SUPPLEMENT III*b* TO BOOK I, CH. IV.

THE INTEGRATION OF LOGARITHMIC AND EXPONENTIAL TERMS.

2) Concerning the value of the integral formula $\int \frac{z^{\lambda-\omega}\pm z^{\lambda+\omega}}{1\pm z^{2\lambda}}\cdot\frac{dz}{z}(lz)^{\mu}$, in the case where after the integration there is put $z=1$.

Novi Commentarii Acad. Imp. Sc. Petropolitanae. Tom XIX. Pag. 30-64.

§. 69. Some time ago now, from the consideration of innumerable circular arcs which have a common sine or tangent, I deduced the sum of the two infinite series, which were seen to be especially noteworthy on account of the great generality [see, e.g. Series One, *Opera Omnia*, Vol. 14, E122, E321, E254]. Indeed if the letters m and n may denote any numbers, with the ratio of the diameter to the periphery of a circle may be put as 1 to π, these two summations may be shown in this manner:

$$\frac{1}{m}+\frac{1}{n-m}-\frac{1}{n+m}-\frac{1}{2n-m}+\frac{1}{2n+m}+\frac{1}{3n-m}-\text{etc.}=\frac{\pi}{n\sin.\frac{m\pi}{n}}$$

and

$$\frac{1}{m}-\frac{1}{n-m}+\frac{1}{n+m}-\frac{1}{2n-m}+\frac{1}{2n+m}-\frac{1}{3n-m}+\text{etc.}=\frac{\pi}{n\,\text{tang}.\frac{m\pi}{n}},$$

now at that time I was elucidating the sums of all these series from these two series, the denominators of which were progressing according to the powers of the natural numbers, just as I have set out further in the introduction to the *Analysis Infinitorum* and elsewhere. But now these same series have led me to the integration of the formulas in the title of the expression, where therefore it may be seen to be worth more attention, because by no means will it be allowed to perform integrations of this kind by other methods.

§. 70. Moreover it is apparent at once, these two infinite series arise from the expansions of certain integral formulas, if after the integration , a certain value such as unity, may be given to the value of the variable quantity; thus the first series is deduced from the resolution of the integration of this formula :

$$\int\frac{z^{m-1}+z^{n-m-1}}{1+z^{n}}dz,$$

truly the latter from the resolution of this :

$$\int\frac{z^{m-1}-z^{n-m-1}}{1-z^{n}}dz,$$

if indeed after the integration there may be put $z=1$. Hence moreover I have shown these from the principles of integral calculus, if indeed there may be put $z=1$, the former to be reduced to this simple formula

$$\frac{\pi}{n\sin.\frac{m\pi}{n}},$$

but for the latter integral, in the same case $z=1$, thus to this

$$\frac{\pi}{n\,\mathrm{tang}.\frac{m\pi}{n}},$$

so that from the principles of integral calculus truly there shall be :

$$\frac{1}{m}+\frac{1}{n-m}-\frac{1}{n+m}-\frac{1}{2n-m}+\frac{1}{2n+m}+\frac{1}{3n-m}-\text{etc.}=\frac{\pi}{n\sin.\frac{m\pi}{n}}$$

and

$$\frac{1}{m}-\frac{1}{n-m}+\frac{1}{n+m}-\frac{1}{2n-m}+\frac{1}{2n+m}-\frac{1}{3n-m}+\text{etc.}=\frac{\pi}{n\,\mathrm{tang}.\frac{m\pi}{n}},$$

if indeed after putting the integration in place thus, so that the integral may vanish on putting $z=0$, it may remain on putting $z=1$.

§. 71. Now so that we may reduce this two-fold integration to the proposed form, we may make $n=2\lambda$ and $m=\lambda-\omega$, from which these two infinite series may adopt this form

$$\frac{1}{\lambda-\omega}+\frac{1}{\lambda+\omega}-\frac{1}{3\lambda-\omega}-\frac{1}{3\lambda+\omega}+\frac{1}{5\lambda-\omega}+\frac{1}{5\lambda+\omega}-\text{etc.}$$

and

$$\frac{1}{\lambda-\omega}-\frac{1}{\lambda+\omega}+\frac{1}{3\lambda-\omega}-\frac{1}{3\lambda+\omega}+\frac{1}{5\lambda-\omega}-\frac{1}{5\lambda+\omega}-\text{etc.}$$

therefore the sum of this first series will be

$$\frac{\pi}{2\lambda\sin.\frac{\pi(\lambda-\omega)}{2\lambda}}=\frac{\pi}{2\lambda\cos.\frac{\pi\omega}{2\lambda}},$$

truly the sum of the latter will be

$$\frac{\pi}{2\lambda\,\mathrm{tang}.\frac{\pi(\lambda-\omega)}{2\lambda}}=\frac{\pi}{2\lambda\,\mathrm{cotang}.\frac{\pi\omega}{2\lambda}}=\frac{\pi\,\mathrm{tang}.\frac{\pi\omega}{2\lambda}}{2\lambda}.$$

So that if, for the sake of brevity, we may put

$$\frac{\pi}{2\lambda\cos.\frac{\pi\omega}{2\lambda}}=\mathrm{S},\ \text{and}\ \frac{\pi}{2\lambda}\mathrm{tang}.\frac{\pi\omega}{2\lambda}=\mathrm{T},$$

we will have the two following integrations :

$$\int \frac{z^{\lambda-\omega}+z^{\lambda+\omega}}{1+z^{2\lambda}}\cdot\frac{dz}{z}=\mathrm{S},$$

and

$$\int \frac{z^{\lambda-\omega}-z^{\lambda+\omega}}{1-z^{2\lambda}}\cdot\frac{dz}{z}=\mathrm{T}.$$

§.72. Before everything else, concerning these two integrations, I note that these questions have to be considered likewise, whether whole numbers or fractions may be taken for the letters λ and ω. For λ and ω may be any fractions which emerge from the integration, which if multiplied by α, from which put in place there becomes $z=x^{\alpha}$, and there will be $\frac{dz}{z}=\frac{\alpha dx}{x}$, and some power $z^{\delta}=x^{\alpha\delta}$; therefore the formula will become

$$\int \frac{x^{\alpha(\lambda-\omega)}+x^{\alpha(\lambda+\omega)}}{1+x^{2\alpha\lambda}}\cdot\frac{\alpha dx}{x},$$

where, since now all the exponents are whole numbers, the value of this formula put in place after integration will be $x=1$, since then also there shall be $z=1$, therefore which only differs from the preceding because here we have $\alpha\lambda$ and $\alpha\omega$ in place of λ and ω, and besides this the factor α shall be present; on account of which the value of this formula will become

$$\alpha\cdot\frac{\pi}{2\alpha\lambda\cos.\frac{\pi\omega}{2\lambda}}=\frac{\pi}{2\lambda\cos.\frac{\pi\omega}{2\lambda}},$$

which value therefore is $=\mathrm{S}$ exactly as before; which identity is evident also in the other formula, from which it is apparent, even if same fractions may be taken for λ and ω, here the integration shown is no less useful; which circumstance deserves to be understood properly, because in the following we are going to be using the letter ω as the variable.

§. 73. Therefore after these two integral formulas were indicated by the letters S and T, thus so that they vanish on putting $z=0$, the integrals can be seen to be not only functions of the quantity z, but also as functions of the two variables z and ω, since the number ω is allowed to be treated as a variable quantity, just as also the exponent λ may be taken as a variable quantity: but because here integral formulas of another kind are going to be produced, and here I have put in place to be considered, I am going to be treating only the quantity ω, as well as the variable z itself, here as a variable quantity.

§.74. Therefore since there shall be

$$\mathrm{S}=\int \frac{z^{\lambda-\omega}+z^{\lambda+\omega}}{1+z^{2\lambda}}\cdot\frac{dz}{z},$$

in which integration only z may be considered as variable, there will be designated following the custom now used :

$$\left(\frac{dS}{dz}\right)=\frac{z^{\lambda-\omega}+z^{\lambda+\omega}}{1+z^{2\lambda}}\cdot\frac{1}{z};$$

this formula may be differentiated again, only by putting the letter ω to be variable, and there will become

$$\left(\frac{ddS}{dzd\omega}\right)=\frac{-z^{\lambda-\omega}+z^{\lambda+\omega}}{1+z^{2\lambda}}\cdot\frac{1}{z}lz,$$

which formula multiplied by dz, and integrated anew only with z taken for the variable, will give

$$\int dz\left(\frac{ddS}{dzd\omega}\right)=\int\frac{-z^{\lambda-\omega}+z^{\lambda+\omega}}{1+z^{2\lambda}}\cdot\frac{dz}{z}lz,$$

where it may be observed, that

$$S=\frac{\pi}{2\lambda\cos.\frac{\pi\omega}{2\lambda}};$$

thus so that hence we may deduce

$$\frac{dS}{d\omega}=\frac{\pi\pi\sin.\frac{\pi\omega}{2\lambda}}{4\lambda\lambda\left(\cos.\frac{\pi\omega}{2\lambda}\right)^2},$$

therefore with this value substituted here, we arrive at this integration:

$$\int\frac{-z^{\lambda-\omega}+z^{\lambda+\omega}}{1+z^{2\lambda}}\cdot\frac{dz}{z}lz=\frac{\pi\pi\sin.\frac{\pi\omega}{2\lambda}}{4\lambda\lambda\left(\cos.\frac{\pi\omega}{2\lambda}\right)^2}.$$

§. 75. So that if now the other form may be treated in a similar manner, since there shall be

$$T=\frac{\pi\omega}{2\lambda}\,\text{tang}\,\frac{\pi\omega}{2\lambda},$$ there becomes

$$\frac{dT}{d\omega}=\frac{\pi\pi}{4\lambda\lambda\left(\cos.\frac{\pi\omega}{2\lambda}\right)^2};$$

moreover from the form of the integral there will be

$$\left(\frac{dT}{d\omega}\right)=\int\frac{-z^{\lambda-\omega}+z^{\lambda+\omega}}{1-z^{2\lambda}}\cdot\frac{dz}{z}lz,$$

from which we deduce the following integration

$$\int\frac{z^{\lambda-\omega}+z^{\lambda+\omega}}{1-z^{2\lambda}}\cdot\frac{dz}{z}lz=\frac{-\pi\pi}{4\lambda\lambda\left(\cos.\frac{\pi\omega}{2\lambda}\right)^2}.$$

§.76. Since we have given also the letters S and T expressed by series , also there will be expressed by similar series :

$$\left(\frac{dS}{d\omega}\right)=\frac{1}{(\lambda-\omega)^2}-\frac{1}{(\lambda+\omega)^2}-\frac{1}{(3\lambda-\omega)^2}+\frac{1}{(3\lambda+\omega)^2}+\frac{1}{(5\lambda-\omega)^2}-\text{etc.}$$
$$=\frac{\pi\pi\sin.\frac{\pi\omega}{2\lambda}}{4\lambda\lambda\left(\cos.\frac{\pi\omega}{2\lambda}\right)^2}.$$

And in a similar manner also, for the other series,

$$\left(\frac{dS}{d\omega}\right)=\frac{1}{(\lambda-\omega)^2}-\frac{1}{(\lambda+\omega)^2}-\frac{1}{(3\lambda-\omega)^2}+\frac{1}{(3\lambda+\omega)^2}+\frac{1}{(5\lambda-\omega)^2}-\text{etc.}$$
$$=\frac{\pi\pi\sin.\frac{\pi\omega}{2\lambda}}{4\lambda\lambda\left(\cos.\frac{\pi\omega}{2\lambda}\right)^2};$$

and thus we have represented the sums of these series also in a twofold manner, clearly resolved by a term involving the quantity π, then truly also by an integral term, which has been prepared thus, so that its integral shall be unable to be assigned by any of the current methods at this time.

§. 77. We will apply these integrations to some special cases: and initially indeed we will assume $\omega=0$, in which case the first integration springs to mind at once, but the latter gives

$$\int\frac{2z^{\lambda}}{1-z^{2\lambda}}\cdot\frac{dz}{z}lz=-\frac{\pi\pi}{4\lambda\lambda},$$
or
$$\int\frac{z^{\lambda-1}dzlz}{1-z^{2\lambda}}=-\frac{\pi\pi}{8\lambda\lambda};$$

and hence likewise we arrive at this same summation

$$\frac{1}{\lambda\lambda}+\frac{1}{\lambda\lambda}+\frac{1}{9\lambda\lambda}+\frac{1}{9\lambda\lambda}+\frac{1}{25\lambda\lambda}+\frac{1}{25\lambda\lambda}+\text{etc.}=\frac{\pi\pi}{4\lambda\lambda},$$
or
$$1+\frac{1}{9}+\frac{1}{25}+\frac{1}{49}+\frac{1}{81}+\text{etc.}=\frac{\pi\pi}{8},$$

that which formerly has been demonstrated by me.

§. 78. This is apparent at once to be the same, whatever number may be taken for λ ; therefore let $\lambda = 1$, and this equation will be had:

$$\int \frac{dzlz}{1-z^2} = -\frac{\pi\pi}{8};$$

from which the simpler integrations

$$\int \frac{dzlz}{1-z} \text{ and } \int \frac{dzlz}{1+z}$$

can be derived with the aid of this reasoning; there may be put

$$\int \frac{zdzlz}{1-zz} = \mathrm{P},$$

and on putting $zz = v$, so that there shall be $zdz = \frac{dv}{2}$ and $lz = \frac{1}{2}lv$, there will be produced

$$\frac{1}{4}\int \frac{dvlv}{1-v} = \mathrm{P},$$

evidently if after the integration there may become $v = 1$, certainly since in that case also there shall be $z = 1$; thus there will become

$$\int \frac{dvlv}{1-v} = 4\mathrm{P}:$$

now that first form may be added to that found, and there will be

$$\int \frac{dzlz + zdzlz}{1-zz} = \mathrm{P} - \frac{\pi\pi}{8},$$

but this term is reduced at once to this

$$\int \frac{dzlz}{1-z} = \mathrm{P} - \frac{\pi\pi}{8},$$

moreover in the manner we have seen

$$\int \frac{dvlv}{1-v} \text{ or } \int \frac{dzlz}{1-z} = 4\mathrm{P}, \text{ thus so that there shall be } 4\mathrm{P} = \mathrm{P} - \frac{\pi\pi}{8},$$

from which it shall be evident that $\mathrm{P} = -\frac{\pi\pi}{24}$, from which it follows there shall be

$$\int \frac{dzlz}{1-z} = -\frac{\pi\pi}{6};$$

in a similar manner there will be

$$\int \frac{dzlz - zdzlz}{1-zz} = -\mathrm{P} - \frac{\pi\pi}{8} = -\frac{\pi\pi}{12},$$

which, on being divided above and below by $1-z$, there becomes

$$\int \frac{dzlz}{1+z} = -\frac{\pi\pi}{12},$$

whereby now I have acquired three most noteworthy :

$$\text{I. } \int \frac{dzlz}{1+z} = -\frac{\pi\pi}{12},$$
$$\text{II. } \int \frac{dzlz}{1-z} = -\frac{\pi\pi}{6},$$
$$\text{III. } \int \frac{dzlz}{1-zz} = -\frac{\pi\pi}{8},$$

to which it can be added :

$$\text{IV. } \int \frac{zdzlz}{1-zz} = -\frac{\pi\pi}{24}.$$

§.79. Therefore just as these formulas have been deduced from these principles of the integral calculus, thus also the truth of these can be proven easily by the resolution into series; since indeed there shall be

$$\frac{1}{1+z} = 1 - z + zz - z^3 + z^4 - z^5 + \text{etc.},$$

and in general

$$\int z^n dzlz = \frac{z^{n+1}}{n+1} lz - \frac{z^{n+1}}{(n+1)^2},$$

which value on putting $z=1$ is reduced to $\frac{1}{(n+1)^2}$, it is evident to become

$$\int \frac{dzlz}{1+z} = -1 + \frac{1}{4} - \frac{1}{9} + \frac{1}{16} - \frac{1}{25} + \text{etc.} = -\frac{\pi\pi}{12},$$

or

$$1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \frac{1}{25} - \text{etc.} = \frac{\pi\pi}{12},$$

in a similar manner on account of

$$\frac{1}{1-z} = 1 + z + zz + z^3 + z^4 + z^5 + \text{etc.},$$

there will be

$$\int \frac{dzlz}{1-z} = -1 - \frac{1}{4} - \frac{1}{9} - \frac{1}{16} - \frac{1}{25} - \text{etc.} = -\frac{\pi\pi}{12},$$

or

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \text{etc.} = \frac{\pi\pi}{6},$$

In the same manner also:

$$\int \frac{zdzlz}{1-zz} = -\frac{1}{4} - \frac{1}{16} - \frac{1}{36} - \text{etc.} = -\frac{\pi\pi}{24},$$

or

$$\frac{1}{4} + \frac{1}{16} + \frac{1}{36} + \frac{1}{64} + \text{etc.} = \frac{\pi\pi}{24},$$

indeed which summations now are most noteworthy. Nor yet at this stage can any be shown by a direct method :

$$\int \frac{zdzlz}{1+zz} == -\frac{\pi\pi}{12}.$$

§. 80. Now we may put $\omega = 1$, and our integrations adopt these forms:

$$1^0. \int \frac{-z^{\lambda-2}(1-zz)dzlz}{1+z^{2\lambda}} \cdot \frac{dz}{z} lz = \frac{\pi\pi \sin.\frac{\pi}{2\lambda}}{4\lambda\lambda\left(\cos.\frac{\pi}{2\lambda}\right)^2}$$

and

$$2^0. \int \frac{-z^{\lambda-2}(1+zz)dzlz}{1-z^{2\lambda}} \cdot \frac{dz}{z} lz = +\frac{\pi\pi}{4\lambda\lambda\left(\cos.\frac{\pi}{2\lambda}\right)^2},$$

from which for diverse values of λ, which indeed cannot be accepted for values less than two, the following integrations may be obtained :

I^0. if $\lambda = 2,$ there will be

$$1^0 \int \frac{-(1-zz)dzlz}{1+z^4} = \frac{\pi\pi}{8\sqrt{2}},$$

$$2^0 \int \frac{-(1+zz)dzlz}{1-z^4} = +\frac{\pi\pi}{8}, \text{ or } \int \frac{-dzlz}{1-zz} = +\frac{\pi\pi}{8}.$$

II^0. If $\lambda = 3,$ we will have

$$1^0 \int \frac{-z(1-zz)dzlz}{1+z^6} = \frac{\pi\pi}{54}, \text{ and}$$

$$2^0 \int \frac{-z(1+zz)dzlz}{1-z^6} = \int \frac{-zdzlz}{1-zz+z^4} = \frac{\pi\pi}{27}.$$

Moreover these two forms on putting $zz = v$, will be changed into the following :

$$1^0 \int \frac{-dv(1-v)lv}{1+v^3} = \frac{2\pi\pi}{27},$$

and

$$2^0 \int \frac{dvlv}{1-v+vv} = \frac{4\pi\pi}{27}.$$

III^0. Let $\lambda = 4$ and the following arise :

$$1^0 \int \frac{-zz(1-zz)dzlz}{1+z} = \frac{\pi\pi\sqrt{\frac{\sqrt{2}-1}{2\sqrt{2}}}}{16\left(2+\sqrt{2}\right)} = \frac{\pi\pi\sqrt{\left(2-\sqrt{2}\right)}}{32\left(2+\sqrt{2}\right)}$$

and

$$2^0 \int \frac{-zz(1+zz)dzlz}{1-z^8} = \int \frac{-zzdzlz}{(1-zz)\left(1+z^4\right)} = \frac{\pi\pi}{16\left(2+\sqrt{2}\right)},$$

which latter form is reduced to this

$$\int -\frac{dzlz}{(1-zz)}+\int\frac{(1-zz)dzlz}{(1+z^4)}=\frac{\pi\pi}{8(2+\sqrt{2})},$$

truly there is $\int\frac{-dzlz}{1-zz}=\frac{\pi\pi}{8}$, from which there is found :

$$\int\frac{(1-zz)dzlz}{1+z^4}=-\frac{\pi\pi(1+\sqrt{2})}{8(2+\sqrt{2})}=-\frac{\pi\pi}{8\sqrt{2}},$$

which value has now been found in the case above, $\lambda=2$.

§. 81. But nothing prevents us from making also λ less than 1, while the integration thus may be taken so that it vanishes on putting $z=0$, but then we will find

$$1^0\ \int\frac{-(1-zz)dzlz}{z(1+zz)}=\infty$$

and

$$2^0\ \int\frac{-(1+zz)dzlz}{z(1-zz)}=\infty,$$

hence from which nothing can be concluded. In addition also our series found above indicate clearly that the sums of these are infinite, since the first term of each $\frac{1}{(\lambda-\omega)^2}$ shall be infinite , as we have assumed $\lambda=1$ and $\omega=1$ to be used.

