CHAPTER III

# CONCERNING THE INTEGRATION OF HOMOGENEOUS EQUATIONS WHERE THE INDIVIDUAL TERMS CONTAIN DIFFERENTIAL FORMULAS OF THE SAME ORDER

## PROBLEM 69

**416.** *To investigate the integral or the nature of the function z of the homogeneous equation of the second order*

$$A\left(\frac{ddz}{dx^2}\right)+B\left(\frac{ddz}{dxdy}\right)+C\left(\frac{ddz}{dy^2}\right)=0$$

*with the letters A, B, C denoting some constant quantities.*

## SOLUTION

I call this equation homogeneous, because it is in agreement with the formulas of the second order differentials nor in addition does it involve other variable quantities. In order that this may be resolved, I observe that it satisfies a homogeneous equation of the first order of this kind

$$\left(\frac{dz}{dx}\right)+\alpha\left(\frac{dz}{dy}\right)=\Delta=\text{Const.};$$

for from this, on differentiating in a twofold manner by $x$ and $y$, there arises

$$\text{I.}\quad \left(\frac{ddz}{dx^2}\right)+\alpha\left(\frac{ddz}{dxdy}\right)=0,$$

$$\text{II.}\quad \left(\frac{ddz}{dxdy}\right)+\alpha\left(\frac{ddz}{dy^2}\right)=0.$$

Now with the former multiplied by $A$, and truly the latter multiplied by $\frac{C}{\alpha}$ jointly produce the proposed equation, if there should be

$$A\alpha+\frac{C}{\alpha}=B \text{ or } A\alpha\alpha-B\alpha+C=0,$$

from which a twofold value for $\alpha$ results, each of which by the assumed equation will give a part of the function sought $z$. Therefore since there shall be $\left(\frac{dz}{dx}\right)=\Delta-\alpha\left(\frac{dz}{dy}\right)$, there will be

[from $dz=\left(\frac{dz}{dx}\right)dx+\left(\frac{dz}{dy}\right)dy$ : ]

$$dz=\Delta dx+\left(dy-\alpha dx\right)\left(\frac{dz}{dy}\right);$$

it is clear that $\left(\frac{dz}{dy}\right)$ must be a function of $y-\alpha x$, from which on putting this $=\Gamma':(y-ax)$ there will be $z=fx+\Gamma:(y-\alpha x)$ with $f$ denoting some constant.

On account of which the solution of the proposed equation thus will be had itself. From the first equation there may be formed an algebraic equation

$Auu+Bu+C=0$, the simple factors of which shall be $u+\alpha$ and $u+\beta$, thus so that there becomes

$$Auu+Bu+C=A(u+\alpha)(u+\beta);$$

then the integral sought will be :

$$z=fx+\Gamma:(y-\alpha x)+\Delta:(y-\beta x);$$

where since the first $fx$ now may be agreed to be contained in the two indefinite functions on account of

$$fx=\frac{f(y-\alpha x)-f(y-\beta x)}{\beta-\alpha}$$

thus it may be expressed more succinctly

$$z=\Gamma:(y-\alpha x)+\Delta:(y-\beta x),$$

because on account the two arbitrary functions certainly it may be considered for completion, with a single case excepted, with which there shall be $\beta=\alpha$. For which case we may put in place $\beta=\alpha+d\alpha$, and since there shall be

$$\Delta:(y-(\alpha+d\alpha)x)=\Delta:(y-\alpha x)-xd\alpha\Delta':(y-ax),$$

because the first part now may be contained in the former member and in place of the latter it is allowed to write $x\Delta:(y-\alpha x)$, there will be for the case $\beta=\alpha$ or $BB=4AC$, the integral

$$z=\Gamma:(y-\alpha x)+x\Delta:(y-\alpha x).$$

## COROLLARY 1

**417**. For the case $\beta=\alpha$ it is evident that the integral can also be expressed in this manner

$$z=\Gamma:(y-\alpha x)+y\Delta:(y-\alpha x),$$

as which form do not disagree with that above.

