CHAPTER II

CONCERNING THE FINDING OF FUNCTIONS OF THREE VARIABLES FROM THE VALUE OF A CERTAIN DIFFERENTIAL FORMULA

PROBLEM 74

445. *From the value of a certain differential formula of the first order to find a function of three variables, which arises from that formula of the differential.*

SOLUTION

Let v be the function sought of the three variables x, y and z and let S be some given function of the same, to which the formula of the differential $\left(\frac{dv}{dx}\right)$ must be equal. Therefore since there shall be $\left(\frac{dv}{dx}\right)=S$, there will be with x put as the only quantity to be variable, with the two remaining truly y and z regarded as constants

$$dv=Sdx \quad \text{and thus} \quad v=\int Sdx+\text{Const.},$$

where it is to be noted in the integration of the formula Sdx, both the quantities y and z are to be regarded as constants, and in place of the Const. some function of y and z must be written, from which the function sought can be written thus

$$v=\int Sdx+T:(y \text{ and } z);$$

evidently here $T:(y \text{ and } z)$ may denote some quantity composed from the two variables y and z together with some constants.

In a similar manner if there is proposed $\left(\frac{dv}{dy}\right)=S$, then there will be

$$v=\int Sdy+T:(x \text{ and } z)$$

and this equation $\left(\frac{dv}{dz}\right)=S$ integrated gives

$$v=\int Sdz+T:(x \text{ and } y)$$

COROLLARY 1

446. Now here it is made abundantly clear that in the integration of functions of this kind, that in place of constants, there are to be introduced arbitrary functions of the two variable quantities and indeed to be established in the character of this integration.

COROLLARY 2

447. Therefore here we have given that solution of the problem, in which there is sought a function v of the three variables x, y, z, so that on putting

$$dv = pdx + qdy + rdz$$

there is made either $p = S$, $q = S$, or $r = S$ with some function S given arising from the same variables, or involving either two or a single one.

COROLLARY 3

448. Therefore if there should be $\left(\frac{dv}{dx}\right) = 0$ or $p = 0$, the function sought will be $v = \Gamma:(y \text{ and } z)$, and in order that there becomes $\left(\frac{dv}{dy}\right) = 0$, there will be $v = \Gamma:(x \text{ and } z)$, then truly so that there becomes $\left(\frac{dv}{dz}\right) = 0$, it is necessary that there shall be $v = \Gamma:(x \text{ and } y)$.

SCHOLIUM 1

449. Just as in the preceding part arbitrary functions of one variable are able to be represented by the applied lines of some curves, either regular or irregular, thus in this part arbitrary functions of two variables are able to be represented as it pleases by describing a surface. Thus if above a plane, in which the two coordinates x and y are taken in the customary manner, some extended surface is considered, the third coordinate will represent the distance of any point on the surface from that plane designating some function of the two variables x and y. And in this manner most readily truly the idea of functions of this kind may be consider to be established, since from that not only an account of regular functions, but also of irregular functions can be examined.

SCHOLIUM 2

450. Here also it is agreed to note also that functions of this kind of two variables can be designated in boundless ways. For in the plane mentioned with the two coordinates x and y changed into the two others t and u, so that there shall be $t = \alpha x + \beta y$ and $u = \gamma x + \delta y$, it is evident that t and u is a function of the two variables or $\Gamma: (t \text{ and } u)$ to agree with the function of x and y or $\Gamma:(x \text{ and } y)$; for if in place of t and u these values for x and y may be substituted, certainly a function is produced involving the two variables x and y. And more generally if t is equal to some given function of x and y and equally u is of this kind for the other function, then $\Gamma:(t \text{ and } u)$ with the substitution made will change into a function of x and y thus being expressed by $\Delta:(x \text{ et } y)$; for it is not necessary, that the same letter Γ denoting as it were an account of the composition of the function should each be the same, since this in general is done concerning functions of any kind. Whereby if in the following perhaps functions of this kind occur $\Gamma:(ax+by \text{ and } fxx+gyy)$ or $\Gamma:\left(\sqrt{(xx+yy)} \text{ and } l\frac{x}{y}\right)$ etc., this simple form $\Gamma:(x \text{ and } y)$ always can be written in place of these.

SCHOLIUM 3

451. The consideration of the solutions that we have given rests on the following reflections. In the first place on putting

$$dv = pdx + qdy + rdz$$

if there should be $p = \left(\frac{dv}{dx}\right) = 0$, there becomes

$$dv = qdy + rdz,$$

from which it is apparent that v is a quantity of this kind, the differential of this shall have this form $qdy + rdz$; because it cannot happen, unless the quantity v should be a function of the two variables y and z only with the third x completely excluded; and because no condition is been prescribed concerning the quantities q and r, we pronounce correctly that in place of the quantity v it is possible to take some function of the two variables y and z or to be $\Gamma:(y \text{ and } z)$, as a consideration of the formula $\left(\frac{dv}{dx}\right) = 0$ suggests the same solution.

Then if more generally there should be $\left(\frac{dv}{dx}\right) = p = S$ with S denoting some quantity from the variables x, y, z combined together, we will have

$$dv = Sdx + qdy + rdz,$$

which equation thus is resolved. In the first place the integral of the formula Sdx is sought with the quantity x to be considered as the variable, which shall be $= V$; and this quantity may be given by all three variables differentiated:

$$dV = Sdx + Qdy + Rdz;$$

from which since there shall be $Sdx = dV - Qdy - Rdz$, there will be

$$dv = dV + (q-Q)dy + (r-R)dz \quad \text{or} \quad d.(v-V) = (q-Q)dy + (r-R)dz,$$

from which as before it is apparent that the quantity $v - V$ can be put equal to some function of the two variables y and z. Whereby on account of $V = \int Sdx$ there is produced as before

$$v = \int Sdx + \Gamma:(y \text{ and } z);$$

and this reasoning, by which we have arrived here, deserves to be noted carefully, since also it was possible to give excellent use in the first part.

Indeed with the proposed equation [§ 296]

$$\left(\frac{ddz}{dy^2}\right) = aa\left(\frac{ddz}{dx^2}\right),$$

because here there is

$$d.\left(\frac{dz}{dx}\right) = dx\left(\frac{ddz}{dx^2}\right) + dy\left(\frac{ddz}{dxdy}\right) \quad \text{and} \quad d.\left(\frac{dz}{dy}\right) = dx\left(\frac{ddz}{dxdy}\right) + dy\left(\frac{ddz}{dy^2}\right),$$

there will be

$$ad.\left(\frac{dz}{dx}\right) + d.\left(\frac{dz}{dy}\right) = \left(\frac{ddz}{dx^2}\right)(adx + aady) + \left(\frac{ddz}{dxdy}\right)(ady + dx),$$

or

$$ad.\left(\frac{dz}{dx}\right) + d.\left(\frac{dz}{dy}\right) = (dx + ady)\left(a\left(\frac{ddz}{dx^2}\right) + \left(\frac{ddz}{dxdy}\right)\right),$$

