FINAL BOOK OF THE INTEGRAL CALCULUS

FIRST PART

THE INVESTIGATION OF FUNCTIONS OF TWO VARIABLES FROM A GIVEN RELATION OF THE VARIABLES OF ANY ORDER.

SECTION TWO

THE INVESTIGATION OF FUNCTIONS OF TWO VARIABLES FROM A RELATION OF THE DIFFERENTIALS OF THE SECOND ORDER.

CHAPTER I

CONCERNING DIFFERENTIAL FORMULAS OF THE SECOND ORDER IN GENERAL

PROBLEM 38

220. *If z shall be some function of the two variables x and y, to show the differential formulas of the second order of this.*

SOLUTION

Since z shall be a function of the two variables x and y, the differential of this will have a form of this kind $dz = pdx + qdy$, from which p and q are differential formulas of the first order, which we are accustomed to denote thus

$$p = \left(\frac{dz}{dx}\right) \quad \text{and} \quad q = \left(\frac{dz}{dy}\right).$$

Now since also p and q shall be some functions of x and y, the differential formulas thence arising shall be differential formulas of the second order of z, from which it is understood four formulas of this kind arise

$$\left(\frac{dp}{dx}\right), \quad \left(\frac{dp}{dy}\right), \quad \left(\frac{dq}{dx}\right), \quad \left(\frac{dq}{dy}\right),$$

but of which the second and the third, as it has been shown in the differential calculus, are equal to each other. [*Institutionum calculi differentialis* §228 *et seq*.] But since there shall be $p = \left(\frac{dz}{dx}\right)$, on account of writing in a like manner, there will be $\left(\frac{dp}{dx}\right) = \left(\frac{ddz}{dx^2}\right)$, the significance of writing [in this way] is at once apparent. Then in the same manner there will be $\left(\frac{dp}{dy}\right) = \left(\frac{ddz}{dxdy}\right)$ and on account of $q = \left(\frac{dz}{dy}\right)$ we will have $\left(\frac{dq}{dx}\right) = \left(\frac{ddz}{dydx}\right)$ and $\left(\frac{dq}{dy}\right) = \left(\frac{ddz}{dy^2}\right)$. Therefore because there is

$$\left(\frac{ddz}{dydx}\right) = \left(\frac{ddz}{dxdy}\right),$$

the functions z agree on three differential formulas of the second order, which are

$$\left(\frac{ddz}{dx^2}\right), \quad \left(\frac{ddz}{dxdy}\right), \quad \left(\frac{ddz}{dy^2}\right).$$

COROLLARY 1

221. Therefore in order that the function *z* of the two variables *x* and *y* will have two differential formulas of the first order,

$$\left(\frac{dz}{dx}\right) \quad \text{and} \quad \left(\frac{dz}{dy}\right),$$

thus it has three differential formulas of the second order

$$\left(\frac{ddz}{dx^2}\right), \quad \left(\frac{ddz}{dxdy}\right), \quad \text{and} \quad \left(\frac{ddz}{dy^2}\right).$$

COROLLARIUM 2

222. Therefore these formulas arise by a twofold differentiation yet only on taking a single quantity as variable. Clearly in the first the same variable *x* is taken twice, in the second truly in the first differentiation *x*, but in the second the variable *y* is taken, but in the third *y* twice.

COROLLARY 3

223. In a similar manner it is apparent that four differential formulas of the same function *z* of the third order are to be given , evidently

$$\left(\frac{d^3z}{dx^3}\right), \quad \left(\frac{d^3z}{dx^2dy}\right), \quad \left(\frac{d^3z}{dxdy^2}\right), \quad \left(\frac{ddz}{dy^2}\right),$$

moreover five of the fourth order, six of the fifth, etc.

SCHOLION

224. These differential formulas of the second order at any rate are able to be returned to the first order. Just as the formula $\left(\frac{ddz}{dx^2}\right)$, if there is put $\left(\frac{dz}{dx}\right) = p$, will be transformed into $\left(\frac{dp}{dx}\right)$, moreover the formula $\left(\frac{ddz}{dxdy}\right)$ by the same substitution is transformed into this $\left(\frac{dq}{dy}\right)$. But on putting $\left(\frac{dz}{dy}\right) = q$ the formula $\left(\frac{ddz}{dxdy}\right)$ is transformed into this $\left(\frac{dq}{dx}\right)$, but the formula $\left(\frac{ddz}{dy^2}\right)$ into this $\left(\frac{dq}{dy}\right)$. Moreover as from the equality $p = \left(\frac{dz}{dx}\right)$ we were led to

$$\left(\frac{dp}{dx}\right) = \left(\frac{ddz}{dx^2}\right) \quad \text{and} \quad \left(\frac{dp}{dy}\right) = \left(\frac{ddz}{dxdy}\right),$$

thus from these on progressing further we may deduce

$$\left(\frac{ddp}{dx^2}\right) = \left(\frac{d^3z}{dx^3}\right), \quad \left(\frac{ddp}{dxdy}\right) = \left(\frac{d^3z}{dx^2dy}\right), \quad \left(\frac{ddp}{dy^2}\right) = \left(\frac{d^3z}{dxdy^2}\right).$$

Then truly also if we put $\left(\frac{dp}{dx}\right)=\left(\frac{dz}{dy}\right)$, hence these equalities follow $\left(\frac{ddp}{dx^2}\right)=\left(\frac{ddz}{dxdy}\right)$ and $\left(\frac{ddp}{dxdy}\right)=\left(\frac{ddz}{dy^2}\right)$. And this is as if a new algorithm, the principles thus by themselves are evident, so that there is no need for further illustration.

EXAMPLE 1

225. *If there shall be $z=xy$, to show the differential formulas of the second order of this.*

Since there shall be

$$\left(\frac{dz}{dx}\right)=y \quad \text{and} \quad \left(\frac{dz}{dy}\right)=x,$$

there will be

$$\left(\frac{ddz}{dx^2}\right)=0, \quad \left(\frac{ddz}{dxdy}\right)=1 \quad \text{and} \quad \left(\frac{ddz}{dy^2}\right)=0.$$

EXAMPLE 2

226. *If there shall be $z=x^m y^n$, to show the second order differential formulas of this.*

Since there shall be

$$\left(\frac{dz}{dx}\right)=mx^{m-1}y^n \quad \text{and} \quad \left(\frac{dz}{dy}\right)=nx^m y^{n-1}$$

there will be

$$\left(\frac{ddz}{dx^2}\right)=m(m-1)x^{m-2}y^n, \quad \left(\frac{ddz}{dxdy}\right)=mnx^{m-1}y^{n-1}, \quad \left(\frac{ddz}{dy^2}\right)=n(n-1)x^m y^{n-2}.$$

EXAMPLE 3

227. *If there shall be $z=\sqrt{(xx+yy)}$, to show the second order differential formulas of this.*

Since there shall be

$$\left(\frac{dz}{dx}\right)=\frac{x}{\sqrt{(xx+yy)}} \quad \text{and} \quad \left(\frac{dz}{dy}\right)=\frac{y}{\sqrt{(xx+yy)}},$$

there will be

$$\left(\frac{ddz}{dx^2}\right)=\frac{yy}{(xx+yy)^{\frac{3}{2}}}, \quad \left(\frac{ddz}{dxdy}\right)=\frac{-xy}{(xx+yy)^{\frac{3}{2}}}, \quad \left(\frac{ddz}{dy^2}\right)=\frac{xx}{(xx+yy)^{\frac{3}{2}}}.$$

SCHOLIUM

228. Just as two differential formulas of the first grade of any function *z* thus have been prepared, so that there shall be

$$dz = dx\left(\frac{dz}{dx}\right) + dy\left(\frac{dz}{dy}\right)$$

and on integrating

$$z = \int\left(dx\left(\frac{dz}{dx}\right) + dy\left(\frac{dz}{dy}\right)\right),$$

thus also with formulas of the second order there will be

$$\left(\frac{dz}{dx}\right) = \int\left(dx\left(\frac{ddz}{dx^2}\right) + dy\left(\frac{ddz}{dxdy}\right)\right) \quad \text{and} \quad \left(\frac{dz}{dy}\right) = \int\left(dx\left(\frac{ddz}{dxdy}\right) + dy\left(\frac{ddz}{dy^2}\right)\right).$$

Therefore three formulas of the second order are to be prepared always, so that they produce a twin integration, evidently if they may be combined duly with the differentials *dx* and *dy* ; and this property, which is to be noted properly, will produce significant help in the following.

PROBLEM 39

229. *If z shall be a function of the two variables x and y, in place of x and y there may be introduced two new variables t and u, thus so that x as well as y is equal to a certain function of t and u; to define the differential formulas of the second order of z with respect to these new variables.*

SOLUTION

As far as *z* is given by *x* and *y* , the differential formulas of this are given as for the first order $\left(\frac{dz}{dx}\right)$, $\left(\frac{dz}{dy}\right)$ as of the second order $\left(\frac{ddz}{dx^2}\right)$, $\left(\frac{ddz}{dxdy}\right)$, $\left(\frac{ddz}{dy^2}\right)$; from which it is required to define how the differential formulas in respect of the new variables *t* and *u* are determined.