### COROLLARY 2

**418.** If $C=0$, so that there shall be

$$A\left(\frac{ddz}{dx^2}\right)+B\left(\frac{ddz}{dxdy}\right)=0$$

and hence $Auu+Bu=Au\left(u+\frac{B}{A}\right)$, there becomes $\alpha=0$ and $\beta=\frac{B}{A}$ and the integral

$$z=\Gamma{:}y+\Delta{:}\left(y-\frac{B}{A}x\right)=\Gamma{:}y+\Delta{:}(Ay-Bx)$$

In a similar manner the integral of the equation

$$B\left(\frac{ddz}{dxdy}\right)+C\left(\frac{ddz}{dy^2}\right)=0$$

is

$$z=\Gamma{:}x+\Delta{:}(Cx-By).$$

### COROLLARY 3

**419.** Again the integral of this equation

$$aa\left(\frac{ddz}{dx^2}\right)+2ab\left(\frac{ddz}{dxdy}\right)+bb\left(\frac{ddz}{dy^2}\right)=0,$$

on account of $aauu+2abu+bb=aa\left(u+\frac{b}{a}\right)^2$, is

$$z=\Gamma{:}(ay-bx)+x\Delta{:}(ay-bx).$$

### SCHOLIUM

**420.** The form of these integrals labours under no difficulty, as long as the equation

$$Auu+Bu+C=0$$

it has two real roots, either unequal or equal; but when these roots become imaginary, so that there shall be

$$\alpha=\mu+\nu\sqrt{-1} \text{ and } \beta=\mu-\nu\sqrt{-1},$$

then the use of nearly all arbitrary functions has to be abandoned. For if the nature of the functions $\Gamma$ and $\Delta$ is represented by some curved lines drawn, so that $\Gamma{:}v$ and $\Delta{:}v$ denote the applied lines [*i.e.* the *y* coordinates ] agreeing with the abscissas *v* of these, in no way is it evident, how the values

$$\Gamma:\left(p+q\sqrt{-1}\right) \text{ and } \Delta:\left(p-q\sqrt{-1}\right)$$

can be shown, even if the imaginary parts mutually cancel. In which a huge distinction is perceived between continuous and discontinuous functions, since in the former the values really are always able to be expressed thus

$$\Gamma:\left(p+q\sqrt{-1}\right)+\Gamma:\left(p-q\sqrt{-1}\right) \text{ and } \frac{\Delta:\left(p+q\sqrt{-1}\right)-\Delta:\left(p-q\sqrt{-1}\right)}{\sqrt{-1}},$$

because that will not succeed in any manner, if $\Gamma$ and $\Delta$ signify discontinuous functions. Therefore from these cases the general solution found here is seen to be restricted to continuous functions, since they are opposed to the application and performance of discontinuous functions.

## PROBLEM 70

**421.** *With this proposed homogeneous equation of the third order,*

$$A\left(\frac{d^3z}{dx^3}\right)+B\left(\frac{d^3z}{dx^2dy}\right)+C\left(\frac{d^3z}{dxdy^2}\right)+D\left(\frac{d^3z}{dy^3}\right)=0$$

*to find the complete integral of this.*

## SOLUTION

It is observed, as in the preceding problem, for this equation also to satisfy very well a differential equation of the first order, from which the particular integral will have such a form

$$z=\Gamma:(y+nx);$$

hence these individual differential formulas of the third order are deduced, which will be

$$\left(\frac{d^3z}{dx^3}\right)=n^3\Gamma''':(y+nx),\quad \left(\frac{d^3z}{dx^2dy}\right)=n^2\Gamma''':(y+nx),$$
$$\left(\frac{d^3z}{dxdy^2}\right)=n\Gamma''':(y+nx),\quad \left(\frac{d^3z}{dy^3}\right)=\Gamma''':(y+nx),$$

with which substituted, since division by $\Gamma''':(y+nx)$ is possible, this equation arises

$$An^3+Bn^2+Cn+D=0;$$

of which the three roots, if they should be $n=\alpha$, $n=\beta$, $n=\gamma$, it is evident for the proposed equation to satisfy this form

$$z=\Gamma:(y+\alpha x)+\Delta:(y+\beta x)+\Sigma:(y+\gamma x)\ ;$$

which since three arbitrary functions are involved, there is no reason to doubt why this should not be the complete integral.

