INTEGRAL CALCULUS
FINAL BOOK

SECOND PART

THE INVESTIGATION OF FUNCTIONS OF THREE

VARIABLES FROM A GIVEN RELATION OF THE DIFFERENTIALS

CHAPTER I

CONCERNING THE DIFFERENTIAL FORMULAS OF FUNCTIONS INVOLVING THREE VARIABLES

PROBLEM 72

430. *If v shall be some function of the three variable quantities x, y and z, to show the differential formulas of this of the first order.*

SOLUTION

Since v shall be a function of the three variables x, y and z, if that may be differentiated in the usual customary way, the differential of this generally thus may be found expressed

$$dv = pdx + qdy + rdz\,.$$

Evidently that will be agreed with three parts, of which the first pdx separately is found, if in the differentiation only the quantity x may be treated as variable with the two remaining variables y and z considered as constants. In a similar manner the second part qdy is obtained from the differentiation of the function v thus put in place, so that with only the quantity y for a variable, truly the two remaining parts x and z may be considered as constants, as likewise is to be understood for the third part rdz, which is the differential of v on account of the variability had of the quantity z. Hence it is apparent, how by differentiation the quantities themselves p, q and r may be found separately, which here I will call the differential formulas of the first order of the function v, and there shall not be a need for the introduction of new letters in the calculation, the natures of those I will indicate suitably thus :

$$p = \left(\tfrac{dv}{dx}\right), \quad q = \left(\tfrac{dv}{dy}\right), \quad r = \left(\tfrac{dv}{dz}\right).$$

Therefore any function v of the three variables x, y et z has three differential formulas of the first order thus designated in whatever one of the variables taken into account, as long as the two remaining variables may be considered as constant, and since the differentials are removed on division, these differential formulas are referred to as a finite class of quantities.

COROLLARY 1

431. Thus the differential of v is composed in the usual customary manner from the three differential formulas found, so that there shall be

$$dv = dx\left(\tfrac{dv}{dx}\right) + dy\left(\tfrac{dv}{dy}\right) + dz\left(\tfrac{dv}{dz}\right);$$

therefore in turn the integral of this form is that function itself *v*, or also with the same quantity either augmented or diminished by some amount.

COROLLARY 2

432. If *v* should be a given function of the three variables *x, y* and *z*, the individual differential formulas of this

$$\left(\frac{dv}{dx}\right),\quad \left(\frac{dv}{dy}\right),\quad \left(\frac{dv}{dz}\right)$$

again will be certain functions of the same variables *x, y* and *z* easily found by differentiation. Yet meanwhile it can come about, that one or several of the variables depart completely from differential formulas of this kind.

SCHOLIUM 1

433. Also nothing prevents the quantity *v* from being considered as a function of three variables *x, y* and *z*, even if perhaps it involves only two, clearly provided an account of the composition has thus been prepared, as if in case the third variable should vanish, as that is less to be wondered at, since the same can eventuate in functions both of one and of two variables. As indeed functions of one variable are accustomed to be represented most conveniently by the applied lines of some curved line [*i.e.* the *y*-axis]; for if indeed from the nature of the applied lines of this curve, as the abscissas *x* can be considered for the case in which the curved line becomes a right line parallel to the axis, then the applied line of the quantity is equal to a constant, and again therefore from that, the general idea can by no means be excluded, from which that variable is considered as a function of the abscissa *x*; and nor indeed, if there shall be such a function *y* of *x*, then this corresponding inconsistency is agreed upon, which calls this function *y* to be equal to a constant quantity.

Because from that it may pertain to functions of the two variables *x* and *y*, which it is always allowed to represent by the intervals, by which the individual points of a certain surface are distant from a certain plane, provided the two variables *x* and *y* are taken in this plane, it is evident that certainly a surface is able to be thus prepared, so that a function can be determined there by *x* or *y* alone. Also why not if the surface should be a plane and with that parallel to that plane, then that function at that stage will become a constant quantity and therefore it must be considered no less as a function of two variables. On which account also when the treatment concerning functions of three variables is worked on, in these generally functions of this kind also are present, which are determined by either two or one of the three variables *y* and *z*, or which at this point shall themselves be constant quantities.

