Preliminary Notes Regarding Integral Calculus in General

DEFINITION 1

1. *Integral calculus is the method by which the relation is to be found between quantities from the given relation of the differentials, and the operation, which is presented here, is accustomed to be called integration.*

COROLLARY 1

2. Therefore since differential calculus teaches how to investigate the relation of the differentials from the given relation of the magnitudes of the variables, integral calculus is the inverse method.

COROLLARY 2

3. Clearly just as in analysis there are always two operations opposing each other, such as subtraction to addition, division to multiplication, the extraction of roots and the raising to powers, thus also by similar reasoning integral and differential calculus oppose each other.

COROLLARY 3

4. With any relation proposed between the two variable quantities x and y in differential calculus the ratio of the differentials $dy:dx$ is to be investigated ; but if in turn from this ratio of the differentials the relation between the quantities x and y is itself to be defined, this work is assigned to integral calculus.

SCHOLIUM 1

5. Now in differential calculus I have observed that an investigation about differentials is to be understood as not absolute but relative, thus in order that, if y were some function of x, but it is to be defined not so much as the differential of this dy but as the ratio of this differential to the differential dx. For since all differentials *per se* are equal to zero, whatever the function y should be of x, there is always $dy = 0$ nor thus can any greater absolute amount able to be sought. Now the question must duly be proposed, in order that, while the increment x is taken infinitely small and thus with dx vanishing, the ratio of the increment of the function y is to be defined, which thus must be taken, to that dx; and even if each is equal to 0, yet a certain ratio exists between these, which is properly investigated in differential calculus. Thus if it should be that $y = xx$, in differential calculus it is shown to be $\frac{dy}{dx} = 2x$, and neither is this ratio of the increments true, unless the increment dx, from which dy arises, is placed equal to zero. Now nevertheless here in a true inquiry of differentials, the common talk about differentials as if they were absolute [quantities] can be tolerated, provided that always in the mind at least they can be referred to the truth. Hence we say correctly, if $y = xx$, there could be $dy=2xdx$, even if

it is not false, if anyone says $dy = 3xdx$ or $dy=4xdx$, as on account of $dx = 0$ and $dy = 0$ there equalities might stand also; but only the true ratio $\frac{dy}{dx} = 2x$ is to be agreed upon.

SCHOLIUM 2

6. Just as the differential calculus may be called by the English the method of fluxions, thus the integral calculus is accustomed to be called by them the inverse method of fluxions, whenever it reverts from fluxion to fluent quantities. Indeed what we call variable quantities, these the English call by the more suitable name fluent quantities, and the infinitely small increments or vanishing quantities of these they name fluxions, thus in order that fluxions are the same as we call differentials. This difference in the way of saying things thus now has increased in use, so that agreement can scarcely be expected at any time ; indeed I might willingly follow the English in the formulas to be discussed, but the signs which we use are seen to be preferred by far. Now since so many books have been produced written by each method, no use is to be had from an acceptance of this [other] kind.

DEFINITION 2

7. *Since the differential of any kind of function of x has the form Xdx, for the proposed form of such a differential Xdx, in which X is some function of x, that function, the differential of which is equal Xdx, is called the integral of this and prefixed by the sign usually indicated by $\int$, thus in order that $\int Xdx$ denotes the same variable quantity, the differential of which is equal to Xdx.*

COROLLARY 1

8. Hence just as the integral of the proposed differential formula Xdx or that function of x, of which the differential is equal to Xdx, which is to be indicated by writing $\int Xdx$, must be investigated, and is to be set forth in the integral calculus.

COROLLARY 2

9. Hence as the letter d is sign of differentiation, thus we use the letter $\int$ for the sign of integration, and thus these two signs mutually oppose each other and as if they destroy each other, clearly $\int dX$ is equal to X, because this denotes a quantity, of which the differential is dX, which certainly is X.

COROLLARY 3

10. Therefore since the differentials of these functions of x : x^2, x^n, $\sqrt{(aa - xx)}$ are $2xdx$, $nx^{n-1}dx$, $\frac{-xdx}{\sqrt{(aa-xx)}}$, with the sign of integration $\int$ employed, it is apparent that

$$\int 2xdx = x^2,\ \int nx^{n-1}dx = x^n,\ \int \frac{-xdx}{\sqrt{(aa-xx)}} = \sqrt{(aa-xx)}.$$

SCHOLIUM 1

11. Here only a single variable quantity is seen to enter into the calculation, since nevertheless we establish always the ratio of two or more differentials to be considered, both in differential as well as in integral calculus. Now even if here only a single variable quantity *x* appears, yet in fact two are considered ; for the other is that function itself, the differential of which we take to be *Xdx;* which if we designate with the letter *y,* becomes $dy = Xdx$ or $\frac{dy}{dx} = X$, thus as here generally is the ratio of the differentials *dy: dx* proposed, which is equal to X, and thus there becomes $y = \int Xdx$; but this integral is to be considered to arise not from the differential *Xdx* itself, since that is everywhere equal to 0, but as from the ratio of this to *dx*. This other sign $\int$ is accustomed to be designated to the *sum* to be carried out, which arises from a suitable understanding of the equal parts, *i.e.* of X and $\frac{dy}{dx}$, from which the integral is to be considered as the sum of all the differentials ; nor can anything more be justly admitted, as the lines are accustomed generally to be considered as put in place from the points.

SCHOLIUM 2

12. But integral calculus is much broader than is apparent from integral formulas of this kind, which are embracing a single variable only. For just as here a function of a single variable *x* is investigated from given differential forms, thus integral calculus ought to be extended also to functions of two or more variables to be investigated, when a certain relation of the differentials has been proposed. Then the integral calculus not only is tied up with differentials of the first order, but also it must deliver the instructions, with the help of which functions of one as well as two or more variables can be investigated, since a certain relation of the differentials of the second or of higher orders of the same has been given. And hence on account of this we have thus put in place a definition of the integral calculus, in order that it might embrace in itself all investigations of this kind ; for differentials of any order must be understood, and by an expression of the *relation* which is proposed between those that I have used, as it appears broader than the expression of the *ratio,* which is seen to indicate a comparison between two differential only. Hence from these we are able to put in place a division of the integral calculus.