But for the first order since there shall be

$$dz = dx\left(\frac{dz}{dx}\right) + dy\left(\frac{dz}{dy}\right),$$

since both *x* as well as *y* is given by *t* and *u*, there will be

$$dx = dt\left(\frac{dx}{dt}\right) + du\left(\frac{dx}{du}\right) \quad \text{and} \quad dy = dt\left(\frac{dy}{dt}\right) + du\left(\frac{dy}{du}\right),$$

with which values substituted there will be had the complete differential of *z* arising from the variation of each of *t* and *u*

$$dz = dt\left(\frac{dx}{dt}\right)\left(\frac{dz}{dx}\right) + du\left(\frac{dx}{du}\right)\left(\frac{dz}{dx}\right) + dt\left(\frac{dy}{dt}\right)\left(\frac{dz}{dy}\right) + du\left(\frac{dy}{du}\right)\left(\frac{dz}{dy}\right).$$

Now if either *t* only or *u* only is taken as the variable, there will be produced differential formulas of the first order

$$\left(\frac{dz}{dt}\right)=\left(\frac{dx}{dt}\right)\left(\frac{dz}{dx}\right)+\left(\frac{dy}{dt}\right)\left(\frac{dz}{dy}\right),\quad \left(\frac{dz}{du}\right)=\left(\frac{dx}{du}\right)\left(\frac{dz}{dx}\right)+\left(\frac{dy}{du}\right)\left(\frac{dz}{dy}\right).$$

In a similar manner on progressing further we may differentiate the formulas

$$\left(\frac{dz}{dx}\right)=p \quad \text{and} \quad \left(\frac{dz}{dy}\right)=q$$

in the first place generally, then truly we may introduce also *t* and *u* in place of *x* and *y* ; and hence we find

$$\left(\frac{dp}{dt}\right)=\left(\frac{dx}{dt}\right)\left(\frac{dp}{dx}\right)+\left(\frac{dy}{dt}\right)\left(\frac{dp}{dy}\right),\quad \left(\frac{dp}{du}\right)=\left(\frac{dx}{du}\right)\left(\frac{dp}{dx}\right)+\left(\frac{dy}{du}\right)\left(\frac{dp}{dy}\right),$$
$$\left(\frac{dq}{dt}\right)=\left(\frac{dx}{dt}\right)\left(\frac{dq}{dx}\right)+\left(\frac{dy}{dt}\right)\left(\frac{dq}{dy}\right),\quad \left(\frac{dq}{du}\right)=\left(\frac{dx}{du}\right)\left(\frac{dq}{dx}\right)+\left(\frac{dy}{du}\right)\left(\frac{dq}{dy}\right).$$

From which we can assign the formulas of the differential quantities $\left(\frac{dz}{dt}\right)$ and $\left(\frac{dz}{du}\right)$ for the variability of *t* alone as well as of *u* alone; clearly since there shall be

$$\left(\frac{dz}{dt}\right)=p\left(\frac{dx}{dt}\right)+q\left(\frac{dy}{dt}\right),\quad \text{and} \quad \left(\frac{dz}{du}\right)=p\left(\frac{dx}{du}\right)+q\left(\frac{dy}{du}\right),$$

we may find

$$\left(\frac{ddz}{dt^2}\right)=\left(\frac{ddx}{dt^2}\right)\left(\frac{dz}{dx}\right)+\left(\frac{ddy}{dt^2}\right)\left(\frac{dz}{dy}\right)+\left(\frac{dx}{dt}\right)^2\left(\frac{ddz}{dx^2}\right)+2\left(\frac{dx}{dt}\right)\left(\frac{dy}{dt}\right)\left(\frac{ddz}{dxdy}\right)+\left(\frac{dy}{dt}\right)^2\left(\frac{ddz}{dy^2}\right),$$
$$\left(\frac{ddz}{dtdu}\right)=\left(\frac{ddx}{dtdu}\right)\left(\frac{dz}{dx}\right)+\left(\frac{ddy}{dtdu}\right)\left(\frac{dz}{dy}\right)+\left(\frac{dx}{dt}\right)\left(\frac{dx}{du}\right)\left(\frac{ddz}{dx^2}\right)+\left(\frac{dx}{dt}\right)\left(\frac{dy}{du}\right)\left(\frac{ddz}{dxdy}\right)$$
$$+\left(\frac{dy}{dt}\right)\left(\frac{dx}{du}\right)\left(\frac{ddz}{dxdy}\right)+\left(\frac{dy}{dt}\right)\left(\frac{dy}{du}\right)\left(\frac{ddz}{dy^2}\right),$$
$$\left(\frac{ddz}{du^2}\right)=\left(\frac{ddx}{du^2}\right)\left(\frac{dz}{dx}\right)+\left(\frac{ddy}{du^2}\right)\left(\frac{dz}{dy}\right)+\left(\frac{dx}{du}\right)^2\left(\frac{ddz}{dx^2}\right)+2\left(\frac{dx}{du}\right)\left(\frac{dy}{du}\right)\left(\frac{ddz}{dxdy}\right)+\left(\frac{dy}{du}\right)^2\left(\frac{ddz}{dy^2}\right).$$

COROLLARY 1

230. Therefore with a certain condition proposed between the differential formulas of the function *z*, as far as it is defined by the variables *t* and *u*, the same condition for the same function *z* is transferred to the two other variables *x* and *y* depending on these in some manner.

COROLLARY 2

231. Indeed the formulas

$$\left(\frac{dx}{dt}\right),\ \left(\frac{dy}{dt}\right),\ \left(\frac{dx}{du}\right),\ \left(\frac{dy}{du}\right)\ \text{etc.}$$

are expressed by t and u from a relation which is assumed between x, y and t, u, truly in the same place the same formulas can be recalled according to the variables x and y.

SCHOLION

232. Just as this variability of the quantities t and u has been expressed through the differential formulas arising from the variables x and y, thus in turn, if the variables t and u are proposed, from which in a certain manner the others x and y may be determined, the following reductions will be had with only a permutation made of the variables.
Clearly in the first place for the formulas of the first order

$$\left(\frac{dz}{dx}\right)=\left(\frac{dt}{dx}\right)\left(\frac{dz}{dt}\right)+\left(\frac{du}{dx}\right)\left(\frac{dz}{du}\right),\quad \left(\frac{dz}{dy}\right)=\left(\frac{dt}{dy}\right)\left(\frac{dz}{dt}\right)+\left(\frac{du}{dy}\right)\left(\frac{dz}{du}\right);$$

but for the differential formulas of the second order

$$\left(\frac{ddz}{dx^2}\right)=\left(\frac{ddt}{dx^2}\right)\left(\frac{dz}{dt}\right)+\left(\frac{ddu}{dx^2}\right)\left(\frac{dz}{du}\right)+\left(\frac{dt}{dx}\right)^2\left(\frac{ddz}{dt^2}\right)+2\left(\frac{dt}{dx}\right)\left(\frac{du}{dx}\right)\left(\frac{ddz}{dtdu}\right)+\left(\frac{du}{dx}\right)^2\left(\frac{ddz}{du^2}\right),$$

$$\left(\frac{ddz}{dxdy}\right)=\left(\frac{ddt}{dxdy}\right)\left(\frac{dz}{dt}\right)+\left(\frac{ddu}{dxdy}\right)\left(\frac{dz}{du}\right)+\left(\frac{dt}{dx}\right)\left(\frac{dt}{dy}\right)\left(\frac{ddz}{dt^2}\right)+\left(\frac{dt}{dx}\right)\left(\frac{du}{dy}\right)\left(\frac{ddz}{dtdu}\right)$$
$$+\left(\frac{du}{dx}\right)\left(\frac{dt}{dy}\right)\left(\frac{ddz}{dtdu}\right)+\left(\frac{du}{dx}\right)\left(\frac{du}{dy}\right)\left(\frac{ddz}{du^2}\right),$$

$$\left(\frac{ddz}{dy^2}\right)=\left(\frac{ddt}{dy^2}\right)\left(\frac{dz}{dt}\right)+\left(\frac{ddu}{dy^2}\right)\left(\frac{dz}{du}\right)+\left(\frac{dt}{dy}\right)^2\left(\frac{ddz}{dt^2}\right)+2\left(\frac{dt}{dy}\right)\left(\frac{du}{dy}\right)\left(\frac{ddz}{dtdu}\right)+\left(\frac{du}{dy}\right)^2\left(\frac{ddz}{dy^2}\right).$$

where the determination of the letters t and u by the others x and y must be considered. Evidently since we are accustomed to use x and y for the two variables, in place of these we are able to use some other variables t and u by introducing in place of these differential formulas these new related forms according to the variables t and u, where henceforth the relation between the variables x, y and t, u thus has to be put in place, so that the question comes out easier with a solution. Therefore we set out examples with various relations of this kind.

EXAMPLE 1

233. *If this relation is put in place between the variables x, y and t, u, so that there shall be*

$$t = \alpha x + \beta y \ \text{et} \ u = \gamma x + \delta y,$$

to show the reduction of the differential formulas.