SCHOLIUM 2

434. In the differential calculus now it has been shown of functions involving several variables the differentials are to be found, if each one of the variables separately is seen to be the only variable and all the differentials arise from that may be put together in a single sum. Therefore if the differentiation is put in place in this manner, these single operations give rise to the formulas, only with the differentials deleted, which we may indicate by these signs,

$$\left(\frac{dv}{dx}\right),\quad \left(\frac{dv}{dy}\right)\quad \text{and}\quad \left(\frac{dv}{dz}\right)$$

and likewise it is understood, how also the differential formulas of functions of four or more variables are to be found. But we attach several examples concerning functions of the three variables *x, y* and *z,* for which we shall show the differential formulas of these.

EXEMPLE 1

435. *If* $v = \alpha x + \beta y + \gamma z$ *shall be a function of the three variables, the differential formulas of this may be had:*

Since by differentiation there is produced $dv = \alpha dx + \beta dy + \gamma dz$, it is evident that there becomes :

$$\left(\frac{dv}{dx}\right) = \alpha,\quad \left(\frac{dv}{dy}\right) = \beta,\quad \left(\frac{dv}{dz}\right) = \gamma\ ,$$

and thus all three differential formulas are constant.

EXAMPLE 2

436. *If* $v = x^{\lambda} y^{\mu}\ z^{\nu}$ *shall be a function of the three variables, the differential formulas of this themselves will be had :*

By differentiation carried out in the accustomed manner there becomes

$$dv = \lambda x^{\lambda-1} y^{\mu} z^{\nu} + \mu x^{\lambda} y^{\mu-1} z^{\nu} + \nu x^{\lambda} y^{\mu} z^{\nu-1}$$

from which there is seen to be the differential formulas

$$\left(\frac{dv}{dx}\right) = \lambda x^{\lambda-1} y^{\mu} z^{\nu},\quad \left(\frac{dv}{dy}\right) = \mu x^{\lambda} y^{\mu-1} z^{\nu},\quad \left(\frac{dv}{dz}\right) = \nu x^{\lambda} y^{\mu} z^{\nu-1},$$

which hence are individual new functions of all three variables *x, y, z,* unless the exponents λ, μ, ν are either zero or equal to unity.

EXAMPLE 3

437. *If the function v involves only two variables x and y, with the third z not entering into the arrangement of this, the differential formulas thus may themselves be had :*

Because the function v involves only the two variables x and y, the differential of this adopts a form of this kind $dv = pdx + qdy + 0.dz$, clearly with the third part arising from the variability of z vanishing, from which we will have

$$\left(\frac{dv}{dx}\right) = p, \quad \left(\frac{dv}{dy}\right) = q, \quad \left(\frac{dv}{dz}\right) = 0.$$

COROLLARY

438. Hence therefore in turn is apparent, if there should be $\left(\frac{dv}{dz}\right) = 0$, then v is to be some function of the two variables x and y, as in the last part we have indicated thus $v = \Gamma:(x, y)$, with $\Gamma:(x, y)$ denoting some function of the two variables x and y.

SCHOLIUM

439. Soon we will show, when a function of the three variables is proposed from a certain given relation, or from the condition of the differential formulas to be investigated, in some manner some arbitrary function of the two variables is to be introduced into the integration and thus in this a criterion is put in place, from which this part of the integral calculus can be distinguished from the preceding. For just as, while the nature of the function is found from a given condition of the differential of a single variable, with which the whole first book has been occupied, through the integration some arbitrary constant as you please is brought into the calculation; thus in the preceding part of this following book we have seen that, if functions of two variables are to be investigated from a given relation of the differential formulas, then it pertains to the essence of this treatment, that whatever may be introduced by the integration, it is not a constant quantity, but to sum up rather a function of one variable, that may be introduced into the calculation; and if indeed just as generally these functions $\Gamma:(\alpha x + \beta y)$ will implicate both variables x and y, yet there the whole quantity $\alpha x + \beta y$ is regarded as one variable, and that formula $\Gamma:(\alpha x + \beta y)$ denotes some function of which. Therefore now, where functions of three variables are to be set in place, it is to be noted properly that whatever the integration an arbitrary function of two variables indeed is to be introduced into the calculation, from which likewise the nature of the integration can be deduced, which is concerned with functions of several variables.