DEFINITION 3

13. The integral calculus is divided into two parts, of which the first part method treats the function of one variable to be found from a certain given relation between the differential of this, so of the first and of higher orders.

But the second part of the method contains functions of two or more variables to be discovered, since a relation may have been proposed between the differential of that, either of the first or of a certain higher degree.

COROLLARY 1

14. Hence as the function to be found from a given relation of the differentials either embraces a single variable or two or more, thus the integral calculus can be separated into two principal parts, for which we are resolved to publish two books.

COROLLARY 2

15. Therefore the integral calculus is directed towards the discovery of functions either of one or of several variables, since clearly a certain relation between the differential of this or of some higher order should be proposed.

SCHOLIUM

16. Since here we establish the first part of the integral calculus in the investigation of functions of a single variable from a given relation of the differentials, it is considered that more parts must be put in place for the number of variables entering the function , thus so that the second part considers functions of two variables, the third of three, the fourth of four, etc. Now almost the same method is required for these latter parts, thus so that if the finding of the functions involving two variables should be in a power, the way to these which involves more variables, should be revealed easily enough; and the discovery of the same kind of functions, which contain two or more variables, we can join together conveniently and thus we can set up for a single part of the calculus to be treated in the following book.

This other remaining part on the elements has not been treated anywhere at this time, even if the use of this in mechanics and especially in the teaching of fluids is of the greatest use. On this account since in this generally in addition to the first rudiments hardly anything shall be investigated, our second book on integral calculus is excessively sterile and besides remembering these things, which are desired at this stage, little is to be anticipated ; but truly this book is itself considered to gather together much to the growth of knowledge.

DEFINITION 4

17. Each book concerned with the integral calculus is subdivided conveniently into sections for the degree of the differentials, from which the relation of the function sought is required to be investigated. Thus the first part is concerned with the relation of differentials of the first order, the second with the relation of differentials of the second degree, also with a view to referring to differentials of higher degrees which are to be investigated at this point, on account of the sparseness of these.

COROLLARY 1

18. Hence each book is presented in two parts, in the first of which the relation between the differentials of the first degree proposed are to be considered, now in the second part integrations of the same kind occur, where the relation between the differentials of the second or higher degrees are proposed.

COROLLARY 2

19. Hence in the first part of the first book a function of this kind of the variable x is proposed to be found, so that on putting that function equal to y et $\frac{dy}{dx}=p$ some given relation between the three quantities x, y and p may be fulfilled, or some proposed equation between these three quantities can be elicited, in order that the nature of the function y or an equation between x and y only, with p excluded.

COROLLARY 3

20. But in the latter part of the first book questions are thus to be composed, so that on putting $\frac{dy}{dx}=p, \frac{dp}{dx}=q, \frac{dq}{dx}=$ r etc., if some equation is proposed between the quantities x, y, p, q, r etc., the nature of the functions of y by x or an equation between x and y is elicited.

SCHOLIUM 1

21. The greater first part of the first book, which at this stage have been elaborated on in the integral calculus which are to be referred to, in which in the first place the geometers have gathered together their work with the developments; there are a few, which are better fitted for the second part, and the other book, which we have composed following, even now is left almost empty. But the first part of the first book, in which chiefly our treatment is our main concern, again is to be divided into several sections according to the kind of the relation, which is proposed between the quantities x, y and $p=\frac{dy}{dx}$. For the most simple relation before the rest is, when $p=\frac{dy}{dx}$ is equal to a function of some kind of x; in which on putting it equal to X, in order that there becomes $\frac{dy}{dx}=X$ or $dy =$

Xdx, the whole business in the integration of the differential formula *Xdx* is completed; now above we have made mention of operations of this kind, which commonly under the title of integration of simple differential formulas or of involving a singe variable are accustomed to be treated. In the same way the thing reappears, if $p = \frac{dy}{dx}$ is equal to a function of *y* only, whenever the quantities *x* and *y* thus between themselves are reciprocated, so that it is possible to regard the one as a function of the other ; hence these may be referred to in the first section. But if $p = \frac{dy}{dx}$ is equal to an expression involving both the quantities *x* and *y*, the equation may have differentials of this form $Pdx + Qdy = 0$, where *P* and *Q* are some expressions composed from *x, y* and constants. But though many geometers have exerted themselves in the integration of equations of this kind, nevertheless they have scarcely progressed far enough beyond certain special cases. But if *p* is determined through *x* and *y* in a more complicated way, so that the value of this cannot be shown explicitly, just as if it should be

$$p^5 = xxp^3 - xyp + x^5 - y^5,$$

indeed no way can be agreed upon to be attempted, how thus the relation between *x* and *y* can be investigated; hence there is only a few which can be treated here, and which occupy with the preceding, the second section of the first part of the first book. Thus from our general treatment it becomes more apparent, which besides in integral calculus may be desired, as what can be brought about now, since this before that is to be considered as a certain minimum small amount.

SCHOLIUM 2

22. In the individual parts which we have describes, it is also accustomed to become, that not only one single function, but also likewise several are to be investigated, thus as neither without the rest can be defined, just as in algebra it arrives form general use, that to the solution of a problem several unknowns will have to be introduced into the calculation, which hence may be determined by all the equations. Just as if two functions *y* and *z* of *x* of this kind are required to be found, in order that

$$xdy + azzdx = 0 \text{ and } xxdz + bxydy = cdy,$$

hence new subdivisions of our treatment can be put in place. Now since here as in Algebra the whole task can be reduced to the common feature towards the elimination of one letter, as hence only two variables become present in one equation, and hence the treatment is considered to become less.