§. 82. From these cases established, we can progress further and put in place the integral forms found

$$\int\frac{-z^{\lambda-\omega}+z^{\lambda+\omega}}{1+z^{2\lambda}}\cdot\frac{dz}{z}lz=\mathrm{S'}$$

and

$$\int\frac{-z^{\lambda-\omega}-z^{\lambda+\omega}}{1-z^{2\lambda}}\cdot\frac{dz}{z}lz=\mathrm{T'}$$

thus so that there shall become

$$S'=\frac{\pi\pi\sin.\frac{\pi}{2\lambda}}{4\lambda\lambda\left(\cos.\frac{\pi}{2\lambda}\right)^2}\ \text{and}\ \mathrm{T'}=\frac{\pi\pi}{4\lambda\lambda\left(\cos.\frac{\pi}{2\lambda}\right)^2},$$

and as before now we differentiate by the number ω alone to be made variable ; with which done we obtain the following integrals :

$$\int \frac{z^{\lambda-\omega}+z^{\lambda+\omega}}{1+z^{2\lambda}} \cdot \frac{dz}{z}(lz)^2 = \left(\frac{dS'}{d\omega}\right),$$

and

$$\int \frac{z^{\lambda-\omega}-z^{\lambda+\omega}}{1-z^{2\lambda}} \cdot \frac{dz}{z}(lz)^2 = \left(\frac{dT'}{d\omega}\right).$$

Hence finally we may put for brevity the angle $\frac{\pi\omega}{2\pi} = \varphi$, so that there becomes

$$S' = \frac{\pi\pi \sin.\varphi}{4\lambda\lambda\left(\cos.^2\varphi\right)} = \frac{\pi\pi}{4\lambda\lambda} \cdot \frac{\sin.\varphi}{\cos.^2\varphi},$$

and

$$\mathrm{T}' = \frac{\pi\pi}{4\lambda\lambda} \cdot \frac{1}{\cos.^2\varphi},$$

and we will find:

$$d \cdot \frac{\sin.\varphi}{\cos.^2\varphi} = \frac{\cos.^2\varphi + 2\sin.^2\varphi}{\cos.^3\varphi} d\varphi = \frac{1+\sin.^2\varphi}{\cos.^3\varphi} d\varphi,$$

where $d\varphi = \frac{\pi d\omega}{2\pi}$; from which we deduce

$$\left(\frac{dS'}{d\omega}\right) = \frac{\pi^3}{8\lambda^2}\left(\frac{1+\left(\sin.\frac{\pi\omega}{2\lambda}\right)^2}{\left(\cos.\frac{\pi\omega}{2\lambda}\right)^3}\right) = \frac{\pi^3}{8\lambda^2}\left(\frac{2}{\left(\cos.\frac{\pi\omega}{2\lambda}\right)^3} - \frac{1}{\cos.\frac{\pi\omega}{2\lambda}}\right);$$

in a like manner because $\mathrm{T}' = \frac{\pi\pi}{4\lambda\lambda} \cdot \frac{1}{\cos.^2\varphi}$, there will be

$$d \cdot \frac{1}{\cos.^2\varphi} = \frac{2d\varphi \sin.\varphi}{\cos.^3\varphi},$$

and hence

$$\left(\frac{dT}{d\omega}\right) = \frac{\pi^3}{8\lambda^2} \cdot \frac{2\sin.\frac{\pi\omega}{2\lambda}}{\left(\cos.\frac{\pi\omega}{2\lambda}\right)^3}.$$

Consequently these integrations will arise :

$$\int \frac{z^{\lambda-\omega}+z^{\lambda+\omega}}{1+z^{2\lambda}} \cdot \frac{dz}{z}(lz)^2 = \frac{\pi^3}{8\lambda^2}\left(\frac{2}{\left(\cos.\frac{\pi\omega}{2\lambda}\right)^3} - \frac{1}{\cos.\frac{\pi\omega}{2\lambda}}\right),$$

$$\int \frac{z^{\lambda-\omega}-z^{\lambda+\omega}}{1-z^{2\lambda}} \cdot \frac{dz}{z}(lz)^2 = \frac{\pi^3}{8\lambda^2} \cdot \frac{2\sin.\frac{\pi\omega}{2\lambda}}{\left(\cos.\frac{\pi\omega}{2\lambda}\right)^3}.$$

§. 83. If now, in the same manner of series §.76 we may differentiate again, with only ω taken to be variable, we will arrive at the following summations :

$$\frac{\pi^3}{8\lambda^3}\left(\frac{2}{\left(\cos.\frac{\pi\omega}{2\lambda}\right)^3}-\frac{1}{\cos.\frac{\pi\omega}{2\lambda}}\right)=+\frac{2}{(\lambda-\omega)^3}+\frac{2}{(\lambda+\omega)^3}-\frac{2}{(3\lambda-\omega)^3}-\frac{2}{(3\lambda+\omega)^3}+\frac{2}{(5\lambda-\omega)^3}+\frac{2}{(5\lambda+\omega)^3}-\text{etc.}$$

$$\frac{\pi^3}{8\lambda^3}\cdot\frac{2\sin.\frac{\pi\omega}{2\lambda}}{\left(\cos.\frac{\pi\omega}{2\lambda}\right)^3}=\frac{2}{(\lambda-\omega)^3}-\frac{2}{(\lambda+\omega)^3}+\frac{2}{(3\lambda-\omega)^3}-\frac{2}{(3\lambda+\omega)^3}+\frac{2}{(5\lambda-\omega)^3}-\text{etc.}$$

§. 84. Now if here we may assume $\omega=0$ and $\lambda=1$, the first integration will adopt this form :

$$\int\frac{2dz(lz)^2}{1+zz}=\frac{\pi^3}{8}=\frac{2}{1^3}+\frac{2}{1^3}-\frac{2}{3^3}-\frac{2}{3^3}+\frac{2}{5^3}+\frac{2}{5^3}-\frac{2}{7^3}-\frac{2}{7^3}+\text{etc.}$$

thus so that there shall be

$$\frac{1}{1^3}-\frac{1}{3^3}+\frac{1}{5^3}-\frac{1}{7^3}+\frac{1}{9^3}-\frac{1}{11^3}+\text{etc.}=\frac{\pi^3}{32},$$

just as formerly I have shown now . But the other integration in this case will go to zero. Truly from the previous integral,

$$\int\frac{dz(lz)^2}{1+zz}=\frac{\pi^3}{16},$$

it is not possible to derive the other, as we have done from the form

$$\int\frac{dzlz}{1-zz}=-\frac{\pi\pi}{8},$$

since here the denominator $1+zz$ does not have real factors.

§. 85. Therefore we may take $\lambda=2$ and $\omega=1$, and the first integration will give

$$\int\frac{(1+zz)dz(lz)^2}{1+z^4}=\frac{5\pi^3}{32\sqrt{2}};$$

hence moreover the series will become:

$$\frac{2}{1^3}+\frac{2}{3^3}-\frac{2}{5^3}-\frac{2}{7^3}+\frac{2}{9^3}+\frac{2}{11^3}-\text{etc.},$$

thus so that there shall become :

$$\frac{1}{1^3}+\frac{1}{3^3}-\frac{1}{5^3}-\frac{1}{7^3}+\frac{1}{9^3}+\frac{1}{11^3}-\text{etc.}=\frac{3\pi^3}{64\sqrt{2}},$$

which added to the above gives :

$$\frac{1}{1^3}-\frac{1}{7^3}+\frac{1}{9^3}-\frac{1}{15^3}+\frac{1}{17^3}-\frac{1}{25^3}+\text{etc.}=\frac{\pi^3\left(5+\sqrt{2}\right)}{128\sqrt{2}}.$$

Truly, the other integration in this case gives :

$$\int\frac{dz(lz)^2}{1+zz}=\frac{\pi^3}{16},$$

which agrees perfectly with the preceding paragraph, just as hence the series as arisen also :

$$\frac{2}{1^3}-\frac{2}{3^3}+\frac{2}{5^3}-\frac{2}{7^3}+\frac{2}{9^3}-\frac{2}{11^3}+\frac{2}{15^3}-\text{etc.}$$

§86. But so that we may be able to deduce the following integrations more easily by continued differentiation, we may represent these in a general manner ; and since initially there shall be :

$$\mathrm{S}=\frac{\pi}{2\lambda\cos.\frac{\pi\omega}{2\lambda}},$$

hence the integrations will proceed in the following form :

$$\begin{array}{ll}
\text{I.} & \int\frac{z^{\lambda-\omega}+z^{\lambda+\omega}}{1+z^{2\lambda}}\cdot\frac{dz}{z}=\mathrm{S},\\
\text{II.} & \int\frac{-z^{\lambda-\omega}+z^{\lambda+\omega}}{1+z^{2\lambda}}\cdot\frac{dz}{z}lz=\left(\frac{d\mathrm{S}}{d\omega}\right),\\
\text{III.} & \int\frac{z^{\lambda-\omega}+z^{\lambda+\omega}}{1+z^{2\lambda}}\cdot\frac{dz}{z}(lz)^2=\left(\frac{dd\mathrm{S}}{d\omega^2}\right),\\
\text{IV.} & \int\frac{-z^{\lambda-\omega}+z^{\lambda+\omega}}{1+z^{2\lambda}}\cdot\frac{dz}{z}(lz)^3=\left(\frac{d^3\mathrm{S}}{d\omega^3}\right),\\
\text{V.} & \int\frac{z^{\lambda-\omega}+z^{\lambda+\omega}}{1+z^{2\lambda}}\cdot\frac{dz}{z}(lz)^4=\left(\frac{d^4\mathrm{S}}{d\omega^4}\right),\\
\text{VI.} & \int\frac{-z^{\lambda-\omega}+z^{\lambda+\omega}}{1+z^{2\lambda}}\cdot\frac{dz}{z}(lz)^5=\left(\frac{d^5\mathrm{S}}{d\omega^5}\right),\\
\text{VII.} & \int\frac{z^{\lambda-\omega}+z^{\lambda+\omega}}{1+z^{2\lambda}}\cdot\frac{dz}{z}(lz)^6=\left(\frac{d^6\mathrm{S}}{d\omega^6}\right),\\
 & \text{etc.}\qquad\text{etc.}\qquad\text{etc.}
\end{array}$$

§.87. In order to resolve these continued differentiations more easily, we may put for brevity hence : $\frac{\pi}{2\lambda}=\alpha$, so that there shall become

$$\mathrm{S}=\frac{\alpha}{\cos.\alpha\omega};$$

then truly there shall be

$$\sin.\alpha\omega=p \text{ and } \cos.\alpha\omega=q,$$

and there shall be

$$dp = \alpha q d\omega \text{ and } dq = -\alpha p d\omega.$$

Truly besides there may be observed :

$$d \cdot \frac{p^n}{q^{n+1}} = \alpha d\omega \left\{ \frac{np^{n-1}}{q^n} + \frac{(n+1)p^{n+1}}{q^{n+2}} \right\}.$$

With these premises in place there will become, on account of $\mathrm{S} = \alpha \cdot \frac{1}{q}$,

$$\left(\frac{d\mathrm{S}}{d\omega}\right) = \alpha^2 \cdot \frac{p}{qq}, \quad \text{thence}$$

$$\left(\frac{dd\mathrm{S}}{d\omega^2}\right) = \alpha^3 \left(\frac{1}{q} + \frac{2pp}{q^3}\right), \text{ again}$$

$$\left(\frac{d^3\mathrm{S}}{d\omega^3}\right) = \alpha^4 \left(\frac{5p}{qq} + \frac{6p^3}{q^4}\right),$$

$$\left(\frac{d^4\mathrm{S}}{d\omega^4}\right) = \alpha^5 \left(\frac{5}{q} + \frac{28pp}{q^3} + \frac{24p^4}{q^5}\right),$$

$$\left(\frac{d^5\mathrm{S}}{d\omega^5}\right) = \alpha^6 \left(\frac{61p}{qq} + \frac{180p^3}{q^4} + \frac{120p^5}{q^6}\right),$$

$$\left(\frac{d^6\mathrm{S}}{d\omega^6}\right) = \alpha^7 \left(\frac{61p}{q} + \frac{662pp}{q^3} + \frac{1320p^4}{q^5} + \frac{720p^6}{q^7}\right),$$

$$\left(\frac{d^7\mathrm{S}}{d\omega^7}\right) = \alpha^8 \left(\frac{1385p}{qq} + \frac{7266p^3}{q^4} + \frac{10920p^5}{q^6} + \frac{5040p^7}{q^8}\right), \text{etc.}$$

but these values, because $pp = 1 - qq$, are reduced to the following

$$S = \alpha \cdot \frac{1}{q},$$

$$\left(\frac{d\mathrm{S}}{d\omega}\right) = \alpha^2 p \cdot \frac{1}{qq},$$

$$\left(\frac{dd\mathrm{S}}{d\omega^2}\right) = \alpha^3 \left(\frac{1.2}{q^3} - \frac{1}{q}\right),$$

$$\left(\frac{d^3\mathrm{S}}{d\omega^3}\right) = \alpha^4 p \left(\frac{1.2.3}{q^4} - \frac{1}{qq}\right),$$

$$\left(\frac{d^4\mathrm{S}}{d\omega^4}\right) = \alpha^5 \left(\frac{1.2.3.4}{q^5} - \frac{20}{q^3} + \frac{1}{q}\right),$$

$$\left(\frac{d^5\mathrm{S}}{d\omega^5}\right) = \alpha^6 p \left(\frac{1.2.3.4.5}{q^6} - \frac{60}{q^4} + \frac{1}{qq}\right),$$

$$\left(\frac{d^6\mathrm{S}}{d\omega^6}\right) = \alpha^7 \left(\frac{1.....6}{q^7} - \frac{840}{q^5} + \frac{182}{q^3} - \frac{1}{q}\right), \text{etc.}$$

§. 88. These latter forms can be found with the aid of these two lemmas :

$$\text{I.}\quad d\cdot\frac{1}{q^{n+1}}=\alpha d\omega\frac{(n+1)p}{q^{n+2}},$$

and

$$\text{II.}\quad d\cdot\frac{p}{q^{n+1}}=\alpha d\omega\left\{\frac{n+1}{q^{n+2}}-\frac{n}{q^{n}}\right\}.$$

hence indeed we will find

$$S=\alpha\cdot\frac{1}{q},$$

$$\left(\frac{dS}{d\omega}\right)=\alpha^{2}p\cdot\frac{1}{qq},$$

$$\left(\frac{ddS}{d\omega^{2}}\right)=\alpha^{3}\left(\frac{1.2}{q^{3}}-\frac{1}{q}\right),$$

$$\left(\frac{d^{3}S}{d\omega^{3}}\right)=\alpha^{4}\left(\frac{1.2.3}{q^{4}}-\frac{p}{qq}\right),$$

$$\left(\frac{d^{4}S}{d\omega^{4}}\right)=\alpha^{5}\left(\frac{1.2.3.4}{q^{5}}-\frac{20}{q^{3}}+\frac{1}{q}\right),$$

$$\left(\frac{d^{5}S}{d\omega^{5}}\right)=\alpha^{6}\left(\frac{1.2.3.4.5p}{q^{6}}-\frac{3.20p}{q^{4}}+\frac{p}{qq}\right),$$

$$\left(\frac{d^{6}S}{d\omega^{6}}\right)=\alpha^{7}\left(\frac{1.2.3.4.4.6p}{q^{7}}-\frac{840}{q^{5}}+\frac{182}{q^{3}}-\frac{1}{q}\right),$$

$$\left(\frac{d^{7}S}{d\omega^{7}}\right)=\alpha^{8}\left(\frac{1.2......7p}{q^{8}}-\frac{5.840p}{q^{6}}+\frac{3.182}{q^{4}}-\frac{p}{qq}\right),\text{etc.}$$

§. 89. Moreover the series corresponding to these terms will be :

$$S=\frac{1}{\lambda-\omega}+\frac{1}{\lambda+\omega}-\frac{1}{3\lambda-\omega}-\frac{1}{3\lambda+\omega}+\frac{1}{5\lambda-\omega}+\frac{1}{5\lambda+\omega}-\text{etc.},$$

$$\left(\frac{dS}{d\omega}\right)=\frac{1}{(\lambda-\omega)^2}+\frac{1}{(\lambda+\omega)^2}-\frac{1}{(3\lambda-\omega)^2}-\frac{1}{(3\lambda+\omega)^2}+\frac{1}{(5\lambda-\omega)^2}+\frac{1}{(5\lambda+\omega)^2}-\text{etc.},$$

$$\left(\frac{ddS}{d\omega^2}\right)=\frac{1.2}{(\lambda-\omega)^3}+\frac{1.2}{(\lambda+\omega)^3}-\frac{1.2}{(3\lambda-\omega)^3}-\frac{1.2}{(3\lambda+\omega)^3}+\frac{1.2}{(5\lambda-\omega)^3}+\text{etc.},$$

$$\left(\frac{d^3S}{d\omega^3}\right)=\frac{1.2.3}{(\lambda-\omega)^4}+\frac{1.2.3}{(\lambda+\omega)^4}-\frac{1.2.3}{(3\lambda-\omega)^4}-\frac{1.2.3}{(3\lambda+\omega)^4}+\frac{1.2.3}{(5\lambda-\omega)^4}-\text{etc.},$$

$$\left(\frac{d^4S}{d\omega^4}\right)=\frac{1.2.3.4}{(\lambda-\omega)^5}+\frac{1.2.3.4}{(\lambda+\omega)^5}-\frac{1.2.3.4}{(3\lambda-\omega)^5}-\frac{1.2.3.4}{(3\lambda+\omega)^5}+\frac{1.2.3.4}{(5\lambda-\omega)^5}+\text{etc.},$$

$$\left(\frac{d^5S}{d\omega^5}\right)=\frac{1.2.3.4.5}{(\lambda-\omega)^6}+\frac{1.2.3.4.5}{(\lambda+\omega)^6}-\frac{1.2.3.4.5}{(3\lambda-\omega)^6}-\frac{1.2.3.4.5}{(3\lambda+\omega)^6}+\frac{1.2.3.4.5}{(5\lambda-\omega)^6}-\text{etc.},$$

$$\left(\frac{d^6S}{d\omega^6}\right)=\frac{1.2.3.4.5.6}{(\lambda-\omega)^7}+\frac{1.2.3.4.5.6}{(\lambda+\omega)^7}-\frac{1.2.3.4.5.6}{(3\lambda-\omega)^7}-\frac{1.2.3.4.5.6}{(3\lambda+\omega)^7}+\frac{1.2.3.4.5.6}{(5\lambda-\omega)^7}+\text{etc.},$$

$$\left(\frac{d^7S}{d\omega^7}\right)=\frac{1......7}{(\lambda-\omega)^8}+\frac{1......7}{(\lambda+\omega)^8}-\frac{1......7}{(3\lambda-\omega)^8}-\frac{1......7}{(5\lambda+\omega)^8}+\frac{1......7}{(5\lambda-\omega)^8}-\text{etc.},$$

Moreover, it is required to remember properly the values associated with these, to be

$$\alpha=\frac{\pi}{2\lambda},\ p=\sin.\alpha\omega=\sin.\frac{\pi\omega}{2\lambda},\text{ and } q=\cos.\alpha\omega=\cos.\frac{\pi\omega}{2\lambda}.$$

§. 90. In the same manner we may extricate the values or the integral formulas of the other kind, for which there is :

$$T=\frac{\pi}{2\lambda}\,\text{tang}.\frac{\pi\omega}{2\lambda},$$

from which by differentiating continually the following integrations arise :

$$\text{I.} \quad \int \frac{z^{\lambda-\omega}-z^{\lambda+\omega}}{1-z^{2\lambda}} \cdot \frac{dz}{z} = \mathrm{T},$$

$$\text{II.} \quad \int \frac{-z^{\lambda-\omega}-z^{\lambda+\omega}}{1-z^{2\lambda}} \cdot \frac{dz}{z}\, lz = \left(\frac{d\mathrm{T}}{d\omega}\right),$$

$$\text{III.} \quad \int \frac{z^{\lambda-\omega}-z^{\lambda+\omega}}{1-z^{2\lambda}} . \frac{dz}{z} (lz)^2 = \left(\frac{dd\mathrm{T}}{d\omega^2}\right),$$

$$\text{IV.} \quad \int \frac{-z^{\lambda-\omega}-z^{\lambda+\omega}}{1-z^{2\lambda}} \cdot \frac{dz}{z} (lz)^3 = \left(\frac{d^3\mathrm{T}}{d\omega^3}\right),$$

$$\text{V.} \quad \int \frac{z^{\lambda-\omega}-z^{\lambda+\omega}}{1-z^{2\lambda}} \cdot \frac{dz}{z} (lz)^4 = \left(\frac{d^4\mathrm{T}}{d\omega^4}\right),$$

$$\text{VI.} \quad \int \frac{-z^{\lambda-\omega}-z^{\lambda+\omega}}{1-z^{2\lambda}} \cdot \frac{dz}{z} (lz)^5 = \left(\frac{d^5\mathrm{T}}{d\omega^5}\right),$$

$$\text{VII.} \quad \int \frac{z^{\lambda-\omega}-z^{\lambda+\omega}}{1-z^{2\lambda}} \cdot \frac{dz}{z} (lz)^6 = \left(\frac{d^6\mathrm{T}}{d\omega^6}\right),$$

etc.

§. 91. Again there may be put $\frac{\pi}{2\lambda} = \alpha$, $\sin.\alpha\omega = p$, and $\cos.\alpha\omega = q$, so that there shall be :

$$\mathrm{T} = \frac{\alpha p}{q},$$

which formula continually differentiated, following the lemma §.88., will give

$$\mathrm{T} = \alpha \cdot \frac{1}{q},$$

$$\left(\frac{d\mathrm{T}}{d\omega}\right) = \alpha^2 \cdot \frac{1}{qq},$$

$$\left(\frac{dd\mathrm{T}}{d\omega^2}\right) = \alpha^3 \cdot \frac{2p}{q^3},$$

$$\left(\frac{d^3\mathrm{T}}{d\omega^3}\right) = \alpha^4 \left(\frac{6}{q^4} - \frac{4}{qq}\right),$$

$$\left(\frac{d^4\mathrm{T}}{d\omega^4}\right) = \alpha^5 \left(\frac{24p}{q^5} - \frac{8p}{q^3}\right),$$

$$\left(\frac{d^5\mathrm{T}}{d\omega^5}\right) = \alpha^6 \left(\frac{120}{q^6} - \frac{120p}{q^4} + \frac{16}{qq}\right),$$

$$\left(\frac{d^6\mathrm{T}}{d\omega^6}\right) = \alpha^7 \left(\frac{720p}{q^7} - \frac{480p}{q^5} + \frac{32p}{q^3}\right),$$

$$\left(\frac{d^7\mathrm{T}}{d\omega^7}\right) = \alpha^8 \left(\frac{5040}{q^8} - \frac{6720}{q^6} + \frac{2016}{q^4} - \frac{64}{qq}\right), \text{etc.}$$

§. 92. Moreover the infinite series, which hence arise, will become

$$\mathrm{T}=\frac{1}{\lambda-\omega}-\frac{1}{\lambda+\omega}+\frac{1}{3\lambda-\omega}-\frac{1}{3\lambda+\omega}+\frac{1}{5\lambda-\omega}-\frac{1}{5\lambda+\omega}+\text{etc.},$$

$$\left(\frac{d\mathrm{T}}{d\omega}\right)=\frac{1}{(\lambda-\omega)^2}+\frac{1}{(\lambda+\omega)^2}+\frac{1}{(3\lambda-\omega)^2}+\frac{1}{(3\lambda+\omega)^2}+\frac{1}{(5\lambda-\omega)^2}+\text{etc.},$$

$$\left(\frac{dd\mathrm{T}}{d\omega^2}\right)=\frac{1.2}{(\lambda-\omega)^3}-\frac{1.2}{(\lambda+\omega)^3}+\frac{1.2}{(3\lambda-\omega)^3}-\frac{1.2}{(3\lambda+\omega)^3}+\frac{1.2}{(5\lambda-\omega)^3}-\text{etc.},$$

$$\left(\frac{d^3\mathrm{T}}{d\omega^3}\right)=\frac{1.2.3}{(\lambda-\omega)^4}+\frac{1.2.3}{(\lambda+\omega)^4}+\frac{1.2.3}{(3\lambda-\omega)^4}+\frac{1.2.3}{(3\lambda+\omega)^4}+\frac{1.2.3}{(5\lambda-\omega)^4}+\text{etc.},$$

$$\left(\frac{d^4\mathrm{T}}{d\omega^4}\right)=\frac{1.2.3.4}{(\lambda-\omega)^5}-\frac{1.2.3.4}{(\lambda+\omega)^5}+\frac{1.2.3.4}{(3\lambda-\omega)^5}-\frac{1.2.3.4}{(3\lambda+\omega)^5}+\frac{1.2.3.4}{(5\lambda-\omega)^5}-\text{etc.},$$

$$\left(\frac{d^5\mathrm{T}}{d\omega^5}\right)=\frac{1.2.3.4.5}{(\lambda-\omega)^6}+\frac{1.2.3.4.5}{(\lambda+\omega)^6}+\frac{1.2.3.4.5}{(3\lambda-\omega)^6}+\frac{1.2.3.4.5}{(3\lambda+\omega)^6}+\frac{1.2.3.4.5}{(5\lambda-\omega)^6}+\text{etc.},$$

$$\left(\frac{d^6\mathrm{T}}{d\omega^6}\right)=\frac{1.....6}{(\lambda-\omega)^7}-\frac{1.....6}{(\lambda+\omega)^7}+\frac{1.....6}{(3\lambda-\omega)^7}-\frac{1.....6}{(3\lambda+\omega)^7}+\frac{1.....6}{(5\lambda-\omega)^7}-\text{etc.},$$

etc. etc. etc.

