CHAPTER II

IN WHICH A SINGLE FORMULA OF THE SECOND ORDER DIFFERENTIAL IS GIVEN IN TERMS OF SOME OTHER REMAINING QUANTITIES

PROBLEM 41

245. *If z should be a function of this kind of x and y, so that the formula of the second order* $\left(\frac{ddz}{dx^2}\right)$ *is equal to given functions of x and y, to investigate the nature of the function z.*

SOLUTION

Let P be that given function of x and y, thus so that there must be

$$\left(\frac{ddz}{dx^2}\right)=P.$$

Now y is taken constant, and since there shall be $d.\left(\frac{dz}{dx}\right)=dx\left(\frac{ddz}{dx^2}\right)$, there will be

$$d.\left(\frac{dz}{dx}\right)=Pdx,$$

from which on integrating there is produced

$$\left(\frac{dz}{dx}\right)=\int Pdx+\text{Const.},$$

where in the integration $\int Pdx$ the quantity y may be considered constant and some constant function of y will be denoted that is required to be added, thus so that this first integration gives

$$\left(\frac{dz}{dx}\right)=\int Pdx+f:y$$

Now in turn with the quantity y regarded as constant there will be $dz=dx\left(\frac{dz}{dx}\right)$ or

$$dz=dx\int Pdx+dx\, f:y;$$

where since $\int Pdx$ shall be some function of x and y, of which here y is assumed constant, the integration again put in place will give

$$z=\int dx\int Pdx+x\, f:y+F:y,$$

which is the complete integral of the second order differential equation proposed $\left(\frac{ddz}{dx^2}\right) = P$, therefore so that it includes the two arbitrary functions $f{:}\,y$ and $F{:}\,y$, each of which can be taken as you please, so that discontinuous functions also are not excluded.

COROLLARY 1

246. But if therefore this condition $\left(\frac{ddz}{dx^2}\right) = 0$ may be proposed, the complete integral of this will be given

$$z = x\, f{:}y + F{:}y$$

on account of $P = 0$, the truth of this is seen from differentiation, from which there becomes in the first place $\left(\frac{dz}{dx}\right) = f{:}y$, then truly $\left(\frac{ddz}{dx^2}\right) = 0$.

COROLLARY 2

247. In the same way the integral found in general can be proved by differentiation. Since indeed we have found

$$z = \int dx \int Pdx + x\, f{:}y + F{:}y,$$

by the first differentiation there is given

$$\left(\frac{dz}{dx}\right) = \int Pdx + f{:}y\,,$$

truly with that repeated $\left(\frac{ddz}{dx^2}\right) = P$.

COROLLARY 3

248. In a similar manner if this condition is proposed $\left(\frac{ddz}{dy^2}\right) = Q$, with some function Q of x and y present, the complete integral is found

$$z = \int dy \int Qdy + y\, f{:}x + F{:}x,$$

where in the twin integral $\int dy \int Qdy$ the quantity x may be considered constant.

SCHOLIUM

249. Hence an account of the complete integrals which arise from differential formulas of the second order can be seen in general, which have been put in place, so that two arbitrary functions may be introduced; where again it is to be observed that these functions are able to be discontinuous as well as continuous . Therefore unless through the whole section the integrals involve two arbitrary functions of this kind, these cannot be considered as complete. For however often a problem leads to an equation of this kind $\left(\frac{ddz}{dx^2}\right) = P$, the nature of this must always be prepared thus, so that by attributing to x a certain value $x = a$ both the formula $\left(\frac{dz}{dx}\right)$ as well as the magnitude z will be equal to some function of y. Whereby if both the integral $\int Pdx$ as well as this $\int dx \int Pdx$ may be taken thus, so that each vanishes on putting $x = a$, there will be for the same case $x = a$ the value

$$\left(\frac{dz}{dx}\right) = f{:}y \quad \text{and} \quad z = af{:}y + F{:}y,$$

from which from the nature of the problem each function $f{:}\,y$ and $F{:}\,y$ is defined. But this application cannot be made to all cases, unless the complete integral may be had ; on account of which with this it is to be applied especially, that the complete integral of all problems of this kind be considered.

Moreover I teach by warning always here, that whenever an integral formula of this kind $\int Pdx$ arises, always it is to be understood that a single variable x be taken, accordingly, if also y may be taken as a variable, the formula $\int Pdx$ should not indeed even admit the significance of this. In a similar manner in the formula $\int dx \int Pdx$ it must be understood in each integration only x be assumed variable. But if such a form $\int dy \int Pdx$ arises, it is to be understood that the integral $\int Pdx$ is to be deduced from the variability of x only; which if it is put $= R$, so that there may be considered $\int Rdy$, now here only y will be considered as variable.

EXAMPLE 1

250. *There is sought a function z of the two variables x and y of this kind, so that there shall be*

$$\left(\frac{ddz}{dx^2}\right)=\frac{xy}{a}.$$

Since here there shall be $P=\frac{xy}{a}$, then there will be

$$\int Pdx=\frac{xxy}{2a} \text{ and } \int dx\int Pdx=\frac{x^3y}{6a}$$

and thus there will be considered from the first integration

$$\left(\frac{dz}{dx}\right)=\frac{xxy}{2a}+f:y,$$

thus so that on putting $x=a$ the formula $\left(\frac{dz}{dx}\right)$ can be equal to some function of *y* or corresponding to the abscissa *y* of some applied curve. Then truly the other integration put in place will be

$$z=\frac{x^3y}{6a}+xf:y+F:y,$$

which value in the case $x=a$ can be equal to the value of some new function of y.

EXAMPLE 2

251. *A function z of the two variables x et y of this kind is sought, so that there shall be*

$$\left(\frac{ddz}{dx^2}\right)=\frac{ax}{\sqrt{(xx+yy)}}.$$

On account of $P=\frac{ax}{\sqrt{(xx+yy)}}$ there will be

$$\int Pdx=a\sqrt{(xx+yy)}$$

and

$$\int dx\int Pdx=a\int dx\sqrt{(xx+yy)}=\tfrac{1}{2}ax\sqrt{(xx+yy)}+\tfrac{1}{2}ayyl\left(x+\sqrt{(xx+yy)}\right),$$

from which the first integration gives

$$\left(\frac{dz}{dx}\right)=a\sqrt{(xx+yy)}+f:y$$

and truly the other

$$z=\tfrac{1}{2}ax\sqrt{(xx+yy)}+\tfrac{1}{2}ayyl\left(x+\sqrt{(xx+yy)}\right)+xf:y+F:y\,.$$

EXAMPLE 3

252. *A function z is sought of the two variables x and y of this kind, so that there shall be*

$$\left(\frac{ddz}{dx^2}\right)=\frac{1}{\sqrt{(aa-xx-yy)}}$$

Since there shall be $P=\frac{1}{\sqrt{(aa-xx-yy)}}$, then there shall be

$$\int Pdx=\text{Ang.sin.}\frac{x}{\sqrt{(aa-yy)}}\,,$$

then truly

$$\int dx\int Pdx=x\text{Ang.sin.}\frac{x}{\sqrt{(aa-yy)}}-\int\frac{xdx}{\sqrt{(aa-xx-yy)}}.$$

Whereby the first integration gives

$$\left(\frac{dz}{dx}\right)=\text{Ang.sin.}\frac{x}{\sqrt{(aa-yy)}}+f:y$$

and hence the function sought itself will be

$$z=x\text{Ang.sin.}\frac{x}{\sqrt{(aa-yy)}}+\sqrt{(aa-xx-yy)}+xf:y+F:y.$$

EXAMPLE 4

253. *A function z is sought of the two variables x and y such that there shall be*

$$\left(\frac{ddz}{dx^2}\right)=x\sin.(x+y).$$

On account of $P=x\sin.(x+y)$ there will be

$$\int Pdx=\int xdx\sin.(x+y)=-x\cos.(x+y)+\int dx\cos.(x+y)=-x\cos.(x+y)+\sin.(x+y).$$

Then truly there will be

$$\int xdx\cos.(x+y) = x\sin.(x+y) + \cos.(x+y)$$

and thus

$$\int dx \int Pdx = -\,2\cos.(x+y) - x\sin.(x+y).$$

On account of which both our integrals will be

$$\left(\tfrac{dz}{dx}\right) = \sin.(x+y) - \cos.(x+y) + f:y$$

and

$$z = -\,2\cos.(x+y) - x\sin.(x+y) + xf:y + F:y.$$

PROBLEM 42

254. *If z should be a function of the two variables x and y of this kind, so that there shall be*

$$\left(\tfrac{ddz}{dx^2}\right) = P\left(\tfrac{dz}{dx}\right) + Q$$

with some functions present P and Q of x and y, to investigate the nature of the function z in general.

SOLUTION

Here we may put $\left(\tfrac{dz}{dx}\right) = v$, so that there shall be $\left(\tfrac{ddz}{dx^2}\right) = \left(\tfrac{dv}{dx}\right)$; our equation to be integrated will be

$$\left(\tfrac{dv}{dx}\right) = Pv + Q.$$

Therefore only x may be considered as a variable and on account of $dv = dx\left(\tfrac{dv}{dx}\right)$ there will be

$$dv = Pvdx + Qdx,$$

which multiplied by $e^{-\int Pdx}$ and integrated gives

$$e^{-\int Pdx}v = \int e^{-\int Pdx}Qdx + f:y.$$

and thus

$$\left(\tfrac{dz}{dx}\right) = e^{\int Pdx}\int e^{-\int Pdx}Qdx + e^{\int Pdx}f:y.$$

Only x will be retained variable, with y regarded as constant which on account of $dz = dx\left(\frac{dz}{dx}\right)$ there will be

$$z = \int e^{\int Pdx} dx \int e^{-\int Pdx} Qdx + f:y \int e^{\int Pdx} dx + F:y,$$

which on account of the two arbitrary functions *f: y* and *F: y* is the complete integral.

COROLLARY 1

255. This problem can be extended much wider than the preceding, since the condition proposed since involves a formula of the first order $\left(\frac{dz}{dx}\right)$; happier still truly the solution has been successful.

COROLLARY 2

256. Therefore here a fourfold integration is needed. Clearly in the first place the integral $\int Pdx$ must be sought; which if it is put = lR, again the integral must be sought

$$\int e^{\int Pdx} dx = \int Rdx;$$

which if we put $= S$, the integral remains

$$\int Rdx \int \frac{Qdx}{R} = \int dS \int \frac{Qdx}{R},$$

which becomes

$$S \int \frac{Qdx}{R} - \int \frac{QSdx}{R},$$

thus so that these two forms above must be integrated.

COROLLARY 3

257. The problem can be solved entirely in the same way, where there must be

$$\left(\frac{ddz}{dy^2}\right) = P\left(\frac{dz}{dy}\right) + Q,$$

if P and Q were some given functions of x and y. Indeed there is found

$$\left(\frac{dz}{dy}\right) = e^{\int Pdy} \int e^{-\int Pdy} Qdy + e^{\int Pdy} f:x$$

and

$$z=\int e^{\int Pdy}dy\int e^{-\int Pdy}Qdy+f:x\int e^{\int Pdy}dy+F:x.$$

EXAMPLE 1

258. *A function z is sought of the two variables x and y of this kind, such that there shall be*

$$\left(\frac{ddz}{dx^2}\right)=\frac{n}{x}\left(\frac{dz}{dx}\right).$$

On putting $\left(\frac{dz}{dx}\right)=v$ and with x only taken to be variable there shall be $\left(\frac{dv}{dx}\right)=\frac{nv}{x}$ and thus

$\frac{dv}{v}=\frac{ndx}{dx}$, of which the integration gives

$$v=\left(\frac{dz}{dx}\right)=x^n f:y.$$

Now again only x will be considered to be variable

$$dz=x^n dx\, f:y,$$

the complete integral of this is

$$z=\frac{1}{n+1}x^{n+1}f:y+F:y.$$

Moreover in the case $n=-1$ or $\left(\frac{ddz}{dx^2}\right)=\frac{-1}{x}\left(\frac{dz}{dx}\right)$ there will be

$$\left(\frac{dz}{dx}\right)=\frac{1}{x}f:y \quad \text{and} \quad z=lx\cdot f:y+F:y.$$

EXAMPLE 2

259. *A function z is sought of the two variables x and y of this kind, so that there shall be*

$$\left(\frac{ddz}{dx^2}\right)=\frac{n}{x}\left(\frac{dz}{dx}\right)+\frac{a}{xy}.$$

On putting $\left(\frac{dz}{dx}\right)=v$ and on taking x alone to be variable, there will be

$$dv=\frac{nvdx}{x}+\frac{adx}{xy},$$

which equation divided by x^n and integrated gives

$$\frac{v}{x^n}=\frac{a}{y}\int\frac{dx}{x^{n+1}}=\frac{-a}{nx^n y}+f:y$$

or

$$v=\left(\frac{dz}{dx}\right)=\frac{-a}{ny}+x^{n}f:y.$$

Again only *x* shall be variable, so that there shall be had

$$dz=\frac{-adx}{ny}+x^{n}dx\, f:y,$$

and the complete integral will be produced

$$z=\frac{-ax}{ny}+\frac{1}{n+1}x^{n+1}\, f:y+F:y.$$

EXAMPLE 3

260. *A function z is sought of the two variables x and y of this kind, such that there shall be*

$$\left(\frac{ddz}{dx^{2}}\right)=\frac{2nx}{xx+yy}\left(\frac{dz}{dx}\right)+\frac{x}{ay}\ .$$

On putting $\left(\frac{dz}{dx}\right)=v$ there will be on taking *y* constant

$$dv=\frac{2nxv}{xx+yy}+\frac{xdx}{ay}$$

which equation on division by $(xx+yy)^{n}$ and integrated gives

$$\frac{v}{(xx+yy)^{n}}=\frac{1}{ay}\int\frac{xdx}{(xx+yy)^{n}}=-\frac{1}{2(n-1)ay(xx+yy)^{n-1}}+f:y$$

or

$$v=\left(\frac{dz}{dx}\right)=\frac{-(xx+yy)}{2(n-1)ay}+(xx+yy)^{n}\ f:y.$$

Hence on taking *y* constant again there becomes

$$z=\frac{-x(xx+3yy)}{6(n-1)ay}+f:y\int(xx+yy)^{n}dx+F:y$$

In the case, in which $n=1$ or

$$\left(\frac{ddz}{dx^{2}}\right)=\frac{2x}{xx+yy}\left(\frac{dz}{dx}\right)+\frac{x}{ay}$$

there will be

$$\frac{v}{(xx+yy)}=\frac{1}{ay}\int\frac{xdx}{(xx+yy)}=\frac{1}{2ay}l(xx+yy)+f:y,$$

hence

$$\left(\frac{dz}{dx}\right)=\frac{xx+yy}{2ay}l\left(xx+yy\right)+\left(xx+yy\right)f:y$$

and

$$z=\frac{x(xx+3yy)}{6ay}\ l\left(xx+yy\right)-\frac{1}{9ay}\left(x^3+6xy^2-6xy^3\text{Ang.tang.}\frac{x}{y}\right)+\frac{1}{3}x\left(xx+3yy\right)f:y+F:y.$$

PROBLEMA 43

261. *If z must be a function of the two variables x and y, so that there becomes*

$$\left(\frac{ddz}{dx^2}\right)=P\left(\frac{dz}{dx}\right)+Q$$

with some given function present P and Q of all the three variables x, y and z, to investigate the nature of the function z.