SCHOLIUM 3

23. In the second book of the integral calculus, in which a function of two or more variables is investigated from a given relation of the differentials, a much greater variety of questions can be put in place. For let *z* be a function of two variables *x* and *t* to be

investigated, and since $\left(\frac{dz}{dx}\right)$ denotes the ratio of the differential to *dx*, if *x* alone is taken for a variable, but $\left(\frac{dz}{dt}\right)$ is the ratio of this differential to *dt*, if *t* is taken as the only variable, the first part contains questions of this kind, in which a certain relation is proposed between the quantities *x*, *t*, *z* and $\left(\frac{dz}{dx}\right), \left(\frac{dz}{dt}\right)$, and the equation is reduced here, so that an equation between the quantities *x*, *t* and *z* only can be elicited ; for hence it is apparent that *z* is a function of such a kind of *x* and *t* . In the second part in addition to these formulas $\left(\frac{dz}{dx}\right)$ and $\left(\frac{dz}{dt}\right)$ even these $\left(\frac{ddz}{dxdx}\right)$, $\left(\frac{ddz}{dxdt}\right)$ and $\left(\frac{ddz}{dtdt}\right)$ can enter into the computation, the meaning of which is to be understood thus, so that on putting initially $\left(\frac{dz}{dx}\right) = p$ and $\left(\frac{dz}{dt}\right) = q$, where *p* and *q* in turn are certain functions of *x* and *t;* which become in a similar manner of expression

$$\left(\frac{ddz}{dxdx}\right) = \left(\frac{dp}{dx}\right),\ \left(\frac{ddz}{dxdt}\right) = \left(\frac{dp}{dt}\right) = \left(\frac{dq}{dx}\right),\ \left(\frac{ddz}{dtdt}\right) = \left(\frac{dq}{dt}\right).$$

Hence for the proposed relation between these formulas and likewise the preceding quantities *x*, *t* and *z*, there must be elicited an equation between these three quantities alone *x*, *t* and *z*. Questions of this kind occur frequently in mechanics and hydraulics, when the motion of flexible bodies and fluids is investigated ; from which especially there should be chosen [such questions], so that these can be developed with all care in the second section of the second book. Now nor is there a need that we may extend this investigation to higher differentials, since thus far no questions of this kind are to be treated, which is augmented only to the calculus desired.

DEFINITION 5

24. If functions, which are sought in integral calculus from a relation of differentials, cannot be shown algebraically, then these are called transcending, since an account of these transcends the powers of common analysis.

COROLLARY 1

25. Hence as often as an integration does not succeed, so hence the function which is sought by integration is to be taken as transcendental. Thus if the formula of the differential *Xdx* does not admit to integration, the integral of this, which thus be indicated only by $\int Xdx$, is a transcending function of *x*.

COROLLARY 2

26. Hence it is understood, if y should be a transcending function of x, in turn x is a transcending function of y and from this inversion new transcending functions arise.

COROLLARY 3

27. Also more general kinds of transcending functions arise for the various parts and sections of the integral calculus, the number of which thus rises to infinity; from which it is apparent that the great abundance of all the possible quantities is unknown to us at present.

SCHOLIUM 1

28. Now before we penetrate into the analysis of the infinites, it is evident that we become acquainted with certain kinds of transcending functions. First the principles of logarithms is made available ; for if y denotes the logarithm of x, so that there becomes $y = lx$, then y certainly is a transcending function of x and thus the logarithms establish as it were the first kind of transcending function. Then since from the equation $y = lx$ there shall be in turn $x = e^y$, then x certainly also is a transcending function of y and such functions are called exponentials. While again the consideration of the angles uncovers a different kind ; just as if the angle, the sine of which is s, is put equal to φ, so that $\varphi = arc\,sin\,s$, there is no doubt that φ is a transcending function of s and indeed of an infinite form ; and hence since on changing it produces $s = sin.\varphi$, then also the sine function s is a transcending function of the angle φ. But though these transcending functions have been acknowledged without the help of integral calculus, yet they are in themselves as if on the threshold of integral calculus, we can deduce these and also the innate character of these has become evident to us, as they are almost able to be added to the algebraic functions. Whereby also always in integral calculus, as often as transcending functions there found are allowed to be reduced to logarithms or angles, these we are accustomed to consider as algebraic.

SCHOLIUM 2

29. Since integral calculus can arise as the inverse of differential calculus, likewise the remaining methods of inversion lead us to noting new kinds of quantities. Thus as if with the beginner of the first elements [of mathematics] we should postulate nothing besides the positive integers, with addition to be understood, clearly at once he is led to the inverse operation of subtraction, and the idea of negative numbers is understood. Then with multiplication treated, when he progresses to division, there he can accept the notion of fractions. Again after learning how to raise numbers to powers, if he should undertake the inverse operation of the extraction of roots, as often the calculation does not succeed, then the idea of irrational numbers is come upon, and this is considered as being sufficiently common knowledge throughout all analysis. Hence in a like manner the integral calculus, as far as it does not succeed in an integration, uncovers for us a new kind of transcendental quantity. For it is not possible that all the differentials can be

furnished, but thus in turn the integrals of all the differentials are permitted to be furnished.

SCHOLIUM 3

30. Now in the initial attempt at solving an integral, the functions sought should not at once be first be taken as transcendental ; for it is often happens that even an algebraic integral is only able to be obtained through skilful operations. Then when the function sought should be transcending, it is to be considered with care, whether perhaps it can be reduced to these most simple kinds of integrals of logarithms or angles, in which case the algebraic solution is to be compared equally. But if this has not succeeded, nevertheless it is agreed that the most simple form of transcendental functions be investigated, to which the sought integral can to be reduced. But by far the most suitable method, in order that the closest values of transcending functions can be produced, and to which in the end a significant part of the integral calculus is devoted, is in the investigation of infinite series, and which values of these functions they may contain.

THEOREM

31. *All functions found through the calculation of an integral are indeterminate, they require a determination required from the nature of the question, of which they supply the solution.*

DEMONSTRATION

Since an infinitely of functions can be given, of which the differential is the same, if indeed of the function P + C, whatever value is attributed to the constant C, the same differential is equal dP, in turn also the proposed integral of the differential dP is $P + C$, where for the constant C some constant amount can be put in place ; from which it is apparent that the function, the differential of which is given equal to dP, is an indeterminate, since an arbitrary constant amount is involved within it. Also the same must come about, if the function is to be determined from some relation of the differentials, and always enfolds an arbitrary constant quantity, of which no trace has appeared in the relation of the differentials. Hence a function of this kind is determined through the calculated integral found, while a value is attributed to that arbitrary constant, which always is provided by the nature of the question, the solution of which has led to that function.

