CHAPTER VI

CONCERNING THE RESOLUTION OF EQUATIONS IN WHICH SOME RELATION IS GIVEN BETWEEN THE TWO DIFFERENTIAL FORMULAS $\left(\frac{dz}{dx}\right)$, $\left(\frac{dz}{dy}\right)$ AND ALL THREE VARIABLES x, y, z

PROBLEM 30

174. *If on putting* $dz = pdx + qdy$ *there must become* $nz = px + qy$ *, to investigate the nature of the function z in general.*

SOLUTION

With the aid of the given relation either p or q is removed; evidently since there shall be $q = \frac{nz}{y} - \frac{px}{y}$, then there becomes

$$dz = pdx + \frac{nzdy}{y} - \frac{pxdy}{y},$$

which equation is transformed into this form

$$dz - \frac{nzdy}{y} = p\left(dx - \frac{xdy}{y}\right) = pyd.\frac{x}{y}.$$

So that the first part $dz - \frac{nzdy}{y}$ may be returned integrable, the equation may be multiplied by $\frac{1}{z}\text{funct.}\frac{z}{y^n}$ or in particular by $\frac{1}{y^n}$ and there will be

$$d.\frac{z}{y^n} = py^{1-n}d.\frac{x}{y}.$$

With which accomplished, it is evident there must be put in place $py^{1-n} = f':\frac{x}{y}$, so that there becomes

$$\frac{z}{y^n} = f:\frac{x}{y} \quad \text{or} \quad z = y^n f:\frac{x}{y}$$

From which it becomes apparent that z shall be a homogeneous function of x and y, the number of the dimensions present being $= n$.

If in general the equation should be multiplied by $\frac{1}{z}\text{funct.}\frac{z}{y^n}$, $F:\frac{z}{y^n}$ will be the integral of the first part, but for the second part if there is put $\frac{py}{z}\text{funct.}\frac{z}{y^n} = f':\frac{x}{y}$, there will be

$$F:\tfrac{z}{y^n} = f:\tfrac{x}{y}$$

and as before, $\frac{z}{y^n}$ will be equal to some function of $\frac{x}{y}$.

COROLLARY 1

175. Since z is equal to a homogeneous function of n dimensions of x and y, p and q will be functions of $n-1$ dimensions. Evidently since there shall be $z = y^n f:\frac{x}{y}$, there will be

$$p = y^{n-1} f':\tfrac{x}{y} \quad \text{and} \quad q = ny^{n-1} f:\tfrac{x}{y} - xy^{n-2} f':\tfrac{x}{y},$$

from which clearly there shall be $nz = px + qy$.

COROLLARY 2

176. If p and q were functions of $n-1$ dimensions of x and y and the formula $pdx + qdy$ shall be integrable or $\left(\frac{dp}{dy}\right) = \left(\frac{dq}{dx}\right)$, then the integral certainly will be $\frac{px+qy}{n}$, which property sometimes has a conspicuous use.

SCHOLIUM

177. The basis of this solution rests on this, that the equation to be integrated can be resolved into two parts, of which with each by the aid of a certain multiplier can be rendered integrable, from which then a single variable quantity, the differential of which is not present in the equation, can be determined. Hence our equation

$$dz - \frac{nzdy}{y} = p\left(dx - \frac{xdy}{y}\right)$$

thus also can be represented

$$\frac{dx}{y} - \frac{xdy}{yy} = \frac{1}{py}\left(dz - \frac{nzdy}{y}\right) = \frac{y^{n-1}}{p}\left(\frac{dz}{y^n} - \frac{nzdy}{y^{n+1}}\right)$$

or

$$d.\frac{x}{y} = \frac{y^{n-1}}{p}\, d.\frac{z}{y^n}.$$

Therefore let $\frac{y^{n-1}}{p} = F':\frac{z}{y^n}$ and there shall be $\frac{x}{y} = F:\frac{z}{y^n}$ and in turn $\frac{z}{y^n} = f:\frac{x}{y}$ as before.

Also we are able at once to remove z from the calculation; since indeed there shall be $nz = px + qy$, then there will be

$$ndz = pdx + qdy + xdp + ydq.$$

But there is

$$ndz = npdx + nqdy$$

per hypothesis and thus

$$(n-1)\,pdx - xdp + (n-1)\,qdy - ydq = 0$$

or

$$x^n\left(\tfrac{(n-1)pdx}{x^n} - \tfrac{dp}{x^{n-1}}\right) + y^n\left(\tfrac{(n-1)qdy}{y^n} - \tfrac{dq}{y^{n-1}}\right) = 0\,,$$

which is reduced to this form

$$-x^n d.\tfrac{p}{x^{n-1}} - y^n d.\tfrac{q}{y^{n-1}} = 0 \quad \text{or} \quad d.\tfrac{q}{y^{n-1}} = -\tfrac{x^n}{y^n}\,d.\tfrac{p}{x^{n-1}}\,.$$

There is put in place

$$\tfrac{x^n}{y^n} = -f':\tfrac{p}{x^{n-1}}\,;$$

and there will become

$$\tfrac{q}{y^{n-1}} = f:\tfrac{p}{x^{n-1}}\,.$$

Or on putting $\tfrac{x}{y} = v$, if on account of $v^n = -f':\tfrac{p}{x^{n-1}}$, in turn there is put

$$\tfrac{p}{x^{n-1}} = u = \tfrac{1}{v^{n-1}}F':v\,,$$

so that there becomes $f':u = -v^n$, there will be found

$$\int du\, f':u = f:u = nF:v - vF':v\,.$$

Hence

$$p = \tfrac{x^{n-1}}{v^{n-1}}F':v = y^{n-1}F':\tfrac{x}{y}$$

and

$$q = y^{n-1}f:u = ny^{n-1}F:\tfrac{x}{y} - xy^{n-2}F':\tfrac{x}{y}$$

and thus

$$nz = px + qy = ny^n F:\tfrac{x}{y} \quad \text{or} \quad z = y^n F:\tfrac{x}{y}$$

as before.

PROBLEM 31

178. *If on putting* $dz = pdx + qdy$ *there must become* $\alpha px + \beta qy = nz$ *, to investigate the nature of the function z.*

SOLUTION

From the prescribed condition as before $q = \frac{nz}{\beta y} - \frac{\alpha px}{\beta y}$ is removed and there will be

$$dz - \frac{nzdy}{\beta y} = pdx - \frac{\alpha pxdy}{\beta y},$$

which equation divided by $y^{\frac{n}{\beta}}$ gives

$$d.\frac{z}{y^{n:\beta}} = \frac{p}{y^{n:\beta}}\left(dx - \frac{\alpha xdy}{\beta y}\right) = \frac{py^{\alpha:\beta}}{y^{n:\beta}}d.\frac{x}{y^{\alpha:\beta}}$$

From which if we put

$$py^{(\alpha-n):\beta} = f':\frac{x}{y^{\alpha:\beta}}$$

we will have the solution

$$z = y^{n:\beta}f:\frac{x}{y^{\alpha:\beta}}$$

But the function of $\frac{x}{y^{\alpha:\beta}}$ is reduced to a function of $\frac{x^{\beta}}{y^{\alpha}}$, from which thus z also is determined by x and y, so that there shall be

$$z = y^{n:\beta}f:\frac{x^{\beta}}{y^{\alpha}} \quad \text{or also} \quad z^{1:n} = y^{1:\beta}f:\frac{x^{1:\alpha}}{y^{1:\beta}}$$

But if hence the quantities $x^{1:\alpha}$ and $y^{1:\beta}$ are considered to have a single dimension, $z^{1:n}$ will be equal to a function of the same single dimension, but the same quantity *z* to be a function of *n* dimensions. Or on taking for *z* some homogeneous function of *n* dimensions of the two variables *t* and *u,* then there is written $t = x^{1:\alpha}$ and $u = y^{1:\beta}$ and a function will be produced appropriate for *z.*

PROBLEM 32

179. *If on putting $dz = pdx + qdy$ there must be $Z = pX + qY$ with Z denoting some function of z, X of x and Y of y, to investigate the nature of the function z in general.*

SOLUTIO

From the prescribed condition, $q = \frac{Z}{Y} - \frac{pX}{Y}$ is removed, which value substituted gives

$$dz - \frac{Zdy}{Y} = p\left(dx - \frac{Xdy}{Y}\right)$$

and hence

$$\frac{dz}{Z} - \frac{dy}{Y} = \frac{p}{Z}\left(dx - \frac{Xdy}{Y}\right) = \frac{pX}{Z}\left(\frac{dx}{X} - \frac{dy}{Y}\right),$$

where now the resolution is clear. Evidently there may be put in place

$$\frac{pX}{Z} = f':\left(\int \frac{dx}{X} - \int \frac{dy}{Y}\right)$$

and there will be

$$\int \frac{dz}{Z} - \int \frac{dy}{Y} = f:\left(\int \frac{dx}{X} - \int \frac{dy}{Y}\right),$$

from which the value of z is defined by x and y .

COROLLARY 1

180. Therefore here z thus must be defined by x and y, so that, if X, Y and Z shall be given functions in turn of x, y and z, there becomes

$$X\left(\frac{dz}{dx}\right) + Y\left(\frac{dz}{dy}\right) = Z ,$$

therefore the resolution of this equation we have found contained here in this finite equation

$$\int \frac{dz}{Z} = \int \frac{dy}{Y} + f:\left(\int \frac{dx}{X} - \int \frac{dy}{Y}\right)$$

COROLLARY 2

181. But just as here the value satisfies the condition of the problem is apparent at once from the differentiation of this. For since indeed there shall be

$$\frac{dz}{Z} = \frac{dy}{Y} + \left(\frac{dx}{X} - \frac{dy}{Y}\right) f':\left(\int \frac{dx}{X} - \int \frac{dy}{Y}\right)$$

there will be

$$\left(\frac{dz}{dx}\right) = \frac{Z}{X} f':\left(\int \frac{dx}{X} - \int \frac{dy}{Y}\right) \quad \text{and} \quad \left(\frac{dz}{dy}\right) = \frac{Z}{Y} - \frac{Z}{Y} f':\left(\int \frac{dx}{X} - \int \frac{dy}{Y}\right)$$

from which there becomes

$$X\left(\frac{dz}{dx}\right) + Y\left(\frac{dz}{dy}\right) = Z .$$

SCHOLIUM

182. Therefore the solution can be completed in the same manner as we have found, without the introduction of the new letters p and q, and on retaining in place the differential values of these $\left(\frac{dz}{dx}\right)$ and $\left(\frac{dz}{dy}\right)$, but the individual letters are easier written and the calculation shall be shorter.

Moreover from this problem generally, where all three variables x, y and z besides the two differential values p and q are present in the determination, just a few [problems] are allowed to be resolved ; and besides this which we have treated, scarcely are we able to adjoin one or another in addition. From which at present significant advances in the calculus are desired. But so that the strength of this problem may be considered thoroughly, we shall add a few examples.

EXAMPLE 1

183. *If on putting* $dz = pdx + qdy$ *there must become* $zz = pxx + qyy$ *, to investigate the nature of the function z in general.*

Here therefore there will be $Z = zz$, $X = xx$ and $Y = yy$, from which we will have

$$\int \frac{dx}{X} = -\frac{1}{x}, \quad \int \frac{dy}{Y} = -\frac{1}{y}, \quad \text{and} \quad \int \frac{dz}{Z} = -\frac{1}{z},$$

with which values substituted for the solution we come upon

$$-\frac{1}{z} = -\frac{1}{y} + f:\left(\frac{1}{y} - \frac{1}{x}\right) \quad \text{or} \quad z = \frac{y}{1 - yf:\left(\frac{1}{y} - \frac{1}{x}\right)} .$$

Therefore some function is taken of the quantity $\frac{1}{y} - \frac{1}{x} = \frac{x-y}{xy}$ which if it is put V, there will be

$$z = \frac{y}{1-Vy} .$$

Just as if we put $V=\frac{n}{y}-\frac{n}{x}$, there will be $\frac{1}{z}=\frac{1}{y}-\frac{n}{y}+\frac{n}{x}=\frac{ny-(n-1)x}{xy}$ and hence

$$z=\frac{xy}{ny-(n-1)x},$$

from which there becomes

$$p=\left(\frac{dz}{dx}\right)=\frac{nxy}{\left(ny-(n-1)x\right)^2} \quad \text{and} \quad q=\left(\frac{dz}{dy}\right)=\frac{-(n-1)xx}{\left(ny-(n-1)x\right)^2}$$

and thus

$$pxx+qyy=\frac{xxyy}{\left(ny-(n-1)x\right)^2}=zz.$$

EXAMPLE 2

184. *If on putting $dz=pdx+qdy$ there must become $\frac{n}{z}=\frac{p}{x}+\frac{q}{y}$, to investigate the nature of the function z.*

Since here there shall be $X=\frac{1}{x}$, $Y=\frac{1}{y}$ and $Z=\frac{n}{z}$, there will be

$$\int\frac{dx}{X}=\frac{1}{2}xx, \quad \int\frac{dy}{Y}=\frac{1}{2}yy, \quad \text{and} \quad \int\frac{dz}{Z}=\frac{1}{2n}zz,$$

from which the solution will be prepared thus

$$\frac{1}{2n}zz=\frac{1}{2}yy+f:(xx-yy) \quad \text{or} \quad zz=nyy+f:(xx-yy);$$

indeed it is not necessary that the function be multiplied by *2n*, since that now actually involves all the operations themselves.