SOLUTION

On putting y constant there will be

$$\left(\frac{ddz}{dx^2}\right)=\frac{ddz}{dx^2}\quad\text{and}\quad\left(\frac{dz}{dx}\right)=\frac{dz}{dx}$$

and thus there will be had a differential equation of the second order relating to the previous book

$$ddz=Pdxdz+Qdx^2$$

which is to be considered to involve the two variables x and z, because in that y is considered as a constant. Therefore the integration of this equation may be tested by the methods set out there ; which if it should succeed, in place of the two constants, which the twofold integration brings in, the indefinite functions of y f:y and F: y may be written , which thus can be accepted as discontinuous, and thus the complete integral of the proposed equation will be had.

COROLLARY 1

262. Therefore the solution of this equation is reduced to the method of integration treated in the above book, where a function of one variable was required to be investigated from a given relation of a second order differential.

COROLLARY 2

263. But if therefore the resolution of all differential equations of the second order, which involve only two variables, we may demand here to be conceded by us, that the solution of our problem is to be considered as accomplished.

COROLLARY 3

264. Following my own advise, it is not understood that the equation

$$\left(\frac{ddz}{dx^2}\right)=P\left(\frac{dz}{dx}\right)+Q$$

is required to be treated in the same way, and that the solution of this can be regarded as accomplished, whatever the functions P and Q should be of x, y and z.

SCHOLIUM 1

265. From the account of the solution it is understood that a much wider problem appears to be resolved in a similar manner. If indeed the formula $\left(\frac{ddz}{dx^2}\right)$ may be determined in some manner through the principal quantities x, y, z and in addition the formula $\left(\frac{dz}{dx}\right)$ is determined, thus so that if the powers of this formula $\left(\frac{dz}{dx}\right)$ or some other functions are introduced, a solution will be recalled always according to the above book, because on putting y constant there becomes

$$\left(\frac{dz}{dx}\right)=\frac{dz}{dx} \text{ and } \left(\frac{ddz}{dx^2}\right)=\frac{ddz}{dx^2}$$

and thus there results a differential equation of the second order of the accustomed form involving only the two variables x and z . Only this may be tentative, for in place of the constants with each integration, it is required to write the forms $f: y$ and $F: y$. Therefore we have extricated a part of our proposition, clearly since either $\left(\frac{ddz}{dx^2}\right)$ and $\left(\frac{dz}{dx}\right)$ is determined in some manner by x, y, z, or $\left(\frac{ddz}{dy^2}\right)$ and $\left(\frac{dz}{dy}\right)$ is determined in some manner by x, y, z ; certainly in the former the formula of the first order $\left(\frac{dz}{dy}\right)$ is excluded, now truly the formula $\left(\frac{dz}{dx}\right)$ for the latter. Which if it should be approached, by no means is the question able to be treated by this method, or just as from this simplest case $\left(\frac{ddz}{dx^2}\right)=\left(\frac{dz}{dy}\right)$ is it allowed to be understood that the resolution of this has to be thought out with the greatest hardship.

SCHOLIUM 2

266. Therefore since the first and the third of the three formulas of second order differentials $\left(\frac{ddz}{dx^2}\right)$, $\left(\frac{ddz}{dxdy}\right)$, $\left(\frac{ddz}{dy^2}\right)$ that I may consider at this stage, in as much as the determination of these by the remaining quantities indeed allows here a resolution by some method to be used, there remains, that we must consider also the following formula $\left(\frac{ddz}{dxdy}\right)$, and from which determinations we will investigate if a solution may be able to be resolved from the remaining quantities *x, y, z*, $\left(\frac{dz}{dx}\right)$, $\left(\frac{dz}{dy}\right)$, in which undertaking it is agreed to begin with the simplest cases.

PROBLEM 44

267. *If z should be a function of this kind of the two variables x and y, so that there becomes* $\left(\frac{ddz}{dxdy}\right)=P$ *with some given function P present of x and y, to determine the nature of the function z.*

SOLUTION

There is put $\left(\frac{dz}{dx}\right)=v$ and there will be $\left(\frac{ddz}{dxdy}\right)=\left(\frac{dv}{dy}\right)$ and thus there will be considered $\left(\frac{dv}{dy}\right)=P$. Now the quantity *x* may be regarded as constant, thus so that *P* may contain the variable *y* only, and there will be $dv=Pdy$, from which by the hypothesis of the constant quantity *x* on integrating there will be produced

$$v=\left(\frac{dz}{dx}\right)=\int Pdy+f':x,$$

where $\int Pdy$ will be a given function of *x* et *y*. Now again *x* may be regarded as variable, *y* truly as constant, so that we may arrive at this differential equation

$$dz=dx\int Pdy+dx\,f':x,$$

which integrated gives

$$z=\int dx\int Pdy+f:x+F:y\,;$$

where since there may be considered two arbitrary functions, this indicates that the integral is complete.

COROLLARIUM 1

268. If with the order inverted we may put first *y*, then truly *x* constant, we come upon

$$\left(\frac{dz}{dy}\right)=\int Pdx+f':y \quad \text{and} \quad z=\int dy\int Pdx+f:y+F:x,$$

which value and the preceding satisfy equally.

COROLLARY 2

269. Therefore it is apparent that either $\int dx \int Pdy = \int dy \int Pdx$ or at any rate the difference can be expressed by an aggregate from a function of *x* and a function of *y*. Which also thence may be apparent, since on putting

$$\int dx \int Pdy = \int dy \int pdx = V$$

each becomes $P = \left(\frac{ddV}{dxdy}\right)$.

COROLLARY 3

270. If there shall be $P = 0$ or there must become $\left(\frac{ddz}{dxdy}\right) = 0$, this form $z = f:x + F:y.$ is found for the nature of the function *z*.

SCHOLIUM

271. Here a case occurs frequently in the teaching of volumes. For if the nature of a surface is expressed by an equation between the three coordinates *x*, *y* and *u*, the volume $= \int dx \int udy$; whereby if the volume is expressed by *z*, then there will be $\left(\frac{ddz}{dxdy}\right) = u$, clearly normal to the two ordinates *x* and *y*. Indeed if there is put

$$du = pdx + qdy,$$

the surface [area] of this solid will be

$$\int dx \int dy \sqrt{(1 + pp + qq)};$$

[This well-known expression relates the area of an increment on the surface $dS(x,y,u)$ to a corresponding increment of area *dxdy* in the xy-plane in terms of the angle ϑ between the normal to the surface at the point and the normal to the *xy* plane. Thus, in terms of vectors, $dS(x,y,u)\hat{n}\cdot\hat{k} = dS(x,y,u)\cos\vartheta = dxdy$. If $r(x,y,u)$, where $u = u(x,y)$ is any point on the surface, then $du = \frac{\partial u}{dx}dx + \frac{\partial u}{dy}dy = pdx + qdy$ and $0 = \frac{\partial u}{dx}dx + \frac{\partial u}{dy}dy - 1du$. Treating this last equation as a zero scalar product, since $dr = \hat{i}dx + \hat{j}dy + \hat{k}du$ lies in the surface, then $\frac{\partial u}{dx}\hat{i} + \frac{\partial u}{dy}\hat{j} - \hat{k}$ is a normal to the surface, and hence in terms of unit normals, $\hat{n}\cdot\hat{k} = \frac{1}{\sqrt{(1+p^2+q^2)}} = \cos\vartheta$, leading to the required result.

If you should know if and where Euler established this result, you might like to let me know, so that a reference can be given for it.]

which surface, if it is expressed by the letter *z*, will be

$$\left(\frac{ddz}{dxdy}\right)=\sqrt{\left(1+pp+qq\right)}\,.$$

Therefore when in our problem of this kind the function *z* of *x* and *y* is sought, so that there shall be $\left(\frac{ddz}{dxdy}\right)=P$, it is the same, as if the volume of the corresponding surface is sought, the nature of which is expressed by the equation between the three coordinates *x*, *y* and *P*. Hence we will illustrate this calculation by some examples.

EXAMPLE 1

272. *A function z of this kind of the two variables x and y is sought, so that there shall be*

$$\left(\frac{ddz}{dxdy}\right)=\alpha x+\beta y\,.$$

Since here there shall be $P=\alpha x+\beta y$, there will be

$$\int Pdy=\alpha xy+\tfrac{1}{2}\beta yy \quad \text{and} \quad \int dx\int Pdy=\tfrac{1}{2}\alpha xxy+\tfrac{1}{2}\beta xyy=\tfrac{1}{2}xy\left(\beta x+\beta y\right),$$

from which the function sought *z* thus is expressed, so that there shall be

$$z=\tfrac{1}{2}xy\left(\beta x+\beta y\right)+f{:}x+F{:}y\,.$$

EXAMPLE 2

273. *A function z of the two variables x and y is sought of this kind, so that there shall be*

$$\left(\frac{ddz}{dxdy}\right)=\sqrt{\left(aa-yy\right)}\,.$$

Here there is $P=\sqrt{\left(aa-yy\right)}$, therefore

$$\int Pdx=x\sqrt{\left(aa-yy\right)}\,,$$

where, because it is likewise, I begin from the variability of *x*. Hence therefore there becomes

$$\int dy\int Pdx=x\int dy\sqrt{\left(aa-yy\right)}=\tfrac{1}{2}xy\sqrt{\left(aa-yy\right)}+\tfrac{1}{2}aax\int\frac{dy}{\sqrt{\left(aa-yy\right)}}$$

from which the complete integral will be

$$z=\tfrac{1}{2}xy\sqrt{(aa-yy)}+\tfrac{1}{2}aax\,\text{Ang.sin.}\tfrac{y}{a}+f:x+F:y.$$

EXAMPLE 3

274. *A function of this kind z is sought from the two variables x and y, so that there shall be*

$$\left(\frac{ddz}{dxdy}\right)=\frac{a}{\sqrt{(aa-xx-yy)}}$$

On account of $P=\frac{a}{\sqrt{(aa-xx-yy)}}$ there will be

$$\int Pdy=a\text{Ang.sin.}\frac{y}{\sqrt{(aa-xx)}}$$

hence

$$\int dx\int Pdy=a\int dx\text{Ang.sin.}\frac{y}{\sqrt{(aa-xx)}}\,.$$

For the sake of brevity there is put

$$\text{Ang.sin.}\frac{y}{\sqrt{(aa-xx)}}=\varphi\,;$$

there will be

$$\int dx\int Pdy=a\int\varphi dx=ax\varphi-a\int xdx\left(\frac{d\varphi}{dx}\right);$$

for in this integration y may be considered constant. Whereby on account of $\frac{y}{\sqrt{(aa-xx)}}=\text{sin.}\varphi$ there will be

$$\frac{yx}{(aa-xx)^{\frac{3}{2}}}=\left(\frac{d\varphi}{dx}\right)\text{cos.}\varphi\,.$$

But truly there is $\text{cos.}\varphi=\frac{\sqrt{(aa-xx-yy)}}{\sqrt{(aa-xx)}}$ and hence

$$\left(\frac{d\varphi}{dx}\right)=\frac{yx}{(aa-xx)\sqrt{(aa-xx-yy)}}$$

and

$$\int xdx\left(\frac{d\varphi}{dx}\right)=y\int\frac{xxdx}{(aa-xx)\sqrt{(aa-xx-yy)}}$$

from which the integral found will be

$$z = ax\text{Ang.sin.}\frac{y}{\sqrt{(aa-xx)}} - ay\int\frac{xxdx}{(aa-xx)\sqrt{(aa-xx-yy)}} + f:x + F:x$$

which form by the integration performed is reduced to this

$$z = ax\text{Ang.sin.}\frac{y}{\sqrt{(aa-xx)}} + ay\text{Ang.sin.}\frac{x}{\sqrt{(aa-yy)}}$$
$$-aa\text{Ang.sin.}\frac{xy}{\sqrt{(aa-xx)(aa-xx-yy)}} + f:x + F:x.$$

Indeed the integral is easily elicited thus for the formula

$$\int\frac{aadx}{(aa-xx)\sqrt{(aa-xx-yy)}}$$

There is put

$$\frac{x}{\sqrt{(aa-xx-yy)}} = p;$$

then there will be $xx = \frac{pp(aa-yy)}{1+pp}$ and on account of constant y, by differentiating the logarithms,

$$\frac{dx}{x} = \frac{dp}{p} - \frac{pdp}{1+pp} = \frac{dp}{p(1+pp)}$$

then on multiplying by that formula

$$\frac{dx}{\sqrt{(aa-xx-yy)}} = \frac{dp}{1+pp}.$$

Again there is $aa - xx = \frac{aa+ppyy}{1+pp}$, from which formula the integral will be

$$\int\frac{xxdx}{(aa-xx)\sqrt{(aa-xx-yy)}} = \int\frac{aadp}{aa+ppyy} = \frac{aa}{yy}\int\frac{dp}{\frac{aa}{yy}+pp}$$

$$= \frac{a}{y}\text{Ang.tang.}\frac{py}{a} = \frac{a}{y}\text{Ang. tang.}\frac{xy}{a\sqrt{(aa-xx-yy)}} = \frac{a}{y}\text{Ang.sin.}\frac{xy}{\sqrt{(aa-xx)(aa-yy)}}.$$

PROBLEM 45

275. *If z should be a function of x and y of this kind, so that there will be*

$$\left(\frac{ddz}{dxdy}\right)=P\left(\frac{dz}{dx}\right)+Q$$

with some functions present P and Q of x and y, to investigate the nature of the function z.