PROBLEM 73

440. *If there shall be some function v of the three variables x, y and z whatever, to show the differential formulas of the second and higher orders of this.*

SOLUTION

Since of this there shall be three differential formulas of the first order:

$$\left(\frac{dv}{dx}\right), \quad \left(\frac{dv}{dy}\right), \quad \left(\frac{dv}{dz}\right),$$

whatever the nature of the new function considered again three differential formulas will be supplied, but which on account of

$$\left(\frac{ddv}{dxdy}\right) = \left(\frac{ddv}{dydx}\right)$$

this will be reduced to the six following :

$$\left(\frac{ddv}{dx^2}\right), \quad \left(\frac{ddv}{dy^2}\right), \quad \left(\frac{ddv}{dz^2}\right), \quad \left(\frac{ddv}{dxdy}\right), \quad \left(\frac{ddv}{dydz}\right) \quad \left(\frac{ddv}{dzdx}\right),$$

from the denominators of which it is understood, from which of the three quantities x, y, z a variable alone must be taken in each differentiation. In a similar manner it is evident that the differential formulas of the third order give the following ten :

$$\left(\frac{ddv}{dx^3}\right), \quad \left(\frac{ddv}{dx^2dy}\right), \quad \left(\frac{ddv}{dxdy^2}\right),$$
$$\left(\frac{ddv}{dy^3}\right), \quad \left(\frac{ddv}{dy^2dz}\right), \quad \left(\frac{ddv}{dydz^2}\right), \quad \left(\frac{ddv}{dxdzdx}\right),$$
$$\left(\frac{ddv}{dz^3}\right), \quad \left(\frac{ddv}{dz^2dx}\right), \quad \left(\frac{ddv}{dzdx^2}\right),$$

Again the number of formulas of the differentials of the fourth order is 15, of the fifth 21 etc., following triangular numbers, and likewise from the form of each it is evident, how this value must be elicited from a given function v by repeated differentiation by considering some single variable.

COROLLARY 1

441. Behold therefore all the differential formulas of each order, which it is allowed to derive by differentiation from some function of the three variables, which again are able to be considered as functions of the three variables.

COROLLARY 2

442. Therefore just as from a given function of this kind, all the differential formulas are discovered with the aid of differential calculus, thus in turn it is necessary to find that function itself with the aid of integral calculus, from some given formula of the differentials or from two or more differential relations from a certain relation, from which these arise.

SCHOLION 1

443. Indeed in a calculation with the differential it matters little, whether the function being differentiated involves one or several variables, since the precepts of differentiation for whatever number of variables remains the same; as also, on account of this reason, it is not worth the effort to distinguish the differential calculations following this kind of functions into different parts. But otherwise by far it happens in the integral calculus, which it is necessary to divide into parts following this variety of functions, clearly which parts, both by reason of their own nature as on account of the precepts, greatly disagreeing between each other. Just as hence it is agreed that this part of the theory occupied by functions of three variables is seen to require to be explained .

And in the first place indeed these cases will be set out most conveniently, in which the value of one of certain differential formulas is given, from which the nature of the function sought is required to be defined, because this investigation labours under no difficulty. Then I may approach questions of this kind, in which a certain relation between two or more differential formulas is proposed; where it matters very much indeed, of which order they should be, if indeed it is possible to set out more cases from the first order, while from the higher orders scarcely any at this point can be reported to the public. Therefore I will observe this order in that discussion.

SCHOLIUM 2

444. Here it may be possible to consider admitting as many as two conditions or relations between the differential formulas in defining functions of three variables and that it would not be possible to be determined from a single prescribed question. For if there is put

$$dv = pdx + qdy + rdz\,,$$

where the letters *p, q, r* in turn are the bearers of differential formulas of the first order, and for example these two conditions may be proposed, so that there shall be $q = p$ and $r = p$, and therefore

$$dv = p\left(dx + dy + dz\right),$$

it is evident a solution could be given, clearly

$$v = \Gamma{:}\left(x + y + z\right).$$

[Thus, the equation could be made integrable in this manner.]

