A Fuller Explanation Concerning the Comparison of the Quantities Contained in the Integral Formula $\int \frac{Z\partial z}{\sqrt{\left(1+mzz+nz^4\right)}}$, with the Function Z Denoting some Rational Function of zz

[E581] Acta academiae scientiarum Petropolitanae 1781: II (1785), p. 3-22.

§.1. Even if I have often examined this argument, and the most illustrious Lagrange has published several outstanding observations on the formulas of this kind, yet that by no means satisfies the investigation at this stage, much less can it be considered to be exhausted, but many very-well hidden properties may be observed, which require the deepest investigation and promise significant analytical advances. But initially these analytical operations, which led me in the first place to this investigation, are to be prepared thus, so that the whole business may be constructed, but only by going down several winding paths, from which even now the direct method leading to the same relations deserves merit, and especially is to be desired. Truly besides, this whole investigation extends much more widely than to these integral formulas, which I had considered originally, where for the letter Z, I assumed only a constant quantity or an integral function of zz of this form : $\mathrm{F}+\mathrm{G}zz+\mathrm{H}z^4+\mathrm{I}z^6+\mathrm{K}z^8+\text{etc.}$, from which proposed cases I have shown from any two quantities of this kind always a third of this same kind can be found, which may differ from the sum of these by an algebraic quantity, which indeed may vanish in the case where Z is only a constant quantity.

§. 2. But now I have observed the same comparisons can be put in place, if some rational function of zz can be taken for Z, which clearly may have a form of this kind :

$$\frac{\mathrm{F}+\mathrm{G}zz+\mathrm{H}z^4+\mathrm{I}z^6+\mathrm{K}z^8+\text{etc.}}{\mathfrak{F}+\mathfrak{G}zz+\mathfrak{H}z^4+\mathfrak{I}z^6+\mathfrak{K}z^8+\text{etc.}},$$

where indeed this distinction occurs, so that the difference between the sum of two formulas of this kind and a third formula of the same kind requiring to be found shall no further be an algebraic quantity, yet truly always may be able to be shown by logarithms and circular arcs, thus so that now the same investigation may extend much more widely, that I was constructing at this point. And hence perhaps, if all the operations, which lead to this goal, must be considered with great attention, they will be able to find an easier path for the direct method to be come upon and this whole very abstruse argument to be scrutinized more happily with success.

§.3. But so that all these may be seen clearer, this character $\Pi : z$ will denote that transcending quantity, [*i.e.* the indefinite integral] which arises from the integration of the proposed formula,

$$\int \frac{Z\partial z}{\sqrt{\left(1+mzz+nz^4\right)}},$$

while it is assumed, with the integral taken thus, so that it may vanish on putting $z=0$; from which it is evident at once also to become $\Pi:0=0$. Then since Z may involve only even powers of z, also powers of this kind are present in the root formulas, it is clear, if in place of z there may be written $-z$, then the value both of this integral formula, and thus also of the expression $\Pi:z$, turn into their negative, thus so that there shall be $\Pi:(-z)=-\Pi:z$. With these prescribed, if any two quantities of this kind $\Pi:p$ and $\Pi:q$ may be proposed, a third quantity of this same kind $\Pi:r$ always can be found, which may differ from the sum of these formulas $\Pi:p+\Pi:q$ either by an algebraic quantity or perhaps to be assigned by logarithms and circular arcs. Truly the rules, by which the third r is elicited from the given letters p and q, always remains the same, whatever function may be designated by the letter Z; indeed there will be always :

$$r=\frac{p\sqrt{\left(1+mqq+nq^4\right)}+q\sqrt{\left(1+mpp+np^4\right)}}{1-nppqq}.$$

Hence moreover for the following combinations it will be a help to observe :

$$\sqrt{\left(1+mrr+nr^4\right)}=\frac{\left(mpq+\sqrt{\left(1+mpp+np^4\right)}\right)\left(\sqrt{\left(1+mqq+nq^4\right)}\right)(1+nppqq)+2npq(pp+qq)}{(1-nppqq)^2}.$$

[These results are shown below.]

§.4. But not only is this investigation restricted to formulas of this kind $\Pi:p$ and $\Pi:q$ requiring to be taken arbitrarily, but thus it can be extended to any number of given formulas, thus so that, whatever number of formulas of this kind proposed, evidently :

$$\Pi:f+\Pi:g+\Pi:h+\Pi:i+\Pi:k+\text{etc.},$$

a new formula of this kind can always be assigned $\Pi:r$, which may differ from the sum of those by a quantity to be designated either algebraically, or perhaps by logarithms or circular arcs. But also these formulas, which we have considered as given, it will be allowed to be defined, as that difference, either algebraic or depending on logarithms or circular arcs, therefore may vanish, thus so that there shall become :

$$\Pi:r=\Pi:f+\Pi:g+\Pi:h+\Pi:i+\Pi:k+\text{etc.},$$

And these are almost, according to which this more general investigation, which here it has been agreed to set out, indeed it may be permitted by me to extend ; on account of which the operations, which have been deduced by me here, I am going to propose briefly.

OPERATION 1

§.5. Generally, I have began this investigation from the consideration of this algebraic equation :

$$\alpha+\gamma\left(xx+yy\right)+2\delta xy+\zeta xxyy=0,$$

from which, since it shall be a quadratic both for *x* as well as *y*, by extracting the root, there is deduced, either

$$y=\frac{-\delta x+\sqrt{\left(-\alpha\gamma+(\delta\delta-\gamma\gamma-\alpha\zeta)xx-\gamma\zeta x^4\right)}}{\gamma+\zeta xx}$$

or

$$x=\frac{-\delta y+\sqrt{\left(-\alpha\gamma+(\delta\delta-\gamma\gamma-\alpha\zeta)yy-\gamma\zeta y^4\right)}}{\gamma+\zeta yy},$$

thus so that hence there becomes :

$$\sqrt{\left(-\alpha\gamma+\left(\delta\delta-\gamma\gamma-\alpha\zeta\right)xx-\gamma\zeta x^4\right)}=\gamma y+\delta x+\zeta xxy$$

and

$$\sqrt{\left(-\alpha\gamma+\left(\delta\delta-\gamma\gamma-\alpha\zeta\right)yy-\gamma\zeta y^4\right)}=\gamma x+\delta y+\zeta xyy.$$

§.6. Now I define the letters $\alpha,\ \gamma,\ \delta,\ \zeta$ thus, so that both the root formulas may be reduced to the form

$$\sqrt{\left(1+mxx+nx^4\right)}\ \text{ and }\ \sqrt{\left(1+myy+ny^4\right)},$$

which to this end, I make:

$$1.\ -\alpha\gamma=k,\ \ 2.\ \delta\delta-\gamma\gamma-\alpha\zeta=mk\ \text{ and }\ 3.\ -\gamma\zeta=nk;$$

from the first of which equalities there becomes $\alpha=-\frac{k}{\gamma}$, from the third $\zeta=\frac{-nk}{r}$, which values substituted into the second give

$$\delta\delta = \gamma\gamma + \tfrac{nkk}{rr} + mk$$

and thus

$$\delta = \sqrt{\left(\gamma\gamma + \tfrac{nkk}{rr} + mk\right)} = \tfrac{1}{\gamma}\sqrt{\left(\gamma^4 + m\gamma\gamma k + nkk\right)};$$

from which our equation now will be :

$$-k + \gamma\gamma\left(xx + yy\right) + 2xy\sqrt{\left(\gamma^4 + m\gamma\gamma k + nkk\right)} - nkxxyy = 0;$$

hence therefore our two irrational formulas will become

$$\sqrt{k\left(1 + mxx + nx^4\right)} = \gamma y + \tfrac{1}{\gamma}x\sqrt{\left(\gamma^4 + m\gamma\gamma k + nkk\right)} - \tfrac{nk}{\gamma}xxy,$$
$$\sqrt{k\left(1 + myy + ny^4\right)} = \gamma x + \tfrac{1}{\gamma}y\sqrt{\left(\gamma^4 + m\gamma\gamma k + nkk\right)} - \tfrac{nk}{\gamma}xyy.$$

§.7. Now since both the quantities x and y thus shall depend on each other in turn, just as the assumed equation shall specify, we may define at this point the indefinite quantities r and k, so that on putting $x = 0$ there may become $y = a$. Therefore it will be necessary for $-k + \gamma\gamma aa = 0$ and thus $k = \gamma\gamma aa$, with which value substituted our equation will be :

$$0 = \gamma\gamma\left(xx + yy - aa\right) + 2\gamma\gamma xy\sqrt{\left(1 + maa + na^4\right)} - n\gamma\gamma aaxxyy,$$

and hence there will be on dividing by $\gamma\gamma$:

$$0 = \left(xx + yy - aa\right) + 2xy\sqrt{\left(1 + maa + na^4\right)} - naaxxyy.$$

Then truly both our roots of the formula thus will be expressed :

$$\sqrt{\left(1 + mxx + nx^4\right)} = \tfrac{y}{a} + \tfrac{x}{a}\sqrt{\left(1 + maa + na^4\right)} - naxxy,$$
$$\sqrt{\left(1 + myy + ny^4\right)} = \tfrac{x}{a} + \tfrac{y}{a}\sqrt{\left(1 + maa + na^4\right)} - naxyy.$$

§.8. So that we may return these formulas more easily handled, we may put

$$\sqrt{\left(1 + maa + na^4\right)} = \mathfrak{A}$$

and in a similar manner:

$$\sqrt{\left(1+mxx+nx^4\right)}=\mathfrak{X} \text{ and } \sqrt{\left(1+myy+ny^4\right)}=\mathfrak{Y}$$

and our equation will be

$$xx+yy-aa+2\mathfrak{A}xy-naaxxyy=0,$$

from which there is found

$$y=-\frac{\mathfrak{A}x-a\mathfrak{X}}{1-naaxx} \text{ and } x=-\frac{\mathfrak{A}y-a\mathfrak{Y}}{1-naayy};$$

from which it is apparent, if there were $y=0$, to become $x=a$; then truly the root formulas will be :

$$\sqrt{\left(1+mxx+nx^4\right)}=\mathfrak{X}=\frac{y}{a}+\frac{\mathfrak{A}x}{a}-naxxy,$$
$$\sqrt{\left(1+myy+ny^4\right)}=\mathfrak{Y}=\frac{y}{a}+\frac{\mathfrak{A}y}{a}-naxyy.$$

