CHAPTER II

CONCERNING THE INTEGRATION OF EQUATIONS OF HIGHER ORDERS BY REDUCTION TO LOWER ORDERS

PROBLEM 64

395. *With this proposed equation of the third order* $\left(\frac{d^3z}{dx^3}\right)=a^3z$, *to investigate the nature of the function z.*

SOLUTION

To satisfy this equation this simpler equation of the first order is devised

$$\left(\frac{dz}{dx}\right)=nz\,,$$

and since hence on being differentiated there is obtained

$$\left(\frac{ddz}{dx^2}\right)=n\left(\frac{dz}{dx}\right)=nnz$$

and hence again

$$\left(\frac{d^3z}{dx^3}\right)=nn\left(\frac{dz}{dx}\right)=n^3z\,,$$

it is evident for the question to be satisfied, provided there shall be $n^3=a^3$, and that can come about in three ways:

$$\text{I.}\quad n=a,\ \text{II.}\quad n=\frac{-1+\sqrt{-3}}{2}a,\quad \text{III.}\quad n=\frac{-1-\sqrt{-3}}{2}a\,.$$

Therefore for any value the complete integral of the equation $\left(\frac{dz}{dx}\right)=nz$ is sought, and these three integrals taken together will give the complete integral of the proposed equation. But since in the equation $\left(\frac{dz}{dx}\right)=nz$ the quantity y is taken constant, then there will be

$$dz=nzdx \quad \text{or} \quad \frac{dz}{z}=ndx,$$

from which there becomes

$$lz=nx+l\Gamma{:}y \quad \text{or} \quad z=e^{nx}\Gamma{:}y\,.$$

Now the three values of n are given, and for the proposed equation there will be

$$z = e^{ax}\Gamma{:}y + e^{\frac{-1+\sqrt{-3}}{2}ax}\varDelta{:}y + e^{\frac{-1-\sqrt{-3}}{2}ax}\Sigma{:}y.$$

But since there shall be

$$e^{m\sqrt{-1}} = \cos.m + \sqrt{-1}.\sin.m,$$

on changing the form of the arbitrary functions there will be

$$z = e^{ax}\Gamma{:}y + e^{-\frac{1}{2}ax}\cos.\tfrac{ax\sqrt{3}}{2}\cdot\varDelta{:}y + e^{-\frac{1}{2}ax}\cdot\sin.\tfrac{ax\sqrt{3}}{2}\Sigma{:}y.$$

COROLLARY 1

396. This integral can be represented also in the form

$$z = e^{ax}\Gamma{:}y + e^{-\frac{1}{2}ax}\varDelta{:}y\cdot\cos.\left(\tfrac{ax\sqrt{3}}{2} + Y\right)$$

with Y denoting some function of y.

COROLLARY 2

397. Because there is a need for three integrations and in the individual cases the quantity y it treated as a constant, following the precepts of the first book, this equation $d^3z = a^3zdx^3$ may be resolved and in place of the three constants some functions of y may be introduced, from which the same solution is elicited.

PROBLEM 65

398. *With this proposed equation of any order*

$$Pz + Q\left(\tfrac{dz}{dx}\right) + R\left(\tfrac{ddz}{dx^2}\right) + S\left(\tfrac{d^3z}{dx^3}\right) + T\left(\tfrac{d^4z}{dx^4}\right) + \text{etc.} = 0,$$

where the letters P, Q, R, S, T etc. denote some functions of the two variables x and y, to define the nature of the function z.

SOLUTION

Since in all the integrations being put in place the quantity y shall be regarded always as constant, and this equation may be agreed to depend only on the two variables x and z. Whereby by the precepts of the first book, the equation being treated will be

$$Pz + \frac{Qdz}{dx} + \frac{Rddz}{dx^2} + \frac{Sd^3z}{dx^3} + \frac{Td^4z}{dx^4} + \text{etc.} = 0;$$

the resolution of which should succeed, only if there is a requirement that in place of the constants introduced by the individual integrations, some functions of *y* may be written down; and thus that desired complete integral may be had, if indeed this equation should be permitted to be integrated completely.

COROLLARY 1

399. Therefore if the letters *P, Q, R, S* etc. shall be constants or involve the variable *y* only, the integration will succeed always, because in the first book we have shown how to integrate equations of this kind.

COROLLARY 2

400. Then also the resolution of this equation will succeed [Ch. V, Section II]

$$Az + Bx\left(\frac{dz}{dx}\right) + Cx^2\left(\frac{ddz}{dx^2}\right) + Dx^3\left(\frac{d^3z}{dx^3}\right) + \text{etc.} = 0,$$

if the letters *A, B, C* etc. shall be constants, or functions of *y* only.

COROLLARY 3

401. Then also truly, if these forms shall not be equal to zero, but equal to some function of *x* and *y*, the resolution is no less successful for those, which have been set out in the last chapters of the first book.

SCHOLIUM

402. These also can be extended much wider clearly to all equations, in which no other differential formulas occur except these

$$\left(\frac{dz}{dx}\right), \quad \left(\frac{ddz}{dx^2}\right), \quad \left(\frac{d^3z}{dx^3}\right) \quad \text{etc.},$$

which only involve *x* as the variable. However indeed should these formulas be involved with the finite quantities *x, y* and *z*, the equation pertaining to the first book always is to be understood, because in all the integrations put in place the quantity *y* is treated always as a constant. Finally with the integrations in this accomplished a distinction is put in place, as in place of the arbitrary constants arbitrary functions of *y* are introduced into the calculation. It would be superfluous to advise here, what these functions of the other variable *y* have been called, also what is to be understood about these functions of the other variable *x*.

PROBLEM 66

403. *With this equation proposed*

$$\left(\frac{ddz}{dx^2}\right)+bx\left(\frac{ddz}{dxdy}\right)-2a\left(\frac{dz}{dx}\right)-ab\left(\frac{dz}{dy}\right)+aaz=0$$

to investigate the nature of the function z.