of which the integral of the last part clearly is $F:(x+ay)$, and hence

$$\left(\frac{dz}{dy}\right) = -a\left(\frac{dz}{dx}\right) + a\Gamma':(x+ay),$$

from which a single absolute integration is agreed upon. Whereby since there shall be

$$dz = dx\left(\frac{dz}{dx}\right) + dy\left(\frac{dz}{dy}\right),$$

there will be had

$$dz = \left(\frac{dz}{dx}\right)(dx - ady) + ady\Gamma':(x+ay).$$

Let $\left(\frac{dz}{dx}\right) = p$ and $x - ay = t$,so that there becomes

$$dz = pdt + ady\Gamma':(t+2ay)$$

for the two variables *t* and *y*, and hence

$$z = \tfrac{1}{2}\Gamma:(t+2ay) + \int dt\left(p - \tfrac{1}{2}\Gamma':(t+2ay)\right) = \Gamma:(x+ay) + \Delta:(x-ay),$$

because

$$z = \tfrac{1}{2}\Gamma:(t+2ay) + \Delta:t = \Delta:(x-ay) \quad \text{and} \quad \Gamma:(t+2ay) = \Gamma:(x+ay).$$

PROBLEM 75

452. *To investigate the nature of the function of the three variables x, y, z, the formula of which is equal to some differential of a certain given function S.*

SOLUTION

Let v denote the function sought, and since six differential formulas of this of the second order may be given, in the first place we may put this to be $\left(\frac{ddv}{dx^2}\right)=S$ and on integrating once there emerges established

$$\left(\frac{dv}{dx}\right)=\int Sdx+\Gamma:(y \text{ and } z)$$

and on integrating again

$$v=\int dx\int Sdx+x\Gamma:(y \text{ and } z)+\Delta:(y \text{ and } z),$$

where in the twofold integration of the formula $\int dx\int Sdx$ only the quantity x is considered as variable, just as had been established above in [§ 249]. Moreover entirely alike is the integration of the equation

$$\left(\frac{ddv}{dy^2}\right)=S \text{ and } \left(\frac{ddv}{dz^2}\right)=S .$$

With the remaining differential formulas of the second order it suffices that one be resolved, $\left(\frac{ddv}{dxdy}\right)=S$; which integrated by the variable x alone will give initially

$$\left(\frac{dv}{dy}\right)=\int Sdx+f:(y \text{ and } z).$$

Then with the other integration by the variable y put in place there is deduced

$$v=\int dy\int Sdx+\int dyf:(y \text{ and } z)+\Delta:(x \text{ and } z),$$

where in the first place I observe that there is no distinction of the order to be had between the two variables x and y in the first part and thus it can be expressed by $\int\int Sdxdy$. Then [in the second part] there should be some function $f:(y \text{ and } z)$ of y and z that may be integrated, if that is multiplied by dy and with z considered as constant, and it is evident that a new function of y and z is produced, and because that cannot be determined in any way, also this is to be indeterminate and thus arbitrary, from which we may put in place :

$$v = \iint Sdxdy + \Gamma:(y \text{ and } z) + \Delta:(x \text{ and } z).$$

COROLLARY 1

453. Here I note now that through the integration of the formula $\int dyf:(y \text{ and } z)$ at once the formula $\Delta:(x \text{ and } z)$ is to be brought in; since indeed there, only the quantity y may be considered as variable, and in place of the constant quantity on integration there can be written to be added some function of x and z.

COROLLARY 2

454. But if that given function S vanishes, the following integrations come about:

If $\left(\frac{ddv}{dx^2}\right) = 0$, there will be $v = x\Gamma:(y \text{ and } z) + \Delta:(y \text{ and } z)$,

if $\left(\frac{ddv}{dy^2}\right) = 0$, there will be $v = y\Gamma:(x \text{ and } z) + \Delta:(x \text{ and } z)$,

if $\left(\frac{ddv}{dz^2}\right) = 0$, there will be $v = z\Gamma:(x \text{ and } y) + \Delta:(x \text{ and } y)$,

if $\left(\frac{ddv}{dxdy}\right) = 0$, there will be $v = \Gamma:(x \text{ and } z) + \Delta:(y \text{ and } z)$,

if $\left(\frac{ddv}{dxdz}\right) = 0$, there will be $v = \Gamma:(x \text{ and } y) + \Delta:(y \text{ and } z)$,

if $\left(\frac{ddv}{dydz}\right) = 0$, there will be $v = \Gamma:(x \text{ and } y) + \Delta:(x \text{ and } z)$.

COROLLARY 3

455. Because here there is a need for a twofold integration and also two arbitrary functions, each of the two variables have been placed in the calculation, this most certainly is the criterion that the integrals found are complete.

SCHOLIUM

456. These same integrals can be elicited in another way, which depends on the principle indicated above (§ 451), because if there should be $dv = Sdx + qdy + rdz$, it becomes

$$v = \int Sdx + f:(y \text{ and } z).$$

Therefore following this principle, if there should be $\left(\frac{ddv}{dx^2}\right) = S$, there will be

$$d.\left(\frac{dv}{dx}\right) = Sdx + dy\left(\frac{ddv}{dxdy}\right) + dz\left(\frac{ddv}{dxdz}\right),$$

which form with that deduced in place of v we may consider $\left(\frac{dv}{dx}\right)$ and in place of q and r these formulas $\left(\frac{ddv}{dxdy}\right)$ and $\left(\frac{ddv}{dxdz}\right)$ from which the integral will be

$$\left(\frac{dv}{dx}\right)=\int Sdx+f:(y \text{ and } z).$$

Now since again there will be

$$dv=\left(\frac{dv}{dx}\right)dx+\left(\frac{dv}{dy}\right)dy+\left(\frac{dv}{dz}\right)dz,$$

there will be

$$dv=dx\int Sdx+dxf:(y \text{ and } z)+dy\left(\frac{dv}{dy}\right)+dz\left(\frac{dv}{dz}\right),$$

from which equally it evidently follows that

$$v=\int dx\int Sdx+xf:(y \text{ and } z)+\Delta:(y \text{ and } z).$$

In a like manner the operation can be established for the equation $\left(\frac{ddv}{dxdy}\right)=S$; from which indeed there becomes

$$d.\left(\frac{dv}{dy}\right)=Sdx+dy\left(\frac{ddv}{dy^2}\right)+dz\left(\frac{ddv}{dydz}\right),$$

the integral of which is

$$\left(\frac{dv}{dy}\right)=\int Sdx+f:(y \text{ and } z);$$

the other integration is put in place in this form

$$dv=dy\int Sdx+dyf:(y \text{ and } z)+dx\left(\frac{dv}{dx}\right)+dz\left(\frac{dv}{dz}\right),$$

from which on account of $\int dyf:(y \text{ and } z)=\Gamma:(y \text{ and } z)$ there will be found as before

$$v=\iint Sdxdy+\Gamma:(y \text{ and } z)+\Delta:(x \text{ and } z).$$

PROBLEM 76

457. *To investigate the nature of functions of the three variables x, y and z, of which a certain differential formula of the third order is equal to some given quantity S with the variables and constants composed in some manner from this.*

SOLUTION

With the function sought put = v we may run through not only the individual differential formulas of this of the third order as well as these, of which the account is different.