§.9. But just as it has been allowed to express both *y* by *x* as well as *x* by *y*, thus also $\mathfrak{Y}$ will be allowed to be expressed by *x* alone, and $\mathfrak{X}$ by *y* only. Moreover with the calculation put in place there will be found to be :

$$\mathfrak{X}=\frac{(-may+\mathfrak{A}\mathfrak{Y})(1+naayy)-2nay(aa+yy)}{(1-naayy)^2},$$
$$\mathfrak{Y}=\frac{(-max+\mathfrak{A}\mathfrak{X})(1+naaxx)-2nax(aa+xx)}{(1-naaxx)^2}.$$

§.10. But it will be especially noteworthy concerning our equation

$$xx+yy-aa+2\mathfrak{A}xy-naaxxyy=0,$$

because the three quantities *xx, yy, aa* shall be perfectly permutable between themselves. If indeed the irrational member may be transferred to the other side, so that there shall be

$$xx+yy-aa-naaxxyy=-2\mathfrak{A}xy,$$

and squares may be taken, with its value $1+maa+na^4$ restored for $\mathfrak{A}^2$, its value will produce this equation :

$$\left.\begin{array}{l} +x^4-2xxyy-4maaxxyy-2na^4xxyy+nna^4x^4\ y^4 \\ +y^4-2aaxx \qquad\qquad\qquad -2naax^4yy \\ +a^4-2aayy \qquad\qquad\qquad -2naaxxy^4 \end{array}\right\}=0,$$

where the permutations of the letters *a, x, y* is evident at once. Indeed in these above formulas, where that quantity *a* itself enters, the permutability thus is not evident, but certainly it will be made clear, if in place of *a* we may write $-b$ and likewise $\mathfrak{B}$ in place of $\mathfrak{A}$; then indeed, just as there was

$$y=-\frac{x\mathfrak{B}+b\mathfrak{X}}{1-nbbxx} \text{ and } x=-\frac{y\mathfrak{B}+b\mathfrak{Y}}{1-nbbyy},$$

thus there will be

$$b=-\frac{x\mathfrak{Y}+y\mathfrak{X}}{1-nxxyy}$$

and in a similar manner, for the formulas with the roots or capital letters, there will be :

$$\mathfrak{Y}=\frac{(mbx+\mathfrak{B}\mathfrak{X})(1+nbbxx)+2nbx(aa+xx)}{(1-nbbxx)^2},$$
$$\mathfrak{X}=\frac{(mby+\mathfrak{B}\mathfrak{Y})(1+nbbyy)+2nby(bb+yy)}{(1-nbbyy)^2},$$
$$\mathfrak{B}=\frac{(mxy+\mathfrak{X}\mathfrak{Y})(1+nxxyy)+2nxy(xx+yy)}{(1-nxxyy)^2},$$

and thus is seen with perfect permutability .

OPERATION 2

§.11. Now we will differentiate our algebraic equation assumed, which is

$$xx+yy-aa+2\mathfrak{A}xy-naaxxyy=0,$$

and the equation of the differential will be

$$\partial x\left(x+\mathfrak{A}y-naaxyy\right)+\partial y\left(y+\mathfrak{A}x-naaxxy\right)=0$$

or

$$\frac{\partial x}{y+\mathfrak{A}x-naaxxy}+\frac{\partial y}{x+\mathfrak{A}y-naaxyy}=0.$$

But it is agreed from the above :

$$y+\mathfrak{A}x-naaxxy=a\mathfrak{X} \text{ and } x+\mathfrak{A}y-naaxyy=a\mathfrak{Y},$$

from which the differential equation will adopt this form :

$$\frac{\partial x}{a\mathfrak{X}}+\frac{\partial y}{a\mathfrak{Y}}=0$$

or

$$\frac{\partial x}{\sqrt{(1+mxx+nx^4)}}+\frac{\partial y}{\sqrt{(1+myy+ny^4)}}=0.$$

§.12. Therefore with this equation found the integral $\int\frac{\partial x}{\mathfrak{X}}$ may be denoted by the symbol $\Gamma:x$ and with the integral $\int\frac{\partial y}{\mathfrak{Y}}$ by the symbol $\Gamma:y$ and with each integral thus taken, so that it may vanish on putting either $x=0$ or $y=0$, and by integrating that equation there will become $\Gamma:x+\Gamma:y=\text{C}$. But on taking $x=0$ there becomes also $\Gamma:x=0$ and $y=a$, therefore it is agreed that $\text{C}=\Gamma:a$, thus so that we may have this equation $\Gamma:x+\Gamma:y=\Gamma:a.$

§.13. Since here no further ratio of the variables may be considered, it is evident with the two letters x and y taken as it pleases it will be possible always to define the letter a, so that there becomes

$$\Gamma:a=\Gamma:x+\Gamma:y.$$

Indeed if in §.10 there may be written $-a$ in place of b, there must be taken

$$a=\frac{x\mathfrak{Y}+y\mathfrak{X}}{1-nxxyy},$$

which preparation that we have undertaken now constitutes a special case of the general investigation. For if in place of x and y we may write p and q, moreover r in place of a, then truly $\mathfrak{P}$, $\mathfrak{Q}$ and $\mathfrak{R}$ in place of $\mathfrak{X}$, $\mathfrak{Y}$ et $\mathfrak{A}$ (and if with the quantities p, q assumed as it pleases there may be taken $r=\frac{p\mathfrak{Q}+q\mathfrak{P}}{1-nppqq}$, then certainly there will be $\Gamma:r=\Gamma:p+\Gamma:q$, thus so that in this case that distinction between $\Gamma:r$ and the sum $\Gamma:p+\Gamma:q$ plainly may vanish. And thus now we set out the case, where in our general form

$$\int\frac{\text{Z}\partial z}{\sqrt{(1+mzz+nz^4)}},$$

a constant quantity is assumed for Z.

OPERATION 3

§.14. Now so that we may approach closer to our design, X and Y shall be such functions of x and y, as we wish to be Z of z, and since just as we find :

$$\frac{\partial x}{\sqrt{\left(1+mxx+nx^4\right)}}+\frac{\partial y}{\sqrt{\left(1+myy+ny^4\right)}}=0,$$

we may put to be

$$\frac{\mathrm{X}\partial x}{\sqrt{\left(1+mxx+nx^4\right)}}+\frac{\mathrm{Y}\partial y}{\sqrt{\left(1+myy+ny^4\right)}}=\partial\mathrm{V},$$

thus so that, if X and Y were constant magnitudes, there would become $\partial\mathrm{V}=0$. Hence therefore, if in place of $\frac{\partial y}{\sqrt{\left(1+myy+ny^4\right)}}$ we may write $\frac{-\partial x}{\sqrt{\left(1+mxx+nx^4\right)}}$, there will become

$$\partial\mathrm{V}=\frac{(\mathrm{X}-\mathrm{Y})\partial x}{\sqrt{\left(1+mxx+nx^4\right)}} \text{ or also } \frac{(\mathrm{Y}-\mathrm{X})\partial y}{\sqrt{\left(1+myy+ny^4\right)}}.$$

But if in place of the roots we may write their rational values, there will be

$$\partial\mathrm{V}=\frac{a(\mathrm{X}-\mathrm{Y})\partial x}{y+\mathfrak{A}x-naaxxy} \text{ or } \partial\mathrm{V}=\frac{a(\mathrm{Y}-\mathrm{X})\partial y}{x+\mathfrak{A}y-naaxyy}.$$

§.15. But when there shall be no reason, why we may express that differential ∂V by ∂x rather than by ∂y, it will be decided to introduce a new quantity into the calculation, which may refer equally to both *x* and *y*. To this end we may make the product $xy=u$ and we may put

$$\frac{\partial x}{y+\mathfrak{A}x-naaxxy}=-\frac{\partial y}{x+\mathfrak{A}y-naaxyy}=s\partial u.$$

Hence there will be therefore

$$\partial x=s\partial u\left(y+\mathfrak{A}x-naaxxy\right) \text{ and } \partial y=-s\partial u\left(x+\mathfrak{A}y-naaxyy\right),$$

from which we deduce

$$ydx+xdy=s\partial u\left(yy-\ xx\right)=\partial u,$$

and thus we will have $s=\frac{1}{yy-xx}$, thus so that we will have

$$\frac{\partial x}{y+\mathfrak{A}x-naaxxy}=-\frac{\partial y}{x+\mathfrak{A}y-naaxyy}=\frac{\partial u}{yy-xx}.$$

Therefore with this value substituted , we obtain :

$$\partial \mathrm{V} = \frac{a(\mathrm{X}-\mathrm{Y})\partial u}{yy-xx} = \frac{-a\partial u(\mathrm{X}-\mathrm{Y})}{xx-yy}.$$

§.16. But since X and Y themselves shall be even rational functions of x and y, in which only even powers of these letters shall be present, it is readily understood the formula $\mathrm{X}-\mathrm{Y}$ always to be divisible by $xx-yy$ and besides however many times the above product $xy=u$ involves the sum of the two squares $xx+yy$; on account of which we may put $xx+yy=t$, and since our fundamental equation becomes

$$t-aa+2\mathfrak{A}u-naauu=0,$$

from that there becomes

$$t=aa-2\mathfrak{A}u+naauu,$$

thus so that t shall be equal to a rational function of u. But if therefore this value may be written in place of t, our sought differential $\partial \mathrm{V}$ may be expressed by the variable u alone, thus so that by putting $\partial \mathrm{V}=\mathrm{U}\partial u$, U shall always be a rational function of u; which therefore if it were whole, then V will be equal to an algebraic function of u, but if it may be a fractional function, then the integral $\mathrm{V}=\int \mathrm{U}\partial u$ always will be able to be expressed by logarithms and circular arcs. Therefore this integral, if it may be taken thus, so that it may vanish on putting $u=xy=0$, it also vanishes on putting $x=0$ or $y=0$. And hence on integrating we will obtain :

$$\int \frac{\mathrm{X}\partial x}{\sqrt{(1+mxx+nx^4)}} + \int \frac{\mathrm{Y}\partial y}{\sqrt{(1+myy+ny^4)}} = \mathrm{C}+\mathrm{V} = \mathrm{C}+\int \mathrm{U}\partial u.$$