If $\alpha(xx-yy)$ is taken for this function, the particular solution will be considered

$$zz=\alpha xx+(n-\alpha)yy \quad \text{and} \quad z=\sqrt{\left(\alpha xx+(n-\alpha)yy\right)}$$

and hence

$$p=\left(\frac{dz}{dx}\right)=\frac{\alpha x}{\sqrt{\left(\alpha xx+(n-\alpha)yy\right)}} \quad \text{and} \quad q=\left(\frac{dz}{dy}\right)=\frac{(n-\alpha)y}{\sqrt{\left(\alpha xx+(n-\alpha)yy\right)}}$$

or $\frac{p}{x}=\frac{\alpha}{z}$ and $\frac{q}{y}=\frac{n-\alpha}{z}$ and thus from which

$$\frac{p}{x}+\frac{q}{y}=\frac{n}{z}$$

PROBLEM 32a [32 Repeated.]

185. *If on putting* $dz = pdx + qdy$ *there must become* $q = pT + V$ *, with T being some function of x and y, and with V a function of y and z, to investigate the nature of the function z.*

SOLUTION

With the prescribed value substituted in place of *q*, the form taken by the equation becomes

$$dz - Vdy = p\left(dx + Tdy\right).$$

Since now *V* involves only the two variables *y* and *z*, a multiplier *M* will be given returning the first part $dz - Vdy$ integrable ; therefore there is put in place

$$M\left(dz - Vdy\right) = dS.$$

In a similar manner because *T* contains only *x* and *y*, a multiplier *L* will be given rendering the second part $dx + Tdy$ integrable also ; therefore let there be

$$L\left(dx + Tdy\right) = dR,$$

thus so that now there shall be the known functions *R* and *S*, the one of *x* and *y*, now truly the other of *y* and *z*. Hence our equation adopts this form

$$\frac{dS}{M} = \frac{pdR}{L} \quad \text{or} \quad dS = \frac{pMdR}{L},$$

the integrability of which by necessity demands that $\frac{pM}{L}$ shall be a function of *R*. Therefore we may put

$$\frac{pM}{L} = f':R:$$

and there will be

$$S = f:R,$$

from which relation an equation is defined between *z* and *x*, *y*.

COROLLARY 1

186. As a particular case may be contained in this preceding problem ; since indeed there must be $Z = pX + qY$ everywhere, then there will be $q = -\frac{X}{Y}p + \frac{Z}{Y}$ and thus by an application of this problem there becomes $T = -\frac{X}{Y}$ and $V = \frac{Z}{Y}$.

COROLLARY 2

187. But nevertheless this problem appears to be of much greater extent than the previous, yet at this stage it is held within the closest limits and neither with the help of this or that simplest case is it possible for $z = py + qx$ to be resolved.

SCHOLIUM

188. This form $z = py + qx$ is altogether worthy of note, because by no known means so far is it considered possible to be resolved. For if from that there may be elicited $q = \frac{z-py}{x}$,
from which there becomes

$$dz - \frac{zdy}{x} = p\left(dx - \frac{ydy}{x}\right),$$

or p in a similar manner, no approach to a solution is evident; the cause of this difficulty clearly is situated in this, because the formula $dz - \frac{zdy}{x}$ cannot be rendered integrable by any multiplier or because this equation $dz - \frac{zdy}{x} = 0$ plainly is impossible, since x and likewise , y, and z shall be variables. Evidently now above in [§ 6] I have noted that not all differential equations between three variables are possible and likewise I have shown that the character of the possibility [of such a reduction], which here is to be reduced to such a form

$$dz + Pdx + Qdy = 0,$$

so that there shall be

$$P\left(\frac{dQ}{dz}\right) - Q\left(\frac{dP}{dz}\right) = \left(\frac{dQ}{dx}\right) - \left(\frac{dP}{dy}\right).$$

Now in our case there is $P = 0$ and $Q = \frac{-z}{x}$, from which here the character gives $0 = \frac{z}{xx}$; since that is false, also that equation shall be false $dz - \frac{zdy}{x} = 0$ is impossible, which indeed is evident by itself.

Yet truly for this case $z = py + qx$ a particular solution stands out, clearly $z = n(x + y)$, from which there becomes $p = q = n$. Moreover next we will give a general method [§ 195] from which a particular solution of this kind may be elicited.

EXAMPLE 1

189. *If on putting* $dz = pdx + qdy$ *there has to be* $py + qx = \frac{nxz}{y}$ *, to investigate the nature of the function z.*

Since hence there shall be $q = -\frac{py}{x} + \frac{nz}{y}$, then there shall be

$$T = \frac{-y}{x} \quad \text{and} \quad V = \frac{nz}{y},$$

from which there shall be

$$dS = M\left(dz - \frac{nzdy}{x}\right) \quad \text{and} \quad dR = L\left(dx - \frac{ydy}{x}\right).$$

Therefore there is taken $M = \frac{1}{y^n}$ so that there becomes $S = \frac{z}{y^n}$ and $L = 2x$, so that there becomes $R = xx - yy$. On which account we reach this solution

$$\frac{z}{y^n} = f:(xx - yy) \text{ or } z = y^n f:(xx - yy).$$

EXEMPLUM 2

190. *If on putting* $dz = pdx + qdy$ *there has to be* $pxx + qyy = nyz$ *, to define the nature of the function z.*

Therefore since there shall be $q = -\frac{pxx}{yy} + \frac{nz}{y}$ then there will be

$$T = \frac{-xx}{yy} \text{ and } V = \frac{nz}{y}$$

and thus this case is contained in our problem. From which it is required to deduce that

$$dR = L\left(dx - \frac{xxdy}{yy}\right) \text{ and } dS = M\left(dz - \frac{nzdy}{y}\right).$$

Whereby on assuming $L = \frac{1}{xx}$ there becomes $R = \frac{1}{y} - \frac{1}{x} = \frac{x-y}{xy}$ and on assuming $M = \frac{1}{y^n}$ there becomes $S = \frac{z}{y^n}$ and thus this solution itself is produced

$$\frac{z}{y^n} = f:\frac{x-y}{xy} \quad \text{and} \quad z = y^n f:\frac{x-y}{xy}.$$

PROBLEM 33

191. *If on putting* $dz = pdx + qdy$ *there must become* $p = qT + V$ *with a function* T *arising of* x *and y, but with V a function of x and z, to investigate the nature of the function z.*

SOLUTION

In a similar way as before if in place of *p* the prescribed value is substituted, there will be obtained

$$dz - Vdx = q(dy + Tdx).$$

Now on account of the nature of the functions *V* and *T* the following integrations can be put in place

$$M(dz - Vdx) = dS \quad \text{and} \quad N(dy + Tdx) = dR,$$

from which there becomes

$$\frac{dS}{M} = \frac{qdR}{N} \quad \text{or} \quad dS = \frac{Mq}{N}dR.$$

And hence this solution is easily deduced

$$\frac{Mq}{N} = f':R \quad \text{et} \quad S = f:R.$$

PROBLEM 34

192. *If on putting* $dz = pdx + qdy$ *there must be* $z = Mp + Nq$ *with some functions M and N of the two variables x and y present, from a certain particular solution, as agreed to be* $z = V$ *, to determine the nature of the function z in general.*

SOLUTIO

That particular value *V* which is a function of *x* and *y*, may be differentiated and there will be

$$dV = Pdx + Qdy;$$

which value, because it satisfies the equation on being substituted in place of *z*, where it becomes $p = P$ and $q = Q$, will be by hypothesis

$$V = MP + NQ.$$

Now generally there is put $z = Vf:T$ and there shall be

$$dT = Rdx + Sdy$$

and now it is required to find this function *T*. But from differentiation we elicit

$$p=\left(\tfrac{dz}{dx}\right)=Pf:T+VRf':T \quad \text{and} \quad q=\left(\tfrac{dz}{dy}\right)=Qf:T+VSf':T .$$

Whereby since there shall be $z=Mp+Nq=Vf:T$, then

$$Vf:T=\left(MP+NQ\right)f:T+V\left(MR+NS\right)f':T$$

and on account of $V=MP+NQ$ there will be had by hypothesis $MR+NS=0$, hence

$$dT=R\left(dx-\tfrac{Mdy}{N}\right).$$

Now it is not required to know *R*, but it suffices that the formula $Ndx-Mdy$ be considered, which can be rendered integrable with the help of a certain multiplier. Therefore the solution here can be returned easily, as from the prescribed condition $z=Mp+Nq$ there may be formed the actual equation

$$dT=R\left(Ndx-Mdy\right);$$

indeed with a suitable multiplier *R* the quantity *T* may be found by integration, with which found there shall be $z=Vf:T$.

OTHERWISE

The general value can be found more easily in this way. On account of the known value of *z* the known *V* may be put in place $z=Vv$ and there shall be $dv=rdx+sdy$; there will be

$$p=Pv+Vr \quad \text{and} \quad q=Qv+Vs$$

and thus

$$z=Mp+Nq=\left(MP+NQ\right)v+V\left(Mr+Ns\right)=Vv .$$

But there is $V=MP+NQ$, therefore

$$Mr+Ns=0 \quad \text{or} \quad s=-\tfrac{Mr}{N},$$

from which there becomes

$$dv=r\left(dx-\tfrac{Mdy}{N}\right)=\tfrac{r}{N}\left(Ndx-Mdy\right).$$

Therefore a suitable multiplier may be put in place by considering

$$R\left(Ndx-Mdy\right)=dT;$$

there will be $dv=\tfrac{r}{NR}dT$, from which it is deduced that

$$\frac{r}{NR} = f':T \quad \text{and} \quad v = f:T,$$

thus so that in general there shall be as before $z = Vv$.

COROLLARIUM 1

193. Therefore form the proposed condition $z = Mp + Nq$ as there shall be $dz = pdx + qdy$, the differential equation can be considered at once $R(Ndx - Mdy) = dT$, from which both the multiplier R as well as the integral T is found from that; and this operation does not depend on a particular known value V.

COROLLARY 2

194. But with the quantity T found if in every respect a solution becomes known particularly satisfying $z = V$, the general solution will be $z = Vf:T$. But it may be properly noted that it is not possible to elicit the general solution from a particular solution, unless there shall be a prescribed condition of this kind $z = Mp + Nq$.

EXAMPLE 1

195. *If on putting $dz = pdx + qdy$ there must become $z = py + qx$, from the particular value $z = x + y$ to define the general.*

Since here there shall be $M = y$ and $N = x$, we will have this equation

$$R(xdx - ydy) = dT$$

and hence

$$T = f:(xx - yy);$$

therefore the general solution will be

$$z = (x + y)f:(xx - yy).$$

EXAMPLE 2

196. *If on putting* $dz = pdx + qdy$ *there must become* $z = p(x+y) + q(y-x)$, *from the particular value* $z = \sqrt{(xx+yy)}$ *to define the general.*

On account of $M = x + y$ and $N = y - x$ the formula $Ndx - Mdy$ leads to this equation

$$R(ydx - xdx - xdy - ydy) = dT.$$

There is taken $R = \frac{1}{xx+yy}$, so that there shall be

$$dT = \frac{ydx - xdy}{xx+yy} - \frac{xdx + ydy}{xx+yy};$$

then

$$T = \text{Ang.tang.}\frac{x}{y} - \frac{1}{2}l(xx+yy).$$

And from this two-fold transcending value there will be

$$z = (xx+yy)\, f:T$$

and likewise it is apparent that no other particular value is to be given which shall be algebraic, in addition to the given $z = \sqrt{(xx+yy)}$.

EXAMPLE 3

197. *If on putting* $dz = pdx + qdy$ *there must become* $z = p(\alpha x + \beta y) + q(\gamma x + \delta y)$, *from the particular value found* $z = V$, *to define the nature of the function z in general.*

Here there is $M = \alpha x + \beta y$ and $N = \gamma x + \delta y$, from which we are led to this equation

$$R((\gamma x + \delta y)dx - (\alpha x + \beta y)dy) = dT,$$

where on account of the homogeneous form, there must be

$$R = \frac{1}{\gamma xx + (\delta - \alpha)xy - \beta yy},$$

so that there shall be

$$dT = \frac{(\gamma x + \delta y)dx - (\alpha x + \beta y)dy}{\gamma xx + (\delta - \alpha)xy - \beta yy},$$

in order to find the integral there is put $y = ux$, and there will be produced

$$dT = \frac{dx}{x} - \frac{(\alpha+\beta u)du}{\gamma+(\delta-\alpha)u-\beta uu}.$$

There shall be

$$\int \frac{(\alpha+\beta u)du}{\gamma+(\delta-\alpha)u-\beta uu} = lU;$$

then there will be $T = lx - lU$, and since the function of T shall be a function of $\frac{x}{U}$ also, in general there will be $z = Vf:\frac{x}{U}$. Moreover it is apparent, since U shall be a function of $u = \frac{y}{x}$, U shall be a homogeneous function of zero dimension of x and y and thus $\frac{x}{U}$ shall be a function of one dimension.