SOLUTION

There is put $\left(\frac{dz}{dx}\right)=v$, so that this equation itself arises $\left(\frac{dv}{dx}\right)=Pv+Q$, which contains the quantities *x*, *y* et *v;* therefore there is put in place *x* constant and there will be

$$dv=Pvdy+Qdy\,,$$

which multiplied by $e^{-\int Pdy}$ gives

$$e^{-\int Pdy}v=\int e^{-\int Pdy}Qdy+f':x$$

and thus

$$v=e^{\int Pdy}\int e^{-\int Pdy}Qdy+e^{\int Pdy}f':x$$

Now since these integral determined contain *x* and *y*, there is considered *y* as constant and the following integration gives

$$z=\int e^{\int Pdy}dx\int e^{-\int Pdy}Qdy+\int e^{\int Pdy}dxf':x+F:y,$$

which integrals become evident in any case worked out.

COROLLARY 1

276. Therefore accordingly the problem being resolved by integration in the first place *R* is sought, so that there shall be $\int Pdy=lR$; then *S* is sought, so that there shall be $\int\frac{Qdy}{R}=S$; finally there shall be $\int RSdx=T$, thus so that in the former only the quantity *y*, in the latter truly only *x* may be considered variable. With which accomplished there will be our complete integral

$$z=T+\int Rdxf':x+F:y\,.$$

COROLLARY 2

277. Therefore here the arbitrary function *f: x* is involved in the formula of the integral ; which yet if by the application of the abscissa *x* of any corresponding curve may be shown, this integral $\int Rdxf':x$ for any value of *y* can be constructed separately, if indeed in this equation the quantity *y* can be regarded as constant.

SCHOLIUM

278. Clearly in the same manner this problem can be solved by interchanging the variables *x* and *y*, when the function *z* is sought, so that there shall be

$$\left(\frac{ddz}{dxdy}\right)=P\left(\frac{dz}{dy}\right)+Q,$$

as long as *P* and *Q* shall be functions of *x* and *y* only not implicating the function *z* itself ; for the solution thus itself will be had

$$z=\int e^{\int Pdx}dy\int e^{-\int Pdx}Qdx+\int e^{\int Pdx}dyf':y+F:x.$$

Moreover each problem can be extended more widely, and the first solution is allowed if the formula $\left(\frac{ddz}{dxdy}\right)$ is equal to some function of the three variables *x, y* and $\left(\frac{dz}{dx}\right)$, truly the latter, if $\left(\frac{ddz}{dxdy}\right)$ is equal to some function of the three quantities *x, y* and $\left(\frac{dz}{dy}\right)$; for in each case the problem is reduced to a differential equation of the first order. Truly neither does this method of solving succeed, if each formula of the first order $\left(\frac{dz}{dx}\right)$ and $\left(\frac{dz}{dy}\right)$ likewise is present, nor if the functions *P* and *Q* also are involved with the quantity *z* itself.

EXAMPLE 1

279. *A function z is sought of the two variables x and y, so that there shall be*

$$\left(\frac{ddz}{dxdy}\right)=\frac{n}{y}\left(\frac{dz}{dx}\right)+\frac{m}{x}$$

Let $\left(\frac{dz}{dx}\right)=v$; there will be $\left(\frac{dv}{dy}\right)=\frac{nv}{y}+\frac{m}{x}$ and on considering *x* as constant there will be

$$dv=\frac{nvdy}{y}+\frac{mdy}{x},$$

from which divided by y^n there will be

$$\frac{v}{y^n}=\frac{m}{x}\int\frac{dy}{y^n}=\frac{-m}{(n-1)xy^{n-1}}+f':x,$$

thus so that there shall be

$$v=\left(\frac{dz}{dx}\right)=\frac{-my}{(n-1)x}+y^{n}f':x;$$

now there is taken y constant and on integrating again there is obtained

$$z=\frac{-m}{n-1}ylx+y^{n}f:x+F:y.$$

EXAMPLE 2

280. *A function z of the two variables x and y is sought, so that there shall be*

$$\left(\frac{ddz}{dxdy}\right)=\frac{y}{xx+yy}\left(\frac{dz}{dx}\right)+\frac{a}{xx+yy}.$$

On putting $\left(\frac{dz}{dx}\right)=v$ and on taking x constant there will be

$$dv=\frac{vydy}{xx+yy}+\frac{ady}{xx+yy},$$

which equation divided by $\sqrt{(xx+yy)}$ gives

$$\frac{dv}{\sqrt{(xx+yy)}}=a\int\frac{dy}{(xx+yy)^{\frac{3}{2}}}=\frac{ay}{xx\sqrt{(xx+yy)}}+f:x.$$

Therefore

$$v=\left(\frac{dz}{dx}\right)=\frac{ay}{xx}+\sqrt{(xx+yy)}\cdot f:x;$$

now let y be constant and there will be found

$$z=\frac{-ay}{x}+\int f:xdx\sqrt{(xx+yy)}+F:y,$$

where some integral $\int f:xdx\sqrt{(xx+yy)}$ on account of the indeterminate function f: x, even if y is put constant, in general cannot be expressed, thus so that it could be shown explicitly by y and functions of x.

SCHOLIUM

281. Hence the formula of the second order $\left(\frac{ddz}{dxdy}\right)$ does not admit to as lavish an abundance of soluble cases as the two remaining $\left(\frac{ddz}{dx^2}\right)$ and $\left(\frac{ddz}{dy^2}\right)$, since in these a solution may succeed, even if the quantity z is introduced in the determination of these, because in turn this comes about, since the method may not be extended to the resolution of an equation of this kind $\left(\frac{ddz}{dxdy}\right) = P\left(\frac{dz}{dx}\right) + Q$, when the letters P and Q contain the quantity z ; nor also can a solution be found, when in addition a formula of the first order $\left(\frac{dz}{dx}\right)$ likewise is present with the other. Yet meanwhile cases are given, in which particular solutions can be shown and these to such an infinite extent, which taken jointly are considered to be equivalent to a general solution, even if in the application to practical use they bring little of help generally; yet it pleases to note the forms of solutions of this kind.

PROBLEM 46

282. *If z should be a function of the two variables x and y, so that there becomes* $\left(\frac{ddz}{dxdy}\right) = az$ *, to investigate at least the nature of the particular functions of z.*

SOLUTION

Since the quantity z maintains one dimension everywhere, it is evident, if there is put $z = e^p q$, that the exponential quantity e^p vanishes from the calculation. Therefore we may put $z = e^{\alpha x} Y$, thus so that the function Y contains only y, and there will be

$$\left(\frac{dz}{dx}\right) = \alpha e^{\alpha x} Y \quad \text{and} \quad \left(\frac{ddz}{dxdy}\right) = \alpha e^{\alpha x} \frac{dY}{dy} = a e^{\alpha x} Y$$

from which there becomes

$$\frac{\alpha dY}{Y} = a dy \quad \text{and} \quad Y = e^{\frac{ay}{\alpha}}$$

and thus we now have the particular solution

$$z = A e^{\alpha x + \frac{ay}{\alpha}},$$

but which appears general enough, since both A as well as α can be taken as it pleases. Moreover many values of z separately satisfying the equation taken together also are satisfactory, from which we deduce a more general expression of this kind

$$z = A e^{\alpha x + \frac{ay}{\alpha}} + B e^{\beta x + \frac{ay}{\beta}} + C e^{\gamma x + \frac{ay}{\gamma}} + D e^{\delta x + \frac{ay}{\delta}} + \text{etc.};$$

where since *A, B, C,* etc., likewise α, β, γ etc. are able to receive possible values, this form has to be taken for the most universal and not, if to the extent we consider, any are to go beyond the most general solution, which involve two arbitrary functions, therefore since here twofold arbitrary coefficients occur ; yet meanwhile it is not clear how discontinuous functions are able to be represented.

COROLLARY 1

283. Therefore for a particular solution to be found the two numbers *m* and *n* are taken, so that the product of these shall be $mn = a$, and there will be $z = Ae^{mx+ny}$. And also from the same numbers interchanged there will be $z = Ae^{nx+my}$.

COROLLARY 2

284. From such a pair of numbers *m* and *n*, so that there shall be $mn = a$, solutions in terms of sines and cosines also can be shown; for there will be

$$z = B\sin.(mx - ny) \text{ or } z = B\cos.(mx - ny)$$

and also on permuting

$$z = B\sin.(nx - my) \text{ or } z = B\cos.(nx - my).$$

COROLLARY 3

285. Therefore since innumerable formulas of this kind are able to be shown, the individual ones multiplied by some constants and collected together into one sum will give the general solution of the problem.

SCHOLIUM

286. Yet neither this solution, even if infinitely often it receives an infinitude of determinations, has been prepared thus, so that it can be reckoned as equivalent to solutions of this kind, which involve two arbitrary functions ; therefore since it is not apparent, how the individual letters are required to be taken, so that for a given case, for argument's sake $y = 0$, the quantity *z* or $\left(\frac{dz}{dx}\right)$ or $\left(\frac{dz}{dy}\right)$ becomes equal to a given function of *x*, also of whatever nature this function should be. But always the solution must be capable of a twofold general determination of this kind. But when such a solution is not permitted to be obtained, we have to be content at any rate with solutions of this kind, as we have come upon here. And indeed we can obtain such solutions in a similar manner, if an equation of this kind is put in place

$$\left(\frac{ddz}{dxdy}\right) = P\left(\frac{dz}{dx}\right) + Q\left(\frac{dz}{dy}\right) + Rz = 0,$$

but only if the letters *P, Q, R* only denote functions of x. For on putting

$z = e^{\alpha y} X$, so that X shall be a function of x only, on account of

$$\left(\frac{dz}{dx}\right) = e^{\alpha y} \frac{dX}{dx} \quad \text{and} \quad \left(\frac{dz}{dy}\right) = \alpha e^{\alpha y} X$$

there will be

$$\frac{\alpha dX}{dx} + \frac{PdX}{dx} + \alpha QX + RX = 0,$$

from which there is found

$$\frac{dX}{X} = \frac{-dx(\alpha Q + R)}{\alpha + P},$$

and thus for any suitable number α the value of X is elicited. Whereby on taking infinitely many numbers α in this manner, the expression receiving an infinite number of infinite determinations is gathered together

$$z = Ae^{\alpha y} X + Be^{\beta y} X' + Ce^{\gamma y} X'' + \text{etc.}$$

Yet truly also cases of this kind of equation are given, which indeed allow complete solutions, an account of which we will investigate in the following problem.

PROBLEM 47

287. *With the proposed equation requiring to be resolved*

$$\left(\frac{ddz}{dxdy}\right) + P\left(\frac{dz}{dx}\right) + Q\left(\frac{dz}{dy}\right) + Rz + S = 0$$

to be investigated, the quantities P, Q, R, and S are to be functions of this kind of x and y, so that indeed a complete solution is allowed.