§.17. Therefore because if the symbols $\Pi : x$ and $\Pi : y$ may denote the values of these integrals, thus so that each may vanish on assuming either $x=0$ or $y=0$, since by making $x=0$ by the hypothesis there becomes $y=a$, it is evident this constant to become $\Pi : a$ and thus this same finite equation will itself result

$$\Pi : x + \Pi : y = \Pi : a + \int \mathrm{U}\partial u.$$

§.18. But we shall enquire more carefully into the values of this fraction U for some case. And indeed initially, if there may be assumed

$$\mathrm{Z}=\alpha+\beta zz+\gamma z^4+\delta z^6+\text{etc.},$$

in a similar manner there will be

$$\mathrm{X}=\alpha+\beta xx+\gamma x^4+\delta x^6+\text{etc. and } \mathrm{Y}=\alpha+\beta yy+\gamma y^4+\delta y^6+\text{etc.};$$

whereby since we will have found

$$\partial \mathrm{V} = \mathrm{U}\partial u = -\frac{a\partial u(\mathrm{X-Y})}{xx-yy},$$

there will be

$$\mathrm{U} = -\frac{a(\mathrm{X-Y})}{xx-yy} \text{ and thus } \mathrm{U} = -\frac{a\left(\beta(xx-yy)+\gamma\left(x^4-y^4\right)+\delta\left(x^6-y^6\right)\right)}{xx-yy},$$

from which there becomes :

$$\mathrm{U} = -a\beta - a\gamma\left(xx+yy\right) - a\delta\left(x^4+xxyy+y^4\right).$$

Therefore since there shall be $xx+yy=t$ and $xy=u$, there will become :

$$u = -a\beta - a\gamma t - a\delta\left(tt-uu\right);$$

from which, since there shall be $t = aa - 2\mathfrak{A}u + naauu$, with the calculation performed there will become

$$\int U\partial u = -a\beta u - a\gamma\left(aau - \mathfrak{A}uu + \tfrac{1}{3}naau^3\right)$$
$$-a\delta\left(a^4u - 2aa\mathfrak{A}uu + \tfrac{2}{3}na^4u^3 + \tfrac{4}{3}\mathfrak{A}^2u^3 - \tfrac{1}{3}u^3 - n\mathfrak{A}a^2u^4 + \tfrac{1}{5}n^2a^4u^5\right).$$

And hence it is understood, if the function Z may rise to higher powers, how the value of its integral $\int \mathrm{U}\partial u$ thence may be able to be found.

§.19. But if Z were a fractional function, evidently

$$\mathrm{Z} = \frac{\alpha+\beta zz+\gamma z^4}{\zeta+\eta zz+\theta z^4}$$

and hence

$$\mathrm{X} = \frac{\alpha+\beta xx+\gamma x^4}{\zeta+\eta xx+\theta x^4} \text{ and } \mathrm{Y} = \frac{\alpha+\beta yy+\gamma y^4}{\zeta+\eta yy+\theta y^4},$$

there will be

$$\mathrm{X-Y} = \frac{(\beta\zeta-\alpha\eta)(xx-yy)+(\gamma\zeta-\alpha\theta)\left(x^4-y^4\right)+(\gamma\eta-\beta\theta)x^2y^2\left(x^2-y^2\right)}{\zeta\zeta+\zeta\eta(xx+yy)+\zeta\theta\left(x^4+y^4\right)+\eta^2x^2y^2+\eta\theta x^2y^2(xx+yy)+\theta\theta x^4y^4}.$$

Therefore hence with the letters t and u introduced, there will become

$$\frac{\mathrm{X}-\mathrm{Y}}{xx-yy}=\frac{\beta\zeta-\alpha\eta+(\gamma\zeta-\alpha\theta)t+(\gamma\eta-\beta\theta)uu}{\zeta\zeta+\zeta\eta t+\zeta\theta(tt-2uu)+\eta\eta uu+\eta\theta tuu+\theta\theta u^4};$$

on account of which, since there shall be

$$\mathrm{U}=-\frac{a(\mathrm{X}\text{-}\mathrm{Y})}{xx-yy},$$

on account of which $t=aa-2\mathfrak{A}u+naauu$ evidently is the integral of the formula $\int \mathrm{U}\partial u$, if it were not algebraic, always able to be shown from permitted logarithms or circular arcs. And thus by these three operations we have established everything , for which there is a need for all the problems here being seen requiring to be solved.

PROBLEM 1

§.20. *If* $\Pi:x$ and $\Pi:y$ *may denote the values of the integral formulas,*

$$\int \frac{\mathrm{X}\partial x}{\sqrt{\left(1+mxx+nx^4\right)}} \text{ and } \int \frac{\mathrm{Y}\partial y}{\sqrt{\left(1+myy+ny^4\right)}}$$

where X *and* Y *shall be even similar functions of x and y, and the two formulas of this kind* $\Pi:x$ and $\Pi:y$ *may be given, to find a third formula of this kind* $\Pi:z$, *so that there shall be* $\Pi:z=\Pi:x + \Pi:y+W$ *, thus so that W shall be either an algebraic function or assignable by logarithms or circular arcs.*

SOLUTION

Since the two quantities *x* and *y* may be given, from these the roots may be formed

$$\mathfrak{X}=\sqrt{\left(1+mxx+nx^4\right)} \text{ and } \mathfrak{Y}=\sqrt{\left(1+myy+ny^4\right)},$$

from which the quantity *z* may be defined, in that same way as we have shown above how to define the letter *a* by *x* and *y* , thus so that there shall be

$$z=\frac{x\mathfrak{Y}+y\mathfrak{X}}{1-nxxyy}$$

and the value of its irrational

$$\mathfrak{Z}=\sqrt{\left(1+mzz+nz^4\right)}=\frac{(mxy+\mathfrak{X}\mathfrak{Y})(1+nxxyy)+2nxy(xx+yy)}{(1-nxxyy)^2};$$

where in the upper formulas everywhere we may write z and Z in place of a and $\mathfrak{A}$ and there may be taken $\mathrm{U} = -\frac{z(\mathrm{X}-\mathrm{Y})}{xx-yy}$, the quantity we have seen above always able to be reduced to its function of u proving to be $u = xy$,and there may be put $\mathrm{V} = \int \mathrm{U}\partial u$, in which integration the quantities z and $\mathfrak{Z}$ are required to be considered as constants, thus so that the letter V may be considered to be put only as the function itself $u = xy$, since also z and $\mathfrak{Z}$ may be determined by x and y . But correctly only the quantity u may be considered to be treated as a variable in this formula. Therefore from this quantity V there will be found :

$$\Pi : x + \Pi : y = \Pi : z + \mathrm{V};$$

from which, since there must be

$$\Pi : z = \Pi : x + \Pi : y + \mathrm{W},$$

it is evident to be $\mathrm{W} = -\mathrm{V}$ and thus the quantity assigned either algebraically, or by logarithms and circular arcs.

COROLLARY 1

§.21. Therefore the whole matter here reduces to the integration of the formula $\mathrm{U}\partial u$ with $u = xy$ and $\mathrm{U} = -\frac{z(\mathrm{X}-\mathrm{Y})}{xx-yy}$ arising, as we have seen above always able to be expressed by u , if indeed in this integration the letters z and $\mathfrak{Z}$ may be treated as constant quantities.

COROLLARY 2

§.22. Therefore since this integration may not labour under any difficulty from the given nature of the two functions X and Y, and the integral itself may be expressed by u, that is, by xy, its value may be allowed to be shown always by the given quantities x and y, in place of the quantity V we may write the symbol $\Phi : xy$ in the following, from which for any of the other letters assumed in place of x and y , the symbol $\Phi : pq$, $\Phi : ab$ etc. etc. is understood.

COROLLARY 3

§.23. Therefore with this symbol adopted, if for the given quantities x et y we may take $z = \frac{x\mathfrak{Y} + y\mathfrak{X}}{1-nxxyy}$, from which

$$\mathfrak{Z} = \frac{(mxy+\mathfrak{X}\mathfrak{Y})(1+nxxyy)+2nxy(xx+yy)}{(1-nxxyy)^2},$$

there will be :

$$\Pi : z = \Pi : x + \Pi : y - \Phi : xy .$$

PROBLEM 2

§.24. *With all the symbols attended to, which we have explained at present, if the three formulas may be given* $\Pi : p$, $\Pi : q$, $\Pi : r$, *to find a fourth of the same kind* $\Pi : z$, *so that there may become ,*

$$\Pi : z = \Pi : p + \Pi : q + \Pi : r + W,$$

thus so that W shall be an algebraic quantity, or assignable either by logarithms or circular arcs.

SOLUTION

From the two given quantities p and q and thus also $\mathfrak{P}$ et $\mathfrak{Q}$ thence arising, there may be taken

$$x = \frac{p\mathfrak{Q}+q\mathfrak{P}}{1-nppqq}$$

and likewise

$$\mathfrak{X} = \frac{(mpq+\mathfrak{P}\mathfrak{Q})(1+nppqq)+2npq(pp+qq)}{(1-nppqq)^{2}}.$$

While truly also the value of the symbol $\Phi : pq$ may be deduced and it will be by the preceding :

$$\Pi : x = \Pi : p + \Pi : q - \Phi : pq,$$

or

$$\Pi : p + \Pi : q = \Pi : x + \Phi : pq,$$

with which value substituted there will be

$$\Pi : z = \Pi : x + \Pi : r + \Phi : pq + \mathrm{W}.$$

But from the preceding problem, if here we may write r in place of y and we may take

$$z = \frac{x\mathfrak{R}+r\mathfrak{X}}{1-nrrxx},$$

from which there becomes :

$$\mathfrak{Z} = \frac{(mrx+\mathfrak{R}\mathfrak{X})(1+nrrxx)+2nrx(rr+xx)}{(1-nrrxx)^{2}},$$

there will be

$$\Pi : z = \Pi : x + \Pi : r - \Phi : rx,$$

from which form taken with the preceding, there is deduced :

$$\mathrm{W} = -\Phi : pq - \Phi : rx,$$

thus so that there shall be:

$$\Pi : z = \Pi : p + \Pi : q + \Pi : r - \Phi : pq - \Phi : rx .$$

PROBLEM 3

§.25. *From the proposed formulas of this kind* $\Pi : p,\ \Pi : q,\ \Pi : r,\ \Pi : s$ *to find a fifth of the same kind* $\Pi : z$*, so that there may become*

$$\Pi :\ z = \Pi : p + \Pi : q + \Pi : r + \Pi : s + W ,$$

thus so that W shall be a quantity assignable either algebraically, by logarithms, or by circular arcs.