SCHOLIUM

198. Therefore in this example a difficulty remains, how the particular solution $z = V$ can be found; for unless at least one particular solution of this kind can be agreed upon, the general solution cannot indeed be absolved. But here for this case a particular solution can be elicited in the following manner; which since it may be considered a little unusual, there is no doubt why with the aid of this, this kind of calculation will be helped considerably in what is about to follow.

PROBLEM 35

199. *If on putting $dz = pdx + qdy$ there must become $z = p(\alpha x + \beta y) + q(\gamma x + \delta y)$, to investigate a particular value, which substituted in place of z satisfies this condition.*

SOLUTION

This work will be successful, if we should look for a value of z of this kind, which shall be a function of zero dimensions of x and y, or on putting $y = ux$, which shall be a function of u only. Therefore we may put $z = f:u = f:\frac{y}{x}$ and there will be $f':u = \frac{dz}{du}$; but on account of $du = \frac{dy}{x} - \frac{ydx}{xx}$ there will be

$$dz = \left(\frac{dy}{x} - \frac{udx}{x}\right) f':u$$

hence

$$p = -\frac{u}{x} f':u = -\frac{udz}{xdu} \quad\text{and}\quad q = -\frac{1}{x} f':u = \frac{dz}{xdu}$$

With which values substituted for p and q, the prescribed condition becomes

$$z = x(\alpha+\beta u)p + x(\gamma+\delta u)q = \frac{-udz(\alpha+\beta u)+dz(\gamma+\delta u)}{du},$$

from which there is made

$$\frac{dz}{z}=\frac{du}{\gamma+(\delta-\alpha)u-\beta uu}.$$

We may put

$$\int\frac{du}{\gamma+(\delta-\alpha)u-\beta uu}=lV,$$

so that there becomes $z=V$, and V will be a particular value satisfying z.

COROLLARY 1

200. With the aid of this value V found of the preceding example, the general solution can be easily found. Evidently there will be $z=Vf:\frac{x}{U}$ with

$$\frac{dU}{U}=\frac{(\alpha+\beta u)du}{\gamma+(\delta-\alpha)u-\beta uu}$$ present,

from which it is apparent that the quantity U can be found from the particular value of V itself.

COROLLARY 2

201. Indeed there shall be

$$lU=-l\sqrt{(\gamma+(\delta-\alpha)u-\beta uu)}+\int\frac{\frac{1}{2}(\delta+\alpha)du}{\gamma+(\delta-\alpha)u-\beta uu}$$

and thus

$$lU=-l\sqrt{(\gamma+(\delta-\alpha)u-\beta uu)}+\frac{1}{2}(\delta+\alpha)lV$$

or

$$U=\frac{V^{\frac{1}{2}(\alpha+\delta)}}{\sqrt{(\gamma+(\delta-\alpha)u-\beta uu)}}$$

hence

$$\frac{x}{U}=\frac{\sqrt{(\gamma xx+(\delta-\alpha)xy-\beta yy)}}{V^{\frac{1}{2}(\alpha+\delta)}}$$

COROLLARY 3

202. On which account with the particular value found $z=V$, so that there shall be

$$\frac{dV}{V}=\frac{du}{\gamma+(\delta-\alpha)u-\beta uu}$$

with $u=\frac{y}{x}$ present, the value generally satisfying

$$z = Vf : \frac{\gamma xx + (\delta - \alpha)xy - \beta yy}{V^{\alpha+\delta}} = Vf : \frac{x(\gamma x + \delta y) - y(\alpha x + \beta y)}{V^{\alpha+\delta}}.$$

COROLLARY 4

203. Hence another particular value is deduced, which is algebraic always ; evidently this shall be

$$z = \left(x(\gamma x + \delta y) - y(\alpha x + \beta y) \right)^{\frac{1}{\alpha+\delta}}$$

or some multiplier of this. But unless V shall be an algebraic quantity, all the remaining values will be transcending and are contained in this form

$$z = \left(x(\gamma x + \delta y) - y(\alpha x + \beta y) \right)^{\frac{1}{\alpha+\delta}} f : \frac{x(\gamma x + \delta y) - y(\alpha x + \beta y)}{V^{\alpha+\delta}}.$$

SCHOLIUM

204. There is a single case, in which $\delta = -\alpha$ and the proposed condition

$$z = p(ax + \beta y) + q(\gamma x - \alpha y),$$

demands a special treatment. But first on putting $u = \frac{y}{x}$ for the particular value of $z = V$, there will be

$$lV = \int \frac{du}{\gamma - 2\alpha u - \beta uu}$$

Then indeed on account of $\frac{dU}{U} = \frac{(\alpha + \beta u)du}{\gamma - 2\alpha u - \beta uu}$, there will be

$$U = \frac{1}{\sqrt{(\gamma - 2\alpha u - \beta uu)}} \quad \text{and} \quad \frac{x}{U} = \sqrt{(\gamma xx - 2\alpha xy - \beta yy)},$$

thus so that now the general value will be

$$z = Vf : (\gamma xx - 2\alpha xy - \beta yy).$$

Accordingly indeed it is evident that the formula $f : \sqrt{T}$ can be expressed by $f : T$. Therefore unless V shall be an algebraic function, in this case no particular algebraic solution can be put in place.

EXAMPLE 1

205. *If on putting $dz = pdx + qdy$ there must become $nz = py - qx$, to investigate the nature of the function z.*

On comparison with our general form established there becomes

$$\alpha = 0, \quad \beta = \tfrac{1}{n}, \quad \gamma = -\tfrac{1}{n}, \quad \delta = 0.$$

Therefore this case on account of $\delta = -\alpha$ pertains to the preceding paragraph, from which there becomes

$$lV = \int \tfrac{ndu}{-1-uu} = -n\text{Ang.tang.}u.$$

Therefore since there shall be $u = \frac{y}{x}$, the general form is

$$z = e^{-n\text{Ang. tang.}\frac{y}{x}} f:(xx + yy).$$

EXAMPLE 2

206. *If on putting $dz = pdx + qdy$ there must become $z = p(x + y) - q(x + y)$, to investigate the general nature of the function z.*

With the comparison made there arises

$$\alpha = 1, \quad \beta = 1, \quad \gamma = -1, \quad \delta = -1.$$

and hence

$$lV = \int \tfrac{du}{-1-2u-uu} = \tfrac{1}{1+u} \quad \text{and} \quad V = e^{\frac{1}{1+u}}$$

and the general solution is

$$z = e^{\frac{x}{x+y}} f:(x + y).$$

EXAMPLE 3

207. *If on putting* $dz = pdx + qdy$ *there must become* $z = p(x-2y) + q(2x-3y)$, *to investigate the nature of the function z.*

Therefore since here there shall be

$$\alpha = 1, \quad \beta = -2, \quad \gamma = 2, \quad \delta = -3,$$

there will be initially

$$lV = \int \frac{ndu}{2-4u+2uu} = \frac{1}{2(1-u)} = \frac{x}{2(x-y)},$$

and because it is not the case that $\delta = -\alpha$, the general solution is produced at once

$$z = (2xx - 4xy + 2yy)^{-\frac{1}{2}} f : \frac{2xx-4xy+2yy}{V^{-2}}$$

and on account of $V = e^{\frac{x}{2(x-y)}}$ there will be

$$z = \frac{1}{x-y} f : (x-y)^2 e^{\frac{x}{x-y}}$$

From which the simplest solution is $z = \frac{1}{x-y}$.

SCHOLIUM

208. Here deservedly we may ask, by what agreement this general solution is able to be found at once without the aid of a special solution; but in the following manner that investigation is considered to be put in place.

Since there shall be

$$p(\alpha x + \beta y) = z - q(\gamma x + \delta y) \text{ and } q(\gamma x + \delta y) = z - p(\alpha x + \beta y),$$

if each value separately is substituted into the form

$$dz = pdx + qdy,$$

the two following equations will be produced

$$(\alpha x + \beta y)dz = zdx - q(\gamma x + \delta y)dx + q(\alpha x + \beta y)dy,$$
$$(\gamma x + \delta y)dz = zdy + p(\gamma x + \delta y)dx - p(\alpha x + \beta y)dy.$$

The first may be multiplied by the indefinite *M*, the second by *N*, and the sum of the products will give

$$dz\left(M\left(\alpha x+\beta y\right)+N\left(\gamma x+\delta y\right)\right)-z\left(Mdx+Ndy\right)$$
$$=\left(Np-Mq\right)\left(\left(\gamma x+\delta y\right)dx-\left(ax+\beta y\right)dy\right),$$

where now *M* and *N* must be taken thus, so that the first member admits to integration ; for then the integral of this will be equal to some function of the quantity

$$\int\frac{(\gamma x+\delta y)dx-(ax+\beta y)dy}{\gamma xx+(\delta-\alpha)xy-\beta yy},$$

as we have instructed to define above (§ 197) ; from which it is apparent that the integral becomes $=f:\frac{x}{U}$. But it is evident that the *M* and *N* are required to be functions of the same kind, so that this equation is made possible

$$\frac{dz}{z}=\frac{Mdx+Ndy}{M(\alpha x+\beta y)+N(\gamma x+\delta y)}$$

or so that the latter member admits to integration ; but if the integral of this shall be $=lV$, then there will be $\frac{z}{V}=f:\frac{x}{U}$. For this integration we may put $y=ux$ and both *M* and *N* are functions of *u;* then there will be

$$\frac{dz}{z}=\frac{(M+Nu)dx+Nxdx}{Mx(\alpha+\beta u)+Nx(\gamma+\delta u)},$$

where the integration succeeds on taking $M=-Nu$, so that there shall be

$$\frac{dz}{z}=\frac{du}{\gamma+(\delta-\alpha)u-\beta uu}\quad\text{or}\quad lV=\int\frac{du}{\gamma+(\delta-\alpha)u-\beta uu},$$

in short as before.

PROBLEM 36

209. *If on putting* $dz=pdx+qdy$ *there must become* $Z=pP+qQ$ *with only a function Z of z present, but with some functions P and Q of x and y given, to investigate the nature of the function z.*

SOLUTION

The following equations may be formed from the proposed

$$Ldz=Lpdx+Lqdy,$$

$$MZdx=MpPdx+MqQdx,\ \ NZdy=NpPdy+NqQdy,$$

which gathered into one sum will give

$$Ldz + Z\left(Mdx + Ndy\right) = p\left(\left(L + MP\right)dx + NPdy\right) + q\left(\left(L + NQ\right)dy + MQdx\right).$$

As now the latter part has a factor free from the letters p and q , there becomes

$$L + MP{:}NP = MQ{:}L + NQ,$$

[on finding $\frac{dy}{dx}$ from each coefficient which must be the same] from which here becomes

$$LL + LNQ + LMP = 0 \quad \text{or} \quad L = -MP - NQ\,,$$

with which value introduced there will be

$$-\,dz\left(MP + NQ\right) + Z\left(Mdx + Ndy\right) = \left(Mq - Np\right)\left(Qdx - Pdy\right).$$

Now since P and Q shall be given functions of x and y, a multiplier R will be given, so that there becomes

$$R\left(Qdx - Pdy\right) = dU$$

and thus

$$-\,dz\left(MP + NQ\right) + Z\left(Mdx + Ndy\right) = \tfrac{Mq - Np}{R}\,dU\ .$$

For the first part indefinite functions M and N may be taken thus, so that the formula $\frac{Mdx+Ndy}{MP+NQ}$ emerges integrable, which is permitted to happen always, and there shall be

$$\tfrac{Mdx+Ndy}{MP+NQ} = dV$$

and on account of

$$Mdx + Ndy = \left(MP + NQ\right)dV$$

our equation can adopt this form

$$\left(MP + NQ\right)\left(-dz + ZdV\right) = \tfrac{Mq-Np}{R}\,dU$$

or

$$\tfrac{dz}{Z} - dV = \tfrac{Np-Mq}{RZ\left(MP+NQ\right)}\,dU\ .$$

Now there is put in place

$$\tfrac{Np-Mq}{RZ\left(MP+NQ\right)} = f\,'{:}U$$

and there will be had

$$\int \frac{dz}{Z} - V = f:U \quad \text{seu} \quad \int \frac{dz}{Z} = V + f:U ,$$

from which z is determined by x and y.

COROLLARY 1

210. Therefore for the solution of the problem there is sought initially a multiplier R according to the formula $Qdx - Pdy$ returning that integrable and there is put in place

$$R(Qdx - Pdy) = dU ,$$

from which the quantity U is deduced expressed by the two variables x and y.

