CHAPTER III

CONCERNING THE RESOLUTION OF DIFFERENTIAL EQUATIONS OF THE FIRST ORDER

PROBLEM 79

469. *If, for some function v of the three variables x, y, z ,on putting*

$$dv = pdx + qdy + rdz\,,$$

there should be

$$\alpha p + \beta q + \gamma r = 0,$$

to define the nature of the function v.

SOLUTION

Since there shall be $\gamma dv = \gamma pdx + \gamma qdy - (\alpha p + \beta q)dz$, there will be

$$\gamma dv = p(\gamma dx - \alpha dz) + q(\gamma dy - \beta dz)$$

and thus on putting $\gamma x - \alpha z = t$ and $\gamma y - \beta z = u$ there will be had $\gamma dv = pdt + qdu$, from which it is apparent that the quantity *v* be equal to some function of the two variables of the two variables *t* and *u,* thus so that there shall be $v = \Gamma:(t \text{ and } u)$ and with the assumed values restored,

$$v = \Gamma:\left(\overline{\gamma x - \alpha z} \text{ and } \overline{\gamma y - \beta z}\right),$$

which therefore is the solution of the problem, if this condition is put in place between the differential formulas, so that there shall be

$$\alpha\left(\frac{dv}{dx}\right) + \beta\left(\frac{dv}{dy}\right) + \gamma\left(\frac{dv}{dz}\right) = 0\,,$$

and thus the integral of this equation may be shown more clearly thus

$$v = \Gamma:\left(\overline{\frac{x}{\alpha} - \frac{z}{\gamma}} \quad \text{and} \quad \overline{\frac{y}{\beta} - \frac{z}{\gamma}}\right).$$

COROLLARY 1

470. It is evident that this integral also can be expressed thus :

$$v = \Gamma : \left(\sqrt{\tfrac{x}{\alpha} - \tfrac{z}{\gamma}} \quad \text{and} \quad \sqrt{\tfrac{y}{\beta} - \tfrac{z}{\gamma}} \right),$$

since in general, as we have observed above [§ 450], there is

$$\Gamma : (x \text{ and } y) = \Delta : (t \text{ and } u),$$

if indeed t and u may be determined in some manner by x and y.

COROLLARY 2

471. So that also it is permitted to affirm from these three formulas put in place

$$\tfrac{x}{\alpha} - \tfrac{y}{\beta}, \quad \tfrac{y}{\beta} - \tfrac{z}{\gamma}, \quad \tfrac{z}{\gamma} - \tfrac{x}{\alpha}$$

that the quantity v is to be some function of these three formulas ; if indeed now with each one given by the two remaining, v therefore nevertheless is equal to a function of only two of the variable quantities.

PROBLEM 80

472. *If, on putting $dv = pdx + qdy + rdz$, this condition should be required, that there shall be*

$$px + qy + rz = nv \quad or \quad nv = x\left(\tfrac{dv}{dx}\right) + y\left(\tfrac{dv}{dy}\right) + z\left(\tfrac{dv}{dz}\right),$$

to investigate the nature of this function v.

SOLUTION

From the prescribed condition the value $r = \frac{nv - px - qy}{z}$ is taken, with which substituted there becomes

$$dv - \tfrac{nvdz}{z} = p\left(dx - \tfrac{xdz}{z}\right) + q\left(dy - \tfrac{ydz}{z}\right)$$

or

$$dv - \tfrac{nvdz}{z} = pzd.\tfrac{x}{z} + qzd.\tfrac{y}{z}.$$

So that the first term may be returned integrable, it may be multiplied by $\frac{1}{z^n}$, thus so that now we may consider

$$d.\tfrac{v}{z^n} = \tfrac{pz}{z^n} d.\tfrac{x}{z} + \tfrac{qz}{z^n} d.\tfrac{y}{z}.$$

Now since the quantities p and q shall not be determined, as in general from such an equation $dV = PdX + QdY$ it follows that $V = \Gamma:(X \text{ and } Y)$, for our case we may deduce

$$\tfrac{v}{z^n} = \Gamma:\left(\tfrac{x}{z} \text{ and } \tfrac{y}{z}\right) \quad \text{or} \quad v = z^n \Gamma:\left(\tfrac{x}{z} \text{ and } \tfrac{y}{z}\right).$$

Evidently if some function of the two quantities $\tfrac{x}{z}$ and $\tfrac{y}{z}$ may be multiplied by z^n or also, because it returns the same, by x^n or y^n, a suitable value arises satisfying the prescribed condition for the function *v*.

COROLLARY 1

473. Moreover it is evident that the form $\Gamma:\left(\tfrac{x}{z} \text{ and } \tfrac{y}{z}\right)$ expresses a function of the same kind, in which the three variables *x*, *y*, *z* everywhere constitute a number of zero dimensions, and in turn all the functions of this kind are to be contained in this form.

COROLLARY 2

474. But again with the multiplication made by z^n there arises a homogeneous function of the three variables *x*, *y*, *z*, the number of dimensions of which is = *n;* from which the solution to our problem thus can be enunciated, so that the quantity sought *v* shall be a homogeneous function of the three variables *x*, *y* and *z* with the number of dimensions present = *n*.

COROLLARY 3

475. But if therefore the prescribed condition shall be

$$px + qy + rz = 0 \quad \text{or} \quad x\left(\tfrac{dv}{dx}\right) + y\left(\tfrac{dv}{dy}\right) + z\left(\tfrac{dv}{dz}\right) = 0,$$

the quantity *v* shall be a homogeneous function of zero dimensions of the three variables *x*, *y* and *z*.