SOLUTION

From the two given *p* and *q, x* may be found, so that there shall be

$$x = \frac{p\mathfrak{Q}+q\mathfrak{P}}{1-nppqq}$$

likewise

$$\mathfrak{X} = \frac{(mpq+\mathfrak{P}\mathfrak{Q})(1+nppqq)+2npq(pp+qq)}{(1-nppqq)^2},$$

and there will become :

$$\Pi : x = \Pi : p + \Pi : q - \Phi : pq.$$

In a similar manner, from the two given *r*, *s* , *y* may be found, so that there shall be

$$y = \frac{r\mathfrak{S}+s\mathfrak{R}}{1-nrrss}$$

and there will become

$$\mathfrak{Y} = \frac{(mrs+\mathfrak{R}\mathfrak{S})(1+nrrss)+2nrs(rr+ss)}{(1-nrrss)^2},$$

then truly,

$$\Pi : y = \Pi : r + \Pi : s - \Phi : rs .$$

Now therefore from *x* and *y* found, there may be assumed

$$z = \frac{x\mathfrak{Y}+y\mathfrak{X}}{1-nxxyy} \quad \text{and} \quad \mathfrak{Z} = \frac{(mxy+\mathfrak{X}\mathfrak{Y})(1+nxxyy)+2nxy(xx+yy)}{(1-nxxyy)^2},$$

and there will become

$$\Pi : z = \Pi : x + \Pi : y - \Phi : xy.$$

But if therefore in place of $\Pi : x$ and $\Pi : y$ the values found just now may be substituted, there will become

$$\Pi : z = \Pi : p + \Pi : q + \Pi : r + \Pi : s - \Phi : pq - \Phi : rs - \Phi : xy .$$

COROLLARY I

§.26. Hence now it is abundantly clear, if some number of formulas of this kind may be proposed, just as it may be required to find a new one $\Pi : z$ of the same kind , which from these taken jointly may differ by an algebraic quantity, or to be assigned by logarithms or circular arcs.

COROLLARY 2

§.27. Because if all there formulas were equal to each other, and the number of these $= \lambda ,$ a new formula $\Pi : z$ will always be able to be found, so that there shall be

$$\Pi : z = \lambda \Pi : p + W$$

with the quantity W being either algebraic or assignable by logarithms or circular arcs. Why not also with two formulas proposed of this kind $\Pi : p$ and $\Pi : q$, $\Pi : z$ will be able to be found, so that there shall be

$$\Pi : z = \lambda \Pi : p + \mu \Pi : q + \mathrm{W}.$$

SCHOLIUM

§.28. Therefore in this manner I may consider to have set out, succinctly and clearly, not only the principles and foundations, by which this argument is supported, but also I have enlarged on this argument much more widely, than has been done at this stage. But it is wished always, that a more direct way may be uncovered, which may lead to the same investigations. Certainly indeed no one will doubt, why a great increase in the whole of analysis should not be forthcoming from this.

Application to the Magnitudes Present Transformed into

$$\int \frac{\partial z\left(\alpha+\beta zz\right)}{\sqrt{\left(1+mzz+nz^4\right)}} = \Pi : z .$$

§.29. Therefore since here there shall be $Z = \alpha + \beta zz ,$ and with the two formulas proposed assumed of this kind $\Pi : x$ and $\Pi : y$:

$$z=\frac{x\mathfrak{Y}+y\mathfrak{X}}{1-nxxyy} \quad \text{and hence} \quad \mathfrak{Z}=\frac{(mxy+\mathfrak{X}\mathfrak{Y})(1+nxxyy)+2nxy\left(x^2+y^2\right)}{(1-nxxyy)^2}$$

from § 18, where $u=xy$ and $a=z$, there will be

$$\Pi:z=\Pi:x+\Pi:y+\beta xyz,$$

thus so that the symbol used before $\Phi:xy$ may take the value βxyz in this case. Therefore with the help of this rule proposed from the two formulas $\Pi:x$ and $\Pi:y$ of this kind a third $\Pi:z$ can be found always, which shall differ from the sum of these by the algebraic quantity βxyz.

§.30. Therefore we may put however many formulas of this kind to be proposed

$$\Pi:a,\ \Pi:b,\ \Pi:c,\ \Pi:d,\ \Pi:e,\ \Pi:f,\ \Pi:g \text{ etc.}$$

and from the individual magnitudes *a, b, c, d* etc. the signified values of the irrational Germanic letters to be deduced

$$\mathfrak{A}=\sqrt{\left(1+maa+na^4\right)},\quad \mathfrak{B}=\sqrt{\left(1+mbb+nb^4\right)},$$
$$\mathfrak{C}=\sqrt{\left(1+mcc+nc^4\right)},\quad \mathfrak{D}=\sqrt{\left(1+mdd+nd^4\right)},$$
$$\text{etc.} \qquad\qquad \text{etc.}$$

will always show new formulas of the same kind, which may differ from the sum of these by an algebraic quantity, also however great the number of these given formulas should be. But the operations leading to this conclusion may be put in place most conveniently in the following manner.

§.31. Evidently in the first place, from the two of the given *a* and *b*, *p* may be sought, so that there shall be

$$p=\frac{a\mathfrak{B}+b\mathfrak{A}}{1-naabb} \quad \text{and} \quad \mathfrak{P}=\frac{(mab+\mathfrak{A}\mathfrak{B})(1+naabb)+2nab(aa+bb)}{(1-naabb)^2}.$$

Then from this quantity *p,* with the third of the given *c* adjoining, a third *q* may be defined jointly*,* so that there shall be

$$q=\frac{p\mathfrak{C}+c\mathfrak{P}}{1-nccpp} \quad \text{and} \quad \mathfrak{Q}=\frac{(mcp+\mathfrak{C}\mathfrak{P})(1+nccpp)+2ncp(cc+pp)}{(1-nccpp)^2}.$$

In the third place, from this quantity q with the fourth of the given d adjoining*,* r is sought , so that there shall be :

$$r=\frac{q\mathfrak{D}+d\mathfrak{Q}}{1-nddqq} \quad \text{and} \quad \mathfrak{R}=\frac{(mdq+\mathfrak{D}\mathfrak{Q})(1+nddqq)+2ndq(dd+qq)}{(1-nddqq)^2}.$$

In the fourth place, from this quantity r with the fifth of the given adjoining e, s may be defined, so that there shall be :

$$s=\frac{r\mathfrak{E}+e\mathfrak{R}}{1-neerr} \quad \text{and} \quad \mathfrak{S}=\frac{(mer+\mathfrak{E}\mathfrak{R})(1+neerr)+2ner(ee+rr)}{(1-neerr)^2}.$$

And these operations may be continued, until all the quantities given in the computation will have been considered.

§.32. Moreover, from all these values found, the following required comparisons thus themselves will be obtained in order

$$\text{I. } \Pi:p=\Pi:a+\Pi:b+\beta abp,$$
$$\text{II. } \Pi:q=\Pi:a+\Pi:b+\Pi:c+\beta abp+\beta cpq,$$
$$\text{III. } \Pi:r=\Pi:a+\Pi:b+\Pi:c+\Pi:d+\beta abp+\beta cpq+\beta dqr,$$
$$\text{IV. } \Pi:s=\Pi:a+\Pi:b+\Pi:c+\Pi:d+\Pi:e$$
$$+\beta abp+\beta cpq+\beta dqr+\beta ers,$$
$$\text{V. } \Pi:t=\Pi:a+\Pi:b+\Pi:c+\Pi:d+\Pi:e+\Pi:f$$
$$+\beta abp+\beta cpq+\beta dqr+\beta ers+\beta fst$$

etc.

§.33. Therefore since this transcending formula

$$\Pi:z=\int\frac{\partial z(\alpha+\beta zz)}{\sqrt{(1+mzz+nz^4)}}$$

within itself may contain the arcs of all the conic sections taken from the vertex, of which with the aid of the formulas, however many arcs may be proposed in whatever conic section, which all shall be taken from the vertex, always a new arc will be able to be formed equally from the vertex in that same conic section, which may differ from the sum of these arcs given by an assignable algebraic quantity. Indeed likewise nothing impedes, where the lesser of some given arcs may be taken to be negative, since now we have advised $\Pi:(-z)=-\Pi:z$, thus so that our determination may also be able to be applied to the arc between any intercepted limits. And by being treated in this manner, since recently I have given such a comparison of such arcs, it will be able to be rendered much more general.

§.34. Besides, since in this case, where we have assumed $Z=\alpha+\beta zz$, the symbol used above $\Phi:xy$ will be changed into βxyz, while clearly from the two quantities x and y following the given precepts, the third z may be determined, thus also, whatever other function may be used in place of Z, since we have put :

$$\Phi:xy=-a\int\frac{(\mathrm{X}\text{-}\mathrm{Y})du}{xx-yy}$$

with $u=xy$ being present, thence from the integration the resulting absolute function will contain only the quantity u with the letters a and $\mathfrak{A}$, since the letter t will be expressed thus :

$$t=aa-2\mathfrak{A}u+naauu,$$

since everywhere in the integral in place of u there may be written xy, but truly in place of a and $\mathfrak{A}$ the letters z and $\mathfrak{Z}$; and in this way the value of the symbol $\Phi:xy$ will be obtained for any proposed case, which function, unless it may have been algebraic, will be able to be shown always by logarithms and circular arcs, if indeed, just as we have assumed, the letter Z were an even rational function of z.

PLENIOR EXPLICATIO CIRCA COMPARATIONEM QUANTITATUM IN FORMULA INTEGRALI $\int \frac{Z\partial z}{\sqrt{(1+mzz+nz^4)}}$ CONTENTARUM DENOTANTE *Z* FUNCTIONEM QUAMCUNQUE RATIONALEM IPSIUS *zz*

[E581] Acta academiae scientiarum Petropolitanae 1781: II (1785), p. 3-22.

§.1. Etsi hoc argumentum iam saepius tractavi atque Illustrissimus LA GRANGE plures egregias observationes super huiusmodi formulis cum publico communicavit, id tamen neutiquam adhuc satis exploratum, multo minus exhaustum est censendum, sed plurima adhuc maxime abscondita involvere videtur, quae profundissimam indagationem requirunt atque insignia incrementa Analyseos pollicentur. Imprimis autem ipsae operationes analyticae, quae me primum ad hanc investigationem perduxerunt, ita sunt comparatae, ut non nisi per plures ambages totum negotium conficiant, unde merito etiamnunc methodus directa ad easdem comparationes perducens maxime est desideranda. Praeterea vero universa haec investigatio multo latius patet quam ad eas formulas integrales, quas primo sum contemplatus, ubi pro littera *Z* tantum vel quantitatem constantem vel functionem integram ipsius *zz* huius formae $\mathrm{F}+\mathrm{G}zz+\mathrm{H}z^4+\mathrm{I}z^6+\mathrm{K}z^8+\text{etc.}$ assumsi, quibus casibus ostendi propositis duabus quibuscunque quantitatibus huius generis semper tertiam eiusdem generis inveniri posse, quae a summa illarum discrepet quantitate algebraica, quae quidem evanescat casu, quo Z est tantum quantitas constans.