COROLLARY 1

32. Hence if a function y of x is defined by some relation of the differentials, thus it possible to be determined by the entrance of an arbitrary constant, as on putting $x = a$ there becomes $y = b$; with which done the function is determined and for whatever the value of x attributed, the function y obtains a determined value.

COROLLARY 2

33. If a function y is defined from a relation of the differentials of the second degree, it involves two arbitrary constants and thus a two-fold determination is granted, by which it can be put into effect, so that on putting $x = a$ not only y obtains a given value b, but also the ratio $\frac{dy}{dx}$ is made equal to a given value c.

COROLLARY 3

34. If y is a function of the two variables x and t elicited from the relation of the differentials, it involves an arbitrary constant also, the determination of which can be effected, as on putting $t = a$, a given equation between y and x may be produced or it may express the nature of some given curve.

SCHOLIUM

35. That determination of integral functions, or which have been found through the integral calculus, in whatever case are easily deduced from the nature of the question treated nor is troubled with any difficulty, unless perhaps besides the need that the solution be produced from differentials, since that has to be elicited by common analysis; in which case as likewise in algebra if useless roots are brought in. But since this determination is put in place only in the application to particular cases, here, where we treat the method of integration in general , we will try to elicit the integrations in their full extent, thus so that the constants introduced by integration remain arbitrary, and not unless a certain condition demands that we determine these. The remaining determination of functions of x is the most simple, where these in the case $x = 0$ are themselves made to vanish.

DEFINITION 6

36. The complete integral is said to be shown, when the function sought is represented with every extension and with an arbitrary constant. But when that constant now has been determined in a certain way, the integral is accustomed to be called particular.

COROLLARY 1

37. Hence in any case whatever there is given a single complete integral; but the integrals are able to furnish an infinite number of particular integrals. Thus the complete integral of the differential xdx is $\frac{1}{2}xx + C$, but the particular integrals $\frac{1}{2}xx, \frac{1}{2}xx+1, \frac{1}{2}xx+2$, etc. are in an infinite multitude.

COROLLARY 2

38. Hence the complete integral embraces all the particular integrals within itself and from that all these are able to be formed easily. But in turn from the particular integrals the general integral cannot become known. But many times, as henceforth it becomes

apparent, a method of finding the complete integral from the particular integral is obtained.

SCHOLIUM

39. Sometimes it is easy for a particular integral to be understood from a conjecture or prediction. Just as if a function of this kind of x is sought, which is y, so that there becomes : $dy + yydx = dx + xxdy$, clearly the solution to this equation is satisfied on taking $y = x$, which hence is a particular integral, because there is no arbitrary constant present in that ; but the complete integral may be found $y = \frac{1+Cx}{C+x}$, as that particular one is contained in that by taking $C = \infty$. In a similar manner on taking $C = 0$, hence another integral is obtained $y = \frac{1}{x}$, since likewise it satisfies the above equation and before $y = x$. But all the particular integrals, whatever they satisfy, by necessity are contained in the general formula $y = \frac{1+Cx}{C+x}$, exactly as more and more values are attributed to the arbitrary constant C; thus on taking $C = 1$ there now becomes $y = 1$. But generally it is accustomed to turn out, so that, even if a certain particular integral is algebraic, nevertheless the complete integral is transcending. Just as if this equation is proposed : $dy + ydx = dx + xdx$, at once it is apparent to be satisfied on putting $y = x$, which hence is a particular integral ; now the complete integral involving an arbitrary constant C is $y = x + Ce^{-x}$ with e denoting the number, the logarithm of which is equal to 1; hence unless here there is taken $C = 0$, the function y always is transcending. This is sufficient to be noted in general, before we advance to the treatment of the integral calculus itself, since that is concerned with all integrations; therefore now we set out the form the work is to take, and we go on to the work to be discussed.

PRAENOTANDA DE CALCULO INTEGRALI IN GENERE

DEFINITIO 1

1. *Calculus integralis est methodus ex data differentialium relatione inveniendi relationem ipsarum quantitatum, et operatio, qua hoc praestatur, integratio vocari solet.*

COROLLARIUM 1

2. Cum igitur calculus differentialis ex data relatione quantitatum variabilium relationem differentialium investigare doceat, calculus integralis methodum inversam suppeditat.

COROLLARIUM 2

3. Quemadmodum scilicet in Analysi perpetuo binae operationes sibi opponuntur, veluti subtractio additioni, divisio multiplicationi, extractio radicum evectioni ad potestates, ita etiam simili ratione calculus integralis calculo differentiali opponitur.

COROLLARIUM 3

4. Proposita relatione quacunque inter binas quantitates variabiles x et y in calculo differentiali methodus traditur rationem differentialium $dy:dx$ investigandi; sin autem vicissim ex hac differentialium ratione ipsa quantitatum x et y relatio sit definienda, hoc opus calculo integrali tribuitur.

SCHOLION 1

5. In calculo differentiali iam notavi quaestionem de differentialibus non absolute sed relative esse intelligendam, ita ut, si y fuerit functio quaecunque ipsius x, non tam ipsum eius differentiale dy quam eius ratio ad differentiale dx sit definienda. Cum enim omnia differentialia per se sint nihilo aequalia, quaecunque functio y fuerit ipsius x, semper est $dy = 0$ neque sic quicquam amplius absolute quaeri posset. Verum quaestio ita rite proponi debet, ut, dum x incrementum capit infinite parvum adeoque evanescens dx, definiatur ratio incrementi functionis y, quod inde capiet, ad istud dx; etsi enim utrumque est = 0, tamen ratio certa inter ea intercedit, quae in calculo differentiali proprie investigatur. Ita si fuerit $y = xx$, in calculo differentiali ostenditur esse $\frac{dy}{dx} = 2x$ neque hanc incrementorum rationem esse veram, nisi incrementum dx, ex quo dy nascitur, nihilo aequale statuatur. Verum tamen hac vera differentialium notione observata locutiones communes, quibus differentialia quasi absolute enunciantur, tolerari possunt, dummodo semper in mente saltem ad veritatem referantur. Recta ergo dicimus, si $y = xx$, fore $dy=2xdx$, tametsi falsum non esset, si quis diceret $dy = 3xdx$ vel $dy=4xdx$, quoniam ob $ax = 0$ et $dy = 0$ has aequalitates aeque subsisterent; sed prima sola rationi verae $\frac{dy}{dx} = 2x$ est consentanea.