Now I respond with this objection that arises in this case, that the two conditions can be consistent at the same time ; for with the one condition unchanged for a short while, so that there remains $q = p$ there must be $r = px$ and thus $dv = p(dx + dy + xdz)$,

[We have $\left(\frac{dr}{dy}\right) = \left(\frac{dq}{dz}\right) = \left(\frac{dp}{dz}\right)$,

and $q = p$ implies that $\left(\frac{dp}{dz}\right) = \left(\frac{dr}{dx}\right) = \left(\frac{dq}{dx}\right)$,from which $r = qx = px$.]

it is evident that no value can be shown for *p*, by which the differential formula $dx + dy + xdz$ on being multiplied is returned integrable, because a single example suffices to show that with two prescribed conditions of this kind questions emerge, more than determined nor therefore is a solution admitted except in certain cases, in which as if one condition is involved now and then the other. On account of which a single relation always suffices between the proposed differential formulas in the determination of the problem, because on that account, on integrating an arbitrary indefinite function is introduced, and equally little is to be had for the indeterminates from the common problems of integral calculus, the solution of which introduces an arbitrary constant.

CALCULI INTEGRALIS
LIBER POSTERIOR

PARS ALTERA

INVESTIGATIO FUNCTIONUM TRIUM
VARIABILIUM EX DATA DIFFERENTIALIUM RELATIONE

CAPUT I

DE FORMULIS DIFFERENTIALIBUS FUNCTIONUM TRES VARIABILES INVOLVENTIUM

PROBLEMA 72

430. *Si v sit functio quaecunque trium quantitatum variabilium x, y et z, eius formulas differentiales primi gradus exhibere.*

SOLUTIO

Cum v sit functio trium variabilium x, y et z, si ea more solito differentietur, eius differentiale in genere ita reperietur expressum

$$dv = pdx + qdy + rdz.$$

Tribus scilicet id constabit partibus, quarum prima pdx seorsim invenitur, si in differentiatione sola quantitas x ut variabilis tractetur binis reliquis y et z ut constantibus spectatis. Simili modo pars secunda qdy impetratur differentiatione functionis v ita instituta, ut sola quantitas y pro variabili, binae reliquae vero x et z pro constantibus habeantur, quod idem de parte tertia rdz est tenendum, quae est differentiale ipsius v variabilitatis solius quantitatis z ratione habita. Hinc patet, quomodo per differentiationem quantitates istae p, q et r seorsim sint inveniendae, quas hic formulas differentiales primi gradus functionis v appellabo, et ne novis litteris in calculum introducendis sit opus, eas naturae suae convenienter ita indicabo

$$p = \left(\frac{dv}{dx}\right), \quad q = \left(\frac{dv}{dy}\right), \quad r = \left(\frac{dv}{dz}\right).$$

Quaelibet ergo functio v trium variabilium x, y et z tres habet formulas differentiales primi gradus ita designandas in quarum qualibet unicae variabilis ratio habetur, dum binae reliquae ut constantes spectantur, et quoniam differentialia per divisionem tolluntur, hae formulae differentiales ad classem quantitatum finitarum sunt referendae.

COROLLARIUM 1

431. Ex tribus formulis differentialibus functionis v inventis eius differentiale solito more sumtum ita conflatur, ut sit

$$dv = dx\left(\frac{dv}{dx}\right) + dy\left(\frac{dv}{dy}\right) + dz\left(\frac{dv}{dz}\right);$$

cuius ergo formae vicissim integrale est ipsa illa functio *v* vel etiam eadem quantitate quacunque sive aucta sive minuta.

COROLLARIUM 2

432. Si trium variabilium *x, y* et *z* functio *v* fuerit data, eius formulae differentiales singulae

$$\left(\frac{dv}{dx}\right), \quad \left(\frac{dv}{dy}\right), \quad \left(\frac{dv}{dz}\right)$$

iterum erunt functiones certae earundem variabilium *x, y* et *z* per differentiationem facile inveniendae. Interim tamen evenire potest, ut una pluresve variabilium ex huiusmodi formulis differentialibus prorsus excedant.