Since there shall be

$$\left(\frac{dt}{dx}\right) = \alpha, \quad \left(\frac{dt}{dy}\right) = \beta, \quad \left(\frac{du}{dx}\right) = \gamma, \quad \left(\frac{du}{dy}\right) = \delta$$

and hence the formulas for the second order vanish, we will have for the formulas of the first order

$$\left(\frac{dz}{dx}\right) = \alpha\left(\frac{dz}{dt}\right) + \gamma\left(\frac{dz}{du}\right), \quad \left(\frac{dz}{dy}\right) = \beta\left(\frac{dz}{dt}\right) + \delta\left(\frac{dz}{du}\right),$$

but for the formulas of the second order

$$\left(\frac{ddz}{dx^2}\right) = \alpha\alpha\left(\frac{ddz}{dt^2}\right) + 2\alpha\gamma\left(\frac{ddz}{dtdu}\right) + \gamma\gamma\left(\frac{ddz}{du^2}\right),$$

$$\left(\frac{ddz}{dxdy}\right) = \alpha\beta\left(\frac{ddz}{dt^2}\right) + (\alpha\delta + \beta\gamma)\left(\frac{ddz}{dtdu}\right) + \gamma\delta\left(\frac{ddz}{du^2}\right),$$

$$\left(\frac{ddz}{dy^2}\right) = \beta\beta\left(\frac{ddz}{dt^2}\right) + 2\beta\delta\left(\frac{ddz}{dtdu}\right) + \delta\delta\left(\frac{ddz}{du^2}\right).$$

COROLLARY 1

234. If it is assumed that $t = x$ and $u = x + y$, then there will be

$$\alpha = 1, \quad \beta = 0, \quad \gamma = 1 \ \text{et} \ \delta = 1;$$

and there will be

$$\left(\frac{dz}{dx}\right) = \left(\frac{dz}{dt}\right) + \left(\frac{dz}{du}\right), \quad \left(\frac{dz}{dy}\right) = \left(\frac{dz}{du}\right)$$

and

$$\left(\frac{ddz}{dx^2}\right) = \left(\frac{ddz}{dt^2}\right) + 2\left(\frac{ddz}{dtdu}\right) + \left(\frac{ddz}{du^2}\right),$$

$$\left(\frac{ddz}{dxdy}\right) = \left(\frac{ddz}{dtdu}\right) + \left(\frac{ddz}{du^2}\right),$$

$$\left(\frac{ddz}{dy^2}\right) = \left(\frac{ddz}{du^2}\right).$$

COROLLARY 2

235. Therefore even if here there should be $t = x$, there is still not $\left(\frac{dz}{dt}\right) = \left(\frac{dz}{dx}\right)$, the reason for this is, since the quantity y is taken constant in the form $\left(\frac{dz}{dx}\right)$, in $\left(\frac{dz}{dt}\right)$ truly the quantity $u = x + y$; that which it helps in general to be noted, lest from the equality $t = x$ we may infer the equality of the formulas $\left(\frac{dz}{dx}\right)$ and $\left(\frac{dz}{dt}\right)$.

EXEMPLE 2

236. *If this relation is put in place between the variables t, u and x, y, so that there shall be* $t = \alpha x^m$ *and* $u = \beta y^n$, *to show the reduction.*

Therefore here there shall be

$$\left(\frac{dt}{dx}\right) = m\alpha x^{m-1}, \quad \left(\frac{dt}{dy}\right) = 0, \quad \left(\frac{ddt}{dx^2}\right) = m(m-1)\alpha x^{m-2},$$
$$\left(\frac{du}{dx}\right) = 0, \quad \left(\frac{du}{dy}\right) = n\beta y^{n-1}, \quad \left(\frac{ddu}{dy^2}\right) = n(n-1)\beta y^{n-2},$$

from which we obtain for the formulas of the first order

$$\left(\frac{dz}{dx}\right) = m\alpha x^{m-1}\left(\frac{dz}{dt}\right), \quad \left(\frac{dz}{dy}\right) = n\beta y^{n-1}\left(\frac{dz}{du}\right),$$

moreover for the formulas of the second order

$$\left(\frac{ddz}{dx^2}\right) = m(m-1)\alpha x^{m-2}\left(\frac{dz}{dt}\right) + mm\alpha\alpha x^{2m-2}\left(\frac{ddz}{dt^2}\right),$$
$$\left(\frac{ddz}{dxdy}\right) = mn\alpha\beta x^{m-1} y^{n-1}\left(\frac{ddz}{dtdu}\right),$$
$$\left(\frac{ddz}{dy^2}\right) = n(n-1)\beta x^{n-2}\left(\frac{dz}{du}\right) + nn\beta\beta x^{2n-2}\left(\frac{ddz}{du^2}\right),$$

In which formulas now in place of x and y the values of these in terms of t and u must be introduced.

EXAMPLE 3

237. *If this relation is put in place between the variables t, u and x, y, so that there shall be* $x=t$ *and* $\frac{x}{y}=u$*, to show the reduction of the differentials of the formulas.*

Since there shall be $t=x$ and $\frac{x}{y}=u$ there will be

$$\left(\frac{dt}{dx}\right)=1,\quad \left(\frac{dt}{dy}\right)=0$$

and hence the formulas involving *ddt* vanish. Again

$$\left(\frac{du}{dx}\right)=\frac{1}{y}=\frac{u}{t},\quad \left(\frac{du}{dy}\right)=\frac{-x}{yy}=\frac{-uu}{t},$$
$$\left(\frac{ddu}{dx^2}\right)=0,\quad \left(\frac{ddu}{dxdy}\right)=\frac{-1}{yy}=\frac{-uu}{tt},\quad \left(\frac{ddu}{dy^2}\right)=\frac{2x}{y^3}=\frac{2u^3}{tt},$$

from which for the formulas of the first order we shall have

$$\left(\frac{dz}{dx}\right)=\left(\frac{dz}{dt}\right)+\frac{u}{t}\left(\frac{dz}{du}\right),\quad \left(\frac{dz}{dy}\right)=\frac{-uu}{t}\left(\frac{dz}{du}\right),$$

moreover for the formulas of the second order

$$\left(\frac{ddz}{dx^2}\right)=\left(\frac{ddz}{dt^2}\right)+\frac{2u}{t}\left(\frac{ddz}{dtdu}\right)+\frac{uu}{tt}\left(\frac{ddz}{du^2}\right),$$
$$\left(\frac{ddz}{dxdy}\right)=\frac{-uu}{tt}\left(\frac{dz}{du}\right)-\frac{uu}{t}\left(\frac{ddz}{dtdu}\right)-\frac{u^3}{tt}\left(\frac{ddz}{du^2}\right),$$
$$\left(\frac{ddz}{dy^2}\right)=\frac{2u^3}{tt}\left(\frac{dz}{du}\right)+\frac{u^4}{tt}\left(\frac{ddz}{du^2}\right).$$

EXAMPLE 4

238. *If this relation is put in place between the variables t, u and x, y, so that there shall be* $t=e^x$ *and* $u=e^xy$ *or* $x=lt$ *and* $y=\frac{u}{t}$*, to show the reduction of the differentials of the formulas.*

Therefore here there is

$$\left(\frac{dt}{dx}\right)=e^x,\quad \left(\frac{dt}{dy}\right)=0,\quad \left(\frac{ddt}{dx^2}\right)=e^x=t,\quad \left(\frac{ddt}{dxdy}\right)=0.$$

In next place

$$\left(\frac{du}{dx}\right)=e^{x}y=u,\quad\left(\frac{du}{dy}\right)=e^{x}=t,$$

then truly

$$\left(\frac{ddu}{dx^2}\right)=e^{x}y=u,\quad\left(\frac{ddu}{dxdy}\right)=e^{x}=t,\quad\left(\frac{ddu}{dy^2}\right)=0.$$

Whereby we will have for the formulas of the first order

$$\left(\frac{dz}{dx}\right)=t\left(\frac{dz}{dt}\right)+u\left(\frac{dz}{du}\right),\quad\left(\frac{dz}{dy}\right)=t\left(\frac{dz}{du}\right),$$

moreover for the formulas of the second order

$$\left(\frac{ddz}{dx^2}\right)=t\left(\frac{dz}{dt}\right)+u\left(\frac{dz}{du}\right)+tt\left(\frac{ddz}{dt^2}\right)+2tu\left(\frac{ddz}{dtdu}\right)+uu\left(\frac{ddz}{du^2}\right),$$
$$\left(\frac{ddz}{dxdy}\right)=t\left(\frac{dz}{du}\right)+tt\left(\frac{ddz}{dtdu}\right)+tu\left(\frac{ddz}{du^2}\right),$$
$$\left(\frac{ddz}{dy^2}\right)=tt\left(\frac{ddz}{du^2}\right).$$

SCHOLION

239. In the general formulas given § 232 we have assumed the values of the variables *t* and *u* to be given expressed by *x* and *y* and then with the general setting out made at last the variables *t* and *u* are put in place for *x* and *y*. Therefore it may be observed to be more convenient, if at once the values of the variables *x* and *y* are to be had expressed by *t* and *u* ; thence indeed exceedingly complex values of the formulas $\left(\frac{dt}{dx}\right),\left(\frac{dt}{dy}\right)$ etc. may be expressed, in order that these are allowed to be introduced into the calculation. Clearly if *x* and *y* may be given by *t* and *u*, from this there shall be

$$\left(\frac{dt}{dx}\right)=\frac{\left(\frac{dy}{du}\right)}{\left(\frac{dx}{dt}\right)\left(\frac{dy}{du}\right)-\left(\frac{dx}{du}\right)\left(\frac{dy}{dt}\right)}$$

and the formulas of the second order many more perplexing are to be produced. Therefore whatever the case, by which a reduction of this kind is seen to be used, it may be concluded rather that by some account a suitable change of the variables may be convenient to be deduced.