SOLUTION

It is readily apparent that this simple equation $\left(\frac{dz}{dx}\right)=az$, from which there becomes $z=e^{ax}$, satisfies this equation [above]. Therefore we may put in place $z=e^{ax}v$ and there becomes

$$\left(\frac{dz}{dx}\right)=e^{ax}\left(av+\left(\frac{dv}{dx}\right)\right),\quad \left(\frac{dz}{dy}\right)=e^{ax}\left(\frac{dv}{dy}\right)$$

and hence

$$\left(\frac{ddz}{dx^2}\right)=e^{ax}\left(aav+2a\left(\frac{dv}{dx}\right)+\left(\frac{ddv}{dx^2}\right)\right)$$

and

$$\left(\frac{ddz}{dxdy}\right)=e^{ax}\left(a\left(\frac{dv}{dy}\right)+\left(\frac{ddv}{dxdy}\right)\right),$$

with which values substituted and with the equation divided by e^{ax} we will have

$$\left(\frac{ddv}{dx^2}\right)+b\left(\frac{ddv}{dxdy}\right)=0.$$

Because now there occurs everywhere $\left(\frac{dv}{dx}\right)$, we may make $\left(\frac{dv}{dx}\right)=u$; there will be

$$\left(\frac{du}{dx}\right)+b\left(\frac{du}{dy}\right)=0\,,$$

the integral of which is $f:(y-bx)=u$ [§79]; therefore we may write

$$u=\left(\frac{dv}{dx}\right)=-b\Gamma':(y-bx)$$

so that there emerges $v=\Gamma:(y-bx)+\Delta:y$, and thus the integral sought will be

$$z=e^{ax}\left(\Gamma:(y-bx)+\Delta:y\right),$$

which form is the complete integral on account of the two arbitrary functions.

PROBLEM 67

404. *With this equation proposed*

$$0=(a+2b)z-(2a+3b)\left(\frac{dz}{dx}\right)+c\left(\frac{dz}{dy}\right)+a\left(\frac{ddz}{dx^2}\right)-2c\left(\frac{ddz}{dxdy}\right)+b\left(\frac{d^3z}{dx^3}\right)+c\left(\frac{d^3z}{dx^2dy}\right)$$

to investigate the nature of the function z.

SOLUTION

This equation has been prepared thus, so that plainly it is satisfied by $z=e^x$; therefore we may put in place $z=e^x v$ and there becomes

$$\left(\frac{dz}{dx}\right)=e^x\left(v+\left(\frac{dv}{dx}\right)\right),\quad \left(\frac{dz}{dy}\right)=e^x\left(\frac{dv}{dy}\right)$$

$$\left(\frac{ddz}{dx^2}\right)=e^x\left(v+2\left(\frac{dv}{dx}\right)+\left(\frac{ddv}{dx^2}\right)\right),\quad \left(\frac{ddz}{dxdy}\right)=e^x\left(\left(\frac{dv}{dy}\right)+\left(\frac{ddv}{dxdy}\right)\right),$$

$$\left(\frac{d^3z}{dx^3}\right)=e^x\left(v+3\left(\frac{dv}{dx}\right)+3\left(\frac{ddv}{dx^2}\right)+\left(\frac{d^3v}{dx^3}\right)\right),$$

$$\left(\frac{d^3z}{dx^2dy}\right)=e^x\left(\left(\frac{dv}{dy}\right)+2\left(\frac{ddv}{dxdy}\right)+\left(\frac{d^3v}{dx^2dy}\right)\right),$$

with which values put in place this simple equation appears

$$0=(a+3b)\left(\frac{ddv}{dx^2}\right)+b\left(\frac{d^3v}{dx^3}\right)+c\left(\frac{d^3v}{dx^2dy}\right),$$

In which it comes about conveniently, that the formula $\left(\frac{ddv}{dx^2}\right)$ is contained in the individual terms, whereby on putting $\left(\frac{ddv}{dx^2}\right)=u$ this equation of the first order is produced

$$0=(a+3b)u+b\left(\frac{du}{dx}\right)+c\left(\frac{du}{dy}\right)$$

from which it is apparent, if there is put $du=pdx+qdy$, there must be

$$(a+3b)u+bp+cq=0,$$

which may be resolved thus.

Since on putting $a+3b=f$ there shall be $q=-\frac{bp}{c}-\frac{fu}{c}$, there will be

$$du=pdx-\frac{bpdy}{c}-\frac{fudy}{c}$$

or

$$dx-\frac{bdy}{c}=\frac{1}{p}\left(du+\frac{fudy}{c}\right)=\frac{u}{p}\left(\frac{du}{u}+\frac{fdy}{c}\right)$$

and thus it is necessary that $\frac{u}{p}$ shall be a function of $x-\frac{by}{c}$ [since an integrable quantity arises from $\frac{du}{u}+\frac{fdy}{c}$, which is equal to $\frac{dx-\frac{bdy}{c}}{\left(\frac{u}{p}\right)}$], from which there becomes

$$lu+\frac{fy}{c}=f:(cx-by)$$

and

$$u=e^{-\frac{fy}{c}}\Gamma'':\left(x-\frac{by}{c}\right)=\left(\frac{ddv}{dx^2}\right).$$

Now on account of constant y, considering that the first integration gives

$$\left(\frac{dv}{dx}\right)=e^{-\frac{fy}{c}}\Gamma':\left(x-\frac{by}{c}\right)+\Delta:y$$

and the other

$$v=e^{-\frac{fy}{c}}\Gamma:\left(x-\frac{by}{c}\right)+x\Delta:y+\Sigma:y\,.$$

Whereby on putting $a+3b=f$ the complete integral of the proposed equation is

$$z=e^{x-\frac{fy}{c}}\Gamma:\left(x-\frac{by}{c}\right)+e^{x}x\Delta:y+e^{x}\Sigma:y\,.$$

PROBLEM 67A

405. *With this proposed differential equation of the third order*

$$0 = Pz - 3P\left(\frac{dz}{dx}\right) + 3P\left(\frac{ddz}{dx^2}\right) - P\left(\frac{d^3z}{dx^3}\right) + Q\left(\frac{dz}{dy}\right) - 2Q\left(\frac{ddz}{dxdy}\right) + Q\left(\frac{d^3z}{dx^2dy}\right)$$

where P and Q shall be some functions of x and y themselves, to investigate the nature of the function z.

SOLUTION

With the substitution $z = e^x v$ made, since from the given form it is easily seen to be satisfied by the value e^x put in place of *z*, this equation is arrived at:

$$-P\left(\frac{d^3v}{dx^3}\right) + Q\left(\frac{d^3v}{dx^2dy}\right) = 0,$$

which again on putting $\left(\frac{ddv}{dx^2}\right) = u$, so that there shall be $v = \iint udx^2$, will change into this

$$-P\left(\frac{du}{dx}\right) + Q\left(\frac{du}{dy}\right) = 0$$

We may put $du = pdx + qdy$; there will be $Qq = Pp$, hence $q = \frac{Pp}{Q}$ and thus

$$du = p\left(dx + \frac{P}{Q}dy\right),$$

from which it is understood that the quantity *p* must be prepared thus, so that the formula

$$dx + \frac{P}{Q}dy$$

multiplied by that becomes integrable. Therefore a multiplier *M* is sought rendering the formula $Qdx + Pdy$ integrable, thus so that there shall be

$$\int M\left(Qdx + Pdy\right) = s;$$

therefore so that a function *s* of *x* and *y* can be found, and on account of

$$Qdx + Pdy = \frac{ds}{M}$$

we will have $du = \frac{pds}{MQ}$, from which it is apparent that $\frac{p}{MQ}$ denotes a function of the quantity *s*. Hence on putting $\frac{p}{MQ} = \Gamma':s$ there will be at once $u = \Gamma:s$ and hence $v = \int dx \int dx \Gamma:s$, in which with each integration the quantity *y* is regarded as constant. On account of which the resolution of the problem itself thus will be had:

For the formula of the differential $Qdx + Pdy$ the multiplier *M* is sought rendering that integrable, so that there shall be

$$M\left(Qdx + Pdy\right) = ds,$$

and with this function *s* of *x* and *y* found there will be

$$z = e^x \int dx \int dx \Gamma:s + e^x x \Delta:y + e^x \Sigma:y$$

SCHOLIUM

406. In these equations this suitable method of using has come about, that on making the substitution $z = e^x v$ they adopt a form of this kind, which again is able to be recalled according to the simple kind considered in the first section ; for even if the differentials of the third order may not have vanished, yet the remaining members thus have passed out of the calculation, so that henceforth with a new substitution $\left(\frac{ddv}{dx^2}\right) = u$, by the aid of this it is possible to arrive at a differential equation of the first order. Therefore by a single substitution this is able to fulfill the requirements, if at once we can put $z = e^x \iint udx^2$. Would that the precepts might be had, with the aid of which substitutions of this kind could be discerned readily!

Meanwhile at last [we end] with a problem of much wider extent that can be resolved by calling on the aid of § 209:

PROBLEM 67 B

407. *With this equation of the third order proposed*

$$0=(P+Q)z-(2P+3Q)\left(\frac{dz}{dx}\right)+(P+3Q)\left(\frac{ddz}{dx^2}\right)-Q\left(\frac{d^3z}{dx^3}\right)$$
$$-R\left(\frac{dz}{dy}\right)+2R\left(\frac{ddz}{dxdy}\right)-R\left(\frac{d^3z}{dx^2dy}\right)$$

where P, Q and R shall be some given functions of x and y, to investigate the nature of the function z.

SOLUTION

With the same substitution made $z=e^{x}v$, which we have used until now, the equation proposed is changed into the following

$$0=P\left(\frac{ddv}{dx^2}\right)-Q\left(\frac{d^3v}{dx^3}\right)-R\left(\frac{d^3v}{dx^2dy}\right)$$

where it comes about conveniently, as on putting $\left(\frac{ddv}{dx^2}\right)=u$ that the equation of the first order emerges :

$$0=Pu-Q\left(\frac{du}{dx}\right)-R\left(\frac{du}{dy}\right),$$

from which *u* shall be a function of such a kind of *x* and *y* required to be sought.

We may put to be $du=pdx+qdy$, and because now that condition may be given

$$Pu=Qp+Rq,$$

following the artifice used above in §209, hence we may form these three following equations

$$Ldu=Lpdx+Lqdy,$$
$$MPudx=MQpdx+MRqdx,$$
$$NPudy=NQpdy+NRqdy,$$

which gathered together into one sum will give

$$Ldu+Pu(Mdx+Ndy)=p((L+MQ)dx+NQdy)+q((L+NR)dy+MRdx);$$

where since the three quantities *L*, *M* and *N* may depend on our choice, between these in the first place there is put in place a relation of this kind, so that the two latter members may obtain a common factor, clearly there shall be

$$L+MQ:NQ=MR:L+NR \text{ or } L=-MQ-NR,$$

[$\left(L+MQ\right)\left(L+RN\right)=NMQR, L\left(L+MQ+NR\right)=0,$ or $L=-MQ-NR$]

and we will have

$$-du\left(MQ+NR\right)+Pu\left(Mdx+Ndy\right)=\left(Mq-Np\right)\left(Rdx-Qdy\right).$$

The multiplier T is sought rendering the formula $Rdx-Qdy$ integrable, so that there shall be $T\left(Rdx-Qdy\right)=ds$, from which both the function T as well as s can be considered as known, and because now we have

$$-du\left(MQ+NR\right)+Pu\left(Mdx+Ndy\right)=\left(Mq-Np\right)\tfrac{ds}{T}$$

or

$$\tfrac{du}{u}-\tfrac{P\left(Mdx+Ndy\right)}{MQ+NR}=\tfrac{Np-Mq}{u\left(MQ+NR\right)}\cdot\tfrac{ds}{T},$$

now, since *P*, *Q*, *R* shall be given functions of x and *y*, it is to be noted properly that a relation of this kind can be put in place between the two variables not yet defined *M* and *N*, so that the formula $\tfrac{P\left(Mdx+Ndy\right)}{MQ+NR}$ admits to integration; therefore let the integral of this be $=lw$, thus so that there shall be

$$Mdx+Ndy=\tfrac{MQ+NR}{P}\cdot\tfrac{dw}{w} \text{ and } \tfrac{du}{u}=\tfrac{dw}{w}+\tfrac{Np-Mq}{Tu\left(MQ+NR\right)}ds.$$

Therefore by necessity the quantities *p* and *q* thus shall be prepared, so that there becomes

$$\tfrac{Np-Mq}{Tu\left(MQ+NR\right)}=f':s$$

and hence $lu=lw+f:s$. In place of $f:s$ we may write $l\Gamma:s$, so that there may arise $u=w\Gamma:s$ and therefore

$$v=\int dx\int wdx\Gamma:s+x\Delta:y+\Sigma:y.$$

Consequently

$$z = e^x \int dx \int w dx \Gamma : s + e^x x \Delta : y + e^x \Sigma : y.$$

COROLLARY 1

408. Therefore according to this solution from the proposed form at once in the first place a function of this kind of *x* and *y* is sought to be elicited, which may be called *s*, so that there becomes

$$ds = T\left(Rdx - Qdy\right),$$

because that arranges the multiplier *T* required to be found, from which the formula of the differential $Rdx - Qdy$ is returned integrable.

COROLLARY 2

409. Now in addition it is required to investigate the quantity *w*. To that end it is agreed to find a ratio of this kind between the quantities *M* and *N*, so that there becomes

$$\int \frac{P(Mdx+Ndy)}{MQ+NR} = lw,$$

which investigation always is to be conceded.

SCHOLIUM

410. Since the whole calculation at once shall lead from that, so that the function *u* must be defined from this equation

$$Pu = Q\left(\frac{du}{dx}\right) + R\left(\frac{du}{dy}\right)$$

without ambiguity, from which in the solution I have used, the solution can be absolved much easier by the following way, that which gives a significant addition to the first section [§ 209].