SOLUTION

Let V be some function of x and y and there is put $z = e^{V} v$, thus so that now v shall be an unknown quantity, the value of which it is required to investigate. Since therefore there shall be

$$\left(\frac{dz}{dx}\right) = e^{V}\left(\left(\frac{dv}{dx}\right) + v\left(\frac{dV}{dx}\right)\right), \quad \left(\frac{dz}{dy}\right) = e^{V}\left(\left(\frac{dv}{dy}\right) + v\left(\frac{dV}{dy}\right)\right),$$

with the substitution made and the whole equation divided by e^{V} the following equation will be produced

$$\left.\begin{aligned}
e^{-V}S+\left(\tfrac{ddv}{dxdy}\right)+\left(\tfrac{dV}{dy}\right)\left(\tfrac{dv}{dx}\right)+\left(\tfrac{dV}{dx}\right)\left(\tfrac{dv}{dy}\right)+\left(\tfrac{dV}{dx}\right)\left(\tfrac{dV}{dy}\right)v\\
+\quad P\left(\tfrac{dv}{dx}\right)+\quad Q\left(\tfrac{dv}{dy}\right)+\quad\left(\tfrac{ddV}{dxdy}\right)v\\
+\quad P\left(\tfrac{dV}{dx}\right)v\\
+\quad Q\left(\tfrac{dV}{dx}\right)v\\
+\quad Rv
\end{aligned}\right\}=0.$$

Now this is required to be effected, so that this equation admits a complete solution ; therefore since before we have seen that such an equation

$$\left(\tfrac{ddv}{dxdy}\right)+T\left(\tfrac{dv}{dx}\right)+e^{-V}S=0$$

generally can be resolved, also whatever functions of *x* and *y* are taken for *T*, *S* and *V*, we can reduce that one to this equation. Therefore by necessity there must be put in place

$$P+\left(\tfrac{dV}{dy}\right)=T,\quad Q+\left(\tfrac{dV}{dx}\right)=0$$

and

$$R+Q\left(\tfrac{dV}{dy}\right)+P\left(\tfrac{dV}{dx}\right)+\left(\tfrac{dV}{dx}\right)\left(\tfrac{dV}{dy}\right)+\left(\tfrac{ddV}{dxdy}\right)=0,$$

from which we obtain

$$P=T-\left(\tfrac{dV}{dy}\right),\quad Q=-\left(\tfrac{dV}{dx}\right)\ \text{ and }\ R=\left(\tfrac{dV}{dx}\right)\left(\tfrac{dV}{dy}\right)-T\left(\tfrac{dV}{dx}\right)-\left(\tfrac{ddV}{dxdy}\right).$$

Therefore since by § 275 there is found

$$v=-\int e^{-\int Tdy}dx\int e^{\int Tdy-V}Sdy+\int e^{-\int Tdy}dxf:x+F:y,$$

there will be the complete integral of the proposed equation

$$\left(\tfrac{ddz}{dxdy}\right)+P\left(\tfrac{dz}{dx}\right)+Q\left(\tfrac{dz}{dy}\right)+Rz+S=0,$$

but only if the letters *P*, *Q*, *R* maintain the assigned values,

$$z=-e^{V}\int e^{-\int Tdy}dx\int e^{\int Tdy-V}Sdy+e^{V}\int e^{-\int Tdy}dxf:x+e^{V}F:y,$$

whenever here the forms *f*: *x* and *F*: *y* denote some functions of *x* and of *y*.

COROLLARY 1

288. Therefore whatever functions of *x* and *y* are taken for the letters *T* and *V*, thence there arise suitable values to be assumed for the letters *P, Q, R,* so that the equation admits a complete resolution ; but with our arbitrary function *S* abandoned.

COROLLARY 2

289. Also the proposed undefined functions *P* and *Q* can be relinquished and there will be then

$$V=-\int Qdx \quad \text{and} \quad \left(\frac{dV}{dy}\right)=-\int dx\left(\frac{dQ}{dy}\right) \quad \text{as well as} \quad \left(\frac{ddV}{dxdy}\right)=-\left(\frac{dQ}{dy}\right),$$

from which finally the quantity R thus must be determined, so that there shall be

$$R-PQ-\left(\frac{dQ}{dy}\right)=0 \quad \text{or} \quad R=PQ+\left(\frac{dQ}{dy}\right).$$

COROLLARY 3

290. Since here for $\int Qdx$ it is possible to write $\int Qdx+Y$ with *Y* denoting some function of *y*, on account of $V=-\int Qdx-Y$ the complete integral will be this equation

$$\left(\frac{ddz}{dxdy}\right)+P\left(\frac{dz}{dx}\right)+Q\left(\frac{dz}{dy}\right)+\left(PQ+\left(\frac{dQ}{dy}\right)\right)z+S=0,$$

the integral of which is

$$z=e^{-\int Qdx-Y}v$$

with there being present

$$\left(\frac{ddz}{dxdy}\right)+\left(P-\int dx\left(\frac{dQ}{dy}\right)-\frac{dY}{dy}\right)\left(\frac{dv}{dx}\right)+e^{-V}S=0$$

and with

$$T=P-\int dx\left(\frac{dQ}{dy}\right)-\frac{dY}{dy}$$

and therefore

$$\int Tdy=\int Pdy-\int Qdx-Y,$$

from which the value of *v* is easily determined.

SCHOLIUM

291. In this calculation, in which it is required to take the differentials of the integral formulas, while another variable quantity is assumed and substituted in the integrand, this rule is to be kept in place, for if there should be $V=\int Qdx$, to become $\left(\frac{dV}{dy}\right)=\int dx\left(\frac{dQ}{dy}\right)$. For since there shall be $\left(\frac{dV}{dx}\right)=Q$, then there will be $\left(\frac{ddV}{dxdy}\right)=\left(\frac{dQ}{dy}\right)$. But if therefore there is put in place $\left(\frac{dV}{dy}\right)=S$, there will be $\left(\frac{dS}{dx}\right)=\left(\frac{dQ}{dy}\right)$ and $S=\left(\frac{dV}{dy}\right)=\int dx\left(\frac{dQ}{dy}\right)$; from which in turn it is deduced, if there should be $S=\int dx\left(\frac{dQ}{dy}\right)$, to become, on account of $\int Sdy=V$, on integrating $\int Sdy=\int Qdx$; because as this will be evident from the principles established before [§50 and §51], accordingly I judge that there will be no need for this as if new precept of the algorithm to be treated separately.

But we may see in some examples, the equations of the kind that can be resolved completely with the aid of this method.

EXAMPLE 1

292. *With the proposed second order differential equation*

$$\left(\frac{ddz}{dxdy}\right)+a\left(\frac{dz}{dx}\right)+b\left(\frac{dz}{dy}\right)+Rz+S=0$$

to define the nature of the function R, so that this equation is allowed to be resolved with some function S of x and y present.

Since there shall be $P=a$ and $Q=b$, there will be $R=ab$ and $V=-bx$; for without risk the function Y can be omitted, because in the following integration now two arbitrary functions are introduced; there will be $T=a$. From which on putting $z=e^{-bx}v$ this equation will be had

$$\left(\frac{ddv}{dxdy}\right)+a\left(\frac{dv}{dx}\right)+e^{bx}S=0,$$

and on putting $\left(\frac{dv}{dx}\right)=u$ it becomes

$$\left(\frac{du}{dy}\right)+au+e^{bx}S=0,$$

an on taking x constant

$$e^{ay}u=-\int e^{ay+bx}Sdy+f':x,$$

therefore

$$u=\left(\frac{dv}{dx}\right)=-e^{-ay}\int e^{ay+bx}Sdy+e^{-ay}f':x$$

and now on taking y constant

$$v=-e^{-ay}\int dx\int e^{ay+bx}Sdy+e^{-ay}f':x+F:x$$

on assuming $\int dxf':x = f:x$. But if now for $e^{-bx}f:x$ there is written $f:x$,there will be

$$z = -e^{-ay-bx}\int dx\int e^{ay+bx}Sdy + e^{-ay}f:x + e^{-bx}F:x.$$

OTHERWISE

If we had taken $V = -bx - ay$, then $T = a - a = 0$ would be produced and thus on putting $z = -e^{-bx-ay}v$ the quantity v must be defined from this equation

$$\left(\frac{ddv}{dxdy}\right) + e^{bx+ay}S = 0\,,$$

which gives

$$\left(\frac{dv}{dx}\right) = -\int e^{bx+ay}Sdy + f':x \quad \text{and} \quad v = -\int dx\int e^{bx+ay}Sdy + f:x + F:y$$

and

$$z = e^{-bx-ay}\left(-\int dx\int e^{bx+ay}Sdy + f:x + F:y\right),$$

which is a simpler form than the previous, even if it is reduced to the same, and this is the complete integral of the equation

$$\left(\frac{ddz}{dxdy}\right) + a\left(\frac{dz}{dx}\right) + b\left(\frac{dz}{dy}\right) + abz + S = 0\,.$$

EXAMPLE 2

293. *With the proposed second order differential equation*

$$\left(\frac{ddz}{dxdy}\right) + \frac{a}{y}\left(\frac{dz}{dx}\right) + \frac{b}{x}\left(\frac{dz}{dy}\right) + Rz + S = 0$$

to define the nature of the function R, so that this equation is allowed to be resolved with some function S of x and y present.

Since there shall be $P = \frac{a}{y}$ and $Q = \frac{b}{x}$, then there will be $V = -blx - Y$ and hence $R = \frac{ab}{xy}$ and the integrable equation will be

$$\left(\frac{ddz}{dxdy}\right) + \frac{a}{y}\left(\frac{dz}{dx}\right) + \frac{b}{x}\left(\frac{dz}{dy}\right) + \frac{ab}{xy}z + S = 0.$$

Therefore since there shall be

$$T = P + \left(\frac{dV}{dy}\right) = \frac{a}{y} - \left(\frac{dY}{dy}\right),$$

we may take $Y = +aly$, so that there becomes $T = 0$, and on putting

$$z = e^{-blx-aly}v = x^{-b}y^{-a}v,$$

the quantity v must be defined from this equation

$$\left(\frac{ddz}{dxdy}\right) + x^b y^a S = 0,$$

from which there becomes

$$\left(\frac{dv}{dx}\right) = -x^b \int y^a Sdy + f':x \quad \text{and} \quad v = -\int x^b dx \int y^a Sdy + f:x + F:x$$

and thus

$$z = \frac{-\int x^b dx \int y^a Sdy + f:x + F:x}{x^a y^b}.$$

SCHOLIUM 1

294. Hence therefore it is apparent that this equation can be integrated with the aid of the method in general

$$\left(\frac{ddz}{dxdy}\right) + P\left(\frac{dz}{dx}\right) + Q\left(\frac{dz}{dy}\right) + \left(PQ + \left(\frac{dQ}{dy}\right)\right)z + S = 0,$$

whatever functions of x and y may be taken for P, Q and S. And indeed thus the resolution is itself had, as on putting $z = e^{-\int Qdx - Y}v$ this quantity v may be determined from this equation

$$\left(\frac{ddv}{dxdy}\right) + \left(P - \int dx\left(\frac{dQ}{dy}\right) - \frac{dY}{dy}\right)\left(\frac{dv}{dx}\right) + e^{\int Qdx + Y}S = 0,$$

where now for y such a function of y can be taken, so that the simplest form of this equation emerges, that which chiefly comes about, if the expression

$$P - \int dx\left(\frac{dQ}{dy}\right) - \frac{dY}{dy}$$

is able to be reduced to zero. Moreover in general there is found

$$v = -\int e^{-\int Pdy + \int Qdx + Y}dx \int e^{\int Pdy}Sdy + \int e^{-\int Pdy + \int Qdx + Y}dx\, f:x + F:y,$$

which value therefore multiplied by $e^{-\int Qdx-Y}$ gives the form of the function z. But in this manner the function Y depending on our choice is removed from inside the calculation and there becomes

$$z=-e^{-\int Qdx}\int e^{-\int Pdy+\int Qdx}dx\int e^{\int Pdy}Sdy+e^{-\int Qdx}\int e^{-\int Pdy+\int Qdx}dx\, f:x+e^{-\int Qdx}F:y\,,$$

which is the complete integral of this equation

$$\left(\tfrac{ddz}{dxdy}\right)+P\left(\tfrac{dz}{dx}\right)+Q\left(\tfrac{dz}{dy}\right)+\left(PQ+\left(\tfrac{dQ}{dy}\right)\right)z+S=0.$$

SCHOLIUM 2

295. But with the variables x and y interchanged also this equation to be integrated completely becomes

$$\left(\tfrac{ddz}{dxdy}\right)+P\left(\tfrac{dz}{dx}\right)+Q\left(\tfrac{dz}{dy}\right)+\left(PQ+\left(\tfrac{dP}{dx}\right)\right)z+S=0.$$

of which the integral shall be

$$z=-e^{-\int Pdy}\int e^{-\int Qdx+\int Pdy}dy\int e^{\int Qdx}Sdx+e^{-\int Pdy}\int e^{-\int Qdx+\int Pdy}dy\, f:y+e^{-\int Pdy}F:x\,,$$

where particularly here the case contained in each form is noteworthy, if there should be put $P=Y$ and $Q=X$ with X a function of x and Y of y present only; then indeed the complete integral of this equation

$$\left(\tfrac{ddz}{dxdy}\right)+Y\left(\tfrac{dz}{dx}\right)+X\left(\tfrac{dz}{dy}\right)+XYz+S=0.$$

will be

$$z=-e^{-\int Xdx-\int Ydy}\int e^{\int Xdx}dx\int e^{\int Ydy}Sdy+e^{-\int Xdx-\int Ydy}\left(f:x+F:y\right),$$

since also it may the shown thus,

$$e^{\int Xdx+\int Ydy}z=f:x+F:y-\int e^{\int Xdx}dx\int e^{\int Ydy}Sdy,$$

or also in this manner

$$e^{\int Xdx+\int Ydy}z=f:x+F:y-\int e^{\int Ydy}dy\int e^{\int Xdx}Sdx.$$

CAPUT II

DE UNA FORMULA DIFFERENTIALI SECUNDI GRADUS PER RELIQUAS QUANTITATES UTCUNQUE DATA

PROBLEMA 41

245. *Si z debeat esse eiusmodi functio ipsarum x et y, ut formula secundi gradus* $\left(\frac{ddz}{dx^2}\right)$ *aequetur functioni datae ipsarum x et y, indolem functionis z investigare.*

SOLUTIO

Sit P functio ista data ipsarum x et y, ita ut esse debeat

$$\left(\frac{ddz}{dx^2}\right) = P.$$

Sumatur iam y constans, et cum sit $d.\left(\frac{dz}{dx}\right) = dx\left(\frac{ddz}{dx^2}\right)$, erit

$$d.\left(\frac{dz}{dx}\right) = Pdx\,,$$

unde integrando prodit

$$\left(\frac{dz}{dx}\right) = \int Pdx + \text{Const.},$$

ubi in integratione $\int Pdx$ quantitas y pro constante habetur et constans adiicienda functionem quamcunque ipsius y denotabit, ita ut haec prima integratio praebeat

$$\left(\frac{dz}{dx}\right) = \int Pdx + f:y$$

Nunc iterum quantitate y ut constante spectata erit $dz = dx\left(\frac{dz}{dx}\right)$ seu

$$dz = dx\int Pdx + dx\, f:y\,;$$

ubi cum $\int Pdx$ sit functio ipsarum x et y, quarum haec y constans assumitur, integratio denuo instituta dabit

$$z = \int dx\int Pdx + x\, f:y + F:y,$$

quod est integrale completum aequationis differentio–differentialis propositae

$\left(\frac{ddz}{dx^2}\right) = P$, propterea quod duas functiones arbitrarias *f*: *y* et *F*: *y* complectitur, quarum utramque ita pro lubitu accipere licet, ut etiam functiones discontinuae non excludantur.