§. 93. Hence it will be worth the effort, to set out the simplest case, which arises on putting $\lambda=1$ and $\omega=0$, so that there shall be $\alpha=\frac{\pi}{2}$, $p=0$ and $q=1$, from which we will have :

First form	Second form
$S=\frac{\pi}{2}$	$\mathrm{T}=0$
$\left(\frac{d\mathrm{S}}{d\omega}\right)=0$	$\left(\frac{d\mathrm{T}}{d\omega}\right)=\frac{\pi\pi}{4}$
$\left(\frac{dd\mathrm{S}}{d\omega^2}\right)=\frac{\pi^3}{8}$	$\left(\frac{dd\mathrm{T}}{d\omega^2}\right)=0$
$\left(\frac{d^3\mathrm{S}}{d\omega^3}\right)=0$	$\left(\frac{d^3\mathrm{T}}{d\omega^3}\right)=\frac{\pi^4}{2}$
$\left(\frac{d^4\mathrm{S}}{d\omega^4}\right)=\frac{5\pi^6}{32}$	$\left(\frac{d^4\mathrm{T}}{d\omega^4}\right)=0$
$\left(\frac{d^5\mathrm{S}}{d\omega^5}\right)=0$	$\left(\frac{d^5\mathrm{T}}{d\omega^5}\right)=\frac{\pi^6}{4}$
$\left(\frac{d^6\mathrm{S}}{d\omega^6}\right)=\frac{61\pi^7}{128}$	$\left(\frac{d^6\mathrm{T}}{d\omega^6}\right)=0$
$\left(\frac{d^7\mathrm{S}}{d\omega^7}\right)=0$	$\left(\frac{d^7\mathrm{T}}{d\omega^7}\right)=\frac{79\pi^8}{32}$
etc.	etc.

§. 94. Hence therefore, with the vanishing values omitted, from the first form we will have the following integral formulas with the series thence arising :

$$\int \frac{dz}{1+zz} = \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \frac{1}{13} - \text{etc.}$$

$$\int \frac{dz(lz)^2}{1+zz} = \frac{\pi^3}{16} = \frac{2}{1^3} - \frac{2}{3^3} + \frac{2}{5^3} - \frac{2}{7^3} + \frac{2}{9^3} - \frac{2}{11^3} + \text{etc.}$$

$$\int \frac{dz(lz)^4}{1+zz} = \frac{5\pi^5}{64} = \frac{24}{1^5} - \frac{24}{3^5} + \frac{24}{5^5} - \frac{24}{7^5} + \frac{24}{9^5} - \frac{24}{11^5} + \text{etc.}$$

$$\int \frac{dz(lz)^6}{1+zz} = \frac{61\pi^7}{256} = \frac{720}{1^7} - \frac{720}{3^7} + \frac{720}{5^7} - \frac{720}{7^7} + \frac{720}{9^7} - \text{etc.}$$

etc. etc. etc.

§. 95. But from the other form for the same case there arise

$$\int \frac{-dzlz}{1-zz} = \frac{\pi\pi}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \frac{1}{11^2} + \frac{1}{13^2} + \text{etc.}$$

$$\int \frac{-dz(lz)^3}{1-zz} = \frac{\pi^4}{16} = \frac{6}{1^4} + \frac{6}{3^4} + \frac{6}{5^4} + \frac{6}{7^4} + \frac{6}{9^4} + \frac{6}{11^4} + \frac{6}{13^4} + \text{etc.}$$

$$\int \frac{-dz(lz)^5}{1-zz} = \frac{\pi^6}{8} = \frac{120}{1^6} + \frac{120}{3^6} + \frac{120}{5^6} + \frac{120}{7^6} + \frac{120}{9^6} + \frac{120}{11^6} + \frac{120}{13^6} \text{etc}$$

etc. etc. 120

§. 96. Just as from the first integral of the latter form we may deduce these formulas :

$$\int \frac{dzlz}{1-z} = -\frac{\pi\pi}{6}, \text{ and } \int \frac{dzlz}{1+z} = -\frac{\pi\pi}{12},$$

likewise too the integral formulas can be deduced from the following ; since indeed there shall be

$$\int \frac{dz(lz)^3}{1-zz} = -\frac{\pi^4}{16},$$

we may put

$$\int \frac{zdz(lz)^3}{1-zz} = \text{P},$$

and there becomes

$$\int \frac{dz(lz)^3}{1-z} = \text{P} - \frac{\pi^4}{16},$$

and

$$\int \frac{dz(lz)^3}{1+z} = -\text{P} - \frac{\pi^4}{16},$$

now truly there may be put $zz = v$, so that there shall be $zdz = \frac{1}{2}dv$, and $lz = \frac{1}{2}lv$, and thus $(lz)^3 = \frac{1}{8}(lv)^3$, with which substituted there will be

$$P = \frac{1}{16}\int\frac{dv(lv)^3}{1-v} = \frac{1}{16}\left(P - \frac{\pi^4}{16}\right),$$

from which there shall be

$$16P = P - \frac{\pi^4}{16}, \text{ and thus } P = -\frac{\pi^4}{240},$$

and thus we have these two new integrations

$$\int\frac{dz(lz)^3}{1-z} = -\frac{\pi^4}{15},$$

and

$$\int\frac{dz(lz)^3}{1+z} = -\frac{7\pi^4}{120}:$$

hence moreover there will become in terms of series

$$\int\frac{dz(lz)^3}{1-z} = -\frac{\pi^4}{15} = 6\left(1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \frac{1}{5^4} + \frac{1}{6^4} + \frac{1}{7^4} + \frac{1}{8^4} + \text{etc.}\right)$$

and

$$\int\frac{-dz(lz)^3}{1+z} = +\frac{7\pi^4}{120} = 6\left(1 - \frac{1}{2^4} + \frac{1}{3^4} - \frac{1}{4^4} + \frac{1}{5^4} - \frac{1}{6^4} + \frac{1}{7^4} - \frac{1}{8^4} + \text{etc.}\right).$$

§. 97. Again, since $\int\frac{dz(lz)^5}{1-zz} = -\frac{\pi^6}{8}$, we may put $\int\frac{zdz(lz)^5}{1-zz} = P$, so that hence we may obtain

$$\int\frac{dz(lz)^5}{1-z} = P - \frac{\pi^6}{8}, \text{ and } \int\frac{dz(lz)^5}{1+z} = -P - \frac{\pi^6}{8},$$

now again we may put $zz = v$, and there will become

$$P = \frac{1}{64}\int\frac{dv(lv)^5}{1-v} = \frac{1}{64}\left(P - \frac{\pi^6}{8}\right),$$

from which there shall become

$$P = -\frac{\pi^6}{504},$$

and the new integrations hence deduced are

$$\int \frac{dz(lz)^5}{1-z} = -\frac{8\pi^6}{63},$$

and

$$\int \frac{dz(lz)^5}{1+z} = -\frac{31\pi^6}{252}:$$

and indeed there is found by series :

$$\int \frac{dz(lz)^5}{1-z} = -\frac{8\pi^6}{63} = -120\left(1+\frac{1}{2^6}+\frac{1}{3^6}+\frac{1}{4^6}+\frac{1}{5^6}+\frac{1}{6^6}+\frac{1}{7^6}+\frac{1}{8^6}+\text{etc.}\right)$$

and

$$\int \frac{dz(lz)^5}{1+z} = -\frac{31\pi^6}{252} = -120\left(1-\frac{1}{2^6}+\frac{1}{3^6}-\frac{1}{4^6}+\frac{1}{5^6}-\frac{1}{6^6}+\frac{1}{7^6}-\frac{1}{8^6}-\text{etc.}\right).$$

thus, so that there shall become:

$$1+\frac{1}{2^6}+\frac{1}{3^6}+\frac{1}{4^6}+\frac{1}{5^6}+\frac{1}{6^6}+\frac{1}{7^6}+\frac{1}{8^6}+\text{etc} = \frac{\pi^6}{945},$$

and

$$1-\frac{1}{2^6}+\frac{1}{3^6}-\frac{1}{4^6}+\frac{1}{5^6}-\frac{1}{6^6}+\frac{1}{7^6}-\frac{1}{8^6}-\text{etc.} = \frac{31\pi^6}{30240} = \frac{31\pi^6}{32.945}.$$

§. 98. Now we will consider the case, in which $\lambda = 2$ and $\omega = 1$, thus so that there becomes $\alpha = \frac{\pi}{4}$ and $\alpha\omega = \frac{\pi}{4}$, hence $p = q = \frac{1}{\sqrt{2}}$, from which for each ordered sequence we have the following values :

First form	Second form
$S = \frac{\pi}{2\sqrt{2}}$	$\mathrm{T} = \frac{\pi}{4}$
$\left(\frac{d\mathrm{S}}{d\omega}\right) = \frac{\pi\pi}{8\sqrt{2}}$	$\left(\frac{d\mathrm{T}}{d\omega}\right) = \frac{\pi\pi}{8}$
$\left(\frac{dd\mathrm{S}}{d\omega^2}\right) = \frac{3\pi^3}{32\sqrt{2}}$	$\left(\frac{dd\mathrm{T}}{d\omega^2}\right) = \frac{\pi^3}{16}$
$\left(\frac{d^3\mathrm{S}}{d\omega^3}\right) = \frac{11\pi^4}{128\sqrt{2}}$	$\left(\frac{d^3\mathrm{T}}{d\omega^3}\right) = \frac{\pi^4}{16}$
$\left(\frac{d^4\mathrm{S}}{d\omega^4}\right) = \frac{57\pi^5}{512\sqrt{2}}$	$\left(\frac{d^4\mathrm{T}}{d\omega^4}\right) = \frac{5\pi^5}{64}$
$\left(\frac{d^5\mathrm{S}}{d\omega^5}\right) = \frac{361\pi^6}{2048\sqrt{2}}$	$\left(\frac{d^5\mathrm{T}}{d\omega^5}\right) = \frac{\pi^6}{8}$
$\left(\frac{d^6\mathrm{S}}{d\omega^6}\right) = \frac{2765\pi^7}{8192\sqrt{2}}$	$\left(\frac{d^6\mathrm{T}}{d\omega^6}\right) = \frac{61\pi^7}{256}$
$\left(\frac{d^7\mathrm{S}}{d\omega^7}\right) = \frac{24611\pi^8}{52768\sqrt{2}}$	$\left(\frac{d^7\mathrm{T}}{d\omega^7}\right) = \frac{79\pi^8}{32}$
etc.	etc.

§. 99. Hence therefore the following integrations, with the corresponding series will result; and initially from the first form:

$$\int \frac{(1+zz)dz}{1+z^4} = \frac{\pi}{2\sqrt{2}} = 1 + \frac{1}{3} - \frac{1}{5} - \frac{1}{7} + \frac{1}{9} + \frac{1}{11} - \frac{1}{13} + \text{etc.}$$

$$\int \frac{-(1-zz)dz(lz)^2}{1+z^4} = \frac{\pi\pi}{8\sqrt{2}} = 1 - \frac{1}{3^2} - \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} - \frac{1}{11^2} - \frac{1}{13^2} + \text{etc.}$$

$$\int \frac{(1+zz)dz(lz)^2}{1+z^4} = \frac{5\pi^3}{32\sqrt{2}} = \frac{2}{1^3} + \frac{2}{3^3} - \frac{2}{5^3} - \frac{2}{7^3} + \frac{2}{9^3} + \frac{2}{11^3} - \frac{2}{13^3} - \text{etc.}$$

$$\int \frac{-(1-zz)dz(lz)^3}{1+z^4} = \frac{11\pi^4}{128\sqrt{2}} = \frac{6}{1^4} - \frac{6}{3^4} - \frac{6}{5^4} + \frac{6}{7^4} + \frac{6}{9^4} - \frac{6}{11^4} - \frac{6}{13^4} + \text{etc.}$$

$$\int \frac{(1+zz)dz(lz)^4}{1+z^4} = \frac{57\pi^5}{512\sqrt{2}} = \frac{24}{1^5} + \frac{24}{3^5} - \frac{24}{5^5} - \frac{24}{7^5} + \frac{24}{9^5} + \frac{24}{11^5} - \frac{24}{13^5} - \text{etc.}$$

$$\int \frac{-(1-zz)dz(lz)^5}{1+z^4} = \frac{361\pi^6}{2048\sqrt{2}} = \frac{120}{1^6} - \frac{120}{3^6} - \frac{120}{5^6} + \frac{120}{7^6} + \frac{120}{9^6} - \frac{120}{11^6} - \frac{120}{13^6} + \text{etc.}$$

$$\int \frac{(1+zz)dz(lz)^6}{1+z^4} = \frac{2763\pi^7}{8192\sqrt{2}} = \frac{720}{1^7} + \frac{720}{3^7} - \frac{720}{5^7} - \frac{720}{7^7} + \frac{720}{9^7} + \frac{720}{11^7} - \frac{720}{13^7} - \text{etc.}$$

$$\int \frac{-(1-zz)dz(lz)^7}{1+z^4} = \frac{24611\pi^8}{52768\sqrt{2}} = \frac{5040}{1^8} - \frac{5040}{3^8} - \frac{5040}{5^8} + \frac{5040}{7^8} + \frac{5040}{9^8} - \frac{5040}{11^8} - \frac{5040}{13^8} + \text{etc.}$$

etc. etc. etc.

§. 100. In the same manner the integrations of the other form will be with the series

$$\int \frac{dz}{1+zz} = \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \frac{1}{13} - \text{etc.}$$

$$\int \frac{-dzlz}{1-zz} = \frac{\pi\pi}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \frac{1}{11^2} + \frac{1}{13^2} + \text{etc.}$$

$$\int \frac{dz(lz)^2}{1+zz} = \frac{\pi^3}{16} = \frac{2}{1^3} - \frac{2}{3^3} + \frac{2}{5^3} - \frac{2}{7^3} + \frac{2}{9^3} - \frac{2}{11^3} + \frac{2}{13^3} - \text{etc.}$$

$$\int \frac{-dz(lz)^3}{1-zz} = \frac{\pi^4}{16} = \frac{6}{1^4} + \frac{6}{3^4} + \frac{6}{5^4} + \frac{6}{7^4} + \frac{6}{9^4} + \frac{6}{11^4} + \frac{6}{13^4} + \text{etc.}$$

$$\int \frac{dz(lz)^4}{1+zz} = \frac{5\pi^5}{64} = \frac{24}{1^5} - \frac{24}{3^5} + \frac{24}{5^5} - \frac{24}{7^5} + \frac{24}{9^5} - \frac{24}{11^5} + \frac{24}{13^5} - \text{etc.}$$

$$\int \frac{-dz(lz)^5}{1-zz} = \frac{\pi^6}{16} = \frac{120}{1^6} + \frac{120}{3^6} + \frac{120}{5^6} + \frac{120}{7^6} + \frac{120}{9^6} + \frac{120}{11^6} + \text{etc.}$$

$$\int \frac{dz(lz)^6}{1+zz} = \frac{61\pi^7}{256} = \frac{720}{1^7} - \frac{720}{3^7} + \frac{720}{5^7} - \frac{720}{7^7} + \frac{720}{9^7} - \frac{720}{11^7} + \text{etc.}$$

$$\int \frac{-dz(lz)^7}{1-zz} = \frac{79\pi^8}{32} = \frac{5040}{1^8} + \frac{5040}{3^8} + \frac{5040}{5^8} + \frac{5040}{7^8} + \frac{5040}{9^8} + \frac{5040}{11^8} + \frac{5040}{13^8} + \text{etc.}$$

etc. etc. etc.

But these series are those themselves, which now we have attended to above in §. 94. and §.95.

§. 101. But in addition these cases especially deserve to be noted, in which integral formulas can be resolved into simpler forms. But this resolution is considered only for the fraction

$$\pm\frac{z^{\lambda-\omega} + z^{\lambda+\omega}}{1\pm z^{2\lambda}},$$

with the factor $\frac{dz}{z}(lz)^{\mu}$ omitted ; towards showing which initially we take $\lambda = 3$ and $\omega = 1$, from which there becomes $\alpha = \frac{\pi}{6}$, $p = \sin.\frac{\pi}{6}$, and $q = \cos.\frac{\pi}{6}$, but then, in the first form the sequences of fractions occur alternatively

$$\text{I.}\quad \frac{zz(1+zz)}{1+z^6} = \frac{zz}{1-zz+z^4},$$

which on putting $zz = v$ will become $\frac{v}{1-v+v^2}$; therefore since there shall be $\frac{dz}{z} = \frac{1}{2}\frac{dv}{v}$, and $lz = \frac{1}{2}lv$, hence such a form

$$\frac{1}{2^{2i+1}}\int \frac{dv(lv)^{2i}}{1-v+v^2}$$

will be able to be integrated, clearly in the case $v = 1$.

$$\text{II.}\quad -\frac{zz(1-zz)}{1+z^6}=+\frac{2}{3(1+zz)}-\frac{(1-2zz)}{3\left(1-zz+z^4\right)},$$

which on putting $zz=v$, goes into $\frac{2}{3(1+v)}+\frac{(2-v)}{3\left(1-v+v^2\right)}$, therefore which form multiplied by $\frac{dz}{z}(lz)^{2i+1}$ or by

$$\frac{1}{2^{2i+2}}\frac{dv}{v}(lv)^{2i+1},$$

can always be integrated on putting $v=1$.

§.102. In the same case for the latter form the following resolutions can be supplied:

$$\text{I.}\quad \frac{zz(1-zz)}{1-z^6}=\frac{zz}{1+zz+z^4}=\frac{v}{1+v+v^2},$$

which multiplied by $\frac{dz}{z}(lz)^{2i}$, or by $\frac{1}{2^{2i+1}}\frac{dv}{v}(lv)^{2i}$, is integrable always.

$$\text{II.}\quad \frac{-zz(1+zz)}{1-z^6}=\frac{-2}{3(1-zz)}+\frac{2+zz}{3\left(1+zz+z^4\right)}=\frac{v}{1+v+v^2},$$

which on putting $zz=v$ becomes

$$\frac{-2}{3(1-v)}+\frac{2+v}{3\left(1+v+v^2\right)},$$

therefore which formulas multiplied by $\frac{dv}{v}(lv)^{2i+1}$, shall become integrable; but because in this resolution the numerators are not allowed to be divided by z or v, there is a need for another resolution, which is found to be :

$$\frac{-zz(1+zz)}{1-z^6}=\frac{-2zz}{3(1-zz)}-\frac{zz(1+2zz)}{3\left(1+zz+z^4\right)},\ \text{or}\ \frac{-2v}{3(1-v)}-\frac{v(1+2v)}{3(1+v+vv)},$$

which formulas multiplied by $\frac{dz}{z}(lz)^{2i+1}$, or by $\frac{1}{2^{2i+2}}\frac{dv}{v}(lv)^{2i+1}$, are allowed to be integrated too.

§. 103. Again for the remaining $\lambda=3$ and $\omega=2$, from which there becomes $\alpha=\frac{\pi}{6}$, $p=\sin.\frac{\pi}{3}$, and $q=\cos.\frac{\pi}{3}$, and from the first form the following reductions follow.

$$\text{I.}\quad \frac{z\left(1+z^4\right)}{1+z^6}=\frac{2z}{3(1+zz)}+\frac{z(1+zz)}{3\left(1-zz+z^4\right)},$$

from which by multiplying by $\frac{dz}{z}(lz)^{2i}$ the integration of the formula in the case $z=1$ being allowed.

$$\text{II. } \frac{-z\left(1-z^4\right)}{1+z^6}=-\frac{z(1-zz)}{1-zz+z^4},$$

which multiplied by $\frac{dz}{z}(lz)^{2i+1}$ can be integrated in the case $z=1$. Truly from the latter form the following reductions will be produced.

$$\text{I. } \frac{z\left(1-z^4\right)}{1-z^6}=\frac{z(1+zz)}{1+zz+z^4},$$

which multiplied by $\frac{dz}{z}(lz)^{2i}$ shall become integrable.

$$\text{II. } \frac{-z\left(1+z^4\right)}{1-z^6}=\frac{-2z}{3(1-zz)}-\frac{z(1-zz)}{3\left(1+zz+z^4\right)},$$

which formulas multiplied by $\frac{dz}{z}(lz)^{2i+1}$ become integrable.

§. 104. Now, it will be worth the effort to actually establish this integration, whereby from §.101. where $\omega=1$, and from number I of this we obtain the following integrations

$$1°.\ \frac{1}{2}\int\frac{v}{1-v+v^2}=\alpha\cdot\frac{1}{q}=\frac{\pi}{3\sqrt{3}}$$
$$2°.\ \frac{1}{8}\int\frac{dv(lv)^2}{1-v+v^2}=\alpha^3\left(\frac{2}{q^3}-\frac{1}{q}\right)=\frac{5\pi^3}{324\sqrt{3}},$$

thence truly from number II of the same §, where also this reduction can be used:

$$-\frac{zz(1-zz)}{1+z^6}=-\frac{2zz}{3(1+zz)}-\frac{zz(1-2zz)}{3\left(1+zz+z^4\right)}=-\frac{2v}{3(1+v)}-\frac{v(1-2v)}{3(1-v+vv)},$$

which multiplied by $\frac{1}{4}\cdot\frac{dv}{v}lv$ will give :

$$-\frac{1}{6}\int\frac{dvlv}{1+v}-\frac{1}{12}\int\frac{dv(1-2v)lv}{(1-v+vv)}=\alpha\alpha\frac{p}{qq}=\frac{\pi\pi}{54},$$

the first integration of which terms is allowed, for there is

$$\int\frac{dvlv}{1+v}=-\frac{\pi\pi}{12},$$

from which the latter is found :

$$\int \frac{dv(1-2v)lv}{(1-v+vv)} = -\frac{\pi\pi}{18}.$$

§. 105. From number I of §.102. it follows :

$$1^0.\ \ \frac{1}{2}\int \frac{dv}{(1+v+vv)} = \frac{\alpha p}{q} = \frac{\pi}{6\sqrt{3}}$$
$$2^0.\ \ \frac{1}{8}\int \frac{dv(lv)^2}{(1+v+vv)} = \alpha^3\frac{2p}{q^3} = \frac{\pi^3}{81\sqrt{3}};$$

then truly from number II there becomes :

$$-\frac{1}{6}\int \frac{dvlv}{1-v} - \frac{1}{12}\int \frac{dv(1+2v)lv}{1+v+vv} = \alpha\alpha\frac{1}{qq} = \frac{\pi\pi}{27};$$

but above we have found :

$$\int \frac{dvlv}{1-v} = -\frac{\pi\pi}{6};$$

with which value substituted there becomes

$$\int \frac{dv(1+2v)lv}{1+v+vv} = -\frac{\pi\pi}{9}:$$

therefore the establishment of these latter integrations has been seen to be of the greatest value.

