CHAPTER VI

CONCERNING EXPONENTIAL AND LOGARITHMIC QUANTITIES

96. Though finally the ideas of transcending functions will be considered carefully in the integral calculus, yet before we arrive at that, it will be appropriate to set out certain kinds of functions which arise more frequently, and begin to uncover the way for further investigations. Therefore in the first place, exponential quantities or powers are required to be considered, the exponent of which itself is a variable quantity. For it is evident quantities of this kind cannot be referred to algebraic functions, since in these the exponents do not have a place except as constants. But the multiples are variable quantities, provided either the exponent alone is a variable quantity or besides also that quantity with the raised quantity itself [can be a function]. a^z is of the former kind, of the latter truly y^z; furthermore also the exponent itself may be an exponential quantity, as in these forms a^{a^z}, a^{y^z}, y^{a^z}, x^{y^z}. but we will not put in place more kinds of quantities of this kind, since the nature of these may be understood clearly enough, if we have treated only the first kind a^z.

97. Therefore the proposed exponential function of this kind will be a^z, which is the power if the constant quantity *a* having the variable exponent *z*. Therefore since the exponent itself z is including in itself all the determined numbers, first it is apparent, if all the positive whole numbers may be substituted successively in place of z, in place of a^z these are going to produce the determined values

$$a^1,\quad a^2,\quad a^3,\quad a^4,\quad a^5,\quad a^6 \quad \text{etc.}$$

But if for *z* successively there may be placed the negative numbers $-1,\ -2,\ -3,\ -4$ etc., they will produce

$$\frac{1}{a},\quad \frac{1}{a^2},\quad \frac{1}{a^3},\quad \frac{1}{a^4} \quad \text{etc.}$$

and, if $z = 0$, there will be $a^0 = 1$ always.

But if fractional numbers may be put in place of z, as $\frac{1}{2}$, $\frac{1}{3}$, $\frac{2}{3}$, $\frac{1}{4}$, $\frac{3}{4}$ etc., these values arise

$$\sqrt{a},\quad \sqrt[3]{a},\quad \sqrt[3]{aa},\quad \sqrt[4]{a},\quad \sqrt[4]{a^3} \quad \text{etc.,}$$

which adopt twin or more values between themselves, since the extraction if roots always produces multiform values. Yet meanwhile here certainly only the first real and positive values are allowed to be admitted, because the quantity a^z is considered as a uniform function of *z*. Thus $a^{\frac{5}{2}}$ will hold a certain mean place between a^2 and a^3 and thus it will be a quantity of the same kind ; and although the value $a^{\frac{5}{2}}$ shall be equally $=-aa\sqrt{a}$ and $=+aa\sqrt{a}$, yet only the latter is taken into account. The matter is treated in the same manner, if the exponent *z* accepts irrational values, in which cases, since it shall be difficult to set out the values of the numbers, only actual single numbers will be considered. Thus $a^{\sqrt{7}}$ will be a value understood to be determined between the limits a^2 and a^3.

98. But the values of the exponential quantities a^z will depend especially on the magnitude of the constant number *a*. For if $a=1$, then there will be $a^z=1$ always, whatever the values that are attributed to the exponent *z*. But if $a>1$, then the values of a^z will be greater there, however great the number may be substituted in place of *z*, and thus on putting $z=\infty$ they will increase to infinity ; if $z=0$, the equation becomes $a^z=1$, and if it becomes $z<0$, the values a^z are made less than one, as long as on putting $z=-\infty$ there becomes $a^z=0$. The opposite comes about, if there shall be $a<1$, truly only positive quantities ; for then the values of a^z decrease with *z* above 0; truly the values increase, if negative numbers may be substituted for *z*. For since there shall be $a<1$, there will be $\frac{1}{a}>1$; therefore on putting $\frac{1}{a}=b$ there becomes $a^z=b^{-z}$, from which the latter case can be decided from the former case.

99. If $a=0$, then a huge leap in the values of a^z is taken. For as long as *z* was a positive number or greater than zero, the equation will always be $a^z=0$; if it becomes $z=0$, then $a^0=1$; but if *z* were a negative number, then a^z will obtain an infinitely great value. For if it becomes $z=-3$; then there will be

$$a^z=0^{-3}=\frac{1}{0^3}=\frac{1}{0}$$

and thus it is infinite.

But much greater leaps occur, if the constant quantity *a* may have a negative value, for example –2. For then by putting whole numbers in place of *z* the values of a^z will be positive and negative alternately, as is understood from this series :

$$\begin{array}{cccccccccl} a^{-4}, & a^{-3}, & a^{-2}, & a^{-1}, & a^0, & a^1, & a^2, & a^3, & a^4 & \text{etc.} \\ +\frac{1}{16}, & -\frac{1}{8}, & +\frac{1}{4}, & -\frac{1}{2}, & 1, & -2, & +4, & -8, & +16 & \text{etc.} \end{array}$$

Truly besides, if fractional values may be attributed to the exponent *z*, the power $a^z = (-2)^z$ soon adopts real as well as imaginary values; for $a^{\frac{1}{2}} = \sqrt{-2}$ will be imaginary, but there will be $a^{\frac{1}{3}} = \sqrt[3]{-2} = -\sqrt[3]{2}$, which is real; but whether, if irrational values may be attributed to the exponent *z*, the power a^z may show real or imaginary values, that indeed is not possible to be defined.

