duas tantum variabiles contineret. Hanc circumstantiam si perpendissemus, plenum differentiale formulae $\int e^{\frac{-xx}{zz}}dx$ ex variabilitate utriusque x et z nullam difficultatem peperisset. Postquam enim pervenimus ad aequationem

$$\int e^{\frac{-xx}{zz}}dx = e^{\frac{-xx}{zz}}\frac{zz}{y-x} + f:z,$$

eam ita repraesentemus

$$\int e^{\frac{-xx}{zz}}\frac{dx}{z} = \int e^{\frac{-xx}{zz}}d.\frac{x}{z} = e^{\frac{-xx}{zz}}\frac{z}{y-x} + Z\ ;$$

ubi cum in formulam integralem etiam variabilitas ipsius z sit inducta, si ea differentietur sumtis omnibus x, y et z variabilibus, orietur

$$e^{\frac{-xx}{zz}}\left(\frac{dx}{z} - \frac{xdz}{zz}\right) = e^{\frac{-xx}{zz}}\left(\frac{dz}{y-x} + \frac{zdx-zdy}{(y-x)^2} - \frac{2xdx}{z(y-x)} + \frac{2xxdz}{zz(y-x)}\right) + dZ,$$

seu

$$e^{\frac{-xx}{zz}}\left(\frac{dx(y+x)}{z(y-x)} - \frac{zdx}{(y-x)^2} + \frac{zdy}{(y-x)^2} - \frac{xdz(y+x)}{zz(y-x)} - \frac{dz}{y-x}\right) = dZ,$$

quae reducitur ad hanc formam

$$\frac{e^{\frac{-xx}{zz}}}{z(y-x)^2}\Big(dx\big(yy-xx-zz\big) + zzdy - zdz\big(y-x\big) - \tfrac{xdz}{z}\big(yy-xx\big)\Big) = dZ,$$

unde patet esse debere $dZ = 0$ et $Z = \text{Const.}$, sicque elicitur aequatio integralis ante inventa.

SCHOLION 2

20. Idem integrale prodiisset, si loco z altera reliquarum x vel y pro constante fuisset assumta; ubi in genere notari convenit, si huiusmodi aequationem

$$Pdx + Qdy + Rdz = 0$$

sumta z constante tractare licuerit, etiam resolutionem, quaecunque trium variabilium pro constante assumatur, succedere debere, etiamsi id quandoque minus perspiciatur. Ita in aequatione proposita si y pro constante habeatur, resolvenda erit haec aequatio

$$dx\big(xx+zz-yy\big) - zdz\big(x-y\big) - \tfrac{xdz}{z}\big(xx-yy\big) = 0\ ;$$

quae per z multiplicata cum in hanc formam abeat

$$\left(zdx-xdz\right)\left(xx+zz-yy\right)+yzzdz=0\,,$$
facile patet eam simpliciorem reddi ponendo $x=pz;$ tum enim ob

$$zdx-xdz=zzdp$$
prodibit

$$dp\left(ppzz+zz-yy\right)+ydz=0\,.$$
Sit porro $z=qy$ fietque

$$dp\left(ppqq+qq-1\right)+dq=0\,,$$
cui cum satisfaciat $q=\frac{1}{p}$, statuatur $q=\frac{1}{p}+\frac{1}{r}$ habebiturque

$$dp\left(\frac{2p}{r}+\frac{pp}{rr}+\frac{1}{pp}+\frac{2}{pr}+\frac{1}{rr}\right)-\frac{dp}{pp}-\frac{dr}{rr}=0$$
seu
$$dp\left(2ppr+p^3+2r+p\right)-pdr=0 \text{ vel } dr-\frac{2rdp\left(pp+1\right)}{p}=dp\left(pp+1\right),$$

quae multiplicata per $\frac{1}{pp}e^{-pp}$ et integrata dat
$$e^{-pp}\frac{r}{pp}=\int e^{-pp}\frac{dp\left(pp+1\right)}{pp}.$$
At
$$\int e^{-pp}\frac{dp}{pp}=-e^{-pp}\frac{1}{p}-2\int e^{-pp}dp$$
unde
$$e^{-pp}\left(\frac{r}{pp}+\frac{1}{p}\right)=-\int e^{-pp}dp.$$