§.106. Because if both the terms of the integral

$$\int \frac{dv(1-2v)lv}{1-v+vv} \text{ et } \int \frac{dv(1+2v)lv}{1+v+vv}$$

may be changed into series, there will be found

$$\int \frac{dv(1-2v)lv}{1-v+vv} = -1 + \frac{1}{4} + \frac{2}{9} + \frac{1}{16} - \frac{1}{25} - \frac{2}{36} - \frac{1}{49} + \text{etc. et}$$
$$\int \frac{dv(1+2v)lv}{1+v+vv} = -1 - \frac{1}{4} + \frac{2}{9} - \frac{1}{16} - \frac{1}{25} + \frac{2}{36} - \frac{1}{49} - \text{etc.}$$

from these two summations worthy of our attention we pursue further :

$$\text{I.}\ \ 1 - \frac{1}{4} - \frac{2}{9} - \frac{1}{16} + \frac{1}{25} + \frac{2}{36} + \frac{1}{49} - \frac{1}{64} - \frac{2}{81} - \frac{1}{100} + \text{etc.} = \frac{\pi\pi}{18},$$
$$\text{II.}\ \ 1 + \frac{1}{4} - \frac{2}{9} + \frac{1}{16} + \frac{1}{25} - \frac{2}{36} + \frac{1}{49} + \frac{1}{64} - \frac{2}{81} + \frac{1}{100} + \text{etc.} = \frac{\pi\pi}{9},$$

with the first of these taken from the second gives

$$\frac{2}{4}+\frac{2}{16}-\frac{4}{36}+\frac{2}{64}+\frac{2}{100}\text{ etc.}=\frac{\pi\pi}{18},$$

the double of which leads to this

$$1+\frac{1}{4}-\frac{2}{9}+\frac{1}{16}+\frac{1}{25}-\frac{2}{36}+\text{etc.}=\frac{\pi\pi}{9},$$

which since it agrees with the second, the truth of each summation is confirmed well enough. Because truly if the second may be take from twice the first, this memorable series will remain :

$$1-\frac{3}{4}-\frac{2}{9}-\frac{3}{16}+\frac{1}{25}+\frac{6}{36}+\frac{1}{49}-\frac{3}{64}-\frac{2}{81}-\frac{3}{100}+\text{etc.}=0$$

which is distributed into periods containing six terms, clearly the order indicated by the enumeration of the terms, which is clearly

$$1-3-2-3+1+6\ .$$

[This most unusual expression is the outcome of equating two different series to each other, which have the same sum.]

An Addition.

§. 107. Just as we have deduced the above integrations through the continued differentiation of the formulas S and T, thus also by integration we will obtain other individual integrations in a straight forwards manner; if indeed as above there were $\mathrm{S}=\int\frac{\mathrm{T}dz}{z}$, with that term T being

$$+\frac{z^{\lambda-\omega}\pm z^{\lambda+\omega}}{1\pm z^{2\lambda}},$$

which besides z also may be taken to involve the exponent variable ω, by the nature of the integral involving two variables there will be

$$\int \mathrm{S}d\omega=\int\frac{dz}{z}\int \mathrm{T}d\omega,$$

where in the prior form of the integral $\int \mathrm{S}d\omega$, where z may be taken as constant, $z=1$ can be written at once ; therefore from this lemma brought forwards, which is

$$\int \mathrm{T}d\omega=\frac{-z^{\lambda-\omega}\pm z^{\lambda+\omega}}{\left(1\pm z^{2\lambda}\right)lz},$$

both forms treated above, clearly we have set out both S and T in this manner, and because we have given each expressed in three ways; the first of course by an infinite series, the second by a finite form, and the third by an integral formula, also the magnitudes, which will result for the integrals $\int \mathrm{S}d\omega$ and $\int \mathrm{T}d\omega,$ will be equal amongst themselves.

§. 108. We may begin from the form S, and since it was by a series

$$S=\frac{1}{\lambda-\omega}+\frac{1}{\lambda+\omega}-\frac{1}{3\lambda-\omega}-\frac{1}{3\lambda+\omega}+\frac{1}{5\lambda-\omega}+\frac{1}{5\lambda+\omega}-\text{etc.},$$

there will be

$$\int \mathrm{S}d\omega=-l(\lambda-\omega)+l(\lambda+\omega)+l(3\lambda-\omega)-l(3\lambda+\omega)-\text{etc.}+\mathrm{C},$$

it is appropriate to define this constant, so that the integral may vanish on putting $\omega=0$, with which done there will be :

$$\int \mathrm{S}d\omega=l\,\frac{\lambda+\omega}{\lambda-\omega}+l\,\frac{5\lambda-\omega}{5\lambda+\omega}+l\,\frac{5\lambda+\omega}{5\lambda-\omega}+l\,\frac{7\lambda-\omega}{7\lambda+\omega}+\text{etc.}$$

which expression is reduced to the following :

$$\int \mathrm{S}d\omega=l\,\frac{(\lambda+\omega)(5\lambda-\omega)(5\lambda+\omega)(7\lambda-\omega)(9\lambda+\omega)\text{etc.}}{(\lambda-\omega)(5\lambda+\omega)(5\lambda-\omega)(7\lambda+\omega)(9\lambda-\omega)\text{etc.}}.$$

Then because by the finite form there was

$$\mathrm{S}=\frac{\pi}{2\lambda\cos.\frac{\pi\omega}{2\lambda}},\ \text{there will become}\ \int \mathrm{S}d\omega=\int\frac{\pi d\omega}{2\lambda\cos.\frac{\pi\omega}{2\lambda}},$$

where, if for the sake of brevity, there may be put $\frac{\pi\omega}{2\lambda}=\varphi$, so that there shall be

$$d\omega=\frac{2\lambda d\varphi}{\pi},\ \text{there will be}\ \int \mathrm{S}d\omega=\int\frac{d\varphi}{\cos.\varphi};$$

therefore because we know

$$\int\frac{d\theta}{\sin.\theta}=l\ \text{tang.}\tfrac{1}{2}\theta,$$

we may assume $\sin.\theta=\cos.\varphi,$ or $\theta=90^\circ-\varphi=\frac{\pi}{2}-\varphi,$ and there will be $d\theta=-d\varphi,$ from which there becomes :

$$\int\frac{-d\varphi}{\cos.\varphi}=l\ \text{tang.}\left(\tfrac{\pi}{4}-\tfrac{1}{2}\varphi\right);$$

but since

$$\varphi=\frac{\pi\omega}{2\lambda},\ \text{there will be}\ \frac{\pi}{4}-\frac{1}{2}\varphi=\frac{\pi(\lambda-\omega)}{2\lambda},$$

from which our integral will become

$$\int \mathrm{S}d\omega = -l \text{ tang.}\frac{\pi(\lambda-\omega)}{4\lambda} = +l \text{ tang.}\frac{\pi(\lambda+\omega)}{4\lambda}.$$

But from the third form of the integral

$$\mathrm{S} = \int \frac{z^{\lambda-\omega}+z^{\lambda+\omega}}{1+z^{2\lambda}} \cdot \frac{dz}{z}$$

there is deduced to be

$$\int \mathrm{S}d\omega = \int \frac{-z^{\lambda-\omega}+z^{\lambda+\omega}}{1+z^{2\lambda}} \cdot \frac{dz}{zlz}$$

which integral is assumed to extend from the end $z=0$ as far as to the end $z=1$; and thus these three values will be equal to each other. And on account perhaps of constants being added, lest any doubt shall remain, these individual expressions vanish at once in the case $\omega=0$.

§. 109. Hence initially we may consider the equality between the first and the second forms: and because each is a logarithm, there will be :

$$\text{tang}\frac{\pi(\lambda+\omega)}{4\lambda} = \frac{(\lambda+\omega)(3\lambda-\omega)(5\lambda+\omega)(7\lambda-\omega)\text{ etc.}}{(\lambda-\omega)(3\lambda+\omega)(5\lambda-\omega)(7\lambda+\omega)\text{ etc.}}$$

since therefore the numerator of this fraction vanishes in the cases where either $\omega=-\lambda,$ or $\omega=+3\lambda,$ or $\omega=-5\lambda,$ or $\omega=+7\lambda$ etc. it is evident in these same cases the tangent also becomes $=0$; truly the denominator vanishes in the cases where either $\omega=\lambda,$ or $\omega=-3\lambda,$ or $\omega=+5\lambda,$ or $\omega=-7\lambda$ etc. in which cases the tangent must increase indefinitely, which also happens most beautifully. In addition this expression agrees with that, which I have found now a little time ago, and established in the introduction.

§. 110. But that infinite product can be reduced to integral formulas by other well established principles with the help of this most general well-known lemma

$$\frac{a(c+b)(a+k)(c+b+k)(a+2k)(c+b+2k)\text{ etc.}}{b(c+a)(b+k)(c+a+k)(b+2k)(c+a+2k)\text{ etc}} = \frac{\int z^{c-1}dz\left(1-z^k\right)^{\frac{b-k}{k}}}{\int z^{c-1}dz\left(1-z^k\right)^{\frac{a-k}{k}}},$$

if indeed after each integration there may be become $z=1$. In our case there will be $a=\lambda+\omega,\ b=\lambda-\omega,\ c=2\lambda,$ et $k=4\lambda;$ from which the value of our product will become

$$\frac{\int z^{2\lambda-1}dz\left(1-z^{2\lambda}\right)^{\frac{-3\lambda-\omega}{4\lambda}}}{\int z^{2\lambda-1}dz\left(1-z^{4\lambda}\right)^{\frac{-3\lambda+\omega}{4\lambda}}}=\text{tang.}\frac{\pi(\lambda+\omega)}{4\lambda}:$$

but the formulas of these integrals may be made much neater, by putting in place $z^{2\lambda}=y,$ for then there will become

$$\text{tang.}\frac{\pi(\lambda+\omega)}{4\lambda}=\frac{\int dy(1-yy)^{\frac{-3\lambda-\omega}{4\lambda}}}{\int dy(1-yy)^{\frac{-3\lambda+\omega}{4\lambda}}},$$

which expression certainly may be considered to be worthy of all attention. Finally from the formula found of the integral there will be also :

$$\int\frac{-z^{\lambda-\omega}+z^{\lambda+\omega}}{1+z^{2\lambda}}\cdot\frac{dz}{zlz}=l\,\text{tang.}\frac{\pi(\lambda+\omega)}{4\lambda}.$$

§. 111. It will be worthwhile also to set out some special cases : therefore initially there shall be $\lambda=2$ and $\omega=1$, and by means of the infinite expression there will be :

$$\int \text{S}d\omega=l\,\frac{3.5}{1.7}\cdot\frac{11.13}{9.15}\cdot\frac{19.21}{17.23}\cdot\frac{27.29}{25.31}\cdot\frac{35.37}{33.39}\cdot\text{etc.}$$

then by the finite expression we will have :

$$\int \text{S}d\omega=l\,\text{tang.}\frac{3\pi}{8},$$

but by means of the integral formula :

$$\int \text{S}d\omega=\int\frac{-(1-zz)}{1+z^{4}}\cdot\frac{dz}{lz}.$$

Then truly from the equality of the two first expressions :

$$\text{tang.}\frac{3\pi}{8}=\frac{3.5}{1.7}\cdot\frac{11.13}{9.15}\cdot\frac{19.21}{17.23}\cdot\text{etc.}$$

and hence by means of the integral formulas

$$\text{tang.}\frac{3\pi}{8}=\frac{\int dy(1-yy)^{-\frac{7}{8}}}{\int dy(1-yy)^{-\frac{5}{8}}}.$$

§. 112. Now we may put $\lambda = 3$ and $\omega = 1$, and by means of the infinite expression there will be :

$$\int \mathrm{S}d\omega = l\,\tfrac{2}{1}\cdot\tfrac{4}{5}\cdot\tfrac{8}{7}\cdot\tfrac{10}{11}\cdot\tfrac{14}{15}\cdot\tfrac{16}{17}\cdot\tfrac{20}{19}\cdot\tfrac{22}{23}\,\text{etc.};$$

by the second, through the finite expression :

$$\int \mathrm{S}d\omega = l\,\text{tang.}\tfrac{\pi}{5} = l\sqrt{3} = \tfrac{1}{2}l3,$$

thus, so that there is going to become :

$$\sqrt{3} = \tfrac{2.4}{1.5}\cdot\tfrac{8.10}{7.11}\cdot\tfrac{14.16}{13.17}\ \text{etc.}$$

and the value of this product by the integral formula will be :

$$\frac{\int dy(1-yy)^{-\frac{5}{6}}}{\int dy(1-yy)^{-\frac{2}{3}}}.$$

Finally, the formula of the integral will furnish :

$$\int \mathrm{S}d\omega = \int \frac{-(1-zz)}{1+z^6}\cdot\frac{dz}{lz}.$$

§. 113. We may set out the other formula T in the same manner also, of which the value according to the series was :

$$\mathrm{T} = \frac{1}{\lambda-\omega} - \frac{1}{\lambda+\omega} + \frac{1}{3\lambda-\omega} - \frac{1}{3\lambda+\omega} + \frac{1}{5\lambda-\omega} - \frac{1}{5\lambda+\omega} + \text{etc.},$$

from which there shall be

$$\int \mathrm{T}d\omega = -l(\lambda-\omega) - l(\lambda+\omega) - l(3\lambda-\omega) - l(3\lambda+\omega) - \text{etc.}$$

which expression, in order that it may vanish on putting $\omega = 0$, will be

$$\int \mathrm{T}d\omega = l\,\frac{\lambda\lambda}{\lambda\lambda-\omega\omega}\cdot\frac{9\lambda\lambda}{9\lambda\lambda-\omega\omega}\cdot\frac{25\lambda\lambda}{25\lambda\lambda-\omega\omega}\ \text{etc.}$$

then truly since the finite form was expressed by :

$\mathrm{T} = \frac{\pi}{2\lambda} \operatorname{tang}.\frac{\pi\omega}{2\lambda}$, there will be

$\int \mathrm{T}d\omega = \int \frac{\pi d\omega}{2\lambda} \operatorname{tang}.\frac{\pi\omega}{2\lambda}$, where on putting $\frac{\pi\omega}{2\lambda} = \varphi$, there will become

$\int \mathrm{T}d\omega = \int d\varphi \operatorname{tang}.\varphi = -l\cos.\varphi$, thus so that there shall become
$\int \mathrm{T}d\omega = -l\cos.\frac{\pi\omega}{2\lambda}$;

the value of which in the case $\omega = 0$ shall be at once $= 0$; and finally by the integral formula we will have

$$\int \mathrm{T}d\omega = \frac{z^{\lambda-\omega} - z^{\lambda+\omega}}{1-z^{2\lambda}} \cdot \frac{dz}{zlz},$$

were the integral likewise must be extended from the term $z = 0$ as far as to the term $z = 1$.

§. 114. Now from a comparison of the first two values, this equation is given :

$$\frac{1}{\cos.\frac{\pi\omega}{2\lambda}} = \frac{\lambda\lambda}{\lambda\lambda-\omega\omega} \cdot \frac{9\lambda\lambda}{9\lambda\lambda-\omega\omega} \cdot \frac{25\lambda\lambda}{25\lambda\lambda-\omega\omega} \cdot \frac{49\lambda\lambda}{49\lambda\lambda-\omega\omega} \text{ etc.}$$

or

$$\cos.\frac{\pi\omega}{2\lambda} = \left(1-\frac{\omega\omega}{\lambda\lambda}\right)\cdot\left(1-\frac{\omega\omega}{9\lambda\lambda}\right)\left(1-\frac{\omega\omega}{25\lambda\lambda}\right)\left(1-\frac{\omega\omega}{49\lambda\lambda}\right) \text{ etc.}$$

or again if the individual factors may be set out in simple form :

$$\cos.\frac{\pi\omega}{2\lambda} = \left(\frac{\lambda+\omega}{\lambda}\right)\cdot\left(\frac{\lambda-\omega}{\lambda}\right)\cdot\left(\frac{3\lambda+\omega}{3\lambda}\right)\cdot\left(\frac{3\lambda-\omega}{3\lambda}\right)\cdot\left(\frac{5\lambda+\omega}{5\lambda}\right)\cdot\left(\frac{5\lambda-\omega}{5\lambda}\right) \text{ etc.}$$

which formula compared with the general reduction advanced above gives ,
$a = \lambda + \omega$, $b = \lambda$, $c = -\omega$, and $k = 2\lambda$, from which we deduce :

$$\cos.\frac{\pi\omega}{2\lambda} = \frac{\int z^{-\omega-1}dz\left(1-z^{2\lambda}\right)^{-\frac{1}{2}}}{\int z^{-\omega-1}dz\left(1-z^{2\lambda}\right)^{\frac{\omega-\lambda}{2\lambda}}}.$$

But so that we may avoid the negative exponent $z^{-\omega-1}$, we may represent the above product thus :

$$\cos.\tfrac{\pi\omega}{2\lambda} = \tfrac{\lambda-\omega}{\lambda}\cdot\tfrac{\lambda+\omega}{\lambda}\cdot\left(\tfrac{3\lambda-\omega}{3\lambda}\right)\cdot\tfrac{3\lambda+\omega}{3\lambda}\text{ etc.}$$

and with the comparison made there will become
$a=\lambda-\omega$, $b=\lambda,\ c\ =+\omega,$ and $k=2\lambda,$ and thus according to the integral formulas there will be :

$$\cos.\tfrac{\pi\omega}{2\lambda} = \frac{\int z^{\omega-1}dz\left(1-z^{2\lambda}\right)^{-\frac{1}{2}}}{\int z^{\omega-1}dz\left(1-z^{2\lambda}\right)^{\frac{-\lambda-\omega}{2\lambda}}},$$

which expression cannot be reduced to a simpler form.

§. 115. If now also $\lambda=2,$ and $\omega=1,$ and our three expressions will become

$$\text{I. } \int \mathrm{T}d\omega = l\,\tfrac{4}{3}\cdot\tfrac{36}{35}\cdot\tfrac{100}{99}\cdot\tfrac{196}{195}\text{ etc.}$$

or

$$\int \mathrm{T}d\omega = l\,\tfrac{2.2}{1.3}\cdot\tfrac{6.6}{5.7}\cdot\tfrac{10.10}{9.11}\cdot\tfrac{14.14}{13.15}\text{ etc.}$$

$$\text{II. } \int \mathrm{T}d\omega = -l\cos.\tfrac{\pi}{4} = +\tfrac{1}{2}l2,\text{ thus}$$

so that there shall become

$$\sqrt{2} = l\,\tfrac{2.2}{1.3}\cdot\tfrac{6.6}{5.7}\cdot\tfrac{10.10}{9.11}\cdot\tfrac{14.14}{13.15}\text{ etc.}$$

which product may be expressed thus through the integral formulas

$$\frac{\int dz\left(1-z^{4}\right)^{-\frac{1}{2}}}{\int dz\left(1-z^{4}\right)^{-\frac{5}{4}}} = \tfrac{1}{2}\sqrt{2}:$$

$$\text{III. } \int \mathrm{T}d\omega = \int \frac{-(1+zz)}{1-z^{4}}\cdot\frac{dz}{lz} = \int \frac{-dz}{(1-zz)lz};$$

which therefore with the integral extending from the term $z=0$ as far as to $z=1$ the same value $+\frac{1}{2}\sqrt{2}$ will be produced, an account of which equality certainly appears most difficult.

§. 116. Finally, as above $\lambda=3,$ and $\omega=1,$ and thus the three formulas will be had :

I. $\int \mathrm{T}d\omega = l\frac{9}{8}\cdot\frac{81}{80}\cdot\frac{225}{221}$ etc. $= l\frac{3.3}{2.4}\cdot\frac{9.9}{8.10}\cdot\frac{15.15}{14.16}\cdot\frac{21.21}{20.22}$ etc.

II. $\int \mathrm{T}d\omega = -l\cos.\frac{\pi}{6} = -l\frac{\sqrt{3}}{2} = l\frac{2}{\sqrt{3}}$, thus so that

$$\frac{2}{\sqrt{3}} = \frac{3.3}{2.4}\cdot\frac{9.9}{8.10}\cdot\frac{15.15}{14.16}\cdot\frac{21.21}{20.22};$$

and thus according to the two integral formulas :

$$\frac{3}{4}\cdot\frac{2}{\sqrt{3}} = \frac{\int dz\left(1-z^6\right)^{-\frac{1}{2}}}{\int dz\left(1-z^6\right)^{-\frac{1}{5}}}.$$

III. $\int \mathrm{T}d\omega = \int \frac{-(1+zz)}{1-z^6}\cdot\frac{dz}{lz}$,

which on putting $zz = v$

it will change into this

$$\int \mathrm{T}d\omega = \int \frac{-dv(1+v)}{\left(1-v^3\right)lv}.$$

Hence it is therefore clear, clearly by this new method to come upon integral formulas, which by methods hitherto known in now way to be allowed to be set out, or at least to be compared between themselves.

SUPPLEMENTUM III. AD TOM.I. CAP. IV.

DE
INTEGRATIONE FORMULARUM LOGARITHMICARUM
ET EXPONENTIALIUM .

2) De valore formulae integralis $\int \frac{z^{\lambda-\omega} \pm z^{\lambda+\omega}}{1 \pm z^{2\lambda}} \cdot \frac{dz}{z} (lz)^{\mu}$, casu quo post integrationem ponitur $z = 1$.

Novi Commentarii Acad. Imp. Sc. Petropolitanae. Tom XIX. Pag. 30-64.

§. 69. Ex consideratione innumerabilium arcuum circularium, qui communem habent vel sinum vel tangentem; jam olim summationem duarum serierum infinitarum deduxi, quae ob summam generalitatem maxime memoratu dignae videbantur. Si enim litterae *m* et *n* numeros quoscunque denotant, posita diametri ratione ad peripheriam ut 1 ad π, illae duae summationes hoc modo se habebant

$$\frac{1}{m} + \frac{1}{n-m} - \frac{1}{n+m} - \frac{1}{2n-m} + \frac{1}{2n+m} + \frac{1}{3n-m} - \text{etc.} = \frac{\pi}{n \sin.\frac{m\pi}{n}} \quad \text{et}$$

$$\frac{1}{m} - \frac{1}{n-m} + \frac{1}{n+m} - \frac{1}{2n-m} + \frac{1}{2n+m} - \frac{1}{3n-m} + \text{etc.} = \frac{\pi}{n \,\text{tang}.\frac{m\pi}{n}},$$

atque ex his duabus seriebus jam tum temporis elicueram summationes omnium serierum illarum, quarum denominatores secundum potestates numerorum naturalium progrediuntur, quemadmodum in introductione in analysin infinitorum et alibi fusius exposui. Nunc autem eaedem series me perduxerunt ad integrationem formulae in titulo expressae, quae eo magis attentione digna videtur, quod hujusmodi integrationes aliis methodis neutiquam exsequi liceat.