There is put in place

$$\left(\frac{du}{dx}\right) = LMu \text{ and } \left(\frac{du}{dy}\right) = LNu;$$

initially there will be $P = L\left(MQ + NR\right)$, hence $L = \frac{P}{MQ+NR}$, then on account of

$$du = dx\left(\frac{du}{dx}\right) + dy\left(\frac{du}{dy}\right)$$

we will have

$$\frac{du}{u} = L\left(Mdx + Ndy\right) = \frac{P(Mdx+Ndy)}{MQ+NR}$$

where it is required thus to take *M* and *N*, in order that the integration may succeed, which since it can be done in innumerable ways, hence the complete solution is considered to be obtained.

Truly provided that the case of the particular integral may be in agreement, the complete solution may be elicited by a much more convenient method. Clearly on putting

$$\frac{dw}{w} = \frac{P(Mdx+Ndy)}{MQ+NR},$$

thus so that the value of w for u now taken is satisfied particularly, and there shall be

$$Pw = Q\left(\frac{dw}{dx}\right) + R\left(\frac{dw}{dy}\right),$$

we may put in place $u = w\Gamma{:}s$ for the complete value and with the substitution made we follow

$$pw\Gamma{:}s = Q\left(\frac{dw}{dx}\right)\Gamma{:}s + R\left(\frac{dw}{du}\right)\Gamma{:}s + Qw\left(\frac{ds}{dx}\right)\Gamma'{:}s + Rw\left(\frac{ds}{dy}\right)\Gamma'{:}s,$$

which equation at once contracts into this :

$$Q\left(\frac{ds}{dx}\right) + R\left(\frac{ds}{dy}\right) = 0,$$

from which we conclude

$$\left(\frac{ds}{dx}\right) = TR \text{ and } \left(\frac{ds}{dy}\right) = -TQ$$

and therefore

$$ds = T(Rdx - Qdy),$$

from which it is apparent that the quantity s be found from the $Rdx - Qdy$, for which the first factor T to be sought returns that integrable, then truly the integral of this must be taken for s. Therefore in the first place this must be attended to, so that the same solution may be allowed to be elicited neatly, to which we may arrive at by so many roundabout ways.

PROBLEM 68

411. *With this differential equation of the fourth order proposed*

$$\left(\frac{d^4z}{dy^4}\right) = aa\left(\frac{ddz}{dx^2}\right),$$

at least of the function z found to reduce the resolution of simpler equations.

SOLUTION

This equation soon will become apparent on more careful observation for it to satisfy a simpler equation of this kind

$$\left(\frac{ddz}{dy^2}\right) = b\left(\frac{dz}{dx}\right)$$

for hence on differentiation by y it becomes

$$\left(\frac{d^3z}{dy^3}\right) = b\left(\frac{ddz}{dxdy}\right)$$

and again in the same manner

$$\left(\frac{d^4z}{dy^4}\right) = b\left(\frac{d^3z}{dxdy^2}\right)$$

but from that assumed by the differentiation through x there will be produced

$$\left(\frac{d^3z}{dxdy^2}\right) = b\left(\frac{ddz}{dx^2}\right),$$

from which value introduced there, it may be deduced

$$\left(\frac{d^4z}{dy^4}\right) = bb\left(\frac{ddz}{dx^2}\right),$$

which form agrees with the proposition, provided there shall be $bb = aa$; since which is able to come about in a twofold manner, $b = +a$ and $b = -a$, after this we must resolve these simpler equations

$$\left(\frac{ddz}{dy^2}\right) - a\left(\frac{dz}{dx}\right) = 0$$

which gives $z = P$,

$$\left(\frac{ddz}{dy^2}\right)+a\left(\frac{dz}{dx}\right)=0$$

which gives $z=Q$, for the proposed equation there will be $z=P+Q$, and because both P and Q involve two arbitrary functions, the integral found in this way involves four functions of this kind and thus will be the complete integral.

COROLLARY 1

412. Boundless particular solutions may be found easily on putting $z=e^{\mu x+\nu y}$. For with the substitution made it is necessary that there becomes

$$\nu^4=\mu\mu aa \quad \text{and} \quad \mu=\pm\frac{\nu\nu}{a}.$$

Let $\nu=\lambda a$; then there will be $\mu=\pm\lambda\lambda a$ and satisfying the integral $z=e^{\lambda a(y\pm\lambda x)}$.

COROLLARY 2

413. Also there can be put $z=e^{\mu x}\cos.(\nu y+\alpha)$, from which there is made $\nu^4=\mu\mu aa$ as before, thus so that another form of the particular integral shall be

$$z=e^{\pm\lambda\lambda ax}\cos.(\lambda ay+\alpha).$$

Endless formulas of this kind taken together are to be considered as if exhausting the complete integral.

COROLLARY 3

414. The same solutions are found on putting more generally $z=XY$, from which there becomes

$$\frac{Xd^4Y}{dy^4}=\frac{aaYddX}{dx^2}$$

with which equation represented thus

$$\frac{d^4Y}{Ydy^4}=\frac{aaddX}{Xdx^2}$$

each member must be equal to the same constant.

SCHOLIUM

415. But the equation, according to which we have reduced the whole calculation,

$$\left(\frac{ddz}{dy^2}\right)=b\left(\frac{dz}{dx}\right)$$

from these is a number, which in no way in general may be seen to be resolved, thus so that we must be content with particular solutions.

But the proposition has not been posed from pure speculation, for, when small vibrations of an elastic plate are investigated in general, there is come upon an equation of the fourth order requiring to be resolved [E443] , which also is the cause, since this question and likewise the vibrations of strings has not been able to be resolved at this stage in general.

But in a similar manner this equation of the fourth order is readily understood

$$\left(\frac{d^4z}{dy^4}\right)=aa\left(\frac{ddz}{dx^2}\right)+2ab\left(\frac{dz}{dx}\right)+bbz$$

to be reduced to this pair of second order equations

$$\left(\frac{ddz}{dy^2}\right)=\pm a\left(\frac{dz}{dx}\right)\pm bz$$

nor is it difficult to elicit other cases to follow, where a place is found for reductions of this kind to a lesser order.