Yet this may be observed, if two roots should be equal, suppose $\gamma=\beta$, the integral becomes

$$z=\Gamma:(y+\alpha x)+\Delta:(y+\beta x)+x\Sigma:(y+\beta x);$$

but if furthermore all three roots should be equal to each other, $\gamma=\beta=\alpha$, then the integral sought will be

$$z=\Gamma:(y+\alpha x)+x\Delta:(y+\alpha x)+xx\Sigma:(y+\alpha x).$$

But if two roots should be imaginary, likewise the solutions will be understood in the same manner as they have been noted before.

## COROLLARY 1

**422.** The last case, in which the three roots are equal, is also evident in this manner, because, if in place of the variables *x* and *y* two new variables $t=x$ and $u=y+\alpha x$ may be introduced, the proposed equation is contracted into this form $\left(\frac{d^3z}{dt^3}\right)=0$, the integral of which evidently is

$$z=\Gamma:u+x\Delta:u+xx\Sigma:u\,.$$

## COROLLARY 2

**423.** Hence therefore it is understood also, how in homogeneous equations of higher grade, if the algebraic equations thence formed may have several equal roots, the integrals must be prepared, thus so that then also neither the case of equal or imaginary roots shall be the cause of any difficulty.

## SCHOLIUM

**424.** But the case of the two imaginary roots, in which there is seen to be no use for arbitrary functions, on account of the continuous functions which they satisfy, deserve a more copious explanation. But the formulas present in the integral in these cases can be reduced always to this form

$$\Gamma:v\left(\cos.\varphi+\sqrt{-1}\cdot\sin.\varphi\right)+\Delta:v\left(\cos.\varphi-\sqrt{-1}\cdot\sin.\varphi\right),$$

from which at first, if the functions were powers, values of this kind are deduced

$$Av^n \cos.n\varphi + Bv^n \sin.n\varphi \text{ or } Av^n \cos.(n\varphi + \alpha);$$

for however many values of this kind, the constants *A*, *n* and $\alpha$ can be given, on being changed in some manner. Then if the functions denote logarithms, they give rise to such values

$$Alv + B\varphi.$$

In the third case, if the functions were exponentials, these solutions arise

$$e^{v\cos.\varphi}\left(A\cos.(v\sin.\varphi) + B\sin.(v\sin.\varphi)\right) = Ae^{v\cos.\varphi}\cos.(v\sin.\varphi + \alpha),$$

and more generally,

$$Ae^{v^n \cos.n\varphi}\cos.\left(v^n \sin.n\varphi + \alpha\right).$$

But most other formulas of this kind can be elicited from the principles of imaginary numbers, which can be used in some manner with these combined from the part of the integral arising from the two imaginary roots, from which and endless multitude of functions arises, which is seen to pretend to be the complete solution, nor yet likewise can it be had for the complete solution that comes about in practice in these cases in which all the roots are real.

But here is may be observed that no problems of mechanics or physics have arisen, which depend on a case of this kind.

## PROBLEM 71

**425.** *With the proposed homogeneous equation of any order of this kind*

$$A\left(\frac{d^\lambda z}{dx^\lambda}\right) + B\left(\frac{d^\lambda z}{dx^{\lambda-1}dy}\right) + C\left(\frac{d^\lambda z}{dx^{\lambda-2}dy^2}\right) + \text{etc.} = 0,$$

*to find the complete integral of this.*

## SOLUTION

Hence there may be formed an algebraic equation of the order $\lambda$

$$An^\lambda + Bn^{\lambda-1} + C^{\lambda-2} + \text{etc.} = 0,$$

of which the roots for the number $\lambda$ shall be

$$n = \alpha,\ n = \beta,\ n = \gamma,\ n = \delta \text{ etc.};$$

which if all were equal, the complete integral of the proposed equation will be

$$z = \Gamma:(y + \alpha x) + \Delta:(y + \beta x) + \Sigma:(y + \gamma x) + \Theta:(y + \delta x) + \text{etc.},$$

of which the number of the unequal functions will be $=\lambda$. But if it comes about, that between these roots two or more may be found to be equal, evidently $\beta=\alpha$, $\gamma=\alpha$ etc., then the functions involving these equal roots with respect to multiplication must be [expressed] through the terms of this geometric progression $1, x, x^2$ etc ., or of this, $1, y, y^2$ etc., thus so that the number of arbitrary functions is not diminished. But concerning imaginary roots these are always to be noted, which we have observed before [§ 420], unless perhaps we may be unwilling to exclude the arbitrary functions of imaginary formulas.