§. 70. Statim autem patet, has duas series infinitas oriri ex evolutione quarundam formularum integralium, si post integrationem quantitati variabilibili certus valor, veluti unitas tribuatur; ita prior series deducitur ex evolutione hujus formulae integralis

$$\int \frac{z^{m-1} + z^{n-m-1}}{1+z^{n}} dz,$$

posterior vero ex evolutione istius

$$\int \frac{z^{m-1} - z^{n-m-1}}{1-z^{n}} dz,$$

si quidem post integrationem statuatur $z = 1$. Deinceps autem ex ipsis principiis calculi integralis demonstravi, valorem integralis prioris harum duarum formularum, si quidem ponatur $z = 1$, reduci ad hanc formulam simplicem

$$\frac{\pi}{n \sin.\frac{m\pi}{n}},$$

integrale autem posterius, eodem casu $z=1$, ad istam

$$\frac{\pi}{n\,\text{tang.}\frac{m\pi}{n}},$$

ita, ut ex ipsis calculi integralis principiis certum sit esse

$$\frac{1}{m}+\frac{1}{n-m}-\frac{1}{n+m}-\frac{1}{2n-m}+\frac{1}{2n+m}+\frac{1}{3n-m}-\text{etc.}=\frac{\pi}{n\sin.\frac{m\pi}{n}}\ \text{ et}$$

$$\frac{1}{m}-\frac{1}{n-m}+\frac{1}{n+m}-\frac{1}{2n-m}+\frac{1}{2n+m}-\frac{1}{3n-m}+\text{etc.}=\frac{\pi}{n\,\text{tang.}\frac{m\pi}{n}},$$

si quidem post integrationem ita institutam, ut integrale evanescat posito $z=0$, statuatur $z=1$.

§. 71. Quo jam hanc duplicem integrationem ad formam propositam reducamus, faciamus $n=2\lambda$ et $m=\lambda-\omega,$ unde binae illae series infinitae hanc induent formam

$$\frac{1}{\lambda-\omega}+\frac{1}{\lambda+\omega}-\frac{1}{3\lambda-\omega}-\frac{1}{3\lambda+\omega}+\frac{1}{5\lambda-\omega}+\frac{1}{5\lambda+\omega}-\text{etc. et}$$

$$\frac{1}{\lambda-\omega}-\frac{1}{\lambda+\omega}+\frac{1}{3\lambda-\omega}-\frac{1}{3\lambda+\omega}+\frac{1}{5\lambda-\omega}-\frac{1}{5\lambda+\omega}-\text{etc.}$$

harum igitur serierum prioris summa erit

$$\frac{\pi}{2\lambda\sin.\frac{\pi(\lambda-\omega)}{2\lambda}}=\frac{\pi}{2\lambda\cos.\frac{\pi\omega}{2\lambda}},$$

posterioris vero summa erit

$$\frac{\pi}{2\lambda\,\text{tang.}\frac{\pi(\lambda-\omega)}{2\lambda}}=\frac{\pi}{2\lambda\text{cotang.}\frac{\pi\omega}{2\lambda}}=\frac{\pi\,\text{tang.}\frac{\pi\omega}{2\lambda}}{2\lambda}.$$

Quod si ergo brevitatis gratia ponamus

$$\frac{\pi}{2\lambda\cos.\frac{\pi\omega}{2\lambda}}=\text{S},\ \text{ et }\ \frac{\pi}{2\lambda}\text{tang.}\frac{\pi\omega}{2\lambda}=\text{T},$$

habebimus sequentes duas integrationes

$$\int\frac{z^{\lambda-\omega}+z^{\lambda+\omega}}{1+z^{2\lambda}}\cdot\frac{dz}{z}=\text{S, et}$$

$$\int\frac{z^{\lambda-\omega}-z^{\lambda+\omega}}{1-z^{2\lambda}}\cdot\frac{dz}{z}=\text{T}.$$

§.72. Circa has binas integrationes ante omnia observo, eas perinde locum habere, sive pro litteris λ et ω accipiantur numeri integri, sive fracti. Sint enim λ et ω numeri fracti quicunque, qui evadant integri, si multipli per α, quo posito fiat $z = x^{\alpha}$, eritque $\frac{dz}{z} = \frac{\alpha dx}{x}$, et potestat quaecunque $z^{\delta} = x^{\alpha\delta}$; igitur formula erit

$$\int \frac{x^{\alpha(\lambda-\omega)} + x^{\alpha(\lambda+\omega)}}{1+x^{2\alpha\lambda}} \cdot \frac{\alpha dx}{x},$$

ubi, cum jam omnes exponentes sint numeri integri, valor hujus formulae posito post integrationem $x = 1$, quandoquidem tunc etiam sit $z = 1$, a praecedente eo tantum differt quod hic habemus $\alpha\lambda$ et $\alpha\omega$ loco λ et ω, ac praeterea hic adsit factor α; quocirca valor istius formulae erit

$$\alpha \cdot \frac{\pi}{2\alpha\lambda \cos.\frac{\pi\omega}{2\lambda}} = \frac{\pi}{2\lambda \cos.\frac{\pi\omega}{2\lambda}},$$

qui ergo valor est $=$S prorsus ut ante; quae identitas etiam manifesto est in altera formula, unde patet, etiamsi pro λ et ω fractiones quaecunque accipiantur, integrationem hic exhibitam nihilo minus locum esse habituram; quae circumstantia probe notari meretur, quoniam in sequentibus litteram ω tanquam variabilem sumus tractaturi.

§. 73. Postquam igitur binae istae formulae integrales litteris S et T indicatae fuerint integratae, ita ut evanescant posito $z = 0$, integralia spectari poterunt non solum ut functiones quantitatis *z*, sed etiam ut functiones binarum variabilium *z* et ω, quandoquidem numerum ω tanquam quantitatem variabilem tractate licet, quin etiam exponentem λ pro quantitate variabili habere liceret: sed quia hinc formulae integrales alius generis essent proditurae, atque hic contemplari constitui, solam quantitatem ω, praeter ipsam variabilem *z*, hic ut quantitatem variabilem sum tractaturus.

§.74. Cum igitur sit

$$\mathrm{S} = \int \frac{z^{\lambda-\omega} + z^{\lambda+\omega}}{1+z^{2\lambda}} \cdot \frac{dz}{z},$$

in qua integratione sola *z* ut variabilis spectatur, erit utique secundum signandi morem jam satis usu receptum

$$\left(\frac{d\mathrm{S}}{dz}\right) = \frac{z^{\lambda-\omega} + z^{\lambda+\omega}}{1+z^{2\lambda}} \cdot \frac{1}{z};$$

haec jam formula denuo differentietur, posita sola littera ω variabili, eritque

$$\left(\frac{dd\mathrm{S}}{dzd\omega}\right) = \frac{-z^{\lambda-\omega} + z^{\lambda+\omega}}{1+z^{2\lambda}} \cdot \frac{1}{z} lz,$$

quae formula ducta in *dz*, ac denuo integrata sola *z* habita pro variabili, dabit

$$\int dz\left(\frac{ddS}{dzd\omega}\right)=\int\frac{-z^{\lambda-\omega}+z^{\lambda+\omega}}{1+z^{2\lambda}}\cdot\frac{dz}{z}lz,$$

ubi notetur esse

$$S=\frac{\pi}{2\lambda\cos.\frac{\pi\omega}{2\lambda}};$$

ita ut hinc deducamus

$$\frac{dS}{d\omega}=\frac{\pi\pi\sin.\frac{\pi\omega}{2\lambda}}{4\lambda\lambda\left(\cos.\frac{\pi\omega}{2\lambda}\right)^2},$$

hoc igitur valore substituto, nanciscimur hanc integrationem

$$\int\frac{-z^{\lambda-\omega}+z^{\lambda+\omega}}{1+z^{2\lambda}}\cdot\frac{dz}{z}lz=\frac{\pi\pi\sin.\frac{\pi\omega}{2\lambda}}{4\lambda\lambda\left(\cos.\frac{\pi\omega}{2\lambda}\right)^2}.$$

§. 75. Quod si jam altera formula simili modo tractetur, cum sit

$$T=\frac{\pi\omega}{2\lambda}\operatorname{tang}\frac{\pi\omega}{2\lambda},\text{ erit}$$

$$\frac{dT}{d\omega}=\frac{\pi\pi}{4\lambda\lambda\left(\cos.\frac{\pi\omega}{2\lambda}\right)^2};$$

ex formula autem integrali erit

$$\left(\frac{dT}{d\omega}\right)=\int\frac{-z^{\lambda-\omega}+z^{\lambda+\omega}}{1-z^{2\lambda}}\cdot\frac{dz}{z}lz,$$

unde colligimus sequentem integrationem

$$\int\frac{z^{\lambda-\omega}+z^{\lambda+\omega}}{1-z^{2\lambda}}\cdot\frac{dz}{z}lz=\frac{-\pi\pi}{4\lambda\lambda\left(\cos.\frac{\pi\omega}{2\lambda}\right)^2}.$$

§.76. Quoniam litteras S et T etiam per series expressas dedimus, erit etiam per similes series

$$\left(\frac{dS}{d\omega}\right)=\frac{1}{(\lambda-\omega)^2}-\frac{1}{(\lambda+\omega)^2}-\frac{1}{(3\lambda-\omega)^2}+\frac{1}{(3\lambda+\omega)^2}+\frac{1}{(5\lambda-\omega)^2}-\text{etc.}$$
$$=\frac{\pi\pi\sin.\frac{\pi\omega}{2\lambda}}{4\lambda\lambda\left(\cos.\frac{\pi\omega}{2\lambda}\right)^2}.$$

Similique modo etiam pro altera serie

$$\left(\frac{dS}{d\omega}\right)=\frac{1}{(\lambda-\omega)^2}-\frac{1}{(\lambda+\omega)^2}-\frac{1}{(3\lambda-\omega)^2}+\frac{1}{(3\lambda+\omega)^2}+\frac{1}{(5\lambda-\omega)^2}-\text{etc.}$$
$$=\frac{\pi\pi\sin.\frac{\pi\omega}{2\lambda}}{4\lambda\lambda\left(\cos.\frac{\pi\omega}{2\lambda}\right)^2}.$$

sicque summas harum serierum quoque duplici modo repraesentavimus, scilicet per formulam evolutam quantitatem π involventem, tum vero etiam per formulam integralem, quae ita est comparata, ut ejus integrale nulla methodo adhuc consueta assignari possit.

§. 77. Applicemus has integrationes ad aliquot casus particulares: ac primo quidem sumamus $\omega=0$, quo quidem casu prior integratio sponte in oculos incurrit, at posterior praebet

$$\int\frac{2z^{\lambda}}{1-z^{2\lambda}}\cdot\frac{dz}{z}lz=-\frac{\pi\pi}{4\lambda\lambda},\ \text{sive}$$
$$\int\frac{z^{\lambda-1}dzlz}{1-z^{2\lambda}}=-\frac{\pi\pi}{8\lambda\lambda};$$

hincque simul istam summationem adipiscimur

$$\frac{1}{\lambda\lambda}+\frac{1}{\lambda\lambda}+\frac{1}{9\lambda\lambda}+\frac{1}{9\lambda\lambda}+\frac{1}{25\lambda\lambda}+\frac{1}{25\lambda\lambda}+\text{etc.}=\frac{\pi\pi}{4\lambda\lambda},\ \text{sive}$$
$$1+\frac{1}{9}+\frac{1}{25}+\frac{1}{49}+\frac{1}{81}+\text{etc.}=\frac{\pi\pi}{8},$$

id quod jam dudum a me est demonstratum.

§. 78. Hic statim patet, perinde esse, quiam numerus pro λ acciptatur; sit igitur $\lambda=1$, et habebitur ista integratio

$$\int\frac{dzlz}{1-z^2}=-\frac{\pi\pi}{8};$$

ex qua sequentia integralia simpliciora

$$\int\frac{dzlz}{1-z}\ \text{et}\int\frac{dzlz}{1+z}$$

derivare licet ope hujus ratiocinii; statuatur

$$\int\frac{zdzlz}{1-zz}=\text{P},$$

et posito $zz=v,$ ut sit $zdz=\frac{dv}{2}$ et $lz=\frac{1}{2}lv,$; prodibit

$$\frac{1}{4}\int\frac{dvlv}{1-v}=\text{P},$$

si scilicet post integrationem fiat $v=1$, quippe quo casu etiam sit $z=1$;
sic igitur erit

$$\int\frac{dvlv}{1-v}=4\text{P}:$$

nunc prior illa formula addatur ad inventam, eritque

$$\int \frac{dzlz+zdzlz}{1-zz} = \mathrm{P} - \frac{\pi\pi}{8},$$

haec autem formula sponte reducitur ad hanc

$$\int \frac{dzlz}{1-z} = \mathrm{P} - \frac{\pi\pi}{8},$$

modo autem vidimus esse

$$\int \frac{dvlv}{1-v} \text{ sive } \int \frac{dzlz}{1-z} = 4\mathrm{P}, \text{ ita ut sit } 4\mathrm{P} = \mathrm{P} - \frac{\pi\pi}{8},$$

unde manifesto sit $\mathrm{P} = -\frac{\pi\pi}{24}$, ex quo sequitur fore

$$\int \frac{dzlz}{1-z} = -\frac{\pi\pi}{6};$$

simili modo erit

$$\int \frac{dzlz-zdzlz}{1-zz} = -\mathrm{P} - \frac{\pi\pi}{8} = -\frac{\pi\pi}{12},$$

quae, supra et infra per $1-z$ dividendo, praebet

$$\int \frac{dzlz}{1+z} = -\frac{\pi\pi}{12},$$

quare jam adepti sumus tres integrationes memoratu maxime dignas

$$\text{I. } \int \frac{dzlz}{1+z} = -\frac{\pi\pi}{12},$$

$$\text{II. } \int \frac{dzlz}{1-z} = -\frac{\pi\pi}{6},$$

$$\text{III. } \int \frac{dzlz}{1-zz} = -\frac{\pi\pi}{8},$$

quibus adiungi potest

$$\text{IV. } \int \frac{zdzlz}{1-zz} = -\frac{\pi\pi}{24}.$$

§.79. Quemadmodum igitur hae formulae ex ipsis calculi integralis principiis sunt deductae, ita etiam earum veritas per resolutionem in series facile comprobatur; cum enim sit

$$\frac{1}{1+z} = 1 - z + zz - z^3 + z^4 - z^5 + \text{etc.},$$

et in genere

$$\int z^n dzlz = \frac{z^{n+1}}{n+1} lz - \frac{z^{n+1}}{(n+1)^2},$$

qui valor posito $z=1$ reducitur ad $\frac{1}{(n+1)^2}$, patet fore

$$\int \frac{dzlz}{1+z} = -1 + \frac{1}{4} - \frac{1}{9} + \frac{1}{16} - \frac{1}{25} + \text{etc.} = -\frac{\pi\pi}{12}, \text{sive}$$
$$1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \frac{1}{25} - \text{etc.} = \frac{\pi\pi}{12},$$

simili modo ob

$$\frac{1}{1-z} = 1 + z + zz + z^3 + z^4 + z^5 + \text{etc.}, \text{erit}$$
$$\int \frac{dzlz}{1-z} = -1 - \frac{1}{4} - \frac{1}{9} - \frac{1}{16} - \frac{1}{25} - \text{etc.} = -\frac{\pi\pi}{12}, \text{seu}$$
$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \text{etc.} = \frac{\pi\pi}{6},$$

Eodem modo etiam

$$\int \frac{zdzlz}{1-zz} = -\frac{1}{4} - \frac{1}{16} - \frac{1}{36} - \text{etc.} = -\frac{\pi\pi}{24}, \text{sive}$$
$$\frac{1}{4} + \frac{1}{16} + \frac{1}{36} + \frac{1}{64} + \text{etc.} = \frac{\pi\pi}{24},$$

quae quidem summationes jam sunt notissimae. Neque tamen quisquam adhuc methodo directa ostendit esse

$$\int \frac{zdzlz}{1+zz} == -\frac{\pi\pi}{12}.$$

§. 80. Ponamus nunc $\omega = 1$, et nostrae integrationes has induent formas

$$1^0. \int \frac{-z^{\lambda-2}(1-zz)dzlz}{1+z^{2\lambda}} \cdot \frac{dz}{z} lz = \frac{\pi\pi \sin.\frac{\pi}{2\lambda}}{4\lambda\lambda\left(\cos.\frac{\pi}{2\lambda}\right)^2} \text{ et}$$
$$2^0. \int \frac{-z^{\lambda-2}(1+zz)dzlz}{1-z^{2\lambda}} \cdot \frac{dz}{z} lz = +\frac{\pi\pi}{4\lambda\lambda\left(\cos.\frac{\pi}{2\lambda}\right)^2},$$

unde pro diversis valoribus ipsius λ, quos quidem binario non minores accipere licet, sequentes obtinentur integrationes

I^0. si $\lambda = 2$, erit

$$1^0 \int \frac{-(1-zz)dzlz}{1+z^4} = \frac{\pi\pi}{8\sqrt{2}},$$
$$2^0 \int \frac{-(1+zz)dzlz}{1-z^4} = +\frac{\pi\pi}{8}, \text{ sive } \int \frac{-dzlz}{1-zz} = +\frac{\pi\pi}{8}.$$

II^0. si $\lambda = 3$, habebimus

$$1^0 \int \frac{-z(1-zz)dzlz}{1+z^6} = \frac{\pi\pi}{54}, \text{ et}$$
$$2^0 \int \frac{-z(1+zz)dzlz}{1-z^6} = \int \frac{-zdzlz}{1-zz+z^4} = \frac{\pi\pi}{27}.$$

Hae autem duae formulae ponendo $zz = v$, abibunt in sequentes

$$1^0 \int \frac{-dv(1-v)lv}{1+v^3} = \frac{2\pi\pi}{27}, \text{ et}$$

$$2^0 \int \frac{dvlv}{1-v+vv} = \frac{4\pi\pi}{27}.$$

III^0. Sit $\lambda = 4$ et consequemur

$$1^0 \int \frac{-zz(1-zz)dzlz}{1+z} = \frac{\pi\pi\sqrt{\frac{\sqrt{2}-1}{2\sqrt{2}}}}{16\left(2+\sqrt{2}\right)} = \frac{\pi\pi\sqrt{\left(2-\sqrt{2}\right)}}{32\left(2+\sqrt{2}\right)} \text{ et}$$

$$2^0 \int \frac{-zz(1+zz)dzlz}{1-z^8} = \int \frac{-zzdzlz}{(1-zz)\left(1+z^4\right)} = \frac{\pi\pi}{16\left(2+\sqrt{2}\right)},$$

quae postrema forma reducitur ad hanc

$$\int -\frac{dzlz}{(1-zz)} + \int \frac{(1-zz)dzlz}{\left(1+z^4\right)} = \frac{\pi\pi}{8\left(2+\sqrt{2}\right)},$$

est vero $\int \frac{-dzlz}{1-zz} = \frac{\pi\pi}{8}$, unde reperitur

$$\int \frac{(1-zz)dzlz}{1+z^4} = -\frac{\pi\pi\left(1+\sqrt{2}\right)}{8\left(2+\sqrt{2}\right)} = -\frac{\pi\pi}{8\sqrt{2}},$$

qui valor jam in superiori casu $\lambda = 2$ est inventus.

§. 81. Nihil autem impedit, quo minus etiam faciamus $\lambda = 1$, dummodo integralia ita capiantur ut evanescant, posito $z = 0$, tum autem reperiemus

$$1^0 \int \frac{-(1-zz)dzlz}{z(1+zz)} = \infty \text{ et}$$

$$2^0 \int \frac{-(1+zz)dzlz}{z(1-zz)} = \infty,$$

unde hinc nihil concludere licet. Caeterum etiam nostrae series supra inventae manifesta declarant, earum summas esse infinitas, quandoquidem primus terminus utriusque $\frac{1}{(\lambda-\omega)^2}$ fit infinitus, sumto uti fecimus $\lambda = 1$ et $\omega = 1$.

