5) Investigation of the integral formula $\int \frac{x^{m-1}\partial x}{\left(1+x^k\right)^n}$, in the case where there may be put $x=\infty$ after the integration.

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§. 141. Now the integral of this formula is known well-enough, to include in the case $n=1$ logarithmic parts and another part to include circular arcs, and the logarithmic part to constitute this progression :

$$-\tfrac{2}{k}\cos.\tfrac{mx}{k}\,l\sqrt{\left(1-2x\cos.\tfrac{x}{k}+xx\right)}$$
$$-\tfrac{2}{k}\cos.\tfrac{3mx}{k}\,l\sqrt{\left(1-2x\cos.\tfrac{3x}{k}+xx\right)}$$
$$-\tfrac{2}{k}\cos.\tfrac{5mx}{k}\,l\sqrt{\left(1-2x\cos.\tfrac{5x}{k}+xx\right)}$$
$$-\tfrac{2}{k}\cos.\tfrac{7mx}{k}\,l\sqrt{\left(1-2x\cos.\tfrac{7x}{k}+xx\right)}$$
$$\vdots$$
$$-\tfrac{2}{k}\cos.\tfrac{imx}{k}\,l\sqrt{\left(1-2x\cos.\tfrac{ix}{k}+xx\right)},$$

where *i* denotes an odd number not greater than *k*. Hence if *k* were an even number, there will be $i=k-1$; and if *k* were an odd number, this progression would be required to be continued as far as to $i=k$, truly the coefficient must be taken smaller by a factor of two or in place of $-\frac{2}{k}$ only $-\frac{1}{k}$ must be written , the account of which irregularities had been presented in the *Calculo Integrali*. [See §. 77, Ch. I, vol. I, also E462 .]

§. 142. Now since these parts vanish spontaneously on putting $x=0$, we may put $x=\infty$ at once, and since in general there shall become

$$\sqrt{\left(1-2x\cos.\omega+xx\right)}=x-\cos.\omega,$$

there will be

$$l\sqrt{\left(1-2x\cos.\omega+xx\right)}=l\left(x-\cos.\omega\right)=lx-\tfrac{\cos.\omega}{x}=lx,\ \text{on account of } \tfrac{\cos.\omega}{x}=0;$$

therefore all these logarithms are reduced to the same form *lx,* which is required to be multiplied by this series :

$$-\tfrac{2}{k}\cos.\tfrac{m\pi}{k}-\tfrac{2}{k}\cos.\tfrac{3m\pi}{k}-\tfrac{2}{k}\cos.\tfrac{5m\pi}{k}-\tfrac{2}{k}\cos.\tfrac{7m\pi}{k}\cdots-\tfrac{2}{k}\cos.\tfrac{im\pi}{k},$$

where, as we have said, *i* specifies the maximum odd number not greater than *k* itself, yet with this restraint, so that, k were odd and thus $i=k$, the final term must be reduced by half. On account of which if we wish to investigate the sum of this progression, two cases are required to be put in place, the one, where *k* is an even number and $i=k-1$, the other indeed, where *k* is odd and $i=k$.

Establishing the first case, where *k* is an even number and $i=k-1$.

§. 143. Therefore in this case on putting $x=\infty$ the formula $-\frac{2}{k}lx$ is multiplied by this series of cosines :

$$\cos.\tfrac{mx}{k}+\cos.\tfrac{3mx}{k}+\cos.\tfrac{5mx}{k}+\cos.\tfrac{7mx}{k}\cdots+\cos.\tfrac{(k-1)mx}{k},$$

the sum of which we may put $=\mathrm{S}$. We may multiply this series by $\sin.\frac{m\pi}{k}$, and since in general there shall be

$$\sin.\tfrac{m\pi}{k}\cos.\tfrac{im\pi}{k}=\tfrac{1}{2}\sin.\tfrac{(i+1)m\pi}{k}-\tfrac{1}{2}\sin.\tfrac{(i-1)m\pi}{k},$$

with this reduction made we will have :

$$\begin{aligned}&\mathrm{S}\sin.\tfrac{m\pi}{k}\\&=\tfrac{1}{2}\sin.\tfrac{2m\pi}{k}+\tfrac{1}{2}\sin.\tfrac{4m\pi}{k}+\tfrac{1}{2}\sin.\tfrac{6m\pi}{k}+\cdots+\tfrac{1}{2}\sin.\tfrac{(k-2)m\pi}{k}+\tfrac{1}{2}\sin.\tfrac{km\pi}{k}\\&-\tfrac{1}{2}\sin.\tfrac{2m\pi}{k}-\tfrac{1}{2}\sin.\tfrac{4m\pi}{k}-\tfrac{1}{2}\sin.\tfrac{6m\pi}{k}-\cdots-\tfrac{1}{2}\sin.\tfrac{(k-2)m\pi}{k},\end{aligned}$$

where all the terms except the last evidently cancel each other, thus so that there shall be

$$S\sin.\tfrac{m\pi}{k}=\tfrac{1}{2}\sin.m\pi.$$

Now truly, since our coefficients *m* and *k* are supposed to be whole, certainly there will be $\sin.m\pi.=0$ and thus also $S=0$, unless also perhaps there were $\sin.\frac{m\pi}{k}=0$, but which case cannot occur, since in the integration of the proposed formula $\frac{x^{m-1}\partial x}{\left(1+x^{k}\right)^{n}}$ it is assumed always to be $m<k$. Therefore in this manner it has prevailed in this case, where after the integration there may be put $x=\infty$, all the logarithmic parts of the integral cancel out.

Establishing the other case where k is an odd number and $i=k$.

§. 144. Therefore in this case, on taking $x=\infty$, the formula lx is multiplied by this series

$$-\tfrac{2}{k}\cos.\tfrac{m\pi}{k}-\tfrac{2}{k}\cos.\tfrac{3m\pi}{k}-\tfrac{2}{k}\cos.\tfrac{5m\pi}{k}-\tfrac{2}{k}\cos.\tfrac{7m\pi}{k}\cdots-\tfrac{2}{k}\cos.\tfrac{im\pi}{k},$$

where the penultimate term is $-\tfrac{2}{k}\cos.\tfrac{(k-2)m\pi}{k}$, truly for the final term there will become $\cos.m\pi=\pm1$ with the upper sign prevailing, if n were an even number, with the lower sign, if odd ; whereby with the final term removed, for the remaining terms we may put :

$$\cos.\tfrac{m\pi}{k}+\cos.\tfrac{3m\pi}{k}+\cos.\tfrac{5m\pi}{k}+\cdots+\cos.\tfrac{(k-2)m\pi}{k}=S,$$

thus so that the multiplier of the logarithm x shall be :

$$-\tfrac{2S}{k}-\tfrac{1}{k}\cos.m\pi.$$

Hence by proceeding as before there becomes :

$$\begin{aligned}S\sin.\tfrac{m\pi}{k}&\\ &=\tfrac{1}{2}\sin.\tfrac{2m\pi}{k}+\tfrac{1}{2}\sin.\tfrac{4m\pi}{k}+\tfrac{1}{2}\sin.\tfrac{6m\pi}{k}+\cdots+\tfrac{1}{2}\sin.\tfrac{(k-3)m\pi}{k}+\tfrac{1}{2}\sin.\tfrac{(k-1)m\pi}{k}\\ &-\tfrac{1}{2}\sin.\tfrac{2m\pi}{k}-\tfrac{1}{2}\sin.\tfrac{4m\pi}{k}-\tfrac{1}{2}\sin.\tfrac{6m\pi}{k}-\cdots-\tfrac{1}{2}\sin.\tfrac{(k-3)m\pi}{k},\end{aligned}$$

where again all the terms except for the final mutually cancel each other, thus so that hence there may be produced :

$$S\sin.\tfrac{m\pi}{k}=\tfrac{1}{2}\sin.\tfrac{(k-1)m\pi}{k}=\tfrac{1}{2}\sin.\left(m\pi-\tfrac{m\pi}{k}\right);$$

but truly there becomes :

$$\sin.\left(m\pi-\tfrac{m\pi}{k}\right)=\sin.m\pi\cos.\tfrac{m\pi}{k}-\cos.m\pi\sin.\tfrac{m\pi}{k}$$

where there may be observed to be $\sin.m\pi=0$ on account of the whole number m ; therefore we will obtain:

$$S\sin.\tfrac{m\pi}{k}=-\tfrac{1}{2}\cos.\left(m\pi-\tfrac{m\pi}{k}\right)\text{ or }S=-\tfrac{1}{2}\cos.m\pi,$$

consequently the multiplier of lx will be

$$=\tfrac{1}{k}\cos.m\pi-\tfrac{1}{k}\cos.m\pi=0$$

and thus it is evident, whether *k* shall be an even or odd number, all the logarithmic terms in our integral cancel each other, if indeed after the integration we may put $x=\infty$, just as we suppose here always.

§. 145. We will consider now also the parts depending on the circle, from which our integral is composed. But these parts have been ascertained to constitute the following progression :

$$\begin{aligned}&\tfrac{2}{k}\sin.\tfrac{m\pi}{k}\,\text{Atang.}\frac{x\sin.\frac{\pi}{k}}{1-x\cos.\frac{\pi}{k}}+\tfrac{2}{k}\sin.\tfrac{3m\pi}{k}\,\text{Atang.}\frac{x\sin.\frac{3\pi}{k}}{1-x\cos.\frac{3\pi}{k}}\\&+\tfrac{2}{k}\sin.\tfrac{5m\pi}{k}\,\text{Atang.}\frac{x\sin.\frac{5\pi}{k}}{1-x\cos.\frac{5\pi}{k}}+\tfrac{2}{k}\sin.\tfrac{7m\pi}{k}\,\text{Atang.}\frac{x\sin.\frac{7\pi}{k}}{1-x\cos.\frac{7\pi}{k}}\\&+\cdots\cdots+\tfrac{2}{k}\sin.\tfrac{im\pi}{k}\,\text{Atang.}\frac{x\sin.\frac{i\pi}{k}}{1-x\cos.\frac{i\pi}{k}},\end{aligned}$$

where in the final part there is either $i=k-1$ or $i=k$*;* clearly the first prevails, if *i* is an even number, the latter, if odd.

§. 146. Also since all these terms vanish on putting $x=0$, for our situation we may put $x=\infty$. In general therefore there becomes :

$$\text{Atang.}\frac{x\sin.\frac{i\pi}{k}}{1-x\cos.\frac{i\pi}{k}}=\text{Atang.}\left(\text{tang.}\tfrac{i\pi}{k}\right).$$

Truly, we have

$$-\text{ tang.}\tfrac{i\pi}{k}=+\text{ tang.}\frac{(k-i)\pi}{k},$$

from which here the arc shall be $=\frac{(k-i)\pi}{k}$. Hence therefore in place of *i* by writing successively the numbers 1, 3, 5, 7 etc. these parts of our integral sought will be

$$\begin{aligned}&\frac{2(k-1)\pi}{kk}\sin.\tfrac{m\pi}{k}+\frac{2(k-3)\pi}{kk}\sin.\tfrac{3m\pi}{k}+\frac{2(k-5)\pi}{kk}\sin.\tfrac{5m\pi}{k}+\frac{2(k-7)\pi}{kk}\sin.\tfrac{7m\pi}{k}+\\&\frac{2(k-9)\pi}{kk}\sin.\tfrac{9m\pi}{k}+\cdots\cdots\frac{2(k-i)\pi}{kk}\sin.\tfrac{im\pi}{k},\end{aligned}$$

where in the case, where *k* is an even number, it is required to have progressed as far as to $i=k-1$, and if *k* shall be odd, as far as to $i=k$.

§. 147. For brevity's sake, we may put

$$(k-1)\sin.\tfrac{m\pi}{k}+(k-3)\sin.\tfrac{3m\pi}{k}+(k-5)\sin.\tfrac{5m\pi}{k}+\cdots\cdots+(k-i)\sin.\tfrac{im\pi}{k}=\mathrm{S},$$

thus so that the integral sought shall be $\frac{2\pi\mathrm{S}}{kk}$, since the logarithmic parts will have cancelled each other out. Now we will multiply each part by $2\sin.\frac{m\pi}{k}$, and since in general there shall be

$$2\sin.\tfrac{m\pi}{k}\sin.\tfrac{im\pi}{k}=\cos.\tfrac{(i-1)m\pi}{k}-\cos.\tfrac{(i+1)m\pi}{k},$$

with the substitution made there shall become :

$$\begin{aligned}2\mathrm{S}\sin.\tfrac{m\pi}{k}=&(k-1)\cos.\tfrac{0m\pi}{k}+(k-3)\cos.\tfrac{2m\pi}{k}+(k-5)\cos.\tfrac{4m\pi}{k}+\cdots\cdots+(k-i)\cos.\tfrac{(i-1)m\pi}{k}\\&-(k-1)\cos.\tfrac{2m\pi}{k}-(k-3)\cos.\tfrac{4m\pi}{k}-\cdots\cdots-(k-i+2)\cos.\tfrac{(i-1)m\pi}{k}-(k-i)\cos.\tfrac{(i+1)m\pi}{k},\end{aligned}$$

which series clearly is contracted into the following :

$$\begin{aligned}2\mathrm{S}\sin.\tfrac{m\pi}{k}=&k-1-2\cos.\tfrac{2m\pi}{k}-2\cos.\tfrac{4m\pi}{k}-2\cos.\tfrac{6m\pi}{k}-\cdots\cdots-2\cos.\tfrac{(i-1)m\pi}{k}\\&-(k-i)\cos.\tfrac{(i+1)m\pi}{k},\end{aligned}$$

where with the first and last term removed, the intermediate terms constitute a regular series, so that we may find its value we may put

$$\mathrm{T}=\cos.\tfrac{2m\pi}{k}+\cos.\tfrac{4m\pi}{k}+\cos.\tfrac{6m\pi}{k}+\cdots\cdots+\cos.\tfrac{(i-1)m\pi}{k},$$

thus so that there shall become:

$$2\mathrm{S}\sin.\tfrac{m\pi}{k}=k-1-2\mathrm{T}-(k-i)\cos.\tfrac{(i+1)m\pi}{k}.$$

But here again it is agreed to consider two cases, just as *k* were even or odd.

Establishing the first case, where *k* is an even number and $i=k-1$

§. 148. Therefore in this case we will have

$$\mathrm{T}=\cos.\tfrac{2m\pi}{k}+\cos.\tfrac{4m\pi}{k}+\cos.\tfrac{6m\pi}{k}+\cdots\cdots+\cos.\tfrac{(k-2)m\pi}{k}.$$

Again we will multiply by $2\sin.\frac{m\pi}{k}$ and from the above reductions we will have

$$2T\sin.\tfrac{m\pi}{k} = \sin.\tfrac{3m\pi}{k} + \sin.\tfrac{5m\pi}{k} + \cdots + \sin.\tfrac{(k-3)m\pi}{k} + \sin.\tfrac{(k-1)m\pi}{k}$$
$$-\sin.\tfrac{m\pi}{k} - \sin.\tfrac{3m\pi}{k} - \sin.\tfrac{5m\pi}{k} - \cdots - \sin.\tfrac{(k-3)m\pi}{k};$$

therefore with the terms removed mutually cancelling each other there will be :

$$2T\sin.\tfrac{m\pi}{k} = -\sin.\tfrac{m\pi}{k} + \sin.\tfrac{(k-1)m\pi}{k}.$$

Indeed there is :

$$\sin.\tfrac{(k-1)m\pi}{k} = \sin.\left(m\pi - \tfrac{m\pi}{k}\right) = \sin.m\pi\cos.\tfrac{m\pi}{k} - \cos.m\pi\sin.\tfrac{m\pi}{k},$$

where $\sin.m\pi = 0$, on account of which there becomes :

$$2\mathrm{T} = -1 - \cos.m\pi.$$

§. 149. From the value found for T there is deduced to be

$$2S\sin.\tfrac{m\pi}{k} = k \text{ and therefore } S = \frac{k}{2\sin.\frac{m\pi}{k}}.$$

Finally truly the value of our integral formula, which we have found, will be $\frac{2\pi \mathrm{S}}{kk}$, and now it is evident in the case of our integral formula, where *S* is an even number, will be $\frac{\pi}{k\sin.\frac{m\pi}{k}}$, if indeed after the integration there may be put $x = \infty$.

Establishing the other case, where k is an odd number and $i = k$

§. 150. In this case there is :

$$\mathrm{T} = \cos.\tfrac{2m\pi}{k} + \cos.\tfrac{4m\pi}{k} + \cos.\tfrac{6m\pi}{k} + \cdots + \cos.\tfrac{(k-1)m\pi}{k},$$

which series multiplied by $2\sin.\frac{m\pi}{k}$ will produce as before :

$$2\mathrm{T}\sin.\tfrac{m\pi}{k} = \sin.\tfrac{3m\pi}{k} + \sin.\tfrac{5m\pi}{k} + \cdots + \sin.\tfrac{(k-2)m\pi}{k} + \sin.\tfrac{km\pi}{k}$$
$$-\sin.\tfrac{m\pi}{k} - \sin.\tfrac{3m\pi}{k} - \sin.\tfrac{5m\pi}{k} - \cdots - \sin.\tfrac{(k-2)m\pi}{k},$$

from which with the terms deleted mutually cancelling each other there will be found :

$$2\mathrm{T}\sin.\tfrac{m\pi}{k}=-\sin.\tfrac{m\pi}{k}+\sin.m\pi$$

and thus

$$2\mathrm{T}=-1+\frac{\sin.m\pi}{\sin.\frac{m\pi}{k}}=-1$$

on account of $\sin.m\pi=0$, and hence again there becomes :

$$2\mathrm{S}\sin.\tfrac{m\pi}{k}=k;$$

whereby since the value of the integral sought shall be $\frac{2\pi S}{kk}$, also in this case our integral $=\frac{\pi}{k\sin.\frac{m\pi}{k}}$, exactly as in the preceding case. Hence therefore we deduce the following

THEOREM

§. 151. *If this differential formula thus may be integrated*

$$\frac{x^{m-1}\partial x}{1+x^{k}},$$

so that on putting $x=0$ *the integral may vanish, then truly on putting* $x=\infty$, *thence the resulting value will be always*

$$\frac{\pi}{k\sin.\frac{m\pi}{k}},$$

whether k shall be an odd or even number.

The demonstration of this theorem is evident from the proceedings.

§. 152. In the establishment of this formula we have assumed to be $m<k$, because otherwise the logarithmic members shall not cancel out ; but truly there in no longer a need for this limitation. For in the case, where there becomes $m=k$, the integral of the formula $\frac{x^{m-1}\partial x}{1+x^{k}}$ shall be $\frac{1}{k}l\left(1+x^{k}\right)$, which on making $x=\infty$ also will become infinite ∞; truly this likewise indicates our integral $\frac{\pi}{k\sin.\frac{m\pi}{k}}=\infty$. Therefore provided that *m* were not greater than *k,* our formula is agreed to be true always.