SCHOLION 2

6. Quemadmodum calculus differentialis apud Anglos methodus fluxionum appellatur, ita calculus integralis ab iis methodus fluxionum inversa vocari solet, quandoquidem a fluxionibus ad quantitates fluentes revertitur. Quas enim nos quantitates variabiles vocamus, eas Angli nomine magis idoneo quantitates fluentes vocant et earum incrementa infinite parva seu evanescentia fluxiones nominant, ita ut fluxiones ipsis idem sint, quod nobis differentialia. Haec diversitas loquendi ita iam usu invaluit, ut conciliatio vix unquam sit expectanda; equidem Anglos in formulis loquendi lubenter imitarer, sed signa, quibus nos utimur, illorum signis longe anteferenda videntur. Verum cum tot iam libri utraque ratione conscripti prodierint, huiusmodi conciliatio nullum usum esset habitura.

DEFINITIO 2

7. *Cum functionis cuiuscunque ipsius x differentiale huiusmodi habeat formam Xdx, proposita tali forma differentiali Xdx, in qua X sit functio quaecunque ipsius x, illa functio, cuius differentiale est $= Xdx$, huius vocatur integrale et praefixo signo $\int$ indicari solet, ita ut $\int Xdx$ eam denotet quantitatem variabilem,cuius differentiale est $= Xdx$.*

COROLLARIUM 1

8. Quemadmodum ergo propositae formulae differentialis Xdx integrale seu ea functio ipsius x, cuius differentiale est $= Xdx$, quae hac scriptura $\int Xdx$ indicatur, investigari debeat, in calculo integrali est explicandum.

COROLLARIUM 2

9. Uti ergo littera d signum est differentiationis, ita littera $\int$ pro signo integrationis utimur sicque haec duo signa sibi mutuo opponuntur et quasi se destruunt, scilicet $\int dX$ erit $= X$, quia ea quantitas denotatur, cuius differentiale est dX, quae utique est X.

COROLLARIUM 3

10. Cum igitur harum ipsius x functionum x^2, x^n, $\sqrt{(aa-xx)}$ differentialia sint $2xdx$, $nx^{n-1}dx$, $\frac{-xdx}{\sqrt{(aa-xx)}}$, signo integrationis $\int$ adhibendo adhibendo patet fore

$$\int 2xdx = x^2, \int nx^{n-1}dx = x^n, \int \frac{-xdx}{\sqrt{(aa-xx)}} = \sqrt{(aa-xx)}.$$

SCHOLION 1

11. Hic unica tantum quantitas variabilis in computum ingredi videtur, cum tamen statuamus tam in calculo differentiali quam integrali semper rationem duorum pluriumve

differentialium spectari. Verum etsi hie una tantum quantitas variabilis x apparet, tamen revera duae considerantur; altera enim est ipsa ilIa functio, cuius differentiale sumimus esse Xdx; quae si designetur littera y, erit $dy = Xdx$ seu $\frac{dy}{dx} = X$, ita ut hic omnino ratio differentialium $dy : dx$ proponatur, quae est = X, indeque erit $y = \int Xdx$; hoc autem integrale non tam ex ipso differentiali Xdx, quod utique est = 0, quam ex eius ratione ad dx inveniri est censendum. Caeterum hoc signum $\int$ vocabulo *summae* efferri solet, quod ex conceptu parum idoneo, quo integrale tanquam summa omnium differentialium spectatur, est natum; neque maiore iure admitti potest, quam vulgo lineae ex punctis constare concipi solent.

SCHOLION 2

12. At calculus integralis multo latius quam ad huiusmodi formulas integrandas patet, quae unicam variabilem complectuntur. Quemadmodum enim hic functio unius variabilis x ex data differentialis forma investigatur, ita calculus integralis quoque extendi debet ad functiones duarum pluriumve variabilium investigandas, cum relatio quaedam differentialium fuerit proposita. Deinde calculus integralis non solum ad differentialia primi ordinis adstringitur, sed etiam praecepta tradere debet, quorum ope functiones tam unius quam duarum pluriumve variabilium investigari queant, cum relatio quaedam differentialium secundi altiorisve euiusdam ordinis fuerit data. Atque hanc ob rem definitionem calculi integralis ita instruximus, ut omnes huiusmodi investigationes in se complecteretur; differentialia enim cuiusque ordinis intelligi debent et voce *relationis*, quae inter ea proponatur, sum usus, ut latius pateret voce *rationis*, quae tantum duorum differentialium comparationem indicare videatur. Ex his ergo divisionem calculi integralis constituere poterimus.

DEFINITIO 3

13. Calculus integralis dividitur in duas partes, quarum prior tradit methodum functionem unius variabilis inveniendi ex data quadam relatione inter eius differentialia tam primi quam altiorum ordinum.

Pars autem altera methodum continet functionem duarum pluriumve variabilium inveniendi, cum relatio inter eius differentialia sive primi sive altioris cuiusdam gradus fuerit proposita.

COROLLARIUM 1

14. Prout ergo functio ex data differentialium relatione invenienda vel unicam variabilem complectitur vel duas pluresve, inde calculus integralis commode in duas partes principales dispescitur, quibus exponendis duos libros destinamus.