Therefore in the first place, let $\left(\frac{d^3v}{dx^3}\right)=S$ and the first integration gives at once

$$\left(\frac{ddv}{dx^2}\right)=\int Sdx+2\Gamma:(y \text{ and } z),$$

then indeed the other

$$\left(\frac{dv}{dx}\right)=\int dx\int Sdx+2x\Gamma:(y \text{ and } z)+\Delta:(y \text{ and } z),$$

from which again there is deduced

$$v=\int dx\int dx\int Sdx+xx\Gamma:(y \text{ and } z)+x\Delta:(y \text{ and } z)+\Sigma:(y \text{ and } z).$$

In the second place let there be $\left(\frac{d^3v}{dx^2dy}\right)=S$ and the two first integrations give as before

$$\left(\frac{dv}{dy}\right)=\int dx\int Sdx+x\Gamma:(y \text{ and } z)+\Delta:(y \text{ and } z);$$

because now, as we have seen [§ 452], for $dy\Gamma:(y \text{ and } z)$ it is possible to write $\Gamma:(y \text{ and } z)$, by the third integration we come upon

$$v=\int^3 Sdx^2dy+x\Gamma:(y \text{ and } z)+\Delta:(y \text{ and } z)+\Sigma:(x \text{ and } z).$$

But in these two cases all the differential formulas of the third order are contained in the permutations of the variables with this final exception alone $\left(\frac{d^3v}{dxdydz}\right)$, as on that account it must be treated separately.

Therefore let $\left(\frac{d^3v}{dxdydz}\right)=S$ and from the first integration by the variable x alone established there is obtained

$$\left(\frac{ddv}{dxdz}\right)=\int Sdx+f:(y \text{ and } z),$$

now following it may be integrated by the variable *y* alone and there will be found

$$\left(\frac{dv}{dz}\right)=\iint Sdxdy+\Gamma:(y \text{ and } z)+\Delta:(y \text{ and } z)$$

from which finally the third integration by *z* will give

$$v=\int^3 Sdxdydz=\Gamma:(y \text{ and } z)+\Delta:(x \text{ and } z)+\Sigma:(x \text{ and } y),$$

and thus the problem has been completely resolved.

COROLLARY 1

458. Since here there was a need for a threefold integration, the integrals found also embraced three arbitrary functions and each one of these of two variables, just as the nature of complete integrals postulates.

COROLLARY 2

459. If the given quantity *S* vanishes, these integrals can themselves be considered in the following way :
If there should be $\left(\frac{d^3v}{dx^3}\right)=0$, then there will be

$$v=xx\Gamma:(y \text{ and } z)+x\Delta:(y \text{ and } z)+\Sigma:(y \text{ and } z);$$

If there should be $\left(\frac{d^3v}{dx^2dy}\right)=0$, then there will be

$$v=x\Gamma:(y \text{ and } z)+\Delta:(y \text{ and } z)+\Sigma:(x \text{ and } z);$$

if there should be $\left(\frac{d^3v}{dxdydz}\right)=0$, then there will be

$$v=\Gamma:(y \text{ and } z)+\Delta:(x \text{ and } z)+\Sigma:(x \text{ and } y).$$

SCHOLIUM

460. The same integrals are able to be found again by another method explained above [§ 451] and it would be superfluous to put in place the individual operations here. Moreover there is little need to pursue these investigations concerning formulas of higher order, since the rule of the progression of the integrals of the individual arbitrary functions put in place, since both the parts themselves as well as these [precepts] which have been explained above, evidently will be satisfied. Whereby for this chapter, in which a certain formula must be equal to the given quantity of the differential, this has been completely satisfied.

But before I proceed further, I may propose besides two widely applicable cases, the resolution of which is easily reduced to the preceding parts of the integral calculus treated already, as therefore only this concession is permitted to be assumed, if indeed the difficulties which occur in these, not at present in place, are to be referred to.

PROBLEM 77

461. *If in a proposed relation, from which it is required to define the nature of the function of the three variables x, y and z, other differential formulas are not to enter, other than those which arise from the single variable x, which are*

$$\left(\frac{dv}{dx}\right),\quad \left(\frac{ddv}{dx^2}\right),\quad \left(\frac{d^3v}{dx^3}\right)\ \text{etc.},$$

to find the function sought.

SOLUTION

Since the proposed equation containing the relation does not include other differential formulas besides the mentioned ones, in these the two quantities y and z may be considered as constants and thus also in the individual integrations are able to be treated as such. Hence the proposed equation is agreed to involve only the two variables x and v, and with the chains of differential formulas rejected there will be had the differential equation referring to the first book, in which, if it rises to higher powers, the element dx is required to be considered constant. But if hence this equation can be integrated with the aid of the precepts in that place, then in place of the constants entering by the single integration there may be put in place arbitrary functions of the two variables y and z, just as $\Gamma:(y \text{ and } z)$, $\Delta:(y \text{ and } z)$ etc., and thus there will be had a complete solution of the proposed equation.

COROLLARY 1

462. In addition most of the integrations set out in book I also admit the following differential equations to resolution of whatever height of order

$$S = Av + B\left(\frac{dv}{dx}\right) + C\left(\frac{ddv}{dx^2}\right) + D\left(\frac{d^3v}{dx^3}\right) + \text{etc.}$$

and

$$S = Av + Bx\left(\frac{dv}{dx}\right) + Cx^2\left(\frac{ddv}{dx^2}\right) + Dx^3\left(\frac{d^3v}{dx^3}\right) + \text{etc.}$$

COROLLARY 2

463. Indeed with the chains abandoned there may be had differential equations of this kind, such as we have taught how to integrate in the final chapters of book I. Yet there is a need, that in place of the constants introduced by integration such functions may be written

$$\Gamma:(y \text{ and } z),\ \Delta:(y \text{ and } z),\ \Sigma:(y \text{ and } z) \text{ etc.},$$

so that by this agreement the complete integrals may be obtained.

SCHOLIUM

464. Here also proposed relations of this kind can be referred to, in which differential equations involving the two elements *dx* and *dy* thus may be encountered, so that in this *dy* should have the same number of dimensions, of which kind are

$$\left(\frac{dv}{dy}\right),\quad \left(\frac{ddv}{dxdy}\right),\quad \left(\frac{d^3v}{dx^2dy}\right),\quad \left(\frac{d^4v}{dx^3dy}\right) \quad \text{etc.}$$

or

$$\left(\frac{ddv}{dy^2}\right),\quad \left(\frac{d^3v}{dxdy^2}\right),\quad \left(\frac{d^4v}{dx^2dy^2}\right),\quad \left(\frac{d^5v}{dx^3dy^2}\right) \quad \text{etc.},$$

but then the quantity v may never appear. For if then for the former case there is put $\left(\frac{dv}{dy}\right)=u$, for the latter truly $\left(\frac{ddv}{dy^2}\right)=u$, the relation will recall other differential formulas to the case of the problem besides not involving v

$$\left(\frac{du}{dx}\right),\quad \left(\frac{ddu}{dx^2}\right),\quad \left(\frac{d^3u}{dx^3}\right) \text{ etc.}$$