Cum nunc sit $p=\frac{x}{z}$ et $\frac{1}{r}=\frac{z}{y}-\frac{z}{x}=\frac{z\left(x-y\right)}{xy}$ erit

$$r=\frac{xy}{z\left(x-y\right)},\quad \frac{r}{pp}=\frac{yz}{x\left(x-y\right)} \quad \text{et} \quad \frac{r}{pp}+\frac{1}{p}=\frac{z}{x-y}$$

Unde aequatio nostra integralis erit

$$\int e^{\frac{-xx}{zz}}d.\frac{x}{z}=e^{\frac{-xx}{zz}}\frac{z}{y-x}+f:y$$

cuius differentiale, si etiam y pro variabili habeatur, cum aequatione proposita comparatum dabit ut ante $f:y=\text{Const.}$

Caeterum cum in his exemplis variabiles *x, y, z* ubique eundem dimensionum numerum impleant, methodum generalem huiusmodi aequationes tractandi exponam.

PROBLEMA 3

21. *Si in aequatione differentiali*

$$Pdx+Qdy+Rdz=0$$

functiones P, Q, R fuerint homogeneae ipsarum x, y et z eiusdem numeri dimensionum, eius integrationem, siquidem fuerit realis, investigare.

SOLUTIO

Sit *n* numerus dimensionum, quas ternae variabiles *x, y* et *z* in functionibus *P, Q, R* constituunt, ac posito $x=pz$ et $y=qz$ fiet

$$P=z^nS,\ \ Q=z^nT\ \text{ et }\ R=z^nV,$$

ita ut iam S, T, V futurae sint functiones binarum tantum variabilium *p* et *q*. Cum iam sit

$$dx=pdz+zdp\ \text{ et }\ dy=qdz+zdq,$$

aequatio nostra hanc induet formam

$$dz\left(pS+qT+V\right)+Szdp+Tzdq=0\ \text{seu}\ \frac{dz}{z}+\frac{Sdp+Tdq}{pS+qT+V}=0,$$

quae aequatio realis esse nequit, nisi formula differentialis binas variabiles *p* et *q* involvens $\frac{Sdp+Tdq}{pS+qT+V}$ per se fuerit integrabilis; quod eveniet, si fuerit

$$\left(qT+V\right)\left(\frac{dS}{dq}\right)+PT\left(\frac{dS}{dp}\right)-\left(pS+V\right)\left(\frac{dT}{dp}\right)-qS\left(\frac{dT}{dq}\right)-S\left(\frac{dV}{dq}\right)+T\left(\frac{dV}{dp}\right)=0.$$

Quoties ergo hic character locum habet, nostra aequatio erit realis eiusque integrale erit

$$lz+\int\frac{Sdp+Tdq}{pS+qT+V}=\text{Const.},$$

ubi tantum opus est, ut loco litterarum *p* et *q* valores assumti restituantur.

COROLLARIUM 1

22. Ita in nostro primo exemplo (§ 12) cum sit

$$P=y+z,\quad Q=x+z,\quad R=x+y,$$

erit
$$S = q+1, \quad T = p+1, \quad V = p+q$$
et
$$\tfrac{dz}{z} + \tfrac{(q+1)dp+(p+1)dq}{2pq+2p+2q} = 0,$$
cuius integrale est
$$lz + \tfrac{1}{2}l\left(pq+p+q\right) = \tfrac{1}{2}l\left(xy+xz+yz\right) = C$$
seu
$$xy+xz+yz = C.$$