COROLLARY 2

211. Hence the quantities M and N may be taken thus, so that the formula $\frac{Mdx+Ndy}{MP+NQ}$ is made integrable; the integral of which if it is put $=V$, at once will give the general solution of the problem, which gives

$$\int \frac{dz}{Z} = V + f:U$$

EXAMPLE

212. *If P and Q shall be homogeneous functions of x and y, each of a number of dimension = n, to complete the solution of the problem.*

There is put $y = ux$ and P as well as Q may become made from the power x^n into a certain function of u. Therefore let

$$P = x^n S \quad \text{and} \quad Q = x^n T$$

and both S and T will be given functions of u. Then truly on account of $dy = udx + xdu$ the formula $Qdx - Pdy$ will change into

$$x^n Tdx - x^n Sudx - x^{n+1} Sdu = x^n \left((T - Su)dx - Sxdu \right).$$

Therefore there is assumed

$$R = \frac{1}{x^{n+1}(T - Su)}$$

and there becomes

$$dU = \frac{dx}{x} - \frac{Sdu}{T - Su},$$

from which U is deduced.

Hence for the other quantity V we will have this equation

$$dV = \frac{(M+Nu)dx+Nxdu}{x^n(MS+NT)},$$

where now it is easy for functions of u of this kind to be assumed for M and N, so that this formula is allowed to be integrated. Evidently the integral will be

$$V = \frac{-M-Nu}{(n-1)x^{n-1}(MS+NT)}$$

but M and N or $\frac{M}{N} = K$ thus must be accepted, so that there becomes

$$\frac{-1}{(n-1)x^{n-1}} d.\frac{K+u}{KS+T} = \frac{1}{x^{n-1}} \cdot \frac{du}{KS+T}$$

or

$$-KKdS + KSdu - uKdS - uSdK + TdX - KdT + Tdu - udT + (n-1)du(KS+T) = 0,$$

which is reduced to this form

$$(T-Su)dK + K(nSdu - udS - dT) - KKdS + nTdu - udT = 0.$$

With which resolution of the equation permitted the quantity K is known, with which found there shall be

$$V = \frac{-K-u}{(n-1)x^{n-1}(KS+T)}$$

But since with the solution of that equation being difficult, there is put at once

$$\frac{K+u}{KS+T} = v$$

and there becomes

$$K = \frac{Tv-u}{1-Sv} \text{ and } KS+T = \frac{T-Su}{1-Sv},$$

from which there becomes

$$dv + \frac{(n-1)du(1-Sv)}{T-Su} = 0$$

with which resolved there shall be

$$V = \frac{-v}{(n-1)x^{n-1}}.$$

COROLLARY

213. But the case in which $n=1$, requires a singular treatment. But then it is easily extended to assume there must be $M=-Nu$, so that there becomes $dV=\frac{du}{T-Su}$; from which, after the quantity V has been found, there will be always

$$\int \frac{dz}{Z}=V+f:U\,.$$

SCHOLIUM

214. Since the three variables *x, y, z* may be permuted between themselves, it is apparent that this problem can be extended more widely. Evidently if the proposed condition may contain this equation $pP+qQ+R=0$, not only does the method of solving used succeed, if *R* shall be a function of *z* with *P* and *Q* functions of *x* and *y*, but also, if *P* were a function of *x* and both *Q* and *R* functions of *y* and *z*, and then truly also, if *Q* were a function of *y*, but *P* and *R* functions of the two remaining [variables] *x* and *z*. Truly the condition is returned with that treated previously, so that the two differential formulas *p* and *q* shall in turn be separated from each other and not be of more than one dimension, and also if in these cases they agree with this great restriction. But if again the condition becomes more complicated, it is seen that a solution can scarcely be hoped for at any time ; yet meanwhile I will offer a case of this kind, in which it is allowed to set out a solution.

PROBLEM 37

215. *If on putting $dz=pdx+qdy$ there must become $q=Ap^{n}x^{\lambda}y^{\mu}z^{\nu}$, to investigate the nature of the function z in general.*

SOLUTION

On putting this value in place of q we will have

$$dz=pdx+Ap^{n}x^{\lambda}y^{\mu}z^{\nu}dy\,,$$

from which there becomes

$$Ay^{\mu}dy=p^{-n}x^{-\lambda}z^{-\nu}\left(dz-pdx\right).$$

There is put $p^{-n}x^{-\lambda}z^{-\nu}=t$, so that there shall be $p=t^{-\frac{1}{n}}x^{-\frac{\lambda}{n}}z^{-\frac{\nu}{n}}$, and there will be

$$Ay^{\mu}dy=tdz-t^{\frac{n-1}{n}}x^{-\frac{\lambda}{n}}z^{-\frac{\nu}{n}}dx.$$

Again there is put $t^{n-1}z^{-\nu}=u^{n}$ or $t=z^{\frac{\nu}{n-1}}\,u^{\frac{n}{n-1}}$; there will be

$$Ay^{\mu}dy=u^{\frac{n}{n-1}}z^{\frac{\nu}{n-1}}dz-ux^{-\frac{\lambda}{n}}dx.$$

Now for the parts, as long as it can happen, we come upon integrals

$$\frac{A}{\mu+1}y^{\mu+1}=\frac{n-1}{n+\nu-1}u^{\frac{n}{n-1}}z^{\frac{n+\nu-1}{n-1}}-\frac{nu}{n-\lambda}x^{\frac{n-\lambda}{n}}-\int du\left(\frac{n}{n+\nu-1}u^{\frac{1}{n-1}}z^{\frac{n+\nu-1}{n-1}}-\frac{n}{n-\lambda}x^{\frac{n-\lambda}{n}}\right).$$

and now it is allowed to set out the solution by the precepts given; clearly there is put in place

$$\frac{1}{n+\nu-1}u^{\frac{1}{n-1}}z^{\frac{n+\nu-1}{n-1}}-\frac{1}{n-\lambda}x^{\frac{n-\lambda}{n}}=f':u$$

and there becomes

$$\frac{A}{\mu+1}y^{\mu+1}=\frac{n-1}{n+\nu-1}u^{\frac{n}{n-1}}z^{\frac{n+\nu-1}{n-1}}-\frac{nu}{n-\lambda}x^{\frac{n-\lambda}{n}}-nf:u\,,$$

and if u can be elicited from these two equations, certainly z is given by x and y.

COROLLARY1

216. The case $n=1$ demands a special treatment ; for since on putting $p=\frac{1}{t}x^{-\lambda}z^{-\nu}$ there shall be $Ay^{\mu}dy=tdz-x^{-\lambda}z^{-\nu}dx$, there shall be

$$\frac{A}{\mu+1}y^{\mu+1}=\frac{1}{\lambda-1}x^{1-\lambda}z^{-\nu}+\int dz\left(t+\frac{\nu}{\lambda-1}x^{1-\lambda}z^{-\nu-1}\right)$$

and hence at once it is concluded

$$\frac{A}{\mu+1}y^{\mu+1}=\frac{1}{\lambda-1}x^{1-\lambda}z^{-\nu}+f:z.$$

COROLLARY 2

217. But the cases $n+\nu-1=0$ and $n-\lambda=0$ cause no trouble, since there shall be in the first case

$$\frac{n-1}{n+\nu-1}z^{\frac{n+\nu-1}{n-1}}=lz\,,$$

moreover for the second

$$\frac{n}{n-\lambda}x^{\frac{n-\lambda}{n}}=lx\,,$$

which values it is required to introduce into the solution.

EXAMPLE

218. *If on putting $dz = pdx + qdy$ there must become $pqxy = az$ or $q = \frac{az}{pxy}$, to investigate r the function z.*

Therefore there will be

$$dz = pdx + \frac{azdy}{pxy} \quad \text{or} \quad \frac{ady}{y} = \frac{px}{z}(dz - pdx)$$

There is put $\frac{px}{z} = t$ or $p = \frac{tz}{x}$; then there will be

$$\frac{ady}{y} = tdz - \frac{ttzdx}{x}.$$

Again there is put $ttz = uu$ or $t = \frac{u}{\sqrt{z}}$, so that there shall be

$$\frac{ady}{y} = \frac{udz}{\sqrt{z}} - \frac{uudx}{x}$$

and, as long as it can happen, on integrating

$$aly = 2u\sqrt{z} - uulx - \int du\left(2\sqrt{z} - 2ulx\right),$$

thus so that now after the integration sign there is found a single differential *du*. Therefore on putting

$$\sqrt{z} - ulx = f':u$$

there will be

$$aly = 2u\sqrt{z} - uulx - 2f:u = uulx + 2uf':u - 2f:u .$$

For the simplest case there is taken $f':u = 0$ and $f:u = 0$; then there will be $u = \frac{\sqrt{z}}{lx}$ and thus

$$aly = \frac{2z}{lx} - \frac{z}{lx} = \frac{z}{lx},$$

thus so that for the simplest case there shall be $z = alx.ly$.

If there is put $f:u = ulc$ and $f:u = \frac{1}{2}uulc$, there will be

$$u = \frac{\sqrt{z}}{lx+lc} = \frac{\sqrt{z}}{lcx} \quad \text{and} \quad aly = \frac{2z}{lcx} - \frac{zlx}{(lcx)^2} - \frac{zlc}{(lcx)^2} = \frac{z}{lcx}$$

thus so that there shall be $z = aly(lc + lx)$; but more generally there will be

$$z = a(lb + ly)(lc + lx).$$

SCHOLIUM

219. The methods treated at this point will not be much enlarged on, if in place of the two variables *x* and *y*, the function of which must be *z*, two other variables *t* and *u* may be introduced, the relation of which to the others is given. Thus if *z* shall be a function of the two variables *x* and *y*, so that from that there may be produced

$$dz = pdx + qdy\ ,$$

and in place of *x* and *y* other new variables *t* and *u* are introduced, so that now with the differentiation in place there may be produced

$$dz = rdt + sdu\ ,$$

it is sought, how *r* and *s* may be determined in terms of *p* and *q* for the relation between the former variables *x, y* and the new *t* and *u* can be made firm. Hence therefore as *x* as well as *y* will be equal to a certain function of *t* and *u*; since which is given, there will be

$$dx = Pdt + Qdu \quad \text{and} \quad dy = Rdt + Sdu\ ,$$

thus so that with this substitution made *z* now shall be a function of *t* and *u*. Therefore since there shall be $dz = pdx + qdy$, now there will be

$$dz = (Pp + Rq)dt + (Qp + Sq)du\ .$$

Now truly by hypothesis $dz = rdt + sdu$, from which there will be had

$$r = Pp + Rq \quad \text{and} \quad s = Qp + Sq\ .$$

Whereby with this substitution made the values of the new differentials thus may be determined from the preceding , so that there shall be

$$\left(\tfrac{dz}{dt}\right) = P\left(\tfrac{dz}{dx}\right) + R\left(\tfrac{dz}{dy}\right) \quad \text{and} \quad \left(\tfrac{dz}{du}\right) = Q\left(\tfrac{dz}{dx}\right) + S\left(\tfrac{dz}{dy}\right).$$

From which also, since there shall be in turn

$$Qr - Ps = (QR - PS)q \quad \text{and} \quad Sr - Rs = (PS - QR)p,$$

we conclude to become

$$\left(\tfrac{dz}{dx}\right) = \tfrac{S}{PS-QR}\left(\tfrac{dz}{dt}\right) - \tfrac{R}{PS-QR}\left(\tfrac{dz}{du}\right)$$

and

$$\left(\tfrac{dz}{dy}\right) = \tfrac{-Q}{PS-QR}\left(\tfrac{dz}{dt}\right) + \tfrac{P}{PS-QR}\left(\tfrac{dz}{du}\right)$$

Or since x and y and likewise z shall be functions of t and u, this relation can be expressed thus, so that there shall be

$$\left(\frac{dz}{dt}\right)=\left(\frac{dx}{dt}\right)\left(\frac{dz}{dx}\right)+\left(\frac{dy}{dt}\right)\left(\frac{dz}{dy}\right) \quad \text{and} \quad \left(\frac{dz}{du}\right)=\left(\frac{dx}{du}\right)\left(\frac{dz}{dx}\right)+\left(\frac{dy}{du}\right)\left(\frac{dz}{dy}\right).$$

Hence it is brought about, so that problems which are able to be resolved by a certain given relation between p, q, x, y, z, these also are able to be resolved thence with a relation between r, s, t, u and z ; from which problems arise often, for which the solution are considered extremely difficult, from which the aids are not to be scorned, and are able to bring a direction into the analysis; but since the use is seen especially in differential formulas of the second grade, not dwelling further on these here I shall proceed to explaining these equations.