COROLLARIUM 1

246. Quodsi ergo proponatur haec conditio $\left(\frac{ddz}{dx^2}\right) = 0$, eius integratio completa dabit

$$z = x\, f{:}y + F{:}y$$

ob $P = 0$, cuius veritas ex differentiatione perspicitur, unde fit primo $\left(\frac{dz}{dx}\right) = f{:}y$, tum vero $\left(\frac{ddz}{dx^2}\right) = 0$.

COROLLARIUM 2

247. Eodem modo in genere integrale inventum per differentiationem comprobatur. Cum enim invenerimus

$$z = \int dx \int Pdx + x\, f{:}y + F{:}y,$$

prima differentiatio praebet

$$\left(\frac{dz}{dx}\right) = \int Pdx + f{:}y\,,$$

repetita vero $\left(\frac{ddz}{dx^2}\right) = P$.

COROLLARIUM 3

248. Simili modo si haec proponatur conditio $\left(\frac{ddz}{dy^2}\right) = Q$ existente *Q* functione quacunque ipsarum *x* et *y*, integrale completum reperitur

$$z = \int dy \int Qdy + y\, f{:}x + F{:}x,$$

ubi in geminato integrali $\int dy \int Qdy$ quantitas *x* pro constante habetur.

SCHOLION

249. Hinc ratio integralium completorum, quae ex formulis differentialibus secundi gradus nascuntur, in genere perspicitur, quae in hoc est sita, ut duae functiones arbitrariae invehantur; ubi iterum notandum est has functiones tam discontinuas quam continuas esse posse. Nisi ergo per totam hanc sectionem integralia duas huiusmodi functiones arbitrarias involvant, ea pro completis haberi nequeunt. Quotiescunque enim problema ad huiusmodi aequationem $\left(\frac{ddz}{dx^2}\right) = P$ perducit, eius indoles semper ita est comparata, ut tributo ipsi x certo quodam valore $x=a$ tam formula $\left(\frac{dz}{dx}\right)$ quam ipsa quantitas z datae cuipiam functioni ipsius y aequari possit. Quare si tam integrale $\int Pdx$ quam hoc $\int dx\int Pdx$ ita accipiatur, ut posito $x=a$ evanescat, erit pro eodem casu $x=a$ valor

$$\left(\frac{dz}{dx}\right)=f:y \quad \text{et} \quad z=af:y+F:y\,,$$

unde ex problematis natura utraque functio $f:y$ et $F:y$ definitur. Haec autem applicatio ad omnes casus fieri non posset, nisi integrale completum haberetur; quamobrem in hoc praecipue est incumbendum, ut omnium huiusmodi problematum integralia completa habeantur.

Ceterum hic in perpetuum monendum duco, quoties huiusmodi formula integralis $\int Pdx$ occurrit, semper solam quantitatem x variabilem accipi esse intelligendam, siquidem, si etiam y variabilis acciperetur, formula $\int Pdx$ ne significatum quidem admitteret. Simili modo in formula $\int dx\int Pdx$ intelligi debet in utraque integratione solam x variabilem assumi. Sin autem talis forma $\int dy\int Pdx$ occurrat, intelligendum est integrale $\int Pdx$ ex variabilitate solius x colligi debere; quod si ponatur $=R$, ut habeatur $\int Rdy$, hic iam sola y pro variabili erit habenda.

EXEMPLUM 1

250. *Quaeratur binarum variabilium x et y eiusmodi functio z, ut sit*

$$\left(\frac{ddz}{dx^2}\right)=\frac{xy}{a}\,.$$

Cum hic sit $P=\frac{xy}{a}$, erit

$$\int Pdx=\frac{xxy}{2a} \quad \text{et} \quad \int dx\int Pdx=\frac{x^3y}{6a}$$

sicque habebitur ex prima integratione

$$\left(\frac{dz}{dx}\right)=\frac{xxy}{2a}+f:y\,,$$

ita ut posito $x=a$ formula $\left(\frac{dz}{dx}\right)$ functioni cuicunque ipsius y aequari possit seu applicatae curvae cuiuscunque respondenti abscissae y. Tum vero altera integratione instituta erit

$$z=\frac{x^3y}{6a}+xf:y+F:y,$$

qui valor casu $x=a$ denuo functioni cuicunque ipsius y aequari potest.

EXEMPLUM 2

251. *Quaeratur binarum variabilium x et y eiusmodi functio z, ut sit*

$$\left(\frac{ddz}{dx^2}\right)=\frac{ax}{\sqrt{(xx+yy)}}.$$

Ob $P=\frac{ax}{\sqrt{(xx+yy)}}$ erit

$$\int Pdx=a\sqrt{(xx+yy)}$$

et

$$\int dx\int Pdx=a\int dx\sqrt{(xx+yy)}=\tfrac{1}{2}ax\sqrt{(xx+yy)}+\tfrac{1}{2}ayyl\left(x+\sqrt{(xx+yy)}\right),$$

unde prima integratio praebet

$$\left(\frac{dz}{dx}\right)=a\sqrt{(xx+yy)}+f:y$$

altera vero

$$z=\tfrac{1}{2}ax\sqrt{(xx+yy)}+\tfrac{1}{2}ayyl\left(x+\sqrt{(xx+yy)}\right)+xf:y+F:y.$$

EXEMPLUM 3

252. *Quaeratur binarum variabilium x et y eiusmodi functio z, ut sit*

$$\left(\frac{ddz}{dx^2}\right)=\frac{1}{\sqrt{(aa-xx-yy)}}$$

Cum sit $P=\frac{1}{\sqrt{(aa-xx-yy)}}$, erit

$$\int Pdx=\text{Ang. sin.}\frac{x}{\sqrt{(aa-yy)}},$$

tum vero

$$\int dx\int Pdx=x\text{Ang. sin.}\frac{x}{\sqrt{(aa-yy)}}-\int\frac{xdx}{\sqrt{(aa-xx-yy)}}.$$

Quare integratio prima praebet

$$\left(\frac{dz}{dx}\right)=\text{Ang. sin.}\frac{x}{\sqrt{(aa-yy)}}+f:y$$

hincque ipsa functio quaesita erit

$$z=x\text{Ang. sin.}\frac{x}{\sqrt{(aa-yy)}}+\sqrt{(aa-xx-yy)}+xf:y+F:y.$$

EXEMPLUM 4

253. *Quaeratur binarum variabilium x et y eiusmodi functio z, ut sit*

$$\left(\frac{ddz}{dx^2}\right) = x\sin.(x+y).$$

Ob $P = x\sin.(x+y)$ erit

$$\int Pdx = \int xdxsin.(x+y) = -x\cos.(x+y) + \int dxcos.(x+y) = -xcos.(x+y) + \sin.(x+y).$$

Tum vero est

$$\int xdx\cos.(x+y) = x\sin.(x+y) + \cos.(x+y)$$

ideoque

$$\int dx \int Pdx = -\ 2\cos.(x+y) - x\sin.(x+y).$$

Quocirca ambo nostra integralia erunt

$$\left(\frac{dz}{dx}\right) = \sin.(x+y) - \cos.(x+y) + f:y$$

et

$$z = -\ 2\cos.(x+y) - x\sin.(x+y) + xf:y + F:y.$$

PROBLEMA 42

254. *Si z debeat esse eiusmodi functio variabilium x et y, ut sit*

$$\left(\frac{ddz}{dx^2}\right) = P\left(\frac{dz}{dx}\right) + Q$$

existentibus P et Q functionibus quibusvis ipsarum x et y, indolem functionis z in genere investigare.

SOLUTIO

Ponamus hic $\left(\frac{dz}{dx}\right) = v$, ut sit $\left(\frac{ddz}{dx^2}\right) = \left(\frac{dv}{dx}\right)$; erit nostra aequatio integranda

$$\left(\frac{dv}{dx}\right) = Pv + Q.$$

Spectetur ergo sola x ut variabilis et ob $dv = dx\left(\frac{dv}{dx}\right)$ erit

$$dv = Pvdx + Qdx,$$

quae per $e^{-\int Pdx}$ multiplicata et integrata dat

$$e^{-\int Pdx}v = \int e^{-\int Pdx}Qdx + f:y.$$

ideoque

$$\left(\frac{dz}{dx}\right) = e^{\int Pdx}\int e^{-\int Pdx}Qdx + e^{\int Pdx} f:y.$$

Retineatur sola *x* variabilis spectata *y* ut constante et ob $dz = dx\left(\frac{dz}{dx}\right)$ erit

$$z = \int e^{\int Pdx}dx \int e^{-\int Pdx}Qdx + f:y \int e^{\int Pdx}dx + F:y,$$

quod ob binas functiones arbitrarias *f: y* et *F: y* est integrale completum.

COROLLARIUM 1

255. Problema hoc multo latius patet praecedente, cum conditio proposita etiam formulam primi gradus $\left(\frac{dz}{dx}\right)$ involvat; nihilo vero minus solutio feliciter successit.

COROLLARIUM 2

256. Hic ergo quadruplici integratione est opus. Primo scilicet quaeri debet integrale $\int Pdx$; quod si ponatur = *lR*, quaeri porro debet integrale

$$\int e^{\int Pdx}dx = \int Rdx;$$

quod si ponamus $= S$, restat integrale

$$\int Rdx \int \frac{Qdx}{R} = \int dS \int \frac{Qdx}{R},$$

quod abit in

$$S\int \frac{Qdx}{R} - \int \frac{QSdx}{R},$$

ita ut insuper hae duae formae integrari debeant.

COROLLARIUM 3

257. Eodem omnino modo resolvitur problema, quo esse debet

$$\left(\frac{ddz}{dy^2}\right) = P\left(\frac{dz}{dy}\right) + Q\,,$$

si P et Q fuerint functiones quaecunque datae ipsarum x et y. Reperitur enim

$$\left(\frac{dz}{dy}\right) = e^{\int Pdy}\int e^{-\int Pdy}Qdy + e^{\int Pdy} f{:}x$$

et

$$z = \int e^{\int Pdy}dy\int e^{-\int Pdy}Qdy + f{:}x\int e^{\int Pdy}dy + F{:}x\,.$$

EXEMPLUM 1

258. *Quaeratur binarum variabilium x et y eiusmodi functio z, ut sit*

$$\left(\frac{ddz}{dx^2}\right) = \frac{n}{x}\left(\frac{dz}{dx}\right).$$

Posito $\left(\frac{dz}{dx}\right) = v$ sumtaque sola x variabili erit $\left(\frac{dv}{dx}\right) = \frac{nv}{x}$ ideoque

$\frac{dv}{v} = \frac{ndx}{dx}$, cuius integra dat

$$v = \left(\frac{dz}{dx}\right) = x^n f{:}y\,.$$

Iam iterum sola x pro variabili habita erit

$$dz = x^n dx\, f{:}y\,,$$

cuius integrale completum est

$$z = \frac{1}{n+1}x^{n+1} f{:}y + F{:}y.$$

Casu autem $n = -1$ seu $\left(\frac{ddz}{dx^2}\right) = \frac{-1}{x}\left(\frac{dz}{dx}\right)$ erit

$$\left(\frac{dz}{dx}\right) = \frac{1}{x} f{:}y \quad \text{et} \quad z = lx\cdot f{:}y + F{:}y.$$

EXEMPLUM 2

259. *Quaeratur binarum variabilium x et y eiusmodi functio z, ut sit*

$$\left(\frac{ddz}{dx^2}\right)=\frac{n}{x}\left(\frac{dz}{dx}\right)+\frac{a}{xy}.$$

Posito $\left(\frac{dz}{dx}\right)=v$ sumtaque sola x variabili erit

$$dv=\frac{nvdx}{x}+\frac{adx}{xy},$$

quae aequatio per x^n divisa et integrata praebet

$$\frac{v}{x^n}=\frac{a}{y}\int\frac{dx}{x^{n+1}}=\frac{-a}{nx^n y}+f:y$$

seu

$$v=\left(\frac{dz}{dx}\right)=\frac{-a}{ny}+x^n f:y.$$

Sit iterum sola x variabilis, ut habeatur

$$dz=\frac{-adx}{ny}+x^n dx\, f:y,$$

prodibitque integrale completum

$$z=\frac{-ax}{ny}+\frac{1}{n+1}x^{n+1}f:y+F:y.$$

EXEMPLUM 3

260. *Quaeratur binarum variabilium x et y eiusmodi functio z, ut sit*

$$\left(\frac{ddz}{dx^2}\right)=\frac{2nx}{xx+yy}\left(\frac{dz}{dx}\right)+\frac{x}{ay}.$$

Posito $\left(\frac{dz}{dx}\right)=v$ erit sumendo y constans

$$dv=\frac{2nxv}{xx+yy}+\frac{xdx}{ay}$$

quae aequatio per $(xx+yy)^n$ divisa et integrata dat

$$\frac{v}{(xx+yy)^n}=\frac{1}{ay}\int\frac{xdx}{(xx+yy)^n}=-\frac{1}{2(n-1)ay(xx+yy)^{n-1}}+f:y$$

seu

$$v = \left(\frac{dz}{dx}\right) = \frac{-(xx+yy)}{2(n-1)ay} + (xx+yy)^n\ f:y\,.$$

Hinc sumto iterum y constante fit

$$z = \frac{-x(xx+3yy)}{6(n-1)ay} + f:y\int(xx+yy)^n\,dx + F:y$$

Casu, quo $n=1$ seu

$$\left(\frac{ddz}{dx^2}\right) = \frac{2x}{xx+yy}\left(\frac{dz}{dx}\right) + \frac{x}{ay}$$

erit

$$\frac{v}{(xx+yy)} = \frac{1}{ay}\int\frac{xdx}{(xx+yy)} = \frac{1}{2ay}l(xx+yy) + f:y\,,$$

hinc

$$\left(\frac{dz}{dx}\right) = \frac{xx+yy}{2ay}l(xx+yy) + (xx+yy)\,f:y$$

et

$$z = \frac{x(xx+3yy)}{6ay}\ l(xx+yy) - \frac{1}{9ay}\left(x^3 + 6xy^2 - 6xy^3\text{Ang.tang.}\frac{x}{y}\right) + \frac{1}{3}x(xx+3yy)\ f:y + F:y.$$

PROBLEMA 43

261. *Si z debeat esse eiusmodi functio binarum variabilium x et y, ut sit*

$$\left(\frac{ddz}{dx^2}\right) = P\left(\frac{dz}{dx}\right) + Q$$

existentibus P et Q functionibus quibuscunque datis omnium trium variabilium x, y et z, indolem functionis z investigare.