§. 2. Nunc autem observavi easdem comparationes institui posse, si pro Z accipiatur functio quaecunque rationalis ipsius *zz,* quae scilicet habeat huiusmodi formam

$$\frac{\mathrm{F}+\mathrm{G}zz+\mathrm{H}z^4+\mathrm{I}z^6+\mathrm{K}z^8+\text{etc.}}{\mathfrak{F}+\mathfrak{G}zz+\mathfrak{H}z^4+\mathfrak{I}z^6+\mathfrak{K}z^8+\text{etc.}},$$

ubi quidem hoc discrimen occurrit, quod differentia inter summam duarum huiusmodi formularum et tertiam formulam eiusdem generis inveniendam non amplius sit quantitas algebraica, veruntamen per logarithmos et arcus circulares semper exhiberi possit, ita ut nunc ista investigatio multo latius pateat, quam eam adhuc eram complexus. Atque hinc fortasse, si omnes operationes, quae ad hunc scopum manuducunt, debita attentione perpendantur, faciliorem viam aperire poterunt ad methodum directam perveniendito tumque hoc argumentum maxime abstrusum feliciori successu perscrutandi.

§.3. Quo autem haec omnia clarius perspici queant, denotet iste character $\Pi : z$ eam quantitatem transcendentem, quae ex integratione formulae propositae

$$\int \frac{Z\partial z}{\sqrt{\left(1+mzz+nz^4\right)}}$$

nascitur, dum integrale ita capi assumitur, ut evanescat posito z 0; unde statim manifestum est fore quoque $\Pi:0=0$. Deinde cum Z involvat tantum pares potestates ipsius z, cuiusmodi etiam in formula radicali insunt,evidens est, si loco z scribatur $-z$, tum valorem quoque istius formulae integralis ideoque etiam characteris $\Pi:z$ in sui negativum abire, ita ut sit $\Pi:(-z)=-\Pi:z$. His praenotatis si proponantur duae quaecunque huiusmodi quantitates $\Pi:p$ et $\Pi:q$, semper invenire licet tertiam quantitate huiusdem generis $\Pi:r$, quae a summa illarum formularum $\Pi:p+\Pi:q$ differat quantitate vel algebraica vel saltem per logarithmos et arcus circulares assignabili. Regula vero, qua ex datis litteris p et q tertia r elicitur, semper manet eadem, quaecunque functio per litteram Z designetur; semper enim erit

$$r=\frac{p\sqrt{\left(1+mqq+nq^4\right)}+q\sqrt{\left(1+mpp+np^4\right)}}{1-nppqq}.$$

Hinc autem pro sequentibus combinationibus observasse iuvabit fore

$$\sqrt{\left(1+mrr+nr^4\right)}=\frac{\left(mpq+\sqrt{\left(1+mpp+np^4\right)}\right)\left(\sqrt{\left(1+mqq+nq^4\right)}\right)(1+nppqq)+2npq(pp+qq)}{(1-nppqq)^2}.$$

§.4. Non solum autem haec investigatio adstringitur ad huiusmodi formulas $\Pi:p$ et $\Pi:q$ pro arbitrio accipiendas, sed adeo ad quotcunque formulas datas potest extendi, ita ut, quotcunque huiusmodi formulae fuerint propositae, scilicet

$$\Pi:f+\Pi:g+\Pi:h+\Pi:i+\Pi:k+\text{etc.},$$

semper nova huiusmodi formula $\Pi:r$ assignari possit, quae ab illarum summa discrepet quantitate vel algebraica vel saltem per logarithmos et arcus circulares assignabili. Quin etiam formulas illas, quas tanquam datasspectavimus, ita definire licebit, ut discrimen illud sive algebraicum sive a logarithmis arcubusque circularibus pendens prorsus evanescat, ita ut futurum sit

$$\Pi:r=\Pi:f+\Pi:g+\Pi:h+\Pi:i+\Pi:k+\text{etc.},$$

Atque haec fere sunt, ad quae hanc investigationem generaliorem, quam hic exponere constitui, mihi quidem extendere licuit; quamobrem singulas operationes, quae me huc perduxerunt, succincte sum propositurus.

OPERATIO 1

§.5. Universam hanc investigationem inchoavi a consideratione huius aequationis algebraicae

$$\alpha+\gamma\left(xx+yy\right)+2\delta xy+\zeta xxyy=0,$$

ex qua, cum sit quadratica, tam pro *x* quam pro *y* radicem extrahendo colligitur vel

$$y=\frac{-\delta x+\sqrt{\left(-\alpha\gamma+(\delta\delta-\gamma\gamma-\alpha\zeta)xx-\gamma\zeta x^4\right)}}{\gamma+\zeta xx}$$

vel

$$x=\frac{-\delta y+\sqrt{\left(-\alpha\gamma+(\delta\delta-\gamma\gamma-\alpha\zeta)yy-\gamma\zeta y^4\right)}}{\gamma+\zeta yy},$$

ita ut hinc fiat

$$\sqrt{\left(-\alpha\gamma+\left(\delta\delta-\gamma\gamma-\alpha\zeta\right)xx-\gamma\zeta x^4\right)}=\gamma y+\delta x+\zeta xxy$$

et

$$\sqrt{\left(-\alpha\gamma+\left(\delta\delta-\gamma\gamma-\alpha\zeta\right)yy-\gamma\zeta y^4\right)}=\gamma x+\delta y+\zeta xyy.$$

§.6. Nunc litteras $\alpha,\ \gamma,\ \delta,\ \zeta$ ita definio, ut ambae formulae radicales ad formam

$$\sqrt{\left(1+mxx+nx^4\right)} \text{ et } \sqrt{\left(1+myy+ny^4\right)}$$

reducantur, quem in finem facio

$$1.-\alpha\gamma=k,\ \ 2.\ \delta\delta-\gamma\gamma-\alpha\zeta=mk \ \text{ et } \ 3.-\gamma\zeta=nk;$$

ex quarum aequalitatum prima fit $\alpha=-\frac{k}{\gamma}$, ex tertia $\zeta=\frac{-nk}{r}$, qui valores in secunda substituti praebent

$$\delta\delta=\gamma\gamma+\frac{nkk}{rr}+mk$$

ideoque

$$\delta=\sqrt{\left(\gamma\gamma+\frac{nkk}{rr}+mk\right)}=\frac{1}{\gamma}\sqrt{\left(\gamma^4+m\gamma\gamma k+nkk\right)};$$

unde aequatio nostra nunc erit

$$-k+\gamma\gamma\left(xx+yy\right)+2xy\sqrt{\left(\gamma^4+m\gamma\gamma k+nkk\right)}-\ nkxxyy=0;$$

hinc igitur ambae nostrae formulae irrationales erunt

$$\sqrt{k\left(1+mxx+nx^4\right)}=\gamma y+\tfrac{1}{\gamma}x\sqrt{\left(\gamma^4+m\gamma\gamma k+nkk\right)}-\tfrac{nk}{\gamma}xxy,$$
$$\sqrt{k\left(1+myy+ny^4\right)}=\gamma x+\tfrac{1}{\gamma}y\sqrt{\left(\gamma^4+m\gamma\gamma k+nkk\right)}-\tfrac{nk}{\gamma}xyy.$$

§.7. Cum nunc ambae quantitates x et y ita a se invicem pendeant, quemadmodum aequatio assumta declarat, litteras adhuc indefinitas r et k ita definiamus, ut posito $x=0$ fiat $y=a$. Oportebit igitur esse $-k+\gamma\gamma aa=0$ ideoque $k=\gamma\gamma aa$, quo valore substituto aequatio nostra erit

$$0=\gamma\gamma\left(xx+yy-aa\right)+2\gamma\gamma xy\sqrt{\left(1+maa+na^4\right)}-n\gamma\gamma aaxxyy,$$

hincque fiet per $\gamma\gamma$ dividendo

$$0=\left(xx+yy-aa\right)+2xy\sqrt{\left(1+maa+na^4\right)}-naaxxyy.$$

Tum vero ambae nostrae formulae radicales ita exprimentur

$$\sqrt{\left(1+mxx+nx^4\right)}=\tfrac{y}{a}+\tfrac{x}{a}\sqrt{\left(1+maa+na^4\right)}-naxxy,$$
$$\sqrt{\left(1+myy+ny^4\right)}=\tfrac{x}{a}+\tfrac{y}{a}\sqrt{\left(1+maa+na^4\right)}-naxyy.$$

§.8. Quo has formulas tractatu faciliores reddamus, ponamus

$$\sqrt{\left(1+maa+na^4\right)}=\mathfrak{A}$$

similique modo

$$\sqrt{\left(1+mxx+nx^4\right)}=\mathfrak{X}\ \text{ et }\ \sqrt{\left(1+myy+ny^4\right)}=\mathfrak{Y}$$

et aequatio nostra erit

$$xx+yy-aa+2\mathfrak{A}xy-naaxxyy=0,$$

unde reperitur

$$y=-\tfrac{\mathfrak{A}x-a\mathfrak{X}}{1-naaxx}\ \text{ et } x=-\tfrac{\mathfrak{A}y-a\mathfrak{Y}}{1-naayy};$$

unde patet, si fuerit $y = 0$, fore $x = a$; tum vero erunt formulae radicales

$$\sqrt{(1+mxx+nx^4)} = \mathfrak{X} = \tfrac{y}{a} + \tfrac{\mathfrak{A}x}{a} - naxxy,$$
$$\sqrt{(1+myy+ny^4)} = \mathfrak{Y} = \tfrac{y}{a} + \tfrac{\mathfrak{A}y}{a} - naxyy.$$

§.9. Quemadmodum autem tam *y* per *x* quam *x* per *y* exprimere licuit, ita etiam $\mathfrak{Y}$ per solum *x* et $\mathfrak{X}$ per solum *y* exprimere licebit. Calculo autem instituto reperietur fore

$$\mathfrak{X} = \frac{(-may+\mathfrak{A}\mathfrak{Y})(1+naayy)-2nay(aa+yy)}{(1-naayy)^2},$$
$$\mathfrak{Y} = \frac{(-max+\mathfrak{A}\mathfrak{X})(1+naaxx)-2nax(aa+xx)}{(1-naaxx)^2}.$$