SCHOLIUM

476. In a similar manner a solution follows, if the prescribed condition should postulate, that there shall be

$$\alpha px + \beta qy + \gamma rz = nv \quad \text{or} \quad \alpha x\left(\tfrac{dv}{dx}\right) + \beta y\left(\tfrac{dv}{dy}\right) + \gamma z\left(\tfrac{dv}{dz}\right) = nv;$$

then indeed on account of $r = \tfrac{nv - \alpha px - \beta qy}{\gamma z}$ there comes about

$$dv - \tfrac{nvdz}{\gamma z} = p\left(dx - \tfrac{\alpha xdz}{\gamma z}\right) + q\left(dy - \tfrac{\beta ydz}{\gamma z}\right)$$

which equation may be shown in the following form

$$\tfrac{\gamma dv}{v} - \tfrac{ndz}{z} = \tfrac{px}{v}\left(\tfrac{\gamma dx}{x} - \tfrac{\alpha dz}{z}\right) + \tfrac{qy}{v}\left(\tfrac{\gamma dy}{y} - \tfrac{\beta dz}{z}\right),$$

from which we may conclude that the integral of the first term $\gamma lv - nlz$ to be equal to some function of the two quantities $\gamma lx - \alpha lz$ and $\gamma ly - \beta lz$, and with the numbers of the logarithms taken to become

$$\tfrac{v^\gamma}{z^n} = \Gamma:\left(\tfrac{x^\gamma}{z^\alpha} \text{ and } \tfrac{y^\gamma}{z^\beta}\right)$$

We may put $\alpha = \tfrac{1}{\lambda}$, $\beta = \tfrac{1}{\mu}$ and $\gamma = \tfrac{1}{\nu}$, so that the prescribed condition shall be

$$\tfrac{px}{\lambda} + \tfrac{qy}{\mu} + \tfrac{rz}{\nu} = nv$$

but the solution may be reduced to this form

$$v = z^{\nu n}\Gamma:\left(\tfrac{x^\lambda}{z^\nu} \text{ and } \tfrac{y^\mu}{z^\nu}\right).$$

But if again we write $x^\lambda = X$, $y^\mu = Y$ and $z^\nu = Z$, there becomes

$$v = Z^n \Delta:\left(\tfrac{X}{Z} \text{ and } \tfrac{Y}{Z}\right)$$

and thus the quantity sought *v* is a homogeneous function, in which the three variables X, Y and *Z* everywhere fulfill the same number of dimensions = *n*.

PROBLEM 81

477. *If, on putting* $dv = pdx + qdy + rdz$*, this condition may be prescribed, so that there shall be*

$$px + qy + rz = nv + S,$$

with some function given S of the three variables x, y, z, *to investigate the nature of the function sought v.*

SOLUTION

Since the condition prescribed gives $r=\frac{nv+S-px-qy}{z}$, there will be

$$dv-\frac{nvdz}{z}=\frac{Sdz}{z}+p\left(dx-\frac{xdz}{z}\right)+q\left(dy-\frac{ydz}{z}\right)$$

or

$$d.\frac{v}{z^n}=\frac{Sdz}{z^{n+1}}+\frac{p}{z^{n-1}}d.\frac{x}{z}+\frac{q}{z^{n-1}}d.\frac{y}{z}.$$

Let $x=tz$ and $y=uz$, in order that now the function *S* is made of the three variables *t, u* and *z*, and the formula of the differential $\frac{Sdz}{z^{n+1}}$ thus may be integrated, as the quantities *t* and *u* may be considered constants ; with which integral put = *V* there will be

$$v=Vz^n+z^n\Gamma:\left(\frac{x}{z}\text{ and }\frac{y}{z}\right),$$

where the last part indicates a homogeneous function of the three variables *x, y, z* with the number of dimensions present = *n.*

COROLLARY 1

478. If *S* shall be a constant quantity = *C*, there will be

$$V=\int\frac{Cdz}{z^{n+1}}=-\frac{C}{nz^n}$$

and hence the first member of the integral becomes

$$Vz^n=-\frac{C}{n}$$

from which it is evident that the same value would arise with the quantities *x, y, z* interchanged between themselves.

COROLLARY 2

479. If *S* shall be a homogeneous function of *x, y, z* themselves with the number of dimensions present = *m*, which then on putting $x=tz$ and $y=uz$ shall be $S=Mz^m$, thus so that *M* involves only the quantities *t* and *u* , and thus may be considered as constant, then there arises

$$V=\int Mz^{m-n-1}dz=\frac{Mz^{m-n}}{m-n}=\frac{S}{(m-n)z^n},$$

and thus the first member of the integral will be $=\frac{S}{(m-n)}$.

COROLLARY 3

480. But if in this case there shall be $m = n$, there comes about $V = Mlz + C = Mlaz$ and the first member of the integral $= Mz^n laz = Slaz$. Moreover that by the same rule $= Slby$ or $Slax$, because that is evident enough, since the differential of these values becomes a homogeneous function of *n* dimensions and thus may be contained in the other member of the integration.

SCHOLIUM

481. The principle of this solution may be included in this apparently wider lemma, because, if there should be

$$dV = SdZ + PdX + QdY \ ,$$

in which *S* denotes a given function, *P* and *Q* truly indefinite functions, there shall become,

$$V = \int Sdz + \Gamma:(X \text{ and } Y);$$

but here it does not suffice to indicate in the integration of the formula *SdZ* only the quantity *Z* to be considered variable, but in addition it is agreed to note that the two quantities *X* and *Y* must be treated as constants. Whereby if perhaps *S* shall be a proposed function of the other three variables *x, y, z,* from which these X, *Y, Z,* of which here the ratio is being considered, arise in a certain manner, initially in place of *x*, *y*, *z* these *X*, *Y* and *Z* must be introduced, so that *S* becomes a function of these *X*, *Y* and *Z;* then truly at last with the two *X* and *Y* for constants and only *Z* taken to vary, the integral $\int SdZ$ is to be taken. Thus in the case in the case of the problem for the integral $\int \frac{Sdz}{z^{n+1}}$, the quantities $\frac{x}{z}$ and $\frac{y}{z}$ are to be considered as constants with only *z* taken for a variable ; from which there is required to be put in place $x = tz$ and $y = uz$ in the function *S*, so that a function S is made of z, $t = \frac{x}{z}$ and $u = \frac{y}{z}$ themselves, of which the last two are to be considered as constants. Therefore in this, in case a significant error be committed, if from which with *z* to be variable and it may be wished to treat the remaining *x* and *y* as constants, because both *x* and *y* also are considered to involve the variable *z*. But since from the interchange of the variables the first member of the integral must result likewise, so that there shall be

$$z^n \int \frac{Sdz}{z^{n+1}} = x^n \int \frac{Sdx}{x^{n+1}} \ ,$$

from which it is apparent, because on putting $x = tz$ and $dx = tdz$ on account of taking *t* constant there becomes

$$x^n \int \frac{Sdx}{x^{n+1}} = t^n z^n \int \frac{Stdz}{t^{n+1} z^{n+1}} = z^n \int \frac{Sdz}{z^{n+1}}$$

For in each integration the ratios of the variables $\frac{x}{z}, \frac{y}{z}, \frac{x}{y}$ have been considered as constants and hence in the reduction made the quantity $t = \frac{x}{z}$ is correctly considered as constant.