§. 153. Also indeed it is not necessary, that the exponents *m* and *k* shall be whole number, provided that *m* were not $>k$; indeed if there were $m=\frac{\mu}{\lambda}$ and $k=\frac{\chi}{\lambda}$ the value of our formula will be $\frac{\lambda\pi}{\chi\sin.\frac{\mu\pi}{\chi}}$, of which the truth is shown thus. Because in this case the formula requiring to be integrated is :

$$\int \frac{x^{\frac{\mu}{\lambda}}}{1+x^{\frac{\chi}{\lambda}}} \cdot \frac{\partial x}{x},$$

there may be put $x = y^{\lambda}$; there will be $\frac{\partial x}{x} = \frac{\lambda \partial y}{y}$ and the formula becomes

$$\int \frac{y^{\mu}}{1+y^{\chi}} \cdot \frac{\lambda \partial y}{y} = \lambda \int \frac{y^{\mu-1}\partial y}{1+y^{\chi}},$$

of which the value will be without doubt $\frac{\lambda\pi}{\chi \sin.\frac{\mu\pi}{\chi}}$.

Another demonstration of the theorem.

§. 154. P will indicate the value of the integral $\int \frac{x^m}{1+x^k} \cdot \frac{\partial x}{x}$ from the limit $x = 0$ as far as to $x = 1$, but Q the value of the same integral from the limit $x = 1$ as far as to $x = \infty$, thus so that $\mathrm{P} + \mathrm{Q}$ may produce that same value, which is maintained in the theorem. Now for finding the value Q there may be put $x = \frac{1}{y}$, from which there shall be $\frac{\partial x}{x} = -\frac{\partial y}{y}$, and there becomes :

$$\mathrm{Q} = \int \frac{y^{-m}}{1+y^{-k}} \cdot \frac{-\partial y}{y} = -\int \frac{y^{k-m}}{1+y^{k}} \cdot \frac{\partial y}{y}$$

from the limit $y = 1$ as far as to $y = 0$. Hence therefore with the limits interchanged, there will be

$$\mathrm{Q} = +\int \frac{y^{k-m}}{1+y^{k}} \cdot \frac{\partial y}{y}$$

from the limit $y = 0$ as far as to $y = 1$. Now because the letter y may be removed from this calculation, it will be permitted to write x in place of y, thus so that there shall become

$$\mathrm{Q} = \int \frac{x^{k-m}}{1+x^{k}} \cdot \frac{\partial x}{x},$$

with which done we will have

$$\mathrm{P} + \mathrm{Q} = \int \frac{x^m + x^{k-m}}{1+x^k} \cdot \frac{\partial x}{x}$$

from the limit $x=0$ as far as to the limit $x=1$. Truly thus not long ago I have shown the value of this integral formula to be $=\frac{\pi}{k\sin.\frac{m\pi}{k}}$, contained within the limits $x=0$ and $x=1$. [See E 463.] Hence the following theorem, none the less noteworthy, therefore arises.

THEOREM

§. 155. *The value of this integral formula*

$$\int \frac{x^{m}+x^{k-m}}{1+x^{k}}\cdot\frac{\partial x}{x}$$

contained between the limits $x=0$ *et* $x=1$ *is equal to the value of this integral*

$$\int \frac{x^{m}}{1+x^{k}}\cdot\frac{\partial x}{x}$$

contained between the limits $x=0$ *and* $x=\infty$.

§. 156. We may approach the proposed integral formula from these expressed in the title, and so that we may reduce that to the form examined at present, we may call on the aid of the following reduction

$$\int \frac{x^{m-1}\partial x}{\left(1+x^{k}\right)^{\lambda+1}}=\frac{\mathrm{A}x^{m}}{\left(1+x^{k}\right)^{\lambda}}+\mathrm{B}\int \frac{x^{m-1}\partial x}{\left(1+x^{k}\right)^{\lambda}},$$

from which differentiated, the following equation arises:

$$\frac{x^{m-1}\partial x}{\left(1+x^{k}\right)^{\lambda+1}}=\frac{m\mathrm{A}x^{m-1}\partial x}{\left(1+x^{k}\right)^{\lambda}}-\frac{\lambda k\mathrm{A}x^{m+k-1}\partial x}{\left(1+x^{k}\right)^{\lambda+1}}+\mathrm{B}\frac{x^{m-1}\partial x}{\left(1+x^{k}\right)^{\lambda}},$$

which equation divided by $x^{m-1}\partial x$ and multiplied by $\left(1+x^{k}\right)^{\lambda}$, the negative term being transposed from the right to the left, there will become :

$$\frac{1+\lambda k\mathrm{A}x^{k}}{1+x^{k}}=m\mathrm{A}+\mathrm{B},$$

which equation clearly cannot remain in place, unless there shall be $\lambda k\mathrm{A}=1$ or $\mathrm{A}=\frac{1}{\lambda k}$, from which there shall become $1=m\mathrm{A}+\mathrm{B}=\frac{m}{\lambda k}+\mathrm{B}$, and thus $\mathrm{B}=1-\frac{m}{\lambda k}$.

§. 157. With these values found for the letters *A* and *B,* at first we assume the integrals thus to be taken, so that they vanish on putting $x=0$; then truly on putting $x=\infty$, because the exponent *n* is supposed to be less than *k,* the complete term associated with the letter A vanishes at once, thus so that in this case $x=\infty$ there may become :

$$\int \frac{x^{m-1}\partial x}{\left(1+x^k\right)^{\lambda+1}} = \left(1-\frac{m}{\lambda k}\right)\int \frac{x^{m-1}\partial x}{\left(1+x^k\right)^{\lambda}}.$$

But if now initially we may put $\lambda = 1$, because before we have found for the same case $x = \infty$ to be :

$$\int \frac{x^{m-1}\partial x}{1+x^k} = \frac{\pi}{k\sin.\frac{m\pi}{k}},$$

we will have the value of this same integral :

$$\int \frac{x^{m-1}\partial x}{\left(1+x^k\right)^2} = \left(1-\frac{m}{k}\right)\frac{\pi}{k\sin.\frac{m\pi}{k}},$$

if indeed the same may be extended from the limit $x = 0$ as far as to the limit $x = \infty$.

§. 158. But if now in a like manner we may put $\lambda = 2$, there may be found for the same limits of the integration :

$$\int \frac{x^{m-1}\partial x}{\left(1+x^k\right)^3} = \left(1-\frac{m}{k}\right)\left(1-\frac{m}{2k}\right)\frac{\pi}{k\sin.\frac{m\pi}{k}};$$

in the same manner, if greater values may be continually attributed to the letter λ, the following forms of the integrals will be found, all worthy of attention:

$$\int \frac{x^{m-1}\partial x}{\left(1+x^k\right)^4} = \left(1-\frac{m}{k}\right)\left(1-\frac{m}{2k}\right)\left(1-\frac{m}{3k}\right)\frac{\pi}{k\sin.\frac{m\pi}{k}},$$

$$\int \frac{x^{m-1}\partial x}{\left(1+x^k\right)^5} = \left(1-\frac{m}{k}\right)\left(1-\frac{m}{2k}\right)\left(1-\frac{m}{3k}\right)\left(1-\frac{m}{4k}\right)\frac{\pi}{k\sin.\frac{m\pi}{k}},$$

$$\int \frac{x^{m-1}\partial x}{\left(1+x^k\right)^6} = \left(1-\frac{m}{k}\right)\left(1-\frac{m}{2k}\right)\left(1-\frac{m}{3k}\right)\left(1-\frac{m}{4k}\right)\left(1-\frac{m}{5k}\right)\frac{\pi}{k\sin.\frac{m\pi}{k}},$$

etc.

§. 159. Whereby if the letter n may denote some number for the formula expressed in the title, if its integral may be extended from the limit $x = 0$ as far as to $x = \infty$, of which the value will be had in the following manner :

$$\left(1-\frac{m}{k}\right)\left(1-\frac{m}{2k}\right)\left(1-\frac{m}{3k}\right)\left(1-\frac{m}{4k}\right)\cdots\left(1-\frac{m}{(n-1)k}\right)\frac{\pi}{k\sin.\frac{m\pi}{k}},$$

which therefore to be the formula for this integral

$$\int \frac{x^{m-1}\partial x}{\left(1+x^{k}\right)^{n}}.$$

§. 160. Indeed here it is not allowed necessarily to accept all numbers except integers for *n* here ; but truly by the method of interpolation, which now has been explained further elsewhere [See E254], this integration is allowed to be extended to the cases, in which the exponent *n* is a fraction. Because if now the following integral formulas may be extended from the limit $y=0$ as far as to $y=1$, in general the value of our proposed formula will be able to be represented thus:

$$\int \frac{x^{m-1}\partial x}{\left(1+x^{k}\right)^{n}} = \frac{\pi}{k\sin.\frac{m\pi}{k}} \cdot \frac{\int y^{nk-m-1}\partial y\left(1-y^{k}\right)^{\frac{m}{k}-1}}{\int y^{k-m-1}\partial y\left(1-y^{k}\right)^{\frac{m}{k}-1}}.$$

From which, if there were $m=1$ and $k=2$, it follows to become

$$\int \frac{\partial x}{\left(1+x^{k}\right)^{n}} = \frac{\pi}{2} \cdot \int \frac{y^{2(n-1)}\partial y}{\sqrt{(1-yy)}} : \int \frac{\partial y}{\sqrt{(1-yy)}} = \int \frac{y^{2(n-1)}\partial y}{\sqrt{(1-yy)}}.$$

Thus, if $n=\frac{3}{2}$, there will become

$$\int \frac{\partial x}{(1+xx)^{\frac{3}{2}}} = \int \frac{y\partial y}{\sqrt{(1-yy)}}.$$

the truth of which will be apparent at once, because the integral of the first generally is $\frac{x}{\sqrt{(1+xx)}}$, truly of the latter is $=1-\sqrt{(1-yy)}$, which by making $x=\infty$ and $y=1$ certainly shall become equal. Otherwise, it will help to note for the general integration the exponent cannot be taken less than unity, because otherwise the values of both the integrals will increase to infinity.

6)

The investigation of the value of the integral

$$\int \frac{x^{m-1}\partial x}{1-2x^k\cos.\theta+x^{2k}}$$

from the limit $x=0$ to be extended as far as to $x=\infty$.

Opuscula analytica. Tom. II. pag. 55-75. [E589]

§. 161. In the first place we may enquire about the indefinite integral of the proposed formula, and thus return to all the operations of analysis from first principles. And indeed initially, because the denominator cannot be resolved into simple real factors, in general that double factor shall be some $1-2x\cos.\omega+xx$; indeed it is clear the denominator to become the product from *k* duplicate factors of this kind.
Therefore with this factor $=0$ there becomes $x=\cos.\omega\pm\sqrt{-1}\cdot\sin.\omega$, also the denominator must vanish in a twofold manner, or if either there is put

$$x=\cos.\omega+\sqrt{-1}\cdot\sin.\omega \text{ or } x=\cos.\omega-\sqrt{-1}\cdot\sin.\omega.$$

But it is agreed all the powers of these formulas thus may be expressed conveniently, so that there shall be

$$\left(\cos.\omega\pm\sqrt{-1}\cdot\sin.\omega\right)^{\lambda}=\cos.\lambda\omega\pm\sqrt{-1}\cdot\sin.\lambda\omega;$$

hence there will be therefore :

$$x^k=\cos.k\omega\pm\sqrt{-1}\cdot\sin.k\omega \text{ and } x^{2k}=\cos.2k\omega\pm\sqrt{-1}\cdot\sin.2k\omega.$$

Therefore we may substitute these values and our denominator will emerge :

$$1-2\cos.\theta\cos.k\omega+\cos.2k\omega\mp2\sqrt{-1}\cdot\cos\theta\sin.k\omega\pm\sqrt{-1}\cdot\sin.2k\omega..$$

§. 162. Therefore it is evident of this equation both the real as well as the imaginary terms must themselves be removed individually, from which these two equations arise :

$$\text{I. } 1-2\cos.\theta\cos.k\omega+\cos.2k\omega=0,$$
$$\text{II. } -2\cos.\theta\sin.k\omega+\sin.2k\omega=0.$$

Therefore since there shall be

$$\sin.2k\omega = 2\sin.k\omega\cos.k\omega,$$

the latter equation will adopt this form

$$-2\cos.\theta\ \sin.k\omega + 2\sin.k\omega\cos.k\omega = 0\,,$$

which divided by $2\sin.k\omega$ gives:

$$\cos.k\omega = \cos.\theta$$

and thus

$$\cos.2k\omega = \cos.2\theta = \cos.^2\theta - \sin.^2\theta = 2\cos.^2\theta - 1\,,$$

which values substituted into the first equation produce an identity, thus so that each of the equations may be satisfied by assuming $\cos.k\omega = \cos.\theta$.

§. 163. Therefore it is required for an angle ω of this kind to be assumed, so that there may become $\cos.k\omega = \cos.\theta$, from which indeed there is deduced at once $k\omega = \theta$ and thus $\omega = \frac{\theta}{k}$. Truly since infinitely many angles are given having the same cosine, which besides the angle itself θ are $2\pi \pm \theta,\ 4\pi \pm \theta,\ 6\pi \pm \theta$ etc and thus in general $2i\pi \pm \theta$ with i denoting all the whole numbers, with our one sought satisfied by making $k\omega = 2i\pi \pm \theta$, from which it is deduced the angle $\omega = \frac{2i\pi \pm \theta}{k}$, and thus we will obtain innumerable satisfying angles for ω, but of which just so many are to be assumed, as the exponent k will contain one; therefore we may attribute the following values successively to the angle ω:

$$\frac{\theta}{k},\ \frac{2\pi+\theta}{k},\ \frac{4\pi+\theta}{k},\ \frac{6\pi+\theta}{k},\ \frac{8\pi+\theta}{k},\ \ldots \frac{2(k-1)\pi \pm \theta}{k}.$$

Whereby therefore if we may attribute these same successive individual values to the angle ω, of which the number is $= k$, all the duplicate factors will be supplied to the formula $1 - 2x\cos.\omega + xx$ of our denominator $1 - 2x^k\cos.\theta + x^{2k}$, of which the number is $= k$.

§. 164. Now with all the factors of our denominator found, the fraction $\frac{x^{m-1}}{1-2x^k\cos.\theta+x^{2k}}$ must be resolved into just as many partial fractions, of which the denominators themselves shall be these two-fold factors, the number of which is *k*, thus so that in general such a fraction of the parts shall have such a form :

$$\frac{\mathrm{A}+\mathrm{B}x}{1-2x\cos.\omega+xx},$$

as we may resolve above into simple though imaginary pairs, and since there shall be :

$$xx-2x\cos.\omega+1=\left(x-\cos.\omega+\sqrt{-1}\cdot\ \sin.\omega\right)\left(x-\cos.\omega\ -\sqrt{-1}\cdot\ \sin.\omega\right),$$

both these partial fractions may be put in place :

$$\frac{f}{x-\cos.\omega-\sqrt{-1}\cdot\ \sin.\omega}+\frac{g}{x-\cos.\omega+\sqrt{-1}\cdot\ \sin.\omega},$$

thus so that the whole business of resolution may be reduced to this, so that both the numerators *f* and *g* may be determined ; indeed the sum of both the fractions will be found from these

$$=\frac{fx+gx-(f+g)\cos.\omega+\sqrt{-1}\cdot(f-g)\sin.\omega}{xx-2x\cos.\omega+1},$$

where there will be :

$$B=\left(f+g\right)\text{ and }\mathrm{A}=(f-g)\sqrt{-1}\cdot\sin.\omega-(f+g)\cos.\omega.$$

§. 165. Therefore by the method we may put in place any fractions requiring to be resolved into simple fractions :

$$\frac{x^{m-1}}{1-2x^{k}\cos.\theta+x^{2k}}=\frac{f}{x-\cos.\omega-\sqrt{-1}\cdot\ \sin.\omega}+\mathrm{R},$$

where R shall include all the remaining partial fractions. Hence by being multiplied by

$$x-\cos.\omega-\sqrt{-1}\cdot\ \sin.\omega$$

there will be had

$$\frac{x^{m}-x^{m-1}\left(\cos.\omega+\sqrt{-1}\cdot\ \sin.\omega\right)}{1-2x^{k}\cos.\theta+x^{2k}}=f+\mathrm{R}\left(x-\cos.\omega-\sqrt{-1}\cdot\sin.\omega\right);$$

since which equation must be true, any value may be attributed to x , we may put $x=\cos.\omega+\sqrt{-1}\cdot\sin.\omega$, so that the latter member may be taken directly from the calculation; then truly in the left-hand part, because the formula $x-\cos.\omega-\sqrt{-1}\cdot\sin.\omega$ likewise is a factor of the denominator, with this substitution made both the numerator as well as the denominator will disappear into nothing, thus so that it may be seen that nothing can be concluded.

§. 166. Therefore here we may use this well-known rule and we may write the differentials of these in place both of the numerator as well as the denominator, from which our equation may accept the following form :

$$\frac{mx^{m-1}-(m-1)x^{m-2}\left(\cos.\omega+\sqrt{-1}\cdot\sin.\omega\right)}{-2kx^{k-1}\cos.\theta+2kx^{2k-1}}=\frac{mx^{m}-(m-1)x^{m-1}\left(\cos.\omega+\sqrt{-1}\cdot\sin.\omega\right)}{-2kx^{k}\cos.\theta+2kx^{2k}}=f,$$

clearly by putting $x=\cos.\omega+\sqrt{-1}\cdot\sin.\omega$. But then there will be

$$x^m=\cos.m\omega+\sqrt{-1}\cdot\sin.m\omega$$

and

$$x^{m-1}\left(\cos.m\omega+\sqrt{-1}\cdot\sin.m\omega\right)=x^m=\cos.m\omega+\sqrt{-1}\cdot\sin.m\omega$$

and for the denominator

$$x^k=\cos.k\omega+\sqrt{-1}\cdot\sin.k\omega \text{ et } x^{2k}=\cos.2k\omega+\sqrt{-1}\cdot\sin.2k\omega;$$

from which the numerator becomes

$$x^m=\cos.m\omega+\sqrt{-1}\cdot\sin.m\omega$$

and the denominator

$$-2k\cos.\theta\cos.k\omega+2k\cos.2k\omega-2k\sqrt{-1}\cdot\cos.\theta\ \sin.k\omega+2k\sqrt{-1}\cdot\sin.2k\omega.$$

§. 167. For the denominator being reduced we may now record to be found above $\cos.k\omega=\cos.\theta$, from which there becomes $\sin.k\omega=\sin.\theta$, then truly

$$\cos.2k\omega=\cos.2\theta=2\cos.^2\theta-1 \text{ and } \sin.2k\omega=2\sin.\theta\cos.\theta,$$

with which values used our denominator will be

$$2k\cos.^2\theta-2k+2k\sqrt{-1}\cdot\sin.\theta\cos.\theta=-2k\sin^2.\theta+2k\sqrt{-1}\cdot\sin.\theta\cos.\theta$$
$$=-2k\sin.\theta\left(\sin.\theta-\sqrt{-1}\cdot\cos.\theta\right),$$

on account of which, with this value use we will have

$$f=\frac{\cos.m\omega+\sqrt{-1}\cdot\sin.m\omega}{2k\sin.\theta\left(\sqrt{-1}\cdot\cos.\theta-\sin.\theta\right)}.$$

Likewise truly hence without a new calculation we may deduce the value *g,* certainly which only differs from *f* by reason of the sign of $\sqrt{-1}$, and thus there will be

$$g=\frac{\cos.m\omega-\sqrt{-1}\cdot\sin.m\omega}{-2k\sin.\theta\left(\sin.\theta+\sqrt{-1}\cdot\cos.\theta\right)}.$$

§. 168. Moreover with these letters found f and g for A and B, in the first place we may deduce ,

$$f+g=\frac{\cos.\theta\sin.m\omega-\sin.\theta\cos.m\omega}{k\sin.\theta}=\frac{\sin.(m\omega-\theta)}{k\sin.\theta},$$

then truly there will become :

$$f-g=-\frac{\sqrt{-1}\cos.(m\omega-\theta)}{k\sin.\theta}.$$

Therefore from these we will find :

$$\mathrm{B}=\frac{\sin.(m\omega-\theta)}{k\sin.\theta}$$

and

$$\mathrm{A}=\frac{\sin.\omega\cos.(m\omega-\theta)-\cos.\omega\sin.(m\omega-\theta)}{k\sin.\theta}=-\frac{\sin.((m\omega-\theta)-\omega)}{k\sin.\theta},$$

where therefore the imaginary terms at once will have mutually cancel out.