SCHOLION 1

433. Nihil etiam impedit, quominus quantitas *v* ut functio trium variabilium *x, y* et *z* spectari possit, etiamsi forte duas tantum involvat, dum scilicet ratio compositionis ita est comparata, ut tertia quasi casu excesserit, quod eo minus est mirandum, cum idem in functionibus tam unius quam duarum variabilium evenire possit. Quoniam enim functiones unius variabilis commodissime per applicatas cuiuspiam lineae curvae repraesentari solent, siquidem pro curvae natura applicatae eius ut certae functiones abscissae *x* spectari possunt, casu, quo linea curva abit in lineam rectam axi parallelam, etsi tum applicata quantitati constanti aequatur, propterea tamen ex illa idea generali, qua ut functio abscissae *x* spectatur, neutiquam excluditur; neque enim, si quaeratur, qualis sit functio *y* ipsius *x*, incongrue is respondere est censendus, qui dicat hanc functionem *y* aequari quantitati constanti.

Quod deinde ad functiones binarum variabilium *x* et *y* attinet, quas semper per intervalla, quibus singula cuiusdam superficiei puncta a quopiam plano distant, repraesentare licet, dum binae variabiles *x* et *y* in hoc plano accipiuntur, manifestum est utique superficiem ita comparatam esse posse, ut functio illa vel per solam *x* vel per solam *y* determinetur. Quin etiam si superficies fuerit plana ipsique illi plano parallela , functio illa adeo abit in quantitatem constantem neque propterea minus tanquam functio binarum variabilium considerari debebit. Quamobrem etiam quando tractatio circa functiones trium variabilium versatur, in eo genere etiam eiusmodi functiones [continentur], quae tantum vel per binas vel unicam trium variabilium *x, y* et *z* determinantur vel adeo ipsae sunt quantitates constantes.

SCHOLION 2

434. In calculo differentiali iam est ostensum functionum plures variabiles involventium differentialia inveniri, si unaquaeque variabilium seorsim tanquam sola esset variabilis spectetur atque omnia differentialia inde nata in unam summam coniiciantur. Quodsi ergo differentiatio hoc modo instituatur, singulae istae operationes deleto tantum differentiali praebebunt formulas differentiales, quas his signis indicamus,

$$\left(\frac{dv}{dx}\right), \quad \left(\frac{dv}{dy}\right) \quad \text{et} \quad \left(\frac{dv}{dz}\right)$$

simulque intelligitur, quomodo etiam functionum quatuor pluresve variabiles involventium formulae differentiales sint inveniendae. Circa functiones autem trium variabilium *x, y* et *z* exempla aliquot subiungamus, quibus earum ternas formulas differentiales exhibebimus.

EXEMPLUM 1

435. *Si functio trium variabilium sit* $v = \alpha x + \beta y + \gamma z$*, eius formulae differentiales ita se habebunt:*

Cum per differentiationem prodeat $dv = \alpha dx + \beta dy + \gamma dz$, manifestum est fore

$$\left(\frac{dv}{dx}\right) = \alpha, \quad \left(\frac{dv}{dy}\right) = \beta, \quad \left(\frac{dv}{dz}\right) = \gamma$$

sicque omnes tres formulas differentiales esse constantes.

EXEMPLUM 2

436. *Si functio trium variabilium sit* $v = x^{\lambda} y^{\mu} z^{\nu}$*, eius formulae differentiales ita se habebunt:*

Differentiatione more solito peracta fit

$$dv = \lambda x^{\lambda-1} y^{\mu} z^{\nu} + \mu x^{\lambda} y^{\mu-1} z^{\nu} + \nu x^{\lambda} y^{\mu} z^{\nu-1}$$

unde perspicuum est fore formulas differentiales

$$\left(\frac{dv}{dx}\right) = \lambda x^{\lambda-1} y^{\mu} z^{\nu}, \quad \left(\frac{dv}{dy}\right) = \mu x^{\lambda} y^{\mu-1} z^{\nu}, \quad \left(\frac{dv}{dz}\right) = \nu x^{\lambda} y^{\mu} z^{\nu-1},$$

quae ergo singulae sunt novae functiones omnium trium variabilium *x*, *y*, *z*, nisi exponentes λ, μ, ν sint vel nihilo vel unitati aequales.