COROLLARIUM 2

23. In secundo exemplo (§ 13) est
$$P = ay-bz,\ Q = cz-ax,\ R = bx-cy,$$
hinc
$$S = aq-b,\ T = c-ap,\ V = bp-cq.$$
Ergo
$$\tfrac{dz}{z} + \tfrac{(aq-b)dp+(c-ap)dq}{0} = 0$$
hincque
$$\left(aq-b\right)dp + \left(c-ap\right)dq = 0$$
et integrando
$$l\tfrac{aq-b}{c-ap} = l\tfrac{ay-bz}{cz-ax} = C.$$

COROLLARIUM 3

24. In tertio exemplo (§ 15) fit
$$S = qq+q+1,\ \ T = pp+p+1 \ \text{ et } \ V = pp+pq+qq$$
hincque
$$\tfrac{dz}{z} + \tfrac{dp(qq+q+1)+dq(pp+p+1)}{pppq+pqq+pp+3pq+qq+p+q} = 0,$$

qui denominator est $= \left(p+q+1\right)\left(pq+p+q\right)$, unde haec fractio resolvitur in has duas
$$\tfrac{-dp-dq}{p+q+1} + \tfrac{dp(q+1)+dq(p+1)}{pq+p+q},$$

ex quo integrale a logarithmis ad numeros perductum oritur

$$\frac{z(pq+p+q)}{p+q+1}=\frac{xy+xz+yz}{x+y+z}=C.$$

COROLLARIUM 4

25. In exemplo quarto (§ 17) fit

$$S=pp-qq+1,\ T=-1,\ V=q-p+p(qq-pp)$$

hincque

$$\frac{dz}{z}+\frac{dp(pp-qq+1)-dq}{0}=0$$

ideoque

$$dq=dp(pp-qq+1).$$

Cum ergo satisfaciat $q=p$, ponatur $q=p+\frac{1}{r}$; fiet $dr-2prdp=dp$ et integrando

$$e^{-pp}r=\int e^{-pp}dp=e^{-pp}\frac{1}{q-p},$$

ita ut integrale sit

$$e^{\frac{-xx}{zz}}\frac{z}{y-x}=\int e^{\frac{-xx}{zz}}d.\frac{x}{z}+\text{Const.}$$

SCHOLION

26. Cum igitur aequationes differentiales tres variabiles involventes nullam habeant difficultatem sibi propriam, quoniam earum resolutio, siquidem fuerint reales, semper ad aequationes differentiales duarum variabilium reduci potest, hoc argumentum fusius non prosequor. Quod enim ad eiusmodi aequationes differentiales trium variabilium attinet, in quibus ipsa differentialia ad plures dimensiones ascendunt, veluti est

$$Pdx^2+Qdy^2+Rdz^2+2Sdxdy+2Tdxdz+2Vdydz=0,$$

de iis generatim tenendum est, nisi per radicis extractionem ad formam

$$Pdx+Qdy+Rdz=0$$

reduci queant, eas semper esse absurdas. Quomodocunque enim aequatio integralis esset comparata, ex ea valor ipsius *z* ita definiri posset, ut *z* aequetur functioni binarum variabilium *x* et *y*, unde foret $dz=pdx+qdy$, neque hae variabiles *x* et *y* ullo modo a se penderent. Hic ergo valor $pdx+qdy$ loco *dz* in aequatione differentiali substitutus ita satisfacere deberet, ut omnes termini se mutuo destruerent, quod autem fieri non posset, si ex aequationis resolutione *dz* ita definiretur, ut differentialia *dx* et *dy* signis radicalibus essent involuta. Hinc aequatio illa exempli loco allata, cum per resolutionem det

$$dz = \frac{-Tdx - Vdy \pm \sqrt{\left((TT-PR)dx^2 + 2(TV-RS)dxdy + (VV-QR)\,dy^2\right)}}{R},$$

realis esse nequit, nisi radix extrahi queat, hoc est, nisi ipsa aequatio in factores formae

$$Pdx + Qdy + Rdz$$

resolvi possit. Atque etiamsi hoc eveniat at hi factores nihilo aequales statuantur, tamen aequatio non erit realis, nisi criterium supra traditum locum habeat.

Ex his perspicuum est ne eiusmodi quidem aequationes, quae quatuor pluresve variabiles involvant, plus difficultatis habere.