and perhaps the function u itself. Whereby if the equation is to be integrated by the above precepts and hence the function u may be defined, then on restoring in place of u either $\left(\frac{dv}{dy}\right)$ or $\left(\frac{ddv}{dy^2}\right)$, so that there becomes $\left(\frac{dv}{dy}\right)=S$ or $\left(\frac{ddv}{dy^2}\right)=S$, hence also by the precepts of this chapter the function v itself will be determined. So that also in this manner equations of this kind may be able to be resolved including only the differential formulas

$$\left(\frac{d^{\mu+\nu}v}{dy^{\mu}dz^{\nu}}\right),\quad \left(\frac{d^{\mu+\nu+1}v}{dxdy^{\mu}dz^{\nu}}\right),\quad \left(\frac{d^{\mu+\nu+2}v}{dx^2dy^{\mu}dz^{\nu}}\right) \text{ etc.},$$

where all three elements *dx, dy, dz* occur; for on putting the other formulas, the equation will not contain other formulas except

$$\left(\frac{du}{dx}\right),\quad \left(\frac{ddu}{dx^2}\right),\quad \left(\frac{d^3u}{dx^3}\right) \text{ etc.}$$

together with the function u and thus it will be referred to the case of the previous problem; from which the resolution if it may arise $u = S\left(\frac{d^{\mu+\nu}v}{dy^{\mu}dz^{\nu}}\right)$, now for some known function S, and the investigation of the function v now labours under no difficulty.

But besides another reducible case is given according to the first part of book II, that I am about to set out in the following problem.

PROBLEM 78

465. *If in the proposed relation, from which of the three variables x, y, z it is required to define the function v, other differential formulas not being present, except those which arise only from the variability of the two x and y; to find the function v with the third element dz completely excluded.*

SOLUTION

Because in resolving the problem, which the proposed relation holds, the quantity z does not enter as a variable, however many integrations should be put in place, thus in these the quantity z must be treated as a constant. Therefore the resolution of equations of this kind is to be referred to the preceding part, since a function of the two variables only x and y from a given relation of the differential formulas shall be investigated. But if accordingly the work should be successful and thus the integral should be found, in that just as many arbitrary functions of the one variable shall occur clearly constructed in this way from x and y, as there should be a need for in the integrals ; let $\Gamma{:}t$ be a function of this kind, where t is assumed to be given by x and y, and now, so that this solution can be adapted to the present set up, where the quantity z is enumerated with the variables, in place of which arbitrary function $\Gamma{:}t$ there may be written here $\Gamma{:}(t \text{ and } z)$, evidently a function of two variables, and thus a complete integral will be had.

COROLLARY 1

466. Therefore if this equation should be proposed

$$\left(\frac{ddv}{dy^2}\right) = aa\left(\frac{ddv}{dx^2}\right),$$

because we have found for the present case in the preceding part [§ 296], where v must be a function of the three variables x, y and z, the integral thus itself will be had

$$v = \Gamma{:}\left(\overline{x+ay} \text{ and } z\right) + \Delta{:}\left(\overline{x-ay} \text{ and } z\right).$$

COROLLARY 2

467. Here clearly it is required to remember the form

$$\Gamma:\left(\overline{x+ay} \text{ and } z\right)$$

to designate some function of the two variables, of which one shall be $=x+ay$, the other truly $=z$; from which the function itself will be allowed to be represented by the applied line referred to a certain surface.

SCHOLIUM

468. But not only equations in the problem described according to the preceding part of the integral calculus may be reduced, but also innumerable others, which with a certain substitution made are recalled to that form. Just as if in the proposed equation other differential formulas do not occur, unless in which everything is found with the single dimension *dz*, which are

$$\left(\frac{dv}{dz}\right),\quad \left(\frac{ddv}{dxdz}\right),\quad \left(\frac{ddv}{dydz}\right),\quad \left(\frac{d^3v}{dx^2dz}\right),\quad \left(\frac{d^3v}{dxdydz}\right),\quad \left(\frac{d^3v}{dy^2dz}\right) \text{ etc.}$$

it is evident on putting $\left(\frac{dv}{dz}\right)=u$ that equation to be transformed into another, from which, from which now it is required to investigate the function *u*, and that to be referred to the case set out in the problem. Whereby if thence the nature of the function *u* can be defined, so that there shall be $u=S$, there remains, that this equation $\left(\frac{dv}{dz}\right)=S$ is resolved, from which , as we saw before [§ 445], shall become

$$v=\int Sdz+\Gamma:\left(x \text{ and } y\right).$$

This likewise is to be understood, if the proposed equation with the aid of the substitution $\left(\frac{ddv}{dz^2}\right)=u$ or $\left(\frac{d^3v}{dz^3}\right)=u$ etc. is able to be reduced to the case of the problem [proposed above]; so that also it is evident accordingly, if with the aid of some transformation, the proposed equation can be reduced to the case of the problem proposed. But I have set out more such transformations above, provided that either in place of the function sought *v* another *u* is introduced on putting $v=Su$ or the variables themselves *x, y, z* are changed into others *p, q, r*, which maintain a certain relation to these, as I have explained above further in the case of the two variables [§ 232, 240]; and thus from this it is evident, that a similar reduction to this case of three variables can be readily accommodated. Perhaps still in the following, transformations of this kind occur ; hence for the other cases, where formulas of all kinds of differentials occur, I scarcely advance beyond explaining the first elements of how the theory may be set out.

CAPUT II

DE INVENTIONE FUNCTIONUM TRIUM VARIABILIUM EX DATO CUIUSPIAM FORMULAE DIFFERENTIALIS VALORE

PROBLEMA 74

445. *Dato valore cuiuspiam formulae differentialis primi gradus investigare ipsam functionem trium variabilium, ex qua illa formula differentialis nascitur.*

SOLUTIO

Sit v functio quaesita trium variabilium x, y et z et S earundem functio data quaecunque, cui formula differentialis $\left(\frac{dv}{dx}\right)$ debeat esse aequalis. Cum igitur sit $\left(\frac{dv}{dx}\right)=S$, erit posita sola quantitate x variabili, binis reliquis vero y et z ut constantibus spectatis

$$dv=Sdx \quad \text{ideoque} \quad v=\int Sdx+\text{Const.},$$

ubi notandum est in integratione formulae Sdx ambas quantitates y et z pro constantibus haberi et loco Const.functionem quamcunque ipsarum y et z scribi debere, ex quo functio quaesita ita exhiberi poterit

$$v=\int Sdx+T:(y \text{ et } z);$$

hic scilicet $T:(y \text{ et } z)$ quantitatem quamcunque ex binis quantitatibus y et z una cum constantibus utcunque conflatam denotat.

Simili modo si proponatur $\left(\frac{dv}{dy}\right)=S$, erit

$$v=\int Sdy+T:(x \text{ et } z)$$

et haec aequatio $\left(\frac{dv}{dz}\right)=S$ integrata praebet

$$v=\int Sdz+T:(x \text{ et } y)$$

COROLLARIUM 1

446. Hic iam abunde intelligitur integratione huiusmodi functionum loco constantis introduci functionem arbitrariam duarum quantitatum variabilium atque adeo in hoc characterem harum integrationum esse constituendum.