**COROLLARY 1**

**426.** In the case of equal roots likewise we may use another geometric series, if indeed the functions are neither functions of $x$ nor of $y$ only. But if these functions should be either of $x$ or of $y$ only, then it may be required to make use of another different variable with a geometric progression.

**COROLLARY 2**

**427.** If in the algebraic equation the initial terms $A, B, C$ etc. vanish, so that the number of roots is seen to be smaller than the exponent $\lambda$, then the deficient roots are to be taken infinitely great, with which the corresponding functions of $x$ only are to be introduced into the integrals.

**COROLLARIUM 3**

**428.** Thus if there should be $A=0$, $B=0$ and $C=0$, the three roots $\alpha, \beta, \gamma$ are agreed to increase to infinity, from which the part of the integral arises

$$\Gamma{:}x+y\Delta{:}x+y^2\Sigma{:}x\,.$$

**SCHOLIUM**

**429.** Because this part of the integral calculus scarcely has begun to be cultivated and thus the investigations of this kind at this point are in short recondite, from this section it is not allowed to offer more and thus with these I am forced to conclude the first part of the following book, which depends on the investigation of functions of two variables from some given differential relation. But it is conceded to present much less about the other part of this book to the public, where the integral calculus is applied to functions of three variables, and hence on that account it will not indeed be worth the effort to subdivide that part into sections, as the following parts touch briefly on much less.

CAPUT III

# DE INTEGRATIONE AEQUATIONUM HOMOGENEARUM UBI SINGULI TERMINI FORMULAS DIFFERENTIALES EIUSDEM GRADUS CONTINENT

## PROBLEMA 69

**416.** *Aequationis homogeneae secundi gradus*

$$A\left(\frac{ddz}{dx^2}\right)+B\left(\frac{ddz}{dxdy}\right)+C\left(\frac{ddz}{dy^2}\right)=0$$

*integrale seu indolem functionis z investigare denotantibus litteris A, B, C quantitates quascunque constantes.*

## SOLUTIO

Hanc aequationem voco homogeneam, quia formulis differentialibus secundi gradus constat neque praeterea alias quantitates variabiles involvit. Ad hanc resolvendam observo ei satisfacere huiusmodi aequationem homogeneam primi gradus

$$\left(\frac{dz}{dx}\right)+\alpha\left(\frac{dz}{dy}\right)=\Delta=\text{Const.};$$

hac enim duplici modo per $x$ et $y$ differentiata oritur

$$\text{I.}\quad \left(\frac{ddz}{dx^2}\right)+\alpha\left(\frac{ddz}{dxdy}\right)=0,$$

$$\text{II.}\quad \left(\frac{ddz}{dxdy}\right)+\alpha\left(\frac{ddz}{dy^2}\right)=0.$$

Iam illa per $A$, haec vero per $\frac{C}{\alpha}$ multiplicata iunctim propositam producent, si fuerit

$$A\alpha+\frac{C}{\alpha}=B \quad\text{seu}\quad A\alpha\alpha-B\alpha+C=0,$$

unde duplex valor pro $a$ resultat, quorum uterque per aequationem assumtam dabit partem functionis quaesitae $z$. Cum igitur sit $\left(\frac{dz}{dx}\right)=\Delta-\alpha\left(\frac{dz}{dy}\right)$, erit

$$dz=\Delta dx+\left(dy-\alpha dx\right)\left(\frac{dz}{dy}\right)\ ;$$

patet $\left(\frac{dz}{dy}\right)$ functionem esse debere ipsius $y-\alpha x$, qua posita $=\Gamma':(y-ax)$ erit $z=fx+\Gamma:(y-\alpha x)$ denotante *f* constantem quamcunque.