CAPUT VI

DE RESOLUTIONE AEQUATIONUM QUIBUS RELATIO INTER BINAS FORMULAS DIFFERENTIALES $\left(\frac{dz}{dx}\right)$, $\left(\frac{dz}{dy}\right)$ ET OMNES TRES VARIABILES x, y, z QUAECUNQUE DATUR

PROBLEMA 30

174. *Si posito* $dz = pdx + qdy$ *debeat esse* $nz = px + qy$ *, indolem functionis z in genere investigare.*

SOLUTIO

Ope relationis datae elidatur vel p vel q; scilicet cum sit $q = \frac{nz}{y} - \frac{px}{y}$, erit

$$dz = pdx + \frac{nzdy}{y} - \frac{pxdy}{y},$$

quae aequatio in hanc formam transfundatur

$$dz - \frac{nzdy}{y} = p\left(dx - \frac{xdy}{y}\right) = pyd.\frac{x}{y}.$$

Ut prius membrum $dz - \frac{nzdy}{y}$ integrabile reddatur, multiplicetur aequatio per $\frac{1}{z}$ funct. $\frac{z}{y^n}$ seu particulariter per $\frac{1}{y^n}$ eritque

$$d.\frac{z}{y^n} = py^{1-n}d.\frac{x}{y}.$$

Quo facto evidens est poni debere $py^{1-n} = f':\frac{x}{y}$, ut fiat

$$\frac{z}{y^n} = f:\frac{x}{y} \quad \text{seu} \quad z = y^n f:\frac{x}{y}$$

Unde patet fore z functionem homogeneam ipsarum x et y dimensionum numero existente = n. Si in genere aequatio multiplicetur per $\frac{1}{z}$ funct. $\frac{z}{y^n}$, erit partis prioris integrale $F:\frac{z}{y^n}$, pro parte autem altera si ponatur $\frac{py}{z}$ funct. $\frac{z}{y^n} = f':\frac{x}{y}$, erit

$$F:\frac{z}{y^n} = f:\frac{x}{y}$$

atque ut ante $\frac{z}{y^n}$ aequabitur functioni cuicunque ipsius $\frac{x}{y}$.

COROLLARIUM 1

175. Cum *z* aequetur functioni homogeneae *n* dimensionum ipsarum *x* et *y*, erunt *p* et *q* functiones $n-1$ dimensionum. Scilicet cum sit $z = y^n f:\frac{x}{y}$, erit

$$p = y^{n-1} f':\tfrac{x}{y} \quad \text{et} \quad q = ny^{n-1} f:\tfrac{x}{y} - xy^{n-2} f':\tfrac{x}{y},$$

unde fit manifesto $nz = px + qy$.

COROLLARIUM 2

176. Si *p* et *q* fuerint functiones $n-1$ dimensionum ipsarum *x* et *y* ac formula $pdx + qdy$ sit integrabilis seu $\left(\frac{dp}{dy}\right) = \left(\frac{dq}{dx}\right)$, tum integrale certo erit $\frac{px+qy}{n}$, quae proprietas nonnunquam insignem usum habere potest.

SCHOLION

177. Fundamentum huius solutionis in hoc consistit, quod aequatio integranda in duas partes resolvatur, quarum utraque ope certi multiplicatoris integrabilis reddi queat, unde deinceps una quantitas variabilis, cuius differentiale in aequatione non occurrit, determinetur. Hinc aequatio nostra

$$dz - \frac{nzdy}{y} = p\left(dx - \frac{xdy}{y}\right)$$

etiam ita repraesentari potest

$$\frac{dx}{y} - \frac{xdy}{yy} = \frac{1}{py}\left(dz - \frac{nzdy}{y}\right) = \frac{y^{n-1}}{p}\left(\frac{dz}{y^n} - \frac{nzdy}{y^{n+1}}\right)$$

seu

$$d.\frac{x}{y} = \frac{y^{n-1}}{p}\, d.\frac{z}{y^n}.$$

Sit ergo $\frac{y^{n-1}}{p} = F':\frac{z}{y^n}$ eritque $\frac{x}{y} = F:\frac{z}{y^n}$ ac vicissim $\frac{z}{y^n} = f:\frac{x}{y}$ ut ante.

Possumus etiam statim *z* ex calculo elidere; cum enim sit $nz = px + qy$, erit

$$ndz = pdx + qdy + xdp + ydq.$$

At est

$$ndz = npdx + nqdy$$

per hypothesin ideoque

$$(n-1)\,pdx - xdp + (n-1)qdy - ydq = 0$$

seu

$$x^n\left(\frac{(n-1)pdx}{x^n}-\frac{dp}{x^{n-1}}\right)+y^n\left(\frac{(n-1)qdy}{y^n}-\frac{dq}{y^{n-1}}\right)=0,$$

quae reducitur ad hanc formam

$$-x^n d.\frac{p}{x^{n-1}}-y^n d.\frac{q}{y^{n-1}}=0 \quad \text{seu} \quad d.\frac{q}{y^{n-1}}=-\frac{x^n}{y^n}d.\frac{p}{x^{n-1}}.$$

Statuatur

$$\frac{x^n}{y^n}=-f':\frac{p}{x^{n-1}};$$

erit

$$\frac{q}{y^{n-1}}=f:\frac{p}{x^{n-1}}.$$

Vel posito $\frac{x}{y}=v$ si ob $v^n=-f':\frac{p}{x^{n-1}}$ reciproce ponatur

$$\frac{p}{x^{n-1}}=u=\frac{1}{v^{n-1}}F':v,$$

ut sit $f':u=-v^n$, reperietur

$$\int du\, f':u=f:u=nF:v-vF':v.$$

Hinc

$$p=\frac{x^{n-1}}{v^{n-1}}F':v=y^{n-1}F':\frac{x}{y}$$

et

$$q=y^{n-1}f:u=ny^{n-1}F:\frac{x}{y}-xy^{n-2}F':\frac{x}{y}$$

ideoque

$$nz=px+qy=ny^nF:\frac{x}{y} \quad \text{seu} \quad z=y^nF:\frac{x}{y}$$

ut ante.

PROBLEMA 31

178. *Si posito $dz=pdx+qdy$ debeat esse $\alpha px+\beta qy=nz$, indolem functionis z investigare.*

SOLUTIO

Ex conditione praescripta eliciatur ut ante $q=\frac{nz}{\beta y}-\frac{\alpha px}{\beta y}$ eritque

$$dz-\frac{nzdy}{\beta y}=pdx-\frac{\alpha pxdy}{\beta y},$$

quae aequatio per $y^{\frac{n}{\beta}}$ divisa dat

$$d.\frac{z}{y^{n:\beta}} = \frac{p}{y^{n:\beta}}\left(dx - \frac{\alpha x dy}{\beta y}\right) = \frac{py^{\alpha:\beta}}{y^{n:\beta}}d.\frac{x}{y^{\alpha:\beta}}$$

Quodsi ergo ponamus

$$py^{(\alpha-n):\beta} = f':\frac{x}{y^{\alpha:\beta}}$$

habebimus solutionem

$$z = y^{n:\beta} f:\frac{x}{y^{\alpha:\beta}}$$

At functio ipsius $\frac{x}{y^{\alpha:\beta}}$ reducitur ad functionem ipsius $\frac{x^{\beta}}{y^{\alpha}}$, unde *z* etiam ita per *x* et *y* determinatur, ut sit

$$z = y^{n:\beta} f:\frac{x^{\beta}}{y^{\alpha}} \quad \text{vel etiam} \quad z^{1:n} = y^{1:\beta} f:\frac{x^{1:\alpha}}{y^{1:\beta}}$$

Quodsi ergo quantitates $x^{1:\alpha}$ et $y^{1:\beta}$ unam dimensionem constituere censeantur, $z^{1:n}$ aequabitur earundem functioni unius dimensionis, ipsa autem quantitas *z* earundem functioni *n* dimensionum. Vel sumta pro *z* functione quacunque homogenea *n* dimensionum binarum variabilium *t* et *u* scribatur deinde $t = x^{1:\alpha}$ et $u = y^{1:\beta}$ ac prodibit functio conveniens pro *z*.

PROBLEMA 32

179. *Si posito* $dz = pdx + qdy$ *debeat esse* $Z = pX + qY$ *denotante Z functionem ipsius z, X ipsius x et Y ipsius y, indolem functionis z in genere investigare.*

SOLUTIO

Ex conditione praescripta elicitur $q = \frac{Z}{Y} - \frac{pX}{Y}$, qui valor substitutus praebet

$$dz - \frac{Zdy}{Y} = p\left(dx - \frac{Xdy}{Y}\right)$$

hincque

$$\frac{dz}{Z} - \frac{dy}{Y} = \frac{p}{Z}\left(dx - \frac{Xdy}{Y}\right) = \frac{pX}{Z}\left(\frac{dx}{X} - \frac{dy}{Y}\right),$$

ubi iam resolutio est manifesta. Statuatur scilicet

$$\frac{pX}{Z} = f':\left(\int\frac{dx}{X} - \int\frac{dy}{Y}\right)$$

eritque

$$\int\frac{dz}{Z} - \int\frac{dy}{Y} = f:\left(\int\frac{dx}{X} - \int\frac{dy}{Y}\right),$$

unde valor ipsius z per x et y definitur.

COROLLARIUM 1

180. Hic ergo z ita per x et y definiri debet, ut, si X, Y et Z datae sint functiones singillatim ipsarum x, y et z, fiat

$$X\left(\frac{dz}{dx}\right)+Y\left(\frac{dz}{dy}\right)=Z,$$

cuius ergo aequationis resolutionem hic invenimus hac aequatione finita contentam

$$\int\frac{dz}{Z}=\int\frac{dy}{Y}+f:\left(\int\frac{dx}{X}-\int\frac{dy}{Y}\right)$$

COROLLARIUM 2

181. Quemadmodum autem hic valor conditioni problematis satisfaciat, ex eius differentiatione statim patet. Cum enim sit

$$\frac{dz}{Z}=\frac{dy}{Y}+\left(\frac{dx}{X}-\frac{dy}{Y}\right)f':\left(\int\frac{dx}{X}-\int\frac{dy}{Y}\right)$$

erit

$$\left(\frac{dz}{dx}\right)=\frac{Z}{X}f':\left(\int\frac{dx}{X}-\int\frac{dy}{Y}\right)\quad\text{et}\quad\left(\frac{dz}{dy}\right)=\frac{Z}{Y}-\frac{Z}{Y}f':\left(\int\frac{dx}{X}-\int\frac{dy}{Y}\right)$$

unde fit

$$X\left(\frac{dz}{dx}\right)+Y\left(\frac{dz}{dy}\right)=Z.$$

SCHOLION

182. Solutio ergo eodem modo, ut fecimus, sine introductione novarum litterarum p et q absolvi potest retinendo earum loco valores differentiales $\left(\frac{dz}{dx}\right)$ et $\left(\frac{dz}{dy}\right)$ facilius autem singulae litterae scribuntur calculusque fit brevior.

Ceterum ex hoc problematum genere, ubi omnes tres variabiles x, y et z praeter binos valores differentiales p et q in determinationem ingrediuntur, paucissima resolvere licet; ac praeter hoc, quod tractavimus, vix unum aut alterum insuper adiungere poterimus. Unde hic insignia adhuc calculi incrementa desiderantur. Quo autem huius problematis vis penitius inspiciatur, nonnulla exempla subiungamus.

EXEMPLUM 1

183. *Si posito* $dz=pdx+qdy$ *debeat esse* $zz=pxx+qyy$, *indolem functionis z in genere investigare.*

Hic ergo est $Z=zz$, $X=xx$ et $Y=yy$, unde habemus

$$\int\frac{dx}{X}=-\frac{1}{x},\quad\int\frac{dy}{Y}=-\frac{1}{y},\quad\text{et}\quad\int\frac{dz}{Z}=-\frac{1}{z},$$

quibus valoribus substitutis pro solutione adipiscimur

$$-\frac{1}{z}=-\frac{1}{y}+f:\left(\frac{1}{y}-\frac{1}{x}\right) \quad \text{seu} \quad z=\frac{y}{1-yf:\left(\frac{1}{y}-\frac{1}{x}\right)}.$$

Sumatur ergo functio quaecunque quantitatis $\frac{1}{y}-\frac{1}{x}=\frac{x-y}{xy}$ quae si ponatur V, erit

$$z=\frac{y}{1-Vy}.$$

Veluti si ponamus $V=\frac{n}{y}-\frac{n}{x}$, erit $\frac{1}{z}=\frac{1}{y}-\frac{n}{y}+\frac{n}{x}=\frac{ny-(n-1)x}{xy}$
hincque

$$z=\frac{xy}{ny-(n-1)x},$$

unde fit

$$p=\left(\frac{dz}{dx}\right)=\frac{nxy}{\left(ny-(n-1)x\right)^2} \quad \text{et} \quad q=\left(\frac{dz}{dy}\right)=\frac{-(n-1)xx}{\left(ny-(n-1)x\right)^2}$$

sicque

$$pxx+qyy=\frac{xxyy}{\left(ny-(n-1)x\right)^2}=zz.$$

EXEMPLUM 2

184. *Si posito* $dz=pdx+qdy$ *debeat esse* $\frac{n}{z}=\frac{p}{x}+\frac{q}{y}$, *indolem functionis z investigare.*

Cum hic sit $X=\frac{1}{x}$, $Y=\frac{1}{y}$ et $Z=\frac{n}{z}$, erit

$$\int\frac{dx}{X}=\frac{1}{2}xx, \quad \int\frac{dy}{Y}=\frac{1}{2}yy, \quad \text{et} \quad \int\frac{dz}{Z}=\frac{1}{2n}zz,$$

unde solutio ita erit comparata

$$\frac{1}{2n}zz=\frac{1}{2}yy+f:(xx-yy) \quad \text{sive} \quad zz=nyy+f:(xx-yy);$$

non enim est necesse functionem per *2n* multiplicari, cum ea omnes operationes iam per se involvat.