PROBLEM 82

482. *If on putting* $dv = pdx + qdy + rdz$ *this condition is prescribed, so that there must become*

$$pL + qM + rN = 0$$

with the given functions L, M, N with respect of the variables x, y and z, clearly L of x, M of y and N of z only, to define the nature of the function sought v.

SOLUTION

On account of $r = \frac{-pL-qM}{N}$ the principal equation becomes

$$dv = p\left(dx - \tfrac{L}{N}dz\right) + q\left(dy - \tfrac{M}{N}dZ\right)$$

or

$$dv = pL\left(\tfrac{dx}{L} - \tfrac{dz}{N}\right) + qM\left(\tfrac{dy}{M} - \tfrac{dz}{N}\right)$$

There is put in place

$$t = \int \frac{dx}{L} - \int \frac{dz}{N} \quad \text{and} \quad u = \int \frac{dy}{M} - \int \frac{dz}{N},$$

so that there arises $dv = pLdt + qMdu$, and it is evident that the quantity v must be equal to some function of the two variables t and u, which thus also it is permitted to describe, in order that with the three formulas put in place

$$\int \frac{dx}{L}, \quad \int \frac{dy}{M}, \quad \text{and} \quad \int \frac{dz}{N}$$

it is required for t and u to take the differentials between two of these,

SCHOLIUM 1

483. Also the solution might be successful, provided $\frac{L}{N}$ were a function of x and z themselves, and $\frac{M}{N}$ of y and z only; then indeed multipliers P and Q must be sought suitable for the integration, so that there becomes

$$P\left(dx-\tfrac{Ldz}{N}\right)=dt \quad \text{and} \quad Q\left(dy-\tfrac{Mdz}{N}\right)=du,$$

and on account of $dv=\frac{pdt}{P}+\frac{qdu}{Q}$ there becomes $v=\Gamma:(t \text{ and } u)$. Now with the variables *x, y* and *z* permuted also other resolvable cases may arise. But when the quantities *L, M, N* have been prepared otherwise, a certain way may not be apparent for reaching a solution ; which certainly may appear not a little abstruse, since for this simple enough case

$$(y+z)p+(x+z)q+(x+y)r=0,$$

I can only reach a solution to this by several round about ways, as on putting

$$t=(x+y+z)(x-z)^2 \text{ and } u=(x+y+z)(y-z)^2$$

there becomes

$$v=\Gamma:(t \text{ and } u);$$

therefore because the two quantities *t* and *u*, any function *v* of which put in place satisfies the condition, in this case are so much more complicated, that generally it is considered much less likely to expect a solution.

SCHOLIUM 2

484. But the solution can be extended to several other cases. If the given functions *L, M,* N should be prepared thus, so that it is possible to find other *E, F, G, H*, from which there is made

$$E\left(dx-\tfrac{Ldz}{N}\right)+F\left(dy-\tfrac{Mdz}{N}\right)=dt$$

and

$$G\left(dx-\tfrac{Ldz}{N}\right)+H\left(dy-\tfrac{Mdz}{N}\right)=du,$$

for then on putting $p=PE+QG$ and $q=PF+QH$ there becomes

$$dv=Pdt+Qdu,$$

where *P* and *Q* are indefinite functions introduced in place of *p* and *q*, the quantity *v* will be equal to some function of the two variables *t* and *u* or there will be

$$v=\Gamma:(t \text{ and } u).$$

Therefore the whole business is returned to this, that for the given proposed functions *L, M, N,* functions *E, F* and *G, H* may be found, because indeed it may be seen always possible to be done, but this question itself emerges more difficult than that proposed. But it suffices that two functions of this kind *E* and *F* and hence the quantity *t* be investigated, because then on permuting the variables *x, y,* z together with the corresponding functions *L, M, N* a suitable value for *u* may be elicited at once. Thus in the previous example produced,

$$L = y + z,\ M = x + z,\ N = x + y$$

after we have found $t = (x + y + z)(x - z)^2$, by permutation alone at once there is given

$u = (x + y + z)(y - z)^2$ or also $u = (x + y + z)(x - y)^2$; indeed this is likewise, with whatever value we may use.

PROBLEM 83

485. *If on putting* $dv = pdx + qdy + rdz$ *this condition is prescribed, so that there shall be*

$$pqr = 1,$$

to investigate the nature of the function v.

SOLUTIO

On account of $r = \frac{1}{pq}$ there will be

$$dv = pdx + qdy + \frac{dz}{pq},$$

from which we deduce

$$v = px + qy + \frac{z}{pq} - \int \left(xdp + ydq - \frac{zdp}{ppq} - \frac{zdq}{pqq} \right)$$

where we have pursued that transformation, so that the formula of the integral may involve only the two differentials *dp* and *dq*. Therefore with these in the place of the principle formulas introduced, we infer that the form of the integral must be equal to some function of the two variables *p* and *q*. Let *S* be such a function, so that there becomes

$$v = px + qy + \frac{z}{pq} - S,$$

and now there remains that, since the letters *p* and *q* may be retained in the calculation, the other two may be removed, that which has been desired thus, so that there shall be

$$dS = \left(x - \frac{z}{ppq} \right) dp + \left(y - \frac{z}{pqq} \right) dq$$

and thus

$$x - \frac{z}{ppq} = \left(\frac{dS}{dp} \right) \quad \text{and} \quad y - \frac{z}{pqq} = \left(\frac{dS}{dq} \right)$$

Therefore now the solution will thus itself be considered. With these three variables *p*, *q* and *z* introduced ,and with some function *S* of the two *p* and *q* taken,

$$x = \frac{z}{ppq} + \left(\frac{dS}{dp}\right) \quad \text{and} \quad y = \frac{z}{pqq} + \left(\frac{dS}{dq}\right)$$

and then the function *v* sought may thus be defined, so that there shall be

$$v = \frac{3z}{pq} + p\left(\frac{dS}{dp}\right) + q\left(\frac{dS}{dq}\right) - S.$$

Or if we may prefer to express *v* by the three variables themselves *x*, *y*, *z*, the values of *p* and *q* are themselves sought from the two equations

$$x = \frac{z}{ppq} + \left(\frac{dS}{dp}\right) \quad \text{and} \quad y = \frac{z}{pqq} + \left(\frac{dS}{dq}\right),$$

from which on substituting into the function *S* there will be

$$v = px + qy + \frac{z}{pq} - S$$

and thus the question will be satisfied.