§. 169. With these values A and B found, it is required to investigate the partial integral

$$\int\frac{(\mathrm{A}+\mathrm{B}x)\partial x}{1-2x\cos.\omega+xx},$$

where, since the differential of the denominator shall be

$$2x\partial x-2\partial x\cos.\omega=2\partial x(x-\cos.\omega),$$

we may put

$$\mathrm{A}+\mathrm{B}x=\mathrm{B}(x-\cos.\omega)+\mathrm{C}$$

and there will be $\mathrm{C}=\mathrm{A}+\mathrm{B}\cos.\omega$; hence therefore there will be

$$\mathrm{C}=\frac{\cos.\omega\sin.(m\omega-\theta)-\sin.((m\omega-\theta)-\omega)}{k\sin.\theta}.$$

Truly because

$$-\sin.((m\omega-\theta)-\omega)=-\sin.(m\omega-\theta)\cos.\omega+\cos.(m\omega-\theta)\sin.\omega,$$

there will be

$$\mathrm{C}=\frac{\sin.\omega\cos.(m\omega-\theta)}{k\sin.\theta}.$$

Therefore with this form used, the formula being integrated $\frac{(\mathrm{A}+\mathrm{B}x)\partial x}{1-2x\cos.\omega+xx}$ will be separated into these two parts

$$\frac{\mathrm{B}(x-cos.\omega)\partial x}{1-2x\cos.\omega+xx}+\frac{\mathrm{C}\partial x}{1-2x\cos.\omega+xx}.$$

Therefore here the integral of the first part evidently is

$$\mathrm{B}l\sqrt{(1-2x\cos.\omega+xx)},$$

truly the integral of the other part readily apparent to be expressed by the arc of a circle , of which the tangent shall be $\frac{x\sin.\omega}{1-x\cos.\omega}$. Towards finding this integral, we may put

$$\int\frac{\mathrm{C}\partial x}{1-2x\cos.\omega+xx}=\mathrm{D}\,\mathrm{Atang.}\frac{x\sin.\omega}{1-x\cos.\omega}$$

and with the differentials taken, because $\partial.\mathrm{Atang.}t$ is equal to $\frac{dt}{1+tt}$ we will have

$$\frac{\mathrm{C}\partial x}{1-2x\cos.\omega+xx}=\mathrm{D}\frac{\partial x\sin.\omega}{1-2x\cos.\omega+xx},$$

from which it is evident there becomes

$$\mathrm{D}=\frac{\mathrm{C}}{\sin.\omega}=\frac{\cos.(m\omega-\theta)}{k\sin.\theta}.$$

§. 170. Therefore with these values found just now substituted in place of B and D and from the individual factors of the denominator $1-2x^{k}\cos.\theta+x^{2k}$, of which the form is $1-2x\cos.\omega+xx$, an agreed part of the integral arises from the logarithmic member and from the circular arc, which will be

$$\frac{\sin.(m\omega-\theta)}{k\sin.\theta}l\sqrt{(1-2x\cos.\omega+xx)}+\frac{\cos.(m\omega-\theta)}{k\sin.\theta}.\mathrm{Atang.}\frac{x\sin.\omega}{1-x\cos.\omega},$$

which vanishes on taking $x=0$. Therefore there is an need in this form only, so that in place of ω we may write successively the values indicated above, as it were

$$\frac{\theta}{k},\ \frac{2\pi+\theta}{k},\ \frac{4\pi+\theta}{k},\ \frac{6\pi+\theta}{k},\ \text{etc.},$$

until $\frac{2(k-1)\pi+\theta}{k}$ may be arrived at ; then indeed the sum of all these formulas will present the whole indefinite integral of the proposed formula.

§. 171. Therefore after we have elicited the indefinite integral, nothing other remains, except that into that we may make $x=\infty$, with which done the logarithmic part, on account of

$$\sqrt{(1-2x\cos.\omega+xx)}=x-\cos.\omega$$

will become $\mathrm{B}l(x-\cos.\omega)$. Truly there is

$$l(x-\cos.\omega)=lx-\tfrac{\cos.\omega}{x}=lx$$

on account of $\frac{\cos.\omega}{x}=0$; wherefore on making $x=\infty$ any logarithmic part will have this form $\frac{\sin.(m\omega-\theta)}{k\sin.\theta}lx$. Then for the part depending on the circle, on making $x=\infty$ there becomes

$$\tfrac{x\sin.\omega}{1-x\cos.\omega}=-\mathrm{tang}.\omega=-\mathrm{tang}.(\pi-\omega)$$

and thus the arc, of which this is the tangent, will be $=\pi-\omega$ and hence whichever circular part becomes

$$\frac{\cos.(m\omega-\theta)}{k\sin.\theta}(\pi-\omega).$$

§. 172. Since in general any value of the angle ω may have the form $\frac{2i\pi+\theta}{k}$, the angle will be

$$m\omega-\theta=\frac{2im\pi-\theta(k-m)}{k} \text{ and } \pi-\omega=\frac{\pi(k-2i)-\theta}{k}.$$

For the sake of brevity we may put:

$$\frac{\theta(k-m)}{k}=\zeta \text{ and } \frac{m\pi}{k}=\alpha,$$

so that there shall become:

$$m\omega-\theta=2i\alpha-\zeta,$$

where in place of *i* the numbers 0, 1, 2, 3 etc. may be written successively as far as to $k-1$. Hence therefore, if we may gather all the logarithmic parts into one sum, this will be able to be represented thus :

$$\frac{lx}{k\sin.\theta}\left\{\begin{array}{l}-\sin.\zeta+\sin(2\alpha-\zeta)+\sin(4\alpha-\zeta)+\sin(6\alpha-\zeta)\\+\sin(8\alpha-\zeta)+\cdots+\sin(2(k-1)\alpha-\zeta)\end{array}\right\};$$

where from these indeed, which have been treated up to this point, it will be easy to consider the whole progression to be directed towards zero. Truly it is necessary to strengthen this by a firm demonstration.

§. 173. Towards showing this we may put

$$\mathrm{S}=-\sin.\zeta+\sin.\left(2\alpha-\zeta\right)+\sin.\left(4\alpha-\zeta\right)+\cdots+\sin.\left(2\left(k-1\right)\alpha-\zeta\right);$$

and we will multiply each side by 2sin.*a,* and since there shall be:

$$2\sin.\alpha\sin.\varphi=\cos.(\alpha-\varphi)-\cos.(\alpha+\varphi),$$

with the help of this reduction we will obtain the following expression:

$$\begin{aligned}2\mathrm{S}\sin.\alpha=\cos.&\left(\alpha+\zeta\right)\\&-\cos.\left(\alpha-\zeta\right)-\cos.\left(3\alpha-\zeta\right)-\cos.\left(5\alpha-\zeta\right)-\cdots\\&+\cos.\left(\alpha-\zeta\right)+\cos.\left(3\alpha-\zeta\right)+\cos.\left(5\alpha-\zeta\right)+\cdots\\&\qquad-\cos\left(\left(2k-1\right)\alpha-\zeta\right),\end{aligned}$$

from which with the deleted terms mutually cancelling each other there will be had :

$$2\mathrm{S}\sin.\alpha=\cos.\left(\alpha+\zeta\right)-\cos\left(\left(2k-1\right)\alpha-\zeta\right).$$

§. 174. We may now put these two angles, which are left,

$$\alpha+\zeta=p \text{ and } \left(2k-1\right)\alpha-\zeta=q$$

and the sum of these will be $p+q=2\alpha k$. Again since there is $\alpha=\frac{m\pi}{k}$, there will be $p+q=2m\pi$, that is by a multiple of the whole periphery of the circle on account of the whole number *m*. Whereby since there shall be $q=2m\pi-p$, there will be $\cos.q=\cos.p$; from which it is apparent the sum found is equal to nothing, and thus it is evident all its parts are logarithmic parts, which have been introduced into the integral of our formula, mutually cancel each other in the case $x=\infty$.

§. 175. Therefore we may progress to the circular parts, the general form of which, as we have seen, is $\frac{\cos.(m\omega-\theta)}{k\sin.\theta}\left(\pi-\omega\right)$, which on putting $\alpha=\frac{m\pi}{k}$ and $\zeta=\frac{\theta(k-m)}{k}$ becomes

$$\frac{\cos.\left(2i\alpha-\zeta\right)}{k\sin.\theta}\left(\pi-\frac{2i\pi+\theta}{k}\right)=\frac{\cos.\left(2i\alpha-\zeta\right)}{k\sin.\theta}\left(\pi-\frac{2i\pi}{k}-\frac{\theta}{k}\right).$$

Here again there may be put $\frac{\pi}{k}=\beta$ and $\pi-\frac{\theta}{k}=\gamma$, so that the general form shall become t

$$\frac{\cos.(2i\alpha-\zeta)}{k\sin.\theta}(\gamma-2i\beta).$$

Whereby if in place of i we may write in order the values 0, 1, 2, 3, 4 as far as to $k-1$, all the circular parts will constitute this progression :

$$\frac{1}{k\sin.\theta}\left(\gamma\cos.\zeta+(\gamma-2\beta)\cos.(2\alpha-\zeta)+(\gamma-4\beta)\cos.(4\alpha-\zeta)\right)+\cdots$$
$$+\left(\gamma-2(k-1)\beta\right)\cos.\left(2(k-1)\alpha-\zeta\right).$$

Therefore we may put

$$\mathrm{S}=\gamma\cos.\zeta+(\gamma-2\beta)\cos.(2\alpha-\zeta)+(\gamma-4\beta)\cos.(4\alpha-\zeta)+\cdots$$
$$+\left(\gamma-2(k-1)\beta\right)\cos.\left(2(k-1)\alpha-\zeta\right),$$

so that the sum of all these circular parts shall be $\frac{\mathrm{S}}{k\sin.\theta}$, which therefore will produce the value sought of the proposed integral formula in the case, where after the integration we may put in place $x=\infty$, thus so that the whole business shall depend on investigating the value of S.

§. 176. To this end we will multiply each side be $2\sin.\alpha$, and since in general there shall be

$$2\sin.\alpha\cos.\varphi=\sin.(\alpha+\varphi)-\sin.(\varphi-\alpha),$$

with this reduction into individual terms made we will arrive at this equation :

$$2\mathrm{S}\sin.\alpha=\gamma\sin.(\alpha+\zeta)$$
$$+\gamma\sin.(\alpha-\zeta)+(\gamma-2\beta)\sin.(3\alpha-\zeta)+(\gamma-4\beta)\sin.(5\alpha-\zeta)+\cdots$$
$$-(\gamma-2\beta)\sin.(\alpha-\zeta)-(\gamma-4\beta)\sin.(3\alpha-\zeta)-(\gamma-6\beta)\sin.(5\alpha-\zeta)-\cdots$$
$$+\left(\gamma-2(k-1)\beta\right)\sin.\left(2(k-1)\alpha-\zeta\right),$$

where besides the first and last terms all the remaining terms have contracted out, thus so that there may be produced :

$$2\mathrm{S}\sin.\alpha=\gamma\sin.(\alpha+\zeta)+2\beta\sin.(\alpha-\zeta)+2\beta\sin.(3\alpha-\zeta)+2\beta\sin.(5\alpha-\zeta)+\cdots$$
$$+2\beta\sin.\left(2(k-3)\alpha-\zeta\right)+\left(\gamma-2(k-1)\beta\right)\sin.\left(2(k-1)\alpha-\zeta\right).$$

§. 177. Now again for this series requiring to be summed we may put :

$$\mathrm{T} = 2\sin.\left(\alpha-\zeta\right)+2\sin.\left(3\alpha-\zeta\right)+2\sin.\left(5\alpha-\zeta\right)+\cdots+2\sin.\left(\left(2k-3\right)\alpha-\zeta\right),$$

so that we may have

$$2\mathrm{S}\sin.\alpha = \gamma\sin.\left(\alpha+\zeta\right)+\left(\gamma-2\left(k-1\right)\beta\right)\sin.\left(\left(2\mathrm{k}-1\right)\alpha-\zeta\right)+\beta\mathrm{T}.$$

So that now at this point we may multiply again by $\sin.\alpha$, and since there shall be

$$2\sin.\alpha\sin.\varphi = \cos.\left(\varphi-\alpha\right)-\cos.\left(\varphi+\alpha\right),$$

with this reduction made we obtain :

$$\begin{aligned}\mathrm{T}\sin.\alpha = &+\cos.\zeta\\ &+\cos.\left(2\alpha-\zeta\right)+\cos.\left(4\alpha-\zeta\right)+\cdots+\cos.\left(2\left(k-2\right)\alpha-\zeta\right)\\ &-\cos.\left(2\alpha-\zeta\right)-\cos.\left(4\alpha-\zeta\right)+\cdots-\cos.\left(2\left(k-2\right)\alpha-\zeta\right)\\ &\qquad\qquad\qquad\qquad -\cos.\left(2\left(k-1\right)\alpha-\zeta\right),\end{aligned}$$

from which with the terms deleted, which mutually cancel, only this expression will remain:

$$\mathrm{T}\sin.\alpha = \cos.\zeta-\cos.\left(2\left(k-1\right)\alpha-\zeta\right).$$

Therefore since there shall be $\alpha=\frac{m\pi}{k}$, there will $2\left(k-1\right)\alpha = 2m\pi-\frac{2m\pi}{k}$, in place of which it is allowed to write $-\frac{2m\pi}{k}$, from which on account of $\zeta=\frac{\theta(k-m)}{k}$ there will be

$$\mathrm{T}\sin.\alpha = \cos.\tfrac{\theta(k-m)}{k}-\cos.\tfrac{2m\pi+\theta(k-m)}{k}.$$

§. 178. Now truly in general there may be noted to be :

$$\cos.p-\cos.q = 2\sin.\tfrac{q+p}{2}\sin.\tfrac{q-p}{2};$$

whereby since there shall be

$$p=\tfrac{\theta(k-m)}{k} \text{ and } q=\tfrac{2m\pi+\theta(k-m)}{k},$$

there will be

$$\tfrac{q+p}{2}=\tfrac{m\pi+\theta(k-m)}{k} \text{ and } \tfrac{q-p}{2}=\tfrac{m\pi}{k},$$

from which there follows to become:

$$\mathrm{T}\sin.\alpha = 2\sin.\tfrac{m\pi+\theta(k-m)}{k}\sin.\tfrac{m\pi}{k}$$

and thus

$$\mathrm{T} = 2\sin.\frac{m\pi+\theta(k-m)}{k}$$

on account of $\alpha = \frac{m\pi}{k}$.

§. 179. Therefore with this value T found, again we will discover

$$2\mathrm{S}\sin.\alpha = \gamma\sin.(\alpha+\zeta) + (\gamma-2(k-1)\beta)\sin.((2\mathrm{k}-1)\alpha-\zeta) + 2\beta\sin.\frac{m\pi+\theta(k-m)}{k},$$

which on account of $\frac{m\pi+\theta(k-m)}{k} = \alpha+\zeta$ is reduced to the form:

$$2S\sin.\alpha = (\gamma+2\beta)\sin.(\alpha+\zeta) + (\gamma-2(k-1)\beta)\sin.((2k-1)\alpha-\zeta),$$

which can be represented thus:

$$2\mathrm{S}\sin.\alpha = (\gamma+2\beta)\left(\sin.(\alpha+\zeta)+\sin.((2k-1)\alpha-\zeta)\right) - 2k\beta\sin.((2k-1)\alpha-\zeta),$$

where for the first part on account of

$$\sin.p + \sin.q = 2\sin.\tfrac{p+q}{2}\sin.\tfrac{p-q}{2}$$

there will be

$$\tfrac{p+q}{2} = \alpha k \text{ et } \tfrac{p-q}{2} = (k-1)\alpha-\zeta,$$

from which the first part shall itself become

$$2(\gamma+2\beta)\sin.ak\cos.((k-1)\alpha-\zeta);$$

where since there shall be $\alpha k = m\pi$, there will be $\sin.\alpha k = 0$, thus so that only

$$2\mathrm{S}\sin.\alpha = -2k\beta\sin.((2k-1)\alpha-\zeta)$$

shall remain, and hence

$$\mathrm{S} = -\frac{k\beta\sin.((2k-1)\alpha-\zeta)}{\sin.\alpha}$$

Truly there is

$$(2k-1)\alpha-\zeta = 2m\pi - \tfrac{m\pi}{k} - \tfrac{\theta(k-m)}{k};$$

therefore with the term $2m\pi$ omitted, there will be

$$\mathrm{S} = +\frac{\pi\sin.\left(\frac{m\pi}{k}+\frac{\theta(k-m)}{k}\right)}{\sin.\frac{m\pi}{k}}$$

and thus the value sought will be

$$\frac{\mathrm{S}}{k\sin\theta} = +\frac{\pi\sin.\left(\frac{m\pi}{k}+\frac{\theta(k-m)}{k}\right)}{k\sin\theta\sin.\frac{m\pi}{k}},$$

which form is reduced to this

$$\frac{\mathrm{S}}{k\sin\theta} = +\frac{\pi\sin.\left(\frac{m(\pi-\theta)+k\theta}{k}\right)}{k\sin\theta\sin.\frac{m\pi}{k}}.$$

§. 180. We will consider here before all the case, in which $\theta = \frac{\pi}{2}$, and the formula of the integral proposed will change into this

$$\int \frac{x^{m-1}\partial x}{1+x^{2k}},$$

the value of which therefore emerges, if there may be put $x = \infty$ after the integration,

$$= \frac{\pi\sin.\left(\frac{\pi}{2}+\frac{m\pi}{2k}\right)}{k\sin.\frac{m\pi}{k}} = \frac{\pi\cos.\left(\frac{m\pi}{2k}\right)}{k\sin.\frac{m\pi}{k}}.$$

Therefore since there is $\sin.\frac{m\pi}{k} = 2\sin.\frac{m\pi}{2k}\cos.\frac{m\pi}{2k}$, this same value will be produced

$$= \frac{\pi\sin.\left(\frac{\pi}{2}+\frac{m\pi}{2k}\right)}{k\sin.\frac{m\pi}{k}} = \frac{\pi}{2k\sin.\frac{m\pi}{2k}},$$

which outstanding value agrees with that, which we have assigned recently for the formula $\int \frac{x^{m-1}\partial x}{1+x^{2k}}$, if indeed there may be written $2k$ in place of k.