PROBLEMA 4

27. *Si V sit functio quaecunque binarum variabilium x et y, in formula autem integrali* $\int Vdx$ *quantitas y pro constante sit habita, definire huius formae* $\int Vdx$ *differentiale, si praeter x etiam y variabilis assumatur.*

SOLUTIO

Ponatur ista formula integralis

$$\int Vdx = Z$$

eritque Z utique functio ambarum variabilium x et y, etiamsi in ipsa integratione y pro constante habeatur. Evidens autem est, si vicissim in differentiatione y constans sumatur, fore $dZ = Vdx$. Quare si etiam y variabilis statuatur, differentiale ipsius $Z = \int Vdx$ huiusmodi habebit formam

$$dZ = Vdx + Qdy$$

et quaestio huc redit, ut ista quantitas Q determinetur. Quia autem forma $Vdx + Qdy$ est verum differentiale, necesse est sit $\left(\frac{dV}{dy}\right) = \left(\frac{dQ}{dx}\right)$ hincque $dx\left(\frac{dQ}{dx}\right) = dx\left(\frac{dV}{dy}\right)$. At $dx\left(\frac{dQ}{dx}\right)$ est differentiale ipsius Q, si y pro constante habeatur; unde Q reperietur, si formula $dx\left(\frac{dV}{dy}\right)$ ita integretur, ut y tanquam constans tractetur, seu erit

$$Q = \int dx\left(\frac{dV}{dy}\right).$$

Quocirca formulae $Z = \int Vdx$ differentiale ex variabilitate utriusque x et y oriundum erit

$$dZ = Vdx + dy\int dx\left(\frac{dV}{dy}\right).$$

COROLLARIUM 1

28. Quoniam V est functio ipsarum x et y, si ponatur $dV = Rdx + Sdy$, erit $S = \left(\frac{dV}{dy}\right)$, unde fit

$$dZ = d.\int Vdx = Vdx + dy\int Sdx,$$

scilicet in formulae $\int Sdx$ integratione, perinde ac formulae $\int Vdx$, sola quantitas x pro variabili est habenda.

COROLLARIUM 2

29. Si V fuerit functio homogenea ipsarum x et y existente numero dimensionum = n, posito $dV = Rdx + Sdy$ erit $Rx + Sy = nV$ ideoque $S = \frac{nV}{y} - \frac{Rx}{y}$ hinc,

$$\int Sdx = \frac{n}{y}\int Vdx - \frac{1}{y}\int Rxdx.$$

At ob y constans est $Rdx = dV$, hinc

$$\int Rxdx = \int xdV = Vx - \int Vdx \text{ ideoque } \int Sdx = \frac{n+1}{y}\int Vdx - \frac{Vx}{y}$$

et

$$dZ = d.\int Vdx = Vdx - \frac{Vxdy}{y} + \frac{(n+1)dy}{y}\int Vdx.$$

COROLLARIUM 3

30. Idem facilius invenitur ex consideratione, quod functio $Z = \int Vdx$ futura sit homogenea $n+1$ dimensionum, quare posito $dZ = Vdx + Qdy$ erit $Vx + Qy = (n+1)Z$ ideoque $Q = \frac{(n+1)Z}{y} - \frac{Vx}{y}$ ut ante.

SCHOLION

31. Problemate iam ante et in praecedente quidem libro sum usus neque tamen abs re fore putavi, si id data opera hic tractarem, quandoquidem hic liber in functionibus binarum pluriumve variabilium occupatur.