[We may put in place as above $\left(\frac{dz}{dx}\right)=\left(\frac{dt}{dx}\right)\left(\frac{dz}{dt}\right)+\left(\frac{du}{dx}\right)\left(\frac{dz}{du}\right),\quad\left(\frac{dz}{dy}\right)=\left(\frac{dt}{dy}\right)\left(\frac{dz}{dt}\right)+\left(\frac{du}{dy}\right)\left(\frac{dz}{du}\right)$; Also, since *x* and *y* are functions of *u* and *v*, we have :

$$\left(\frac{dz}{dt}\right)=\left(\frac{dx}{dt}\right)\left(\frac{dz}{dx}\right)+\left(\frac{dy}{dt}\right)\left(\frac{dz}{dy}\right),\quad\left(\frac{dz}{du}\right)=\left(\frac{dx}{du}\right)\left(\frac{dz}{dx}\right)+\left(\frac{dy}{du}\right)\left(\frac{dz}{dy}\right)\;;$$

hence,

$$\begin{pmatrix}\left(\frac{dz}{dt}\right)\\ \left(\frac{dz}{du}\right)\end{pmatrix}=\begin{pmatrix}\left(\frac{dx}{dt}\right) & \left(\frac{dy}{dt}\right)\\ \left(\frac{dx}{du}\right) & \left(\frac{dy}{du}\right)\end{pmatrix}\begin{pmatrix}\left(\frac{dz}{dx}\right)\\ \left(\frac{dz}{dy}\right)\end{pmatrix};$$

while

$$\begin{pmatrix}\left(\frac{dz}{dx}\right)\\ \left(\frac{dz}{dy}\right)\end{pmatrix}=\begin{pmatrix}\left(\frac{dt}{dx}\right) & \left(\frac{du}{dx}\right)\\ \left(\frac{dt}{dy}\right) & \left(\frac{du}{dy}\right)\end{pmatrix}\begin{pmatrix}\left(\frac{dz}{dt}\right)\\ \left(\frac{dz}{du}\right)\end{pmatrix},$$

hence ,

$$\begin{pmatrix}\left(\frac{dx}{dt}\right) & \left(\frac{dy}{dt}\right)\\ \left(\frac{dx}{du}\right) & \left(\frac{dy}{du}\right)\end{pmatrix}\begin{pmatrix}\left(\frac{dt}{dx}\right) & \left(\frac{du}{dx}\right)\\ \left(\frac{dt}{dy}\right) & \left(\frac{du}{dy}\right)\end{pmatrix}=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix};$$

giving,

$$\begin{pmatrix}\left(\frac{dx}{dt}\right) & \left(\frac{dy}{dt}\right)\\ \left(\frac{dx}{du}\right) & \left(\frac{dy}{du}\right)\end{pmatrix}\begin{pmatrix}\left(\frac{dt}{dx}\right)\\ \left(\frac{dt}{dy}\right)\end{pmatrix}=\begin{pmatrix}1\\ 0\end{pmatrix} \quad\text{and}\quad \begin{pmatrix}\left(\frac{dx}{dt}\right) & \left(\frac{dy}{dt}\right)\\ \left(\frac{dx}{du}\right) & \left(\frac{dy}{du}\right)\end{pmatrix}\begin{pmatrix}\left(\frac{du}{dx}\right)\\ \left(\frac{du}{dy}\right)\end{pmatrix}=\begin{pmatrix}0\\ 1\end{pmatrix};$$

hence, from Cramer's Rule,

$$\left(\frac{dt}{dx}\right)=\frac{\begin{vmatrix}1 & \left(\frac{dy}{dt}\right)\\ 0 & \left(\frac{dy}{du}\right)\end{vmatrix}}{\begin{vmatrix}\left(\frac{dx}{dt}\right) & \left(\frac{dy}{dt}\right)\\ \left(\frac{dx}{du}\right) & \left(\frac{dy}{du}\right)\end{vmatrix}}=\frac{\left(\frac{dy}{du}\right)}{\begin{vmatrix}\left(\frac{dx}{dt}\right) & \left(\frac{dy}{dt}\right)\\ \left(\frac{dx}{du}\right) & \left(\frac{dy}{du}\right)\end{vmatrix}}=\frac{\left(\frac{dy}{du}\right)}{\left(\frac{dx}{dt}\right)\left(\frac{dy}{du}\right)-\left(\frac{dy}{dt}\right)\left(\frac{dx}{du}\right)},$$

with similar formulas for the other differentials $\left(\frac{dt}{dy}\right),\left(\frac{du}{dx}\right),$ and $\left(\frac{du}{dy}\right)$, as required. A lead-in to Jacobi's Determinants.
Note especially that terms such as $\left(\frac{dt}{dx}\right)$ and $\left(\frac{dx}{dt}\right)$ are not the reciprocals of each other in general, as they are evaluated with the different variables *y* and *u* held constant. We also remind the reader of Euler's apparent indifference to conditions to be satisfied in these derivations, such as the denominator not being zero, the derivatives being defined in a region, etc.]

Truly also a reduction by another way gives often brings something of useful significance, provided that the form of the function *z* sought is changed, just as if there is put $z=Vv$ with *V* denoting some given function of *x* et *y*, thus so that now *v* shall be the function sought ; moreover this new function sought *v* can be involved in different way from that given.

PROBLEM 40

240. *With the proposed function z of the two variables x and y and on putting $z = Pv$, thus so that P shall be some given function of x and y, to express the differential formulas of z in terms of the differential formulas of the new function.*

SOLUTION

Since there shall be $z = Pv$, from the rules of differentiation treated we will have the differential formulas of the first order

$$\left(\frac{dz}{dx}\right)=\left(\frac{dP}{dx}\right)v+P\left(\frac{dv}{dx}\right) \quad \text{and} \quad \left(\frac{dz}{dy}\right)=\left(\frac{dP}{dy}\right)v+P\left(\frac{dv}{dy}\right).$$

And hence following the differential formulas of the second order expressed, thus will be produced

$$\left(\frac{ddz}{dx^2}\right)=\left(\frac{ddP}{dx^2}\right)v+2\left(\frac{dP}{dx}\right)\left(\frac{dv}{dx}\right)+P\left(\frac{ddv}{dx^2}\right),$$
$$\left(\frac{ddz}{dxdy}\right)=\left(\frac{ddP}{dxdy}\right)v+\left(\frac{dP}{dx}\right)\left(\frac{dv}{dy}\right)+\left(\frac{dP}{dy}\right)\left(\frac{dv}{dx}\right)+P\left(\frac{ddv}{dxdy}\right),$$
$$\left(\frac{ddz}{dy^2}\right)=\left(\frac{ddP}{dy^2}\right)v+2\left(\frac{dP}{dy}\right)\left(\frac{dv}{dy}\right)+P\left(\frac{ddv}{dy^2}\right);$$

where since *P* shall be a given function of *x* and *y*, likewise the differential formulas will be given.

COROLLARY 1

241. If *P* were a function of *x* only, for example *X*, then on putting $z= Xv$ there will be

$$\left(\frac{dz}{dx}\right)=\left(\frac{dX}{dx}\right)v+X\left(\frac{dv}{dx}\right) \quad \text{and} \quad \left(\frac{dz}{dy}\right)=X\left(\frac{dv}{dy}\right),$$

then

$$\left(\frac{ddz}{dx^2}\right)=\left(\frac{ddX}{dx^2}\right)v+\frac{2dX}{dx}\left(\frac{dv}{dx}\right)+X\left(\frac{ddv}{dx^2}\right),$$
$$\left(\frac{ddz}{dxdy}\right)=\frac{dX}{dx}\left(\frac{dv}{dy}\right)+X\left(\frac{ddv}{dxdy}\right),$$
$$\left(\frac{ddz}{dy^2}\right)=X\left(\frac{ddv}{dy^2}\right).$$

COROLLARY 2

242. This transformation keeps the same variables *x* and *y* and yet in place of the function *z* another *v* is introduced, as before with the same remaining function *z* of the two variables *x* and *y* to be reduced to the others *t* and *u*. From which these two transformations generally are different.