EXEMPLUM 3

437. *Si functio v duas tantum involvat variabiles x et y tertia z in eius compositionem non ingrediente, formulae differentiales ita se habebunt:*

Quia functio *v* duas tantum variabiles *x* et *y* implicat, eius differentiale huiusmodi formam induet $dv = pdx + qdy + 0.dz$, tertia scilicet parte ex variabilitate ipsius *z* orta evanescente, unde habebimus

$$\left(\frac{dv}{dx}\right)=p, \quad \left(\frac{dv}{dy}\right)=q, \quad \left(\frac{dv}{dz}\right)=0 .$$

COROLLARIUM

438. Hinc ergo vicissim patet, si fuerit $\left(\frac{dv}{dz}\right)=0$, tum fore *v* functionem quamcunque binarum variabilium *x* et *y*, quam in posterum ita indicabimus
$v=\Gamma:(x, y)$ denotante $\Gamma:(x, y)$ functionem quamcunque binarum variabilium *x* et *y*.

SCHOLION

439. Mox ostendemus, quando functio trium variabilium ex data quadam relatione seu conditione formularum differentialium investiganda proponitur, qualibet integratione introduci functionem quamcunque arbitrariam binarum variabilium atque adeo in hoc consistere criterium, quo haec pars calculi integralis a praecedentibus distinguitur. Quemadmodum enim, dum natura functionum unicae variabilis ex data differentialium conditione investigatur, in quo universus liber primus est occupatus, per quamlibet integrationem quantitas constans arbitraria in calculum invehitur, ita in parte praecedente huius secundi libri vidimus, si functiones binarum variabilium ex data formularum differentialium relatione investigari debeant, tum ad essentiam huius tractationis id pertinere, quod qualibet integratione non quantitas constans, sed adeo functio unius variabilis. prorsus arbitraria in calculum introducatur; etsi enim plerumque hae functiones veluti $\Gamma:(\alpha x+\beta y)$ ambas variabiles *x* et *y* implicabant, tamen ibi tota quantitas $\alpha x+\beta y$ ut unica spectatur, cuius functionem quamcunque illa formula $\Gamma:(\alpha x+\beta y)$ denotat. Nunc igitur, ubi de functionibus trium variabilium agitur, probe notandum est qualibet integratione functionem arbitrariam duarum adeo variabilium in calculum introduci, ex quo simul indolem integrationum, quae circa functiones plurium variabilium versantur, colligere licet.

PROBLEMA 73

440. *Si sit v functio quaecunque trium variabilium x, y et z, eius formulas differentiales secundi altiorumque graduum exhibere.*

SOLUTIO

Cum eius formulae differentiales primi gradus sint tres

$$\left(\frac{dv}{dx}\right), \quad \left(\frac{dv}{dy}\right), \quad \left(\frac{dv}{dz}\right),$$

quaelibet instar novae functionis considerata iterum tres suppeditabit formulas differentiales, quae autem ob

$$\left(\frac{ddv}{dxdy}\right)=\left(\frac{ddv}{dydx}\right)$$

reducentur ad sex sequentes

$$\left(\frac{ddv}{dx^2}\right),\quad \left(\frac{ddv}{dy^2}\right),\quad \left(\frac{ddv}{dz^2}\right),\quad \left(\frac{ddv}{dxdy}\right),\quad \left(\frac{ddv}{dydz}\right)\quad \left(\frac{ddv}{dzdx}\right),$$

ex quarum denominatoribus intelligitur, quaenam trium quantitatum *x*, *y*, *z* in utraque differentiatione pro sola variabili haberi debeat. Simili modo evidens est formulas differentiales tertii gradus dari decem sequentes

$$\begin{array}{llll}
\left(\frac{ddv}{dx^3}\right), & \left(\frac{ddv}{dx^2dy}\right), & \left(\frac{ddv}{dxdy^2}\right), & \\
\left(\frac{ddv}{dy^3}\right), & \left(\frac{ddv}{dy^2dz}\right), & \left(\frac{ddv}{dydz^2}\right), & \left(\frac{ddv}{dxdzdx}\right), \\
\left(\frac{ddv}{dz^3}\right), & \left(\frac{ddv}{dz^2dx}\right), & \left(\frac{ddv}{dzdx^2}\right), &
\end{array}$$