COROLLARY 1

486. If the function *S* should accept the constant quantity *C*, on account of $ppq = \frac{z}{x}$ and $pqq = \frac{z}{y}$ there will be

$$pq = \sqrt[3]{\frac{zz}{xy}}$$

and hence

$$p = \sqrt[3]{\frac{yz}{xx}} \quad \text{and} \quad q = \sqrt[3]{\frac{xz}{yy}}$$

from which there comes about

$$v = 3\sqrt[3]{xyz} - C,$$

which is the particular value satisfying the problem.

COROLLARIUM 2

487. Because in the prescribed condition $pqr=1$ or $\left(\frac{dv}{dx}\right)\left(\frac{dv}{dy}\right)\left(\frac{dv}{dz}\right)=1$, only the differentials of the three variables *x*, *y* and *z* occur, it is permitted to increase these by any constants, from which a solution of somewhat wider extent arise

$$v=3\sqrt[3]{(x+a)(y+b)(z+c)}-C.$$

SCHOLIUM 1

488. Besides another case admitting to an easy resolution is given on putting $S=2c\sqrt{pq}$, from which there is deduced

$$p=\frac{\sqrt{y}}{\sqrt{x}}\sqrt[3]{\frac{z}{\sqrt{xy}-c}} \quad \text{and} \quad q=\frac{\sqrt{x}}{\sqrt{y}}\sqrt[3]{\frac{z}{\sqrt{xy}-c}}$$

and thus

$$S=2c\cdot\sqrt[3]{\frac{z}{\sqrt{xy}-c}}$$

Therefore we may follow on with

$$v=3\cdot\sqrt[3]{z\left(\sqrt{xy}-c\right)^2}$$

and on permuting the variables ,we will have in a like manner

$$v=3\cdot\sqrt[3]{y\left(\sqrt{xz}-b\right)^2} \quad \text{and} \quad v=3\cdot\sqrt[3]{x\left(\sqrt{yz}-a\right)^2},$$

where again for *x*, *y*, z it is permitted to write $x+f$, $y+g$, $z+h$.

Moreover it is apparent that a general solution follows likewise, if the quantity *r* should be equal to some function of *p* and *q* themselves, or some equation is proposed between *p*, *q*, *r*.

SCHOLIUM 2

489. Because if indeed on putting $dv=pdx+qdy+rdz$, some equation is proposed between the three formulas

$$p=\left(\frac{dv}{dx}\right),\quad q=\left(\frac{dv}{dy}\right),\quad r=\left(\frac{dv}{dz}\right),$$

which differentials give rise to

$$Pdp+Qdq+Rdr=0,$$

then on making

$$S = \int (xdp + ydq + zdr),$$

so that there shall be

$$v = px + qy + rz - S,$$

some function of the three quantities *p*, *q*, *r*, which shall be *V*, and this differentiated gives

$$dV = Ldp + Mdq + Ndr;$$

then truly there shall be

$$0 = Pudp + Qudq + Rudr$$

and thus

$$dV = (L + Pu)dp + (M + Qu)dq + (N + Ru)dr,$$

which form, on account of the new variable introduced *u* appears wider. Now there is put in place $S = V$ and there will be made

$$x = L + Pu, \quad y = M + Qu, \quad z = N + Ru,$$

thus so that now besides the variables *p*, *q*, *r*, of which one is given by the other two, with the new *u* considered, from which now we may thus define the three *x*, *y* and *z*, so that by these in turn these *p*, *q*, *r* and *u* may be determined; then truly there will be

$$v = px + qy + rz - V.$$

Whereby with some function of the three quantities *p*, *q*, *r* taken for *V*, a condition of some kind is prescribed between these, so that there shall be

$$Pdp + Qdq + Rdr = 0,$$

there may be taken

$$x = Pu + \left(\tfrac{dV}{dp}\right), \quad y = Qu + \left(\tfrac{dV}{dq}\right), \quad z = Ru + \left(\tfrac{dV}{dr}\right)$$

and there will be

$$v = (Pp + Qq + Rr)u + p\left(\tfrac{dV}{dp}\right) + q\left(\tfrac{dV}{dq}\right) + r\left(\tfrac{dV}{dr}\right) - V,$$

which solution is thus to be preferred to the preceding one, because in that the three quantities *p*, *q*, and *r* enter equally.

PROBLEM 84

490. *If on putting* $dv = pdx + qdy + rdz$ *is condition may be prescribed, so that there must be*

$$pqr = \frac{v^3}{xyz},$$

to define the nature of the function v.

SOLUTION

We may put $p = \frac{Pv}{x}$, $q = \frac{Qv}{y}$, $r = \frac{Rv}{z}$ and on account of the prescribed condition there must be $PQR = 1$; then indeed there shall be

$$\frac{dv}{v} = \frac{Pdx}{x} + \frac{Qdy}{y} + \frac{Rdz}{z}.$$

Now we may put in place

$$lv = V, \quad lx = X, \quad ly = Y, \quad lz = Z$$

and we will have this equation

$$dV = PdX + QdY + RdZ,$$

with which there must be $PQR = 1$; which question since it does not disagree with the preceding problem, the same solution will be most easily transferred here too.

SCHOLIUM

491. Several cases, which perhaps it would be in order to present in this chapter, I do not pursue here, because since a use has not yet been observed, then truly in the first place, I have decided to outline only the first principles of this part of the integral calculus, at this stage evidently unknown. But with the formulas of differentials of higher order, which may enter in a prescribed condition, scarcely anything can be mentioned, except certain observations pertaining to homogeneous equations, with which therefore I will soon finish this part of the integral calculus, and likewise set an end to the whole work.

CAPUT III

DE RESOLUTIONE AEQUATIONUM DIFFERENTIALIUM PRIMI GRADUS

PROBLEMA 79

469. *Si pro functione v trium variabilium x, y, z posito*

$$dv = pdx + qdy + rdz$$

fuerit

$$\alpha p + \beta q + \gamma r = 0,$$

indolem functionis v definire.