COROLLARIUM 2

447. Hic ergo istud problema solutum dedimus, quo quaeritur functio *v* trium variabilium *x, y, z*, ut posito

$$dv = pdx + qdy + rdz$$

fiat vel $p = S$ vel $q = S$ vel $r = S$ existente *S* functione quacunque data easdem variabiles vel duas vel unicam involvente.

COROLLARIUM 3

448. Quodsi igitur esse debeat $\left(\frac{dv}{dx}\right) = 0$ seu $p = 0$, functio quaesita erit $v = \Gamma:(y \text{ et } z)$, et ut fiat $\left(\frac{dv}{dy}\right) = 0$, erit $v = \Gamma:(x \text{ et } z)$, tum vera ut fiat $\left(\frac{dv}{dz}\right) = 0$, necesse est sit $v = \Gamma:(x \text{ et } y)$.

SCHOLION 1

449. Quemadmodum in praecedente parte functiones arbitrariae unius variabilis per applicatas curvarum quarumcunque, sive regularium sive etiam irregularium, repraesentari poterant, ita in hac parte functiones binarum variabilium arbitrariae per superficiem pro lubitu descriptam repraesentari possunt. Ita si super plano, in quo binae coordinatae *x* et *y* more solito assumuntur, superficies quaecunque expansa concipiatur, tertia coordinata distantiam cuiusvis superficiei puncti ab illo plano designans functionem quamcunque binarum variabilium *x* et *y* repraesentabit. Hocque modo aptissime vera idea huiusmodi functionum constitui videtur, cum ex ea non solum ratio harum functionum regularium, sed etiam irregularium perspiciatur.

SCHOLION 2

450. Hic etiam notari convenit huiusmodi functiones binarum variabilium infinitis diversis modis etiam designari posse. Variatis enim in plano memorato binis coordinatis *x* et *y* in binas alias *t* et *u*, ut sit $t = \alpha x + \beta y$ et $u = \gamma x + \delta y$, manifestum est functionem binarum variabilium *t* et *u* seu $\Gamma: (t \text{ et } u)$ convenire cum functione ipsarum *x* et *y* seu $\Gamma:(x \text{ et } y)$; si enim loco *t* et *u* illi valores pro *x* et *y* substituantur, utique prodit functio duas tantum variabiles *x* et *y* involvens. Atque multo generalius si *t* aequetur functioni cuipiam datae ipsarum *x* et *y* pariterque *u* huiusmodi alii functioni, tum $\Gamma: (t \text{ et } u)$ facta substitutione abibit in functionem ipsarum *x* et *y* ita exprimendam $\Delta:(x \text{ et } y)$; non enim necesse est, ut idem functionis character Γ rationem compositionis quasi denotans utrinque sit idem, cum hic in genere de functionibus quibuscunque agatur. Quare si in sequentibus forte eiusmodi functiones occurrant $\Gamma:(ax + by \text{ et } fxx + gyy)$ vel $\Gamma:\left(\sqrt{(xx + yy)} \text{ et } l\frac{x}{y}\right)$ etc., earum loco semper haec forma simplex $\Gamma:(x \text{ et } y)$ scribi potest.

SCHOLION 3

451. Solutionis, quam dedimus, consideratio nobis suppeditat sequentes reflexiones. Primo posito

$$dv = pdx + qdy + rdz$$

si debeat esse $p = \left(\frac{dv}{dx}\right) = 0$, fiet

$$dv = qdy + rdz,$$

unde patet *v* eiusmodi esse quantitatem, cuius differentiale hanc habiturum sit formam $qdy + rdz$; quod fieri nequit, nisi quantitas *v* fuerit functio binarum variabilium *y* et *z* tantum tertia *x* penitus exclusa; et quia circa quantitates *q* et *r* nulla conditio praescribitur, recte pronunciamus loco quantitatis *v* accipi posse functionem quamcunque binarum variabilium *y* et *z* seu esse $\Gamma:(y \text{ et } z)$, quam eandem solutionem consideratio formulae $\left(\frac{dv}{dx}\right) = 0$ suggessit.

Deinde si esse debeat generalius $\left(\frac{dv}{dx}\right) = p = S$ denotante *S* quantitatem quamcunque ex variabilibus *x, y, z* conflatam, habebimus

$$dv = Sdx + qdy + rdz,$$

quae aequatio ita resolvitur. Quaeratur primo integrale formulae *Sdx* sola quantitate *x* ut variabili spectata, quod sit = *V;* haecque quantitas per omnes tres variabiles differentiata praebeat

$$dV = Sdx + Qdy + Rdz;$$

ex quo cum sit $Sdx = dV - Qdy - Rdz$, erit

$$dv = dV + (q - Q)dy + (r - R)dz \quad \text{seu} \quad d.(v - V) = (q - Q)dy + (r - R)dz,$$

unde ut ante patet quantitatem $v - V$ functioni cuicunque binarum varabilium *y* et *z* aequari posse. Quare ob $V = \int Sdx$ prodit ut ante

$$v = \int Sdx + \Gamma:(y \text{ et } z);$$

hocque ratiocinium, quo isthuc pervenimus, diligenter notari meretur, cum etiam in parte prima eximium usum praestare possit.

Proposita enim aequatione [§ 296]

$$\left(\frac{ddz}{dy^2}\right) = aa\left(\frac{ddz}{dx^2}\right)$$

quia est

$$d.\left(\frac{dz}{dx}\right) = dx\left(\frac{ddz}{dx^2}\right) + dy\left(\frac{ddz}{dxdy}\right) \quad \text{et} \quad d.\left(\frac{dz}{dy}\right) = dx\left(\frac{ddz}{dxdy}\right) + dy\left(\frac{ddz}{dy^2}\right),$$

erit

$$ad.\left(\frac{dz}{dx}\right) + d.\left(\frac{dz}{dy}\right) = \left(\frac{ddz}{dx^2}\right)(adx + aady) + \left(\frac{ddz}{dxdy}\right)(ady + dx),$$

seu

$$ad.\left(\frac{dz}{dx}\right)+d.\left(\frac{dz}{dy}\right)=(dx+ady)\left(a\left(\frac{ddz}{dx^2}\right)+\left(\frac{ddz}{dxdy}\right)\right),$$

cuius posterioris membri integrale manifesto est $F:(x+ay)$, hincque

$$\left(\frac{dz}{dy}\right)=-a\left(\frac{dz}{dx}\right)+a\Gamma':(x+ay),$$

quo una integratio absoluta est censenda. Quare cum sit

$$dz=dx\left(\frac{dz}{dx}\right)+dy\left(\frac{dz}{dy}\right),$$

habebitur

$$dz=\left(\frac{dz}{dx}\right)(dx-ady)+ady\Gamma':(x+ay).$$

Sit $\left(\frac{dz}{dx}\right)=p$ et $x-ay=t$,ut fiat

$$dz=pdt+ady\Gamma':(t+2ay)$$

pro duabus variabilibus t et y hincque

$$z=\tfrac{1}{2}\Gamma:(t+2ay)+\int dt\left(p-\tfrac{1}{2}\Gamma':(t+2ay)\right)=\Gamma:(x+ay)+\Delta:(x-ay),$$

quia

$$z=\tfrac{1}{2}\Gamma:(t+2ay)+\Delta:t=\Delta:(x-ay)\quad\text{et}\quad\Gamma:(t+2ay)=\Gamma:(x+ay).$$