§. 82. His casibus evolutis, ulterius progrediamur ac ponamus formulas integrales inventas

$$\int \frac{-z^{\lambda-\omega}+z^{\lambda+\omega}}{1+z^{2\lambda}} \cdot \frac{dz}{z} lz = \text{S' et}$$

$$\int \frac{-z^{\lambda-\omega}-z^{\lambda+\omega}}{1-z^{2\lambda}} \cdot \frac{dz}{z} lz = \text{T'}$$

ita ut sit

$$S' = \frac{\pi\pi \sin.\frac{\pi}{2\lambda}}{4\lambda\lambda\left(\cos.\frac{\pi}{2\lambda}\right)^2} \text{ et } \mathrm{T}' = \frac{\pi\pi}{4\lambda\lambda\left(\cos.\frac{\pi}{2\lambda}\right)^2},$$

atque ut ante jam differentiemus solo numero ω pro variabili habito; quo facto sequentes nanciscimur integrationes

$$\int \frac{z^{\lambda-\omega}+z^{\lambda+\omega}}{1+z^{2\lambda}} \cdot \frac{dz}{z}(lz)^2 = \left(\frac{dS'}{d\omega}\right), \text{ et}$$
$$\int \frac{z^{\lambda-\omega}-z^{\lambda+\omega}}{1-z^{2\lambda}} \cdot \frac{dz}{z}(lz)^2 = \left(\frac{dT'}{d\omega}\right).$$

Hunc in finem ponamus brevitatis ergo angulum $\frac{\pi\omega}{2\pi} = \varphi$, ut sit

$$S' = \frac{\pi\pi \sin.\varphi}{4\lambda\lambda\left(\cos.^2\varphi\right)} = \frac{\pi\pi}{4\lambda\lambda} \cdot \frac{\sin.\varphi}{\cos.^2\varphi}, \text{ et}$$
$$\mathrm{T}' = \frac{\pi\pi}{4\lambda\lambda} \cdot \frac{1}{\cos.^2\varphi},$$

ac reperiemus

$$d \cdot \frac{\sin.\varphi}{\cos.^2\varphi} = \frac{\cos.^2\varphi + 2\sin.^2\varphi}{\cos.^3\varphi} d\varphi = \frac{1+\sin.^2\varphi}{\cos.^3\varphi} d\varphi,$$

ubi est $d\varphi = \frac{\pi d\omega}{2\pi}$; unde colligimus

$$\left(\frac{dS'}{d\omega}\right) = \frac{\pi^3}{8\lambda^2}\left(\frac{1+\left(\sin.\frac{\pi\omega}{2\lambda}\right)^2}{\left(\cos.\frac{\pi\omega}{2\lambda}\right)^3}\right) = \frac{\pi^3}{8\lambda^2}\left(\frac{2}{\left(\cos.\frac{\pi\omega}{2\lambda}\right)^3} - \frac{1}{\cos.\frac{\pi\omega}{2\lambda}}\right);$$

simili modo ob $\mathrm{T}' = \frac{\pi\pi}{4\lambda\lambda} \cdot \frac{1}{\cos.^2\varphi}$, erit

$$d \cdot \frac{1}{\cos.^2\varphi} = \frac{2d\varphi \sin.\varphi}{\cos.^3\varphi},$$

hincque

$$\left(\frac{dT}{d\omega}\right) = \frac{\pi^3}{8\lambda^2} \cdot \frac{2\sin.\frac{\pi\omega}{2\lambda}}{\left(\cos.\frac{\pi\omega}{2\lambda}\right)^3}.$$

Consequenter integrationes hinc natae erunt

$$\int \frac{z^{\lambda-\omega}+z^{\lambda+\omega}}{1+z^{2\lambda}} \cdot \frac{dz}{z}(lz)^2 = \frac{\pi^3}{8\lambda^2}\left(\frac{2}{\left(\cos.\frac{\pi\omega}{2\lambda}\right)^3} - \frac{1}{\cos.\frac{\pi\omega}{2\lambda}}\right),$$
$$\int \frac{z^{\lambda-\omega}-z^{\lambda+\omega}}{1-z^{2\lambda}} \cdot \frac{dz}{z}(lz)^2 = \frac{\pi^3}{8\lambda^2} \cdot \frac{2\sin.\frac{\pi\omega}{2\lambda}}{\left(\cos.\frac{\pi\omega}{2\lambda}\right)^3}.$$

§. 83. Si jam, eodem modo series §.76 inventas denuo differentiemus, sumta sola ω variabili, perveniamus ad sequentes summationes

$$\frac{\pi^3}{8\lambda^3}\left(\frac{2}{\left(\cos.\frac{\pi\omega}{2\lambda}\right)^3}-\frac{1}{\cos.\frac{\pi\omega}{2\lambda}}\right)=+\frac{2}{(\lambda-\omega)^3}+\frac{2}{(\lambda+\omega)^3}-\frac{2}{(3\lambda-\omega)^3}-\frac{2}{(3\lambda+\omega)^3}$$
$$+\frac{2}{(5\lambda-\omega)^3}+\frac{2}{(5\lambda+\omega)^3}-\text{etc.}$$
$$\frac{\pi^3}{8\lambda^3}\cdot\frac{2\sin.\frac{\pi\omega}{2\lambda}}{\left(\cos.\frac{\pi\omega}{2\lambda}\right)^3}=\frac{2}{(\lambda-\omega)^3}-\frac{2}{(\lambda+\omega)^3}+\frac{2}{(3\lambda-\omega)^3}-\frac{2}{(3\lambda+\omega)^3}+\frac{2}{(5\lambda-\omega)^3}-\text{etc.}$$

§. 84. Si jam hic sumamus $\omega=0$ et $\lambda=1$, prior integratio hanc induit formam

$$\int\frac{2dz(lz)^2}{1+zz}=\frac{\pi^3}{8}=\frac{2}{1^3}+\frac{2}{1^3}-\frac{2}{3^3}-\frac{2}{3^3}+\frac{2}{5^3}+\frac{2}{5^3}-\frac{2}{7^3}-\frac{2}{7^3}+\text{etc.}$$

ita ut sit

$$\frac{1}{1^3}-\frac{1}{3^3}+\frac{1}{5^3}-\frac{1}{7^3}+\frac{1}{9^3}-\frac{1}{11^3}+\text{etc.}=\frac{\pi^3}{32},$$

quemadmodum jam dudum demonstravi. Altera autem integratio hoc casu in nihilum abit. Ex priori vero integrali,

$$\int\frac{dz(lz)^2}{1+zz}=\frac{\pi^3}{16},$$

alia derivare non licet, uti supra fecimus ex formula

$$\int\frac{dzlz}{1-zz}=-\frac{\pi\pi}{8},$$

propterea quod hic denominator $1+zz$ non habet factores reales.

§. 85. Sumamus igitur $\lambda=2$ et $\omega=1$, ac prior integratio dabit

$$\int\frac{(1+zz)dz(lz)^2}{1+z^4}=\frac{5\pi^3}{32\sqrt{2}};$$

series autem hinc nata erit

$$\frac{2}{1^3}+\frac{2}{3^3}-\frac{2}{5^3}-\frac{2}{7^3}+\frac{2}{9^3}+\frac{2}{11^3}-\text{etc.},$$

ita ut sit

$$\frac{1}{1^3}+\frac{1}{3^3}-\frac{1}{5^3}-\frac{1}{7^3}+\frac{1}{9^3}+\frac{1}{11^3}-\text{etc.}=\frac{3\pi^3}{64\sqrt{2}},$$

quae superiori addita praebet

$$\frac{1}{1^3}-\frac{1}{7^3}+\frac{1}{9^3}-\frac{1}{15^3}+\frac{1}{17^3}-\frac{1}{25^3}+\text{etc.}=\frac{\pi^3\left(5+\sqrt{2}\right)}{128\sqrt{2}}.$$

Altera vero integratio hoc casu dat

$$\int\frac{dz(lz)^2}{1+zz}=\frac{\pi^3}{16},$$

quae cum paragrapho praecedenti perfecte congruit, quemadmodum etiam series hinc nata est

$$\frac{2}{1^3}-\frac{2}{3^3}+\frac{2}{5^3}-\frac{2}{7^3}+\frac{2}{9^3}-\frac{2}{11^3}+\frac{2}{15^3}-\text{etc.}$$

§86. Quo autem facilius sequentes integrationes per continuam differentiationem elicere valeamus, eas in genere repraesentemus; et cum propriore sit

$$\mathrm{S}=\frac{\pi}{2\lambda\cos.\frac{\pi\omega}{2\lambda}},$$

integrationes hinc ortae ita ordine procedent

$$\text{I.}\quad\int\frac{z^{\lambda-\omega}+z^{\lambda+\omega}}{1+z^{2\lambda}}\cdot\frac{dz}{z}=\mathrm{S},$$

$$\text{II.}\quad\int\frac{-z^{\lambda-\omega}+z^{\lambda+\omega}}{1+z^{2\lambda}}\cdot\frac{dz}{z}lz=\left(\frac{d\mathrm{S}}{d\omega}\right),$$

$$\text{III.}\quad\int\frac{z^{\lambda-\omega}+z^{\lambda+\omega}}{1+z^{2\lambda}}\cdot\frac{dz}{z}(lz)^2=\left(\frac{dd\mathrm{S}}{d\omega^2}\right),$$

$$\text{IV.}\quad\int\frac{-z^{\lambda-\omega}+z^{\lambda+\omega}}{1+z^{2\lambda}}\cdot\frac{dz}{z}(lz)^3=\left(\frac{d^3\mathrm{S}}{d\omega^3}\right),$$

$$\text{V.}\quad\int\frac{z^{\lambda-\omega}+z^{\lambda+\omega}}{1+z^{2\lambda}}\cdot\frac{dz}{z}(lz)^4=\left(\frac{d^4\mathrm{S}}{d\omega^4}\right),$$

$$\text{VI.}\quad\int\frac{-z^{\lambda-\omega}+z^{\lambda+\omega}}{1+z^{2\lambda}}\cdot\frac{dz}{z}(lz)^5=\left(\frac{d^5\mathrm{S}}{d\omega^5}\right),$$

$$\text{VII.}\quad\int\frac{z^{\lambda-\omega}+z^{\lambda+\omega}}{1+z^{2\lambda}}\cdot\frac{dz}{z}(lz)^6=\left(\frac{d^6\mathrm{S}}{d\omega^6}\right),$$

etc. etc. etc.

§.87. Pro his differentiationibus continuis facilius absoluendis, ponamus brevitatis ergo $\frac{\pi}{2\lambda}=\alpha$, ut sit

$$\mathrm{S}=\frac{\alpha}{\cos.\alpha\omega};$$

tum vero sit

$$\sin.\alpha\omega=p\ \text{et}\ \cos.\alpha\omega=q,$$

eritque

$$dp = \alpha q d\omega \text{ et } dq = -\alpha p d\omega.$$

Praeterea vero notetur esse

$$d \cdot \frac{p^n}{q^{n+1}} = \alpha d\omega \left\{ \frac{np^{n-1}}{q^n} + \frac{(n+1)p^{n+1}}{q^{n+2}} \right\}.$$

His praemissis ob $\mathrm{S} = \alpha \cdot \frac{1}{q}$ erit

$$\left(\frac{d\mathrm{S}}{d\omega}\right) = \alpha^2 \cdot \frac{p}{qq}, \text{ deinde}$$
$$\left(\frac{dd\mathrm{S}}{d\omega^2}\right) = \alpha^3 \left(\frac{1}{q} + \frac{2pp}{q^3}\right), \text{ porro}$$
$$\left(\frac{d^3\mathrm{S}}{d\omega^3}\right) = \alpha^4 \left(\frac{5p}{qq} + \frac{6p^3}{q^4}\right),$$
$$\left(\frac{d^4\mathrm{S}}{d\omega^4}\right) = \alpha^5 \left(\frac{5}{q} + \frac{28pp}{q^3} + \frac{24p^4}{q^5}\right),$$
$$\left(\frac{d^5\mathrm{S}}{d\omega^5}\right) = \alpha^6 \left(\frac{61p}{qq} + \frac{180p^3}{q^4} + \frac{120p^5}{q^6}\right),$$
$$\left(\frac{d^6\mathrm{S}}{d\omega^6}\right) = \alpha^7 \left(\frac{61}{q} + \frac{662pp}{q^3} + \frac{1320p^4}{q^5} + \frac{720p^6}{q^7}\right),$$
$$\left(\frac{d^7\mathrm{S}}{d\omega^7}\right) = \alpha^8 \left(\frac{1385p}{qq} + \frac{7266p^3}{q^4} + \frac{10920p^5}{q^6} + \frac{5040p^7}{q^8}\right), \text{etc.}$$

hi autem valores ob $pp = 1 - qq$ ad sequentes reducuntur

$$S = \alpha \cdot \frac{1}{q},$$
$$\left(\frac{d\mathrm{S}}{d\omega}\right) = \alpha^2 p \cdot \frac{1}{qq},$$
$$\left(\frac{dd\mathrm{S}}{d\omega^2}\right) = \alpha^3 \left(\frac{1.2}{q^3} - \frac{1}{q}\right),$$
$$\left(\frac{d^3\mathrm{S}}{d\omega^3}\right) = \alpha^4 p \left(\frac{1.2.3}{q^4} - \frac{1}{qq}\right),$$
$$\left(\frac{d^4\mathrm{S}}{d\omega^4}\right) = \alpha^5 \left(\frac{1.2.3.4}{q^5} - \frac{20}{q^3} + \frac{1}{q}\right),$$
$$\left(\frac{d^5\mathrm{S}}{d\omega^5}\right) = \alpha^6 p \left(\frac{1.2.3.4.5}{q^6} - \frac{60}{q^4} + \frac{1}{qq}\right),$$
$$\left(\frac{d^6\mathrm{S}}{d\omega^6}\right) = \alpha^7 \left(\frac{1.....6}{q^7} - \frac{840}{q^5} + \frac{182}{q^3} - \frac{1}{q}\right), \text{etc.}$$

§. 88. Has posteriores formas reperire licet ope horum duarum lemmatum

$$\text{I.}\quad d\cdot\frac{1}{q^{n+1}}=\alpha d\omega\frac{(n+1)p}{q^{n+2}},\text{et}$$

$$\text{II.}\quad d\cdot\frac{p}{q^{n+1}}=\alpha d\omega\left\{\frac{n+1}{q^{n+2}}-\frac{n}{q^{n}}\right\}.$$

hinc enim reperiemus

$$S=\alpha\cdot\frac{1}{q},$$

$$\left(\frac{dS}{d\omega}\right)=\alpha^{2}p\cdot\frac{1}{qq},$$

$$\left(\frac{ddS}{d\omega^{2}}\right)=\alpha^{3}\left(\frac{2}{q^{3}}-\frac{1}{q}\right),$$

$$\left(\frac{d^{3}S}{d\omega^{3}}\right)=\alpha^{4}\left(\frac{2.3}{q^{4}}-\frac{p}{qq}\right),$$

$$\left(\frac{d^{4}S}{d\omega^{4}}\right)=\alpha^{5}\left(\frac{2.3.4}{q^{5}}-\frac{20}{q^{3}}+\frac{1}{q}\right),$$

$$\left(\frac{d^{5}S}{d\omega^{5}}\right)=\alpha^{6}\left(\frac{2.3.4.5p}{q^{6}}-\frac{3.20p}{q^{4}}+\frac{p}{qq}\right),$$

$$\left(\frac{d^{6}S}{d\omega^{6}}\right)=\alpha^{7}\left(\frac{2.3.4.4.6p}{q^{7}}-\frac{840}{q^{5}}+\frac{182}{q^{3}}-\frac{1}{q}\right),$$

$$\left(\frac{d^{7}S}{d\omega^{7}}\right)=\alpha^{8}\left(\frac{2......7p}{q^{8}}-\frac{5.840p}{q^{6}}+\frac{3.182}{q^{4}}-\frac{p}{qq}\right),\text{etc.}$$

§. 89. Ipsae autem series his formulis respondentes erunt

$$S = \frac{1}{\lambda-\omega} + \frac{1}{\lambda+\omega} - \frac{1}{3\lambda-\omega} - \frac{1}{3\lambda+\omega} + \frac{1}{5\lambda-\omega} + \frac{1}{5\lambda+\omega} - \text{etc.},$$

$$\left(\frac{dS}{d\omega}\right) = \frac{1}{(\lambda-\omega)^2} + \frac{1}{(\lambda+\omega)^2} - \frac{1}{(3\lambda-\omega)^2} - \frac{1}{(3\lambda+\omega)^2} + \frac{1}{(5\lambda-\omega)^2} + \frac{1}{(5\lambda+\omega)^2} - \text{etc.},$$

$$\left(\frac{ddS}{d\omega^2}\right) = \frac{1.2}{(\lambda-\omega)^3} + \frac{1.2}{(\lambda+\omega)^3} - \frac{1.2}{(3\lambda-\omega)^3} - \frac{1.2}{(3\lambda+\omega)^3} + \frac{1.2}{(5\lambda-\omega)^3} + \text{etc.},$$

$$\left(\frac{d^3S}{d\omega^3}\right) = \frac{1.2.3}{(\lambda-\omega)^4} + \frac{1.2.3}{(\lambda+\omega)^4} - \frac{1.2.3}{(3\lambda-\omega)^4} - \frac{1.2.3}{(3\lambda+\omega)^4} + \frac{1.2.3}{(5\lambda-\omega)^4} - \text{etc.},$$

$$\left(\frac{d^4S}{d\omega^4}\right) = \frac{1.2.3.4}{(\lambda-\omega)^5} + \frac{1.2.3.4}{(\lambda+\omega)^5} - \frac{1.2.3.4}{(3\lambda-\omega)^5} - \frac{1.2.3.4}{(3\lambda+\omega)^5} + \frac{1.2.3.4}{(5\lambda-\omega)^5} + \text{etc.},$$

$$\left(\frac{d^5S}{d\omega^5}\right) = \frac{1.2.3.4.5}{(\lambda-\omega)^6} + \frac{1.2.3.4.5}{(\lambda+\omega)^6} - \frac{1.2.3.4.5}{(3\lambda-\omega)^6} - \frac{1.2.3.4.5}{(3\lambda+\omega)^6} + \frac{1.2.3.4.5}{(5\lambda-\omega)^6} - \text{etc.},$$

$$\left(\frac{d^6S}{d\omega^6}\right) = \frac{1.2.3.4.5.6}{(\lambda-\omega)^7} + \frac{1.2.3.4.5.6}{(\lambda+\omega)^7} - \frac{1.2.3.4.5.6}{(3\lambda-\omega)^7} - \frac{1.2.3.4.5.6}{(3\lambda+\omega)^7} + \frac{1.2.3.4.5.6}{(5\lambda-\omega)^7} + \text{etc.},$$

$$\left(\frac{d^7S}{d\omega^7}\right) = \frac{1......7}{(\lambda-\omega)^8} + \frac{1......7}{(\lambda+\omega)^8} - \frac{1......7}{(3\lambda-\omega)^8} - \frac{1......7}{(5\lambda+\omega)^8} + \frac{1......7}{(5\lambda-\omega)^8} - \text{etc.},$$

Circa hos autem valores probe meminisse opportet, esse

$$\alpha = \frac{\pi}{2\lambda},\ p = \sin.\alpha\omega = \sin.\frac{\pi\omega}{2\lambda},\ \text{et } q = \cos.\alpha\omega = \cos.\frac{\pi\omega}{2\lambda}.$$

§. 90. Eodem modo expediamus valores seu formula integrales alterius generis, pro quibus est

$$T = \frac{\pi}{2\lambda}\,\text{tang}.\frac{\pi\omega}{2\lambda},$$

unde continuo differentiando oriuntur sequentes integrationes

$$\text{I.}\quad \int \frac{z^{\lambda-\omega}-z^{\lambda+\omega}}{1-z^{2\lambda}}\cdot\frac{dz}{z}=\mathrm{T},$$

$$\text{II.}\quad \int \frac{-z^{\lambda-\omega}-z^{\lambda+\omega}}{1-z^{2\lambda}}\cdot\frac{dz}{z}\,lz=\left(\frac{d\mathrm{T}}{d\omega}\right),$$

$$\text{III.}\quad \int \frac{z^{\lambda-\omega}-z^{\lambda+\omega}}{1-z^{2\lambda}}.\frac{dz}{z}(lz)^2=\left(\frac{dd\mathrm{T}}{d\omega^2}\right),$$

$$\text{IV.}\quad \int \frac{-z^{\lambda-\omega}-z^{\lambda+\omega}}{1-z^{2\lambda}}\cdot\frac{dz}{z}(lz)^3=\left(\frac{d^3\mathrm{T}}{d\omega^3}\right),$$

$$\text{V.}\quad \int \frac{z^{\lambda-\omega}-z^{\lambda+\omega}}{1-z^{2\lambda}}\cdot\frac{dz}{z}(lz)^4=\left(\frac{d^4\mathrm{T}}{d\omega^4}\right),$$

$$\text{VI.}\quad \int \frac{-z^{\lambda-\omega}-z^{\lambda+\omega}}{1-z^{2\lambda}}\cdot\frac{dz}{z}(lz)^5=\left(\frac{d^5\mathrm{T}}{d\omega^5}\right),$$

$$\text{VII.}\quad \int \frac{z^{\lambda-\omega}-z^{\lambda+\omega}}{1-z^{2\lambda}}\cdot\frac{dz}{z}(lz)^6=\left(\frac{d^6\mathrm{T}}{d\omega^6}\right),$$

etc.

§. 91. Ponatur iterum $\frac{\pi}{2\lambda}=\alpha, \sin.\alpha\omega=p, \text{et } \cos.\alpha\omega=q,$ ut sit.

$$\mathrm{T}=\frac{\alpha p}{q},$$

quae formula secundum lemmata §.88. continuo differentiata dabit

$$\mathrm{T}=\alpha\cdot\frac{1}{q},$$

$$\left(\frac{d\mathrm{T}}{d\omega}\right)=\alpha^2\cdot\frac{1}{qq},$$

$$\left(\frac{dd\mathrm{T}}{d\omega^2}\right)=\alpha^3\cdot\frac{2p}{q^3},$$

$$\left(\frac{d^3\mathrm{T}}{d\omega^3}\right)=\alpha^4\left(\frac{6}{q^4}-\frac{4}{qq}\right),$$

$$\left(\frac{d^4\mathrm{T}}{d\omega^4}\right)=\alpha^5\left(\frac{24p}{q^5}-\frac{8p}{q^3}\right),$$

$$\left(\frac{d^5\mathrm{T}}{d\omega^5}\right)=\alpha^6\left(\frac{120}{q^6}-\frac{120p}{q^4}+\frac{16}{qq}\right),$$

$$\left(\frac{d^6\mathrm{T}}{d\omega^6}\right)=\alpha^7\left(\frac{720p}{q^7}-\frac{480p}{q^5}+\frac{32p}{q^3}\right),$$

$$\left(\frac{d^7\mathrm{T}}{d\omega^7}\right)=\alpha^8\left(\frac{5040}{q^8}-\frac{6720}{q^6}+\frac{2016}{q^4}-\frac{64}{qq}\right),\text{etc.}$$

§. 92. Series autem infinitae, quae hinc nascuntur,erunt

$$\mathrm{T} = \frac{1}{\lambda-\omega} - \frac{1}{\lambda+\omega} + \frac{1}{3\lambda-\omega} - \frac{1}{3\lambda+\omega} + \frac{1}{5\lambda-\omega} - \frac{1}{5\lambda+\omega} + \text{etc.},$$

$$\left(\frac{d\mathrm{T}}{d\omega}\right) = \frac{1}{(\lambda-\omega)^2} + \frac{1}{(\lambda+\omega)^2} + \frac{1}{(3\lambda-\omega)^2} + \frac{1}{(3\lambda+\omega)^2} + \frac{1}{(5\lambda-\omega)^2} + \text{etc.},$$

$$\left(\frac{dd\mathrm{T}}{d\omega^2}\right) = \frac{1.2}{(\lambda-\omega)^3} - \frac{1.2}{(\lambda+\omega)^3} + \frac{1.2}{(3\lambda-\omega)^3} - \frac{1.2}{(3\lambda+\omega)^3} + \frac{1.2}{(5\lambda-\omega)^3} - \text{etc.},$$

$$\left(\frac{d^3\mathrm{T}}{d\omega^3}\right) = \frac{1.2.3}{(\lambda-\omega)^4} + \frac{1.2.3}{(\lambda+\omega)^4} + \frac{1.2.3}{(3\lambda-\omega)^4} + \frac{1.2.3}{(3\lambda+\omega)^4} + \frac{1.2.3}{(5\lambda-\omega)^4} + \text{etc.},$$

$$\left(\frac{d^4\mathrm{T}}{d\omega^4}\right) = \frac{1.2.3.4}{(\lambda-\omega)^5} - \frac{1.2.3.4}{(\lambda+\omega)^5} + \frac{1.2.3.4}{(3\lambda-\omega)^5} - \frac{1.2.3.4}{(3\lambda+\omega)^5} + \frac{1.2.3.4}{(5\lambda-\omega)^5} - \text{etc.},$$

$$\left(\frac{d^5\mathrm{T}}{d\omega^5}\right) = \frac{1.2.3.4.5}{(\lambda-\omega)^6} + \frac{1.2.3.4.5}{(\lambda+\omega)^6} + \frac{1.2.3.4.5}{(3\lambda-\omega)^6} + \frac{1.2.3.4.5}{(3\lambda+\omega)^6} + \frac{1.2.3.4.5}{(5\lambda-\omega)^6} + \text{etc.},$$

$$\left(\frac{d^6\mathrm{T}}{d\omega^6}\right) = \frac{1.....6}{(\lambda-\omega)^7} - \frac{1.....6}{(\lambda+\omega)^7} + \frac{1.....6}{(3\lambda-\omega)^7} - \frac{1.....6}{(3\lambda+\omega)^7} + \frac{1.....6}{(5\lambda-\omega)^7} - \text{etc.},$$

etc. etc. etc.