SOLUTIO

Cum sit $\gamma dv = \gamma pdx + \gamma qdy - (\alpha p + \beta q)dz$, erit

$$\gamma dv = p(\gamma dx - \alpha dz) + q(\gamma dy - \beta dz)$$

ideoque ponendo $\gamma x - \alpha z = t$ et $\gamma y - \beta z = u$ habebitur $\gamma dv = pdt + qdu$, unde patet quantitatem *v* aequari functioni cuicunque binarum variabilium *t* et *u*, ita ut sit $v = \Gamma:(t \text{ et } u)$ et restitutis valoribus assumtis

$$v = \Gamma:(\overline{\gamma x - \alpha z} \text{ et } \overline{\gamma y - \beta z}),$$

quae ergo est solutio problematis, si inter formulas differentiales proponatur haec conditio, ut sit

$$\alpha\left(\frac{dv}{dx}\right) + \beta\left(\frac{dv}{dy}\right) + \gamma\left(\frac{dv}{dz}\right) = 0,$$

cuius itaque aequationis integrale clarius ita exhibetur

$$v = \Gamma:\left(\overline{\frac{x}{\alpha} - \frac{z}{\gamma}} \quad \text{et} \quad \overline{\frac{y}{\beta} - \frac{z}{\gamma}}\right).$$

COROLLARIUM 1

470. Evidens est hoc integrale etiam ita exprimi posse

$$v = \Gamma:\left(\overline{\frac{x}{\alpha} - \frac{z}{\gamma}} \quad \text{et} \quad \overline{\frac{y}{\beta} - \frac{z}{\gamma}}\right),$$

quandoquidem in genere, uti supra [§ 450] observavimus, est

$$\Gamma:(x \text{ et } y) = \Delta:(t \text{ et } u),$$

siquidem t et u utcunque per x et y determinentur.

COROLLARIUM 2

471. Quin etiam affirmare licet constitutis his tribus formulis

$$\frac{x}{\alpha} - \frac{y}{\beta}, \quad \frac{y}{\beta} - \frac{z}{\gamma}, \quad \frac{z}{\gamma} - \frac{x}{\alpha}$$

quantitatem v esse functionem quamcunque trium harum formularum; siquidem unaquaeque iam per binas reliquas datur ac propterea v nihilominus functioni duarum tantum quantitatum variabilium aequatur.

PROBLEMA 80

472. *Si, posito* $dv = pdx + qdy + rdz$ *haec conditio requiratur, ut sit*

$$px + qy + rz = nv \quad seu \quad nv = x\left(\frac{dv}{dx}\right) + y\left(\frac{dv}{dy}\right) + z\left(\frac{dv}{dz}\right),$$

indolem huius functionis v investigare.

SOLUTIO

Ex conditione praescripta capiatur valor $r = \frac{nv - px - qy}{z}$, quo substituto fit

$$dv - \frac{nvdz}{z} = p\left(dx - \frac{xdz}{z}\right) + q\left(dy - \frac{ydz}{z}\right)$$

seu

$$dv - \frac{nvdz}{z} = pzd.\frac{x}{z} + qzd.\frac{y}{z}.$$

Quo primum membrum integrabile reddatur, multiplicetur per $\frac{1}{z^n}$, ita ut iam habeamus

$$d.\frac{v}{z^n} = \frac{pz}{z^n}d.\frac{x}{z} + \frac{qz}{z^n}d.\frac{y}{z}.$$

Cum nunc quantitates p et q non sint determinatae, quoniam in genere ex tali aequatione $dV = PdX + QdY$ sequitur $V = \Gamma:(X \text{ et } Y)$, pro nostro casu colligimus

$$\frac{v}{z^n} = \Gamma:\left(\frac{x}{z} \text{ et } \frac{y}{z}\right) \quad \text{seu} \quad v = z^n \Gamma:\left(\frac{x}{z} \text{ et } \frac{y}{z}\right).$$

Si scilicet functio quaecunque binarum quantitatum $\frac{x}{z}$ et $\frac{y}{z}$ per z^n seu etiam, quod eodem redit, per x^n vel y^n multiplicetur, oritur valor idoneus pro functione v conditioni praescriptae satisfaciens.

COROLLARlUM 1

473. Perspicuum autem est formam $\Gamma:\left(\frac{x}{z} \text{ et } \frac{y}{z}\right)$ exprimere eiusmodi functionem, in qua tres variabiles x, y, z ubique constituant nullum dimensionum numerum, ac vicissim omnes huiusmodi functiones in forma ina contineri.

COROLLARlUM 2

474. Multiplicatione autem porro facta per z^n oritur functio homogenea trium variabilium x, y, z, cuius dimensionum numerus est = n; unde solutio nostri problematis ita enunciari potest, ut quantitas quaesita v sit functio homogenea trium variabilium x, y et z dimensionum numero existente = n.

COROLLARlUM 3

475. Quodsi ergo conditio praescripta sit

$$px + qy + rz = 0 \quad \text{seu} \quad x\left(\frac{dv}{dx}\right) + y\left(\frac{dv}{dy}\right) + z\left(\frac{dv}{dz}\right) = 0,$$

quantitas v erit functio homogenea nullius dimensionis trium variabilium x, y et z.

SCHOLION

476. Simili modo solutio succedit, si conditio praescripta postulet, ut sit

$$\alpha px + \beta qy + \gamma rz = nv \quad \text{seu} \quad \alpha x\left(\frac{dv}{dx}\right) + \beta y\left(\frac{dv}{dy}\right) + \gamma z\left(\frac{dv}{dz}\right) = nv;$$

tum enim ob $r = \frac{nv - \alpha px - \beta qy}{\gamma z}$ fit

$$dv - \frac{nvdz}{\gamma z} = p\left(dx - \frac{\alpha xdz}{\gamma z}\right) + q\left(dy - \frac{\beta ydz}{\gamma z}\right)$$

quae aequatio sequenti forma exhibeatur

$$\frac{\gamma dv}{v} - \frac{ndz}{z} = \frac{px}{v}\left(\frac{\gamma dx}{x} - \frac{\alpha dz}{z}\right) + \frac{qy}{v}\left(\frac{\gamma dy}{y} - \frac{\beta dz}{z}\right),$$

ex qua concludimus integrale primi membri $\gamma lv - nlz$ aequari functioni cuicunque binarum quantitatum $\gamma lx - \alpha lz$ et $\gamma ly - \beta lz$ et logarithmorum numeris sumtis fore

$$\frac{v^\gamma}{z^n} = \Gamma : \left(\frac{x^\gamma}{z^\alpha} \text{ et } \frac{y^\gamma}{z^\beta} \right)$$