Si pro hac functione sumatur $\alpha(xx-yy)$, habebitur solutio particularis

$$zz=\alpha xx+(n-\alpha)yy \quad \text{et} \quad z=\sqrt{\left(\alpha xx+(n-\alpha)yy\right)}$$

hincque

$$p=\left(\frac{dz}{dx}\right)=\frac{\alpha x}{\sqrt{\left(\alpha xx+(n-\alpha)yy\right)}} \quad \text{et} \quad q=\left(\frac{dz}{dy}\right)=\frac{(n-\alpha)y}{\sqrt{\left(\alpha xx+(n-\alpha)yy\right)}}$$

seu $\frac{p}{x}=\frac{\alpha}{z}$ et $\frac{q}{y}=\frac{n-\alpha}{z}$ ex ideoque

$$\frac{p}{x}+\frac{q}{y}=\frac{n}{z}$$

PROBLEMA 32 a

185. *Si posito* $dz=pdx+qdy$ *debeat esse* $q=pT+V$ *existente T functione quacunque ipsarum x et y ac V functione ipsarum y et z, investigare indolem functionis z.*

SOLUTIO

Substituto loco q valore praescripto huic aequationi inducatur forma

$$dz-Vdy=p\left(dx+Tdy\right).$$

Cum iam V tantum binas variabiles y et z involvat, dabitur multiplicator M prius membrum $dz-Vdy$ integrabile reddens; ponatur ergo

$$M\left(dz-Vdy\right)=dS.$$

Simili modo quia T tantum x et y continet, dabitur multiplicator L membrum quoque posterius $dx+Tdy$ integrabile efficiens; sit igitur

$$L\left(dx+Tdy\right)=dR,$$

ita ut nunc sint R et S functiones cognitae, illa ipsarum x et y, haec vero ipsarum y et z. Hinc nostra aequatio induct hanc formam

$$\frac{dS}{M}=\frac{pdR}{L} \quad \text{seu} \quad dS=\frac{pMdR}{L},$$

cuius integrabilitas necessario postulat, ut sit $\frac{pM}{L}$ functio ipsius R. Ponamus ergo

$$\frac{pM}{L}=f':R:$$

eritque

$$S=f:R,$$

qua aequatione relatio inter z et x, y definitur.

COROLLARIUM 1

186. In hoc problemate praecedens tanquam casus particularis continetur; cum enim ibi esset $Z=pX+qY$, erit $q=-\frac{X}{Y}p+\frac{Z}{Y}$ ideoque huius problematis applicatione facta fit $T=-\frac{X}{Y}$ et $V=\frac{Z}{Y}$.

COROLLARIUM 2

187. Quanquam autem hoc problema infinite latius patet quam praecedens, arctissimis tamen adhuc limitibus continetur neque eius ope vel hunc casum simplicissimum $z = py + qx$ resolvere licet.

SCHOLION

188. Omnino est haec forma $z = py + qx$ digna notatu, quod nulla ratione hactenus cognita resolvi posse videtur. Sive enim inde eliciatur $q = \frac{z-py}{x}$,
unde fit

$$dz - \frac{zdy}{x} = p\left(dx - \frac{ydy}{x}\right),$$

sive simili modo *p*, nulla via ad solutionem patet; cuius difficultatis causa in hoc manifesto est posita, quod formula $dz - \frac{zdy}{x}$ nullo multiplicatore integrabilis reddi potest seu quod haec aequatio $dz - \frac{zdy}{x} = 0$ plane est impossibilis, cum *x* perinde sit variabilis atque *y* et *z*. Supra scilicet [§ 6] iam notavi non omnes aequationes differentiales inter ternas variabiles esse possibiles simulque characterem possibilitatis exhibui, qui pro tali forma

$$dz + Pdx + Qdy = 0$$

huc reducitur, ut sit

$$P\left(\frac{dQ}{dz}\right) - Q\left(\frac{dP}{dz}\right) = \left(\frac{dQ}{dx}\right) - \left(\frac{dP}{dy}\right).$$

Nostro iam casu est $P = 0$ et $Q = \frac{-z}{x}$, unde hic character dat $0 = \frac{z}{xx}$; quod cum sit falsum, etiam aequatio ina $dz - \frac{zdy}{x} = 0$ est impossibilis, quod quidem per se est manifestum.

Verum tamen pro hoc casu $z = py + qx$ solutio particularis est obvia, scilicet $z = n(x + y)$, unde fit $p = q = n$. Deinceps autem [§ 195] methodum dabimus ex huiusmodi solutione particulari generalem eruendi.

EXEMPLUM 1

189. *Si posito* $dz = pdx + qdy$ *debeat esse* $py + qx = \frac{nxz}{y}$, *indolem functionis z investigare.*

Cum hinc sit $q = -\frac{py}{x} + \frac{nz}{y}$, erit

$$T = \frac{-y}{x} \quad \text{et} \quad V = \frac{nz}{y},$$

unde fit

$$dS = M\left(dz - \frac{nzdy}{x}\right) \quad \text{et} \quad dR = L\left(dx - \frac{ydy}{x}\right).$$

Sumatur ergo $M=\frac{1}{y^n}$ ut fiat $S=\frac{z}{y^n}$ et $L=2x$, ut fiat $R=xx-yy$. Quocirca hanc adipiscimur solutionem

$$\frac{z}{y^n}=f:(xx-yy) \text{ seu } z=y^n f:(xx-yy).$$

EXEMPLUM 2

190. *Si posito* $dz=pdx+qdy$ *debeat esse* $pxx+qyy=nyz$, *definire indolem functionis z.*

Cum ergo sit $q=-\frac{pxx}{yy}+\frac{nz}{y}$ erit

$$T=\frac{-xx}{yy} \text{ et } V=\frac{nz}{y}$$

sicque hic casus in nostro problemate continetur. Unde colligi oportet

$$dR=L\left(dx-\frac{xxdy}{yy}\right) \text{ et } dS=M\left(dz-\frac{nzdy}{y}\right).$$

Quare sumto $L=\frac{1}{xx}$ fit $R=\frac{1}{y}-\frac{1}{x}=\frac{x-y}{xy}$ et sumto $M=\frac{1}{y^n}$ fit $S=\frac{z}{y^n}$ ideoque solutio prodit ista

$$\frac{z}{y^n}=f:\frac{x-y}{xy} \quad \text{et} \quad z=y^n f:\frac{x-y}{xy}.$$

PROBLEMA 33

191. *Si posito* $dz=pdx+qdy$ *debeat esse* $p=qT+V$ *existente T functione ipsarum x et y, at V functione ipsarum x et z, indolem functionis z investigare.*

SOLUTIO

Simili modo ut ante si loco *p* valor praescriptus substituatur, obtinebitur

$$dz-Vdx=q(dy+Tdx).$$

Iam ob indolem functionum *V* et *T* sequentes integrationes instituere licebit

$$M(dz-Vdx)=dS \quad \text{et} \quad N(dy+Tdx)=dR,$$

unde fit

$$\frac{dS}{M}=\frac{qdR}{N} \text{ seu } dS=\frac{Mq}{N}dR$$

.
Atque hinc facillime colligitur haec solutio

$$\frac{Mq}{N}=f':R \quad \text{et} \quad S=f:R.$$

PROBLEMA 34

192. *Si posito* $dz = pdx + qdy$ *debeat esse* $z = Mp + Nq$ *existentibus M et N functionibus quibusvis binarum variabilium x et y, ex quadam solutione particulari,quam constat esse* $z = V$ *indolem functionis z in genere determinare.*

SOLUTIO

Valor iste particularis *V*: qui est functio ipsarum *x* et *y*, differentietur sitque

$$dV = Pdx + Qdy;$$

qui valor quia loco *z* substitutus satisfacit, ubi fit $p = P$ et $q = Q$, erit per hypothesin

$$V = MP + NQ.$$

Iam generatim ponatur $z = Vf:T$ sitque

$$dT = Rdx + Sdy$$

et nunc quaeri oportet hanc functionem *T*. Ex differentiatione autem eruimus

$$p = \left(\frac{dz}{dx}\right) = Pf:T + VRf':T \quad \text{et} \quad q = \left(\frac{dz}{dy}\right) = Qf:T + VSf':T .$$

Quare cum sit $z = Mp + Nq = Vf:T$, erit

$$Vf:T = (MP + NQ)\, f:T + V(MR + NS)\, f':T$$

et ob $V = MP + NQ$ per hypothesin habebitur $MR + NS = 0$, hinc

$$dT = R\left(dx - \frac{Mdy}{N}\right).$$

Iam nosse non oportet *R*, sed sufficit considerari formulam $Ndx - Mdy$, quae ope multiplicatoris cuiusdam integrahilis reddi potest. Solutio ergo facillime huc redit, ut ex conditione praescripta $z = Mp + Nq$ formetur aequatio realis

$$dT = R(Ndx - Mdy);$$

invento enim multiplicatore idoneo *R* per integrationem reperitur quantitas *T*, qua inventa erit $z = Vf:T$.

ALITER

Facilius valor generalis hoc modo invenitur. Ob valorem ipsius *z* cognitum *V* statuatur $z = Vv$ sitque $dv = rdx + sdy$; erit

$$p = Pv + Vr \quad \text{et} \quad q = Qv + Vs$$

ideoque

$$z = Mp + Nq = (MP + NQ)v + V(Mr + Ns) = Vv.$$

At est $V = MP + NQ$, ergo

$$Mr + Ns = 0 \quad \text{seu} \quad s = -\tfrac{Mr}{N},$$

unde fit

$$dv = r\left(dx - \tfrac{Mdy}{N}\right) = \tfrac{r}{N}(Ndx - Mdy).$$

Statuatur ergo idoneum multiplicatorem investigando

$$R(Ndx - Mdy) = dT;$$

erit $dv = \tfrac{r}{NR}dT$, ex quo colligitur

$$\tfrac{r}{NR} = f':T \quad \text{et} \quad v = f:T,$$

ita ut in genere sit ut ante $z = Vv$.

COROLLARIUM 1

193. Proposita ergo conditione $z = Mp + Nq$ ut sit $dz = pdx + qdy$, statim consideretur aequatio differentialis $R(Ndx - Mdy) = dT$, unde tam multiplicator R quam inde integrale T reperitur; haecque operatio non pendet a valore particulari cognito V.

COROLLARIUM 2

194. Inventa autem quantitate T si undecunque innotuerit solutio particulariter satisfaciens $z = V$, erit solutio generalis $z = Vf:T$. Probe autem notetur ex solutione particulari generalem elici non posse, nisi conditio praescripta sit huiusmodi $z = Mp + Nq$.

EXEMPLUM 1

195. *Si posito $dz = pdx + qdy$ debeat esse $z = py + qx$, ex valore particulari $z = x + y$ generalem definire.*

Cum hic sit $M = y$ et $N = x$, habebimus hanc aequationem

$$R(xdx - ydy) = dT$$

hincque

$$T = f:(xx - yy);$$

ergo solutio generalis erit

$$z = (x + y)\, f:(xx - yy).$$

EXEMPLUM 2

196. *Si posito* $dz = pdx + qdy$ *debeat esse* $z = p(x+y) + q(y-x)$, *ex valore particulari* $z = \sqrt{(xx+yy)}$ *generalem invenire.*

Ob $M = x + y$ et $N = y - x$ formula $Ndx - Mdy$ deducit ad hanc aequationem

$$R(ydx - xdx - xdy - ydy) = dT .$$

Sumatur $R = \frac{1}{xx+yy}$, ut sit

$$dT = \frac{ydx-xdy}{xx+yy} - \frac{xdx+ydy}{xx+yy};$$

erit

$$T = \text{Ang.tang.}\frac{x}{y} - \frac{1}{2}l(xx+yy).$$

Atque ex valore hoc dupliciter transcendente erit

$$z = (xx+yy)\, f{:}T$$

simulque patet nullum alium dari valorem partieularem, qui sit algebraicus, praeter datum $z = \sqrt{(xx+yy)}$.

EXEMPLUM 3

197. *Si posito* $dz = pdx + qdy$ *debeat esse* $z = p(\alpha x + \beta y) + q(\gamma x + \delta y)$, *ex invento .valore particulari* $z = V$ *indolem functionis z in genere definire.*

Hic est $M = \alpha x + \beta y$ et $N = \gamma x + \delta y$, unde deducimur ad hanc aequationem

$$R((\gamma x + \delta y)dx - (\alpha x + \beta y)dy) = dT ,$$

ubi ob formam homogeneam debet esse

$$R = \frac{1}{\gamma xx + (\delta - \alpha)xy - \beta yy},$$

ut sit

$$dT = \frac{(\gamma x+\delta y)dx - (\alpha x+\beta y)dy}{\gamma xx + (\delta-\alpha)xy - \beta yy},$$

ad quod integrale inveniendum ponatur $y = ux$ ac prodibit

$$dT = \frac{dx}{x} - \frac{(\alpha+\beta u)du}{\gamma+(\delta-\alpha)u-\beta uu}$$

Sit

$$\int \frac{(\alpha+\beta u)du}{\gamma+(\delta-\alpha)u-\beta uu} = lU$$

erit $T = lx - lU$, et cum functio ipsius T sit etiam functio ipsius $\frac{x}{U}$, erit in genere $z = Vf:\frac{x}{U}$. Patet autem, cum U sit functio ipsius $u = \frac{y}{x}$, fore U functionem homogeneam nullius dimensionis ipsarum x et y ideoque $\frac{x}{U}$ functionem unius dimensionis.