PROBLEMA 75

452. *Investigare indolem functionis trium variabilium x, y, z, cuius formula quaedam differentialis secundi gradus aequetur datae cuipiam functioni S.*

SOLUTIO

Denotet v functionem quaesitam, et cum eius sex dentur formulae differentiales secundi gradus, ponamus primo esse debere $\left(\frac{ddv}{dx^2}\right)=S$ et integratione semel instituta prodit

$$\left(\frac{dv}{dx}\right)=\int Sdx+\Gamma:(y\text{ et }z)$$

iterumque integrando

$$v=\int dx\int Sdx+x\Gamma:(y\text{ et }z)+\Delta:(y\text{ et }z),$$

ubi in formulae $\int dx \int Sdx$ duplici integratione sola quantitas x ut variabilis spectatur, quemadmodum iam supra [§ 249] est inculcatum. Similis autem omnino est integratio aequationum

$$\left(\frac{ddv}{dy^2}\right) = S \text{ et } \left(\frac{ddv}{dz^2}\right) = S .$$

Pro reliquis formulis differentialibus secundi gradus sufficit hanc unam $\left(\frac{ddv}{dxdy}\right) = S$ resolvisse; quae primo per solam variabilem x integrata dabit

$$\left(\frac{dv}{dy}\right) = \int Sdx + f:(y \text{ et } z).$$

Deinde altera integratione per solam variabilem y instituta colligitur

$$v = \int dy \int Sdx + \int dyf:(y \text{ et } z) + \varDelta:(x \text{ et } z),$$

ubi primum observo partem primam nullo discrimine ordinis inter binas variabiles x et y habito ita $\int\int Sdxdy$ exprimi posse. Deinde quaecunque fuerit $f:(y \text{ et } z)$ functio ipsarum y et z, si ea per dy multiplicetur et spectata z ut constante integretur, evidens est denuo functionem ipsarum y et z prodire, et quia illa nullo modo determinatur, etiam hanc fore indeterminatam ideoque arbitrariam, unde statuere poterimus

$$v = \iint Sdxdy + \varGamma:(y \text{ et } z) + \varDelta:(x \text{ et } z).$$

COROLLARIUM 1

453. Hic observo per integrationem formulae $\int dyf:(y \text{ et } z)$ iam sponte formulam $\varDelta:(x \text{ et } z)$ invehi; cum enim ibi sola quantitas y ut variabilis spectetur, loco quantitatis constantis per integrationem adiiciendae functio quaecunque ipsarum x et z scribi poterit.

COROLLARlUM 2

454. Quodsi functio illa data S evanescat, sequentes integrationes provenient:

$$\text{Si}\quad \left(\frac{ddv}{dx^2}\right)=0,\quad \text{erit}\quad v=x\Gamma:(y\text{ et }z)+\Delta:(y\text{ et }z),$$
$$\text{si}\quad \left(\frac{ddv}{dy^2}\right)=0,\quad \text{erit}\quad v=y\Gamma:(x\text{ et }z)+\Delta:(x\text{ et }z),$$
$$\text{si}\quad \left(\frac{ddv}{dz^2}\right)=0,\quad \text{erit}\quad v=z\Gamma:(x\text{ et }y)+\Delta:(x\text{ et }y),$$
$$\text{si}\quad \left(\frac{ddv}{dxdy}\right)=0,\quad \text{erit}\quad v=\Gamma:(x\text{ et }z)+\Delta:(y\text{ et }z),$$
$$\text{si}\quad \left(\frac{ddv}{dxdz}\right)=0,\quad \text{erit}\quad v=\Gamma:(x\text{ et }y)+\Delta:(y\text{ et }z),$$
$$\text{si}\quad \left(\frac{ddv}{dydz}\right)=0,\quad \text{erit}\quad v=\Gamma:(x\text{ et }y)+\Delta:(x\text{ et }z).$$

COROLLARIUM 3

455. Quia hic duplici opus est integratione atque etiam duae functiones arbitrariae, utraque binarum variabilium, in calculum sunt invectae, hoc certissimum est criterium haec integralia inventa esse completa.

SCHOLION

456. Alio etiam modo haec eadem integralia erui possunt, qui nititur principio supra (§ 451) indicato, quodsi fuerit $dv=Sdx+qdy+rdz$, fore

$$v=\int Sdx+f:(y\text{ et }z).$$

Secundum hoc principium ergo, si fuerit $\left(\frac{ddv}{dx^2}\right)=S$, erit

$$d.\left(\frac{dv}{dx}\right)=Sdx+dy\left(\frac{ddv}{dxdy}\right)+dz\left(\frac{ddv}{dxdz}\right),$$

qua forma cum illa collata loco v habemus $\left(\frac{dv}{dx}\right)$ et loco q et r has formulas $\left(\frac{ddv}{dxdy}\right)$ et $\left(\frac{ddv}{dxdz}\right)$ ex quo integrale erit

$$\left(\frac{dv}{dx}\right)=\int Sdx+f:(y\text{ et }z).$$

Cum iam porro sit

$$dv=\left(\frac{dv}{dx}\right)dx+\left(\frac{dv}{dy}\right)dy+\left(\frac{dv}{dz}\right)dz,$$

erit

$$dv = dx\int Sdx + dxf:(y \text{ et } z) + dy\left(\tfrac{dv}{dy}\right) + dz\left(\tfrac{dv}{dz}\right),$$

unde pariter manifesto sequitur

$$v = \int dx\int Sdx + xf:(y \text{ et } z) + \varDelta:(y \text{ et } z).$$

Pari modo operatio est instituenda pro aequatione $\left(\frac{ddv}{dxdy}\right) = S$; inde enim fit

$$d.\left(\tfrac{dv}{dy}\right) = Sdx + dy\left(\tfrac{ddv}{dy^2}\right) + dz\left(\tfrac{ddv}{dydz}\right),$$

cuius integrale est

$$\left(\tfrac{dv}{dy}\right) = \int Sdx + f:(y \text{ et } z);$$

altera integratio instituatur in hac forma

$$dv = dy\int Sdx + dyf:(y \text{ et } z) + dx\left(\tfrac{dv}{dx}\right) + dz\left(\tfrac{dv}{dz}\right),$$

unde ob $\int dyf:(y \text{ et } z) = \varGamma:(y \text{ et } z)$ obtinetur ut ante

$$v = \iint Sdxdy + \varGamma:(y \text{ et } z) + \varDelta:(x \text{ et } z).$$

PROBLEMA 76

457. *Investigare indolem functionis trium variabilium x, y et z, cuius quaedam formula differentialis tertii gradus aequetur datae cuipiam quantitati S ex illisvariabilibus et constantibus utcunque compositae.*

SOLUTIO

Posita functione quaesita = v percurramus non tam singulas eius formulas differentiales tertii gradus quam eas, quarum ratio est diversa.