Quocirca aequationis propositae solutio ita se habebit. Formetur primo aequatio algebraica $Auu+Bu+C=0$, cuius factores simplices sint $u+\alpha$ et $u+\beta$, ita ut sit

$$Auu+Bu+C=A(u+\alpha)(u+\beta);$$

tum integrale quaesitum erit

$$z=fx+\Gamma:(y-\alpha x)+\Delta:(y-\beta x);$$

ubi cum prima pars *fx* iam in binis functionibus indefinitis contineri sit censenda ob

$$fx=\frac{f(y-\alpha x)-f(y-\beta x)}{\beta-\alpha}$$

succinctius ita exprimetur

$$z=\Gamma:(y-\alpha x)+\Delta:(y-\beta x),$$

quod ob binas functiones arbitrarias utique pro completo est habendum, unico casu excepto, quo est $\beta=\alpha$. Pro quo casu statuamus $\beta=\alpha+d\alpha$, et cum sit

$$\Delta:(y-(\alpha+d\alpha)x)=\Delta:(y-\alpha x)-xd\alpha\Delta':(y-ax),$$

quia pars prior iam in membro priori continetur et loco posterioris scribere licet $x\Delta:(y-\alpha x)$, erit pro casu $\beta=\alpha$ seu $BB=4AC$ integrale

$$z=\Gamma:(y-\alpha x)+x\Delta:(y-\alpha x).$$

## COROLLARIUM 1

**417**. Pro casu $\beta=\alpha$ manifestum est integrale etiam hoc modo exprimi posse

$$z=\Gamma:(y-\alpha x)+y\Delta:(y-\alpha x),$$

quae autem forma ab illa non discrepat.

### COROLLARIUM 2

**418.** Si $C=0$, ut sit

$$A\left(\frac{ddz}{dx^2}\right)+B\left(\frac{ddz}{dxdy}\right)=0$$

hincque $Auu+Bu=Au\left(u+\frac{B}{A}\right)$,fit $\alpha=0$ et $\beta=\frac{B}{A}$ et integrale

$$z=\Gamma:y+\Delta:\left(y-\frac{B}{A}x\right)=\Gamma:y+\Delta:\left(Ay-Bx\right)$$

Simili modo aequationis

$$B\left(\frac{ddz}{dxdy}\right)+C\left(\frac{ddz}{dy^2}\right)=0$$

integrale est

$$z=\Gamma:x+\Delta:\left(Cx-By\right).$$

### COROLLARIUM 3

**419.** Porro huius aequationis

$$aa\left(\frac{ddz}{dx^2}\right)+2ab\left(\frac{ddz}{dxdy}\right)+bb\left(\frac{ddz}{dy^2}\right)=0$$

ob $aauu+2abu+bb=aa\left(u+\frac{b}{a}\right)^2$ est integrale

$$z=\Gamma:\left(ay-bx\right)+x\Delta:\left(ay-bx\right).$$

### SCHOLION

**420.** Harum integralium forma nulla laborat difficultate, quamdiu aequatio

$$Auu+Bu+C=0$$

duas habet radices reales, sive sint inaequales sive aequales; quando autem hae radices fiunt imaginariae, ut sit

$$\alpha=\mu+\nu\sqrt{-1} \text{ et } \beta=\mu-\nu\sqrt{-1},$$

tum functiones arbitrariae omni fere usu destituuntur. Etsi enim indoles functionum $\Gamma$ et $\Delta$ lineis curvis utcunque ductis repraesentatur, ut $\Gamma: v$ et $\Delta: v$ denotent in iis applicatas abscissae $v$ convenientes, nullo modo patet, quomodo valores

$$\Gamma:\left(p+q\sqrt{-1}\right) \text{ et } \Delta:\left(p-q\sqrt{-1}\right)$$

exhiberi debeant, etiamsi imaginaria se mutuo tollant. In quo ingens cernitur discrimen inter functiones continuas et discontinuas, cum in illis semper valores ita expressi

$$\Gamma:\left(p+q\sqrt{-1}\right)+\Gamma:\left(p-q\sqrt{-1}\right) \text{ et } \frac{\Delta:\left(p+q\sqrt{-1}\right)-\Delta:\left(p-q\sqrt{-1}\right)}{\sqrt{-1}}$$

realiter exhiberi queant, id quod, si $\Gamma$ et $\Delta$ significent functiones discontinuas, nullo modo succedit. His igitur casibus solutio generalis hic inventa ad solas functiones continuas restringenda videtur, quandoquidem discontinuae applicationi et executioni adversantur.