100. Therefore in place of *a* by recalling these inconvenient negative numbers being substituted, we may put *a* here to be a positive number and indeed greater than one, because this will be easily put in place in that case also, in which *a* is a positive number less than one,. Therefore if there may be put $a^z = y$, by substituting all the real numbers in place of *z*, which will be contained between the limits $+\infty$ and $-\infty$, *y* adopts all the real values contained between $+\infty$ and 0. For if $z = \infty$, then $y = \infty$; if $z = 0$, then $y = 1$, and if $z = -\infty$, it becomes $y = 0$. Therefore whichever positive value may be taken for *y* in turn, also a real value corresponding for *z* will be given, as thus it will be $a^z = y$; but if a negative value may be attributed to *y* itself, the exponent *z* cannot have a real value.

101. Therefore if $a^z = y$, *y* will be a certain function of *z;* and whatever the manner *y* may depend on *z* , is understood easily from the powers ; hence indeed, whichever value is attributed to *z*, the value of *y* itself is determined. For there shall be

$$yy = a^{2z}, \quad y^3 = a^{3z}$$

and generally the equation becomes

$$y^n = a^{nz} \ ;$$

from which it follows that

$$\sqrt{y} = a^{\frac{1}{2}z}, \quad \sqrt[3]{y} = a^{\frac{1}{3}z}, \quad \frac{1}{y} = a^{-z}, \quad \frac{1}{yy} = a^{-2z}, \quad \text{and} \quad \frac{1}{\sqrt{y}} = a^{-\frac{1}{2}z}$$

and thus so forth. Besides, if $v = a^x$, it becomes

$$vy = a^{x+z} \quad \text{and} \quad \frac{v}{y} = a^{x-z},$$

with the benefit of which help the value of *y* itself can be found from the given value of *z*.

EXAMPLE

If the base were $a = 10$ from the arithmetic in which we use tens, the values of *y* will be shown at once, if certain whole numbers may be put for *z*. Indeed there will be

$$10^1 = 10,\ 10^2 = 100,\ 10^3 = 1000,\ 10^4 = 10000 \quad \text{and} \quad 10^0 = 1;$$

likewise

$$10^{-1} = \tfrac{1}{10} = 0{,}1,\ \ 10^{-2} = \tfrac{1}{100} = 0{,}01,\ \ 10^{-3} = \tfrac{1}{1000} = 0{,}001;$$

but if fractions are put for *z*, the values of y may be indicated with the aid of the extraction of the roots; thus there will be

$$10^{\frac{1}{2}} = \sqrt{10} = 3{,}162277 \quad \text{etc.}$$

102. But just as the value of *y* can be found from the given number *a* and some given value of *z* , thus in turn for some given positive value of *y*, the agreeing value of *z* will be given, so that there shall be $a^z = y$; but this value of *z*, in as much as it may be seen to be a function of *y*, is usually called the *Logarithm* of *y*. Therefore the theory of logarithms supposes that a certain constant number is required to be substituted in place of *a*, which may therefore be called the *base* of the logarithms ; with which assumed the logarithm of any number *y* will be the exponent of the power a^z , thus so that the power a^z itself will be equal to that number *y;* moreover the logarithm of the number *y* is accustomed to be indicated in this manner *ly*. And therefore if $a^z = y$, then $z = ly$, from which the base of logarithms is understood, and even if it may depend on our choice, yet it must be a number greater than one, and hence only the logarithms of positive numbers are really able to be shown.

103. Therefore whatever number may be accepted for the logarithmic base *a*, there will always be

$$l1 = 0;$$

for if in the equation $a^z = y$, which agrees with this $z = ly$, on putting $y = 1$, it becomes $z = 0$.

Then the logarithms of numbers greater than unity will be positive, depending on the value of the base *a;* thus there will be

$$la = 1,\quad laa = 2,\quad la^3 = 3,\quad la^4 = 4 \quad \text{etc.},$$

from which it can be understood from the previous discussion, however large a number may be assumed for the base; evidently that number will be the logarithm base, the logarithm of which is $=1$,.

Moreover the logarithms of numbers less than unity, but still positive, are negative ; for there will be

$$l\tfrac{1}{a}=-1,\quad l\tfrac{1}{aa}=-2,\quad l\tfrac{1}{a^3}=-3 \quad \text{etc.}$$

But the logarithms of negative numbers are not real, but imaginary, as we have now noted.