§. 181. Also, we may set out the case, in which $\theta = \pi$, and our integral formula becomes $\int \frac{x^{m-1}\partial x}{\left(1+x^{2k}\right)^{2}}$, the value of which on making $x = \infty$ will be

$$\frac{\pi\sin.\left(\frac{m(\pi-\theta)+k\theta}{k}\right)}{k\sin\theta\sin.\frac{m\pi}{k}} = \frac{\pi}{k\sin.\frac{m\pi}{k}}\cdot\frac{\sin.\left(\frac{m(\pi-\theta)}{k}+\theta\right)}{\sin\theta}.$$

But in the case $\theta=\pi$ both the numerator and the denominator of this latter fraction vanish ; whereby so that its true value may be elicited, in place of each we may write its differential, with which done this same fraction will change into this :

$$\frac{\partial\theta\left(1-\frac{m}{k}\right)\cos.\left(\frac{m(\pi-\theta)}{k}+\theta\right)}{\partial\theta\cos\theta},$$

of which the value on making $\theta=\pi$ now evidently is $1-\frac{m}{k}$; and thus the value of the integral sought will be $\left(1-\frac{m}{k}\right)\frac{\pi}{k\sin.\frac{m\pi}{k}}$, exactly as we have found in the above dissertation.

§. 182. But so that we may return the general value found more conveniently, we may put $\pi-\theta=\eta$ and there becomes $\sin.\theta=\sin.\eta$ and $\cos.\theta=-\cos.\eta$; then truly the angle will be :

$$\frac{m(\pi-\theta)}{k}+\theta=\frac{m\eta}{k}+\pi-\eta,$$

the sine of which is $\sin.(1-\frac{m}{k})\eta$, from which the value sought of our formula will be

$$\frac{\pi\sin.\left(1-\frac{m}{k}\right)\eta}{k\sin\eta\sin.\frac{m\pi}{k}},$$

and hence finally we have arrived at the following

THEOREM

§. 183. *If this formula of the integral*

$$\int\frac{x^{m-1}\partial x}{1-2x^k\cos.\theta+x^{2k}}$$

may be extended from the limit $x=0$ *as far as to the limit* $x=\infty$ *, its value will be*

$$=\frac{\pi\sin.\left(1-\frac{m}{k}\right)\eta}{k\sin\eta\sin.\frac{m\pi}{k}},$$

or if there shall be

$$\sin.\left(1-\frac{m}{k}\right)\eta=\sin\eta\cos.\frac{m\pi}{k}-\cos\eta\sin.\frac{m\pi}{k},$$

this same value also can be expressed in this manner

$$\frac{\pi\cos.\frac{m\eta}{k}}{k\sin.\frac{m\pi}{k}}-\frac{\pi\sin.\frac{m\eta}{k}}{k\operatorname{tang}.\eta\sin.\frac{m\pi}{k}}.$$

§. 184. Now we will consider this integral formula in another way

$$\int\frac{x^{m-1}\partial x}{1-2x^{k}\cos.\theta+x^{2k}},$$

the value of which may be put $=\mathrm{P}$ from the limit $x=0$ as far as to $x=1$, truly its same value from $x=1$ as far as to $x=\infty$ may be put $=\mathrm{Q}$, thus so that $\mathrm{P+Q}$ must show that same value as before. Now truly for the value Q requiring to be found we may put $x=\frac{1}{y}$ and our formula represented thus

$$\frac{x^{m-1}}{1+2x^{k}\cos.\eta+x^{2k}}\cdot\frac{\partial x}{x}$$

on account of $\frac{\partial x}{x}=-\frac{\partial y}{y}$ will adopt this form:

$$-\int\frac{y^{-m}}{1+2y^{-k}\cos.\eta+y^{-2k}}\cdot\frac{\partial y}{y}=-\int\frac{y^{2k-m-1}\partial y}{y^{2k}+2y^{k}\cos.\eta+1},$$

the value of which must be extended from the limit $y=1$ as far as to $y=0$. Therefore with these limits interchanged we will have

$$\mathrm{Q}=+\int\frac{y^{2k-m-1}\partial y}{y^{2k}+2y^{k}\cos.\eta+1}$$

from the limit $y=0$ as far as to $y=1$.

§. 185. Because nothing stands in the way, in each form the same condition of the integration will be prescribed for P and Q, from the term 0 as far as to 1, why in the latter in place of y we may write x, from which we will have this integral formula for $\mathrm{P+Q}$

$$\int\frac{x^{m-1}+x^{2k-m-1}}{1+2x^{k}\cos.\eta+x^{2k}}\partial x,$$

of which the value extended from the limit $x=0$ as far as to $x=1$ will be equal to this expression $\frac{\pi\sin.\left(1-\frac{m}{k}\right)\eta}{k\sin\eta\sin.\frac{m\pi}{k}}$. Therefore from these two integral formulas compared we will obtain the following most noteworthy theorem.

THEOREM

§. 186. *This integral formula*

$$\int \frac{x^{m-1}+x^{2k-m-1}}{1+2x^{k}\cos.\eta+x^{2k}}\partial x,$$

extended from the limit $x=0$ *as far as to the limit* $x=1$ *is equal to this integral formula*

$$\int \frac{x^{m-1}\partial x}{1+2x^{k}\cos.\eta+x^{2k}}$$

extended from the limit $x=0$ *as far as to the limit* $x=\infty$ *; and the value of each will be*

$$\frac{\pi\sin.\left(1-\frac{m}{k}\right)\eta}{k\sin\eta\sin.\frac{m\pi}{k}}.$$

§. 187. So that if we may expand this fraction $\frac{\sin.\eta}{1+2x^{k}\cos.\eta+x^{2k}}$ in an infinite series, which shall become

$$\sin.\eta+\mathrm{A}x^{k}+\mathrm{B}x^{2k}+\mathrm{C}x^{3k}+\mathrm{D}x^{4k}+\mathrm{E}x^{5k}+\text{etc.},$$

on multiplying by the denominator we will arrive at this infinite expression

$$\begin{aligned}\sin.\eta=\sin.\eta+{}&\mathrm{A}x^{k}+\mathrm{B}x^{2k}+\mathrm{C}x^{3k}+\mathrm{D}x^{4k}+\mathrm{E}x^{5k}+\text{etc.},\\ &+2\sin.\eta\cos.\eta+2\mathrm{A}\cos.\eta+2\mathrm{B}\cos.\eta+2\mathrm{C}\cos.\eta+2\mathrm{D}\cos.\eta+\text{etc.},\\ &\qquad+\sin.\eta+\mathrm{A}+\mathrm{B}+\mathrm{C}+\text{etc.}\end{aligned}$$

from which with these individual terms reduced to zero we will find :

1. $\mathrm{A}+2\sin.\eta\cos.\eta=0$ and hence $\mathrm{A}=-\sin.2\eta$,
2. $\mathrm{B}+2\mathrm{A}\cos.\eta+\sin.\eta=0$, from which there becomes $\mathrm{B}=-\sin.3\eta$,
3. $\mathrm{C}+2\mathrm{B}\cos.\eta+\mathrm{A}=0$, from which there becomes $\mathrm{C}=-\sin.4\eta$,
4. $\mathrm{D}+2\mathrm{C}\cos.\eta+\mathrm{B}=0$, from which there becomes $\mathrm{D}=\sin.5\eta$

etc. etc,

thus so that our fraction $\frac{\sin.\eta}{1+2x^{k}\cos.\eta+x^{2k}}$ may be resolved into this series

$$\sin.\eta-x^{k}\sin.2\eta+x^{2k}\sin.3\eta+x^{3k}\sin.4\eta+x^{4k}\sin.5\eta-\text{etc.}$$

§. 188. Now we may multiply this series by

$$x^{m-1}\partial x+x^{2k-m-1}\partial x$$

and after the integration we may make $x=1$, so that we may obtain the value of this formula

$$\sin.\eta\int\frac{x^{m-1}+x^{2k-m-1}}{1+2x^{k}\cos.\eta+x^{2k}}\partial x$$

for the case $x=1$, and in this manner we will arrive at the two following series :

$$\frac{\sin.\eta}{m}-\frac{\sin.2\eta}{m+k}+\frac{\sin.3\eta}{m+2k}-\frac{\sin.4\eta}{m+3k}+\frac{\sin.5\eta}{m+4k}-\text{etc.},$$
$$\frac{\sin.\eta}{2k-m}-\frac{\sin.2\eta}{3k-m}+\frac{\sin.3\eta}{4k-m}-\frac{\sin.4\eta}{5k-m}+\frac{\sin.5\eta}{6k-m}-\text{etc.}$$

Therefore the sum of these two series taken together will be equal to this value :

$$\frac{\pi\sin.\left(1-\frac{m}{k}\right)\eta}{k\sin.\frac{m\pi}{k}},$$

to which we may attach now this theorem.

THEOREM

§. 189. *If η may denote some angle, truly the letters m and k may be taken as it pleases, and from these the two following series may be formed*

$$\mathrm{P}=\frac{\sin.\eta}{m}-\frac{\sin.2\eta}{m+k}+\frac{\sin.3\eta}{m+2k}-\frac{\sin.4\eta}{m+3k}+\frac{\sin.5\eta}{m+4k}-\text{etc.},$$
$$\mathrm{Q}=\frac{\sin.\eta}{2k-m}-\frac{\sin.2\eta}{3k-m}+\frac{\sin.3\eta}{4k-m}-\frac{\sin.4\eta}{5k-m}+\frac{\sin.5\eta}{6k-m}-\text{etc.},$$

indeed neither of each sum can be shown, but the sum of each taken jointly will be

$$\mathrm{P}+\mathrm{Q}=\frac{\pi\sin.\left(1-\frac{m}{k}\right)\eta}{k\sin.\frac{m\pi}{k}}.$$

COROLLARY

§. 190. But if therefore we may take the angle η to be infinitely small, so that there shall become

$$\sin.\eta = \eta, \sin.2\eta = 2\eta,\ \sin.3\eta = 3\eta \text{ etc.},$$

because in the formula of the sum there will become

$$\sin.\left(1-\tfrac{m}{k}\right)\eta = \left(1-\tfrac{m}{k}\right)\eta,$$

so that we may divide each by side by η, we will obtain the two following series :

$$\frac{1}{m} - \frac{2}{m+k} + \frac{3}{m+2k} - \frac{4}{m+3k} + \frac{5}{m+4k} - \text{etc.}$$
$$+\frac{1}{2k-m} - \frac{2}{3k-m} + \frac{3}{4k-m} - \frac{4}{5k-m} + \frac{5}{6k-m} - \text{etc.},$$

the sum of which will be therefore $\left(1-\frac{m}{k}\right)\frac{\pi}{k\sin.\frac{m\pi}{k}}$; where it may be observed both these series are able to be contracted consistently to become

$$\frac{2k}{m(2k-m)} - \frac{8k}{(m+k)(3k-m)} + \frac{18k}{(m+2k)(4k-m)} - \frac{32k}{(m+3k)(5k-m)} + \text{etc.}$$

where the numerators are twice the squares of numbers.

§. 191. But the formulas, the values of which we have just found, are able to be expressed much more concisely and elegantly, if in place of the exponent *m* we may write $k-n$; then indeed in the value of the integral found there will become $\left(1-\frac{m}{k}\right)\eta = \frac{n\eta}{k}$, but truly for the denominator there will become $\frac{m\pi}{k} = \pi - \frac{n\pi}{k}$ the sine of which will be $\sin.\frac{n\pi}{k}$; and thus our formula found will adopt this form $\frac{\pi\sin.\frac{n\eta}{k}}{k\sin.\eta\sin.\frac{n\pi}{k}}$, which therefore will express the value of this integral formula

$$\int \frac{x^{k-n-1}\partial x}{1+2x^{k}\cos.\eta + x^{2k}}$$

from $x=0$ to $x=\infty$, as of this formula

$$\int \frac{x^{k-n-1}+x^{k+n-1}}{1+2x^{k}\cos.\eta + x^{2k}}\partial x$$

from the limit $x=0$ up to the limit $x=1$; and because the value of each is

$$\frac{\pi\sin.\frac{n\eta}{k}}{k\sin.\eta\sin.\frac{n\pi}{k}},$$

it is evident that to remain the same, and if in place of n there may be written $-n,$ from which the first formula will be able to be represented thus

$$\int\frac{x^{k\pm n-1}\partial x}{1+2x^{k}\cos.\eta+x^{2k}};$$

but the latter formula plainly no change is allowed on account of this ambiguity.

§. 192. By putting $m=k-n$ also our double series will be taken to be made prettier; indeed there will become :

$$\begin{aligned}&\frac{\sin.\eta}{k-n}-\frac{\sin.2\eta}{2k-n}+\frac{\sin.3\eta}{3k-n}-\frac{\sin.4\eta}{4k-n}+\text{etc.}\\+&\frac{\sin.\eta}{k+n}-\frac{\sin.2\eta}{2k+n}+\frac{\sin.3\eta}{3k+n}-\frac{\sin.4\eta}{4k+n}+\text{etc.}\end{aligned}$$

therefore the sum of which will be $\frac{\pi\sin.\frac{n\eta}{k}}{k\sin.\frac{n\pi}{k}}$. Then truly if these two series may be contracted into one and each side may be divided by *2k*, the following summation worthy of note will be obtained :

$$\frac{\pi\sin.\frac{n\eta}{k}}{2kk\sin.\frac{n\pi}{k}}=\frac{\sin.\eta}{kk-nn}-\frac{2\sin.2\eta}{4kk-nn}+\frac{3\sin.3\eta}{9kk-nn}-\frac{\sin.4\eta}{16kk-nn}+\text{etc.}$$

§. 193. But if this last series may be differentiated by taking only the angle η variable on account of $\partial\sin.\frac{n\eta}{k}=\frac{nd\eta}{k}\cos.\frac{n\eta}{k}$ we will have :

$$\frac{\pi n\cos.\frac{n\eta}{k}}{2k^{3}\sin.\frac{n\pi}{k}}=\frac{\cos.\eta}{kk-nn}-\frac{4\cos.2\eta}{4kk-nn}+\frac{9\cos.3\eta}{9kk-nn}-\frac{16\cos.4\eta}{16kk-nn}+\text{etc.}$$

From which if there shall be taken $\eta=0$, this summation may arise :

$$\frac{\pi n}{2k^{3}\sin.\frac{n\pi}{k}}=\frac{1}{kk-nn}-\frac{4}{4kk-nn}+\frac{9}{9kk-nn}-\frac{16}{16kk-nn}+\text{etc.};$$

but if there may be taken $\eta=90^{\circ}=\frac{\pi}{2}$, there will be

$$\cos.\eta = 0,\ \ \cos.2\eta = -1,\ \ \cos.3\eta = 0,\ \ \cos.4\eta = +1 \text{ etc.},$$

from which the following series arises :

$$\frac{n\pi\cos.\frac{n\pi}{2k}}{2k^3\sin.\frac{n\pi}{k}} = \frac{4}{4kk-nn} - \frac{16}{16kk-nn} + \frac{36}{36kk-nn} - \frac{64}{64kk-nn} + \text{etc.}$$

But since $\sin.\frac{n\pi}{k} = 2\sin.\frac{n\pi}{2k}\cos.\frac{n\pi}{2k}$, the sum of this same series will be $\frac{n\pi}{4k^3\sin.\frac{n\pi}{2k}}$.

§. 194. But if that series shown in § 192 may be multiplied by $\partial\eta$ and integrated, on account of $\int\partial\eta\sin.\frac{n\eta}{k} = -\frac{k}{n}\cos.\frac{n\eta}{k}$ there will be

$$\text{C} - \frac{\pi\cos.\frac{n\eta}{k}}{2nk\sin.\frac{n\pi}{k}} = -\frac{\cos.\eta}{kk-nn} + \frac{\cos.2\eta}{4kk-nn} - \frac{\cos.3\eta}{9kk-nn} + \frac{\cos.4\eta}{16kk-nn} - \text{etc.}$$

But as here we may define the constant C requiring to be added, we may assume $\eta = 0$ and there will become

$$\text{C} - \frac{\pi}{2nk\sin.\frac{n\pi}{k}} = -\frac{1}{kk-nn} + \frac{1}{4kk-nn} - \frac{1}{9kk-nn} + \text{etc.};$$

whereby if the sum of this series may appear from elsewhere, the constant C will be able to be defined.
But the series in the following double series will be able to be resolved :

$$2n\text{C} - \frac{\pi}{k\sin.\frac{n\pi}{k}} = \frac{1}{k+n} - \frac{1}{2k+n} + \frac{1}{3k+n} - \frac{1}{4k+n} + \text{etc.}$$
$$-\frac{1}{k-n} + \frac{1}{2k+n} - \frac{1}{3k+n} + \frac{1}{4k-n} + \text{etc.}$$

35. Therefore since in the *Introductione in Analysin lnfinitorum* pag. 142 we may come upon the series

$$\frac{1}{kk-nn} - \frac{1}{4kk-nn} + \frac{1}{9kk-nn} - \frac{1}{16kk-nn} + \text{etc.} = \frac{\pi}{2kn\sin.\frac{n\pi}{k}} - \frac{1}{2nn}$$

(in place of the letters *m* and *n* used there, here evidently to have written *n* and *k*), with this value used our equation will be

$$\text{C} - \frac{\pi}{2nk\sin.\frac{n\pi}{k}} = \frac{1}{2nn} - \frac{\pi}{2nk\sin.\frac{n\pi}{k}},$$

from which there becomes $\text{C} = \frac{1}{2nn}$. Hence therefore we will have this summation :

$$\frac{\pi\cos.\frac{n\eta}{k}}{2kn\sin.\frac{n\pi}{k}} - \frac{1}{2nn} = \frac{\cos.\eta}{kk-nn} - \frac{\cos.2\eta}{4kk-nn} + \frac{\cos.3\eta}{9kk-nn} - \frac{\cos.4\eta}{16kk-nn} + \text{etc.},$$

which series certainly may be seen to worthy of all attention.

5) Investigatio formulae integralis $\int \frac{x^{m-1}\partial x}{\left(1+x^{k}\right)^{n}}$, casu quo post integrationem statuitur $x=\infty$.

Opuscula Analytica. Tom. II. Pag. 42-54.

Commentatio 588 indicis ENESTROEMIANI
Opuscula. analytica 2, 1785, p. 42-54

§. 141. Iam satis notum est huius formulae integrale [casu, quo $n = 1$] partim logarithmos, partim arcus circulares complecti et partes logarithmicas hanc progressionem constituere

$$-\tfrac{2}{k}\cos.\tfrac{m\pi}{k}\, l\sqrt{\left(1-2x\cos.\tfrac{\pi}{k}+xx\right)}$$
$$-\tfrac{2}{k}\cos.\tfrac{3m\pi}{k}\, l\sqrt{\left(1-2x\cos.\tfrac{3\pi}{k}+xx\right)}$$
$$-\tfrac{2}{k}\cos.\tfrac{5m\pi}{k}\, l\sqrt{\left(1-2x\cos.\tfrac{5\pi}{k}+xx\right)}$$
$$-\tfrac{2}{k}\cos.\tfrac{7m\pi}{k}\, l\sqrt{\left(1-2x\cos.\tfrac{7\pi}{k}+xx\right)}$$
.
.
.
$$-\tfrac{2}{k}\cos.\tfrac{im\pi}{k}\, l\sqrt{\left(1-2x\cos.\tfrac{i\pi}{k}+xx\right)},$$

ubi *i* denotat numerum imparem non maiorem quam *k*. Hinc si *k* fuerit numerus par, erit $i=k-1$; ac si *k* fuerit numerus impar, hanc progressionem continuari oportet usque ad $i=k$, eius vero coefficiens duplo minor capi debet seu loco $-\frac{2}{k}$ tantum scribi debet $-\frac{1}{k}$, cuius irregularitatis ratio in *Calculo Integrali* est exposita.