SCHOLION

198. Hoc ergo exemplo difficultas restat, quomodo solutio particularis $z = V$ obtineri queat; nisi enim una saltem huiusmodi solutio particularis constet, solutio generalis ne absolvi quidem potest. Pro hoc autem casu solutionem particularem sequenti modo elicere licet; qui cum aliquid singulare habeat, nullum est dubium, quin eius ope hoc calculi genus haud parum adiumenti sit consecuturum.

PROBLEMA 35

199. *Si posito $dz = pdx + qdy$ debeat esse $z = p(\alpha x + \beta y) + q(\gamma x + \delta y)$, valorem particularem investigare, qui loco z substitutus huic conditioni satisfaciat.*

SOLUTIO

Negotium hoc succedet, si pro z eiusmodi valorem quaeramus, qui sit functio nullius dimensionis ipsarum x et y, seu posito $y = ux$, qui sit functio ipsius u tantum. Ponamus ergo $z = f:u = f:\frac{y}{x}$ eritque $f':u = \frac{dz}{du}$; at ob $du = \frac{dy}{x} - \frac{ydx}{xx}$ erit

$$dz = \left(\frac{dy}{x} - \frac{udx}{x}\right) f':u$$

hinc

$$p = -\frac{u}{x} f':u = -\frac{udz}{xdu} \quad \text{et} \quad q = -\frac{1}{x} f':u = \frac{dz}{xdu}$$

Quibus valoribus pro p et q substitutis conditio praescripta praebet

$$z = x(\alpha + \beta u)p + x(\gamma + \delta u)q = \frac{-udz(\alpha+\beta u)+dz(\gamma+\delta u)}{du},$$

unde fit

$$\frac{dz}{z} = \frac{du}{\gamma+(\delta-\alpha)u-\beta uu}.$$

Ponamus

$$\int \frac{du}{\gamma+(\delta-\alpha)u-\beta uu} = lV ,$$

ut fiat $z = V$, eritque V valor particularis pro z satisfaciens.

COROLLARIUM 1

200. Invento hoc valore V praecedentis exempli ope solutio generalis facile invenitur. Erit scilicet $z = Vf : \frac{x}{U}$ existente

$$\frac{dU}{U} = \frac{(\alpha+\beta u)du}{\gamma+(\delta-\alpha)u-\beta uu} ,$$

unde patet quantitatem U ex ipso valore particulari V inveniri posse.

COROLLARIUM 2

201. Erit enim

$$lU = -l\sqrt{(\gamma+(\delta-\alpha)u-\beta uu)} + \int \frac{\frac{1}{2}(\delta+\alpha)du}{\gamma+(\delta-\alpha)u-\beta uu}$$

ideoque

$$lU = -l\sqrt{(\gamma+(\delta-\alpha)u-\beta uu)} + \frac{1}{2}(\delta+\alpha)lV$$

sive

$$U = \frac{V^{\frac{1}{2}(\alpha+\delta)}}{\sqrt{(\gamma+(\delta-\alpha)u-\beta uu)}}$$

hinc

$$\frac{x}{U} = \frac{\sqrt{(\gamma xx+(\delta-\alpha)xy-\beta yy)}}{V^{\frac{1}{2}(\alpha+\delta)}} .$$

COROLLARIUM 3

202. Quocirca invento valore particulari $z = V$, ut sit

$$\frac{dV}{V} = \frac{du}{\gamma+(\delta-\alpha)u-\beta uu}$$

existente $u = \frac{y}{x}$, erit valor generaliter satisfaciens

$$z = Vf : \frac{\gamma xx+(\delta-\alpha)xy-\beta yy}{V^{\alpha+\delta}} = Vf : \frac{x(\gamma x+\delta y)-y(\alpha x+\beta y)}{V^{\alpha+\delta}} .$$

COROLLARIUM 4

203. Hinc colligitur alius valor particularis, qui semper est algebraicus; erit is scilicet

$$z=\left(x(\gamma x+\delta y)-y(\alpha x+\beta y)\right)^{\frac{1}{\alpha+\delta}}$$

vel eius multiplum quodcunque. Nisi autem V sit quantitas algebraica, omnes reliqui valores erunt transcendentes et in hac forma contenti

$$z=\left(x(\gamma x+\delta y)-y(\alpha x+\beta y)\right)^{\frac{1}{\alpha+\delta}} f:\frac{x(\gamma x+\delta y)-y(\alpha x+\beta y)}{V^{\alpha+\delta}}.$$

SCHOLION

204. Unicus casus, quo $\delta=-\alpha$ et conditio proposita

$$z=p(ax+\beta y)+q(\gamma x-\alpha y),$$

peculiarem evolutionem postulat. Primo autem posito $u=\frac{y}{x}$ pro valore particulari $z=V$ erit

$$lV=\int\frac{du}{\gamma-2\alpha u-\beta uu}$$

Tum vero ob $\frac{dU}{U}=\frac{(\alpha+\beta u)du}{\gamma-2\alpha u-\beta uu}$ erit

$$U=\frac{1}{\sqrt{(\gamma-2\alpha u-\beta uu)}} \quad \text{et} \quad \frac{x}{U}=\sqrt{(\gamma xx-2\alpha xy-\beta yy)}$$

ita ut iam valor generalis sit

$$z=Vf:(\gamma xx-2\alpha xy-\beta yy)$$

.

Per se enim manifestum est formam $f:\sqrt{T}$ exprimi posse per $f:T$. Nisi ergo V sit functio algebraica, hoc casu nulla solutio particularis algebraica locum habet.

EXEMPLUM 1

205. *Si posito $dz=pdx+qdy$ debeat esse $nz=py-qx$, indolem functionis z investigare.*

Comparatione cum forma nostra generali instituta fit

$$\alpha=0,\quad \beta=\frac{1}{n},\quad \gamma=-\frac{1}{n},\quad \delta=0.$$

Hic ergo casus ob $\delta=-\alpha$ pertinet ad paragraphum praecedentem, unde fit

$$lV = \int \frac{ndu}{-1-uu} = -n\text{Ang.tang.}u.$$

Cum igitur sit $u = \frac{y}{x}$, forma generalis est

$$z = e^{-n\text{Ang. tang.}\frac{y}{x}} f:(xx+yy).$$

EXEMPLUM 2

206. *Si posito* $dz = pdx + qdy$ *debeat esse* $z = p(x+y) - q(x+y)$, *indolem functionis z investigare.*

Comparatione facta fit

$$\alpha = 1, \quad \beta = 1, \quad \gamma = -1, \quad \delta = -1.$$

hincque

$$lV = \int \frac{du}{-1-2u-uu} = \frac{1}{1+u} \quad \text{et} \quad V = e^{\frac{1}{1+u}}$$

et solutio generalis est

$$z = e^{\frac{x}{x+y}} f:(x+y).$$

EXEMPLUM 3

207. *Si posito* $dz = pdx + qdy$ *debeat esse* $z = p(x-2y) + q(2x-3y)$, *indolem functionis z investigare.*

Cum ergo hic sit

$$\alpha = 1, \quad \beta = -2, \quad \gamma = 2, \quad \delta = -3,$$

erit primo

$$lV = \int \frac{ndu}{2-4u+2uu} = \frac{1}{2(1-u)} = \frac{x}{2(x-y)},$$

et quia non est $\delta = -\alpha$, solutio generalis statim prodit

$$z = (2xx - 4xy + 2yy)^{-\frac{1}{2}} f:\frac{2xx-4xy+2yy}{V^{-2}}$$

et ob $V = e^{\frac{x}{2(x-y)}}$ erit

$$z = \frac{1}{x-y} f:(x-y)^2 e^{\frac{x}{x-y}}$$

Unde solutio simplicissima est $z = \frac{1}{x-y}$.

SCHOLION

208. Hic merito quaerimus, quo pacto haec solutio generalis statim sine adiumento solutionis specialis inveniri potuisset; sequenti autem modo ista investigatio instituenda videtur.

Cum sit

$$p(\alpha x+\beta y)=z-q(\gamma x+\delta y) \text{ et } q(\gamma x+\delta y)=z-p(\alpha x+\beta y),$$

si uterque valor seorsim in forma

$$dz=pdx+qdy$$

substituatur, prodibunt binae sequentes aequationes

$$(\alpha x+\beta y)dz=zdx-q(\gamma x+\delta y)dx+q(\alpha x+\beta y)dy,$$
$$(\gamma x+\delta y)dz=zdy+p(\gamma x+\delta y)dx-p(\alpha x+\beta y)dy.$$

MultipIicetur prior indefinite per *M*, posterior per *N*, et productorum summa dabit

$$dz\big(M(\alpha x+\beta y)+N(\gamma x+\delta y)\big)-z(Mdx+Ndy)$$
$$=(Np-Mq)\big((\gamma x+\delta y)dx-(ax+\beta y)dy\big),$$

ubi iam *M* et *N* ita capi debent, ut prius membrum integrationem admittat; tum enim eius integrale aequabitur functioni cuicunque quantitatis

$$\int\frac{(\gamma x+\delta y)dx-(ax+\beta y)dy}{\gamma xx+(\delta-\alpha)xy-\beta yy},$$

quam supra (§ 197) definire docuimus; unde patet integrale fieri $=f:\frac{x}{U}$. Manifestum autem est *M* et *N* eiusmodi functiones esse oportere, ut haec aequatio fiat possibilis

$$\frac{dz}{z}=\frac{Mdx+Ndy}{M(\alpha x+\beta y)+N(\gamma x+\delta y)}$$

seu ut membrum posterius integrationem admittat; quodsi enim eius integrale sit $=lV$, erit $\frac{z}{V}=f:\frac{x}{U}$. Pro hac integrabilitate ponamus $y=ux$ et *M* et *N* functiones ipsius *u;* erit

$$\frac{dz}{z}=\frac{(M+Nu)dx+Nxdx}{Mx(\alpha+\beta u)+Nx(\gamma+\delta u)},$$

ubi integratio succedit sumendo $M=-Nu$, ut sit

$$\frac{dz}{z} = \frac{du}{\gamma+(\delta-\alpha)u-\beta uu} \quad \text{seu} \quad lV = \int \frac{du}{\gamma+(\delta-\alpha)u-\beta uu}$$

prorsus ut ante.

PROBLEMA 36

209. *Si posito* $dz = p\,dx + qdy$ *debeat esse* $Z = pP + qQ$ *existente Z functione ipsius z tantum, P et Q autem functionibus ipsarum x et y quibusvis datis, indolem functionis z investigare.*

SOLUTIO

Formentur sequentes aequationes ex propositis

$$Ldz = Lpdx + Lqdy\,,$$

$$MZdx = MpPdx + MqQdx,\;\; NZdy = NpPdy + NqQdy\,,$$

quae in unam summam collectae dabunt

$$Ldz + Z\left(Mdx + Ndy\right) = p\left(\left(L + MP\right)dx + NPdy\right) + q\left(\left(L + NQ\right)dy + MQdx\right).$$

Ut iam pars posterior habeat factorem a litteris *p* et *q* liberum, fiat

$$L + MP : NP = MQ : L + NQ,$$

unde fit

$$LL + LNQ + LMP = 0 \quad \text{seu} \quad L = -MP - NQ\,,$$

quo valore inducto erit

$$-\,dz\left(MP + NQ\right) + Z\left(Mdx + Ndy\right) = \left(Mq - Np\right)\left(Qdx - Pdy\right).$$

Cum nunc *P* et *Q* sint functiones datae ipsarum *x* et *y*, dabitur multiplicator *R*, ut fiat

$$R\left(Qdx - Pdy\right) = dU$$

ideoque

$$-\,dz\left(MP + NQ\right) + Z\left(Mdx + Ndy\right) = \frac{Mq-Np}{R}\,dU\,.$$

Pro parte priori capiantur functiones indefinitae *M* et N ita, ut formula $\frac{Mdx+Ndy}{MP+NQ}$ integrabilis evadat, id quod semper fieri licet, sitque

$$\frac{Mdx+Ndy}{MP+NQ}=dV$$

et ob

$$Mdx+Ndy=(MP+NQ)dV$$

aequatio nostra hanc induet formam

$$(MP+NQ)(-dz+ZdV)=\frac{Mq-Np}{R}dU$$

seu

$$\frac{dz}{Z}-dV=\frac{Np-Mq}{RZ(MP+NQ)}dU\ .$$

Statuatur iam

$$\frac{Np-Mq}{RZ(MP+NQ)}=f':U$$

atque habebitur

$$\int\frac{dz}{Z}-V=f:U \quad \text{seu} \quad \int\frac{dz}{Z}=V+f:U\ ,$$

unde z determinatur per x et y.

COROLLARIUM 1

210. Pro solutione ergo problematis quaeratur primo ad formulam $Qdx-Pdy$ multiplicator R eam reddens integrabilem statuaturque

$$R(Qdx-Pdy)=dU\ ,$$

unde colligitur quantitas U per binas variabiles x et y expressa.