§. 93. Operae pretium erit, hinc casus simplissimos evolvere, qui oriuntur ponendo. $\lambda = 1$ et $\omega = 0$, ita ut sit $\alpha = \frac{\pi}{2}$, $p = 0$ et $q = 1$,
unde habebimus

Pro ordine priore	Pro ordine posteriore
$S = \frac{\pi}{2}$	$\mathrm{T} = 0$
$\left(\frac{d\mathrm{S}}{d\omega}\right) = 0$	$\left(\frac{d\mathrm{T}}{d\omega}\right) = \frac{\pi\pi}{4}$
$\left(\frac{dd\mathrm{S}}{d\omega^2}\right) = \frac{\pi^3}{8}$	$\left(\frac{dd\mathrm{T}}{d\omega^2}\right) = 0$
$\left(\frac{d^3\mathrm{S}}{d\omega^3}\right) = 0$	$\left(\frac{d^3\mathrm{T}}{d\omega^3}\right) = \frac{\pi^4}{2}$
$\left(\frac{d^4\mathrm{S}}{d\omega^4}\right) = \frac{5\pi^6}{32}$	$\left(\frac{d^4\mathrm{T}}{d\omega^4}\right) = 0$
$\left(\frac{d^5\mathrm{S}}{d\omega^5}\right) = 0$	$\left(\frac{d^5\mathrm{T}}{d\omega^5}\right) = \frac{\pi^6}{4}$
$\left(\frac{d^6\mathrm{S}}{d\omega^6}\right) = \frac{61\pi^7}{128}$	$\left(\frac{d^6\mathrm{T}}{d\omega^6}\right) = 0$
$\left(\frac{d^7\mathrm{S}}{d\omega^7}\right) = 0$	$\left(\frac{d^7\mathrm{T}}{d\omega^7}\right) = \frac{79\pi^8}{32}$
etc.	etc.

§. 94. Hinc ergo, omissis valoribus evanescentibus, ex priore ordine habebimus sequentes formulas integrales cum seriebus inde natis

$$\int\frac{dz}{1+zz}=\frac{\pi}{4}=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\frac{1}{11}+\frac{1}{13}-\text{etc.}$$

$$\int\frac{dz(lz)^2}{1+zz}=\frac{\pi^3}{16}=\frac{2}{1^3}-\frac{2}{3^3}+\frac{2}{5^3}-\frac{2}{7^3}+\frac{2}{9^3}-\frac{2}{11^3}+\text{etc.}$$

$$\int\frac{dz(lz)^4}{1+zz}=\frac{5\pi^5}{64}=\frac{24}{1^5}-\frac{24}{3^5}+\frac{24}{5^5}-\frac{24}{7^5}+\frac{24}{9^5}-\frac{24}{11^5}+\text{etc.}$$

$$\int\frac{dz(lz)^6}{1+zz}=\frac{61\pi^7}{256}=\frac{720}{1^7}-\frac{720}{3^7}+\frac{720}{5^7}-\frac{720}{7^7}+\frac{720}{9^7}-\text{etc.}$$

etc. etc. etc.

§. 95. Ex altero autem ordine pro eodem casu oriuntur

$$\int\frac{-dzlz}{1-zz}=\frac{\pi\pi}{8}=\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\frac{1}{9^2}+\frac{1}{11^2}+\frac{1}{13^2}+\text{etc.}$$

$$\int\frac{-dz(lz)^3}{1-zz}=\frac{\pi^4}{16}=\frac{6}{1^4}+\frac{6}{3^4}+\frac{6}{5^4}+\frac{6}{7^4}+\frac{6}{9^4}+\frac{6}{11^4}+\frac{6}{13^4}+\text{etc.}$$

$$\int\frac{-dz(lz)^5}{1-zz}=\frac{\pi^6}{8}=\frac{120}{1^6}+\frac{120}{3^6}+\frac{120}{5^6}+\frac{120}{7^6}+\frac{120}{9^6}+\frac{120}{11^6}+\frac{120}{13^6}\text{etc}$$

etc. etc. 120

§. 96. Quemadmodum ex primo integrali ordinis posterioris deduximus has formulas

$$\int\frac{dzlz}{1-z}=-\frac{\pi\pi}{6},\ \text{et}\int\frac{dzlz}{1+z}=-\frac{\pi\pi}{12},$$

similes quoque formulac integrales ex sequentibus deduci possunt; cum enim sit

$$\int\frac{dz(lz)^3}{1-zz}=-\frac{\pi^4}{16},$$

ponamus esse

$$\int\frac{zdz(lz)^3}{1-zz}=\text{P},\ \text{eritque}$$

$$\int\frac{dz(lz)^3}{1-z}=\text{P}-\frac{\pi^4}{16},\ \text{et}$$

$$\int\frac{dz(lz)^3}{1+z}=-\text{P}-\frac{\pi^4}{16},$$

nunc vero statuatur $zz=v$, ut sit $zdz=\frac{1}{2}dv$, et $lz=\frac{1}{2}lv$, ideoque $(lz)^3=\frac{1}{8}(lv)^3$, quibus subtitutis erit

$$P = \frac{1}{16}\int \frac{dv(lv)^3}{1-v} = \frac{1}{16}\left(P - \frac{\pi^4}{16}\right),$$

unde fit

$$16P = P - \frac{\pi^4}{16}, \text{ ideoque } P = -\frac{\pi^4}{240},$$

sicque has duas habebimus integrationes novas

$$\int \frac{dz(lz)^3}{1-z} = -\frac{\pi^4}{15}, \text{ et}$$

$$\int \frac{dz(lz)^3}{1+z} = -\frac{7\pi^4}{120}:$$

hinc autem per series erit

$$\int \frac{dz(lz)^3}{1-z} = -\frac{\pi^4}{15} = 6\left(1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \frac{1}{5^4} + \frac{1}{6^4} + \frac{1}{7^4} + \frac{1}{8^4} + \text{etc.}\right) \text{ et}$$

$$\int \frac{-dz(lz)^3}{1+z} = +\frac{7\pi^4}{120} = 6\left(1 - \frac{1}{2^4} + \frac{1}{3^4} - \frac{1}{4^4} + \frac{1}{5^4} - \frac{1}{6^4} + \frac{1}{7^4} - \frac{1}{8^4} + \text{etc.}\right).$$

§. 97. Porro cum $\int \frac{dz(lz)^5}{1-zz} = -\frac{\pi^6}{8}$, ponamus esse $\int \frac{dz(lz)^5}{1-zz} = P$, ut hinc obtineamus

$$\int \frac{dz(lz)^5}{1-z} = P - \frac{\pi^6}{8}, \text{ et } \int \frac{dz(lz)^5}{1+z} = -P - \frac{\pi^6}{8},$$

nunc igitur statuamus $zz = v$, eritque

$$P = \frac{1}{64}\int \frac{dv(lv)^5}{1-v} = \frac{1}{64}\left(P - \frac{\pi^6}{8}\right),$$

unde sit

$$P = -\frac{\pi^6}{504},$$

novaeque integrationes hinc deductae sunt

$$\int \frac{dz(lz)^5}{1-z} = -\frac{8\pi^6}{63}, \text{ et}$$

$$\int \frac{dz(lz)^5}{1+z} = -\frac{31\pi^6}{252}:$$

et vero per series reperitur

$$\int \frac{dz(lz)^5}{1-z} = -\frac{8\pi^6}{63} = -120\left(1+\frac{1}{2^6}+\frac{1}{3^6}+\frac{1}{4^6}+\frac{1}{5^6}+\frac{1}{6^6}+\frac{1}{7^6}+\frac{1}{8^6}+\text{etc.}\right) \text{ et}$$

$$\int \frac{dz(lz)^5}{1+z} = -\frac{31\pi^6}{252} = -120\left(1-\frac{1}{2^6}+\frac{1}{3^6}-\frac{1}{4^6}+\frac{1}{5^6}-\frac{1}{6^6}+\frac{1}{7^6}-\frac{1}{8^6}-\text{etc.}\right).$$

ita ut sit

$$1+\frac{1}{2^6}+\frac{1}{3^6}+\frac{1}{4^6}+\frac{1}{5^6}+\frac{1}{6^6}+\frac{1}{7^6}+\frac{1}{8^6}+\text{etc} = \frac{\pi^6}{945}, \text{ et}$$

$$1-\frac{1}{2^6}+\frac{1}{3^6}-\frac{1}{4^6}+\frac{1}{5^6}-\frac{1}{6^6}+\frac{1}{7^6}-\frac{1}{8^6}-\text{etc.} = \frac{31\pi^6}{30240} = \frac{31\pi^6}{32.945}.$$

§. 98. Consideremus etiam casus, quibus $\lambda = 2$ et $\omega = 1$, ita ut sit $\alpha = \frac{\pi}{4}$ et $\alpha\omega = \frac{\pi}{4}$, hinc $p = q = \frac{1}{\sqrt{2}}$, unde pro utroque ordine sequentes habebimus valores

Pro ordine priore	Pro ordine posteriore
$S = \frac{\pi}{2\sqrt{2}}$	$T = \frac{\pi}{4}$
$\left(\frac{dS}{d\omega}\right) = \frac{\pi\pi}{8\sqrt{2}}$	$\left(\frac{dT}{d\omega}\right) = \frac{\pi\pi}{8}$
$\left(\frac{ddS}{d\omega^2}\right) = \frac{3\pi^3}{52\sqrt{2}}$	$\left(\frac{ddT}{d\omega^2}\right) = \frac{\pi^3}{16}$
$\left(\frac{d^3S}{d\omega^3}\right) = \frac{11\pi^4}{128\sqrt{2}}$	$\left(\frac{d^3T}{d\omega^3}\right) = \frac{\pi^4}{16}$
$\left(\frac{d^4S}{d\omega^4}\right) = \frac{57\pi^5}{512\sqrt{2}}$	$\left(\frac{d^4T}{d\omega^4}\right) = \frac{5\pi^5}{64}$
$\left(\frac{d^5S}{d\omega^5}\right) = \frac{361\pi^6}{2048\sqrt{2}}$	$\left(\frac{d^5T}{d\omega^5}\right) = \frac{\pi^6}{8}$
$\left(\frac{d^6S}{d\omega^6}\right) = \frac{2765\pi^7}{8192\sqrt{2}}$	$\left(\frac{d^6T}{d\omega^6}\right) = \frac{61\pi^7}{256}$
$\left(\frac{d^7S}{d\omega^7}\right) = \frac{24611\pi^8}{52768\sqrt{2}}$	$\left(\frac{d^7T}{d\omega^7}\right) = \frac{79\pi^8}{32}$
etc.	etc.

§. 99. Hinc igitur sequentes integrationes, cum seriebus respondentibus resultant; ac primo quidem ex ordine primo

$$\int\frac{(1+zz)dz}{1+z^4}=\frac{\pi}{2\sqrt{2}}=1+\frac{1}{3}-\frac{1}{5}+\frac{1}{7}-\frac{1}{9}+\frac{1}{11}-\frac{1}{13}+\text{etc.}$$
$$\int\frac{-(1-zz)dz(lz)^2}{1+z^4}=\frac{\pi\pi}{8\sqrt{2}}=1-\frac{1}{3^2}-\frac{1}{5^2}+\frac{1}{7^2}+\frac{1}{9^2}-\frac{1}{11^2}-\frac{1}{13^2}+\text{etc.}$$
$$\int\frac{(1+zz)dz(lz)^2}{1+z^4}=\frac{5\pi^3}{32\sqrt{2}}=\frac{2}{1^3}+\frac{2}{3^3}-\frac{2}{5^3}-\frac{2}{7^3}+\frac{2}{9^3}+\frac{2}{11^3}-\frac{2}{13^3}-\text{etc.}$$
$$\int\frac{-(1-zz)dz(lz)^3}{1+z^4}=\frac{11\pi^4}{128\sqrt{2}}=\frac{6}{1^4}-\frac{6}{3^4}-\frac{6}{5^4}+\frac{6}{7^4}+\frac{6}{9^4}-\frac{6}{11^4}-\frac{6}{13^4}+\text{etc.}$$
$$\int\frac{(1+zz)dz(lz)^4}{1+z^4}=\frac{57\pi^5}{512\sqrt{2}}=\frac{24}{1^5}+\frac{24}{3^5}-\frac{24}{5^5}-\frac{24}{7^5}+\frac{24}{9^5}+\frac{24}{11^5}-\frac{24}{13^5}-\text{etc.}$$
$$\int\frac{-(1-zz)dz(lz)^5}{1+z^4}=\frac{361\pi^6}{2048\sqrt{2}}=\frac{120}{1^6}-\frac{120}{3^6}-\frac{120}{5^6}+\frac{120}{7^6}+\frac{120}{9^6}-\frac{120}{11^6}-\frac{120}{13^6}+\text{etc.}$$
$$\int\frac{(1+zz)dz(lz)^6}{1+z^4}=\frac{2763\pi^7}{8192\sqrt{2}}=\frac{720}{1^7}+\frac{720}{3^7}-\frac{720}{5^7}-\frac{720}{7^7}+\frac{720}{9^7}+\frac{720}{11^7}-\frac{720}{13^7}-\text{etc.}$$
$$\int\frac{-(1-zz)dz(lz)^7}{1+z^4}=\frac{24611\pi^8}{52768\sqrt{2}}=\frac{5040}{1^8}-\frac{5040}{3^8}-\frac{5040}{5^8}+\frac{5040}{7^8}+\frac{5040}{9^8}-\frac{5040}{11^8}-\frac{5040}{13^8}+\text{etc.}$$
etc. etc. etc.

§. 100. Eodem modo integrationes alterius ordinis cum seriebus erunt

$$\int\frac{dz}{1+zz}=\frac{\pi}{4}=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\frac{1}{11}+\frac{1}{13}-\text{etc.}$$
$$\int\frac{-dzlz}{1-zz}=\frac{\pi\pi}{8}=\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\frac{1}{9^2}+\frac{1}{11^2}+\frac{1}{13^2}+\text{etc.}$$
$$\int\frac{dz(lz)^2}{1+zz}=\frac{\pi^3}{16}=\frac{2}{1^3}-\frac{2}{3^3}+\frac{2}{5^3}-\frac{2}{7^3}+\frac{2}{9^3}-\frac{2}{11^3}+\frac{2}{13^3}-\text{etc.}$$
$$\int\frac{-dz(lz)^3}{1-zz}=\frac{\pi^4}{16}=\frac{6}{1^4}+\frac{6}{3^4}+\frac{6}{5^4}+\frac{6}{7^4}+\frac{6}{9^4}+\frac{6}{11^4}+\frac{6}{13^4}+\text{etc.}$$
$$\int\frac{dz(lz)^4}{1+zz}=\frac{5\pi^5}{64}=\frac{24}{1^5}-\frac{24}{3^5}+\frac{24}{5^5}-\frac{24}{7^5}+\frac{24}{9^5}-\frac{24}{11^5}+\frac{24}{13^5}-\text{etc.}$$
$$\int\frac{-dz(lz)^5}{1-zz}=\frac{\pi^6}{16}=\frac{120}{1^6}+\frac{120}{3^6}+\frac{120}{5^6}+\frac{120}{7^6}+\frac{120}{9^6}+\frac{120}{11^6}+\text{etc.}$$
$$\int\frac{dz(lz)^6}{1+zz}=\frac{61\pi^7}{256}=\frac{720}{1^7}-\frac{720}{3^7}+\frac{720}{5^7}-\frac{720}{7^7}+\frac{720}{9^7}-\frac{720}{11^7}+\text{etc.}$$
$$\int\frac{-dz(lz)^7}{1-zz}=\frac{79\pi^8}{32}=\frac{5040}{1^8}+\frac{5040}{3^8}+\frac{5040}{5^8}+\frac{5040}{7^8}+\frac{5040}{9^8}+\frac{5040}{11^8}+\frac{5040}{13^8}+\text{etc.}$$
etc. etc. etc.

Haec antem series sunt eae ipsae, quas jam supra §§. 94. et 95. sumus consecuti.

§. 101. Praeterea autem ii casus imprimis notari merentur, quibus formulae integrales in formas simpliciores resolvi possunt. Haec autem resolutio tantum spectat ad fractionem

$$\pm\frac{z^{\lambda-\omega}+z^{\lambda+\omega}}{1\pm z^{2\lambda}},$$

omisso facture $\frac{dz}{z}(lz)^{\mu}$; ad quod ostendendum sumamus primo $\lambda=3$ et $\omega=1$, unde fit $\alpha=\frac{\pi}{6}$, $p=\sin.\frac{\pi}{6}$, et $q=\cos.\frac{\pi}{6}$, tum autem, in priori ordine occurrunt alternatim sequentes fractiones

$$\text{I.}\quad \frac{zz(1+zz)}{1+z^6}=\frac{zz}{1-zz+z^4},$$

quae posito $zz=v$ abit in $\frac{v}{1-v+v^2}$; ergo cum sit $\frac{dz}{z}=\frac{1}{2}\frac{dv}{v}$, et $lz=\frac{1}{2}lv$,
hinc talis forma

$$\frac{1}{2^{2i+1}}\int\frac{dv(lv)^{2i}}{1-v+v^2}$$

integrari poterit, casu scilicet $v=1$.

$$\text{II.}\quad -\frac{zz(1-zz)}{1+z^6}=+\frac{2}{3(1+zz)}-\frac{(1-2zz)}{3\left(1-zz+z^4\right)},$$

quae posito $zz=v$, abit in $\frac{2}{3(1+v)}+\frac{(2-v)}{3\left(1-v+v^2\right)}$, quae ergo forma ducta in

in $\frac{dz}{z}(lz)^{2i+1}$ vel in

$$\frac{1}{2^{2i+2}}\frac{dv}{v}(lv)^{2i+1},$$

semper integrari potest posito $v=1$.

§.102. Eodem casu ordo posterior sequentes suppeditat resolutiones

$$\text{I.}\quad \frac{zz(1-zz)}{1-z^6}=\frac{zz}{1+zz+z^4}=\frac{v}{1+v+v^2},$$

quae in $\frac{dz}{z}(lz)^{2i}$, vel in $\frac{1}{2^{2i+2}}\frac{dv}{v}(lv)^{2i}$, ducta semper est integrabilis.

$$\text{II.}\quad \frac{-zz(1+zz)}{1-z^6}=\frac{-2}{3(1-zz)}+\frac{2+zz}{3\left(1+zz+z^4\right)}=\frac{v}{1+v+v^2},$$

quae facto $zz=v$ fit

$$\frac{-2}{3(1-v)}+\frac{2+v}{3\left(1+v+v^2\right)},$$

quae ergo formulae in $\frac{dv}{v}(lv)^{2i+1}$, ductae fiunt integrabiles; quia autem in hac resolutione numeratores per *z* vel *v* dividere non licet, alia resolutione est opus, quae reperitur

$$\frac{-zz(1+zz)}{1-z^6}=\frac{-2zz}{3(1-zz)}-\frac{zz(1+2zz)}{3\left(1+zz+z^4\right)},\ \text{sive}\ \frac{-2v}{3(1-v)}-\frac{v(1+2v)}{3(1+v+vv)},$$

quae formulae ductae in $\frac{dz}{z}(lz)^{2i+1}$, vel in $\frac{1}{2^{2i+2}}\frac{dv}{v}(lv)^{2i+1}$, integrationem quoque admittunt.

§. 103. Porro manente $\lambda=3$ et $\omega=2$, unde fit $\alpha=\frac{\pi}{6}$, $p=\sin.\frac{\pi}{3}$, et $q=\cos.\frac{\pi}{3}$, et ex ordine priore oriuntur sequentes reductiones.

$$\text{I.}\ \frac{z\left(1+z^4\right)}{1+z^6}=\frac{2z}{3(1+zz)}+\frac{z(1+zz)}{3\left(1-zz+z^4\right)},$$

unde multiplicando per $\frac{dz}{z}(lz)^{2i}$ oriuntur formulae integrationem admittentes casu $z=1$.

$$\text{II.}\ \frac{-z\left(1-z^4\right)}{1+z^6}=-\frac{z(1-zz)}{1-zz+z^4},$$

quae per $\frac{dz}{z}(lz)^{2i+1}$ multiplicata integrari poterit casu $z=1$. Ex ordine vero posteriori sequentes prodibunt reductiones.

$$\text{I.}\ \frac{z\left(1-z^4\right)}{1-z^6}=\frac{z(1+zz)}{1+zz+z^4},$$

quae ducta in $\frac{dz}{z}(lz)^{2i}$ fit integrabilis.

$$\text{II.}\ \frac{-z\left(1+z^4\right)}{1-z^6}=\frac{-2z}{3(1-zz)}-\frac{z(1-zz)}{3\left(1+zz+z^4\right)},$$

quae formulae in $\frac{dz}{z}(lz)^{2i+1}$ ductae fiunt integrabiles.