CAPUT II

DE INTEGRATIONE AEQUATIONUM ALTIORUM PER REDUCTIONEM AD INFERIORES

PROBLEMA 64

395. *Proposita hac aequatione tertii gradus* $\left(\frac{d^3z}{dx^3}\right) = a^3z$ *indolem functionis z investigare.*

SOLUTIO

Fingatur huic aequationi satisfacere haec simplicior primi gradus

$$\left(\frac{dz}{dx}\right) = nz\,,$$

et cum hinc differentiando obtineatur

$$\left(\frac{ddz}{dx^2}\right) = n\left(\frac{dz}{dx}\right) = nnz$$

hincque porro

$$\left(\frac{d^3z}{dx^3}\right) = nn\left(\frac{dz}{dx}\right) = n^3z\,,$$

evidens est quaesito satisfieri, dum sit $n^3 = a^3$, id quod triplici modo evenire potest:

$$\text{I.}\quad n = a,\ \text{II.}\quad n = \frac{-1+\sqrt{-3}}{2}a,\quad \text{III.}\quad n = \frac{-1-\sqrt{-3}}{2}a\,.$$

Pro quolibet ergo valore quaeratur integrale completum aequationis $\left(\frac{dz}{dx}\right) = nz$ et tria haec integralia coniuncta praebebunt integrale completum aequationis propositae. Cum autem in aequatione $\left(\frac{dz}{dx}\right) = nz$ quantitas y constans sumatur, erit

$$dz = nzdx \quad \text{seu} \quad \frac{dz}{z} = ndx,$$

unde fit

$$lz = nx + l\mathit{\Gamma}:y \quad \text{seu} \quad z = e^{nx}\mathit{\Gamma}:y\,.$$

Tribuantur iam ipsi n terni valores eritque pro aequatione proposita

$$z = e^{ax}\mathit{\Gamma}:y + e^{\frac{-1+\sqrt{-3}}{2}ax}\mathit{\Delta}:y + e^{\frac{-1-\sqrt{-3}}{2}ax}\mathit{\Sigma}:y\,.$$

Cum autem sit

$$e^{m\sqrt{-1}} = \cos.m + \sqrt{-1}.\sin.m\,,$$

erit functionum arbitrariarum formam mutando

$$z = e^{ax}\Gamma{:}y + e^{-\frac{1}{2}ax}\cos.\tfrac{ax\sqrt{3}}{2}\cdot \Delta{:}y + e^{-\frac{1}{2}ax}\cdot \sin.\tfrac{ax\sqrt{3}}{2}\,\Sigma{:}y\,.$$

COROLLARIUM 1

396. lntegrale hoc etiam ita repraesentari potest

$$z = e^{ax}\Gamma{:}y + e^{-\frac{1}{2}ax}\Delta{:}y\cdot \cos.\left(\tfrac{ax\sqrt{3}}{2} + Y\right)$$

denotante *Y* functionem quamcunque ipsius *y*.

COROLLARIUM 2

397. Quia tribus integrationibus est opus et in singulis quantitas *y* ut constans tractatur, secundum praecepta libri primi haec aequatio $d^3z = a^3zdx^3$ resolvatur et loco trium constantium functiones quaecunque ipsius *y* introducantur, unde eadem solutio elicitur.

PROBLEMA 65

398. *Proposita hac aequatione cuiuscunque gradus*

$$Pz + Q\left(\tfrac{dz}{dx}\right) + R\left(\tfrac{ddz}{dx^2}\right) + S\left(\tfrac{d^3z}{dx^3}\right) + T\left(\tfrac{d^4z}{dx^4}\right) + \text{etc.} = 0\,,$$

ubi litterae P, Q, R, S, T etc. functiones denotant quascunque binarum variabilium x et y, indolem functionis z definire.

SOLUTIO

Cum in omnibus integrationibus instituendis quantitas *y* perpetuo ut constans spectetur, haec aequatio inter duas tantum variabiles *x* et z consistere est censenda. Quare per praecepta libri primi haec tractanda erit aequatio

$$Pz + \frac{Qdz}{dx} + \frac{Rddz}{dx^2} + \frac{Sd^3z}{dx^3} + \frac{Td^4z}{dx^4} + \text{etc.} = 0\;;$$

cuius resolutio si succedat, tantum opus est, ut loco constantium per singulas integrationes invectarum functiones quaecunque ipsius *y* scribantur; sicque habebitur integrale desideratum idque completum, siquidem hanc aequationem complete integrare licuerit.

COROLLARIUM 1

399. Si ergo litterae *P*, *Q*, *R*, *S* etc. sint constantes vel solam variabilem *y* involvant, integratio semper succedit, quoniam in primo libro huiusmodi aequationes in genere integrare docuimus.

COROLLARIUM 2

400. Deinde etiam resolutio succedit huius aequationis

$$Az+Bx\left(\frac{dz}{dx}\right)+Cx^2\left(\frac{ddz}{dx^2}\right)+Dx^3\left(\frac{d^3z}{dx^3}\right)+\text{etc.}=0\,,$$

sive litterae *A*, *B*, *C* etc. sint constantes sive functiones ipsius *y* tantum.

COROLLARIUM 3

401. Tum vero etiam, si hae formae non sint aequales nihilo, sed functioni cuicunque ipsarum *x* et *y* aequentur, resolutio nihilo minus succedit per ea, quae in postremis capitibus libri primi sunt exposita.

SCHOLION

402. Haec etiam multo latius extendi possunt ad omnes plane aequationes, in quibus nullae aliae formulae differentiales praeter has

$$\left(\frac{dz}{dx}\right),\quad \left(\frac{ddz}{dx^2}\right),\quad \left(\frac{d^3z}{dx^3}\right)\quad \text{etc.},$$

quae solam *x* ut variabilem implicant, occurrunt. Quomodocunque enim istae formulae cum quantitatibus finitis *x*, *y* et z fuerint complicatae, aequatio semper ad librum primum pertinere est censenda, quoniam in omnibus integrationibus instituendis quantitas *y* perpetuo ut constans tractatur. Confectis demum integrationibus discrimen in hoc consistit, ut loco constantium arbitrariarum functiones arbitrariae ipsius *y* in calculum introducantur. Superfluum foret hic monere, quae de altera variabilium *y* sunt dicta, etiam de altera *x* esse intelligenda.

PROBLEMA 66

403. *Proposita hac aequatione*

$$\left(\frac{ddz}{dx^2}\right)+bx\left(\frac{ddz}{dxdy}\right)-2a\left(\frac{dz}{dx}\right)-ab\left(\frac{dz}{dy}\right)+aaz=0$$

investigare indolem functionis z.

SOLUTIO

Facile patet huic aequationi satisfacere hanc aequationem simplicem

$\left(\frac{dz}{dx}\right)=az$, unde fit $z=e^{ax}$. Statuamus ergo $z=e^{ax}v$ eritque

$$\left(\frac{dz}{dx}\right)=e^{x}\left(v+\left(\frac{dv}{dx}\right)\right),\quad \left(\frac{dz}{dy}\right)=e^{x}\left(\frac{dv}{dy}\right)$$

hincque

$$\left(\frac{ddz}{dx^2}\right)=e^{ax}\left(aav+2a\left(\frac{dv}{dx}\right)+\left(\frac{ddv}{dx^2}\right)\right)$$

et

$$\left(\frac{ddz}{dxdy}\right)=e^{ax}\left(a\left(\frac{dv}{dy}\right)+\left(\frac{ddv}{dxdy}\right)\right),$$

quibus valoribus substitutis et divisa aequatione per e^{ax} habebimus

$$\left(\frac{ddv}{dx^2}\right)+b\left(\frac{ddv}{dxdy}\right)=0.$$

Quia nunc hic ubique occurrit $\left(\frac{dv}{dx}\right)$, faciamus $\left(\frac{dv}{dx}\right)=u$; erit

$$\left(\frac{du}{dx}\right)+b\left(\frac{du}{dy}\right)=0,$$

cuius integrale est $f:(y-bx)=u$ [§ 79]; scribamus ergo

$$u=\left(\frac{dv}{dx}\right)=-b\Gamma':(y-bx)$$

ut prodeat $v=\Gamma:(y-bx)+\Delta:y$, ideoque integrale quaesitum erit

$$z=e^{ax}\left(\Gamma:(y-bx)+\Delta:y\right),$$

quae forma ob duas functiones arbitrarias utique est integrale completum.