## PROBLEMA 70

**421.** *Proposita hac aequatione tertii gradus homogenea*

$$A\left(\frac{d^3z}{dx^3}\right)+B\left(\frac{d^3z}{dx^2dy}\right)+C\left(\frac{d^3z}{dxdy^2}\right)+D\left(\frac{d^3z}{dy^3}\right)=0$$

*eius integrale completum invenire.*

## SOLUTIO

Huic quoque aequationi uti in praecedente problemate satisfacere aequationem differentialem simplicem primi gradus satis luculenter perspicitur, ex quo integrale particulare talem habebit formam

$$z=\Gamma:(y+nx);$$

colligantur hinc singulae formulae differentiales tertii gradus, quae erunt

$$\left(\frac{d^3z}{dx^3}\right)=n^3\Gamma''':(y+nx),\quad \left(\frac{d^3z}{dx^2dy}\right)=n^2\Gamma''':(y+nx),$$
$$\left(\frac{d^3z}{dxdy^2}\right)=n\Gamma''':(y+nx),\quad \left(\frac{d^3z}{dy^3}\right)=\Gamma''':(y+nx),$$

quibus substitutis, quoniam divisio per $\Gamma''':(y+nx)$ succedit, nascitur ista aequatio

$$An^3+Bn^2+Cn+D=0;$$

cuius tres radices si fuerint $n=\alpha$, $n=\beta$, $n=\gamma$, evidens est aequationi propositae satisfacere hanc formam

$$z=\Gamma:(y+\alpha x)+\Delta:(y+\beta x)+\Sigma:(y+\gamma x)\;;$$

quae cum tres functiones arbitrarias complectatur, dubium non est, quin ea sit integrale completum.

Hoc tantum notetur, si duae radices sint aequales, puta $\gamma=\beta$, integrale fore

$$z=\Gamma:(y+\alpha x)+\Delta:(y+\beta x)+x\Sigma:(y+\beta x);$$

sin autem adeo omnes tres fuerint inter se aequales, $\gamma=\beta=\alpha$, tum erit integrale quaesitum

$$z=\Gamma:(y+\alpha x)+x\Delta:(y+\alpha x)+xx\Sigma:(y+\alpha x).$$

Quodsi duae radices fuerint imaginariae, eadem erunt tenenda, quae modo ante sunt observata.

### COROLLARIUM 1

**422.** Ultimus casus, quo tres radices sunt aequales, etiam inde est manifestus, quod, si loco variabilium *x* et *y* binae novae $t=x$ et $u=y+\alpha x$ introducantur, aequatio proposita contrahatur in hanc formam $\left(\frac{d^3z}{dt^3}\right)=0$, cuius integrale manifesto est

$$z=\Gamma:u+x\Delta:u+xx\Sigma:u\,.$$

### COROLLARIUM 2

**423.** Hinc ergo etiam intelligitur, quomodo in aequationibus homogeneis altioris gradus, si aequationes algebraicae inde formatae plures habeant radices aequales, integralia futura sint comparata, ita ut etiam tum neque casus radicum aequalium neque imaginariarum ulli difficultati sit obnoxius.

### SCHOLION

**424.** Casus autem binarum radicum imaginariarum, quibus functiones arbitrariae nullum usum habere videntur, ratione functionum continuarum, quae satisfaciunt, uberiorem evolutionem merentur. Formulae autem his casibus in integrale ingredientes semper ad hanc formam reduci possunt

$$\Gamma:v\left(\cos.\varphi+\sqrt{-1}\cdot\sin.\varphi\right)+\Delta:v\left(\cos.\varphi-\sqrt{-1}\cdot\sin.\varphi\right),$$

unde primum, si functiones sint potestates, huiusmodi valores colliguntur

$$Av^n\cos.n\varphi + Bv^n\sin.n\varphi \text{ seu } Av^n\cos.(n\varphi+\alpha);$$

quotcunque enim huiusmodi valores, constantes *A*, *n* et $\alpha$ utcunque mutando, adhiberi possunt. Deinde si functiones denotent logarithmos, prodeunt tales valores