§. 142. Cum hae partes sponte iam evanescant posito $x=0$, statuamus statim $x=\infty$, et cum in genere sit

$$\sqrt{\left(1-2x\cos.\omega+xx\right)}=x-\cos.\omega \text{ ,erit}$$

$$l\sqrt{\left(1-2x\cos.\omega+xx\right)}=l\left(x-\cos.\omega\right)=lx-\tfrac{\cos.\omega}{x}=lx, \text{ ob } \tfrac{\cos.\omega}{x}=0;$$

omnes ergo illi logarithmi reducuntur ad eandem formam *lx,* quae multiplicanda est per hanc seriem

$$-\tfrac{2}{k}\cos.\tfrac{m\pi}{k}-\tfrac{2}{k}\cos.\tfrac{3m\pi}{k}-\tfrac{2}{k}\cos.\tfrac{5m\pi}{k}-\tfrac{2}{k}\cos.\tfrac{7m\pi}{k}\cdots-\tfrac{2}{k}\cos.\tfrac{im\pi}{k},$$

ubi, ut diximus, i denotat maximum numerum imparem ipso k non maiorem, hac tamen restrictione, ut, si k fuerit impar ideoque $i=k,$ ultimum membrum ad dimidium reduci debeat. Quamobrem si huius progressionis summam investigare velimus, duo casus erunt constituendi, alter, quo k est numerus par et $i=k-1$, alter vero, quo k est impar et $i=k$.

EVOLUTIO CASUS PRIORIS QUO k EST NUMERUS PAR ET $i=k-1$

§. 143. Hoc ergo casu posito $x=\infty$ formula $-\frac{2}{k}lx$ multiplicatur per hanc cosinuum seriem

$$\cos.\tfrac{mx}{k}+\cos.\tfrac{3mx}{k}+\cos.\tfrac{5mx}{k}+\cos.\tfrac{7mx}{k}\cdots+\cos.\tfrac{(k-1)mx}{k},$$

cuius summam statuamus $=$ S . Ducamus hanc seriem in $\sin.\frac{m\pi}{k}$, et cum in genere sit

$$\sin.\tfrac{m\pi}{k}\cos.\tfrac{im\pi}{k}=\tfrac{1}{2}\sin.\tfrac{(i+1)m\pi}{k}-\tfrac{1}{2}\sin.\tfrac{(i-1)m\pi}{k},$$

facta hac reductione habebimus

$$\begin{aligned}&S\sin.\tfrac{m\pi}{k}\\ &=\tfrac{1}{2}\sin.\tfrac{2m\pi}{k}+\tfrac{1}{2}\sin.\tfrac{4m\pi}{k}+\tfrac{1}{2}\sin.\tfrac{6m\pi}{k}+\cdots+\tfrac{1}{2}\sin.\tfrac{(k-2)m\pi}{k}+\tfrac{1}{2}\sin.\tfrac{km\pi}{k}\\ &-\tfrac{1}{2}\sin.\tfrac{2m\pi}{k}-\tfrac{1}{2}\sin.\tfrac{4m\pi}{k}-\tfrac{1}{2}\sin.\tfrac{6m\pi}{k}-\cdots-\tfrac{1}{2}\sin.\tfrac{(k-2)m\pi}{k},\end{aligned}$$

ubi omnes termini praeter ultimum manifesto se destruunt, ita ut sit

$$S\sin.\tfrac{m\pi}{k}=\tfrac{1}{2}\sin.m\pi.$$

Iam vero, quia nostri coefficientes m et k supponuntur integri, utique erit $\sin.m\pi.=0$ ideoque etiam $S=0$, nisi forte etiam fuerit $\sin.\frac{m\pi}{k}=0,$ qui autem casus locum habere nequit, quoniam in integratione formulae propositae $\frac{x^{m-1}\partial x}{\left(1+x^k\right)^n}$ semper assumi solet esse $m<k$. Hoc igitur modo evictum est casu, quo post integrationem statuitur $x=\infty$, omnes partes logarithmicas integralis se destruere.

EVOLUTIO CASUS ALTERIUS QUO EST k NUMERUS IMPAR ET $i = k$.

§. 144. Hoc ergo casu sumto $x = \infty$ formula lx multiplicatur per hanc seriem

$$-\tfrac{2}{k}\cos.\tfrac{m\pi}{k} - \tfrac{2}{k}\cos.\tfrac{3m\pi}{k} - \tfrac{2}{k}\cos.\tfrac{5m\pi}{k} - \tfrac{2}{k}\cos.\tfrac{7m\pi}{k} \cdots - \tfrac{2}{k}\cos.\tfrac{im\pi}{k},$$

ubi terminus penultimus est $-\tfrac{2}{k}\cos.\frac{(k-2)m\pi}{k}$, pro ultimo vero termino erit $\cos.m\pi = \pm 1$ signo superiore valente, si n sit numerus par, inferiore, si impar; quare remoto termino ultimo pro reliquis ponamus

$$\cos.\tfrac{m\pi}{k} + \cos.\tfrac{3m\pi}{k} + \cos.\tfrac{5m\pi}{k} + \cos.\frac{(k-2)m\pi}{k} = S,$$

ita ut multiplicator ipsius logarithmi x sit

$$-\tfrac{2S}{k} - \tfrac{1}{k}\cos.m\pi.$$

Hinc procedendo ut ante fiet

$$\begin{aligned} &\mathrm{S}\sin.\tfrac{m\pi}{k} \\ &= \tfrac{1}{2}\sin.\tfrac{2m\pi}{k} + \tfrac{1}{2}\sin.\tfrac{4m\pi}{k} + \tfrac{1}{2}\sin.\tfrac{6m\pi}{k} + \cdots + \tfrac{1}{2}\sin.\frac{(k-3)m\pi}{k} + \tfrac{1}{2}\sin.\frac{(k-1)m\pi}{k} \\ &-\tfrac{1}{2}\sin.\tfrac{2m\pi}{k} - \tfrac{1}{2}\sin.\tfrac{4m\pi}{k} - \tfrac{1}{2}\sin.\tfrac{6m\pi}{k} - \cdots - \tfrac{1}{2}\sin.\frac{(k-3)m\pi}{k}, \end{aligned}$$

ubi iterum omnes termini praeter ultimum se mutuo tollunt, ita ut hinc prodeat

$$\mathrm{S}\sin.\tfrac{m\pi}{k} = \tfrac{1}{2}\sin.\frac{(k-1)m\pi}{k} = \tfrac{1}{2}\sin.\left(m\pi - \tfrac{m\pi}{k}\right);$$

at vero est

$$\sin.\left(m\pi - \tfrac{m\pi}{k}\right) = \sin.m\pi\cos.\tfrac{m\pi}{k} - \cos.m\pi\sin.\tfrac{m\pi}{k}$$

ubi notetur esse $\sin.m\pi = 0$ ob m numerum integrum; habebimus ergo

$$\mathrm{S}\sin.\tfrac{m\pi}{k} = -\tfrac{1}{2}\cos.\left(m\pi - \tfrac{m\pi}{k}\right) \text{ sive } \mathrm{S} = -\tfrac{1}{2}\cos.m\pi,$$

consequenter multiplicator ipsius lx erit

$$= \tfrac{1}{k}\cos.m\pi - \tfrac{1}{k}\cos.m\pi = 0$$

sicque manifestum est, sive k sit numerus par sive impar, omnia membra logarithmica in nostro integrali se mutuo destruere, si quidem post integrationem statuamus $x=\infty$, quemadmodum hic semper supponimus.

§. 145. Consideremus nunc etiam partes a circulo pendentes, ex quibus integrale nostrae formulae componitur. Hae autem partes sequentem progressionem constituere sunt compertae:

$$\tfrac{2}{k}\sin.\tfrac{m\pi}{k}\,\text{Atang.}\frac{x\sin.\frac{\pi}{k}}{1-x\cos.\frac{\pi}{k}}+\tfrac{2}{k}\sin.\tfrac{3m\pi}{k}\,\text{Atang.}\frac{x\sin.\frac{3\pi}{k}}{1-x\cos.\frac{3\pi}{k}}$$

$$+\cdots\cdots+\tfrac{2}{k}\sin.\tfrac{im\pi}{k}\,\text{Atang.}\frac{x\sin.\frac{i\pi}{k}}{1-x\cos.\frac{i\pi}{k}},$$

ubi in ultimo membro est vel $i=k-1$ vel $i=k$*;* prius scilicet valet, si i est numerus par, posterius, si impar.

§. 146. Cum etiam omnia haec membra evanescant posito $x=0$, faciamus pro instituto nostro $x=\infty$. In genere igitur fiet

$$\text{Atang.}\frac{x\sin.\frac{i\pi}{k}}{1-x\cos.\frac{i\pi}{k}}=\text{Atang.}\left(\text{tang.}\tfrac{i\pi}{k}\right).$$

Est vero

$$-\ \text{tang.}\tfrac{i\pi}{k}=+\ \text{tang.}\frac{(k-i)\pi}{k},$$

ex quo hic arcus fit $=\frac{(k-i)\pi}{k}$. Hinc ergo loco i scribendo successive numeros 1, 3, 5, 7 etc. istae partes nostri integralis quaesiti erunt

$$\tfrac{2(k-1)\pi}{kk}\sin.\tfrac{m\pi}{k}+\tfrac{2(k-3)\pi}{kk}\sin.\tfrac{3m\pi}{k}+\tfrac{2(k-5)\pi}{kk}\sin.\tfrac{5m\pi}{k}+\tfrac{2(k-7)\pi}{kk}\sin.\tfrac{7m\pi}{k}+$$
$$\tfrac{2(k-9)\pi}{kk}\sin.\tfrac{9m\pi}{k}+\cdots\cdots\tfrac{2(k-i)\pi}{kk}\sin.\tfrac{im\pi}{k},$$

ubi casu, quo k est numerus par, progredi oportet usque ad $i=k-1$, ac si k sit numerus impar, usque ad $i=k$.

§. 147. Statuamus brevitatis gratia

$$\left(k-1\right)\sin.\tfrac{m\pi}{k}+\left(k-3\right)\sin.\tfrac{3m\pi}{k}+\left(k-5\right)\sin.\tfrac{5m\pi}{k}+\cdots\cdots+\left(k-i\right)\sin.\tfrac{im\pi}{k}=\text{S},$$

ita ut integrale quaesitum sit $\frac{2\pi S}{kk}$, quandoquidem partes logarithmicae se mutuo destruxerunt. Multiplicemus nunc utrinque per $2\sin.\frac{m\pi}{k}$, et cum in genere sit

$$2\sin.\tfrac{m\pi}{k}\sin.\tfrac{im\pi}{k} = \cos.\tfrac{(i-1)m\pi}{k} - \cos.\tfrac{(i+1)m\pi}{k},$$

facta substitutione erit

$$\begin{aligned}2S\sin.\tfrac{m\pi}{k} = &(k-1)\cos.\tfrac{0m\pi}{k} + (k-3)\cos.\tfrac{2m\pi}{k} + (k-5)\cos.\tfrac{4m\pi}{k} + \cdots + (k-i)\cos.\tfrac{(i-1)m\pi}{k}\\ &-(k-1)\cos.\tfrac{2m\pi}{k} - (k-3)\cos.\tfrac{4m\pi}{k} - \cdots - (k-i+2)\cos.\tfrac{(i-1)m\pi}{k} - (k-i)\cos.\tfrac{(i+1)m\pi}{k},\end{aligned}$$

quae series manifesto contrahitur in sequentem

$$\begin{aligned}2S\sin.\tfrac{m\pi}{k} = k-1-2\cos.\tfrac{2m\pi}{k} - 2\cos.\tfrac{4m\pi}{k} - 2\cos.\tfrac{6m\pi}{k} - \cdots - 2\cos.\tfrac{(i-1)m\pi}{k}\\ -(k-i)\cos.\tfrac{(i+1)m\pi}{k},\end{aligned}$$

ubi primo et ultimo membro sublatis regularem termini intermedii constituunt seriem, pro cuius valore investigando ponamus

$$T = \cos.\tfrac{2m\pi}{k} + \cos.\tfrac{4m\pi}{k} + \cos.\tfrac{6m\pi}{k} + \cdots + \cos.\tfrac{(i-1)m\pi}{k},$$

ita ut sit

$$2S\sin.\tfrac{m\pi}{k} = k-1-2T-(k-i)\cos.\tfrac{(i+1)m\pi}{k}.$$

Hic autem iterum convenit duos casus perpendere, prout k fuerit par vel impar.

EVOLUTIO CASUS PRIORIS QUO k EST NUMERUS PAR ET $i = k-1$

§. 148. Hoc ergo casu habebimus

$$T = \cos.\tfrac{2m\pi}{k} + \cos.\tfrac{4m\pi}{k} + \cos.\tfrac{6m\pi}{k} + \cdots + \cos.\tfrac{(k-2)m\pi}{k}.$$

Multiplicemus denuo per $2\sin.\frac{m\pi}{k}$ et per reductiones supra indicatas habebimus

$$\begin{aligned}2T\sin.\tfrac{m\pi}{k} = \sin.\tfrac{3m\pi}{k} + \sin.\tfrac{5m\pi}{k} + \cdots + \sin.\tfrac{(k-3)m\pi}{k} + \sin.\tfrac{(k-1)m\pi}{k}\\ -\sin.\tfrac{m\pi}{k} - \sin.\tfrac{3m\pi}{k} - \sin.\tfrac{5m\pi}{k} - \cdots - \sin.\tfrac{(k-3)m\pi}{k};\end{aligned}$$

deletis igitur terminis se mutuo tollentibus erit

$$2\mathrm{T}\sin.\tfrac{m\pi}{k} = -\sin.\tfrac{m\pi}{k} + \sin.\tfrac{(k-1)m\pi}{k}.$$

Est vero

$$\sin.\tfrac{(k-1)m\pi}{k} = \sin.\left(m\pi - \tfrac{m\pi}{k}\right) = \sin.m\pi\cos.\tfrac{m\pi}{k} - \cos.m\pi\sin.\tfrac{m\pi}{k},$$

ubi $\sin.m\pi = 0$, quamobrem fiet

$$2\mathrm{T} = -1 - \cos.m\pi.$$

§. 149. Invento valore pro T colligitur fore

$$2S\sin.\tfrac{m\pi}{k} = k \text{ ideoque } S = \frac{k}{2\sin.\frac{m\pi}{k}}.$$

Denique vero ipse valor formulae nostrae integralis, quem quaerimus, erit $\frac{2\pi S}{kk}$ et nunc manifestum est integrale nostrae formulae casu, quo S est numerus par, fore $\frac{\pi}{k\sin.\frac{m\pi}{k}}$, siquidem post integrationem statuatur $x = \infty$.

EVOLUTIO ALTERIUS CASUS QUO k EST NUMERUS IMPAR ET $i = k$

§. 150. Hoc ergo casu est

$$\mathrm{T} = \cos.\tfrac{2m\pi}{k} + \cos.\tfrac{4m\pi}{k} + \cos.\tfrac{6m\pi}{k} + \cdots + \cos.\tfrac{(k-1)m\pi}{k},$$

quae series multiplicata per $2\sin.\frac{m\pi}{k}$ producet ut ante

$$\begin{aligned} 2\mathrm{T}\sin.\tfrac{m\pi}{k} &= \sin.\tfrac{3m\pi}{k} + \sin.\tfrac{5m\pi}{k} + \cdots + \sin.\tfrac{(k-2)m\pi}{k} + \sin.\tfrac{km\pi}{k} \\ &\quad -\sin.\tfrac{m\pi}{k} - \sin.\tfrac{3m\pi}{k} - \sin.\tfrac{5m\pi}{k} - \cdots - \sin.\tfrac{(k-2)m\pi}{k}, \end{aligned}$$

unde deletis terminis se mutuo tollentibus reperietur

$$2\mathrm{T}\sin.\tfrac{m\pi}{k} = -\sin.\tfrac{m\pi}{k} + \sin.m\pi$$

ideoque

$$2\mathrm{T} = -1 + \frac{\sin.m\pi}{\sin.\frac{m\pi}{k}} = -1$$

ob $\sin.m\pi = 0$, hincque porro fiet

$$2S\sin.\tfrac{m\pi}{k}=k;$$

quare cum valor integralis quaesitus sit $\frac{2\pi S}{kk}$, erit etiam hoc casu integrale nostrum $=\frac{\pi}{k\sin.\frac{m\pi}{k}}$ prorsus uti praecedente casu. Hinc ergo deducimus sequens

THEOREMA

§. 151. *Si haec formula differentialis*

$$\frac{x^{m-1}\partial x}{1+x^k}$$

ita integretur, ut posito $x=0$ *integrale evanescat, tum vero statuatur* $x=\infty$, *valor inde resultans semper erit*

$$\frac{\pi}{k\sin.\frac{m\pi}{k}},$$

sive k sit numerus par sive impar.

Huius theorematis demonstratio ex praecedentibus est manifesta.

§. 152. In evolutione huius formulae assumsimus esse $m<k$, quia alioquin membra logarithmica se non destruissent; at vero ne hac quidem limitatione nunc amplius est opus. Casu enim, quo foret $m=k$, integrale formulae $\frac{x^{m-1}\partial x}{1+x^k}$ esset $\frac{1}{k}l\left(1+x^k\right)$, quod facto $x=\infty$ fieret etiam ∞; verum hoc idem indicat nostrum integrale esse $\frac{\pi}{k\sin.\frac{m\pi}{k}}=\infty$. Dummodo ergo *m* non fuerit maius quam *k,* nostra formula veritati semper est consentanea.

§. 153. Quin etiam ne quidem necesse est, ut exponentes *m* et *k* sint numeri integri, dummodo non fuerit $m>k$; si enim fuerit $m=\frac{\mu}{\lambda}$ et $k=\frac{\chi}{\lambda}$ erit valor per nostram formulam $\frac{\lambda\pi}{\chi\sin.\frac{\mu\pi}{\chi}}$, cuius veritas ita ostenditur. Quia hoc casu formula integranda est

$$\int\frac{x^{\frac{\mu}{\lambda}}}{1+x^{\frac{\chi}{\lambda}}}\cdot\frac{\partial x}{x},$$

statuatur $x=y^{\lambda}$; erit $\frac{\partial x}{x}=\frac{\lambda\partial y}{y}$ et formula fiet

$$\int \frac{y^\mu}{1+y^\chi} \cdot \frac{\lambda \partial y}{y} = \lambda \int \frac{y^{\mu-1}\partial y}{1+y^\chi},$$

cuius valor utique erit $\frac{\lambda\pi}{\chi \sin.\frac{\mu\pi}{\chi}}$.