Ponamus $\alpha = \frac{1}{\lambda}$, $\beta = \frac{1}{\mu}$ et $\gamma = \frac{1}{\nu}$, ut conditio praescripta sit

$$\frac{px}{\lambda} + \frac{qy}{\mu} + \frac{rz}{\nu} = nv$$

at solutio reducetur ad hanc formam

$$v = z^{\nu n} \Gamma : \left(\frac{x^\lambda}{z^\nu} \text{ et } \frac{y^\mu}{z^\nu} \right).$$

Quodsi porro scribamus $x^\lambda = X$, $y^\mu = Y$ et $z^\nu = Z$, fiet

$$v = Z^n \Delta : \left(\frac{Z}{Z} \text{ et } \frac{Y}{Z} \right)$$

ideoque quantitas quaesita v est functio homogenea, in qua tres variabiles X, Y et Z ubique eundem dimensionum numerum = n adimplent.

PROBLEMA 81

477. *Si posito* $dv = pdx + qdy + rdz$ *haec conditio praescribatur, ut sit*

$$px + qy + rz = nv + S$$

existente S functione quacunque data variabilium x, y, z, *investigare naturam functionis quaesitae v.*

SOLUTIO

Cum conditio praescripta praebeat $r = \frac{nv + S - px - qy}{z}$, erit

$$dv - \frac{nvdz}{z} = \frac{Sdz}{z} + p\left(dx - \frac{xdz}{z} \right) + q\left(dy - \frac{ydz}{z} \right)$$

seu

$$d.\frac{v}{z^n} = \frac{Sdz}{z^{n+1}} + \frac{p}{z^{n-1}} d.\frac{x}{z} + \frac{q}{z^{n-1}} d.\frac{y}{z}.$$

Sit $x = tz$ et $y = uz$, ut iam S fiat functio trium variabilium t, u et z, et formula differentialis $\frac{Sdz}{z^{n+1}}$ ita integretur, ut quantitates t et u constantes habeantur; quo integrali posito = V erit

$$v = Vz^n + z^n \Gamma : \left(\frac{x}{z} \text{ et } \frac{y}{z} \right),$$

ubi pars posterior significat functionem homogeneam trium variabilium *x, y, z* numero dimensionum existente = *n.*

COROLLARIUM 1

478. Si *S* sit quantitas constans = *C*, erit

$$V=\int\frac{Cdz}{z^{n+1}}=-\frac{C}{nz^n}$$

hincque primum integralis membrum

$$Vz^n=-\frac{C}{n}$$

ex quo perspicuum est eundem valorem proditurum fuisse quantitatibus *x, y, z* inter se permutatis.

COROLLARIUM 2

479. Si *S* sit functio homogenea ipsarum *x, y, z* dimensionum numero existente = *m*, quia tum posito $x=tz$ et $y=uz$ sit $S=Mz^m$, ita ut *M* tantum quantitates *t* et *u* involvat ideoque pro constante sit habenda, prodit

$$V=\int Mz^{m-n-1}dz=\frac{Mz^{m-n}}{m-n}=\frac{S}{(m-n)z^n}$$

sicque primum integralis membrum erit $=\frac{S}{(m-n)}$

COROLLARIUM 3

480. At si hoc casu sit $m=n$, fit $V=Mlz+C=Mlaz$ et primum integralis membrum $=Mz^n laz=Slaz$. Pari iure id autem erit $=Slby$ vel $Slax$, id quod satis est manifestum, cum horum valorum differentia fiat functio homogenea *n* dimensionum ideoque in altero integralis membro contineatur.

SCHOLION

481. Principium huius solutionis in hoc lemmate latissime patente continetur, quod, si fuerit

$$dV=SdZ+PdX+QdY\ ,$$

ubi *S* denotat functionem datam, *P* et *Q* vero functiones indefinitas, futurum sit

$$V=\int Sdz+\Gamma:(X\text{ et }Y);$$

at hic non sufficit indicasse in integratione formulae *SdZ* solam quantitatem *Z* pro variabili haberi, sed insuper notari convenit binas *X* et *Y* tanquam constantes tractari debere. Quare si forte *S* sit proposita functio aliarum trium variabilium *x, y, z*, ex quibus hae X, *Y, Z*, quarum ratio hic est habenda, certo modo nascantur, primum loco *x, y, z* istae *X, Y* et *Z* introduci debent, ut fiat *S* functio harum *X, Y* et *Z;* tum vero demum binis *X* et *Y* pro constantibus solaque *Z* pro variabili sumta integrale $\int SdZ$ est capiendum. Ita in casu problematis pro integrali $\int \frac{Sdz}{z^{n+1}}$ quantitates $\frac{x}{z}$ et $\frac{y}{z}$ ut constantes sunt spectandae sola *z* pro variabili sumta; ex quo in functione *S* statui oportet $x = tz$ et $y = uz$, ut S fiat functio ipsarum *z*, $t = \frac{x}{z}$ et $u = \frac{y}{z}$ quarum binae posteriores pro constantibus sunt habendae. Hoc ergo casu insignis error committeretur, si quis sumta *z* variabili reliquas *x* et *y* ut constantes tractare voluerit, quoniam ambae *x* et *y* etiam variabilem *z* involvere sunt censendae. Quod autem variabilibus permutatis primum integralis membrum idem resultare debeat, ut sit

$$z^n \int \frac{Sdz}{z^{n+1}} = x^n \int \frac{Sdx}{x^{n+1}},$$

inde patet, quod posito $x = tz$ et $dx = tdz$ ob *t* constantem sumendam fiat

$$x^n \int \frac{Sdx}{x^{n+1}} = t^n z^n \int \frac{Stdz}{t^{n+1} z^{n+1}} = z^n \int \frac{Sdz}{z^{n+1}}$$

In utraque enim integratione rationes variabilium $\frac{x}{z}, \frac{y}{z}, \frac{x}{y}$ pro constantibus sunt habendae hincque in reductione facta quantitas $t = \frac{x}{z}$ recte ut constans spectatur.

PROBLEMA 82

482. *Si posito* $dv = pdx + qdy + rdz$ *haec conditio praescribatur, ut esse debeat*

$$pL + qM + rN = 0$$

existentibus L, M, N functionibus datis respective variabilium x, y et z, nempeL ipsius x, M ipsius y et N ipsius z tantum, naturam functionis quaesitae v definire.