SOLUTIO

Posita quantitate y constante erit

$$\left(\frac{ddz}{dx^2}\right) = \frac{ddz}{dx^2} \quad \text{et} \quad \left(\frac{dz}{dx}\right) = \frac{dz}{dx}$$

ideoque habebitur aequatio differentialis secundi gradus ad librum praecedentem pertinens

$$ddz = Pdxdz + Qdx^2$$

quae duas tantum variabiles x et z involvere est censenda, quia y in ea tanquam constans spectatur. Tentetur ergo integratio huius aequationis per methodos ibi expositas; quae si successerit, loco binarum constantium, quas duplex integratio invehit, scribantur ipsius y functiones indefinitae $f:y$

et $F : y$, quae adeo discontinuae accipi possunt, sicque habebitur aequationis propositae integrale completum.

COROLLARIUM 1

262. Reducitur ergo solutio huius problematis ad methodum integrandi in superiori libro traditam, ubi functionem unius variabilis ex data differentialium secundi gradus relatione investigari oportebat.

COROLLARIUM 2

263. Quodsi ergo resolutionem omnium aequationum differentialium secundi gradus, quae binas tantum variabiles involvunt, hic nobis concedi postulemus, solutio nostri problematis pro confecta est censenda.

COROLLARIUM 3

264. Me non monente intelligitur eodem modo aequationem

$$\left(\frac{ddz}{dx^2}\right) = P\left(\frac{dz}{dx}\right) + Q$$

tractari oportere eiusque solutionem tanquam confectam spectari posse, quaecunque fuerint P et Q functiones ipsarum x, y et z.

SCHOLION 1

265. Ex solutionis ratione intelligitur problema multo latius patens simili modo resolvi posse. Si enim formula $\left(\frac{ddz}{dx^2}\right)$ quomodocunque per quantitates principales x, y, z ac praeterea formulam $\left(\frac{dz}{dx}\right)$ determinetur, ita ut etiam huius formulae $\left(\frac{dz}{dx}\right)$ potestates aliaeve functiones quaecunque ingrediantur, solutio semper ad librum superiorem revocabitur, quia ponendo y constans fit

$$\left(\frac{dz}{dx}\right) = \frac{dz}{dx} \text{ et } \left(\frac{ddz}{dx^2}\right) = \frac{ddz}{dx^2}$$

ideoque resultat aequatio differentialis secundi gradus formae consuetae duas tantum variabiles x et z involvens. Hoc tantum teneatur loco constantium per utramque integrationem ingredientium scribi oportere formas $f : y$ et $F : y$. Satis igitur notabilem partem propositi nostri expedivimus, scilicet cum vel $\left(\frac{ddz}{dx^2}\right)$ utcunque per x, y, z et $\left(\frac{dz}{dx}\right)$, vel $\left(\frac{ddz}{dy^2}\right)$ utcunque per x, y, z et $\left(\frac{dz}{dy}\right)$ determinatur; ibi nempe excluditur formula primi gradus $\left(\frac{dz}{dy}\right)$, hic vero formula $\left(\frac{dz}{dx}\right)$. Quae si accederet, quaestio hac methodo neutiquam tractari posset, quemadmodum vel ex hoc casu simplicissimo $\left(\frac{ddz}{dx^2}\right) = \left(\frac{dz}{dy}\right)$ intelligere licet, cuius resolutio maxime ardua est putanda.

SCHOLION 2

266. Cum igitur trium formularum differentialium secundi gradus $\left(\frac{ddz}{dx^2}\right)$, $\left(\frac{ddz}{dxdy}\right)$, $\left(\frac{ddz}{dy^2}\right)$ primam ac tertiam hactenus sim contemplatus, quatenus earum per reliquas quantitates determinatio resolutionem admittit methodo quidem hic adhibita, superest, ut formulam quoque secundam $\left(\frac{ddz}{dxdy}\right)$ consideremus et, quibusnam determinationibus per reliquas quantitates *x, y, z,* $\left(\frac{dz}{dx}\right)$, $\left(\frac{dz}{dy}\right)$ solutio absolvi queat, investigemus, in quo negotio a casibus simplicissimis exordiri conveniet.

PROBLEMA 44

267. *Si z eiusmodi debeat esse functio binarum variabilium x et y, ut fiat* $\left(\frac{ddz}{dxdy}\right)=P$ *existente P functione quacunque data ipsarum x et y, indolem functionis z generaliter determinare.*

SOLUTIO

Ponatur $\left(\frac{dz}{dx}\right)=v$ eritque $\left(\frac{ddz}{dxdy}\right)=\left(\frac{dv}{dy}\right)$ ideoque habebitur $\left(\frac{dv}{dy}\right)=P$. Iam spectetur quantitas *x* ut constans, ita ut *P* solam variabilem *y* contineat, eritque $dv=Pdy$, unde in hypothesi quantitatis *x* constantis integrando prodit

$$v=\left(\frac{dz}{dx}\right)-\int Pdy+f':x,$$

ubi $\int Pdy$ erit functio data ipsarum *x* et *y*. Nunc porro spectetur *x* ut variabilis, *y* vero ut constans, ut adipiscamur hanc aequationem differentialem

$$dz=dx\int Pdy+dx\, f':x,$$

quae integrata dat

$$z=\int dx\int Pdy+f:x+F:y\ ;$$

ubi cum habeantur duae functiones arbitrariae, id indicio est hoc integrale esse completum.

COROLLARY 1

268. Si ordine inverso primum *y*, tum vero *x* constans posuissemus, invenissemus

$$\left(\frac{dz}{dy}\right)=\int Pdx+f':y \quad \text{et} \quad z=\int dy\int Pdx+f:y+F:x,$$

qui valor aeque satisfacit ac praecedens.

COROLLARIUM 2

269. Patet ergo vel fore $\int dx \int Pdy = \int dy \int Pdx$ vel differentiam saltem exprimi per aggregatum ex functione ipsius *x* et functione ipsius *y*. Quod etiam inde patet, quod posito

$$\int dx \int Pdy = \int dy \int pdx = V$$

fiat utrinque $P = \left(\frac{ddV}{dxdy}\right)$.

COROLLARIUM 3

270. Si sit $P = 0$ seu debeat esse $\left(\frac{ddz}{dxdy}\right) = 0$, reperitur pro indole functionis *z* haec forma $z = f:x + F:y$.

SCHOLION

271. Hic casus in doctrina solidorum frequenter occurrit. Si enim natura superficiei exprimatur aequatione inter ternas coordinatas *x*, *y* et *u*, erit soliditas $= \int dx \int udy$; quare si soliditas exprimatur per *z*, erit $\left(\frac{ddz}{dxdy}\right) = u$, ordinatae scilicet ad binas *x* et *y* normali. Tum vero si ponatur

$$du = pdx + qdy,$$

superficies huius solidi erit

$$\int dx \int dy \sqrt{(1 + pp + qq)};$$

quae superficies si exprimatur littera *z*, erit

$$\left(\frac{ddz}{dxdy}\right) = \sqrt{(1 + pp + qq)}.$$

Quando ergo in nostro problemate eiusmodi functio *z* ipsarum *x* et *y* quaeritur, ut sit $\left(\frac{ddz}{dxdy}\right) = P$, idem est, ac si quaeratur soliditas respondens superficiei, cuius natura aequatione inter ternas coordinatas *x*, *y* et *P* exprimitur. Exemplis igitur aliquot hunc calculum illustremus.

EXEMPLUM 1

272. *Quaeratur binarum variabilium x et y eiusmodi functio z, ut sit* $\left(\frac{ddz}{dxdy}\right)=\alpha x+\beta y$.

Cum hic sit $P=\alpha x+\beta y$, erit

$$\int Pdy=\alpha xy+\tfrac{1}{2}\beta yy \quad \text{et} \quad \int dx\int Pdy=\tfrac{1}{2}\alpha xxy+\tfrac{1}{2}\beta xyy=\tfrac{1}{2}xy(\beta x+\beta y),$$

unde functio quaesita *z* ita exprimitur, ut sit

$$z=\tfrac{1}{2}xy(\beta x+\beta y)+f:x+F:y.$$

EXEMPLUM 2

273. *Quaeratur binarum variabilium x et y eiusmodi functio z, ut sit* $\left(\frac{ddz}{dxdy}\right)=\sqrt{(aa-yy)}$.

Hic est $P=\sqrt{(aa-yy)}$, ergo

$$\int Pdx=x\sqrt{(aa-yy)},$$

ubi, quia perinde est, a variabilitate ipsius *x* incipio. Hinc igitur fit

$$\int dy\int Pdx=x\int dy\sqrt{(aa-yy)}=\tfrac{1}{2}xy\sqrt{(aa-yy)}+\tfrac{1}{2}aax\int\frac{dy}{\sqrt{(aa-yy)}}$$

ex quo integrale completum erit

$$z=\tfrac{1}{2}xy\sqrt{(aa-yy)}+\tfrac{1}{2}aax\,\text{Ang.sin.}\frac{y}{a}+f:x+F:y.$$

EXEMPLUM 3

274. *Quaeratur binarum variabilium x et y eiusmodi functio z, ut sit* $\left(\frac{ddz}{dxdy}\right)=\frac{a}{\sqrt{(aa-xx-yy)}}$

Ob $P=\frac{a}{\sqrt{(aa-xx-yy)}}$ erit

$$\int Pdy=a\text{Ang.sin.}\frac{y}{\sqrt{(aa-xx)}}$$

hinc

$$\int dx\int Pdy=a\int dx\text{Ang.sin.}\frac{y}{\sqrt{(aa-xx)}}.$$

Ponatur brevitatis gratia

$$\text{Ang.sin.}\frac{y}{\sqrt{(aa-xx)}}=\varphi$$

erit

$$\int dx\int Pdy=a\int\varphi dx=ax\varphi-a\int xdx\left(\frac{d\varphi}{dx}\right);$$

in hac enim integratione *y* pro constante habetur. Quare ob $\frac{y}{\sqrt{(aa-xx)}}=\sin.\varphi$
erit

$$\frac{yx}{(aa-xx)^{\frac{3}{2}}}=\left(\frac{d\varphi}{dx}\right)\cos.\varphi\,.$$

At vero est $\cos.\varphi=\frac{\sqrt{(aa-xx-yy)}}{\sqrt{(aa-xx)}}$ hincque

$$\left(\frac{d\varphi}{dx}\right)=\frac{yx}{(aa-xx)\sqrt{(aa-xx-yy)}}$$

et

$$\int xdx\left(\frac{d\varphi}{dx}\right)=y\int\frac{xxdx}{(aa-xx)\sqrt{(aa-xx-yy)}}$$

quo integrali invento erit

$$z=ax\text{Ang.sin.}\frac{y}{\sqrt{(aa-xx)}}-ay\int\frac{xxdx}{(aa-xx)\sqrt{(aa-xx-yy)}}+f:x+F:x$$

quae forma per integrationem evoluta reducitur ad hanc

$$z=ax\text{Ang.sin.}\frac{y}{\sqrt{(aa-xx)}}+ay\text{Ang.sin.}\frac{x}{\sqrt{(aa-yy)}}$$
$$-aa\text{Ang.sin.}\frac{xy}{\sqrt{(aa-xx)(aa-xx-yy)}}+f:x+F:x.$$

Formulae enim

$$\int\frac{aadx}{(aa-xx)\sqrt{(aa-xx-yy)}}$$

integrale ita facillime elicitur. Ponatur

$$\frac{x}{\sqrt{(aa-xx-yy)}}=p\,;$$

erit $xx=\frac{pp(aa-yy)}{1+pp}$ et ob y constans per logarithmos differentiando

$$\frac{dx}{x}=\frac{dp}{p}-\frac{pdp}{1+pp}=\frac{dp}{p(1+pp)}$$

tum per illam formulam multiplicando

$$\frac{dx}{\sqrt{(aa-xx-yy)}}=\frac{dp}{1+pp}$$

Porro est $aa-xx=\frac{aa+ppyy}{1+pp}$, unde formula integralis fit

$$\int\frac{xxdx}{(aa-xx)\sqrt{(aa-xx-yy)}}=\int\frac{aadp}{aa+ppyy}=\frac{aa}{yy}\int\frac{dp}{\frac{aa}{yy}+pp}$$

$$=\frac{a}{y}\text{Ang.tang.}\frac{py}{a}=\frac{a}{y}\text{Ang. tang.}\frac{xy}{a\sqrt{(aa-xx-yy)}}=\frac{a}{y}\text{Ang.sin.}\frac{xy}{\sqrt{(aa-xx)(aa-yy)}}.$$

PROBLEMA 45

275. *Si z eiusmodi esse debeat functio binarum variabilium x et y, ut sit*

$$\left(\frac{ddz}{dxdy}\right)=P\left(\frac{dz}{dx}\right)+Q$$

existentibus P et Q functionibus quibuscunque ipsarum x et y, investigare indolem functionis z.