ALIA DEMONSTRATIO THEOREMATIS

§. 154. Denotet P valorem integralis $\int \frac{x^m}{1+x^k} \cdot \frac{\partial x}{x}$ a termino $x=0$ usque ad $x=1$, at Q valorem eiusdem integralis a termino $x=1$ usque ad $x=\infty$, ita ut $\mathrm{P}+\mathrm{Q}$ praebeat eum ipsum valorem, qui in theoremate continetur. Nunc pro valore *Q* inveniendo statuatur $x=\frac{1}{y}$, unde fit $\frac{\partial x}{x}=-\frac{\partial y}{y}$, fietque

$$\mathrm{Q} = \int \frac{y^{-m}}{1+y^{-k}} \cdot \frac{-\partial y}{y} = -\int \frac{y^{k-m}}{1+y^k} \cdot \frac{\partial y}{y}$$

a termino $y=1$ usque ad $y=0$. Hinc igitur commutatis terminis erit

$$\mathrm{Q} = +\int \frac{y^{k-m}}{1+y^k} \cdot \frac{\partial y}{y}$$

a termino $y=0$ usque ad $y=1$. Iam quia hoc integrali expedito littera *y* ex calculo egreditur, loco *y* scribere licebit *x,* ita ut sit

$$\mathrm{Q} = \int \frac{x^{k-m}}{1+x^k} \cdot \frac{\partial x}{x},$$

quo facto habebimus

$$\mathrm{P}+\mathrm{Q} = \int \frac{x^m + x^{k-m}}{1+x^k} \cdot \frac{\partial x}{x}$$

a termino $x=0$ usque ad terminum $x=1$. Verum non ita pridem demonstravi valorem huius formulae integralis intra terminos $x=0$ et $x=1$ contentum esse $=\frac{\pi}{k\sin.\frac{m\pi}{k}}$.. Hinc igitur nascitur sequens theorema non minus notatu dignum.

THEOREMA

§. 155. *Valor huius formulae integralis*

$$\int \frac{x^m + x^{k-m}}{1+x^k} \cdot \frac{\partial x}{x}$$

intra terminos $x=0$ *et* $x=1$ *contentus aequalis est valori istius integralis*

$$\int \frac{x^m}{1+x^k}\cdot\frac{\partial x}{x}$$

intra terminos $x=0$ *et* $x=\infty$ *contento.*

§. 156. His expensis formulam integralem in titulo propositam aggrediamur, et quo eam ad formam hactenus tractatam reducamus, in subsidium vocemus sequentem reductionem

$$\int \frac{x^{m-1}\partial x}{\left(1+x^k\right)^{\lambda+1}} = \frac{\mathrm{A}x^m}{\left(1+x^k\right)^{\lambda}} + \mathrm{B}\int \frac{x^{m-1}\partial x}{\left(1+x^k\right)^{\lambda}},$$

unde facta differentiatione prodit sequens aequatio

$$\frac{x^{m-1}\partial x}{\left(1+x^k\right)^{\lambda+1}} = \frac{m\mathrm{A}x^{m-1}\partial x}{\left(1+x^k\right)^{\lambda}} - \frac{\lambda k\mathrm{A}x^{m+k-1}\partial x}{\left(1+x^k\right)^{\lambda+1}} + \mathrm{B}\frac{x^{m-1}\partial x}{\left(1+x^k\right)^{\lambda}},$$

quae aequatio per $x^{m-1}\partial x$ divisa ac per $\left(1+x^k\right)^{\lambda}$ multiplicata terminum negativum a dextra ad sinistram transponendo erit

$$\frac{1+\lambda k\mathrm{A}x^k}{1+x^k} = m\mathrm{A}+\mathrm{B},$$

quae aequatio manifesto subsistere nequit, nisi sit $\lambda k\mathrm{A}=1$ sive $\mathrm{A}=\frac{1}{\lambda k}$, unde erit $1=m\mathrm{A}+\mathrm{B}=\frac{m}{\lambda k}+\mathrm{B}$, sicque erit $\mathrm{B}=1-\frac{m}{\lambda k}$.

§. 157. Inventis his valoribus pro litteris *A* et *B* primum assumimus integralia ita capi, ut evanescant posito $x=0$; tum vero posito $x=\infty$, quia exponens *n* minor supponitur quam *k,* membrum absolutum littera A affectum sponte evanescit, ita ut hoc casu $x=\infty$ fiat

$$\int \frac{x^{m-1}\partial x}{\left(1+x^k\right)^{\lambda+1}} = \left(1-\frac{m}{\lambda k}\right)\int \frac{x^{m-1}\partial x}{\left(1+x^k\right)^{\lambda}}.$$

Quodsi iam primo capiamus $\lambda=1$, quia ante invenimus pro eodem casu $x=\infty$ esse

$$\int \frac{x^{m-1}\partial x}{1+x^k} = \frac{\pi}{k\sin.\frac{m\pi}{k}},$$

habebimus valorem istius integralis

$$\int \frac{x^{m-1}\partial x}{\left(1+x^{k}\right)^{2}} = \left(1-\frac{m}{k}\right)\frac{\pi}{k\sin.\frac{m\pi}{k}},$$

siquidem integrale etiam a termino $x=0$ usque ad terminum $x=\infty$ extendatur.

§. 158. Quodsi iam simili modo ponamus $\lambda = 2$, reperietur pro iisdem terminis integrationis

$$\int \frac{x^{m-1}\partial x}{\left(1+x^{k}\right)^{3}} = \left(1-\frac{m}{k}\right)\left(1-\frac{m}{2k}\right)\frac{\pi}{k\sin.\frac{m\pi}{k}};$$

eodem modo si litterae λ continuo maiores valores tribuantur, reperientur sequentes integralium formae omni attentione dignae

$$\int \frac{x^{m-1}\partial x}{\left(1+x^{k}\right)^{4}} = \left(1-\frac{m}{k}\right)\left(1-\frac{m}{2k}\right)\left(1-\frac{m}{3k}\right)\frac{\pi}{k\sin.\frac{m\pi}{k}},$$

$$\int \frac{x^{m-1}\partial x}{\left(1+x^{k}\right)^{5}} = \left(1-\frac{m}{k}\right)\left(1-\frac{m}{2k}\right)\left(1-\frac{m}{3k}\right)\left(1-\frac{m}{4k}\right)\frac{\pi}{k\sin.\frac{m\pi}{k}},$$

$$\int \frac{x^{m-1}\partial x}{\left(1+x^{k}\right)^{6}} = \left(1-\frac{m}{k}\right)\left(1-\frac{m}{2k}\right)\left(1-\frac{m}{3k}\right)\left(1-\frac{m}{4k}\right)\left(1-\frac{m}{5k}\right)\frac{\pi}{k\sin.\frac{m\pi}{k}},$$

etc.

§. 159. Quare si littera *n* denotet numerum quemcunque integrum pro formula in titulo expressa, si eius integrale a termino $x=0$ usque ad $x=\infty$ extendatur, eius valor sequenti modo se habebit:

$$\left(1-\frac{m}{k}\right)\left(1-\frac{m}{2k}\right)\left(1-\frac{m}{3k}\right)\left(1-\frac{m}{4k}\right)\cdots\left(1-\frac{m}{(n-1)k}\right)\frac{\pi}{k\sin.\frac{m\pi}{k}},$$

qui ergo conveniet huic formulae integrali

$$\int \frac{x^{m-1}\partial x}{\left(1+x^{k}\right)^{n}}.$$

§. 160. Hic quidem necessario pro *n* alii numeri praeter integros accipi non licet; at vero per methodum interpolationum, quae fusius iam passim est explicata, hanc integrationem etiam ad casus, quibus exponens *n* est numerus fractus, extendere licet. Quodsi enim sequentes formulae integrales a termino $y=0$ usque ad $y=1$ extendantur, in genere valor nostrae formulae propositae ita repraesentari poterit

$$\int\frac{x^{m-1}\partial x}{\left(1+x^{k}\right)^{n}}=\frac{\pi}{k\sin.\frac{m\pi}{k}}\cdot\frac{\int y^{nk-m-1}\partial y\left(1-y^{k}\right)^{\frac{m}{k}-1}}{\int y^{k-m-1}\partial y\left(1-y^{k}\right)^{\frac{m}{k}-1}}.$$

Unde, si fuerit $m=1$ et $k=2$, sequitur fore

$$\int\frac{\partial x}{\left(1+x^{k}\right)^{n}}=\frac{\pi}{2}\cdot\int\frac{y^{2(n-1)}\partial y}{\sqrt{(1-yy)}}:\int\frac{\partial y}{\sqrt{(1-yy)}}=\int\frac{y^{2(n-1)}\partial y}{\sqrt{(1-yy)}}.$$

Ita, si $n=\frac{3}{2}$, erit

$$\int\frac{\partial x}{(1+xx)^{\frac{3}{2}}}=\int\frac{y\partial y}{\sqrt{(1-yy)}}.$$

cuius veritas sponte elucet, quia integrale prius generatim est $\frac{x}{\sqrt{(1+xx)}}$, posterius vero $=1-\sqrt{(1-yy)}$, quae facto $x=\infty$ et $y=1$ utique fiunt aequalia. Caeterum pro hac integratione generali notasse iuvabit exponentem unitate minorem accipi non posse, quia alioquin valores amborum integralium in infinitum excrescerent.

6) Investigatio valoris integralis

$$\int\frac{x^{m-1}\partial x}{1-2x^{k}\cos.\theta+x^{2k}}$$

a termino $x=0$ usque ad $x=\infty$ extensi.

Opuscula analytica. Tom. II. pag. 55-75.
[Commentatio 589 indicis ENESTROEMIANI]

§. 161. Quaeramus primo integrale formulae propositae indefinitum atque adeo omnes operationes ex primis Analyseos principiis repetamus. Ac primo quidem, quoniam denominator in factores reales simplices resolvi nequit, sit in genere eius factor duplicatus quicunque $1-2x\cos.\omega+xx$; evidens enim est denominatorem fore productum ex k huiusmodi factoribus duplicatis.
Cum igitur posito hoc factore $=0$ fiat $x=\cos.\omega\pm\sqrt{-1}\cdot\sin.\omega$, etiam ipse denominator duplici modo evanescere debebit, sive si ponatur

$$x=\cos.\omega+\sqrt{-1}\cdot\sin.\omega \text{ sive } x=\cos.\omega-\sqrt{-1}\cdot\sin.\omega.$$

Constat autem omnes potestates harum formularum ita commode exprimi posse, ut sit

$$\left(\cos.\omega \pm \sqrt{-1}\cdot \sin.\omega\right)^{\lambda} = \cos.\lambda\omega \pm \sqrt{-1}\cdot \sin.\lambda\omega;$$

hinc igitur erit

$$x^k = \cos.k\omega \pm \sqrt{-1}\cdot \sin.k\omega \text{ et } x^{2k} = \cos.2k\omega \pm \sqrt{-1}\cdot \sin.2k\omega.$$

Substituamus ergo hos valores et denominator noster evadet

$$1-2\cos.\theta\cos.k\omega + \cos.2k\omega \mp 2\sqrt{-1}\cdot \cos\theta \sin.k\omega \pm \sqrt{-1}\cdot \sin.2k\omega..$$

§. 162. Perspicuum igitur est huius aequationis tam terminos reales quam imaginarios seorsim se mutuo tollere debere, unde nascuntur hae duae aequationes

$$\text{I. } 1-2\cos.\theta\cos.k\omega + \cos.2k\omega = 0,$$
$$\text{II. } -2\cos.\theta\sin.k\omega + \sin.2k\omega = 0.$$

Cum igitur sit

$$\sin.2k\omega = 2\sin.k\omega\cos.k\omega,$$

posterior aequatio induet hanc formam

$$-2\cos.\theta \sin.k\omega + 2\sin.k\omega\cos.k\omega = 0,$$

quae per $2\sin.k\omega$ divisa dat

$$\cos.k\omega = \cos.\theta$$

ideoque

$$\cos.2k\omega = \cos.2\theta = \cos.^2\theta - \sin.^2\theta = 2\cos.^2\theta - 1,$$

qui valores in aequatione priore substituti praebent aequationem identicam, ita ut utrique aequationi satisfiat sumendo $\cos.k\omega = \cos.\theta$.

§. 163. Pro ω igitur eiusmodi angulum assumi oportet, ut fiat $\cos.k\omega = \cos.\theta$, unde quidem statim. deducitur $k\omega = \theta$ ideoque $\omega = \frac{\theta}{k}$. Verum quia infiniti dantur anguli eundem cosinum habentes, qui praeter ipsum angulum θ sunt $2\pi \pm \theta,\ 4\pi \pm \theta,\ 6\pi \pm \theta$ etc atque adeo in genere $2i\pi \pm \theta$ denotante i omnes numeros integros, quaesito nostro satisfiet faciendo $k\omega = 2i\pi \pm \theta$, unde colligitur angulus $\omega = \frac{2i\pi \pm \theta}{k}$, sicque pro ω nancisceremur innumerabiles angulos satisfacientes, quorum autem sufficiet tot assumsisse, quot exponens k continet unitates; successive igitur angulo ω sequentes tribuamus valores

$$\frac{\theta}{k},\ \frac{2\pi+\theta}{k},\ \frac{4\pi+\theta}{k},\ \frac{6\pi+\theta}{k},\ \frac{8\pi+\theta}{k},\ \ldots \frac{2(k-1)\pi\pm\theta}{k}.$$

Quodsi ergo angulo ω successive singulos istos valores, quorum numerus est $=k$, tribuamus, formula $1-2x\cos.\omega+xx$ omnes suppeditabit factores duplicatos nostri denominatoris $1-2x^{k}\cos.\theta+x^{2k}$, quorum numerus erit $=k$.

§. 164. Inventis iam omnibus factoribus duplicatis nostri denominatoris

fractio $\frac{x^{m-1}}{1-2x^{k}\cos.\theta+x^{2k}}$ resolvi debet in tot fractiones partiales, quarum denominatores sint ipsi isti factores duplicati, quorum numerus est k, ita ut in genere talis fractio partialis habitura sit talem formam

$$\frac{\mathrm{A}+\mathrm{B}x}{1-2x\cos.\omega+xx},$$

quam insuper resolvamus in binas simplices etsi imaginarias, et cum sit

$$xx-2x\cos.\omega+1=\left(x-\cos.\omega+\sqrt{-1}\cdot\ \sin.\omega\right)\left(x-\cos.\omega\ -\sqrt{-1}\cdot\ \sin.\omega\right),$$

statuantur ambae istae fractiones partiales

$$\frac{f}{x-\cos.\omega-\sqrt{-1}\cdot\ \sin.\omega}+\frac{g}{x-\cos.\omega+\sqrt{-1}\cdot\ \sin.\omega},$$

ita ut totum resolutionis negotium huc redeat, ut ambo numeratores f et g determinentur; iis enim inventis habebitur summa ambarum fractionum

$$=\frac{fx+gx-(f+g)\cos.\omega+\sqrt{-1}\cdot(f-g)\sin.\omega}{xx-2x\cos.\omega+1},$$

ubi igitur erit

$$\mathrm{B}=\left(f+g\right)\ \text{et}\ \mathrm{A}=(f-g)\sqrt{-1}\cdot\sin.\omega-(f+g)\cos.\omega.$$

§. 165. Per methodum igitur fractiones quascunque in fractiones simplices resolvendi statuamus

$$\frac{x^{m-1}}{1-2x^{k}\cos.\theta+x^{2k}}=\frac{f}{x-\cos.\omega-\sqrt{-1}\cdot\ \sin.\omega}+\mathrm{R},$$

ubi R complectatur omnes reliquas fractiones partiales. Hinc per

$$x-\cos.\omega-\sqrt{-1}\cdot\ \sin.\omega$$

multiplicando habebitur

$$\frac{x^{m}-x^{m-1}\left(\cos.\omega+\sqrt{-1}\cdot\ \sin.\omega\right)}{1-2x^{k}\cos.\theta+x^{2k}}=f+\mathrm{R}\left(x-\cos.\omega-\sqrt{-1}\cdot\sin.\omega\right);$$

quae aequatio cum vera esse debeat, quicunque valor ipsi x tribuatur, statuamus $x=\cos.\omega+\sqrt{-1}\cdot\sin.\omega$, ut membrum postremum prorsus e calculo tollatur; tum vero in parte sinistra, quia formula $x-\cos.\omega-\sqrt{-1}\cdot\sin.\omega$ simul est factor denominatoris, facta hac substitutione tam numerator quam denominator in nihilum abibunt, ita ut hinc nihil concludi posse videatur.

§. 166. Hic igitur utamur regula notissima et loco tam numeratoris quam denominatoris eorum differentialia scribamus, unde nostra aequatio accipiet sequentem formam

$$\frac{mx^{m-1}-(m-1)x^{m-2}\left(\cos.\omega+\sqrt{-1}\cdot\sin.\omega\right)}{-2kx^{k-1}\cos.\theta+2kx^{2k-1}}=\frac{mx^{m}-(m-1)x^{m-1}\left(\cos.\omega+\sqrt{-1}\cdot\sin.\omega\right)}{-2kx^{k}\cos.\theta+2kx^{2k}}=f,$$

posito scilicet $x=\cos.\omega+\sqrt{-1}\cdot\sin.\omega$. Tum autem erit

$$x^m=\cos.m\omega+\sqrt{-1}\cdot\sin.m\omega$$

et

$$x^{m-1}\left(\cos.m\omega+\sqrt{-1}\cdot\sin.m\omega\right)=x^m=\cos.m\omega+\sqrt{-1}\cdot\sin.m\omega$$

et pro denominatore

$$x^k=\cos.k\omega+\sqrt{-1}\cdot\sin.k\omega \text{ et } x^{2k}=\cos.2k\omega+\sqrt{-1}\cdot\sin.2k\omega;$$

unde fit numerator

$$x^m=\cos.m\omega+\sqrt{-1}\cdot\sin.m\omega$$

et denominator

$$-2k\cos.\theta\cos.k\omega+2k\cos.2k\omega-2k\sqrt{-1}\cdot\cos.\theta\ \sin.k\omega+2k\sqrt{-1}\cdot\sin.2k\omega.$$

§. 167. Pro denominatore reducendo recordemur iam supra inventum esse $\cos.k\omega=\cos.\theta$, unde fit $\sin.k\omega=\sin.\theta$, tum vero

$$\cos.2k\omega=\cos.2\theta=2\cos.^2\theta-1 \text{ et } \sin.2k\omega=2\ \sin.\theta\cos.\theta,$$

quibus valoribus adhibitis denominator noster erit

$$2k\cos.^2\theta-2k+2k\sqrt{-1}\cdot\sin.\theta\cos.\theta=-2k\sin^2.\theta+2k\sqrt{-1}\cdot\sin.\theta\cos.\theta$$
$$=-2k\sin.\theta\left(\sin.\theta-\sqrt{-1}\cdot\cos.\theta\right),$$

quamobrem hoc valore adhibito habebimus

$$f=\frac{\cos.m\omega+\sqrt{-1}\cdot\sin.m\omega}{2k\sin.\theta\left(\sqrt{-1}\cdot\cos.\theta-\sin.\theta\right)}.$$

Simul vero hinc sine novo calculo deducemus valorem *g,* quippe qui ab *f* ratione signi $\sqrt{-1}$ tantum discrepat, sicque erit

$$g=\frac{\cos.m\omega-\sqrt{-1}\cdot\sin.m\omega}{-2k\sin.\theta\left(\sin.\theta+\sqrt{-1}\cdot\cos.\theta\right)}.$$

§. 168. Inventis autem his litteris *f* et *g* pro litteris A et B colligemus primo

$$f+g=\frac{\cos.\theta\sin.m\omega-\sin.\theta\cos.m\omega}{k\sin.\theta}=\frac{\sin.(m\omega-\theta)}{k\sin.\theta},$$

tum vero erit

$$f-g=-\frac{\sqrt{-1}\cos.(m\omega-\theta)}{k\sin.\theta}.$$

Ex his igitur reperiemus

$$\mathrm{B}=\frac{\sin.(m\omega-\theta)}{k\sin.\theta}$$

et

$$\mathrm{A}=\frac{\sin.\omega\cos.(m\omega-\theta)-\cos.\omega\sin.(m\omega-\theta)}{k\sin.\theta}=-\frac{\sin.((m\omega-\theta)-\omega)}{k\sin.\theta},$$

ubi ergo imaginaria sponte se mutuo destruxerunt.