SCHOLIUM 1

243. A simpler case would be, if by addition we could put $z = P + v$, so that P would be some given function of x and y; truly then the transformation shall become obvious, so that there is no need for an investigation ; indeed it is evident that

$$\left(\frac{dz}{dx}\right) = \left(\frac{dP}{dx}\right) + \left(\frac{dv}{dx}\right), \quad \left(\frac{dz}{dy}\right) = \left(\frac{dP}{dy}\right) + \left(\frac{dv}{dy}\right),$$

$$\left(\frac{ddz}{dx^2}\right) = \left(\frac{ddP}{dx^2}\right) + \left(\frac{ddv}{dx^2}\right),$$
$$\left(\frac{ddz}{dxdy}\right) = \left(\frac{ddP}{dxdy}\right) + \left(\frac{ddv}{dxdy}\right),$$
$$\left(\frac{ddz}{dy^2}\right) = \left(\frac{ddP}{dy^2}\right) + \left(\frac{ddv}{dy^2}\right).$$

Nor also is it necessary for forms with more parts to be worked, just as if we put, $z = \sqrt{(PP + vv)}$, since such a form scarcely ever will have a use.

SCHOLIUM 2

244. From these principles presented, and with the transformations of the work we have approached, both the methods we have found from a given relation between the differential formulas of the second order and of the first order, and likewise the relation of the principal quantities themselves are required to be investigated. Here clearly besides these quantities themselves x, y and z and the differential formulas of these of the first order $\left(\frac{dz}{dx}\right)$ and $\left(\frac{dz}{dy}\right)$ to be considered, there comes about three differential formulas of the second order $\left(\frac{ddz}{dx^2}\right)$, $\left(\frac{ddz}{dxdy}\right)$ and $\left(\frac{ddz}{dy^2}\right)$, of which either one or two or all are to be present in a proposed relation, where above they constitute a huge distinction between the formulas of the first order either in the relation they enter or otherwise. Moreover not only shall the combinations be exceedingly long to pursue, as we have accomplished in the preceding section, but also the lack of convenient methods hinders the kinds of individual question here of concern that we come across. Therefore we have thus put in place the chapters to be treated, as the method of solution allows, and those [problems] where nothing evidently is outstanding, we have completely passed over.

CALCULI INTEGRALIS
LIBER POSTERIOR.
PARS PRIMA

SEU

INVESTIGATIO FUNCTIONUM DUARUM

VARIABILIUM EX DATA DIFFERENTIALIVM
CUIUSVIS GRADUS RELATIONE

SECTIO SECUNDA

INVESTIGATIO DUARUM VARIABILIUM
lUNCTIONUM EX DATA DIFFERENTIALIUM
SECUNDI GRADUS RELATIONE

CAPUT I

DE FORMULIS DIFFERENTIALIBUS SECUNDI GRADUS IN GENERE

PROBLEMA 38

220. *Si z sit functio quaecunque binarum variabilium x et y, eius formulas differentiales secundi gradus exhibere.*

SOLUTIO

Cum z sit functio binarum variabilium x et y, eius differentiale huiusmodi habebit formam $dz = pdx + qdy$, ex qua p et q sunt formulae differentiales primi gradus, quas ita denotare solemus

$$p = \left(\frac{dz}{dx}\right) \quad \text{et} \quad q = \left(\frac{dz}{dy}\right).$$

Cum nunc sint quoque p et q functiones ipsarum x et y, formulae differentiales inde natae erunt formulae differentiales secundi gradus ipsius z, unde intelligitur quatuor huiusmodi formulas nasci

$$\left(\frac{dp}{dx}\right), \quad \left(\frac{dp}{dy}\right), \quad \left(\frac{dq}{dx}\right), \quad \left(\frac{dq}{dy}\right),$$

quarum autem secundam ac tertiam inter se congruere in Calculo differentiali est demonstratum. Sed cum sit $p = \left(\frac{dz}{dx}\right)$, simili scribendi ratione erit $\left(\frac{dp}{dx}\right) = \left(\frac{ddz}{dx^2}\right)$, cuius scripturae significatus hinc sponte patet. Deinde eodem modo erit $\left(\frac{dp}{dy}\right) = \left(\frac{ddz}{dxdy}\right)$ atque ob $q = \left(\frac{dz}{dy}\right)$ habebimus $\left(\frac{dq}{dx}\right) = \left(\frac{ddz}{dydx}\right)$ et $\left(\frac{dq}{dy}\right) = \left(\frac{ddz}{dy^2}\right)$. Quia ergo est

$$\left(\frac{ddz}{dydx}\right) = \left(\frac{ddz}{dxdy}\right),$$

functioni z convenient tres formulae differentiales secundi gradus, quae sunt

$$\left(\frac{ddz}{dx^2}\right), \quad \left(\frac{ddz}{dxdy}\right), \quad \left(\frac{ddz}{dy^2}\right).$$

COROLLARIUM 1

221. Ut ergo functio z duarum variabilium x et y duas habet formulas differentiales primi gradus

$$\left(\frac{dz}{dx}\right) \quad \text{et} \quad \left(\frac{dz}{dy}\right),$$

ita habet tres formulas differentiales secundi gradus

$$\left(\frac{ddz}{dx^2}\right), \quad \left(\frac{ddz}{dxdy}\right), \quad \text{et} \quad \left(\frac{ddz}{dy^2}\right).$$

COROLLARIUM 2

222. Hae ergo formulae per duplicem differentiationem nascuntur unicam tantum quantitatem pro variabili accipiendo. In prima scilicet bis eadem *x* variabilis sumitur, in secunda vero in altera differentiatione *x*, in altera autem *y* variabilis accipitur, in tertia autem bis *y*.

COROLLARIUM 3

223. Simili modo patet eiusdem functionis *z* quatuor dari formulas differentiales tertii gradus , scilicet

$$\left(\frac{d^3z}{dx^3}\right), \quad \left(\frac{d^3z}{dx^2dy}\right), \quad \left(\frac{d^3z}{dxdy^2}\right), \quad \left(\frac{ddz}{dy^2}\right),$$

quarti autem gradus quinque, quinti sex etc.

SCHOLION

224. Formulae hae differentiales secundi gradus ope substitutionis saltem ad formam primi gradus revocari possunt. Veluti formula $\left(\frac{ddz}{dx^2}\right)$, si ponatur $\left(\frac{dz}{dx}\right)=p$, transformabitur in $\left(\frac{dp}{dx}\right)$, formula autem $\left(\frac{ddz}{dxdy}\right)$ eadem substitutione in hanc $\left(\frac{dq}{dy}\right)$. At posito $\left(\frac{dz}{dy}\right)=q$ formula $\left(\frac{ddz}{dxdy}\right)$ transmutatur in hanc $\left(\frac{dq}{dx}\right)$, formula autem $\left(\frac{ddz}{dy^2}\right)$ in hanc $\left(\frac{dq}{dy}\right)$. Vicissim autem uti ex aequalitate $p=\left(\frac{dz}{dx}\right)$ deduximus

$$\left(\frac{dp}{dx}\right)=\left(\frac{ddz}{dx^2}\right) \quad \text{et} \quad \left(\frac{dp}{dy}\right)=\left(\frac{ddz}{dxdy}\right),$$

ita ex his ulterius progrediendo colligemus

$$\left(\frac{ddp}{dx^2}\right)=\left(\frac{d^3z}{dx^3}\right), \quad \left(\frac{ddp}{dxdy}\right)=\left(\frac{d^3z}{dx^2dy}\right), \quad \left(\frac{ddp}{dy^2}\right)=\left(\frac{d^3z}{dxdy^2}\right).$$

Tum vero etiam si ponamus $\left(\frac{dp}{dx}\right)=\left(\frac{dz}{dy}\right)$, hinc sequentur istae aequalitates $\left(\frac{ddp}{dx^2}\right)=\left(\frac{ddz}{dxdy}\right)$ et $\left(\frac{ddp}{dxdy}\right)=\left(\frac{ddz}{dy^2}\right)$. Hicque est quasi novus algorithmus, cuius principia per se ita sunt manifesta, ut maiore illustratione non indigeant.