§.10. Praecipue autem circa nostram aequationem

$$xx + yy - aa + 2\mathfrak{A}xy - naaxxyy = 0$$

notari meretur, quod ternae quantitates *xx, yy, aa* perfecte inter se sint permutabiles. Si enim membrum irrationale ad alteram partem transferatur, ut sit

$$xx + yy - aa - naaxxyy = -2\mathfrak{A}xy\,,$$

et quadrata sumantur, restituendo pro $\mathfrak{A}^2$ valorem suum $1 + maa + na^4$ prodibit ista aequatio

$$\left.\begin{array}{llll} +x^4 - 2xxyy - 4maaxxyy & - 2na^4xxyy + nna^4x^4\ y^4 \\ +y^4 - 2aaxx & - 2naax^4yy \\ +a^4 - 2aayy & - 2naaxxy^4 \end{array}\right\} = 0,$$

ubi permutabilitas litterarum *a, x, y* manifesto in oculos incurrit. In ipsis quidem formulis superioribus, ubi ipsa quantitas *a* ingreditur, permutabilitas non adeo est manifesta, sed prorsus elucebit, si loco *a* scribamus $-b$ itemque $\mathfrak{B}$ loco $\mathfrak{A}$; tum enim, quemadmodum erat

$$y = -\tfrac{x\mathfrak{B}+b\mathfrak{X}}{1-nbbxx} \text{ et } x = -\tfrac{y\mathfrak{B}+b\mathfrak{Y}}{1-nbbyy},$$

ita erit

$$b = -\tfrac{x\mathfrak{Y}+y\mathfrak{X}}{1-nxxyy}$$

similique modo pro formulis radicalibus seu litteris maiusculis erit

$$\mathfrak{Y} = \frac{(mbx+\mathfrak{B}\mathfrak{X})(1+nbbxx)+2nbx(aa+xx)}{(1-nbbxx)^2},$$
$$\mathfrak{X} = \frac{(mby+\mathfrak{B}\mathfrak{Y})(1+nbbyy)+2nby(bb+yy)}{(1-nbbyy)^2},$$
$$\mathfrak{B} = \frac{(mxy+\mathfrak{X}\mathfrak{Y})(1+nxxyy)+2nxy(xx+yy)}{(1-nxxyy)^2},$$

sicque perfecta permutabilitas perspicitur.

OPERATIO 2

§.11. Differentiemus nunc nostram aequationem algebraicam assumtam, quae est

$$xx+yy-aa+2\mathfrak{A}xy-naaxxyy=0,$$

et aequatio differentialis erit

$$\partial x\left(x+\mathfrak{A}y-naaxyy\right)+\partial y\left(y+\mathfrak{A}x-naaxxy\right)=0$$

sive

$$\frac{\partial x}{y+\mathfrak{A}x-naaxxy}+\frac{\partial y}{x+\mathfrak{A}y-naaxyy}=0.$$

Ex superioribus autem constat esse

$$y+\mathfrak{A}x-naaxxy=a\mathfrak{X} \;\text{ et }\; x+\mathfrak{A}y-naaxyy=a\mathfrak{Y},$$

unde aequatio differentialis hanc induet formam

$$\frac{\partial x}{a\mathfrak{X}}+\frac{\partial y}{a\mathfrak{Y}}=0$$

sive

$$\frac{\partial x}{\sqrt{(1+mxx+nx^4)}}+\frac{\partial y}{\sqrt{(1+myy+ny^4)}}=0.$$

§.12. Inventa igitur hac aequatione differentiali denotet iste character $\Gamma:x$ integrale $\int\frac{\partial x}{\mathfrak{X}}$ et character $\Gamma:y$ integrale $\int\frac{\partial y}{\mathfrak{Y}}$ utroque integrali ita sumto, ut evanescat posito vel $x=0$ vel $y=0$, atque aequationem illam differentialem integrando fiet $\Gamma:x+\Gamma:y=\mathrm{C}$. Cum autem sumto $x=0$ fiat etiam $\Gamma:x=0$ et $y=a$, erit constans illa $\mathrm{C}=\Gamma:a$, ita ut habeamus hanc aequationem $\Gamma:x+\Gamma:y=\Gamma:a$.

§.13. Quoniam hic nulla amplius variabilitatis ratio tenetur, patet sumtis binis litteris *x* et *y* pro lubitu litteram *a* ita semper definiri posse, ut fiat

$$\Gamma : a = \Gamma : x + \Gamma : y.$$

Si enim in § 10 loco *b* scribatur $-a$, sumi debet

$$a = \frac{x\mathfrak{Y} + y\mathfrak{X}}{1-nxxyy},$$

quae comparatio iam casum constituit specialem investigationis generalis, quam suscepimus. Si enim loco *x* et *y* scribamus *p* et *q*, at *r* loco *a*, tum vero $\mathfrak{P}$, $\mathfrak{Q}$ et $\mathfrak{R}$ loco $\mathfrak{X}$, $\mathfrak{Y}$ et $\mathfrak{A}$ (atque si sumtis pro lubitu quantitatibus *p, q* capiatur $r = \frac{p\mathfrak{Q}+q\mathfrak{P}}{1-nppqq}$, tum utique erit $\Gamma : r = \Gamma : p + \Gamma : q$, ita ut hoc casu discrimen illud inter $\Gamma : r$ et summam $\Gamma : p + \Gamma : q$ plane evanescat. Sicque iam evolvimus casum, quo in nostra forma generali

$$\int \frac{Z\partial z}{\sqrt{(1+mzz+nz^4)}}$$

pro Z sumitur quantitas constans.

OPERATIO 3

§.14. Quo nunc propius ad nostrum institutum accedamus, sint X et *Y* tales functiones ipsarum *x* et *y,* qualem volumus esse Z ipsius *z,* et quoniam modo invenimus

$$\frac{\partial x}{\sqrt{(1+mxx+nx^4)}} + \frac{\partial y}{\sqrt{(1+myy+ny^4)}} = 0,$$

ponamus esse

$$\frac{\mathrm{X}\partial x}{\sqrt{(1+mxx+nx^4)}} + \frac{\mathrm{Y}\partial y}{\sqrt{(1+myy+ny^4)}} = \partial \mathrm{V},$$

ita ut, si *X* et *Y* essent quantitates constantes, foret $\partial \mathrm{V} = 0$. Hinc ergo, si loco $\frac{\partial y}{\sqrt{(1+myy+ny^4)}}$ scribamus $\frac{-\partial x}{\sqrt{(1+mxx+nx^4)}}$, fiet

$$\partial \mathrm{V} = \frac{(\mathrm{X}-\mathrm{Y})\partial x}{\sqrt{(1+mxx+nx^4)}} \text{ vel etiam } \frac{(\mathrm{Y}-\mathrm{X})\partial y}{\sqrt{(1+myy+ny^4)}}.$$

At si loco radicalium suos valores rationales scribamus, erit

$$\partial \mathrm{V} = \frac{a(\mathrm{X}-\mathrm{Y})\partial x}{y+\mathfrak{A}x-naaxxy} \text{ vel } \partial \mathrm{V} = \frac{a(\mathrm{Y}-\mathrm{X})\partial y}{x+\mathfrak{A}y-naaxyy}.$$

§.15. Cum autem nulla sit ratio, cur istud differentiale ∂V potius per ∂x quam per ∂y exprimamus, consultum erit novam quantitatem in calculum introducere, quae aeque referatur ad *x* et ad *y.* Hunc in finem faciamus productum $xy = u$ ac statuamus

$$\frac{\partial x}{y+\mathfrak{A}x-naaxxy}=-\frac{\partial y}{x+\mathfrak{A}y-naaxyy}=s\partial u.$$

Hinc igitur erit

$$\partial x=s\partial u\left(y+\mathfrak{A}x-naaxxy\right) \text{ et } \partial y=-s\partial u\left(x+\mathfrak{A}y-naaxyy\right),$$

ex quibus colligimus

$$ydx+xdy=s\partial u\left(yy-\ xx\right)=\partial u\,,$$

sicque habebimus $s=\frac{1}{yy-xx}$, ita ut habeamus

$$\frac{\partial x}{y+\mathfrak{A}x-naaxxy}=-\frac{\partial y}{x+\mathfrak{A}y-naaxyy}=\frac{\partial u}{yy-xx}.$$

Hoc igitur valore substituto nanciscimur

$$\partial \mathrm{V}=\frac{a(\mathrm{X}-\mathrm{Y})\partial u}{yy-xx}=\frac{-a\partial u(\mathrm{X}-\mathrm{Y})}{xx-yy}.$$

§.16. Cum autem X et Y sint functiones rationales pares ipsarum *x* et *y,* in quibus tantum insunt potestates pares harum litterarum, facile intelligitur formulam $\mathrm{X}-\mathrm{Y}$ semper esse divisibilem per $xx-yy$ et quotum praeter productum $xy=u$ insuper involvere summam quadratorum $xx+yy$; quamobrem statuamus $xx+yy=t$, et cum aequatio nostra fundamentalis fiat

$$t-aa+2\mathfrak{A}u-naauu=0,$$

ex ea fit

$$t=aa-2\mathfrak{A}u+naauu\,,$$

ita ut *t* aequetur functioni rationali ipsius *u.* Quod si ergo hic valor ubique loco *t* scribatur, differentiale nostrum quaesitum $\partial\mathrm{V}$ per solam variabilem *u* exprimetur, ita ut posito $\partial\mathrm{V}=\mathrm{U}\partial u$ semper sit U functio rationalis ipsius *u*; quae ergo si fuerit integra, tum V aequabitur functioni algebraicae ipsius *u,* sin autem sit functio fracta, tum integrale $\mathrm{V}=\int \mathrm{U}\partial u$ semper per logarithmos et arcus circulares exhiberi poterit. Hoc ergo integrale si ita capiatur,ut evanescat posito $u=xy=0$, id etiam evanescet posito $x=0$ vel $y=0$. Atque hinc integrando impetrabimus

$$\int\frac{\mathrm{X}\partial x}{\sqrt{\left(1+mxx+nx^4\right)}}+\int\frac{\mathrm{Y}\partial y}{\sqrt{\left(1+myy+ny^4\right)}}=\mathrm{C}+\mathrm{V}=\mathrm{C}+\int\mathrm{U}\partial u.$$