§. 169. Inventis his valoribus A et B investigari oportet integrale partiale

$$\int\frac{(\mathrm{A}+\mathrm{B}x)\partial x}{1-2x\cos.\omega+xx},$$

ubi, cum denominatoris differentiale sit

$$2x\partial x-2\partial x\cos.\omega=2\partial x(x-\cos.\omega),$$

statuamus

$$\mathrm{A}+\mathrm{B}x=\mathrm{B}(x-\cos.\omega)+\mathrm{C}$$

eritque $\mathrm{C}=\mathrm{A}+\mathrm{B}\cos.\omega$; hinc igitur erit

$$\mathrm{C}=\frac{\cos.\omega\sin.(m\omega-\theta)-\sin.((m\omega-\theta)-\omega)}{k\sin.\theta}.$$

Quia vero

$$-\sin.\left((m\omega-\theta)-\omega\right)=-\sin.\left(m\omega-\theta\right)\cos.\omega+\cos.\left(m\omega-\theta\right)\sin.\omega,$$

erit

$$\mathrm{C}=\frac{\sin.\omega\cos.(m\omega-\theta)}{k\sin.\theta}.$$

Hac ergo forma adhibita formula integranda $\frac{(\mathrm{A}+\mathrm{B}x)\partial x}{1-2x\cos.\omega+xx}$ discerpatur in has duas partes

$$\frac{\mathrm{B}(x-\cos.\omega)\partial x}{1-2x\cos.\omega+xx}+\frac{\mathrm{C}\partial x}{1-2x\cos.\omega+xx}.$$

Hic igitur prioris partis integrale manifesto est

$$Bl\sqrt{(1-2x\cos.\omega+xx)}$$

,

alterius vero partis facile patet integrale per arcum circuli expressum iri, cuius tangens sit $\frac{x\sin.\omega}{1-x\cos.\omega}$. Ad hoc integrale inveniendum ponamus

$$\int\frac{\mathrm{C}\partial x}{1-2x\cos.\omega+xx}=\mathrm{D}\,\mathrm{Atang.}\frac{x\sin.\omega}{1-x\cos.\omega}$$

et sumtis differentialibus, quia $\partial.\mathrm{Atang.}t$ aequale est $\frac{dt}{1+tt}$ habebimus

$$\frac{\mathrm{C}\partial x}{1-2x\cos.\omega+xx}=\mathrm{D}\frac{\partial x\sin.\omega}{1-2x\cos.\omega+xx},$$

unde manifesto fit

$$\mathrm{D}=\frac{\mathrm{C}}{\sin.\omega}=\frac{\cos.(m\omega-\theta)}{k\sin.\theta}.$$

§. 170. Substituamus igitur loco B et D valores modo inventos et ex singulis factoribus denominatoris $1-2x^{k}\cos.\theta+x^{2k}$, quorum forma est $1-2x\cos.\omega+xx$, oritur pars integralis constans ex membro logarithmico et arcu circulari, quae erit

$$\frac{\sin.(m\omega-\theta)}{k\sin.\theta}l\sqrt{(1-2x\cos.\omega+xx)}+\frac{\cos.(m\omega-\theta)}{k\sin.\theta}.\mathrm{Atang.}\frac{x\sin.\omega}{1-x\cos.\omega},$$

quae evanescit sumto $x=0$. In hac igitur forma tantum opus est, ut loco ω successive scribamus valores supra indicatos, scilicet

$$\frac{\theta}{k},\ \frac{2\pi+\theta}{k},\ \frac{4\pi+\theta}{k},\ \frac{6\pi+\theta}{k},\ \text{etc.},$$

donec perveniatur ad $\frac{2(k-1)\pi+\theta}{k}$; tum enim summa omnium harum formarum praebebit totum integrale indefinitum formulae propositae.

§. 171. Postquam igitur integrale indefinitum elicuimus, nihil aliud superest, nisi ut in eo faciamus $x=\infty$, quo facto pars logarithmica ob

$$\sqrt{(1-2x\cos.\omega+xx)}=x-\cos.\omega$$

erit $\mathrm{B}l(x-\cos.\omega)$. Est vero

$$l(x-\cos.\omega)=lx-\tfrac{\cos.\omega}{x}=lx$$

ob $\frac{\cos.\omega}{x}=0$; quamobrem facto $x=\infty$ quaelibet pars logarithmica habebit hanc formam $\frac{\sin.(m\omega-\theta)}{k\sin.\theta}lx$. Deinde pro partibus a circulo pendentibus facto $x=\infty$
fit

$$\tfrac{x\sin.\omega}{1-x\cos.\omega}=-\text{tang}.\omega=-\text{tang}.(\pi-\omega)$$

sicque arcus, cuius haec est tangens, erit $=\pi-\omega$ hincque pars circularis quaecunque fiet

$$\tfrac{\cos.(m\omega-\theta)}{k\sin.\theta}(\pi-\omega).$$

§. 172. Cum quilibet valor anguli ω in genere hanc habeat formam $\frac{2i\pi+\theta}{k}$, erit angulus

$$m\omega-\theta=\tfrac{2im\pi-\theta(k-m)}{k} \text{ et } \pi-\omega=\tfrac{\pi(k-2i)-\theta}{k}.$$

Ponamus brevitatis gratia

$$\tfrac{\theta(k-m)}{k}=\zeta \text{ et } \tfrac{m\pi}{k}=\alpha,$$

ut sit

$$m\omega-\theta=2i\alpha-\zeta,$$

ubi loco i scribi debent successive numeri 0, 1, 2, 3 etc. usque ad $k-1$. Hinc igitur, si omnes partes logarithmicas in unam summam colligamus, ea ita repraesentari poterit

$$\tfrac{lx}{k\sin.\theta}\left\{\begin{array}{l}-\sin.\zeta+\sin(2\alpha-\zeta)+\sin(4\alpha-\zeta)+\sin(6\alpha-\zeta)\\+\sin(8\alpha-\zeta)+\cdots+\sin(2(k-1)\alpha-\zeta)\end{array}\right\};$$

ubi quidem ex iis, quae hactenus sunt tradita, facile suspicari licet totam hanc progressionem ad nihilum redigi. Verum hoc ipsum firma demonstratione muniri necesse est.

§. 173. Ad hoc ostendendum ponamus

$$\mathrm{S}=-\sin.\zeta+\sin.\left(2\alpha-\zeta\right)+\sin.\left(4\alpha-\zeta\right)+\cdots+\sin.\left(2\left(k-1\right)\alpha-\zeta\right);$$

multiplicemus utrinque per 2sin.*a,* et cum sit

$$2\sin.\alpha\sin.\varphi=\cos.(\alpha-\varphi)-\cos.(\alpha+\varphi),$$

huius reductionis ope obtinebimus sequentem expressionem

$$\begin{aligned}2\mathrm{S}\sin.\alpha=\cos.&\left(\alpha+\zeta\right)\\ &-\cos.\left(\alpha-\zeta\right)-\cos.\left(3\alpha-\zeta\right)-\cos.\left(5\alpha-\zeta\right)-\cdots\\ &+\cos.\left(\alpha-\zeta\right)+\cos.\left(3\alpha-\zeta\right)+\cos.\left(5\alpha-\zeta\right)+\cdots\\ &\qquad-\cos\left(\left(2k-1\right)\alpha-\zeta\right),\end{aligned}$$

unde deletis terminis se mutuo destruentibus habebitur

$$2\mathrm{S}\sin.\alpha=\cos.\left(\alpha+\zeta\right)-\cos\left(\left(2k-1\right)\alpha-\zeta\right).$$

§. 174. Ponamus hos duos angulos, qui sunt relicti,

$$\alpha+\zeta=p \text{ et } \left(2k-1\right)\alpha-\zeta=q$$

eritque eorum summa $p+q=2\alpha k$. Quia porro est $\alpha=\frac{m\pi}{k}$, erit $p+q=2m\pi$, hoc est multiplico totius circuli peripheriae ob *m* numerum integrum. Quare cum sit $q=2m\pi-p$, erit $\cos.q=\cos.p$; unde patet summam inventam nihilo esse aequalem sicque manifestum est omnes partes logarithmicas, quae in integrale formulae nostrae ingrediuntur, casu $x=\infty$ se mutuo destruere.

§. 175. Progrediamur igitur ad partes circulares, quarum forma generalis, ut vidimus, est $\frac{\cos.(m\omega-\theta)}{k\sin.\theta}\left(\pi-\omega\right)$, quae posito $\alpha=\frac{m\pi}{k}$ et $\zeta=\frac{\theta(k-m)}{k}$ fit

$$\frac{\cos.\left(2i\alpha-\zeta\right)}{k\sin.\theta}\left(\pi-\frac{2i\pi+\theta}{k}\right)=\frac{\cos.\left(2i\alpha-\zeta\right)}{k\sin.\theta}\left(\pi-\frac{2i\pi}{k}-\frac{\theta}{k}\right).$$

Hic ponatur porro $\frac{\pi}{k}=\beta$ et $\pi-\frac{\theta}{k}=\gamma$, ut forma generalis sit

$$\frac{\cos.(2i\alpha-\zeta)}{k\sin.\theta}(\gamma-2i\beta).$$

Quare si loco *i* scribamus ordine valores 0, 1, 2, 3, 4 usque ad $k-1$, omnes partes circulares hanc progressionem constituent

$$\frac{1}{k\sin.\theta}\left(\gamma\cos.\zeta+(\gamma-2\beta)\cos.(2\alpha-\zeta)+(\gamma-4\beta)\cos.(4\alpha-\zeta)\right)+\cdots$$
$$+\left(\gamma-2(k-1)\beta\right)\cos.\left(2(k-1)\alpha-\zeta\right).$$

Ponamus igitur

$$\mathrm{S}=\gamma\cos.\zeta+(\gamma-2\beta)\cos.(2\alpha-\zeta)+(\gamma-4\beta)\cos.(4\alpha-\zeta)+\cdots$$
$$+\left(\gamma-2(k-1)\beta\right)\cos.\left(2(k-1)\alpha-\zeta\right),$$

ut summa omnium partium circularium sit $\frac{\mathrm{S}}{k\sin.\theta}$, quae ergo praebebit valorem quaesitum formulae integralis propositae casu, quo post integrationem statuitur $x=\infty$, ita ut totum negotium in investigando valore ipsius S versetur.

§. 176. Hunc in finem multiplicemus utrinque per $2\sin.\alpha$, et cum in genere sit

$$2\sin.\alpha\cos.\varphi=\sin.(\alpha+\varphi)-\sin.(\varphi-\alpha),$$

hac reductione in singulis terminis facta perveniemus ad hanc aequationem

$$\begin{aligned}2\mathrm{S}\sin.\alpha=\gamma\sin.(\alpha+\zeta)&\\ +\gamma\sin.(\alpha-\zeta)+(\gamma-2\beta)\sin.(3\alpha-\zeta)+(\gamma-4\beta)\sin.(5\alpha-\zeta)+\cdots&\\ -(\gamma-2\beta)\sin.(\alpha-\zeta)-(\gamma-4\beta)\sin.(3\alpha-\zeta)-(\gamma-6\beta)\sin.(5\alpha-\zeta)-\cdots&\\ +\left(\gamma-2(k-1)\beta\right)\sin.\left(2(k-1)\alpha-\zeta\right),&\end{aligned}$$

ubi praeter primum et ultimum terminum omnes reliqui contrahi possunt, ita ut prodeat

$$2\mathrm{S}\sin.\alpha=\gamma\sin.(\alpha+\zeta)+2\beta\sin.(\alpha-\zeta)+2\beta\sin.(3\alpha-\zeta)+2\beta\sin.(5\alpha-\zeta)+\cdots$$
$$+2\beta\sin.\left(2(k-3)\alpha-\zeta\right)+\left(\gamma-2(k-1)\beta\right)\sin.\left(2(k-1)\alpha-\zeta\right).$$

§. 177. Iam pro hac serie summanda ponamus porro

$$\mathrm{T} = 2\sin.(\alpha-\zeta)+2\sin.(3\alpha-\zeta)+2\sin.(5\alpha-\zeta)+\cdots+2\sin.((2k-3)\alpha-\zeta),$$

ut habeamus

$$2\mathrm{S}\sin.\alpha = \gamma\sin.(\alpha+\zeta)+(\gamma-2(k-1)\beta)\sin.((2\mathrm{k}-1)\alpha-\zeta)+\beta\mathrm{T}.$$

Iam multiplicemus ut hactenus per $\sin.\alpha$, et cum sit

$$2\sin.\alpha\sin.\varphi = \cos.(\varphi-\alpha)-\cos.(\varphi+\alpha),$$

facta hac reductione nanciscimur

$$\begin{aligned}\mathrm{T}\sin.\alpha = &+\cos.\zeta\\ &+\cos.(2\alpha-\zeta)+\cos.(4\alpha-\zeta)+\cdots+\cos.(2(k-2)\alpha-\zeta)\\ &-\cos.(2\alpha-\zeta)-\cos.(4\alpha-\zeta)+\cdots-\cos.(2(k-2)\alpha-\zeta)\\ &\qquad\qquad-\cos.(2(k-1)\alpha-\zeta),\end{aligned}$$

unde deletis terminis, quae se mutuo destruunt, remanebit tantum ista expressio

$$\mathrm{T}\sin.\alpha = \cos.\zeta-\cos.(2(k-1)\alpha-\zeta).$$

Cum igitur sit $\alpha=\frac{m\pi}{k}$, erit $2(k-1)\alpha = 2m\pi-\frac{2m\pi}{k}$, cuius loco scribere licet $-\frac{2m\pi}{k}$, unde ob $\zeta=\frac{\theta(k-m)}{k}$ erit

$$\mathrm{T}\sin.\alpha = \cos.\frac{\theta(k-m)}{k}-\cos.\frac{2m\pi+\theta(k-m)}{k}.$$

§. 178. Nunc vero notetur in genere esse

$$\cos.p-\cos.q = 2\sin.\tfrac{q+p}{2}\sin.\tfrac{q-p}{2};$$

quare cum sit

$$p=\frac{\theta(k-m)}{k} \quad \text{et} \quad q=\frac{2m\pi+\theta(k-m)}{k},$$

erit

$$\frac{q+p}{2}=\frac{m\pi+\theta(k-m)}{k} \quad \text{et} \quad \frac{q-p}{2}=\frac{m\pi}{k},$$

unde sequitur fore

$$\mathrm{T}\sin.\alpha = 2\sin.\frac{m\pi+\theta(k-m)}{k}\sin.\frac{m\pi}{k}$$

ideoque

$$\mathrm{T} = 2\sin.\frac{m\pi+\theta(k-m)}{k}$$

ob $\alpha = \frac{m\pi}{k}$.

§. 179. Hoc igitur valore T invento reperiemus porro

$$2\mathrm{S}\sin.\alpha = \gamma\sin.(\alpha+\zeta) + (\gamma - 2(k-1)\beta)\sin.((2\mathrm{k}-1)\alpha - \zeta) + 2\beta\sin.\frac{m\pi+\theta(k-m)}{k},$$

quae ob $\frac{m\pi+\theta(k-m)}{k} = \alpha + \zeta$ reducitur ad hanc formam

$$2S\sin.\alpha = (\gamma + 2\beta)\sin.(\alpha+\zeta) + (\gamma - 2(k-1)\beta)\sin.((2k-1)\alpha - \zeta),$$

quae ita repraesentari potest

$$2\mathrm{S}\sin.\alpha = (\gamma+2\beta)\big(\sin.(\alpha+\zeta) + \sin.((2k-1)\alpha-\zeta)\big) - 2k\beta\sin.((2k-1)\alpha-\zeta),$$

ubi pro parte priore ob

$$\sin.p + \sin.q = 2\sin.\tfrac{p+q}{2}\sin.\tfrac{p-q}{2}$$

erit

$$\tfrac{p+q}{2} = \alpha k \quad \text{et} \quad \tfrac{p-q}{2} = (k-1)\alpha - \zeta,$$

unde pars ipsa prior fit

$$2(\gamma+2\beta)\sin.ak\cos.((k-1)\alpha-\zeta);$$

ubi cum sit $\alpha k = m\pi$, erit $\sin.\alpha k = 0$, ita ut tantum supersit

$$2\mathrm{S}\sin.\alpha = -2k\beta\sin.((2k-1)\alpha-\zeta)$$

hincque

$$\mathrm{S} = \frac{-k\beta\sin.((2k-1)\alpha-\zeta)}{\sin.\alpha}$$

Est vero

$$(2k-1)\alpha - \zeta = 2m\pi - \frac{m\pi}{k} - \frac{\theta(k-m)}{k};$$

omisso termino $2m\pi$ erit igitur

$$S=+\frac{\pi \sin.\left(\frac{m\pi}{k}+\frac{\theta(k-m)}{k}\right)}{\sin.\frac{m\pi}{k}}$$

ideoque valor quaesitus erit

$$\cdot\frac{S}{k\sin\theta}=+\frac{\pi \sin.\left(\frac{m\pi}{k}+\frac{\theta(k-m)}{k}\right)}{k\sin\theta\sin.\frac{m\pi}{k}},$$

quae forma reducitur ad hanc

$$\frac{S}{k\sin\theta}=+\frac{\pi \sin.\left(\frac{m(\pi-\theta)+k\theta}{k}\right)}{k\sin\theta\sin.\frac{m\pi}{k}}.$$

§. 180. Contemplemur hic ante omnia casum, quo $\theta=\frac{\pi}{2}$, et formula integralis proposita abit in hanc

$$\int\frac{x^{m-1}\partial x}{1+x^{2k}},$$

cuius ergo valor, si post integrationem ponatur $x=\infty$, evadet

$$=\frac{\pi \sin.\left(\frac{\pi}{2}+\frac{m\pi}{2k}\right)}{k\sin.\frac{m\pi}{k}}=\frac{\pi\cos.\left(\frac{m\pi}{2k}\right)}{k\sin.\frac{m\pi}{k}}.$$

Quia igitur est $\sin.\frac{m\pi}{k}=2\sin.\frac{m\pi}{2k}\cos.\frac{m\pi}{2k}$, prodabit iste valor

$$=\frac{\pi \sin.\left(\frac{\pi}{2}+\frac{m\pi}{2k}\right)}{k\sin.\frac{m\pi}{k}}=\frac{\pi}{2k\sin.\frac{m\pi}{2k}},$$

qui valor egregie convenit cum eo, quem non ita pridem pro formula $\int\frac{x^{m-1}\partial x}{1+x^{2k}}$ assignavimus, siquidem loco k scribatur $2k$.