SOLUTIO

Ob $r = \frac{-pL-qM}{N}$ aequatio principalis fit

$$dv = p\left(dx - \frac{L}{N}dz\right) + q\left(dy - \frac{M}{N}dZ\right)$$

vel

$$dv = pL\left(\frac{dx}{L} - \frac{dz}{N}\right) + qM\left(\frac{dy}{M} - \frac{dz}{N}\right)$$

Statuatur

$$t=\int\tfrac{dx}{L}-\int\tfrac{dz}{N} \quad \text{et} \quad u=\int\tfrac{dy}{M}-\int\tfrac{dz}{N},$$

ut fiat $dv=pLdt+qMdu$, et manifestum est quantitatem *v* aequari debere functioni cuicunque binarum variabilium *t* et *u*, quas ita quoque describere licet, ut positis formulis tribus integralibus

$$\int\tfrac{dx}{L},\quad \int\tfrac{dy}{M},\quad \text{et}\quad \int\tfrac{dz}{N}$$

pro *t* et *u* sumi oporteat differentias inter binas earum.

SCHOLION 1

483. Solutio etiam successisset, dummodo $\frac{L}{N}$ fuisset functio ipsarum *x* et *z* et $\frac{M}{N}$ ipsarum *y* et *z* tantum; tum enim multiplicatores *P* et *Q* ad integrationem apti quaeri debuissent, ut fieret

$$P\left(dx-\tfrac{Ldz}{N}\right)=dt \quad \text{et} \quad Q\left(dy-\tfrac{Mdz}{N}\right)=du,$$

et ob $dv=\frac{pdt}{P}+\frac{qdu}{Q}$ foret $v=\Gamma:(t \text{ et } u)$. Permutandis vero variabilibus *x, y* et *z* etiam alii casus resolubiles prodeunt. Quando autem quantitates *L*, *M*, *N* aliter sunt comparatae, via non patet certa ad solutionem perveniendi; quae certe haud parum abstrusa videtur, cum pro hoc casu satis simplici

$$(y+z)p+(x+z)q+(x+y)r=0$$

per plures ambages tandem ad hanc pervenerim solutionem, ut posito

$$t=(x+y+z)(x-z)^2 \quad \text{et} \quad u=(x+y+z)(y-z)^2$$

fiat

$$v=\Gamma:(t \text{ et } u);$$

quoniam igitur binae quantitates *t* et *u*, quarum functio quaecunque loco *v* posita conditioni satisfacit, hoc casu tantopere sunt complicatae, generaliter multo minus solutionem expectare licebit.

SCHOLION 2

484. Ad plures autem alios casus solutio extendi potest. Si functiones datae *L*, *M*, N ita fuerint comparatae, ut alias *E*, *F*, *G*, *H* reperire liceat, quibus fiat

$$E\left(dx-\tfrac{Ldz}{N}\right)+F\left(dy-\tfrac{Mdz}{N}\right)=dt$$

et

$$G\left(dx-\tfrac{Ldz}{N}\right)+H\left(dy-\tfrac{Mdz}{N}\right)=du,$$

tum enim posito $p=PE+QG$ et $q=PF+QH$ fiet

$$dv=Pdt+Qdu\ ,$$

ubi P et Q sunt functiones indefinitae loco p et q introductae, quantitas v aequabitur functioni cuicunque binarum variabilium t et u seu erit

$$v=\Gamma:(t \text{ et } u).$$

Totum ergo negotium huc redit, ut pro datis functionibus L, M, N functiones E, F et G, H inveniantur, quod quidem semper praestari posse videtur, sed haec ipsa quaestio plerumque difficilior evadit quam ipsa proposita. Sufficit autem binas eiusmodi functiones E et F indeque quantitatem t investigasse, quia deinceps permutandis variabilibus x, y, z una cum respondentibus functionibus L, M, N sponte idoneus valor pro u elicitur. Ita in exemplo ante allato

$$L=y+z,\ M=x+z,\ N=x+y$$

postquam invenerimus $t=(x+y+z)(x-z)^2$, sola permutatio statim praebet

$u=(x+y+z)(y-z)^2$ vel etiam $u=(x+y+z)(x-y)^2$; perinde enim est, utro valore utamur.

PROBLEMA 83

485. *Si posito $dv=pdx+qdy+rdz$ haec conditio praescribatur, ut sit*

$$pqr=1,$$

naturam functionis v investigare.

SOLUTIO

Ob $r=\frac{1}{pq}$ erit

$$dv=pdx+qdy+\tfrac{dz}{pq},$$

unde colligimus

$$v=px+qy+\tfrac{z}{pq}-\int\left(xdp+ydq-\tfrac{zdp}{ppq}-\tfrac{zdq}{pqq}\right)$$

qua transformatione id sumus assecuti, ut formula integralis bina tantum differentialia dp et dq involvat. His igitur in locum principalium inductis concludimus illam formulam integralem aequari debere functioni cuicunque binarum variabilium p et q. Sit S talis functio, ut fiat

$$v = px + qy + \tfrac{z}{pq} - S,$$

et iam superest, ut, cum litterae p et q in calculo retineantur, aliae duae elidantur, id quod iude est petendum, quod sit

$$dS = \left(x - \tfrac{z}{ppq}\right)dp + \left(y - \tfrac{z}{pqq}\right)dq$$

ideoque

$$x - \tfrac{z}{ppq} = \left(\tfrac{dS}{dp}\right) \quad \text{et} \quad y - \tfrac{z}{pqq} = \left(\tfrac{dS}{dq}\right)$$

Nunc igitur solutio ita se habebit. Introductis his ternis variabilibus p, q et z sumtaque binarum p et q functione quacunque S capiatur

$$x = \tfrac{z}{ppq} + \left(\tfrac{dS}{dp}\right) \quad \text{et} \quad y = \tfrac{z}{pqq} + \left(\tfrac{dS}{dq}\right)$$

ac tum functio quaesita v ita definietur, ut sit

$$v = \tfrac{3z}{pq} + p\left(\tfrac{dS}{dp}\right) + q\left(\tfrac{dS}{dq}\right) - S.$$

Vel si malimus v per ipsas tres variabiles x, y, z exprimere, ex binis aequationibus

$$x = \tfrac{z}{ppq} + \left(\tfrac{dS}{dp}\right) \quad \text{et} \quad y = \tfrac{z}{pqq} + \left(\tfrac{dS}{dq}\right)$$

quaerantur valores ipsarum p et q, quibus in functione S substitutis erit

$$v = px + qy + \tfrac{z}{pq} - S$$

sicque quaesito erit satisfactum.