$$Alv + B\varphi.$$

Tertio si functiones sint exponentiales, oriuntur hi

$$e^{v\cos.\varphi}\left(A\cos.(v\sin.\varphi)+B\sin.(v\sin.\varphi)\right) = Ae^{v\cos.\varphi}\cos.(v\sin.\varphi+\alpha)$$

et generalius

$$Ae^{v^n\cos.n\varphi}\cos.\left(v^n\sin.n\varphi+\alpha\right).$$

Plurimae autem aliae huiusmodi formulae ex doctrina imaginariorum elici possunt, quae utcunque cum his combinatae pro parte integrali ex binis radicibus imaginariis nata usurpari poterunt, unde infinita functionum multitudo nascitur, quae solutionem completam mentiri videtur neque tamen pro completa perinde haberi potest atque usu venit iis casibus, quibus omnes radices sunt reales. Hic autem observetur nullum adhuc problema mechanicum seu physicum occurrisse, quod ab huiusmodi casu penderet.

## PROBLEMA 71

**425.** *Proposita huiusmodi aequatione homogenea gradus cuiuscunque*

$$A\left(\frac{d^\lambda z}{dx^\lambda}\right)+B\left(\frac{d^\lambda z}{dx^{\lambda-1}dy}\right)+C\left(\frac{d^\lambda z}{dx^{\lambda-2}dy^2}\right)+\text{etc.}=0$$

*eius integrale completum invenire.*

## SOLUTIO

Formetur hinc aequatio algebraica ordinis $\lambda$

$$An^\lambda + Bn^{\lambda-1} + C^{\lambda-2} + \text{etc.} = 0,$$

cuius radices numero $\lambda$ sint

$$n=\alpha,\ n=\beta,\ n=\gamma,\ n=\delta \text{ etc.};$$

quae si omnes fuerint inaequales, integrale completum aequationis propositae erit

$$z = \Gamma:(y+\alpha x)+\Delta:(y+\beta x)+\Sigma:(y+\gamma x)+\Theta:(y+\delta x)+\text{etc.},$$

quarum functionum disparium numerus erit $=\lambda$. Sin autem eveniat, ut inter has radices duae pluresve reperiantur aequales, scilicet $\beta=\alpha$, $\gamma=\alpha$ etc., tum functiones has radices aequales involventes respective multiplicari debent per terminos progressionis geometricae huius

$1, x, x^2$ etc. vel huius $1, y, y^2$ etc., ita ut functionum arbitrariarum numerus non minuatur. De radicibus autem imaginariis perpetuo ea sunt notanda, quae ante [§ 420] observavimus, nisi forte functiones arbitrarias formularum imaginariarum excludere nolimus.

### COROLLARIUM 1

**426.** Casu radicum aequalium perinde est, utra serie geometrica utamur, siquidem functiones neque sint ipsius $x$ neque ipsius $y$ tantum. Sin autem hae functiones fuerint vel ipsius $x$ vel ipsius $y$ tantum, tum alterius variabilis diversae progressione geometrica uti oportet.

### COROLLARIUM 2

**427.** Si in aequatione algebraica termini initiales $A$, $B$, $C$ etc. evanescant, ut radicum numerus exponente $\lambda$ minor esse videatur, tum radices deficientes pro infinite magnis sunt habendae, quibus functiones ipsius $x$ tantum respondebunt in integrale introducendae.

### COROLLARIUM 3

**428.** Ita si fuerit $A = 0, B = 0$ et $C = 0$, tres radices $\alpha, \beta, \gamma$ in infinitum excrescere sunt censendae, ex quibus nascetur pars integralis

$$\Gamma{:}x + y\Delta{:}x + y^2\Sigma{:}x .$$

### SCHOLION

**429.** Quoniam haec pars calculi integralis vix excoli coepit ideoque huius generis investigationes adhuc prorsus sunt reconditae, de hac sectione plura proferre non licet ideoque his partem primam libri secundi, quae in investigatione functionum binarum variabilium ex data quadam differentialium relatione versatur, concludere cogor. Multo autem pauciora circa partem alteram huius libri in medium afferre conceditur, ubi calculus integralis ad functiones trium variabilium accommodatur, hancque ob causam ne operae quidem erit pretium istam partem in sectiones subdividere, multo minus sequentes partes attingere.