COROLLARIUM 2

15. Semper igitur calculus integralis in inventione functionum vel unius vel plurium variabilium versatur, cum scilicet relatio quaepiam inter eius differentialia sive altioris cuiuspiam ordinis fuerit proposita.

SCHOLION

16. Cum hic primam partem calculi integralis in investigatione functionum unicae variabilis ex data differentialium relatione constituamus, plures partes pro numero variabilium functionem ingredientium constitui debere videantur, ita ut pars secunda functiones duarum variabilium, tertia trium, quarta quatuor etc. complectatur. Verum pro his posterioribus partibus methodus fere eadem requiritur, ita ut, si inventio functionum duas variabiles involventium fuerit in potestate, via ad eas, quae plures varabiles implicant, satis sit patefacta; unde inventionem eiusmodi functionum, quae duas pluresve variabiles continent, commode coniungimus indeque unicam partem calculi integralis constituimus posteriori libro tractandam.

Caeterum haec altera pars in elementis adhuc nusquam est tractata, etiamsi eius usus in Mechanica ac praecipue in doctrina fluidorum maximi sit usus. Quocirca cum in hoc genere praeter prima rudimenta via quicquam sit exploratum, noster secundus liber de calculo integrali admodum erit sterilis ac praeter commemorationem eorum, quae adhuc desiderantur, parum erit expectandum; verum hoc ipsum ad scientiae incrementum multum conferre videtur.

DEFINITIO 4

17. Uterque de calculo integrali liber commode subdividitur in partes pro gradu differentialium, ex quorum relatione functionem quaesitam investigari oportet. Ita prima pars versatur in relatione differentialium primi gradus, secunda in relatione differentialium secundi gradus, quorsum etiam differentialia altiorum graduum ob tenuitatem eorum, quae adhuc sunt investigata, referri possunt.

COROLLARIUM 1

18. Uterque ergo liber constabit duabus partibus, in quarum priore relatio inter differentialia primi gradus proposita considerabitur, in posteriore vero eiusmodi integrationes occurrent, ubi relatio inter differentialia secundi altiorumve graduum proponitur.

COROLLARIUM 2

19. In primi ergo libri parte prima eiusmodi functio variabilis x invenienda proponitur, ut posita ea functione $= y$ et $\frac{dy}{dx} = p$ relatio quaecunque data inter has tres quantitates x, y et p adimpleatur, seu proposita quacunque aequatione inter has ternas quantitates ut indoles functionis y seu aequatio inter x et y tantum, exclusa p, eruatur.

COROLLARIUM 3

20. Posterioris autem partis primi libri quaestiones ita erunt comparatae, ut posito $\frac{dy}{dx} = p, \frac{dp}{dx} = q, \frac{dq}{dx} =$ r etc., si proponatur aequatio quaecunque inter quantitates *x, y, p, q, r* etc., indoles functionis *y* per *x* seu aequatio inter *x* et *y* eliciatur.

SCHOLION 1

21. Quae adhuc in calculo integrali sunt elaborata, maximam partem ad libri primi partem primam sunt referenda, in qua excolenda Geometrae imprimis operam suam collocarunt; pauca sunt, quae in parte posteriore sunt praestita, et alter liber, quem secundum fecimus, etiamnunc fere vacuus est relictus. Prima autem pars libri primi, in qua potissimum nostra tractatio consumetur, denuo in plures sectiones distinguitur pro modo relationis, quae inter quantitates x, y et $p = \frac{dy}{dx}$ proponitur. Relatio enim prae caeteris simplicissima est, quando $p = \frac{dy}{dx}$ aequatur functioni cuipiam ipsius *x;* qua posita = *X*, ut sit $\frac{dy}{dx} = X$ seu *dy* = *Xdx*, totum negotium in integratione formulae differentialis *Xdx* absolvitur; huius operationis iam supra mentionem fecimus, quae vulgo sub titulo integrationis formularum differentialium simplicium seu unicam variabilem involventium tractari solet. Eodem res rediret, si $p = \frac{dy}{dx}$ aequaretur functioni ipsius *y* tantum, quandoquidem quantitates *x* et *y* ita inter se reciprocantur, ut altera tanquam functio alterius spectari possit; haec ergo ad sectionem primam referentur. Sin autem $p = \frac{dy}{dx}$ aequetur expressioni ambas quantitates *x* et *y* involventi, aequatio habetur differentialis huius formae *Pdx* + *Qdy* = 0, ubi *P* et *Q* sunt expressiones quaecunque ex *x*, *y* et constantibus conflatae. Quanquam autem Geometrae multum in huiusmodi aequationum integratione desudarunt, tamen vix ultra quosdam casus satis particulares sunt progressi. Sin autem *p* magis complicate per *x* et *y* determinatur, ut eius valor explicite exhiberi nequeat, veluti si fuerit

$$p^5 = xxp^3 - xyp + x^5 - y^5,$$

ne via quidem constat tentanda, quomodo inde relatio inter *x* et *y* investigari queat; pauca ergo, quae hic tradere licebit, cum praecedentibus secundam sectionem primae partis libri primi occupabunt. Ita ex universa nostra tractatione magis patebit, quid adhuc in calculo integrali desideretur, quam quid iam sit expeditum, cum hoc prae illo ut minima quaedam particula sit spectandum.

SCHOLION 2

22. In singulis partibus, quas enarravimus, fieri etiam solet, ut non solum una quaedam functio, sed etiam simul plures investigentur, ita ut neutra sine reliquis definiri possit, quemadmodum in Algebra communi usu venit, ut ad solutionem prablematis plures

incognitae in calculum. sint introducendae, quae deinceps per totidem aequationes determinentur. Veluti si eiusmodi binae functiones y et z ipsius x sint inveniendae, ut sit

$$xdy + azzdx = 0 \text{ et } xxdz + bxydy = cdy,$$

hinc novae subdivisiones nostrae tractationis constitui possant. Verum quia hic ut in Algebra communi totum negotium. ad eliminationem unius litterae revocatur, ut deinceps duae tantum variabiles in una aequatione supersint, hinc tractatio non multiplicanda videtur.