COROLLARIUM 1

486. Si functio S sumatur quantitas constans C, ob $ppq = \tfrac{z}{x}$ et $pqq = \tfrac{z}{y}$ erit

$$pq = \sqrt[3]{\tfrac{zz}{xy}}$$

hincque

$$p = \sqrt[3]{\tfrac{yz}{xx}} \quad \text{et} \quad q = \sqrt[3]{\tfrac{xz}{yy}}$$

unde fit

$$v = 3\sqrt[3]{xyz} - C,$$

qui est valor particularis problemati satisfaciens.

COROLLARIUM 2

487. Quoniam in conditione praescripta $pqr = 1$ seu $\left(\frac{dv}{dx}\right)\left(\frac{dv}{dy}\right)\left(\frac{dv}{dz}\right) = 1$ tantum differentialia trium variabilium *x, y* et *z* occurrunt, eas quantitatibus constantibus quibusvis augere licet, unde nascitur solutio aliquanto latius patens

$$v = 3\sqrt[3]{(x+a)(y+b)(z+c)} - C.$$

SCHOLION 1

488. Alius datur praeterea casus facilem evolutionem admittens ponendo $S = 2c\sqrt{pq}$, unde colligitur

$$p = \frac{\sqrt{y}}{\sqrt{x}}\sqrt[3]{\frac{z}{\sqrt{xy}-c}} \quad \text{et} \quad q = \frac{\sqrt{x}}{\sqrt{y}}\sqrt[3]{\frac{z}{\sqrt{xy}-c}}$$

ideoque

$$S = 2c \cdot \sqrt[3]{\frac{z}{\sqrt{xy}-c}}$$

Assequimur ergo

$$v = 3 \cdot \sqrt[3]{z\left(\sqrt{xy} - c\right)^2}$$

et permutandis variabilibus simili modo habebimus

$$v = 3 \cdot \sqrt[3]{y\left(\sqrt{xz} - b\right)^2} \quad \text{et} \quad v = 3 \cdot \sqrt[3]{x\left(\sqrt{yz} - a\right)^2},$$

ubi porro pro *x, y,* z scribere licet $x + f$, $y + g$, $z + h$.

Ceterum patet solutionem generalem perinde succedere, si quantitas *r* functioni cuicunque ipsarum *p* et *q* aequari debeat seu si inter *p, q, r* aequatio quaecunque proponatur.

SCHOLION 2

489. Quodsi enim posito $dv = pdx + qdy + rdz$ inter ternas formulas

$$p = \left(\frac{dv}{dx}\right), \quad q = \left(\frac{dv}{dy}\right), \quad r = \left(\frac{dv}{dz}\right)$$

aequatio proponatur quaecunque, quae differentiata praebeat

$$Pdp + Qdq + Rdr = 0,$$

tum facto

$$S=\int\left(xdp+ydq+zdr\right),$$

ut sit

$$v=px+qy+rz-S,$$

sumatur functio quaecunque trium quantitatum *p, q, r,* quae sit *V*, haecque differentiata praebeat

$$dV=Ldp+Mdq+Ndr;$$

tum vero est

$$0=Pudp+Qudq+Rudr$$

ideoque

$$dV=\left(L+Pu\right)dp+\left(M+Qu\right)dq+\left(N+Ru\right)dr,$$

quae forma ob novam introductam variabilem *u* latissime patet. Statuatur iam $S=V$ fietque

$$x=L+Pu,\quad y=M+Qu,\quad z=N+Ru,$$

ita ut nunc praeter variabiles *p, q, r,* quarum una per binas reliquas datur, nova habeatur *u*, ex quibus iam tres *x*, *y* et *z* ita definivimus, ut per eas vicissim hae *p*, *q*, *r* et *u* determinentur; tum vero erit

$$v=px+qy+rz-V.$$

Quare pro *V* sumta quacunque functione trium quantitatum *p, q, r*, inter quas eiusmodi conditio praescribitur, ut sit

$$Pdp+Qdq+Rdr=0,$$

sumatur

$$x=Pu+\left(\tfrac{dV}{dp}\right),\quad y=Qu+\left(\tfrac{dV}{dq}\right),\quad z=Ru+\left(\tfrac{dV}{dr}\right)$$

eritque

$$v=\left(Pp+Qq+Rr\right)u+p\left(\tfrac{dV}{dp}\right)+q\left(\tfrac{dV}{dq}\right)+r\left(\tfrac{dV}{dr}\right)-V,$$

quae solutio praecedenti ideo est anteferenda, quod in hanc tres quantitates *p, q, r* aequaliter ingrediuntur.

PROBLEMA 84

490. *Si posito* $dv = pdx + qdy + rdz$ *haec conditio praescribatur, ut esse debeat*

$$pqr = \frac{v^3}{xyz},$$

naturam functionis v definire.

SOLUTIO

Ponamus $p = \frac{Pv}{x}$, $q = \frac{Qv}{y}$, $r = \frac{Rv}{z}$ et ob conditionem praescriptam debet esse $PQR = 1$; tum vero erit

$$\frac{dv}{v} = \frac{Pdx}{x} + \frac{Qdy}{y} + \frac{Rdz}{z}.$$

Statuamus nunc

$$lv = V, \quad lx = X, \quad ly = Y, \quad lz = Z$$

et habebimus hanc aequationem

$$dV = PdX + QdY + RdZ,$$

pro qua esse debet $PQR = 1$; quae quaestio cum non discrepet a problemate praecedente, eadem solutio huc quoque facillime transferetur.

SCHOLION

491. Plures casus, quos forte in hoc capite expedire liceat, hic non evolvo, cum quia usus nondum perspicitur, tum vero imprimis, quoniam huius partis calculi integralis prorsus adhuc incognitae prima tandum principia adumbrare constitui. Pro formulis autem differentialibus altiorum graduum, quae in conditionem praescriptam ingrediantur, vix quicquam proferre licet praeter quasdam observationes ad aequationes homogeneas pertinentes, quibus ergo hanc partern calculi integralis sum finiturus simulque toti operi finem impositurus.