Sit igitur primo $\left(\frac{d^3v}{dx^3}\right)=S$ et prima integratio statim dat

$$\left(\frac{ddv}{dx^2}\right)=\int Sdx+2\Gamma:(y \text{ et } z),$$

tum vero altera

$$\left(\frac{dv}{dx}\right)=\int dx\int Sdx+2x\Gamma:(y \text{ et } z)+\Delta:(y \text{ et } z),$$

unde tandem colligitur

$$v=\int dx\int dx\int Sdx+xx\Gamma:(y \text{ et } z)+x\Delta:(y \text{ et } z)+\Sigma:(y \text{ et } z).$$

Sit secundo $\left(\frac{d^3v}{dx^2dy}\right)=S$ et binae priores integrationes ut ante dant

$$\left(\frac{dv}{dy}\right)=\int dx\int Sdx+x\Gamma:(y \text{ et } z)+\Delta:(y \text{ et } z);$$

quia nunc, ut vidimus [§ 452], pro $dy\Gamma:(y \text{ et } z)$ scribere licet $\Gamma:(y \text{ et } z)$, per tertiam integrationem invenimus

$$v=\int^3 Sdx^2dy+x\Gamma:(y \text{ et } z)+\Delta:(y \text{ et } z)+\Sigma:(x \text{ et } z).$$

In his autem duobus casibus omnes formulae differentiales tertii gradus variabilibus permutandis continentur sola excepta ultima hac $\left(\frac{d^3v}{dxdydz}\right)$, quam idcirco seorsim tractari oportet.

Sit igitur $\left(\frac{d^3v}{dxdydz}\right)=S$ et prima integratione per solam variabilem x instituta obtinetur

$$\left(\frac{ddv}{dxdz}\right)=\int Sdx+f:(y \text{ et } z),$$

nunc secunda integretur per solam yariabilem *y* ac reperietur

$$\left(\frac{dv}{dz}\right)=\iint Sdxdy+\Gamma:(y \text{ et } z)+\Delta:(y \text{ et } z)$$

unde tandem tertia integratio per *z* dabit

$$v=\int^3 Sdxdydz=\Gamma:(y \text{ et } z)+\Delta:(x \text{ et } z)+\Sigma:(x \text{ et } y),$$

sicque problema perfecte est resolutum.

COROLLARIUM 1

458. Quoniam hic triplici opus erat integratione, integralia inventa etiam tres functiones arbitrarias complectuntur easque singulas binarum variabilium, quemadmodum natura integralium completorum postulat.

COROLLARIUM 2

459. Si quantitas data *S* evanescat, integralia haec sequenti modo se habebunt:
Si fuerit $\left(\frac{d^3v}{dx^3}\right)=0$, erit

$$v=xx\Gamma:(y \text{ et } z)+x\Delta:(y \text{ et } z)+\Sigma:(y \text{ et } z);$$

si fuerit $\left(\frac{d^3v}{dx^2dy}\right)=0$, erit

$$v=x\Gamma:(y \text{ et } z)+\Delta:(y \text{ et } z)+\Sigma:(x \text{ et } z);$$

si fuerit $\left(\frac{d^3v}{dxdydz}\right)=0$, erit

$$v=\Gamma:(y \text{ et } z)+\Delta:(x \text{ et } z)+\Sigma:(x \text{ et } y).$$

SCHOLION

460. Eadem integralia etiam altera methodo supra [§ 451] exposita inveniri possunt superfluumque foret singulas operationes hic apponere. Aeque parum autem opus erit has investigationes ad formulas differentiales altiorum graduum prosequi, cum lex progressionis functionum arbitrariarum singulas integralium partes constituentium cum per se tum per ea, quae supra sunt exposita, satis sit manifesta. Quare huic capiti, quo una quaedam formula differentialis quantitati datae aequari debet, plene est satisfactum.

Antequam autem ulterius progredior, duos adhuc casus satis late patentes proponam, quorum resolutio facile ad praecedentes iam tractatas calculi integralis partes reducitur, quam propterea hic tanquam concessam assumere licet, siquidem difficultates, quae in iis occurrunt, non ad praesens institutum sunt referendae.

PROBLEMA 77

461. *Si in relationem propositam, ex qua naturam functionis trium variabilium x, y et z definiri oportet, aliae formulae differentiales non ingrediantur, nisi quae ex unica variabili x oriuntur, quae sunt*

$$\left(\frac{dv}{dx}\right),\quad \left(\frac{ddv}{dx^2}\right),\quad \left(\frac{d^3v}{dx^3}\right) \text{ etc.,}$$

functionem quaesitam investigare.

SOLUTIO

Cum aequatio propositam continens relationem alias formulas differentiales praeter memoratas non comprehendat, in ea binae quantitates y et z pro constantibus habentur ideoque etiam in singulis integrationibus tanquam tales tractari possunt. Hinc aequatio proposita duas tantum variabiles x et v involvere est censenda et reiectis formularum differentialium vinculis habebitur aequatio differentialis ad librum primum referenda, in qua, si ad altiores gradus exsurgat, elementum dx constans sumtum est putandum. Quodsi ergo praeceptorum ibidem traditorum ope haec aequatio integrari queat, tum loco constantium per singulas integrationes ingressarum substituantur functiones arbitrariae binarum variabilium y et z, veluti $\varGamma:(y \text{ et } z)$, $\varDelta:(y \text{ et } z)$ etc., sicque habebitur solutio completa problematis propositi.

COROLLARIUM 1

462. Praeter plurimos igitur integrabilitatis casus in libro I expositos etiam sequentes aequationes differentiales quamtumvis alti gradus resolutionem admittent

$$S = Av + B\left(\frac{dv}{dx}\right) + C\left(\frac{ddv}{dx^2}\right) + D\left(\frac{d^3v}{dx^3}\right) + \text{etc.}$$

et

$$S = Av + Bx\left(\frac{dv}{dx}\right) + Cx^2\left(\frac{ddv}{dx^2}\right) + Dx^3\left(\frac{d^3v}{dx^3}\right) + \text{etc.}$$

COROLLARIUM 2

463. Vinculis enim abiectis eiusmodi habentur aequationes differentiales, quales in extremis capitibus libri I integrare docuimus. Tantum opus est, ut loco constantium per integrationes ingressarum scribantur tales functiones

$$\varGamma:(y \text{ et } z),\ \varDelta:(y \text{ et } z),\ \varSigma:(y \text{ et } z) \text{ etc.,}$$

ut hoc pacto integralia completa obtineantur.