SCHOLION 3

23. In secundo libro calculi integralis, quo functio duarum pluriumve variabilium ex data differentialium relatione investigatur, multo maior quaestionum varietas locum habet. Sit enim z functio binarum variabilium x et t investiganda, et cum $\left(\frac{dz}{dx}\right)$ denotet rationem eius differentialis ad dx, si sola x pro variabili habeatur, at $\left(\frac{dz}{dt}\right)$ rationem eius differentialis ad dt, si sola t variabilis sumatur, prima pars eiusmodi continebit quaestiones, in quibus certa quaedam relatio inter quantitates x, t, z et $\left(\frac{dz}{dx}\right)$, $\left(\frac{dz}{dt}\right)$ proponitur, et quaestio huc redit, ut hinc aequatio inter solas quantitates x, t et z eruatur; inde enim qualis z sit functio ipsarum x et t patebit. In secunda parte praeter has formulas $\left(\frac{dz}{dx}\right)$ et $\left(\frac{dz}{dt}\right)$ etiam istae $\left(\frac{ddz}{dxdx}\right)$, $\left(\frac{ddz}{dxdt}\right)$ et $\left(\frac{ddz}{dtdt}\right)$ in computum ingredientur, quarum significatio ita est intelligenda, ut positis prioribus $\left(\frac{dz}{dx}\right) = p$ et $\left(\frac{dz}{dt}\right) = q$, ubi p et q iterum certae erunt functiones ipsorum x et t; futurum sit simili expressionis modo

$$\left(\frac{ddz}{dxdx}\right) = \left(\frac{dp}{dx}\right), \left(\frac{ddz}{dxdt}\right) = \left(\frac{dp}{dt}\right) = \left(\frac{dq}{dx}\right), \left(\frac{ddz}{dtdt}\right) = \left(\frac{dq}{dt}\right).$$

Proposita ergo relatione inter has formulas et praecedentes simulque ipsas quantitates x, t et z, aequatio inter ternas istas quantitates solas x, t et z erui debet. Huiusmodi quaestiones frequenter occurrunt in Mechanica et Hydraulica, quando motus corporum flexibilium et fluidorum indagatur; ex quo maxime est optandum, ut haec altera sectio secundi libri calculi integralis omni cura excolatur. Neque vero opus erit, ut hanc investigationem ad differentialia altiora extendamus, cum nullae adhuc quaestiones sint traetatae, quae tanta calculi incrementa desiderent.

DEFINITIO 5

24. Si functiones, quae in calculo integrali ex relatione differentialium quaeruntur, algebraice exhiberi nequeant, tum eae vocantur transcendentes, quandoquidem earum ratio vires Analyseos communis transcendit.

COROLLARIUM 1

25. Quoties ergo integratio non succedit, toties functio, quae per integrationem quaeritur, pro transcendente est habenda. Ita si formula differentialis Xdx integrationem non admittit, eius integrale, quod ita indicari solet $\int Xdx$, est functio transcendens ipsius x.

COROLLARIUM 2

26. Hinc intelligitur, si y fuerit functio transcendens ipsius x, vicissim fore x functionem transcendentem ipsius y atque ex hac conversione novae functiones transcendentes oriuntur.

COROLLARIUM 3

27. Pro variis partibus et sectionibus calculi integralis nascuntur etiam plura genera functionum transcendentium, quorum adeo numerus in infinitum exsurgit; unde patet, quanta copia omnium quantitatum possibilium nobis adhuc sit ignota.

SCHOLION 1

28. Iam antequam in Analysin infinitorum penetravimus, species quasdam functionum transcendentium cognoscere licuit. Primam suppeditavit doctrina logarithmorum; si enim y denotet logarithmum ipsius x, ut sit $y = lx$,erit y utique functio transcendens ipsius x sicque logarithmi quasi primam speciem functionum transcendentium constituunt. Deinde cum ex aequatione $y = lx$ vicissim sit $x = e^{y}$, erit x utique etiam functio transcendens ipsius y ac tales functiones vocantur exponantiales. Porro autem consideratio angulorum aliud genus aperuit; veluti si angulus, cuius sinus est s, ponatur $= \varphi$, ut sit $\varphi = Arc.\sin s$, nullum est dubium, quin φ sit functio transcendens ipsius s et quidem infinitiformis; hincque cum convertendo prodeat $s = \sin.\varphi$, erit etiam sinus s functio transcendens anguli φ. Quanquam autem hae functiones transcendentes sine subsidio calculi integralis sunt agnitae, tamen in ipso quasi limine calculi integralis ad eas deducimur earumque indoles ita nobis iam est perspecta, ut propemodum functionibus algebraicis accenseri queant. Quare etiam perpetuo in calculo integrali, quoties functiones transcendentes ibi repertas ad logarithmos vel angulos revocare licet, eas tanquam algebraicas spectare solemus

SCHOLION 2

29. Cum calculus integralis ex inversione calculi differentialis oriatur, perinde ac reliquae methodi inversae ad notitiam novi generis quantitatum nos perducit. Ita si a tirone primorum elementorum nihil praeter notitiam numerorum integrorum positivorum postulemus, apprehensa additione, statim atque ad operationem inversam, subtractionem scilicet, ducitur, notionem numerorum negativorum assequetur. Deinde multiplicatione tradita, cum ad divisionem progreditur, ibi notionem fractionum accipiet'. Porro postquam evectionem ad potestates didicerit, si per operationem inversam extractionem radicum suscipiat, quoties negotium non succedit, ideam numerorum irrationalium

adipiscetur haecque cognitio per totam Analysin communem sufficiens censetur. Simili ergo modo calculus integralis, quatenus integratio non succcedit, novum nobis genus quantitatum transeendentium aperit. Non enim, uti omnium differentialia exhiberi possunt, ita vicissim omnium differentialium integralia exhibere licet.