Formularum porro differentialium quarti gradus numerus est 15, quinti 21 etc., secundum numeros triangulares, simulque ex cuiusque forma perspicuum est, quomodo eius valor ex data functione *v* per repetitam differentiationem in qualibet unicam variabilem considerando elici debeat.

COROLLARIUM 1

441. En ergo omnes formulas differentiales cuiusque gradus, quas ex qualibet functione trium variabilium derivare licet per differentiationem, quae porro ut functiones trium variabilium spectari possunt.

COROLLARIUM 2

442. Quemadmodum ergo ex huiusmodi functione data omnes eius formulae differentiales ope calculi differentialis inveniuntur, ita vicissim ex data quapiam formula differentiali vel duarum pluriumve relatione quadam ope calculi integralis ipsa illa functio, unde eae nascuntur, investigari debet.

SCHOLION 1

443. In calculo quidem differentiali parum refert, utrum functio differentianda unam pluresve variabiles involvat, cum praecepta differentiandi pro quovis variabilium numero maneant eadem; quam ob causam etiam calculum differentialem secundum hanc functionum varietatem in diversas partes distingui non erat opus. Longe secus autem accidit in calculo integrali, quem secundum hanc functionum varietatem omnino in partes dividi necesse est, quippe quae partes tam ratione propriae indolis quam ratione praeceptorum maxime inter se discrepant. Quemadmodum igitur hanc partem circa functiones trium variabilium occupatam tractari conveniat, exponendum videtur. Ac primo quidem ii casus commodissime evolventur, quibus unius cuiusdam formulae differentialis valor datur, ex quo indolem functionis quaesitae definiri oporteat, quoniam haec investigatio nulla laborat difficultate. Deinde huiusmodi quaestiones aggrediar, quibus relatio quaepiam inter duas pluresve formulas differentiales proponitur; ubi quidem plurimum refert, cuiusnam gradus ea fuerint, siquidem ex primo gradu plures casus expedire licet, dum ex altioribus vix adhuc quicquam in medium afferri potest. Hunc ergo ordinem in ista tractatione observabo.

SCHOLION 2

444. Videri hic posset ad functiones trium variabilium definiendas duas adeo conditiones seu relationes inter formulas differentiales admitti posse neque unica praescripta quaestionem esse determinatam. Quodsi enim ponatur

$$dv = pdx + qdy + rdz\,,$$

ubi litterae *p*, *q*, *r* vicem gerunt formularum differentialium primi gradus, atque verbi gratia hae duae proponantur conditiones, ut sit $q = p$ et $r = p$ ac propterea

$$dv = p\left(dx + dy + dz\right),$$

manifestum est solutionem dari posse, scilicet

$$v = \Gamma:\left(x + y + z\right).$$

Verum ad hanc obiectionem respondeo in hoc exemplo casu evenire, ut binae conditiones simul consistere possint; altera enim parumper immutata, ut manente $q = p$ esse debeat $r = px$ ideoque $dv = p\left(dx + dy + xdz\right)$, perspicuum est nullum pro *p* valorem exhiberi posse, per quem formula differentialis $dx + dy + xdz$ multiplicata integrabilis reddatur, quod unicum exemplum sufficit ad demonstrandum duabus conditionibus praescribendis huiusmodi quaestiones evadere plus quam determinatas neque propterea solutionem admittere nisi certis casibus, quibus quasi altera conditio iam in altera involvitur. Quocirca semper unica relatio inter formulas differentiales proposito omnino sufficit problemati determinando, quod idcirco, quia per integrationem functio arbitraria indefinita ingreditur, aeque parum pro indeterminato est habendum ac problemata calculi integralis communis, quorum solutio constantem arbitrariam introducit.