EXEMPLUM 1

225. *Si sit $z = xy$, eius formulas differentiales secundi gradus exhibere.*
Cum sit

$$\left(\frac{dz}{dx}\right) = y \quad \text{et} \quad \left(\frac{dz}{dy}\right) = x,$$

erit

$$\left(\frac{ddz}{dx^2}\right) = 0, \quad \left(\frac{ddz}{dxdy}\right) = 1 \;\text{ et }\; \left(\frac{ddz}{dy^2}\right) = 0.$$

EXEMPLUM 2

226. *Si sit $z = x^m y^n$, eius formulas differentiales secundi gradus exhibere.*
Cum sit

$$\left(\frac{dz}{dx}\right) = mx^{m-1}y^n \quad \text{et} \quad \left(\frac{dz}{dy}\right) = nx^m y^{n-1}$$

erit

$$\left(\frac{ddz}{dx^2}\right) = m(m-1)x^{m-2}y^n, \quad \left(\frac{ddz}{dxdy}\right) = mnx^{m-1}y^{n-1}, \quad \left(\frac{ddz}{dy^2}\right) = n(n-1)x^m y^{n-2}.$$

EXEMPLUM 3

227. *Si sit $z = \sqrt{(xx+yy)}$, eius formulas differentiales secundi gradus exhibere.*
Cum sit

$$\left(\frac{dz}{dx}\right) = \frac{x}{\sqrt{(xx+yy)}} \quad \text{et} \quad \left(\frac{dz}{dy}\right) = \frac{y}{\sqrt{(xx+yy)}},$$

erit

$$\left(\frac{ddz}{dx^2}\right) = \frac{yy}{(xx+yy)^{\frac{3}{2}}}, \quad \left(\frac{ddz}{dxdy}\right) = \frac{-xy}{(xx+yy)^{\frac{3}{2}}}, \quad \left(\frac{ddz}{dy^2}\right) = \frac{xx}{(xx+yy)^{\frac{3}{2}}}.$$

SCHOLION

228. Quemadmodum binae formulae differentiales primi gradus cuiusque functionis z ita sunt comparatae, ut sit

$$dz = dx\left(\frac{dz}{dx}\right) + dy\left(\frac{dz}{dy}\right)$$

et integrando

$$z=\int\left(dx\left(\frac{dz}{dx}\right)+dy\left(\frac{dz}{dy}\right)\right),$$

ita quoque in formulis secundi gradus erit

$$\left(\frac{dz}{dx}\right)=\int\left(dx\left(\frac{ddz}{dx^2}\right)+dy\left(\frac{ddz}{dxdy}\right)\right)\quad\text{et}\quad\left(\frac{dz}{dy}\right)=\int\left(dx\left(\frac{ddz}{dxdy}\right)+dy\left(\frac{ddz}{dy^2}\right)\right).$$

Tres igitur formulae secundi gradus semper ita sunt comparatae, ut geminam integrationem praebeant, si scilicet cum differentialibus dx et dy rite combinentur; haecque proprietas, quae probe notetur, in sequentibus insigne adiumentum afferet.

PROBLEMA 39

229. *Si z sit functio binarum variabilium x et y, loco x et y introducantur binae novae variabiles t et u, ita ut tam x quam y aequetur certae functioni ipsarum t et u; formulas differentiales secundi gradus ipsius z respectu harum novarum variabilium definire.*

SOLUTIO

Quatenus z per x et y datur, datae sunt eius formulae differentiales tam primi gradus $\left(\frac{dz}{dx}\right)$, $\left(\frac{dz}{dy}\right)$ quam secundi gradus $\left(\frac{ddz}{dx^2}\right)$, $\left(\frac{ddz}{dxdy}\right)$, $\left(\frac{ddz}{dy^2}\right)$; ex quibus, quomodo formulae differentiales respectu novarum variabilium t et u determinentur, definiri oportet.

Pro primo gradu autem cum sit

$$dz=dx\left(\frac{dz}{dx}\right)+dy\left(\frac{dz}{dy}\right),$$

quia tam x quam y datur per t et u, erit

$$dx=dt\left(\frac{dx}{dt}\right)+du\left(\frac{dx}{du}\right)\quad\text{et}\quad dy=dt\left(\frac{dy}{dt}\right)+du\left(\frac{dy}{du}\right),$$

quibus valoribus substitutis habebitur ipsius z differentiale plenum ex variatione utriusque, t et u ortum

$$dz=dt\left(\frac{dx}{dt}\right)\left(\frac{dz}{dx}\right)+du\left(\frac{dx}{du}\right)\left(\frac{dz}{dx}\right)+dt\left(\frac{dy}{dt}\right)\left(\frac{dz}{dy}\right)+du\left(\frac{dy}{du}\right)\left(\frac{dz}{dy}\right).$$

Quodsi iam vel sola t variabilis sumatur vel sola u, prodibunt formulae differentiales primi gradus

$$\left(\frac{dz}{dt}\right)=\left(\frac{dx}{dt}\right)\left(\frac{dz}{dx}\right)+\left(\frac{dy}{dt}\right)\left(\frac{dz}{dy}\right),\quad\left(\frac{dz}{du}\right)=\left(\frac{dx}{du}\right)\left(\frac{dz}{dx}\right)+\left(\frac{dy}{du}\right)\left(\frac{dz}{dy}\right).$$

Simili modo ulterius progrediendo differentiemus formulas

$$\left(\frac{dz}{dx}\right)=p \quad \text{et} \quad \left(\frac{dz}{dy}\right)=q$$

primo generaliter, tum vero loco x et y, etiam t et u introducamus; hincque nanciscemur

$$\left(\frac{dp}{dt}\right)=\left(\frac{dx}{dt}\right)\left(\frac{dp}{dx}\right)+\left(\frac{dy}{dt}\right)\left(\frac{dp}{dy}\right), \quad \left(\frac{dp}{du}\right)=\left(\frac{dx}{du}\right)\left(\frac{dp}{dx}\right)+\left(\frac{dy}{du}\right)\left(\frac{dp}{dy}\right),$$
$$\left(\frac{dq}{dt}\right)=\left(\frac{dx}{dt}\right)\left(\frac{dq}{dx}\right)+\left(\frac{dy}{dt}\right)\left(\frac{dq}{dy}\right), \quad \left(\frac{dq}{du}\right)=\left(\frac{dx}{du}\right)\left(\frac{dq}{dx}\right)+\left(\frac{dy}{du}\right)\left(\frac{dq}{dy}\right).$$

Unde poterimus formulas [differentiles qllantitatum] $\left(\frac{dz}{dt}\right)$ et $\left(\frac{dz}{du}\right)$ pro variabilitate tam solius t quam solius u assignare; scilicet cum sit

$$\left(\frac{dz}{dt}\right)=p\left(\frac{dx}{dt}\right)+q\left(\frac{dy}{dt}\right), \quad \text{et} \quad \left(\frac{dz}{du}\right)=p\left(\frac{dx}{du}\right)+q\left(\frac{dy}{du}\right),$$

inveniemus

$$\left(\frac{ddz}{dt^2}\right)=\left(\frac{ddx}{dt^2}\right)\left(\frac{dz}{dx}\right)+\left(\frac{ddy}{dt^2}\right)\left(\frac{dz}{dy}\right)+\left(\frac{dx}{dt}\right)^2\left(\frac{ddz}{dx^2}\right)+2\left(\frac{dx}{dt}\right)\left(\frac{dy}{dt}\right)\left(\frac{ddz}{dxdy}\right)+\left(\frac{dy}{dt}\right)^2\left(\frac{ddz}{dy^2}\right),$$
$$\left(\frac{ddz}{dtdu}\right)=\left(\frac{ddx}{dtdu}\right)\left(\frac{dz}{dx}\right)+\left(\frac{ddy}{dtdu}\right)\left(\frac{dz}{dy}\right)+\left(\frac{dx}{dt}\right)\left(\frac{dx}{du}\right)\left(\frac{ddz}{dx^2}\right)+\left(\frac{dx}{dt}\right)\left(\frac{dy}{du}\right)\left(\frac{ddz}{dxdy}\right)+\left(\frac{dy}{dt}\right)\left(\frac{dx}{du}\right)\left(\frac{ddz}{dxdy}\right)+\left(\frac{dy}{dt}\right)\left(\frac{dy}{du}\right)\left(\frac{ddz}{dy^2}\right),$$
$$\left(\frac{ddz}{du^2}\right)=\left(\frac{ddx}{du^2}\right)\left(\frac{dz}{dx}\right)+\left(\frac{ddy}{du^2}\right)\left(\frac{dz}{dy}\right)+\left(\frac{dx}{du}\right)^2\left(\frac{ddz}{dx^2}\right)+2\left(\frac{dx}{du}\right)\left(\frac{dy}{du}\right)\left(\frac{ddz}{dxdy}\right)+\left(\frac{dy}{du}\right)^2\left(\frac{ddz}{dy^2}\right).$$

COROLLARIUM 1

230. Proposita ergo conditione quadam inter formulas differentiales functionis z, quatenus per variabiles t et u definitur, eadem conditio pro eadem functione z transfertur ad alias binas variabiles x et y ab illis utcunque pendentes.

COROLLARIUM 2

231. Formulae quidem

$$\left(\frac{dx}{dt}\right), \quad \left(\frac{dy}{dt}\right), \quad \left(\frac{dx}{du}\right), \quad \left(\frac{dy}{du}\right) \quad \text{etc.}$$

per t et u exprimuntur ex relatione, quae inter x, y et t, u assumitur, verum indidem eaedem formulae ad variabiles x et y revocari possunt.