COROLLARIUM 2

211. Deinde quantitates M et N ita capiantur, ut formula $\frac{Mdx+Ndy}{MP+NQ}$ fiat integrabilis; cuius integrale si statuatur $=V$, statim habetur solutio generalis problematis, quae dat

$$\int\frac{dz}{Z}=V+f:U$$

EXEMPLUM

212. *Si P et Q sint functiones homogeneae ipsarum x et y, utraque dimensionum numeri = n, solutionem problematis perficere.*

Ponatur $y=ux$ et tam P quam Q fiet productum ex potestate x^n in functionem quandam ipsius u. Sit ergo

$$P = x^n S \quad \text{et} \quad Q = x^n T$$

eruntque S et T functiones datae ipsius u. Tum vero ob $dy = udx + xdu$ formula $Qdx - Pdy$ abit in

$$x^n Tdx - x^n Sudx - x^{n+1} Sdu = x^n \left((T - Su) dx - Sxdu \right).$$

Sumatur ergo

$$R = \frac{1}{x^{n+1}(T - Su)}$$

fietque

$$dU = \frac{dx}{x} - \frac{Sdu}{T - Su},$$

unde colligjtur U.

Deinde pro altera quantitate V habebimus hanc aequationem

$$dV = \frac{(M + Nu)dx + Nxdu}{x^n(MS + NT)},$$

ubi iam facile est pro M et N eiusmodi functiones ipsius u assumere, ut haec formula integrationem admittat. Integrale scilicet erit

$$V = \frac{-M - Nu}{(n-1)x^{n-1}(MS + NT)}$$

at M et N seu $\frac{M}{N} = K$ ita accipi debet, ut fiat

$$\frac{-1}{(n-1)x^{n-1}} d.\frac{K + u}{KS + T} = \frac{1}{x^{n-1}} \cdot \frac{du}{KS + T}$$

seu

$$-KKdS + KSdu - uKdS - uSdK + TdX - KdT + Tdu - udT + (n-1)du(KS + T) = 0,$$

quae ad hanc formam reducitur

$$(T - Su)dK + K(nSdu - udS - dT) - KKdS + nTdu - udT = 0.$$

Ex qua concessa aequationum resolutione cognoscitur quantitas K, qua inventa erit

$$V = \frac{-K - u}{(n-1)x^{n-1}(KS + T)}$$

Cum autem illa aequatio solutu difficilis videatur, ponatur statim

$$\frac{K+u}{KS+T}=v$$

eritque

$$K=\frac{Tv-u}{1-Sv} \text{ et } KS+T=\frac{T-Su}{1-Sv},$$

unde fit

$$dv+\frac{(n-1)du(1-Sv)}{T-Su}=0$$

qua resoluta erit

$$V=\frac{-v}{(n-1)x^{n-1}}.$$

COROLLARIUM

213. Casus autem, quo $n=1$, singulari evolutione eget. Facile autem patet tum sumi debere $M=-Nu$, ut fiat $dV=\frac{du}{T-Su}$; unde, postquam quantitas V fuerit inventa, erit semper

$$\int\frac{dz}{Z}=V+f:U.$$

SCHOLION

214. Cum ternae variabiles *x, y, z* sint inter se permutabiles, patet hoc problema multo latius extendi posse. Scilicet si conditio proposita hac contineatur aequatione $pP+qQ+R=0$, non solum solvendi methodus adhibita succedit, si *R* sit functio ipsius *z* et *P* cum *Q* functiones ipsarum *x* et *y*, sed etiam, si fuerit *P* functio ipsius *x* et *Q* et *R* functiones ipsarum *y* et *z*,tum vero etiam, si *Q* functio ipsius *y*, at *P* et *R* functiones binarum reliquarum *x* et *z*. Haec vero conditio cum ante tractatis eo redit, ut binae formulae differentiales *p* et *q* sint a se invicem separatae neque plus una dimensione occupent, etiamsi et his casibus ingens restrictio accedat. Quodsi autem conditio magis sit complicata, solutio vix unquam sperari posse videtur; interim tamen casum eiusmodi proferam, quo solutionem expedire licet.

PROBLEMA 37

215. *Si posito* $dz=pdx+qdy$ *debeat esse* $q=Ap^{n}x^{\lambda}y^{\mu}z^{\nu}$, *indolem functionis z in genere investigare.*

SOLUTIO

Posito hoc valore loco *q* habebimus

$$dz=pdx+Ap^{n}x^{\lambda}y^{\mu}z^{\nu}dy,$$

unde fit

$$Ay^{\mu}dy=p^{-n}x^{-\lambda}z^{-\nu}(dz-pdx).$$

Ponatur $p^{-n}x^{-\lambda}z^{-\nu}=t$, ut sit $p=t^{-\frac{1}{n}}x^{-\frac{\lambda}{n}}z^{-\frac{\nu}{n}}$, eritque

$$Ay^{\mu}dy=tdz-t^{\frac{n-1}{n}}x^{-\frac{\lambda}{n}}z^{-\frac{\nu}{n}}dx.$$

Statuatur porro $t^{n-1}z^{-\nu}=u^{n}$ seu $t=z^{\frac{\nu}{n-1}}\,u^{\frac{n}{n-1}}$; erit

$$Ay^{\mu}dy=u^{\frac{n}{n-1}}z^{\frac{\nu}{n-1}}dz-ux^{-\frac{\lambda}{n}}dx.$$

Iam partibus, quoad fieri licet, integratis adipiscimur

$$\tfrac{A}{\mu+1}y^{\mu+1}=\tfrac{n-1}{n+\nu-1}u^{\frac{n}{n-1}}z^{\frac{n+\nu-1}{n-1}}-\tfrac{nu}{n-\lambda}x^{\frac{n-\lambda}{n}}-\int du\left(\tfrac{n}{n+\nu-1}u^{\frac{1}{n-1}}z^{\frac{n+\nu-1}{n-1}}-\tfrac{n}{n-\lambda}x^{\frac{n-\lambda}{n}}\right).$$

ac nunc solutionem per praecepta supra data expedire licet; scilicet statuatur

$$\tfrac{1}{n+\nu-1}u^{\frac{1}{n-1}}z^{\frac{n+\nu-1}{n-1}}-\tfrac{1}{n-\lambda}x^{\frac{n-\lambda}{n}}=f':u$$

eritque

$$\tfrac{A}{\mu+1}y^{\mu+1}=\tfrac{n-1}{n+\nu-1}u^{\frac{n}{n-1}}z^{\frac{n+\nu-1}{n-1}}-\tfrac{nu}{n-\lambda}x^{\frac{n-\lambda}{n}}-nf:u\,,$$

atque ex his binis aequationibus si elidatur *u*, dabitur utique *z* per *x* et *y*.

COROLLARIUM 1

216. Casus $n=1$ peculiarem postulat tractationem; cum enim posito $p=\frac{1}{t}x^{-\lambda}z^{-\nu}$ sit $Ay^{\mu}dy=tdz-x^{-\lambda}z^{-\nu}dx$, erit

$$\tfrac{A}{\mu+1}y^{\mu+1}=\tfrac{1}{\lambda-1}x^{1-\lambda}z^{-\nu}+\int dz\left(t+\tfrac{\nu}{\lambda-1}x^{1-\lambda}z^{-\nu-1}\right)$$

atque hinc statim concluditur

$$\tfrac{A}{\mu+1}y^{\mu+1}=\tfrac{1}{\lambda-1}x^{1-\lambda}z^{-\nu}+f:z.$$

COROLLARIUM 2

217. Casus autem $n+\nu-1=0$ et $n-\lambda=0$ nullam facessunt molestiam, cum sit priori casu

$$\tfrac{n-1}{n+\nu-1}z^{\frac{n+\nu-1}{n-1}}=lz$$

posteriori autem

$$\frac{n}{n-\lambda}x^{\frac{n-\lambda}{n}}=lx,$$

quos valores in solutionem introduci oportet.

EXEMPLUM

218. *Si posito* $dz=pdx+qdy$ *debeat esse* $pqxy=az$ *seu* $q=\frac{az}{pxy}$, *functionem z investigare.*

Erit ergo

$$dz=pdx+\frac{azdy}{pxy}\quad\text{seu}\quad\frac{ady}{y}=\frac{px}{z}\left(dz-pdx\right)$$

Ponatur $\frac{px}{z}=t$ seu $p=\frac{tz}{x}$; erit

$$\frac{ady}{y}=tdz-\frac{ttzdx}{x}.$$

Statuatur porro $ttz=uu$ seu $t=\frac{u}{\sqrt{z}}$, ut sit

$$\frac{ady}{y}=\frac{udz}{\sqrt{z}}-\frac{uudx}{x}$$

et, quoad fieri potest, integrando

$$aly=2u\sqrt{z}-uulx-\int du\left(2\sqrt{z}-2ulx\right),$$

ita ut iam post signum integrale unicum differentiale *du* reperiatur. Posito ergo

$$\sqrt{z}-ulx=f':u$$

erit

$$aly=2u\sqrt{z}-uulx-2f:u=uulx+2uf':u-2f:u\,.$$

Pro casu simplicissimo sumatur $f':u=0$ et $f:u=0$; erit $u=\frac{\sqrt{z}}{lx}$ ideoque

$$aly=\frac{2z}{lx}-\frac{z}{lx}=\frac{z}{lx}$$

ita ut pro casu simplicissimo sit $z=alx.ly$.

Si ponatur $f:u=ulc$ et $f:u=\frac{1}{2}uulc$, erit

$$u=\frac{\sqrt{z}}{lx+lc}=\frac{\sqrt{z}}{lcx}\quad\text{et}\quad aly=\frac{2z}{lcx}-\frac{zlx}{(lcx)^2}-\frac{zlc}{(lcx)^2}=\frac{z}{lcx}$$

ita ut sit $z=aly\left(lc+lx\right)$; magis generaliter autem erit

$$z=a\left(lb+ly\right)\left(lc+lx\right).$$

SCHOLION

219. Methodi hactenus traditae haud mediocriter amplificabuntur, si loco binarum variabilium x et y, quarum functio esse debet z, binae aliae variabiles t et u introducantur, quarum relatio ad illas detur. Ita si z sit functio binarum variabilium x et y, ut inde prodeat $dz = pdx + qdy$, ac loco x et y aliae novae variabiles t et u introducantur, ut iam differentiatione instituta prodeat $dz = rdt + sdu$, quaeritur, quomodo r et s per p et q determinentur pro relatione inter pristinas variabiles x, y et novas t et u stabilita. Hinc ergo tam x quam y certae cuidam functioni ipsarum t et u aequabitur; quae cum detur, sit

$$dx = Pdt + Qdu \quad \text{et} \quad dy = Rdt + Sdu,$$

ita ut facta hac substitutione z iam sit functio ipsarum t et u. Cum igitur esset $dz = pdx + qdy$, erit nunc

$$dz = (Pp + Rq)dt + (Qp + Sq)du.$$

Est vero per hypothesin $dz = rdt + sdu$, unde habebitur

$$r = Pp + Rq \quad \text{et} \quad s = Qp + Sq.$$

Quare facta hac substitutione valores differentiales novi ex praecedentibus ita determinabuntur, ut sit

$$\left(\frac{dz}{dt}\right) = P\left(\frac{dz}{dx}\right) + R\left(\frac{dz}{dy}\right) \quad \text{et} \quad \left(\frac{dz}{du}\right) = Q\left(\frac{dz}{dx}\right) + S\left(\frac{dz}{dy}\right).$$

Unde etiam, cum sit vicissim

$$Qr - Ps = (QR - PS)q \quad \text{et} \quad Sr - Rs = (PS - QR)p,$$

concludimus fore

$$\left(\frac{dz}{dx}\right) = \frac{S}{PS-QR}\left(\frac{dz}{dt}\right) - \frac{R}{PS-QR}\left(\frac{dz}{du}\right)$$

et

$$\left(\frac{dz}{dy}\right) = \frac{-Q}{PS-QR}\left(\frac{dz}{dt}\right) + \frac{P}{PS-QR}\left(\frac{dz}{du}\right)$$

Vel cum x et y perinde ac z sint functiones ipsarum t et u, haec relatio ita exprimi potest, ut sit

$$\left(\frac{dz}{dt}\right) = \left(\frac{dx}{dt}\right)\left(\frac{dz}{dx}\right) + \left(\frac{dy}{dt}\right)\left(\frac{dz}{dy}\right) \quad \text{et} \quad \left(\frac{dz}{du}\right) = \left(\frac{dx}{du}\right)\left(\frac{dz}{dx}\right) + \left(\frac{dy}{du}\right)\left(\frac{dz}{dy}\right).$$

Hinc efficitur, ut, quae problemata pro data quadam relatione inter p, q, x, y, z resolvi possunt, ea quoque pro relatione inde resultante inter r, s, t, u et z resolvi queant; unde saepe problemata nascuntur, quae solutu vehementer difficilia videantur, ex quo non contemnenda subsidia in hanc Analyseos partem inferri possent; sed quia usus praeeipue in formulis differentialibus secundi gradus spectatur, his non fusius immorans ad eas evolvendas progredior.