Praecipuum autem negotium non in eiusmodi aequationibus differentialibus, quales in hoc capite integrare docui, versatur, quod quidem brevi esset absolutum, sed cum differentiatio functionis binarum variabilium x et y duplices formulas $\left(\frac{dV}{dx}\right)$ et $\left(\frac{dV}{dy}\right)$ suppeditet existente V huiusmodi functione, hoc loco eiusmodi quaestiones potissimum contemplabimur, quibus talis functio V ex data quacunque relatione harum duarum formularum $\left(\frac{dV}{dx}\right)$ et $\left(\frac{dV}{dy}\right)$ est definienda. Relatio autem haec per aequationem inter istas formulas et binas variabiles x et y, quam etiam ipsa functio quaesita V ingredi potest, exprimitur, ex cuius aequationis indole divisio tractationis erit petenda. Problema scilicet generale, in quo solvendo ista sectio est occupata, ita se habet, ut ea binarum variabilium x et y functio V inveniatur, quae satisfaciat aequationi cuicunque inter quantitates x, y, V, $\left(\frac{dV}{dx}\right)$ et $\left(\frac{dV}{dy}\right)$ propositae. Quodsi in hanc aequationem altera tantum binarum formularum differentialium $\left(\frac{dV}{dx}\right)$ vel $\left(\frac{dV}{dy}\right)$ ingrediatur, resolutio non est difficilis atque ad casum aequationum differentialium duas tantum variabiles involventium reducitur; quando autem ambae istae formulae in aequatione proposita insunt, quaestio multo magis est ardua ac saepenumero ne resolvi quidem potest, etiamsi resolutio aequationum differentialium duas tantum variabiles complectentium admittatur; in hoc enim negotio, quoties resolutionem ad integrationem aequationum differentialium inter duas variabiles reducere licet, problema pro resoluto erit habendum.

Cum igitur ex aequatione proposita formula $\left(\frac{dV}{dy}\right)$ aequetur functioni utcunque ex quantitatibus x, y, V et $\left(\frac{dV}{dx}\right)$ confiatae, ex indole huius functionis, prout fuerit simplicior et vel solam formulam $\left(\frac{dV}{dx}\right)$ vel praeter earn unicam ex reliquis vel etiam binas vel adeo omnes comprehendat, tractationem sequentem distribuemus. Hoc enim ordine servato facillime apparebit, quantum adhuc praestare liceat et quantum adhuc desideretur. Praeterea vera nonnulla subsidia circa transformationem binarum formularum differentialium ad alias variabiles exponenda occurrent.

DIVISIO HUIUS SECTIONIS

32. Quo partes, quas in hac sectione pertractari convenit, clarius conspectui exponantur, quoniam hae quaestiones circa functiones binarum variabilium versantur, sint x et y binae variabiles et z earum functio ex data quadam differentialium relatione definienda, ita ut aequatio finita inter x, y et z requiratur. Ponamus autem $dz = pdx + qdy$, ita ut sit recepto signandi modo $p = \left(\frac{dz}{dx}\right)$ et $q = \left(\frac{dz}{dy}\right)$ atque ideo p et q sint formulae differentiales, quae in relationem propositam ingrediantur. In genere ergo relatio ista erit aequatio quaecunque inter quantitates p, q, x, y et z proposita atque haec sectio perfecte absolveretur, si methodus constaret ex data aequatione quacunque inter has quantitates p, q, x, y et z eruendi aequationem inter x, y et z; quod autem cum in genere ne pro functionibus quidem unicae variabilis praestari possit, multo minus hic est expectandum; ex quo eos casus tantum evolvi conveniet, qui resolutionem admittant.

Primo autem resolutio succedit, si in aequatione proposita altera formularum differentialium p vel q plane desit, ita ut aequatio vel inter p, x, y et z vel inter q, x, y et z proponatur.

Deinde aequationes, quae solas binas formulas differentiales p et q continent, ita ut altera debeat esse functio quaecunque alterius, commode resolvere licet.

Tum igitur sequentur aequationes, quae praeter p et q unicam quantitatum finitarum x vel y vel z complectantur, ex quo genere cuiusmodi casus resolvi queant, videamus.

Ordo porro postulat, ut ad aequationes, quae praeter binas formulas differentiales p et q insuper binas quantitatum finitarum vel x et y vel x et z vel y et z involvunt, progrediamur; ac denique de resolutione aequationum omnes litteras p, q, x, y et z implicantium agemus subsidia transformationis deinceps expondituri.