§. 104. Operae jam erit pretium haec integralia actu evolvere, quare ex §. 101. ubi $\omega=1$, ejusque numero I nanciscimur sequentes integrationes

$$1^\circ.\ \frac{1}{2}\int\frac{v}{1-v+v^2}=\alpha\cdot\frac{1}{q}=\frac{\pi}{3\sqrt{3}}$$

$$2^\circ.\ \frac{1}{8}\int\frac{dv(lv)^2}{1-v+v^2}=\alpha^3\left(\frac{2}{q^3}-\frac{1}{q}\right)=\frac{5\pi^3}{324\sqrt{3}},$$

deinde vero ex ejusdem §. numero II. ubi etiam haec reductio locum habet

$$-\frac{zz(1-zz)}{1+z^6}=-\frac{2zz}{3(1+zz)}-\frac{zz(1-2zz)}{3\left(1+zz+z^4\right)}=-\frac{2v}{3(1+v)}-\frac{v(1-2v)}{3(1-v+vv)},$$

quae ducta in $\frac{1}{4}\cdot\frac{dv}{v}lv$ dabit

$$-\frac{1}{6}\int\frac{dvlv}{(1+v)}-\frac{1}{12}\int\frac{dv(1-2v)lv}{(1-v+vv)}=\alpha\alpha\frac{p}{qq}=\frac{\pi\pi}{54},$$

quarum formularum prior integrationem admittit, est enim

$$\int\frac{dvlv}{1+v}=-\frac{\pi\pi}{12},$$

unde invenitur posterior

$$\int\frac{dv(1-2v)lv}{(1-v+vv)}=-\frac{\pi\pi}{18}.$$

§. 105. Ex §.102. ejusque numero 1 sequitur

$$1^0.\ \frac{1}{2}\int\frac{dv}{(1+v+vv)}=\frac{\alpha p}{q}=\frac{\pi}{6\sqrt{3}}$$

$$2^0.\ \frac{1}{8}\int\frac{dv(lv)^2}{(1+v+vv)}=\alpha^3\frac{2p}{q^3}=\frac{\pi^3}{81\sqrt{3}};$$

deinde vero ex numero II fit

$$-\frac{1}{6}\int\frac{dvlv}{1-v}-\frac{1}{12}\int\frac{dv(1+2v)lv}{1+v+vv}=\alpha\alpha\frac{1}{qq}=\frac{\pi\pi}{27};$$

supra autem invenimus esse

$$\int\frac{dvlv}{1-v}=-\frac{\pi\pi}{6};$$

quo valore substituto fit

$$\int\frac{dv(1+2v)lv}{1+v+vv}=-\frac{\pi\pi}{9}:$$

maxime igitur operae pretium est visum, has postremas integrationes evolvisse.

§.106. Quod si ambae formulae integrales

$$\int\frac{dv(1-2v)lv}{1-v+vv}\text{ et }\int\frac{dv(1+2v)lv}{1+v+vv}$$

in series convertantur, reperitur

$$\int\frac{dv(1-2v)lv}{1-v+vv}=-1+\frac{1}{4}+\frac{2}{9}+\frac{1}{16}-\frac{1}{25}-\frac{2}{36}-\frac{1}{49}+\text{etc. et}$$
$$\int\frac{dv(1+2v)lv}{1+v+vv}=-1-\frac{1}{4}+\frac{2}{9}-\frac{1}{16}-\frac{1}{25}+\frac{2}{36}-\frac{1}{49}-\text{etc.}$$

unde has duas summationes attentione nostra non indignas assequimur

$$\text{I.}\quad 1-\frac{1}{4}-\frac{2}{9}-\frac{1}{16}+\frac{1}{25}+\frac{2}{36}+\frac{1}{49}-\frac{1}{64}-\frac{2}{81}-\frac{1}{100}+\text{etc.}=\frac{\pi\pi}{18},$$
$$\text{II.}\quad 1+\frac{1}{4}-\frac{2}{9}+\frac{1}{16}+\frac{1}{25}-\frac{2}{36}+\frac{1}{49}+\frac{1}{64}-\frac{2}{81}+\frac{1}{100}+\text{etc.}=\frac{\pi\pi}{9},$$

quarum prior a posteriore ablata praebet

$$\frac{2}{4}+\frac{2}{16}-\frac{4}{36}+\frac{2}{64}+\frac{2}{100}\text{etc.}=\frac{\pi\pi}{18},$$

cujus duplum perducit ad hanc

$$1+\frac{1}{4}-\frac{2}{9}+\frac{1}{16}+\frac{1}{25}-\frac{2}{36}+\text{etc.}=\frac{\pi\pi}{9},$$

quae quoniam cum secunda congruit, veritas utriusque summationis satis confirmatur. Quod si vero secunda a duplo primae subtrahatur, remanebit ista series memorabilis

$$1-\frac{3}{4}-\frac{2}{9}-\frac{3}{16}+\frac{1}{25}+\frac{6}{36}+\frac{1}{49}-\frac{3}{64}-\frac{2}{81}-\frac{3}{100}+\text{etc.}=0$$

quae in periodos sex terminas complectentes distributa, manifestum ordinem in numerationibus declarat, quippe qui sunt

$$1-3-2-3+1+6\ .$$

Additamentum.

§. 107. Quemadmodum superiores integrationes per continuam differentiationem formularum S et T deduximus, ita etiam per integrationem alias et prorsus singulares integrationes impetrabimus; si enim ut supra fuerit $\mathrm{S}=\int\frac{\mathrm{T}dz}{z}$ existente T formula illa

$$+\frac{z^{\lambda-\omega}\pm z^{\lambda+\omega}}{1\pm z^{2\lambda}},$$

quae praeter z etiam exponentem variabilem ω involvere concipitur, erit per naturam integralium duas variabiles involventium

$$\int\mathrm{S}d\omega=\int\frac{dz}{z}\int\mathrm{T}d\omega,$$

ubi in priore formula integrali $\int\mathrm{S}d\omega$, ubi z pro constanti habetur, statim scribi potest $z=1$; hoc igitur lemmate praemisso, quia est $\int\mathrm{T}d\omega=\frac{-z^{\lambda-\omega}\pm z^{\lambda+\omega}}{(1\pm z^{2\lambda})lz}$, ambas formulas supra tractatas nempe S et T hoc modo evolvamus, et quia utramque triplici modo expressam dedimus; primo scilicet per seriem infinitam, secundo, per formulam finitam, ac tertio per formulam integralem, etiam qantitates, quae pro integralibus $\int \mathrm{S}d\omega$ et $\int \mathrm{T}d\omega$, resultabunt, erunt inter se aequales.

§. 108. Incipiamus a formula S, et cum per seriem fuerit

$$S = \frac{1}{\lambda-\omega} + \frac{1}{\lambda+\omega} - \frac{1}{3\lambda-\omega} - \frac{1}{3\lambda+\omega} + \frac{1}{5\lambda-\omega} + \frac{1}{5\lambda+\omega} - \text{etc.},$$

erit

$$\int \mathrm{S}d\omega = -l(\lambda-\omega) + l(\lambda+\omega) + l(3\lambda-\omega) - l(3\lambda+\omega) - \text{etc.} + \mathrm{C},$$

quam constantem ita definire decet, ut integrale evanescat posito $\omega = 0$, quo facto erit

$$\int \mathrm{S}d\omega = l\frac{\lambda+\omega}{\lambda-\omega} + l\frac{5\lambda-\omega}{5\lambda+\omega} + l\frac{5\lambda+\omega}{5\lambda-\omega} + l\frac{7\lambda-\omega}{7\lambda+\omega} + \text{etc.}$$

quae expessio reducitur ad sequentem

$$\int \mathrm{S}d\omega = l\frac{(\lambda+\omega)(5\lambda-\omega)(5\lambda+\omega)(7\lambda-\omega)(9\lambda+\omega)\text{etc.}}{(\lambda-\omega)(5\lambda+\omega)(5\lambda-\omega)(7\lambda+\omega)(9\lambda-\omega)\text{etc.}}.$$

Deinde quia per formulam finitam erat

$$\mathrm{S} = \frac{\pi}{2\lambda\cos.\frac{\pi\omega}{2\lambda}}, \text{ erit } \int \mathrm{S}d\omega = \int \frac{\pi d\omega}{2\lambda\cos.\frac{\pi\omega}{2\lambda}},$$

ubi si brevitatis gratia ponatur $\frac{\pi\omega}{2\lambda} = \varphi$, ut sit

$$d\omega = \frac{2\lambda d\varphi}{\pi}, \text{ erit } \int \mathrm{S}d\omega = \int \frac{d\varphi}{\cos.\varphi};$$

quia igitur novimus esse

$$\int \frac{d\theta}{\sin.\theta} = l \text{ tang.} \tfrac{1}{2}\theta,$$

sumamus $\sin.\theta = \cos.\varphi$, sive $\theta = 90° - \varphi = \frac{\pi}{2} - \varphi$, eritque $d\theta = -d\varphi$, unde fit

$$\int \frac{-d\varphi}{\cos.\varphi} = l \text{ tang.}\left(\tfrac{\pi}{4} - \tfrac{1}{2}\varphi\right);$$

quoniam autem est

$$\varphi = \frac{\pi\omega}{2\lambda}, \text{ erit } \frac{\pi}{4} - \frac{1}{2}\varphi = \frac{\pi(\lambda-\omega)}{2\lambda},$$

unde nostrum integrale erit

$$\int \mathrm{S}d\omega = -l \text{ tang.}\frac{\pi(\lambda-\omega)}{4\lambda} = +l \text{ tang.}\frac{\pi(\lambda+\omega)}{4\lambda}.$$

Ex tertia autem formula integrali

$$\mathrm{S} = \int \frac{z^{\lambda-\omega}+z^{\lambda+\omega}}{1+z^{2\lambda}} \cdot \frac{dz}{z}$$

colligitur fore

$$\int \mathrm{S}d\omega = \int \frac{-z^{\lambda-\omega}+z^{\lambda+\omega}}{1+z^{2\lambda}} \cdot \frac{dz}{zlz}$$

quod integrale a termina $z=0$ usque ad terminnm $z=1$ extendi assumitur; sicque tres isti valores inventi inter se erunt aequales. Ac ne ob constantes forte addendas ullum dubium supersit, singulae istae expressiones sponte evanescunt casu $\omega=0$.

§. 109. Consideremus hinc primo aequalitatem inter formulam primam et secundam: et quia utraque est logarithmus, erit

$$\text{tang}\frac{\pi(\lambda+\omega)}{4\lambda} = \frac{(\lambda+\omega)(3\lambda-\omega)(5\lambda+\omega)(7\lambda-\omega) \text{ etc.}}{(\lambda-\omega)(3\lambda+\omega)(5\lambda-\omega)(7\lambda+\omega) \text{ etc.}}$$

cum igitur hujus fractionis numerator evanescat casibus, vel $\omega=-\lambda$, vel $\omega=+3\lambda$, vel $\omega=-5\lambda$, vel $\omega=+7\lambda$ etc. evidens est iisdem casibus quoque tangentem fieri $=0$; denominator vero evanescit casibus vel $\omega=\lambda$, vel $\omega=-3\lambda$, vel $\omega=+5\lambda$, vel $\omega=-7\lambda$ etc. quibus ergo casibus tangens in infinitum excrescere debet, id quod etiam pulcherrime evenit. Caeterum haec expressio congruit cum ea, quam jam dudum inveni et in introductione exposui.

§. 110. Productum autem istud infinitum per principia alibi stabilita ad formulas integrales reduci potest ope hujus lemmatis latissime patentis

$$\frac{a(c+b)(a+k)(c+b+k)(a+2k)(c+b+2k) \text{ etc.}}{b(c+a)(b+k)(c+a+k)(b+2k)(c+a+2k) \text{ etc}} = \frac{\int z^{c-1}dz(1-z^k)^{\frac{b-k}{k}}}{\int z^{c-1}dz(1-z^k)^{\frac{a-k}{k}}},$$

si quidem post utramque integrationem fiat $z=1$. Nostro igitur casu erit $a=\lambda+\omega$, $b=\lambda-\omega$, $c=2\lambda$, et $k=4\lambda$; unde valor nostri producti erit

$$\frac{\int z^{2\lambda-1}dz\left(1-z^{2\lambda}\right)^{\frac{-3\lambda-\omega}{4\lambda}}}{\int z^{2\lambda-1}dz\left(1-z^{4\lambda}\right)^{\frac{-3\lambda+\omega}{4\lambda}}} = \text{tang.}\frac{\pi(\lambda+\omega)}{4\lambda}:$$

formulae autem istae integrales concinniores evadunt, statuendo $z^{2\lambda} = y$, tum enim erit

$$\text{tang.}\frac{\pi(\lambda+\omega)}{4\lambda} = \frac{\int dy(1-yy)^{\frac{-3\lambda-\omega}{4\lambda}}}{\int dy(1-yy)^{\frac{-3\lambda+\omega}{4\lambda}}},$$

quae expressio utique omni attentione digna videtur. Denique ex formula integrali inventa erit quoque

$$\int \frac{-z^{\lambda-\omega}+z^{\lambda+\omega}}{1+z^{2\lambda}}\cdot\frac{dz}{zlz} = l\,\text{tang.}\frac{\pi(\lambda+\omega)}{4\lambda}.$$

§. 111. Operae erit pretium, etiam aliquot casus particulares evolvere: sit igitur primo $\lambda = 2$ et $\omega = 1$, ac per expressionem infinitam erit

$$\int \text{S}d\omega = l\,\frac{3.5}{1.7}\cdot\frac{11.13}{9.15}\cdot\frac{19.21}{17.23}\cdot\frac{27.29}{25.31}\cdot\frac{35.37}{33.39}\cdot\text{etc.}$$

deinde per expressionem finitam habebimus

$$\int \text{S}d\omega = l\,\text{tang.}\frac{3\pi}{8},$$

at per formulam integralem

$$\int \text{S}d\omega = \int \frac{-(1-zz)}{1+z^4}\cdot\frac{dz}{lz}.$$

Tum vero ex aequalitate duarum priorum expressionum

$$\text{tang.}\frac{3\pi}{8} = \frac{3.5}{1.7}\cdot\frac{11.13}{9.15}\cdot\frac{19.21}{17.23}\cdot\text{etc.}$$

hincque per binas formulas integrales

$$\text{tang.}\frac{3\pi}{8} = \frac{\int dy(1-yy)^{-\frac{7}{8}}}{\int dy(1-yy)^{-\frac{5}{8}}}.$$

§. 112. Ponamus nunc esse $\lambda = 3$ et $\omega = 1$, ac per expressionem infinitam erit

$$\int \text{S}d\omega = l\,\frac{2}{1}\cdot\frac{4}{5}\cdot\frac{8}{7}\cdot\frac{10}{11}\cdot\frac{14}{15}\cdot\frac{16}{17}\cdot\frac{20}{19}\cdot\frac{22}{23}\,\text{etc.}$$

secundo, per expressionem finitam

$$\int \mathrm{S}d\omega = l\,\text{tang.}\tfrac{\pi}{5} = l\sqrt{3} = \tfrac{1}{2}l3,$$

ita, ut futurum sit

$$\sqrt{3} = \tfrac{2.4}{1.5}\cdot\tfrac{8.10}{7.11}\cdot\tfrac{14.16}{13.17}\ \text{etc.}$$

hujusque producti valor per formulas integrales erit

$$\frac{\int dy(1-yy)^{-\frac{5}{6}}}{\int dy(1-yy)^{-\frac{2}{3}}}.$$

Denique formula integralis praebebit

$$\int \mathrm{S}d\omega = \int \tfrac{-(1-zz)}{1+z^6}\cdot\tfrac{dz}{lz}.$$

§. 113. Eodem modo etiam evolvamus alteram formulam T, cujus valor per seriem erat

$$\mathrm{T} = \tfrac{1}{\lambda-\omega} - \tfrac{1}{\lambda+\omega} + \tfrac{1}{3\lambda-\omega} - \tfrac{1}{3\lambda+\omega} + \tfrac{1}{5\lambda-\omega} - \tfrac{1}{5\lambda+\omega} + \text{etc.},$$

unde sit

$$\int \mathrm{T}d\omega = -l(\lambda-\omega) - l(\lambda+\omega) - l(3\lambda-\omega) - l(3\lambda+\omega) - \text{etc.}$$

quae expressio, ut evanescat posito $\omega = 0$, erit

$$\int \mathrm{T}d\omega = l\,\tfrac{\lambda\lambda}{\lambda\lambda-\omega\omega}\cdot\tfrac{9\lambda\lambda}{9\lambda\lambda-\omega\omega}\cdot\tfrac{25\lambda\lambda}{25\lambda\lambda-\omega\omega}\ \text{etc.}$$

deinde vero cum per formulam finitam fuerit

$$\mathrm{T} = \tfrac{\pi}{2\lambda}\,\text{tang.}\tfrac{\pi\omega}{2\lambda},\ \text{erit}$$

$$\int \mathrm{T}d\omega = \int \tfrac{\pi d\omega}{2\lambda}\,\text{tang.}\tfrac{\pi\omega}{2\lambda},\ \text{ubi posito}\ \ \tfrac{\pi\omega}{2\lambda} = \varphi,\ \text{erit}$$

$$\int \mathrm{T}d\omega = \int d\varphi\,\text{tang.}\varphi \ = -l\cos.\varphi,\ \text{ita ut sit}$$

$$\int \mathrm{T}d\omega = -l\cos.\tfrac{\pi\omega}{2\lambda};$$

cujus valor casu $\omega = 0$ fit sponte $= 0$; denique per formulam integralem habebimus

$$\int \mathrm{T}d\omega = \frac{z^{\lambda-\omega} - z^{\lambda+\omega}}{1-z^{2\lambda}} \cdot \frac{dz}{zlz},$$

ubi integrale itidem a termina $z=0$ usque ad terminum $z=1$ extendi debet.

§. 114. Jam comparatio duorum priorum valorum hanc praebet aequationem.

$$\frac{1}{\cos.\frac{\pi\omega}{2\lambda}} = \frac{\lambda\lambda}{\lambda\lambda-\omega\omega} \cdot \frac{9\lambda\lambda}{9\lambda\lambda-\omega\omega} \cdot \frac{25\lambda\lambda}{25\lambda\lambda-\omega\omega} \cdot \frac{49\lambda\lambda}{49\lambda\lambda-\omega\omega} \text{ etc. vel}$$

$$\cos.\frac{\pi\omega}{2\lambda} = \left(1-\frac{\omega\omega}{\lambda\lambda}\right)\cdot\left(1-\frac{\omega\omega}{9\lambda\lambda}\right)\left(1-\frac{\omega\omega}{25\lambda\lambda}\right)\left(1-\frac{\omega\omega}{49\lambda\lambda}\right) \text{ etc.}$$

vel si factores singuli iterum in simplices evolvantur,

$$\cos.\frac{\pi\omega}{2\lambda} = \left(\frac{\lambda+\omega}{\lambda}\right)\cdot\left(\frac{\lambda-\omega}{\lambda}\right)\cdot\left(\frac{3\lambda+\omega}{3\lambda}\right)\cdot\left(\frac{3\lambda-\omega}{3\lambda}\right)\cdot\left(\frac{5\lambda+\omega}{5\lambda}\right)\cdot\left(\frac{5\lambda-\omega}{5\lambda}\right) \text{ etc.}$$

quae formula cum reductione generali supra allata comparata dat,
$a=\lambda+\omega$, $b=\lambda$, $c=-\omega$, et $k=2\lambda$, unde colligimus

$$\cos.\frac{\pi\omega}{2\lambda} = \frac{\int z^{-\omega-1}dz\left(1-z^{2\lambda}\right)^{-\frac{1}{2}}}{\int z^{-\omega-1}dz\left(1-z^{2\lambda}\right)^{\frac{\omega-\lambda}{2\lambda}}}.$$

Ut autem exponentes negativos $z^{-\omega-1}$ evitemus, superius productum ita repraesentemus

$$\cos.\frac{\pi\omega}{2\lambda} = \frac{\lambda-\omega}{\lambda}\cdot\frac{\lambda+\omega}{\lambda}\cdot\left(\frac{3\lambda-\omega}{3\lambda}\right)\cdot\frac{3\lambda+\omega}{3\lambda}\text{etc.}$$

eritque facta comparatione $a=\lambda-\omega$, $b=\lambda$, $c=+\omega$, et $k=2\lambda$, sicque per formulas integrales erit

$$\cos.\frac{\pi\omega}{2\lambda} = \frac{\int z^{\omega-1}dz\left(1-z^{2\lambda}\right)^{-\frac{1}{2}}}{\int z^{\omega-1}dz\left(1-z^{2\lambda}\right)^{\frac{-\lambda-\omega}{2\lambda}}},$$

quae expressio ad simpliciorem formam reduci nequit.

§. 115. Sit nunc etiam $\lambda = 2$, et $\omega = 1$, eruntque ternae nostrae expressiones

$$\text{I. } \int \mathrm{T} d\omega = l\, \tfrac{4}{3} \cdot \tfrac{36}{35} \cdot \tfrac{100}{99} \cdot \tfrac{196}{195} \text{ etc. sive}$$

$$\int \mathrm{T} d\omega = l\, \tfrac{2.2}{1.3} \cdot \tfrac{6.6}{5.7} \cdot \tfrac{10.10}{9.11} \cdot \tfrac{14.14}{13.15} \text{ etc.}$$

$$\text{II. } \int \mathrm{T} d\omega = -l \cos.\tfrac{\pi}{4} = +\tfrac{1}{2} l 2, \text{ita ut sit}$$

$$\sqrt{2} = l\, \tfrac{2.2}{1.3} \cdot \tfrac{6.6}{5.7} \cdot \tfrac{10.10}{9.11} \cdot \tfrac{14.14}{13.15} \text{ etc.}$$

quod productum per formulas integrales ita exprimitur

$$\frac{\int dz\left(1-z^4\right)^{-\frac{1}{2}}}{\int dz\left(1-z^4\right)^{-\frac{5}{4}}} = \tfrac{1}{2}\sqrt{2}:$$

$$\text{III. } \int \mathrm{T} d\omega = \int \frac{-(1+zz)}{1-z^4} \cdot \frac{dz}{lz} = \int \frac{-dz}{(1-zz)lz};$$

quod ergo integrale a tcrmino $z = 0$ usque ad $z = 1$ extensum praebet eundem valorem $+\tfrac{1}{2}\sqrt{2}$, cujus aequalitatis ratio utique difficillime patet.

§. 116. Sit denique ut supra $\lambda = 3$, et $\omega = 1$, ac ternae formulae ita se habebunt

$$\text{I. } \int \mathrm{T} d\omega = l\, \tfrac{9}{8} \cdot \tfrac{81}{80} \cdot \tfrac{225}{221} \text{ etc. } = l\, \tfrac{3.3}{2.4} \cdot \tfrac{9.9}{8.10} \cdot \tfrac{15.15}{14.16} \cdot \tfrac{21.21}{20.22} \text{ etc.}$$

$$\text{II. } \int \mathrm{T} d\omega = -l \cos.\tfrac{\pi}{6} = -l\, \tfrac{\sqrt{3}}{2} = l\, \tfrac{2}{\sqrt{3}}, \text{ita ut sit}$$

$$\tfrac{2}{\sqrt{3}} = \tfrac{3.3}{2.4} \cdot \tfrac{9.9}{8.10} \cdot \tfrac{15.15}{14.16} \cdot \tfrac{21.21}{20.22};$$

ideoque per binas formulas integrales

$$\tfrac{3}{4} \cdot \tfrac{2}{\sqrt{3}} = \frac{\int dz\left(1-z^6\right)^{-\frac{1}{2}}}{\int dz\left(1-z^6\right)^{-\frac{1}{5}}}.$$

$$\text{III. } \int \mathrm{T} d\omega = \int \frac{-(1+zz)}{1-z^6} \cdot \frac{dz}{lz},$$

quae posito $zz = v$ abit in hanc

$$\int \mathrm{T} d\omega = \int \frac{-dv(1+v)}{\left(1-v^3\right)lv}.$$

Hinc igitur patet, hac methode plane nova perveniri ad formulas integrales, quas per methodes adhuc cognitas nulle modo evolvere, vel saltem inter se comparare, licuit.