SCHOLION 3

30. Neque vero, statim ac primi conatus in integratione expedienda fuerint initi, functiones quaesitae pro transcendentibus sunt habendae; fieri enim saepe solet, ut integrale etiam algebraicum nonnisi per operationes artificiosas obtineri queat. Deinde quando functio quaesita fuerit transcendens, sollicite videndum est, num forte ad species illas simplicissimas logarithmorum vel angulorum revocari possit, quo casu solutio algebraicae esset aequiparanda. Quod si minus successerit, formam tamen simplicissimam functionum transcendentium, ad quam quaesitam reducere liceat, indagari conveniet. Ad usum autem longe commodissimum est, ut valores functionum transcendentium vero proxime exhibeantur, quem in finem insignis pars calculi integralis in investigationem serierum infinitarum impenditur, quae valores earum functionum contineant.

THEOREMA

31. *Omnes functiones per calculum integralem inventae sunt indeterminatae ac requirunt determinationem ex natura quaestionis, cuius solutionem suppeditant, petendam.*

DEMONSTRATIO

Cum semper infinitae dentur functiones, quarum idem est differentiale, siquidem functionis $P + C$, quicunque valor constanti C tribuatur, differentiale idem est $= dP$, vicissim etiam proposito differentiali dP integrale est $P + C$,ubi pro C quantitatem constantem quamcunque ponere licet; unde patet eam functionem, cuius differentiale datur $= dP$, esse indeterminatam, cum quantitatem constantem arbitrariam in se involvat. Idem etiam eveniat necesse est, si functio ex quacunque differentialium relatione sit determinanda, semperque complectetur quantitatem constantem arbitrariam, cuius nullum vestigium in relatione differentialium apparuit. Determinabitur ergo huiusmodi functio per calculum integralem inventa, dum constanti illi arbitrariae certus valor tribuitur, quem semper natura quaestionis, cuius solutio ad illam functionem perduxerat, suppeditabit.

COROLLARIUM 1

32. Si ergo functio y ipsius x ex relatione quapiam differentialium definitur, per constantem arbitrariam ingressam ita determinari potest, ut posito $x = a$ fiat $y = b;$ quo facto functio erit determinata et pro quovis valore ipsi x tributo functio y determinatum obtinebit valorem.

COROLLARIUM 2

33. Si ex relatione differentialium secundi gradus functio y definiatur, binas involvet constantes arbitrarias ideoque duplicem determinationem admittit, qua effici potest, ut

posito $x = a$ non solum y obtineat datum valorem b, sed etiam ratio $\frac{dy}{dx}$ dato valori c fiat aequalis.

COROLLARIUM 3

34. Si y sit functio binarum variabilium x et t ex relatione differentialium eruta, etiam constantem arbitrariam involvet, cuius determinatione effici poterit, ut posito $t = a$ aequatio inter y et x prodeat data seu naturam datae cuiuspiam curvae exprimat.

SCHOLION

35. Ista functionum integralium, seu quae per calculum integralem sunt inventae, determinatio quovis casu ex natura quaestionis tractatae facile deducitur neque ulla difficultate laborat, nisi forte praeter necessitatem solutio ad differentialia fuerit perducta, cum per Analysin communem erui potuisset; quo casu perinde atque in Algebra quasi radices inutiles ingeruntur. Cum autem haec determinatio tantum in applicatione ad certos casus instituatur, hic, ubi integrandi methodum in genere tradimus, integralia in omni amplitudine eruere conabimur, ita ut constantes per integrationem ingressae maneant arbitrariae, neque, nisi conditio quaedam urgeat, eas determinabimus. Caeterum determinatio functionum ipsius x simplicissima est, qua eae casu $x = 0$ ipsae evanescentes redduntur.

DEFINITIO 6

36. Integrale completum exhiberi dicitur, quando functio quaesita omni extensione cum constante arbitraria repraesentatur. Quando autem ista constans iam certo modo est determinata, integrale vocari solet particulare.

COROLLARIUM 1

37. Quovis ergo casu datur unicum integrale completum; integralia autem particularia infinita exhiberi possunt. Sic differentialis xdx integrale completum est $\frac{1}{2}xx + C$, integralia autem particularia $\frac{1}{2}xx, \frac{1}{2}xx + 1, \frac{1}{2}xx + 2$ etc. multitudine infinita.

COROLLARIUM 2

38. Integrale ergo completum omnia integralia particularia in se complectitur ex eoque haec omnia facile formari possunt. Vicissim antem ex integralibus particularibus integrale completum non innotescit. Saepenumero autem, uti deinceps patebit, habetur methodus ex integrali particulari completum inveniendi.

SCHOLION

39. Interdum facile est integrale particulare coniectura vel divinatione assequi. Veluti si eiusmodi functio ipsius x, quae sit y, quaeritur, ut sit $dy + yydx = dx + xxdy$, huic aequationi manifesto satisfit sumendo $y = x$, quod ergo est integrale particulare, quoniam in eo nulla inest constans arbitraria; at integrale completum reperietur $y = \frac{1+Cx}{C+x}$, quod illud particulare in se continet sumendo $C = \infty$. Simili modo sumendo $C = 0$ hinc aliud integrale obtinetur $y = \frac{1}{x}$, quod superiori aequationi perinde satisfacit ac prius $y = x$. Omnia autem integralia particularia, quaecunque satisfaciunt, contineri necesse est in formula generali $y = \frac{1+Cx}{C+x}$, prouti constanti arbitrariae C alii atque alii valores tribuantur; ita sumto $C = 1$ fit etiam $y = 1$. Plerumque autem evenire solet, ut, etiamsi integrale quoddam particulare sit algebraicum, tamen integrale completum sit transcendens. Veluti si proposita sit haec aequatio $dy + ydx = dx + xdx$, statim patet satisfieri posito $y = x$, quod ergo est integrale particulare; verum integrale completum constantem arbitrariam C involvens est $y = x + Ce^{-x}$ denotante e numerum, cuius logarithmus $= 1$; nisi ergo hic sumatur $C = 0$, functio y semper est transcendens. Haec in genere notasse sufficiat, antequam ad tractationem ipsam calculi integralis aggrediamur, quandoquidem ad omnes integrationes pertinent; nunc igitur forma tractationis exposita ad opus tractandum pergamus.