SCHOLION

232. Quemadmodum hic variabilitas quantitatum t et u per formulas differentiales ex variabilibus x et y natas est expressa, ita vicissim, si variabiles t et u proponantur, ex quibus certo modo alterae x et y determinentur, sequentes reductiones habebuntur facta tantum variabilium permutatione. Primo scilicet pro formulis primi gradus

$$\left(\frac{dz}{dx}\right)=\left(\frac{dt}{dx}\right)\left(\frac{dz}{dt}\right)+\left(\frac{du}{dx}\right)\left(\frac{dz}{du}\right),\quad \left(\frac{dz}{dy}\right)=\left(\frac{dt}{dy}\right)\left(\frac{dz}{dt}\right)+\left(\frac{du}{dy}\right)\left(\frac{dz}{du}\right);$$

pro formulis autem differentialibus secundi gradus

$$\left(\frac{ddz}{dx^2}\right)=\left(\frac{ddt}{dx^2}\right)\left(\frac{dz}{dt}\right)+\left(\frac{ddu}{dx^2}\right)\left(\frac{dz}{du}\right)+\left(\frac{dt}{dx}\right)^2\left(\frac{ddz}{dt^2}\right)+2\left(\frac{dt}{dx}\right)\left(\frac{du}{dx}\right)\left(\frac{ddz}{dtdu}\right)+\left(\frac{du}{dx}\right)^2\left(\frac{ddz}{du^2}\right),$$

$$\left(\frac{ddz}{dxdy}\right)=\left(\frac{ddt}{dxdy}\right)\left(\frac{dz}{dt}\right)+\left(\frac{ddu}{dxdy}\right)\left(\frac{dz}{du}\right)+\left(\frac{dt}{dx}\right)\left(\frac{dt}{dy}\right)\left(\frac{ddz}{dt^2}\right)+\left(\frac{dt}{dx}\right)\left(\frac{du}{dy}\right)\left(\frac{ddz}{dtdu}\right)+\left(\frac{du}{dx}\right)\left(\frac{dt}{dy}\right)\left(\frac{ddz}{dtdu}\right)+\left(\frac{du}{dx}\right)\left(\frac{du}{dy}\right)\left(\frac{ddz}{du^2}\right),$$

$$\left(\frac{ddz}{dy^2}\right)=\left(\frac{ddt}{dy^2}\right)\left(\frac{dz}{dt}\right)+\left(\frac{ddu}{dy^2}\right)\left(\frac{dz}{du}\right)+\left(\frac{dt}{dy}\right)^2\left(\frac{ddz}{dt^2}\right)+2\left(\frac{dt}{dy}\right)\left(\frac{du}{dy}\right)\left(\frac{ddz}{dtdu}\right)+\left(\frac{du}{dy}\right)^2\left(\frac{ddz}{dy^2}\right).$$

ubi determinatio litterarnm t et u per alteras x et y considerari debet. Quoniam scilicet in conditionibus praescriptis binis variabilibus x et y uti solemus, earnm loco alias quascunque t et u introducendo loco illarum formularum differentialium has novas formas ad variabiles t et u relatas adhibere poterimus, ubi deinceps relatio inter variabiles x, y et t, u ita est constituenda, ut quaestio solutu facilior evadat. Pro variis igitur huiusmodi relationibus exempla evolvamus.

EXEMPLUM 1

233. *Si inter variabiles x, y et t, u haec relatio constituatur, ut sit*

$$t=\alpha x+\beta y \text{ et } u=\gamma x+\delta y,$$

reductionem formularum differentialium exhibere.

Cum sit

$$\left(\frac{dt}{dx}\right)=\alpha,\quad \left(\frac{dt}{dy}\right)=\beta,\quad \left(\frac{du}{dx}\right)=\gamma,\quad \left(\frac{du}{dy}\right)=\delta$$

hincque formulae pro secundo gradu evanescant, habebimus pro formulis primi gradus

$$\left(\frac{dz}{dx}\right)=\alpha\left(\frac{dz}{dt}\right)+\gamma\left(\frac{dz}{du}\right),\quad \left(\frac{dz}{dy}\right)=\beta\left(\frac{dz}{dt}\right)+\delta\left(\frac{dz}{du}\right),$$

pro formulis autem secundi gradus

$$\left(\frac{ddz}{dx^2}\right)=\alpha\alpha\left(\frac{ddz}{dt^2}\right)+2\alpha\gamma\left(\frac{ddz}{dtdu}\right)+\gamma\gamma\left(\frac{ddz}{du^2}\right),$$

$$\left(\frac{ddz}{dxdy}\right)=\alpha\beta\left(\frac{ddz}{dt^2}\right)+(\alpha\delta+\beta\gamma)\left(\frac{ddz}{dtdu}\right)+\gamma\delta\left(\frac{ddz}{du^2}\right),$$

$$\left(\frac{ddz}{dy^2}\right)=\beta\beta\left(\frac{ddz}{dt^2}\right)+2\beta\delta\left(\frac{ddz}{dtdu}\right)+\delta\delta\left(\frac{ddz}{du^2}\right).$$

COROLLARIUM 1

234. Si sumatur $t = x$ et $u = x + y$, erit

$$\alpha = 1, \quad \beta = 0, \quad \gamma = 1 \text{ et } \delta = 1 ;$$

erit

$$\left(\frac{dz}{dx}\right) = \left(\frac{dz}{dt}\right) + \left(\frac{dz}{du}\right), \quad \left(\frac{dz}{dy}\right) = \left(\frac{dz}{du}\right)$$

atque

$$\left(\frac{ddz}{dx^2}\right) = \left(\frac{ddz}{dt^2}\right) + 2\left(\frac{ddz}{dtdu}\right) + \left(\frac{ddz}{du^2}\right),$$

$$\left(\frac{ddz}{dxdy}\right) = \left(\frac{ddz}{dtdu}\right) + \left(\frac{ddz}{du^2}\right),$$

$$\left(\frac{ddz}{dy^2}\right) = \left(\frac{ddz}{du^2}\right).$$

COROLLARIUM 2

235. Etsi ergo hic est $t = x$, tamen non est $\left(\frac{dz}{dt}\right) = \left(\frac{dz}{dx}\right)$, cuius rei ratio est, quod in forma $\left(\frac{dz}{dx}\right)$ quantitas y sumitur constans, in $\left(\frac{dz}{dt}\right)$ vero quantitas $u = x + y$; id quod in genere notasse iuvat, ne ex aequalitate $t = x$ ad aequalitatem formularum $\left(\frac{dz}{dx}\right)$ et $\left(\frac{dz}{dt}\right)$ concludamus.

EXEMPLUM 2

236. *Si inter variabiles t, u et x, y haec relatio connstituatur, ut sit $t = \alpha x^m$ et $u = \beta y^n$, reductionem exhibere.*

Hic ergo erit

$$\left(\frac{dt}{dx}\right) = m\alpha x^{m-1}, \quad \left(\frac{dt}{dy}\right) = 0, \quad \left(\frac{ddt}{dx^2}\right) = m(m-1)\alpha x^{m-2},$$

$$\left(\frac{du}{dx}\right) = 0, \quad \left(\frac{du}{dy}\right) = n\beta y^{n-1}, \quad \left(\frac{ddu}{dy^2}\right) = n(n-1)\beta y^{n-2},$$

unde obtinemus pro formulis primi gradus

$$\left(\frac{dz}{dx}\right) = m\alpha x^{m-1}\left(\frac{dz}{dt}\right), \quad \left(\frac{dz}{dy}\right) = n\beta y^{n-1}\left(\frac{dz}{du}\right),$$

pro formulis autem secundi gradus

$$\left(\frac{ddz}{dx^2}\right)=m(m-1)\alpha x^{m-2}\left(\frac{dz}{dt}\right)+mm\alpha\alpha x^{2m-2}\left(\frac{ddz}{dt^2}\right),$$
$$\left(\frac{ddz}{dxdy}\right)=mn\alpha\beta x^{m-1}y^{n-1}\left(\frac{ddz}{dtdu}\right),$$
$$\left(\frac{ddz}{dy^2}\right)=n(n-1)\beta x^{n-2}\left(\frac{dz}{du}\right)+nn\beta\beta x^{2n-2}\left(\frac{ddz}{du^2}\right),$$

In quibus formulis iam loco x et y earum valores per t et u induci debent.

EXEMPLUM 3

237. *Si inter variabiles t, u et x, y haec relatio constituatur, ut sit* $x=t$ *et* $\frac{x}{y}=u$, *formularum differentialium reductionem exhibere.*

Cum sit $t=x$ et $\frac{x}{y}=u$ erit

$$\left(\frac{dt}{dx}\right)=1,\quad \left(\frac{dt}{dy}\right)=0$$

hincque formulae involventes *ddt* evanescunt. Porro

$$\left(\frac{du}{dx}\right)=\frac{1}{y}=\frac{u}{t},\quad \left(\frac{du}{dy}\right)=\frac{-x}{yy}=\frac{-uu}{t},$$
$$\left(\frac{ddu}{dx^2}\right)=0,\quad \left(\frac{ddu}{dxdy}\right)=\frac{-1}{yy}=\frac{-uu}{tt},\quad \left(\frac{ddu}{dy^2}\right)=\frac{2x}{y^3}=\frac{2u^3}{tt},$$

unde pro formulis primi gradus habebimus

$$\left(\frac{dz}{dx}\right)=\left(\frac{dz}{dt}\right)+\frac{u}{t}\left(\frac{dz}{du}\right),\quad \left(\frac{dz}{dy}\right)=\frac{-uu}{t}\left(\frac{dz}{du}\right),$$

pro formulis autem secundi gradus

$$\left(\frac{ddz}{dx^2}\right)=\left(\frac{ddz}{dt^2}\right)+\frac{2u}{t}\left(\frac{ddz}{dtdu}\right)+\frac{uu}{tt}\left(\frac{ddz}{du^2}\right),$$
$$\left(\frac{ddz}{dxdy}\right)=\frac{-uu}{tt}\left(\frac{dz}{du}\right)-\frac{uu}{t}\left(\frac{ddz}{dtdu}\right)-\frac{u^3}{tt}\left(\frac{ddz}{du^2}\right),$$
$$\left(\frac{ddz}{dy^2}\right)=\frac{2u^3}{tt}\left(\frac{dz}{du}\right)+\frac{u^4}{tt}\left(\frac{ddz}{du^2}\right).$$