§.17. Quod si igitur characteres $\Pi:x$ et $\Pi:y$ denotent valores horum integralium, ita ut utrumque evanescat sumto vel $x=0$ vel $y=0$, quoniam facto $x=0$ per hypothesin fit $y=a$, manifestum est constantem hanc fore $\Pi:a$ sicque aequatio finita resultabit ista

$$\Pi:x+\Pi:y=\Pi:a+\int\mathrm{U}\partial u\,.$$

§.18. Accuratius autem in valores huius fractionis U pro quovis casu inquiramus. Ac primo quidem, si sumatur

$$Z = \alpha + \beta zz + \gamma z^4 + \delta z^6 + \text{etc.},$$

erit simili modo

$$X = \alpha + \beta xx + \gamma x^4 + \delta x^6 + \text{et } Y = \alpha + \beta yy + \gamma y^4 + \delta y^6 + \text{etc.};$$

quare cum invenerimus

$$\partial V = U\partial u = -\frac{a\partial u(X\text{-}Y)}{xx-yy},$$

erit

$$U = -\frac{a(X\text{-}Y)}{xx-yy} \text{ ideoque } U = -\frac{a\left(\beta(xx-yy)+\gamma\left(x^4-y^4\right)+\delta\left(x^6-y^6\right)\right)}{xx-yy},$$

unde fit

$$U = -a\beta - a\gamma\left(xx+yy\right) - a\delta\left(x^4 + xxyy + y^4\right).$$

Cum igitur sit $xx + yy = t$ et $xy = u$, erit

$$u = -a\beta - a\gamma t - a\delta\left(tt - uu\right);$$

unde, cum sit $t = aa - 2\mathfrak{A}u + naauu$, calculo subducto fiet

$$\int U\partial u = -a\beta u - a\gamma\left(aau - \mathfrak{A}uu + \tfrac{1}{3}naau^3\right)$$
$$-a\delta\left(a^4u - 2aa\mathfrak{A}uu + \tfrac{2}{3}na^4u^3 + \tfrac{4}{3}\mathfrak{A}^2u^3 - \tfrac{1}{3}u^3 - n\mathfrak{A}a^2u^4 + \tfrac{1}{5}n^2a^4u^5\right).$$

Atque hinc intelligitur, si functio Z ad altiores potestates exsurgat, quomodo valor integralis ipsius $\int U\partial u$ inde inveniri queat.

§.19. Sin autem *Z* fuerit functio fracta, scilicet

$$Z = \frac{\alpha+\beta zz+\gamma z^4}{\zeta+\eta zz+\theta z^4}$$

hincque

$$X = \frac{\alpha+\beta xx+\gamma x^4}{\zeta+\eta xx+\theta x^4} \text{ et } Y = \frac{\alpha+\beta yy+\gamma y^4}{\zeta+\eta yy+\theta y^4},$$

erit

$$X - Y = \frac{(\beta\zeta-\alpha\eta)(xx-yy)+(\gamma\zeta-\alpha\theta)\left(x^4-y^4\right)+(\gamma\eta-\beta\theta)x^2y^2\left(x^2-y^2\right)}{\zeta\zeta+\zeta\eta(xx+yy)+\zeta\theta\left(x^4+y^4\right)+\eta^2x^2y^2+\eta\theta x^2y^2(xx+yy)+\theta\theta x^4y^4}.$$

Hinc igitur introductis litteris *t* et *u* erit

$$\frac{\mathrm{X}-\mathrm{Y}}{xx-yy}=\frac{\beta\zeta-\alpha\eta+(\gamma\zeta-\alpha\theta)t+(\gamma\eta-\beta\theta)uu}{\zeta\zeta+\zeta\eta t+\zeta\theta(tt-2uu)+\eta\eta uu+\eta\theta tuu+\theta\theta u^4};$$

quamobrem, cum sit

$$\mathrm{U}=-\frac{a(\mathrm{X}\text{-}\mathrm{Y})}{xx-yy},$$

ob $t=aa-2\mathfrak{A}u+naauu$ manifestum est integrale formulae $\int \mathrm{U}\partial u$, nisi fuerit algebraicum, semper concessis logarithmis et arcubus circularibus exhiberi posse. Sicque per has tres operationes omnia praestitimus, quibus opus est ad omnia problemata huc spectantia solvenda.

PROBLEMA 1

§.20. *Si* $\Pi:x$ et $\Pi:y$ *denotent valores formularum integralium,*

$$\int\frac{\mathrm{X}\partial x}{\sqrt{\left(1+mxx+nx^4\right)}} \text{ et } \int\frac{\mathrm{Y}\partial y}{\sqrt{\left(1+myy+ny^4\right)}}$$

ubi X *et* Y *sint functiones pares similes ipsarum x et y, atque dentur binae huiusmodi formulae* $\Pi:x$ et $\Pi:y$, *invenire tertiam formulam eiusdem generis* $\Pi:z$, *ut sit* $\Pi:z=\Pi:x+\Pi:y+W$ *, ita ut W sit functio vel algebraica vel per logarithmos et arcus circulares assignabilis.*

SOLUTIO

Cum dentur binae quantitates *x* et *y,* ex iis formentur radicales

$$\mathfrak{X}=\sqrt{\left(1+mxx+nx^4\right)} \text{ et } \mathfrak{Y}=\sqrt{\left(1+myy+ny^4\right)},$$

ex quibus definiatur quantitas *z,* eodem modo quo supra litteram *a* per *x* et *y* definire docuimus, ita ut sit

$$z=\frac{x\mathfrak{Y}+y\mathfrak{X}}{1-nxxyy}$$

eiusque valor irrationalis

$$\mathfrak{Z}=\sqrt{\left(1+mzz+nz^4\right)}=\frac{(mxy+\mathfrak{X}\mathfrak{Y})(1+nxxyy)+2nxy(xx+yy)}{(1-nxxyy)^2};$$

tum in superioribus formulis ubique loco *a* et $\mathfrak{A}$ scribamus *z* et *Z* et capiatur $\mathrm{U}=-\frac{z(\mathrm{X}\text{-}\mathrm{Y})}{xx-yy}$, quantitatem vidimus semper reduci posse ad functionem ipsius *u* existente $u=xy$, ac ponatur $\mathrm{V}=\int \mathrm{U}\partial u$, in qua integratione quantitates *z* et $\mathfrak{Z}$ pro constantibus sunt habendae, ita ut littera V spectari possit tanquam functio ipsius $u = xy$, quandoquidem etiam z et $\mathfrak{Z}$ per x et y determinantur. Probe autem teneatur in ista formula integrali solam quantitatem u ut variabilem esse tractandam. Hac igitur quantitate V inventa erit

$$\Pi : x + \Pi : y = \Pi : z + \mathrm{V};$$

unde, cum debeat esse

$$\Pi : z = \Pi : x + \Pi : y + \mathrm{W},$$

patet esse $\mathrm{W} = -\mathrm{V}$ ideoque quantitatem vel algebraicam vel per logarithmos et arcus circulares assignabilem.

COROLLARIUM 1

§.21. Totum ergo negotium hic redit ad integrationem formulae $\mathrm{U}\partial u$ existente $u = xy$ et $\mathrm{U} = -\frac{z(\mathrm{X}-\mathrm{Y})}{xx-yy}$, quam supra vidimus semper per u exprimi posse, siquidem in hac integratione litterae z et $\mathfrak{Z}$ ut quantitates constantes tractentur.

COROLLARIUM 2

§.22. Cum igitur pro data indole binarum functionum X et Y haec integratio nulla laboret difficultate ipsumque integrale per *u,* hoc est per *xy* exprimatur, cuius valorem ex datis quantitatibus x et y semper exhibere liceat, loco quantitatis V scribemus in posterum characterem $\Phi : xy$, unde pro quibusque aliis litteris loco x et y assumtis intelligitur significatus characterum $\Phi : pq$, $\Phi : ab$ etc. etc.

COROLLARIUM 3

§.23. Hoc igitur charactere recepto si pro datis quantitatibus x et y capiamus $z = \frac{x\mathfrak{Y} + y\mathfrak{X}}{1-nxxyy}$, unde fit

$$\mathfrak{Z} = \frac{(mxy+\mathfrak{X}\mathfrak{Y})(1+nxxyy)+2nxy(xx+yy)}{(1-nxxyy)^2},$$

erit

$$\Pi : z = \Pi : x + \Pi : y - \Phi : xy.$$

PROBLEMA 2

§.24. *Servatis omnibus characteribus, quos hactenus explicavimus, si dentur ternae formulae* $\Pi : p$, $\Pi : q$, $\Pi : r$ *, invenire quartam eiusdem generis* $\Pi : z$*, ut fiat,*

$$\Pi : z = \Pi : p + \Pi : q + \Pi : r + W,$$

ita ut W sit quantitas algebraica vel per logarithmos arcusve circulares assignabilis.

SOLUTIO

Ex datis binis quantitatibus p et q ideoque etiam $\mathfrak{P}$ et $\mathfrak{Q}$ inde oriundis capiatur

$$x = \frac{p\mathfrak{Q}+q\mathfrak{P}}{1-nppqq}$$

simulque

$$\mathfrak{X} = \frac{(mpq+\mathfrak{P}\mathfrak{Q})(1+nppqq)+2npq(pp+qq)}{(1-nppqq)^2}.$$

Tum vero etiam colligatur valor characteris $\Phi : pq$ eritque per praecedentia

$$\Pi : x = \Pi : p + \Pi : q - \Phi : pq\,,$$

sive

$$\Pi : p + \Pi : q = \Pi : x + \Phi : pq,$$

quo valore substituto erit

$$\Pi : z = \Pi : x + \Pi : r + \Phi : pq + \mathrm{W}.$$

Ex praecedente autem problemate, si loco y hic scribamus r et capiamus

$$z = \frac{x\mathfrak{R}+r\mathfrak{X}}{1-nrrxx}\,,$$

unde fit

$$\mathfrak{Z} = \frac{(mrx+\mathfrak{R}\mathfrak{X})(1+nrrxx)+2nrx(rr+xx)}{(1-nrrxx)^2},$$

erit

$$\Pi : z = \Pi : x + \Pi : r - \Phi : rx\,,$$

qua forma cum praecedente collata colligitur

$$\mathrm{W} = -\Phi : pq - \Phi : rx\,,$$

ita ut sit

$$\Pi : z = \Pi : p + \Pi : q + \Pi : r - \Phi : pq - \Phi : rx\,.$$

PROBLEMA 3

§.25. *Propositis huiusmodi formulis* $\Pi : p,\ \Pi : q,\ \Pi : r,\ \Pi : s$ *invenire quintam eiusdem generis* $\Pi : z$*, ut fiat*

$$\Pi :\ z = \Pi : p + \Pi : q + \Pi : r + \Pi : s + W\,,$$

ita ut W sit quantitas algebraica vel per logarithmos arcusve circulares assignabilis.