SCHOLION

464. Huc etiam referri possunt eiusmodi relationes propositae, in quibus formulae differentiales bina elementa dx et dy involventes ita continentur, ut hoc dy ubique eundem habeat dimensionum numerum, cuiusmodi sunt

$$\left(\frac{dv}{dy}\right), \quad \left(\frac{ddv}{dxdy}\right), \quad \left(\frac{d^3v}{dx^2dy}\right), \quad \left(\frac{d^4v}{dx^3dy}\right) \quad \text{etc.}$$

vel

$$\left(\frac{ddv}{dy^2}\right), \quad \left(\frac{d^3v}{dxdy^2}\right), \quad \left(\frac{d^4v}{dx^2dy^2}\right), \quad \left(\frac{d^5v}{dx^3dy^2}\right) \quad \text{etc.,}$$

ipsa autem tum quantitas v nusquam occurrat. Si enim tum pro priori casu ponatur $\left(\frac{dv}{dy}\right) = u$, pro posteriori vero $\left(\frac{ddv}{dy^2}\right) = u$, relatio ad casum problematis revocabitur alias formulas differentiales non continens praeter

$$\left(\frac{du}{dx}\right), \quad \left(\frac{ddu}{dx^2}\right), \quad \left(\frac{d^3u}{dx^3}\right) \text{ etc.}$$

et ipsam forte functionem u. Quare si aequationem per praecepta supra tradita integrare indeque functionem u definire licuerit, tum restituendo loco u vel $\left(\frac{dv}{dy}\right)$ vel $\left(\frac{ddv}{dy^2}\right)$, ut fiat $\left(\frac{dv}{dy}\right) = S$ vel $\left(\frac{ddv}{dy^2}\right) = S$, etiam hinc per praecepta huius capitis ipsa functio v determinabitur. Quin etiam hoc modo resolvi poterunt aequationes huiusmodi tantum formulas differentiales complectentes

$$\left(\frac{d^{\mu+\nu}v}{dy^{\mu}dz^{\nu}}\right), \quad \left(\frac{d^{\mu+\nu+1}v}{dxdy^{\mu}dz^{\nu}}\right), \quad \left(\frac{d^{\mu+\nu+2}v}{dx^2dy^{\mu}dz^{\nu}}\right) \quad \text{etc.,}$$

ubi omnia tria elementa dx, dy, dz occurrunt; posito enim aequatio alias formulas non continebit praeter

$$\left(\frac{du}{dx}\right), \quad \left(\frac{ddu}{dx^2}\right), \quad \left(\frac{d^3u}{dx^3}\right) \text{ etc.}$$

una cum ipsa functione u sicque ad casum huius problematis erit referenda; ex cuius resolutione si prodierit $u = S\left(\frac{d^{\mu+\nu}v}{dy^{\mu}dz^{\nu}}\right)$ existente iam S functione cognita, investigatio ipsius functionis v iam nulla amplius laborat difficultate.

Datur autem praeterea alius casus ad libri II partem priorem reducibilis, quem sequenti problemate sum expediturus.

PROBLEMA 78

465. *Si in relationem propositam, ex qua trium variabilium x, y, z functionem v definiri oportet, aliae formulae differentiales non ingrediuntur, nisi quae ex variabilitate binarum x et y tantum nascuntur tertio elemento dz penitus excluso, functionem v investigare.*

SOLUTIO

Quoniam in aequationem resolvendam, qua relatio proposita continetur, quantitas z non ut variabilis ingreditur, quotcunque integrationes fuerint instituendae, in iis ita quantitas z, tanquam esset constans, tractari debet. Huius ergo aequationis resolutio ad partem praecedentem est referenda, cum functio binarum tantum variabilium x et y ex formularum differentialium relatione data sit investiganda. Quodsi itaque negotium successerit et integrale fuerit inventum, in eo totidem occurrent functiones arbitrariae unius variabilis certo modo ex x et y conflatae, quot integrationibus fuerit opus; sit $\varGamma{:}t$ huiusmodi functio, ubi t per x et y dari assumitur, ac nunc, ut ista solutio ad praesens institutum accommodetur, ubi quantitas z variabilibus annumeratur, loco cuiusque functionis arbitrariae $\varGamma{:}t$ scribatur hic $\varGamma{:}(t \text{ et } z)$, functio scilicet duarum variabilium, sicque habebitur integrale completum.

COROLLARIUM 1

466. Si ergo haec proposita fuerit aequatio

$$\left(\frac{ddv}{dy^2}\right) = aa\left(\frac{ddv}{dx^2}\right),$$

quia in parte praecedente [§ 296] invenimus pro casu praesente, quo v debet esse functio trium variabilium x, y et z, integrale ita se habebit

$$v = \varGamma{:}\left(\overline{x+ay} \text{ et } z\right) + \varDelta{:}\left(\overline{x-ay} \text{ et } z\right).$$

COROLLARIUM 2

467. Hic scilicet meminisse oportet formam

$$\varGamma{:}\left(\overline{x+ay} \text{ et } z\right)$$

designare functionem quamcunque binarum variabilium, quarum altera sit $= x+ay$, altera vero $= z$; unde ipsam functionem per applicatam ad certam superficiem relatam repraesentare licebit.

SCHOLION

468. Non solum autem aequationes in problemate descriptae ad partem praecedentem calculi integralis reducentur, sed etiam innumerabiles aliae, quae facta quadam substitutione ad earn formam revocantur. Veluti si in aequatione proposita aliae formulae differentiales non occurrant, nisi in quibus omnibus unica dimensio elementi *dz* reperitur, quae sunt

$$\left(\frac{dv}{dz}\right), \quad \left(\frac{ddv}{dxdz}\right), \quad \left(\frac{ddv}{dydz}\right), \quad \left(\frac{d^3v}{dx^2dz}\right), \quad \left(\frac{d^3v}{dxdydz}\right), \quad \left(\frac{d^3v}{dy^2dz}\right) \text{ etc.}$$

manifestum est posito $\left(\frac{dv}{dz}\right)=u$ aequationem illam in aliam transformari, ex qua iam functionem *u* investigari oporteat, eamque ad casum in problemate expositum referri. Quare si inde indoles functionis *u* definiri potuerit, ut sit $u=S$, restat, ut haec aequatio $\left(\frac{dv}{dz}\right)=S$ resolvatur, unde, ut ante [§ 445] vidimus, fit

$$v=\int Sdz+\Gamma:\left(x \text{ et } y\right).$$

Hoc idem tenendum est, si aequatio proposita ope substitutionis $\left(\frac{ddv}{dz^2}\right)=u$ vel $\left(\frac{d^3v}{dz^3}\right)=u$ etc. ad casum problematis reduci queat; quin etiam per se est perspicuum, si ope transformationis cuiuscunque aequatio proposita ad casum problematis reduci queat. Tales autem transformationes supra plures exposui, dum vel loco functionis quaesitae valia *u* introducitur ponendo $v=Su$ vel ipsae variabiles *x, y, z* in alias *p, q, r* mutantur, quae ad illas certam teneant rationem, quod negotium pro casu duarum variabilium supra [§ 232, 240] fusius explicavi; hocque ita perspicuum est, ut similis reductio ad hunc casum trium variabilium facile accommodari queat. In sequentibus tamen forte eiusmodi transformationes occurrent; ad alios ergo casus, ubi omnis generis formulae differentiales occurrunt, progredior vix ultra prima elementa rem producturus.