SOLUTIO

Ponatur $\left(\frac{dz}{dx}\right)=v$, ut oriatur ista aequatio $\left(\frac{dv}{dx}\right)=Pv+Q$, quae continet quantitates *x*, *y* et *v;* statuatur ergo *x* constans eritque

$$dv=Pvdy+Qdy\,,$$

quae per $e^{-\int Pdy}$ multiplicata praebet

$$e^{-\int Pdy}v=\int e^{-\int Pdy}Qdy+f':x$$

ideoque

$$v=e^{\int Pdy}\int e^{-\int Pdy}Qdy+e^{\int Pdy}f':x$$

Nunc cum haec integralia determinate contineant *x* et *y*, spectetur *y* ut constans et sequens integratio praebet

$$z=\int e^{\int Pdy}dx\int e^{-\int Pdy}Qdy+\int e^{\int Pdy}dxf':x+F:y,$$

quae integralia quovis casu evoluta fiunt manifesta.

COROLLARIUM 1

276. Ad hoc ergo problema resolvendum per integrationem primo quaeratur *R*, ut sit $\int Pdy = lR$; deinde quaeratur *S*, ut sit $\int \frac{Qdy}{R} = S$; denique sit $\int RSdx = T$, ita ut in illis sola quantitas *y*, hic vero sola *x* pro variabili habeatur. Quo facto erit nostrum integrale completum

$$z = T + \int Rdxf':x + F:y .$$

COROLLARIUM 2

277. Hic ergo functio arbitraria *f: x* in formula integrali est involuta; quae tamen si per applicatam curvae cuiuscunque respondentem abscissae *x* exhibeatur, hoc integrale $\int Rdxf':x$ pro quovis valore ipsius *y* seorsim construi poterit, siquidem in hac integratione quantitas *y* ut constans spectatur.

SCHOLION

278. Eodem plane modo resolvitur permutandis variabilibus *x* et *y* hoc problema, quo functio *z* quaeritur, ut sit

$$\left(\frac{ddz}{dxdy}\right) = P\left(\frac{dz}{dy}\right) + Q ,$$

dummodo *P* et *Q* sint functiones ipsarum *x* et *y* tantum ipsam functionem *z* non implicantes; solutio enim ita, se habebit

$$z = \int e^{\int Pdx} dy \int e^{-\int Pdx} Qdx + \int e^{\int Pdx} dyf':y + F:x.$$

Quin etiam utrumque problema latius extendi potest, ac prius resolutionem admittet, si formula $\left(\frac{ddz}{dxdy}\right)$ aequetur functioni cuicunque trium quantitatum *x, y* et $\left(\frac{dz}{dx}\right)$, posterius vero, si $\left(\frac{ddz}{dxdy}\right)$ aequetur functioni cuicunque harum trium quantitatum *x, y* et $\left(\frac{dz}{dy}\right)$; utroque enim casu res reducitur ad aequationem differentialem primi gradus. Neque vero haec solvendi methodus succedit, si utraque formula primi gradus $\left(\frac{dz}{dx}\right)$ et $\left(\frac{dz}{dy}\right)$ simul ingrediatur vel si functiones *P* et *Q* etiam ipsam quantitatem *z* complectantur.

EXEMPLUM 1

279. *Quaeratur binarum variabilium x et y functio z, ut sit*

$$\left(\frac{ddz}{dxdy}\right)=\frac{n}{y}\left(\frac{dz}{dx}\right)+\frac{m}{x}$$

Sit $\left(\frac{dz}{dx}\right)=v$; erit $\left(\frac{dv}{dy}\right)=\frac{nv}{y}+\frac{m}{x}$ et spectata x ut constante erit

$$dv=\frac{nvdy}{y}+\frac{mdy}{x},$$

unde per y^n dividendo prodit

$$\frac{v}{y^n}=\frac{m}{x}\int\frac{dy}{y^n}=\frac{-m}{(n-1)xy^{n-1}}+f':x,$$

ita ut sit

$$v=\left(\frac{dz}{dx}\right)=\frac{-my}{(n-1)x}+y^n f':x;$$

sumatur iam y constans et denuo integrando obtinetur

$$z=\frac{-m}{n-1}ylx+y^n f:x+F:y.$$

EXEMPLUM 2

280. *Quaeratur binarum variabilium x et y functio z, ut sit*

$$\left(\frac{ddz}{dxdy}\right)=\frac{y}{xx+yy}\left(\frac{dz}{dx}\right)+\frac{a}{xx+yy}.$$

Posito $\left(\frac{dz}{dx}\right)=v$ etsumto x constante erit

$$dv=\frac{vydy}{xx+yy}+\frac{ady}{xx+yy},$$

quae aequatio per $\sqrt{(xx+yy)}$ divisa dat

$$\frac{dv}{\sqrt{(xx+yy)}}=a\int\frac{dy}{(xx+yy)^{\frac{3}{2}}}=\frac{ay}{xx\sqrt{(xx+yy)}}+f:x.$$

Ergo

$$v=\left(\frac{dz}{dx}\right)=\frac{ay}{xx}+\sqrt{(xx+yy)}\cdot f:x;$$

sit iam y constans reperieturque

$$z=\frac{-ay}{x}+\int f:xdx\sqrt{(xx+yy)}+F:y\,,$$

ubi quidem integrale $\int f:xdx\sqrt{(xx+yy)}$ ob functionem indeterminatam *f*: *x*, etsi *y* constans ponitur, in genere exprimi nequit, ita ut explicate per *y* et functiones ipsius *x* exhiberi possit.

SCHOLION

281. Formula ergo secundi gradus $\left(\frac{ddz}{dxdy}\right)$ non tam largam casuum resolubilium copiam admittit quam binae reliquae $\left(\frac{ddz}{dx^2}\right)$ et $\left(\frac{ddz}{dy^2}\right)$, cum in his solutio succedat, etiamsi ipsa quantitas *z* quoque in earum determinationem ingrediatur, quod hic secus evenit, cum methodus non pateat huiusmodi aequationem $\left(\frac{ddz}{dxdy}\right)=P\left(\frac{dz}{dx}\right)+Q$, quando litterae *P* et *Q* quantitatem *z* continent, resolvendi; neque etiam solutio locum habet, quando praeter formulam primi gradus $\left(\frac{dz}{dx}\right)$ simul quoque altera adest. Interim tamen dantur casus, quibus solutiones particulares exhiberi possunt eaeque adeo infinitae, quae iunctim sumtae solutioni generali aequivalere videntur, etiamsi in applicatione ad usum practicum parum subsidii plerumque afferant; formas tamen huiusmodi solutionum notasse iuvabit.

PROBLEMA 46

282. *Si z eiusmodi debeat esse functio binarum variabilium x et* y, *ut fiat* $\left(\frac{ddz}{dxdy}\right)=az$ *, indolem huius functionis z particulariter saltem investigare.*

SOLUTIO

Cum quantitas *z* unam ubique teneat dimensionem, evidens est, si statuatur $z=e^pq$, quantitatem exponentialem e^p ex calculo evanescere. Ponamus igitur $z=e^{\alpha x}Y$, ita ut *Y* functionem ipsius *y* tantum contineat, eritque

$$\left(\frac{dz}{dx}\right)=\alpha e^{\alpha x}Y \quad \text{et} \quad \left(\frac{ddz}{dxdy}\right)=\alpha e^{\alpha x}\frac{dY}{dy}=ae^{\alpha x}Y$$

unde fit

$$\frac{\alpha dY}{Y}=ady \quad \text{et} \quad Y=e^{\frac{ay}{\alpha}}$$

sicque iam solutionem particularem habemus

$$z=Ae^{\alpha x+\frac{ay}{\alpha}},$$

quae autem satis late patet, cum tam *A* quam α pro lubitu assumi possit. Plures autem valores ipsius *z* seorsim satisfacientes etiam iunctim sumti satisfaciunt, unde huiusmodi expressionem multo generaliorem deducimus

$$z = Ae^{\alpha x + \frac{ay}{\alpha}} + Be^{\beta x + \frac{ay}{\beta}} + Ce^{\gamma x + \frac{ay}{\gamma}} + De^{\delta x + \frac{ay}{\delta}} + \text{etc.};$$

ubi cum *A, B, C*, etc., item α, β, γ etc. omnes valores possibiles recipere queant, haec forma pro maxime universali est habenda neque, si ad amplitudinem spectamus, quicquam cedere videtur superioribus solutionibus, quae binas functiones arbitrarias involvunt, propterea quod hic duplicis generis coefficientes arbitrarii occurrunt; interim tamen haud liquet, quomodo functiones discontinuae hac relatione repraesentari queant.

COROLLARIUM 1

283. Pro solutione ergo particulari invenienda sumantur bini numeri *m* et *n*, ut eorum productum sit $mn = a$, eritque $z = Ae^{mx+ny}$. Atque etiam ex iisdem numeris permutatis erit $z = Ae^{nx+my}$.

COROLLARIUM 2

284. Ex tali numerorum *m* et *n* pari, ut sit $mn = a$, solutiones quoque per sinus et cosinus exhiberi possunt; erit enim

$$z = B\sin.(mx - ny) \text{ vel } z = B\cos.(mx - ny)$$

vel etiam permutando

$$z = B\sin.(nx - my) \text{ vel } z = B\cos.(nx - my).$$

COROLLARIUM 3

285. Cum igitur huiusmodi formulae innumerabiles exhiberi queant, singulae per constantes quascunque multiplicatae et in unam summam collectae dabunt solutionem generalem problematis.

SCHOLION

286. Neque tamen haec solutio, etsi infinities infinitas determinationes recipit, ita est comparata, ut eiusmodi solutionibus, quae binas functiones arbitrarias involvunt, aequivalens aestimari possit; propterea quod non patet, quomodo singulas litteras assumi oporteat, ut pro dato casu, verbi gratia $y = 0$, quantitas *z* vel $\left(\frac{dz}{dx}\right)$ seu $\left(\frac{dz}{dy}\right)$ datae functioni ipsius *x* aequalis evadat, cuiuscunque etiam indolis fuerit haec functio. Semper autem solutio generalis duplicis huiusmodi determinationis capax esse debet.

Quando autem talem solutionem impetrare non licet, utique eiusmodi solutionibus, uti hic invenimus, contenti esse debemus. Ac tales quidem solutiones simili modo obtinere possumus, si proponatur eiusmodi aequatio

$$\left(\frac{ddz}{dxdy}\right) = P\left(\frac{dz}{dx}\right) + Q\left(\frac{dz}{dy}\right) + Rz = 0,$$

si modo litterae *P, Q, R* denotent functiones ipsius *x* tantum. Posito enim

$z = e^{\alpha y} X$, ut *X* sit functio solius *x*, ob

$$\left(\frac{dz}{dx}\right) = e^{\alpha y}\frac{dX}{dx} \quad \text{et} \quad \left(\frac{dz}{dy}\right) = \alpha e^{\alpha y} X$$

erit

$$\frac{\alpha dX}{dx} + \frac{PdX}{dx} + \alpha QX + RX = 0,$$

unde reperitur

$$\frac{dX}{X} = \frac{-dx(\alpha Q + R)}{\alpha + P},$$

sicque elicitur pro quovis numero α idoneus valor ipsius X. Quare sumendis infinitis numeris α hoc modo expressio infinities infinitas determinationes recipiens colligitur

$$z = Ae^{\alpha y}X + Be^{\beta y}X' + Ce^{\gamma y}X'' + \text{etc.}$$

Verum tamen dantur etiam casus eiusmodi aequationum, quae solutiones vere completas admittunt, quarum rationem in sequente problemate indagemus.

PROBLEMA 47

287. *Proposita aequatione resolvenda*

$$\left(\frac{ddz}{dxdy}\right) + P\left(\frac{dz}{dx}\right) + Q\left(\frac{dz}{dy}\right) + Rz + S = 0$$

investigare, cuiusmodi functiones ipsarum x et y esse debeant quantitates P, Q, R et S, ut haec aequatio solutionem vere completam admittat.