PROBLEMA 67

404. *Proposita hac aequatione*

$$0=(a+2b)z-(2a+3b)\left(\frac{dz}{dx}\right)+c\left(\frac{dz}{dy}\right)+a\left(\frac{ddz}{dx^2}\right)-2c\left(\frac{ddz}{dxdy}\right)+b\left(\frac{d^3z}{dx^3}\right)+c\left(\frac{d^3z}{dx^2dy}\right)$$

indolem functionis z investigare.

SOLUTIO

Aequatio haec ita est comparata, ut ei manifesto satisfaciat $z = e^x$; statuamus ergo $z = e^x v$ eritque

$$\left(\frac{dz}{dx}\right) = e^{ax}\left(v + \left(\frac{dv}{dx}\right)\right), \quad \left(\frac{dz}{dy}\right) = e^x\left(\frac{dv}{dy}\right)$$

$$\left(\frac{ddz}{dx^2}\right) = e^x\left(v + 2\left(\frac{dv}{dx}\right) + \left(\frac{ddv}{dx^2}\right)\right), \quad \left(\frac{ddz}{dxdy}\right) = e^x\left(\left(\frac{dv}{dy}\right) + \left(\frac{ddv}{dxdy}\right)\right),$$

$$\left(\frac{d^3z}{dx^3}\right) = e^x\left(v + 3\left(\frac{dv}{dx}\right) + 3\left(\frac{ddv}{dx^2}\right) + \left(\frac{d^3v}{dx^3}\right)\right),$$

$$\left(\frac{d^3z}{dx^2dy}\right) = e^x\left(\left(\frac{dv}{dy}\right) + 2\left(\frac{ddv}{dxdy}\right) + \left(\frac{d^3v}{dx^2dy}\right)\right),$$

quibus valoribus substitutis emergit haec satis simplex aequatio

$$0 = (a+3b)\left(\frac{ddv}{dx^2}\right) + b\left(\frac{d^3v}{dx^3}\right) + c\left(\frac{d^3v}{dx^2dy}\right),$$

In qua commode evenit, ut in singulis terminis formula $\left(\frac{ddv}{dx^2}\right)$ contineatur quare posito $\left(\frac{ddv}{dx^2}\right) = u$ prodit haec aequatio primi gradus

$$0 = (a+3b)u + b\left(\frac{du}{dx}\right) + c\left(\frac{du}{dy}\right)$$

ex qua patet, si ponatur $du = pdx + qdy$, esse debere

$$(a+3b)u + bp + cq = 0,$$

quae ita resolvitur.

Cum posito $a + 3b = f$ sit $q = -\frac{bp}{c} - \frac{fu}{c}$, erit

$$du = pdx - \frac{bpdy}{c} - \frac{fudy}{c}$$

seu

$$dx - \frac{bdy}{c} = \frac{1}{p}\left(du + \frac{fudy}{c}\right) = \frac{u}{p}\left(\frac{du}{u} + \frac{fdy}{c}\right)$$

sicque necesse est, ut sit $\frac{u}{p}$ functio ipsius $x - by$, unde fit

$$lu + \frac{fy}{c} = f:(cx - by)$$

et

$$u = e^{-\frac{fy}{c}} \Gamma'' : \left(x - \frac{by}{c} \right) = \left(\frac{ddv}{dx^2} \right).$$

Iam ob y constans spectandum prima integratio dat

$$\left(\frac{dv}{dx} \right) = e^{-\frac{fy}{c}} \Gamma' : \left(x - \frac{by}{c} \right) + \Delta : y$$

et altera

$$v = e^{-\frac{fy}{c}} \Gamma : \left(x - \frac{by}{c} \right) + x\Delta : y + \Sigma : y .$$

Quare posito $a + 3b = f$ aequationis propositae integrale completum est

$$z = e^{x-\frac{fy}{c}} \Gamma : \left(x - \frac{by}{c} \right) + e^x x\Delta : y + e^x \Sigma : y .$$

PROBLEMA 67A

405. *Proposita hac aequatione differentiali tertii gradus*

$$0 = Pz - 3P\left(\frac{dz}{dx} \right) + 3P\left(\frac{ddz}{dx^2} \right) - P\left(\frac{d^3z}{dx^3} \right) + Q\left(\frac{dz}{dy} \right) - 2Q\left(\frac{ddz}{dxdy} \right) + Q\left(\frac{d^3z}{dx^2dy} \right)$$

ubi P et Q sint functiones quaecunque ipsarum x et y, investigare indolem functionis z.

SOLUTIO

Facta substitutione $z = e^x v$, quandoquidem ex data forma facile perspicitur valorem ex loco z positum satisfacere, pervenitur ad hanc aequationem

$$-P\left(\frac{d^3v}{dx^3} \right) + Q\left(\frac{d^3v}{dx^2dy} \right) = 0 ,$$

quae porro posito $\left(\frac{ddv}{dx^2} \right) = u$, ut sit $v = \iint u dx^2$, abit in hanc

$$-P\left(\frac{du}{dx} \right) + Q\left(\frac{du}{dy} \right) = 0$$

Statuamus $du = pdx + qdy$; erit $Qq = Pp$, hinc $q = \frac{Pp}{Q}$ ideoque

$$du = p\left(dx + \tfrac{P}{Q}dy\right),$$

ex quo intelligitur quantitatem p ita comparatam esse debere, ut formula

$$dx + \tfrac{P}{Q}dy$$

per eam multiplicata integrabilis evadat. Quaeratur ergo multiplicator M formulam $Qdx + Pdy$ integrabilem reddens, ita ut sit

$$\int M\left(Qdx + Pdy\right) = s\,;$$

quam ergo functionem s ipsarum x et y inveniri posse assumo et ob

$$Qdx + Pdy = \tfrac{ds}{M}$$

habebimus $du = \frac{pds}{MQ}$, unde patet $\frac{p}{MQ}$ functionem denotare quantitatis s.
Posito ergo $\frac{p}{MQ} = \Gamma':s$ statim erit $u = \Gamma:s$ hincque $v = \int dx\int dx\Gamma:s$, in qua utraque integratione quantitas y ut constans spectatur. Quocirca resolutio problematis ita se habebit:

Pro formula differentiali $Qdx + Pdy$ quaeratur multiplicator M eam reddens integrabilem, ut sit

$$M\left(Qdx + Pdy\right) = ds\,,$$

et inventa hac ipsarum x et y functione s erit

$$z = e^x\int dx\int dx\Gamma:s + e^x x\Delta:y + e^x\Sigma:y$$

SCHOLION

406. In istis aequationibus hoc commodi usu venit, ut facta substitutione $z = e^x v$ eiusmodi induant formam, quae facile porro ad speciem simplicem in prima sectione consideratam revocari queat; etiamsi enim differentialia tertii gradus non sint destructa, tamen reliqua membra ista e calculo excesserunt, ut deinceps nova substitutione $\left(\frac{ddv}{dx^2}\right) = u$ uti eiusque ope ad aequationem differentialem primi gradus pervenire licuerit. Unica igitur substitutio hoc praestitura fuisset, si statim posuissemus $z = e^x \iint u dx^2$. Utinam praecepta haberentur, quorum ope huiusmodi substitutiones facile dignosci possent!

Interim postremo problemate multo latius patente in subsidium vocata § 209 resolvi poterit:

PROBLEMA 67 B

407. *Proposita hac aequatione differentiali tertii gradus*

$$0 = (P+Q)z - (2P+3Q)\left(\frac{dz}{dx}\right) + (P+3Q)\left(\frac{ddz}{dx^2}\right) - Q\left(\frac{d^3z}{dx^3}\right)$$
$$- R\left(\frac{dz}{dy}\right) + 2R\left(\frac{ddz}{dxdy}\right) - R\left(\frac{d^3z}{dx^2dy}\right)$$

ubi P, Q et R sint functiones quaecunque datae ipsarum x et y, investigare indolem functionis z.

SOLUTIO

Eadem adhibita substitutione $z = e^x v$, qua hactenus sumus usi, aequatio proposita transmutatur in sequentem

$$0 = P\left(\frac{ddv}{dx^2}\right) - Q\left(\frac{d^3v}{dx^3}\right) - R\left(\frac{d^3v}{dx^2dy}\right)$$

ubi commode evenit, ut posito $\left(\frac{ddv}{dx^2}\right) = u$ ista resultet aequatio differentialis primi gradus

$$0 = Pu - Q\left(\frac{du}{dx}\right) - R\left(\frac{du}{dy}\right)$$

unde, qualis ipsarum *x* et *y* functio sit *u*, est inquirendum.

Ponamus esse $du = pdx + qdy$, et quia iam illa conditio praebet

$$Pu = Qp + Rq,$$

secundum artificium supra § 209 usurpatum formemus hinc tres sequentes aequationes

$$Ldu = Lpdx + Lqdy,$$
$$MPudx = MQpdx + MRqdx,$$
$$NPudy = NQpdy + NRqdy,$$

quae in unam summam collectae dabunt

$$Ldu + Pu\left(Mdx + Ndy\right) = p\left(\left(L + MQ\right)dx + NQdy\right) + q\left(\left(L + NR\right)dy + MRdx\right);$$

ubi cum tres quantitates *L, M* et *N* ab arbitrio nostro pendeant, inter eas statuatur primo eiusmodi relatio, ut binae partes posterioris membri communem obtineant factorem, sit scilicet

$$L + MQ:NQ = MR:L + NR \text{ seu } L = -MQ - NR,$$

et habebimus

$$-du\left(MQ + NR\right) + Pu\left(Mdx + Ndy\right) = \left(Mq - Np\right)\left(Rdx - Qdy\right).$$

Quaeratur multiplicator *T* formulam $Rdx - Qdy$ reddens integrabilem, ut sit $T\left(Rdx - Qdy\right) = ds$, ex quo tam functio *T* quam *s* ut cognita spectari poterit, et quia nunc habemus

$$-du\left(MQ + NR\right) + Pu\left(Mdx + Ndy\right) = \left(Mq - Np\right)\tfrac{ds}{T}$$

seu

$$\tfrac{du}{u} - \tfrac{P\left(Mdx+Ndy\right)}{MQ+NR} = \tfrac{Np-Mq}{u\left(MQ+NR\right)} \cdot \tfrac{ds}{T},$$

nunc, cum *P, Q, R* sint functiones datae ipsarum *x* et *y*, probe notandum est inter binas nondum definitas *M* et *N* semper eiusmodi relationem statui posse, ut formula $\tfrac{P\left(Mdx+Ndy\right)}{MQ+NR}$ integrationem admittat; sit ergo eius integrale $= lw$, ita ut sit

$$Mdx + Ndy = \tfrac{MQ+NR}{P} \cdot \tfrac{dw}{w} \text{ et } \tfrac{du}{u} = \tfrac{dw}{w} + \tfrac{Np-Mq}{Tu\left(MQ+NR\right)}ds.$$

Necesse ergo est quantitates *p* et *q* ita sint comparatae, ut fiat

$$\tfrac{Np-Mq}{Tu\left(MQ+NR\right)} = f':s$$

hincque $lu = lw + f:s$. Loco $f:s$ scribamus $l\Gamma:s$, ut prodeat $u = w\Gamma:s$ ac propterea

$$v = \int dx \int wdx\Gamma:s + x\Delta:y + \Sigma:y.$$

Consequenter

$$z = e^x \int dx \int wdx\Gamma:s + e^x x\Delta:y + e^x\Sigma:y.$$

COROLLARIUM 1

408. Ad hanc ergo solutionem ex forma proposita statim eruendam primo quaeratur eiusmodi functio ipsarum *x* et *y*, quae vocetur *s*, ut sit

$$ds = T(Rdx - Qdy),$$

id quod expedietur multiplicatorem *T* investigando, quo formula differentialis $Rdx - Qdy$ integrabilis reddatur.

COROLLARIUM 2

409. Praeterea vero quoque quantitatem *w* investigari oportet. In hunc finem inter quantitates *M* et *N* eiusmodi rationem indagari convenit, ut fiat

$$\int \frac{P(Mdx+Ndy)}{MQ+NR} = lw,$$

quae quidem investigatio semper est concedenda.

SCHOLION

410. Cum statim totum negotium eo sit perductum, ut functio *u* ex hac aequatione definiri debeat

$$Pu = Q\left(\frac{du}{dx}\right) + R\left(\frac{du}{dy}\right)$$

sine ambagibus, quibus in solutione sum usus, solutio sequenti modo multo facilius absolvi poterit, id quod insigne supplementum in sectionem primam [§ 209] suppeditat.