§. 181. Evolvamus etiam casum, quo $\theta=\pi$, et formula nostra integralis [abit in hanc] $\int\frac{x^{m-1}\partial x}{\left(1+x^{2k}\right)^2}$, cuius ergo facto $x=\infty$ valor erit

$$\frac{\pi \sin.\left(\frac{m(\pi-\theta)+k\theta}{k}\right)}{k\sin\theta\sin.\frac{m\pi}{k}}=\frac{\pi}{k\sin.\frac{m\pi}{k}}\cdot\frac{\sin.\left(\frac{m(\pi-\theta)}{k}+\theta\right)}{\sin\theta}.$$

Huius autem posterioris fractionis casu $\theta=\pi$ tam numerator quam denominator evanescit; quare ut eius verus valor eruatur, loco utriusque eius differentiale scribamus, quo facto ista fractio abibit in hanc

$$\frac{\partial\theta\left(1-\frac{m}{k}\right)\cos.\left(\frac{m(\pi-\theta)}{k}+\theta\right)}{\partial\theta\cos\theta},$$

cuius valor facto $\theta=\pi$ nunc manifesto est $1-\frac{m}{k}$; sicque valor integralis quaesitus erit $\left(1-\frac{m}{k}\right)\frac{\pi}{k\sin.\frac{m\pi}{k}}$, prorsus uti in superiore dissertatione invenimus.

§. 182. Quo autem valorem generalem inventum commodiorem reddamus, ponamus $\pi-\theta=\eta$ fietque $\sin.\theta=\sin.\eta$ et $\cos.\theta=-\cos.\eta$; tum vero erit angulus

$$\frac{m(\pi-\theta)}{k}+\theta=\frac{m\eta}{k}+\pi-\eta,$$

cuius sinus est $\sin.(1-\frac{m}{k})\eta$, unde valor quaesitus nostrae formulae erit

$$\frac{\pi\sin.\left(1-\frac{m}{k}\right)\eta}{k\sin\eta\sin.\frac{m\pi}{k}},$$

atque hinc tandem sequens adepti sumus

THEOREMA

§. 183. *Si , haec formula integralis*

$$\int\frac{x^{m-1}\partial x}{1-2x^k\cos.\theta+x^{2k}}$$

a termino $x=0$ *usque ad terminum* $x=\infty$ *extendatur, eius valor erit*

$$=\frac{\pi\sin.\left(1-\frac{m}{k}\right)\eta}{k\sin\eta\sin.\frac{m\pi}{k}},$$

sive cum sit

$$\sin.\left(1-\frac{m}{k}\right)\eta=\sin\eta\cos.\frac{m\pi}{k}-\cos\eta\sin.\frac{m\pi}{k},$$

iste valor etiam hoc modo exprimi potest

$$\frac{\pi\cos.\frac{m\eta}{k}}{k\sin.\frac{m\pi}{k}} - \frac{\pi\sin.\frac{m\eta}{k}}{k\tang.\eta\sin.\frac{m\pi}{k}}.$$

§. 184. Consideremus nunc alio modo hanc formulam integralem

$$\int \frac{x^{m-1}\partial x}{1-2x^k\cos.\theta+x^{2k}},$$

cuius valor a termino $x=0$ usque ad $x=1$ ponatur $=\mathrm{P}$, eiusdem vero valor ab $x=1$ usque ad $x=\infty$ ponatur $=\mathrm{Q}$, ita ut $\mathrm{P+Q}$ exhibere debeat ipsum valorem ante inventum. Nunc vero pro valore Q inveniendo ponamus $x=\frac{1}{y}$ et formula nostra ita repraesentata

$$\frac{x^{m-1}}{1+2x^k\cos.\eta+x^{2k}}\cdot\frac{\partial x}{x}$$

ob $\frac{\partial x}{x}=-\frac{\partial y}{y}$ induet hanc formam

$$-\int \frac{y^{-m}}{1+2y^{-k}\cos.\eta+y^{-2k}}\cdot\frac{\partial y}{y} = -\int \frac{y^{2k-m-1}\partial y}{y^{2k}+2y^k\cos.\eta+1},$$

cuius valor a termino $y=1$ usque ad $y=0$ extendi debet. Commutatis igitur his terminis habebimus

$$\mathrm{Q} = +\int \frac{y^{2k-m-1}\partial y}{y^{2k}+2y^k\cos.\eta+1}$$

a termino $y=0$ usque ad $y=1$.

§. 185. Quia in utraque forma pro *P* et *Q* eadem conditio integrationis praescribitur, a termino 0 usque ad 1 nihil impedit, quominus in posteriore loco *y* scribamus *x,* unde pro $\mathrm{P+Q}$ habebimus hanc formam integralem

$$\int \frac{x^{m-1}+x^{2k-m-1}}{1+2x^k\cos.\eta+x^{2k}}\partial x,$$

cuius valor a termino $x=0$ usque ad $x=1$ extensus aequabitur huic expressioni $\frac{\pi\sin.\left(1-\frac{m}{k}\right)\eta}{k\sin\eta\sin.\frac{m\pi}{k}}$. Comparatis igitur his binis formulis integralibus nanciscemur sequens theorema notatu maxime dignum.

THEOREMA

§. 186. *Haec formula integralis*

$$\int \frac{x^{m-1}+x^{2k-m-1}}{1+2x^{k}\cos .\eta+x^{2k}}\partial x,$$

a termino $x=0$ *usque ad terminum* $x=1$ *extensa aequalis est huic formulae integrali*

$$\int \frac{x^{m-1}\partial x}{1+2x^{k}\cos .\eta+x^{2k}}$$

a. termino $x=0$ *usque ad terminum* $x=\infty$ *extensae; utriusque enim valor erit*

$$\frac{\pi\sin .\left(1-\frac{m}{k}\right)\eta}{k\sin\eta\sin .\frac{m\pi}{k}}.$$

§. 187. Quodsi hanc fractionem $\frac{\sin .\eta}{1+2x^{k}\cos .\eta+x^{2k}}$ in seriem infinitam evolvamus, quae sit

$$\sin .\eta+\mathrm{A}x^{k}+\mathrm{B}x^{2k}+\mathrm{C}x^{3k}+\mathrm{D}x^{4k}+\mathrm{E}x^{5k}+\text{etc.},$$

per denominatorem multiplicando perveniemus ad hanc expressionem infinitam

$$\begin{aligned}\sin .\eta=\sin .\eta+{}&\mathrm{A}x^{k}+\mathrm{B}x^{2k}+\mathrm{C}x^{3k}+\mathrm{D}x^{4k}+\mathrm{E}x^{5k}+\text{etc.},\\&+2\sin .\eta\cos .\eta+2\mathrm{A}\cos .\eta+2\mathrm{B}\cos .\eta+2\mathrm{C}\cos .\eta+2\mathrm{D}\cos .\eta+\text{etc.},\\&+\sin .\eta+\mathrm{A}+\mathrm{B}+\mathrm{C}+\text{etc.}\end{aligned}$$

unde singulis terminis ad nihilum reductis reperiemus

$$\begin{aligned}&1.\ \mathrm{A}+2\sin .\eta\cos .\eta=0 \text{ hincque } \mathrm{A}=-\sin .2\eta,\\&2.\ \mathrm{B}+2\mathrm{A}\cos .\eta+\sin .\eta=0, \text{unde fit } \mathrm{B}=-\sin .\ 3\eta,\\&3.\ \mathrm{C}+2\mathrm{B}\cos .\eta+\mathrm{A}=0, \text{unde fit } \mathrm{C}=-\sin .4\eta,\\&4.\ \mathrm{D}+2\mathrm{C}\cos .\eta+\mathrm{B}=0, \text{unde fit } \mathrm{D}=\ \sin .5\eta\\&\qquad\text{etc.}\qquad\qquad\qquad\qquad\text{etc,}\end{aligned}$$

ita ut nostra fractio $\frac{\sin .\eta}{1+2x^{k}\cos .\eta+x^{2k}}$ resolvatur in hanc seriem

$$\sin .\eta-x^{k}\sin .2\eta+x^{2k}\sin .3\eta+x^{3k}\sin .4\eta+x^{4k}\sin .5\eta-\text{etc.}$$

§. 188. Multiplicemus nunc hanc seriem per

$$x^{m-1}\partial x+x^{2k-m-1}\partial x$$

et post integrationem faciamus $x=1$, ut obtineamus valorem huius formulae

$$\sin.\eta\int\frac{x^{m-1}+x^{2k-m-1}}{1+2x^{k}\cos.\eta+x^{2k}}\partial x$$

pro casu $x=1$, hocque modo perveniemus ad geminas sequentes series

$$\frac{\sin.\eta}{m}-\frac{\sin.2\eta}{m+k}+\frac{\sin.3\eta}{m+2k}-\frac{\sin.4\eta}{m+3k}+\frac{\sin.5\eta}{m+4k}-\text{etc.},$$
$$\frac{\sin.\eta}{2k-m}-\frac{\sin.2\eta}{3k-m}+\frac{\sin.3\eta}{4k-m}-\frac{\sin.4\eta}{5k-m}+\frac{\sin.5\eta}{6k-m}-\text{etc.}$$

Aggregatum igitur harum duarum serierum iunctim sumtarum aequabitur huic valori

$$\frac{\pi\sin.\left(1-\frac{m}{k}\right)\eta}{k\sin.\frac{m\pi}{k}},$$

unde subiungamus adhuc istud theorema.

THEOREMA

§. 189. *Si* η *denotet angulum quemcunque, litterae vero m et k pro lubitu accipiantur ex iisque binae sequentes series formentur*

$$\mathrm{P}=\frac{\sin.\eta}{m}-\frac{\sin.2\eta}{m+k}+\frac{\sin.3\eta}{m+2k}-\frac{\sin.4\eta}{m+3k}+\frac{\sin.5\eta}{m+4k}-\text{etc.},$$
$$\mathrm{Q}=\frac{\sin.\eta}{2k-m}-\frac{\sin.2\eta}{3k-m}+\frac{\sin.3\eta}{4k-m}-\frac{\sin.4\eta}{5k-m}+\frac{\sin.5\eta}{6k-m}-\text{etc.},$$

neutrius quidem summa exhiberi potest, utriusque autem iunctim sumtae summa erit

$$\mathrm{P}+\mathrm{Q}=\frac{\pi\sin.\left(1-\frac{m}{k}\right)\eta}{k\sin.\frac{m\pi}{k}}.$$

COROLLARIUM

§. 190. Quodsi ergo angulum η infinite parvum capiamus, ut fiat

$$\sin.\eta=\eta,\sin.2\eta=2\eta,\ \sin.3\eta=3\eta\ \text{etc.},$$

quia in formula summae fiet

$$\sin.\left(1-\frac{m}{k}\right)\eta=\left(1-\frac{m}{k}\right)\eta,$$

si utrinque per η dividamus, obtinebimus sequentem seriem geminatam

$$\frac{1}{m}-\frac{2}{m+k}+\frac{3}{m+2k}-\frac{4}{m+3k}+\frac{5}{m+4k}-\text{etc.}$$
$$+\frac{1}{2k-m}-\frac{2}{3k-m}+\frac{3}{4k-m}-\frac{4}{5k-m}+\frac{5}{6k-m}-\text{etc.},$$

cuius ergo summa erit $\left(1-\frac{m}{k}\right)\frac{\pi}{k\sin.\frac{m\pi}{k}}$; ubi notetur ambas istas series non incongrue in hanc simplicem contrahi posse

$$\frac{2k}{m(2k-m)}-\frac{8k}{(m+k)(3k-m)}+\frac{18k}{(m+2k)(4k-m)}-\frac{32k}{(m+3k)(5k-m)}+\text{etc.}$$

ubi numeratores sunt numeri quadrati duplicati.

§. 191. Formulae autem, quarum valores hactenus invenimus, multo concinnius et elegantius exprimi possunt, si loco exponentis m scribamus $k-n$; tum enim in valore integrali invento fiet $\left(1-\frac{m}{k}\right)\eta=\frac{n\eta}{k}$, at vero pro denominatore fiet $\frac{m\pi}{k}=\pi-\frac{n\pi}{k}$ cuius sinus erit $\sin.\frac{n\pi}{k}$; sicque nostra formula inventa hanc induet formam $\frac{\pi\sin.\frac{n\eta}{k}}{k\sin.\eta\sin.\frac{m\pi}{k}}$, quae ergo exprimet valorem huius formulae integralis

$$\int\frac{x^{k-n-1}\partial x}{1+2x^k\cos.\eta+x^{2k}}$$

ab $x=0$ usque ad $x=\infty$, ut et huius formulae

$$\int\frac{x^{k-n-1}+x^{k+n-1}}{1+2x^k\cos.\eta+x^{2k}}\partial x$$

a termino $x=0$ usque ad terminum $x=1$; et quia utriusque valor est

$$\frac{\pi\sin.\frac{n\eta}{k}}{k\sin.\eta\sin.\frac{n\pi}{k}},$$

perspicuum est eum manere eundem, etsi loco n scribatur $-n$, ex quo prior formula ita repraesentari poterit

$$\int\frac{x^{k\pm n-1}\partial x}{1+2x^k\cos.\eta+x^{2k}};$$

at posterior formula ob hanc ambiguitatem nullam plane mutationem patitur.

§. 192. Ponendo $m=k-n$ etiam series nostra geminata pulchriorem accipiet faciem; habebitur enim

$$\begin{aligned}&\frac{\sin.\eta}{k-n}-\frac{\sin.2\eta}{2k-n}+\frac{\sin.3\eta}{3k-n}-\frac{\sin.4\eta}{4k-n}+\text{etc.}\\+&\frac{\sin.\eta}{k+n}-\frac{\sin.2\eta}{2k+n}+\frac{\sin.3\eta}{3k+n}-\frac{\sin.4\eta}{4k+n}+\text{etc.}\end{aligned}$$

cuius ergo summa erit $\frac{\pi\sin.\frac{n\eta}{k}}{k\sin.\frac{n\pi}{k}}$. Tum vero si hae geminae series in unam contrahantur et utrinque per *2k* dividatur, obtinebitur sequens summatio memoratu digna

$$\frac{\pi\sin.\frac{n\eta}{k}}{2kk\sin.\frac{n\pi}{k}}=\frac{\sin.\eta}{kk-nn}-\frac{2\sin.2\eta}{4kk-nn}+\frac{3\sin.3\eta}{9kk-nn}-\frac{\sin.4\eta}{16kk-nn}+\text{etc.}$$

§. 193. Quodsi haec postrema series differentietur sumendo solum angulum η variabilem ob $\partial\sin.\frac{n\eta}{k}=\frac{nd\eta}{k}\cos.\frac{n\eta}{k}$ habebimus

$$\frac{\pi n\cos.\frac{n\eta}{k}}{2k^3\sin.\frac{n\pi}{k}}=\frac{\cos.\eta}{kk-nn}-\frac{4\cos.2\eta}{4kk-nn}+\frac{9\cos.3\eta}{9kk-nn}-\frac{16\cos.4\eta}{16kk-nn}+\text{etc.}$$

Unde si sumatur $\eta=0$, orietur ista summatio

$$\frac{\pi n}{2k^3\sin.\frac{n\pi}{k}}=\frac{1}{kk-nn}-\frac{4}{4kk-nn}+\frac{9}{9kk-nn}-\frac{16}{16kk-nn}+\text{etc.};$$

sin autem sumatur $\eta=90°=\frac{\pi}{2}$, erit

$$\cos.\eta=0,\ \cos.2\eta=-1,\ \cos.3\eta=0,\ \cos.4\eta=+1\text{ etc.},$$

unde nascitur sequens series

$$\frac{n\pi\cos.\frac{n\pi}{2k}}{2k^3\sin.\frac{n\pi}{k}}=\frac{4}{4kk-nn}-\frac{16}{16kk-nn}+\frac{36}{36kk-nn}-\frac{64}{64kk-nn}+\text{etc.}$$

Quia autem si $\sin.\frac{n\pi}{k}=2\sin.\frac{n\pi}{2k}\cos.\frac{n\pi}{2k}$, erit eiusdem seriei summa $\frac{n\pi}{4k^3\sin.\frac{n\pi}{2k}}$.

§. 194. At si series illa § 32 exhibita in $\partial\eta$ ducatur et integretur, ob $\int\partial\eta\sin.\frac{n\eta}{k}=-\frac{k}{n}\cos.\frac{n\eta}{k}$ erit

$$\mathrm{C}-\frac{\pi\cos.\frac{n\eta}{k}}{2nk\sin.\frac{n\pi}{k}}=-\frac{\cos.\eta}{kk-nn}+\frac{\cos.2\eta}{4kk-nn}-\frac{\cos.3\eta}{9kk-nn}+\frac{\cos.4\eta}{16kk-nn}-\text{etc.}$$

Ut autem hic constantem addendam C definiamus, sumamus $\eta=0$ fietque

$$\mathrm{C}-\frac{\pi}{2nk\sin.\frac{n\pi}{k}}=-\frac{1}{kk-nn}+\frac{1}{4kk-nn}-\frac{1}{9kk-nn}+\text{etc.};$$

quare si huius seriei summa aliunde pateat, constans C definiri poterit.
Series autem haec in sequentem geminatam resolvi potest

$$2n\mathrm{C}-\frac{\pi}{k\sin.\frac{n\pi}{k}}=\frac{1}{k+n}-\frac{1}{2k+n}+\frac{1}{3k+n}-\frac{1}{4k+n}+\text{etc.}$$
$$-\frac{1}{k-n}+\frac{1}{2k+n}-\frac{1}{3k+n}+\frac{1}{4k-n}+\text{etc.}$$

35. Cum igitur in *Introductione in Analysin lnfinitorum* pag. 142 ad hanc pervenissem seriem

$$\frac{1}{kk-nn}-\frac{1}{4kk-nn}+\frac{1}{9kk-nn}-\frac{1}{16kk-nn}+\text{etc.}=\frac{\pi}{2kn\sin.\frac{n\pi}{k}}-\frac{1}{2nn}$$

(hic scilicet loco litterarum ibi adhibitarum *m* et *n* ·scripsi *n* et *k*)*,* hoc valore adhibito nostra aequatio erit

$$\mathrm{C}-\frac{\pi}{2nk\sin.\frac{n\pi}{k}}=\frac{1}{2nn}-\frac{\pi}{2nk\sin.\frac{n\pi}{k}},$$

unde fit $\mathrm{C}=\frac{1}{2nn}$. Hinc ergo habebimus istam summationem

$$\frac{\pi\cos.\frac{n\eta}{k}}{2kn\sin.\frac{n\pi}{k}}-\frac{1}{2nn}=\frac{\cos.\eta}{kk-nn}-\frac{\cos.2\eta}{4kk-nn}+\frac{\cos.3\eta}{9kk-nn}-\frac{\cos.4\eta}{16kk-nn}+\text{etc.},$$

quae series utique omni attentione digna videtur.