SOLUTIO

Sit *V* functio quaecunque ipsarum *x* et *y* ac ponatur $z = e^{V}v$, ita uti iam *v* sit quantitas incognita, cuius valorem investigari oporteat. Cum igitur sit

$$\left(\frac{dz}{dx}\right) = e^{V}\left(\left(\frac{dv}{dx}\right) + v\left(\frac{dV}{dx}\right)\right), \quad \left(\frac{dz}{dy}\right) = e^{V}\left(\left(\frac{dv}{dy}\right) + v\left(\frac{dV}{dy}\right)\right),$$

facta substitutione totaque aequatione per e^{V} divisa prodibit sequens aequatio

$$\left.\begin{array}{r} e^{-V}S+\left(\frac{ddv}{dxdy}\right)+\left(\frac{dV}{dy}\right)\left(\frac{dv}{dx}\right)+\left(\frac{dV}{dx}\right)\left(\frac{dv}{dy}\right)+\left(\frac{dV}{dx}\right)\left(\frac{dV}{dy}\right)v \\ +\quad P\left(\frac{dv}{dx}\right)+\quad Q\left(\frac{dv}{dy}\right)+\quad \left(\frac{ddV}{dxdy}\right)v \\ +\quad P\left(\frac{dV}{dx}\right)v \\ +\quad Q\left(\frac{dV}{dx}\right)v \\ +\quad Rv \end{array}\right\}=0.$$

Efficiendum iam est, ut haec aequatio resolutionem completam admittat; cum igitur ante viderimus talem aequationem

$$\left(\frac{ddv}{dxdy}\right)+T\left(\frac{dv}{dx}\right)+e^{-V}S=0$$

generaliter resolvi posse, qualescunque etiam functiones ipsarum x et y pro T, S et V accipiantur, ad hanc aequationem illam redigamus. Necesse igitur est statui

$$P+\left(\frac{dV}{dy}\right)=T,\quad Q+\left(\frac{dV}{dx}\right)=0$$

et

$$R+Q\left(\frac{dV}{dy}\right)+P\left(\frac{dV}{dx}\right)+\left(\frac{dV}{dx}\right)\left(\frac{dV}{dy}\right)+\left(\frac{ddV}{dxdy}\right)=0,$$

unde obtinemus

$$P=T-\left(\frac{dV}{dy}\right),\quad Q=-\left(\frac{dV}{dx}\right)\ \text{ et }\ R=\left(\frac{dV}{dx}\right)\left(\frac{dV}{dy}\right)-T\left(\frac{dV}{dx}\right)-\left(\frac{ddV}{dxdy}\right).$$

Cum igitur per § 275 reperiatur

$$v=-\int e^{-\int Tdy}dx\int e^{\int Tdy-V}Sdy+\int e^{-\int Tdy}dxf:x+F:y,$$

erit aequationis propositae

$$\left(\frac{ddz}{dxdy}\right)+P\left(\frac{dz}{dx}\right)+Q\left(\frac{dz}{dy}\right)+Rz+S=0,$$

si modo litterae P, Q, R assignatos teneant valores, integrale eompletum

$$z=-e^{V}\int e^{-\int Tdy}dx\int e^{\int Tdy-V}Sdy+e^{V}\int e^{-\int Tdy}dxf:x+e^{V}F:y,$$

quandoquidem hic formae f: x et F: y functiones quascunque ipsius x et ipsius y denotant.

COROLLARIUM 1

288. Quaecunque ergo functiones ipsarum x et y pro litteris T et V accipiantur, inde oriuntur valores idonei pro litteris P, Q, R assumendi, ut aequatio resolutionem completam admittat; functio autem S arbitrio nostro relinquitur.

COROLLARIUM 2

289. Possunt etiam in aequatione proposita functiones P et Q indefinitae relinqui eritque tum

$$V=-\int Qdx \quad \text{et} \quad \left(\frac{dV}{dy}\right)=-\int dx\left(\frac{dQ}{dy}\right) \quad \text{atque} \quad \left(\frac{ddV}{dxdy}\right)=-\left(\frac{dQ}{dy}\right),$$

unde tantum quantitas R ita determinari debet, ut sit

$$R-PQ-\left(\frac{dQ}{dy}\right)=0 \quad \text{seu} \quad R=PQ+\left(\frac{dQ}{dy}\right).$$

COROLLARIUM 3

290. Quia hic pro $\int Qdx$ scribi potest $\int Qdx+Y$ denotante Y functionem quamcunque ipsius y, ob $V=-\int Qdx-Y$ complete integrabilis erit haec aequatio

$$\left(\frac{ddz}{dxdy}\right)+P\left(\frac{dz}{dx}\right)+Q\left(\frac{dz}{dy}\right)+\left(PQ+\left(\frac{dQ}{dy}\right)\right)z+S=0,$$

cuius integrale est

$$z=e^{-\int Qdx-Y}v$$

existente

$$\left(\frac{ddz}{dxdy}\right)+\left(P-\int dx\left(\frac{dQ}{dy}\right)-\frac{dY}{dy}\right)\left(\frac{dv}{dx}\right)+e^{-V}S=0$$

existente

$$T=P-\int dx\left(\frac{dQ}{dy}\right)-\frac{dY}{dy}$$

ac propterea

$$\int Tdy=\int Pdy-\int Qdx-Y,$$

unde valor ipsius v facile definitur.

SCHOLION

291. In hoc calculo, quo differentialia formularum integralium capi oportet, dum alia quantitas variabilis assumitur atque in integratione supponitur, haec regula est tenenda, quodsi fuerit $V=\int Qdx$, fore $\left(\frac{dV}{dy}\right)=\int dx\left(\frac{dQ}{dy}\right)$. Cum enim sit $\left(\frac{dV}{dx}\right)=Q$, erit $\left(\frac{ddV}{dxdy}\right)=\left(\frac{dQ}{dy}\right)$. Quodsi ergo statuatur $\left(\frac{dV}{dy}\right)=S$, erit $\left(\frac{dS}{dx}\right)=\left(\frac{dQ}{dy}\right)$ et $S=\left(\frac{dV}{dy}\right)=\int dx\left(\frac{dQ}{dy}\right)$; unde vicissim colligitur, si fuerit $S=\int dx\left(\frac{dQ}{dy}\right)$, fore ob $\int Sdy=V$ integrando $\int Sdy=\int Qdx$; quod cum ex principiis ante [§50 et §51] stabilitis per se sit manifestum, non opus esse iudico pro hoc quasi novo algorithmi genere praecepta seorsim tradere.

Videamus autem in aliquot exemplis, cuiusmodi aequationes ope huius methodi complete resolvere liceat.

EXEMPLUM 1

292. *Proposita aequatione differentio-differentiali*

$$\left(\frac{ddz}{dxdy}\right)+a\left(\frac{dz}{dx}\right)+b\left(\frac{dz}{dy}\right)+Rz+S=0$$

definire indolem functionis R, ut haec aequatio resolutionem admittat existente S functione quacunque ipsarum x et y.

Cum sit $P=a$ et $Q=b$, erit $R=ab$ et $V=-bx$; tuto enim functio Y omitti potest, quia in sequente integratione iam binae functiones arbitrariae introducuntur; erit $T=a$. Unde posito $z=e^{-bx}v$ habebitur haec aequatio

$$\left(\frac{ddv}{dxdy}\right)+a\left(\frac{dv}{dx}\right)+e^{bx}S=0$$

ac posito $\left(\frac{dv}{dx}\right)=u$ fit

$$\left(\frac{du}{dy}\right)+au+e^{bx}S=0$$

et sumto x constante

$$e^{ay}u=-\int e^{ay+bx}Sdy+f':x,$$

ergo

$$u=\left(\frac{dv}{dx}\right)=-e^{-ay}\int e^{ay+bx}Sdy+e^{-ay}f':x$$

et sumto iam y constante

$$v=-e^{-ay}\int dx\int e^{ay+bx}Sdy+e^{-ay}f':x+F:x$$

sumendo $\int dxf':x=f:x$. Quodsi iam pro $e^{-bx}f:x$ scribatur $f:x$, erit

$$z = -e^{-ay-bx}\int dx\int e^{ay+bx}Sdy + e^{-ay}f:x + e^{-bx}F:x.$$

ALITER

Si sumsissemus $V = -bx - ay$, prodiisset $T = a - a = 0$ ideoque posito $z = -e^{-bx-ay}v$ quantitas v ex hac aequatione

$$\left(\frac{ddv}{dxdy}\right) + e^{bx+ay}S = 0$$

definiri deberet, quae dat

$$\left(\frac{dv}{dx}\right) = -\int e^{bx+ay}Sdy + f':x \quad \text{et} \quad v = -\int dx\int e^{bx+ay}Sdy + f:x + F:y$$

et

$$z = e^{-bx-ay}\left(-\int dx\int e^{bx+ay}Sdy + f:x + F:y\right),$$

quae forma simplicior est praecedente, etiamsi eodem redeat, estque hoc integrale completum aequationis

$$\left(\frac{ddz}{dxdy}\right) + a\left(\frac{dz}{dx}\right) + b\left(\frac{dz}{dy}\right) + abz + S = 0\,.$$

EXEMPLUM 2

293. *Proposita aequatione differentio-differentiali*

$$\left(\frac{ddz}{dxdy}\right) + \frac{a}{y}\left(\frac{dz}{dx}\right) + \frac{b}{x}\left(\frac{dz}{dy}\right) + Rz + S = 0$$

definire indolem functionis R, ut haec aequatio resolutionem admittat existente S functione quacunque ipsarum x et y.

Cum sit $P = \frac{a}{y}$ et $Q = \frac{b}{x}$, erit $V = -blx - Y$ hincque $R = \frac{ab}{xy}$ et aequatio integrabilis erit

$$\left(\frac{ddz}{dxdy}\right) + \frac{a}{y}\left(\frac{dz}{dx}\right) + \frac{b}{x}\left(\frac{dz}{dy}\right) + \frac{ab}{xy}z + S = 0.$$

Quoniam igitur fit

$$T = P + \left(\frac{dV}{dy}\right) = \frac{a}{y} - \left(\frac{dY}{dy}\right),$$

sumamus $Y = +aly$, ut fiat $T = 0$, ac posito

$$z = e^{-blx-aly}v = x^{-b}y^{-a}v$$

quantitas v ex hac aequatione definiri debet

$$\left(\frac{ddz}{dxdy}\right) + x^b y^a S = 0,$$

unde fit

$$\left(\frac{dv}{dx}\right) = -x^b \int y^a S dy + f':x \quad \text{et} \quad v = -\int x^b dx \int y^a S dy + f:x + F:x$$

ideoque

$$z = \frac{-\int x^b dx \int y^a S dy + f:x + F:x}{x^a y^b}.$$

SCHOLION 1

294. Hinc igitur patet hanc aequationem ope istius methodi in genere integrari posse

$$\left(\frac{ddz}{dxdy}\right) + P\left(\frac{dz}{dx}\right) + Q\left(\frac{dz}{dy}\right) + \left(PQ + \left(\frac{dQ}{dy}\right)\right)z + S = 0,$$

quaecunque functiones ipsarum x et y pro P, Q et S accipiantur. Ac resolutio quidem ita se habet, ut posito $z = e^{-\int Qdx - Y}v$ haec quantitas v determinetur hac aequatione

$$\left(\frac{ddv}{dxdy}\right) + \left(P - \int dx\left(\frac{dQ}{dy}\right) - \frac{dY}{dy}\right)\left(\frac{dv}{dx}\right) + e^{\int Qdx + Y}S = 0,$$

ubi iam pro y talis functio ipsius y accipi potest, ut huius aequationis forma simplicissima evadat, id quod potissimum evenit, si expressio

$$P - \int dx\left(\frac{dQ}{dy}\right) - \frac{dY}{dy}$$

ad nihilum redigi queat. In genere autem reperitur

$$v = -\int e^{-\int Pdy + \int Qdx + Y} dx \int e^{\int Pdy} S dy + \int e^{-\int Pdy + \int Qdx + Y} dx f:x + F:y,$$

qui valor ergo per $e^{-\int Qdx - Y}$ multiplicatus praebet formam functionis z. Hoc modo autem functio Y ab arbitrio nostro pendens penitus e calculo egreditur fitque

$$z = -e^{-\int Qdx}\int e^{-\int Pdy + \int Qdx} dx \int e^{\int Pdy} S dy + e^{-\int Qdx}\int e^{-\int Pdy + \int Qdx} dx f:x + e^{-\int Qdx} F:y,$$

quod est integrale completum huius aequationis

$$\left(\frac{ddz}{dxdy}\right)+P\left(\frac{dz}{dx}\right)+Q\left(\frac{dz}{dy}\right)+\left(PQ+\left(\frac{dQ}{dy}\right)\right)z+S=0.$$

SCHOLION 2

295. Permutandis autem variabilibus x et y etiam haec aequatio complete integrari potest

$$\left(\frac{ddz}{dxdy}\right)+P\left(\frac{dz}{dx}\right)+Q\left(\frac{dz}{dy}\right)+\left(PQ+\left(\frac{dP}{dx}\right)\right)z+S=0.$$

cuius integrale erit

$$z=-e^{-\int Pdy}\int e^{-\int Qdx+\int Pdy}dy\int e^{\int Qdx}Sdx+e^{-\int Pdy}\int e^{-\int Qdx+\int Pdy}dy\, f:y+e^{-\int Pdy}F:x,$$

ubi praecipue hic casus in utraque forma contentus notari meretur, si fuerit $P=Y$ et $Q=X$ existente X functione ipsius x et Y ipsius y tantum; tum enim huius aequationis

$$\left(\frac{ddz}{dxdy}\right)+Y\left(\frac{dz}{dx}\right)+X\left(\frac{dz}{dy}\right)+XYz+S=0.$$

integrale completum erit

$$z=-e^{-\int Xdx-\int Ydy}\int e^{\int Xdx}dx\int e^{\int Ydy}Sdy+e^{-\int Xdx-\int Ydy}\left(f:x+F:y\right),$$

quod etiam ita exhiberi potest

$$e^{\int Xdx+\int Ydy}z=f:x+F:y-\int e^{\int Xdx}dx\int e^{\int Ydy}Sdy,$$

vel etiam hoc modo

$$e^{\int Xdx+\int Ydy}z=f:x+F:y-\int e^{\int Ydy}dy\int e^{\int Xdx}Sdx.$$