Statuatur

$$\left(\frac{du}{dx}\right) = LMu \quad \text{et} \quad \left(\frac{du}{dy}\right) = LNu;$$

erit primo $P = L(MQ + NR)$, hinc $L = \frac{P}{MQ+NR}$, deinde ob

$$du = dx\left(\frac{du}{dx}\right) + dy\left(\frac{du}{dy}\right)$$

habebimus

$$\frac{du}{u} = L(Mdx + Ndy) = \frac{P(Mdx+Ndy)}{MQ+NR}$$

ubi *M* et *N* ita accipi oportet, ut integratio succedat, quod cum innumeris modis fieri possit, solutio hinc completa. obtineri est aestimanda.

Verum dum casus integrationis particularis constet, multo commodius inde solutio completa sequenti ratione elicietur. Posito scilicet

$$\frac{dw}{w} = \frac{P(Mdx+Ndy)}{MQ+NR},$$

ita ut valor ipsius w pro u sumtus iam particulariter satisfaciat sitque

$$Pw = Q\left(\frac{dw}{dx}\right) + R\left(\frac{dw}{dy}\right)$$

statuamus pro valore completo $u = w\Gamma{:}s$ et facta substitutione consequimur

$$pw\Gamma{:}s = Q\left(\frac{dw}{dx}\right)\Gamma{:}s + R\left(\frac{dw}{du}\right)\Gamma{:}s + Qw\left(\frac{ds}{dx}\right)\Gamma'{:}s + Rw\left(\frac{ds}{dy}\right)\Gamma'{:}s,$$

quae aequatio subito in hanc contrahitur

$$Q\left(\frac{ds}{dx}\right) + R\left(\frac{ds}{dy}\right) = 0,$$

ex qua concludimus

$$\left(\frac{ds}{dx}\right) = TR \quad \text{et} \quad \left(\frac{ds}{dy}\right) = -TQ$$

ac propterea

$$ds = T(Rdx - Qdy),$$

unde patet hanc quantitatem s inveniri ex formula $Rdx - Qdy$, pro qua primo factor T eam reddens integrabilem quaeri, tum vero eius integrale pro s sumi debet. Imprimis igitur hic attendatur, quam concinne eandem solutionem elicere liceat, ad quam per tantas ambages perveneramus.

PROBLEMA 68

411. *Proposita hac aequatione differentiali quarti gradus*

$$\left(\frac{d^4z}{dy^4}\right) = aa\left(\frac{ddz}{dx^2}\right)$$

functionis z inventionem saltem ad resolutionem aequationis simplicioris reducere.

SOLUTIO

Hanc aequationem attentius contemplanti mox patebit ei satisfacere huiusmodi simpliciorem

$$\left(\frac{ddz}{dy^2}\right) = b\left(\frac{dz}{dx}\right)$$

hinc enim per y differentiando fit

$$\left(\frac{d^3z}{dy^3}\right)=b\left(\frac{ddz}{dxdy}\right)$$

ac denuo eodem modo

$$\left(\frac{d^4z}{dy^4}\right)=b\left(\frac{d^3z}{dxdy^2}\right)$$

at ex ipsa assumta per x differentiata prodit

$$\left(\frac{d^3z}{dxdy^2}\right)=b\left(\frac{ddz}{dx^2}\right),$$

quo valore ibi inducto colligitur

$$\left(\frac{d^4z}{dy^4}\right)=bb\left(\frac{ddz}{dx^2}\right),$$

quae forma cum proposita congruit, dum sit $bb=aa$; quod cum duplici modo evenire queat, $b=+a$ et $b=-a$, postquam has aequationes simpliciores resolverimus

$$\left(\frac{ddz}{dy^2}\right)-a\left(\frac{dz}{dx}\right)=0$$

quae praebeat $z=P$,

$$\left(\frac{ddz}{dy^2}\right)+a\left(\frac{dz}{dx}\right)=0$$

quae praebeat $z=Q$, erit pro aequatione proposita $z=P+Q$, et quia tam P quam Q binas functiones arbitrarias involvit, integrale hoc modo inventum quatuor eiusmodi functiones complectetur ideoque erit completum.

COROLLARIUM 1

412. Solutiones particulares infinitae facile eliciuntur ponendo $z=e^{\mu x+\nu y}$. Facta enim substitutione fieri necesse est

$$\nu^4=\mu\mu aa \quad \text{et} \quad \mu=\pm\frac{\nu\nu}{a}.$$

Sit $\nu=\lambda a$; erit $\mu=\pm\lambda\lambda a$ et integrale satisfaciens $z=e^{\lambda a(y\pm\lambda x)}$.

COROLLARIUM 2

413. Poni etiam potest $z = e^{\mu x}\cos.(\nu y + \alpha)$, unde fit $\nu^4 = \mu\mu aa$ ut ante, ita ut alia forma integralium particularium sit

$$z = e^{\pm\lambda\lambda ax}\cos.(\lambda ay + \alpha).$$

Huiusmodi formulae infinitae coniunctae integrale completum quasi exhaurire sunt putandae.

COROLLARIUM 3

414. Eaedem solutiones reperiuntur ponendo generalius $z = XY$, unde fit

$$\frac{Xd^4Y}{dy^4} = \frac{aaYddX}{dx^2}$$

qua aequatione ita repraesentata

$$\frac{d^4Y}{Ydy^4} = \frac{aaddX}{Xdx^2}$$

utrumque membrum eidem constanti aequari debet.

SCHOLION

415. Aequatio autem, ad quam totum negotium reduximus,

$$\left(\frac{ddz}{dy^2}\right) = b\left(\frac{dz}{dx}\right)$$

ex earum est numero, quae nullo modo in genere resolvi posse videntur, ita ut in solutionibus particularibus acquiescere debeamus.

Aequatio autem proposita non in mera speculatione est posita, sed, quando laminarum elasticarum vibrationes quam minimae in genere investigantur, ad huiusmodi aequationem quarti gradus resolvendam pervenitur, quae etiam causa est, quod haec quaestio non perinde atque cordarum vibrantium in genere adhuc resolvi potuerit.

Simili autem modo facile intelligitur hanc aequationem quarti gradus

$$\left(\frac{d^4z}{dy^4}\right) = aa\left(\frac{ddz}{dx^2}\right) + 2ab\left(\frac{dz}{dx}\right) + bbz$$

reduci ad hanc geminatam secundi gradus

$$\left(\frac{ddz}{dy^2}\right) = \pm a\left(\frac{dz}{dx}\right) \pm bz$$

neque difficile est alios casus a posteriori eruere, ubi huiusmodi reductiones ad gradum inferiorem locum inveniunt.