EXEMPLUM 4

238. *Si inter variabiles t, u et x, y haec relatio constituatur, ut sit* $t=e^x$ *et* $u=e^x y$ *seu* $x=lt$ *et* $y=\frac{u}{t}$*, reductionem formularum differentialium exhibere.*

Hic ergo est

$$\left(\frac{dt}{dx}\right)=e^x,\quad \left(\frac{dt}{dy}\right)=0,\quad \left(\frac{ddt}{dx^2}\right)=e^x=t,\quad \left(\frac{ddt}{dxdy}\right)=0.$$

Deinde

$$\left(\frac{du}{dx}\right)=e^x y=u,\quad \left(\frac{du}{dy}\right)=e^x=t,$$

tum vero

$$\left(\frac{ddu}{dx^2}\right)=e^x y=u,\quad \left(\frac{ddu}{dxdy}\right)=e^x=t,\quad \left(\frac{ddu}{dy^2}\right)=0.$$

Quare pro formulis primi gradus habebimus

$$\left(\frac{dz}{dx}\right)=t\left(\frac{dz}{dt}\right)+u\left(\frac{dz}{du}\right),\quad \left(\frac{dz}{dy}\right)=t\left(\frac{dz}{du}\right),$$

pro formulis autem secundi gradus

$$\left(\frac{ddz}{dx^2}\right)=t\left(\frac{dz}{dt}\right)+u\left(\frac{dz}{du}\right)+tt\left(\frac{ddz}{dt^2}\right)+2tu\left(\frac{ddz}{dtdu}\right)+uu\left(\frac{ddz}{du^2}\right),$$
$$\left(\frac{ddz}{dxdy}\right)=t\left(\frac{dz}{du}\right)+tt\left(\frac{ddz}{dtdu}\right)+tu\left(\frac{ddz}{du^2}\right),$$
$$\left(\frac{ddz}{dy^2}\right)=tt\left(\frac{ddz}{du^2}\right).$$

SCHOLION

239. In formulis generalibus § 232 datis assumsimus valores variabilium *t* et *u* per *x* et *y* expressos dari et universa evolutione facta tum demum pro *x* et *y* variabiles *t* et *u* restitui. Commodius ergo videatur, si statim variabilium *x* et *y* valores per *t* et *u* expressi habeantur; verum inde valores formularum $\left(\frac{dt}{dx}\right)$, $\left(\frac{dt}{dy}\right)$ etc. nimis complicate exprimerentur, quam ut eas in calculum introducere liceret. Scilicet si *x* et *y* per *t* et *u* dentur, inde fit

$$\left(\frac{dt}{dx}\right)=\frac{\left(\frac{dy}{du}\right)}{\left(\frac{dx}{dt}\right)\left(\frac{dy}{du}\right)-\left(\frac{dx}{du}\right)\left(\frac{dy}{dt}\right)}$$

ac formulae secundi gradus multo magis prodituræ sunt perplexae. Quovis ergo casu, quo huiusmodi reductione utendum videtur, coniectura potius quam certa ratione idoneam variabilium immutationem colligi conveniet.

Alia vero etiam datur reductio saepe insignem utilitatem afferens, dum ipsius functionis z quaesitae forma mutatur, veluti si ponatur $z = Vv$ denotante V functionem datam ipsarum x et y, ita ut iam v sit functio quaesita; quin etiam haec nova quaesita v alio modo cum datis implicari potest.

PROBLEMA 40

240. *Proposita functione z binarum variabilium x et y ac posito $z = Pv$, ita ut P sit data quaedam functio ipsarum x et y, formulas differentiales ipsius z per formulas differentiales novae functionis v exprimere.*

SOLUTIO

Cum sit $z = Pv$, ex regulis differentiandi traditis habebimus primo formulas differentiales primi gradus

$$\left(\frac{dz}{dx}\right) = \left(\frac{dP}{dx}\right)v + P\left(\frac{dv}{dx}\right) \quad \text{et} \quad \left(\frac{dz}{dy}\right) = \left(\frac{dP}{dy}\right)v + P\left(\frac{dv}{dy}\right).$$

Atque hinc deinceps formulae differentiales secundi ordinis ita prodibunt expressae

$$\left(\frac{ddz}{dx^2}\right) = \left(\frac{ddP}{dx^2}\right)v + 2\left(\frac{dP}{dx}\right)\left(\frac{dv}{dx}\right) + P\left(\frac{ddv}{dx^2}\right),$$
$$\left(\frac{ddz}{dxdy}\right) = \left(\frac{ddP}{dxdy}\right)v + \left(\frac{dP}{dx}\right)\left(\frac{dv}{dy}\right) + \left(\frac{dP}{dy}\right)\left(\frac{dv}{dx}\right) + P\left(\frac{ddv}{dxdy}\right),$$
$$\left(\frac{ddz}{dy^2}\right) = \left(\frac{ddP}{dy^2}\right)v + 2\left(\frac{dP}{dy}\right)\left(\frac{dv}{dy}\right) + P\left(\frac{ddv}{dy^2}\right);$$

ubi cum P sit functio data ipsarum x et y, eius formulae differentiales simul habentur.

COROLLARIUM 1

241. Si P esset functio ipsius x tantum, puta X, tum posito $z = Xv$ erit

$$\left(\frac{dz}{dx}\right) = \left(\frac{dX}{dx}\right)v + X\left(\frac{dv}{dx}\right) \quad \text{et} \quad \left(\frac{dz}{dy}\right) = X\left(\frac{dv}{dy}\right),$$

tum

$$\left(\frac{ddz}{dx^2}\right) = \left(\frac{ddX}{dx^2}\right)v + \frac{2dX}{dx}\left(\frac{dv}{dx}\right) + X\left(\frac{ddv}{dx^2}\right),$$
$$\left(\frac{ddz}{dxdy}\right) = \frac{dX}{dx}\left(\frac{dv}{dy}\right) + X\left(\frac{ddv}{dxdy}\right),$$
$$\left(\frac{ddz}{dy^2}\right) = X\left(\frac{ddv}{dy^2}\right).$$

COROLLARIUM 2

242. Transformatio haec easdem variabiles x et y servat et tantum loco functionis z alia v introducitur, cum ante manente eadem functione z binae variabiles x et y ad alias t et u sint reductae. Ex quo hae duae transformationes genere sunt diversae.

SCHOLION 1

243. Casus simplicior fuisset, si per additionem posuissemus $z = P + v$, ut esset P functio quaedam data ipsarum x et y; verum tum transformatio ita fit obvia, ut investigatione non indigeat; est enim manifesto

$$\left(\frac{dz}{dx}\right) = \left(\frac{dP}{dx}\right) + \left(\frac{dv}{dx}\right), \quad \left(\frac{dz}{dy}\right) = \left(\frac{dP}{dy}\right) + \left(\frac{dv}{dy}\right),$$

$$\left(\frac{ddz}{dx^2}\right) = \left(\frac{ddP}{dx^2}\right) + \left(\frac{ddv}{dx^2}\right),$$
$$\left(\frac{ddz}{dxdy}\right) = \left(\frac{ddP}{dxdy}\right) + \left(\frac{ddv}{dxdy}\right),$$
$$\left(\frac{ddz}{dy^2}\right) = \left(\frac{ddP}{dy^2}\right) + \left(\frac{ddv}{dy^2}\right).$$

Neque vero etiam formas magis compositas evolvi necesse est, veluti si ponamus, $z = \sqrt{(PP + vv)}$, quandoquidem talis forma vix unquam usum foret habitura.

SCHOLION 2

244. Praemissis his principiis et transformationibus negotium aggrediamur et methodos aperiamus ex data relatione inter formulas differentiales secundi gradus et primi gradus itemque ipsas quantitates principales harum ipsarum relationem investigandi. Hic scilicet praeter ipsas quantitates x, y et z earumque formulas differentiales primi gradus $\left(\frac{dz}{dx}\right)$ et $\left(\frac{dz}{dy}\right)$ considerandae veniunt tres formulae differentiales secundi gradus $\left(\frac{ddz}{dx^2}\right)$, $\left(\frac{ddz}{dxdy}\right)$ et $\left(\frac{ddz}{dy^2}\right)$, quarum vel una,vel bina vel omnes tres in relationem propositam ingredi possunt, ubi insuper ingens discrimen formulae primi gradus, sive in relationem ingrediantur sive secus, constituunt. Non solum autem nimis longum foret omnes combinationes, uti in praecedente sectione fecimus, prosequi, sed etiam defectus idonearum methodorum impedit, quominus singula quaestionum huc pertinentiuni genera percurramus. Capita igitur pertractanda ita instituamus, prout methodus solvendi patietur, ea, ubi nihil praestare licet, penitus praetermissuri.