SOLUTIO

Ex datis binis *p* et *q* quaeratur *x,* ut sit

$$x = \frac{p\mathfrak{Q}+q\mathfrak{P}}{1-nppqq}$$

item

$$\mathfrak{X} = \frac{(mpq+\mathfrak{P}\mathfrak{Q})(1+nppqq)+2npq(pp+qq)}{(1-nppqq)^2},$$

eritque

$$\Pi : x = \Pi : p + \Pi : q - \Phi : pq.$$

Simili modo ex binis datis *r* et *s* quaeratur *y,* ut sit

$$y = \frac{r\mathfrak{S}+s\mathfrak{R}}{1-nrrss}$$

eritque

$$\mathfrak{Y} = \frac{(mrs+\mathfrak{R}\mathfrak{S})(1+nrrss)+2nrs(rr+ss)}{(1-nrrss)^2},$$

tum vero

$$\Pi : y = \Pi : r + \Pi : s - \Phi : rs .$$

Nunc denique ex inventis *x* et *y* sumatur

$$z = \frac{x\mathfrak{Y}+y\mathfrak{X}}{1-nxxyy} \quad \text{and} \quad \mathfrak{Z} = \frac{(mxy+\mathfrak{X}\mathfrak{Y})(1+nxxyy)+2nxy(xx+yy)}{(1-nxxyy)^2},$$

eritque

$$\Pi : z = \Pi : x + \Pi : y - \Phi : xy.$$

Quodsi ergo loco $\Pi : x$ et $\Pi : y$ valores modo inventi substituantur, fiet

$$\Pi : z = \Pi : p + \Pi : q + \Pi : r + \Pi : s - \Phi : pq - \Phi : rs - \Phi : xy .$$

COROLLARIUM I

§.26. Hinc iam abunde intelligitur, si proponantur quotcunque huiusmodi formulae, quemadmodum novam eiusdem generis $\Pi : z$ investigari oporteat, quae ab illis iunctim sumtis discrepet quantitate algebraica vel per logarithmos arcusve circulares assignabili.

COROLLARIUM 2

§.27. Quod si omnes illae formulae fuerint inter se aequales earumque numerus $=\lambda$, semper nova formula $\Pi : z$ inveniri poterit, ut sit

$$\Pi : z = \lambda \Pi : p + W$$

existente W quantitate vel algebraica vel per logarithmos arcusve circulares assignabili. Quin etiam duabus huiusmodi formulis $\Pi : p$ et $\Pi : q$ propositis inveniri poterit $\Pi : z$, ut sit

$$\Pi : z = \lambda \Pi : p + \mu \Pi : q + \mathrm{W}.$$

SCHOLION

§.28. Hoc igitur modo non solum principia et fundamenta, quibus hoc argumentum innititur, succincte ac dilucide mihi quidem exposuisse videor, sed hoc argumentum etiam multo latius amplificavi, quam adhuc est factum. Semper autem maxime est optandum, ut via magis directa detegatur, quae ad easdem investigationes perducat. Nemo enim certe dubitabit, quin hinc maxima in universam Analysin incrementa essent redundatura.

APPLICATIO AD QUANTITATES TRANSCENDENTES IN FORMA $\int \frac{\partial z(\alpha+\beta zz)}{\sqrt{\left(1+mzz+nz^4\right)}} = \Pi : z$ CONTENTAS

§.29. Cum igitur hic sit $Z = \alpha + \beta zz$, propositis duabus formulis huius generis $\Pi : x$ et $\Pi : y$ sumtoque

$$z = \frac{x\mathfrak{Y}+y\mathfrak{X}}{1-nxxyy} \quad \text{hincque} \quad \mathfrak{Z} = \frac{(mxy+\mathfrak{X}\mathfrak{Y})(1+nxxyy)+2nxy\left(x^2+y^2\right)}{(1-nxxyy)^2}$$

ex § 18, ubi $u = xy$ et $a = z$, erit

$$\Pi : z = \Pi : x + \Pi : y + \beta xyz,$$

ita ut character ante adhibitus $\Phi : xy$ hoc casu accipiat valorem βxyz. Huius igitur regulae ope propositis duabus huiusmodi formulis $\Pi : x$ et $\Pi : y$ tertia $\Pi : z$ semper reperiri potest, quae a summa illarum differat quantitate algebraica βxyz.

§.30. Ponamus igitur quotcunque huiusmodi formulas transcendentes proponi

$$\Pi : a,\ \Pi : b,\ \Pi : c,\ \Pi : d,\ \Pi : e,\ \Pi : f,\ \Pi : g \text{ etc.}$$

et ex singulis quantitatibus *a, b, c, d* etc. colligi valores irrationales litteris germanicis insignitas

$$\mathfrak{A}=\sqrt{\left(1+maa+na^4\right)},\quad \mathfrak{B}=\sqrt{\left(1+mbb+nb^4\right)},$$
$$\mathfrak{C}=\sqrt{\left(1+mcc+nc^4\right)},\quad \mathfrak{D}=\sqrt{\left(1+mdd+nd^4\right)},$$
$$\text{etc.}\qquad\qquad \text{etc.}$$

semper nova formula eiusdem generis exhiberi poterit, quae a summa earum discrepet quantitate algebraica, quantuscunque etiam fuerit earum formularum datarum numerus. Operationes autem ad hunc finem perducentes commodissime sequenti modo instituentur.

§.31. Primo scilicet ex binis datarum *a* et *b* quaeratur *p,* ut sit

$$p=\frac{a\mathfrak{B}+b\mathfrak{A}}{1-naabb}\quad \text{et}\quad \mathfrak{P}=\frac{\left(mab+\mathfrak{A}\mathfrak{B}\right)\left(1+naabb\right)+2nab\left(aa+bb\right)}{\left(1-naabb\right)^2}.$$

Deinde ex hac quantitate *p* cum datarum tertia *c* iuncta definiatur *q,* ut sit

$$q=\frac{p\mathfrak{C}+c\mathfrak{P}}{1-nccpp}\quad \text{and}\quad \mathfrak{Q}=\frac{\left(mcp+\mathfrak{C}\mathfrak{P}\right)\left(1+nccpp\right)+2ncp\left(cc+pp\right)}{\left(1-nccpp\right)^2}.$$

Tertio ex hac quantitate *q* cum quarta datarum *d* iuncta quaeratur *r,* ut sit

$$r=\frac{q\mathfrak{D}+d\mathfrak{Q}}{1-nddqq}\quad \text{et}\quad \mathfrak{R}=\frac{\left(mdq+\mathfrak{D}\mathfrak{Q}\right)\left(1+nddqq\right)+2ndq\left(dd+qq\right)}{\left(1-nddqq\right)^2}.$$

Quarto ex ista quantitate *r* cum quinta datarum *e* iuncta definiatur *s,* ut sit

$$s=\frac{r\mathfrak{E}+e\mathfrak{R}}{1-neerr}\quad \text{et}\quad \mathfrak{S}=\frac{\left(mer+\mathfrak{E}\mathfrak{R}\right)\left(1+neerr\right)+2ner\left(ee+rr\right)}{\left(1-neerr\right)^2}.$$

Haeque operationes continuentur, donec omnes quantitates datae in computum fuerint ductae.

§.32. His autem omnibus valoribus inventis sequentes comparationes desideratae ordine ita se habebunt

$$\text{I. } \Pi : p = \Pi : a + \Pi : b + \beta abp,$$
$$\text{II. } \Pi : q = \Pi : a + \Pi : b + \Pi : c + \beta abp + \beta cpq,$$
$$\text{III. } \Pi : r = \Pi : a + \Pi : b + \Pi : c + \Pi : d + \beta abp + \beta cpq + \beta dqr,$$
$$\text{IV. } \Pi : s = \Pi : a + \Pi : b + \Pi : c + \Pi : d + \Pi : e$$
$$+ \beta abp + \beta cpq + \beta dqr + \beta ers,$$
$$\text{V. } \Pi : t = \Pi : a + \Pi : b + \Pi : c + \Pi : d + \Pi : e + \Pi : f$$
$$+ \beta abp + \beta cpq + \beta dqr + \beta ers + \beta fst$$
etc.

§.33. Cum igitur ista formula transcendens

$$\Pi : z = \int \frac{\partial z(\alpha + \beta zz)}{\sqrt{\left(1+mzz+nz^4\right)}}$$

in se contineat arcus omnium sectionum conicarum a vertice sumtos, harum formularum ope, quotcunque proponantur arcus in quavis sectione conica, qui omnes a vertice sint sumti, semper novus in eadem sectione coni ea arcus pariter a vertice abscindi poterit, qui a summa illorum arcuum datarum discrepet quantitate algebraice assignabili. Quin etiam nihil impedit, quo minus aliqui inter arcus datos capiantur negativi, quandoquidem iam annotavimus esse $\Pi : (-z) = -\Pi : z$, ita ut nostra determinatio etiam accommodari possit ad arcus inter terminos quoscunque interceptos. Hocque modo tractatio, quam nuper circa comparationem talium arcuum dedi, multo generalior reddi poterit.

§.34. Ceterum, quemadmodum hoc casu, quo sumsimus $Z = \alpha + \beta zz$, character supra usurpatus $\Phi : xy$ abiit in βxyz, dum scilicet ex binis quantitatibus *x* et *y* secundum praecepta data tertia *z* determinatur, ita etiam, quaecunque alia functio loco *Z* adhibeatur, quoniam posuimus

$$\Phi : xy = -a \int \frac{(\mathrm{X}\text{-}\mathrm{Y})du}{xx - yy}$$

existente $u = xy$, integratione absoluta functio inde resultans tantum quantitatem *u* cum litteris *a* et $\mathfrak{A}$ continebit, quandoquidem littera *t* ita exprimebatur

$$t = aa - 2\mathfrak{A}u + naauu,$$

cum invento integrali ubique loco *u* scribatur *xy,* at vero loco *a* et $\mathfrak{A}$ litterae z et $\mathfrak{Z}$; atque hoc modo obtinebitur valor characteris *t:P* : *xy* pro quovis casu proposito, quae functio, nisi fuerit algebraica, semper per logarithmos et arcus circulares exhiberi poterit, siquidem, uti assumsimus, littera *Z* fuerit functio rationalis par ipsius *z.*