104. In a similar manner if $ly=z$, then $lyy=2z$, $ly^3=3z$ and generally $ly^n=nz$ or $ly^n=nly$, on account of $z=ly$. Therefore the logarithm of any power of *y* is equal to the logarithm of *y* itself multiplied by the exponent of the power ; thus there will be

$$l\sqrt{y}=\tfrac{1}{2}z=\tfrac{1}{2}ly,\quad l\tfrac{1}{\sqrt{y}}=ly^{-\frac{1}{2}}=-\tfrac{1}{2}ly$$

and thus so on ; hence from the given logarithm of any number, the logarithms of any powers can be found.

But if now the two logarithms shall be found, surely

$$ly=z \quad \text{and} \quad lv=x\,,$$

because $y=a^z$ and $v=a^x$, there will be

$$lvy=x+z=lv+ly\,;$$

hence the logarithm of the product of two numbers is equal to the sum of the logarithms made; in a like manner there will be

$$l\tfrac{y}{v}=z-x=ly-lv$$

and hence the logarithm of a fraction is equal to the logarithm of the numerator with the logarithm of the denominator taken away ; which rules now take care of finding the logarithms of several numbers from some known logarithms.

105. But from these, it is apparent that the rational logarithms of other numbers are not given unless they are a power of the base *a;* for unless the other number *b* were a power of the base *a*, the number of its logarithm cannot be expressed by a rational number. Nor indeed also will the logarithm of *b* be a rational number. For if $lb=\sqrt{n}$, then there must become $a^{\sqrt{n}}=b$ *;* which cannot happen, if the numbers *a* and *b* indeed shall be rationals put in place. But the logarithms of rational and of whole numbers are wanted especially,

because from these the logarithms of fractions and of surds can be found. Therefore since the logarithms of numbers, which are not powers of the base *a*, neither rationally nor irrationally, are able to be shown, deservedly they are referred to transcending quantities and hence the logarithms are accustomed to be enumerated by transcending quantities.

106. On account of this, the logarithms of numbers truly can only be expressed approximately by decimal fractions, which the more figures there should be accurate, the less the logarithms will differ from the true values. And in this manner by the extraction of square roots alone, the logarithm of any number will be able to be determined.
For since on putting

$$ly = z \quad \text{and} \quad lv = x$$

there shall be

$$l\sqrt{vy} = \tfrac{x+z}{2},$$

if the proposed number *b* may be contained within the limits a^2 and a^3, the logarithms of which are 2 and 3, then the value is sought of $a^{2\frac{1}{2}}$ or $a^2\sqrt{a}$, and *b* will be contained within the limits *a* and $a^{2\frac{1}{2}}$ or $a^{2\frac{1}{2}}$ and a^3; whichever is found, by taking the mean of the proportions, new closer limits will be produced, and in this manner the limits desired will be arrived at, the interval of which will become less than with the quantity given, and with which the proposed number *b* may be combined without error. Because truly the logarithms of these individual limits are given, finally the logarithm of the number *b* will be found.

EXAMPLE

The base of logarithms $a = 10$ may be put in place, which is accustomed to happen in using received tables, and truly the logarithm of the number 5 is sought as an approximation ; because this is held between the limits 1 and 10, the logarithms of which are 0 and 1, the continued extraction of the root can be put in place, so that no further discrepancies to the limits may arise for the proposed number 5.

$A = 1,000000$,	$lA = 0,0000000$,	shall be
$B = 10,000000$,	$lB = 1,0000000$,	$C = \sqrt{AB}$,
$C = 3,162277$,	$lC = 0,5000000$,	$D = \sqrt{BC}$,
$D = 5,623413$,	$lD = 0,7500000$,	$E = \sqrt{CD}$,
$E = 4,216964$,	$lE = 0,6250000$,	$F = \sqrt{DE}$,
$F = 4,869674$,	$lF = 0,6875000$,	$G = \sqrt{DF}$,
$G = 5,232991$,	$lG = 0,7187500$,	$H = \sqrt{FG}$,
$H = 5,048065$,	$lH = 0,7031250$,	$I = \sqrt{FH}$,
$1 = 4,958069$,	$lI = 0,6953125$,	$K = \sqrt{HI}$,
$K = 5,002865$,	$lK = 0,6992187$,	$L = \sqrt{IK}$,
$L = 4,980416$,	$lL = 0,6972656$,	$M = \sqrt{KL}$,
$M = 4,991627$,	$lM = 0,6982421$,	$N = \sqrt{KM}$,
$N = 4,997242$,	$lN = 0,6987304$,	$O = \sqrt{KN}$,
$O = 5,000052$,	$lO = 0,6989745$,	$P = \sqrt{NO}$,
$P = 4,998647$,	$lP = 0,6988525$,	$Q = \sqrt{OP}$,
$Q = 4,999350$,	$lQ = 0,6989135$,	$R = \sqrt{OQ}$,
$R = 4,999701$,	$lR = 0,6989440$,	$S = \sqrt{OR}$,
$S = 4,999876$,	$lS = 0,6989592$,	$T = \sqrt{OS}$,
$T = 4,999963$,	$lT = 0,6989668$,	$V = \sqrt{OT}$,
$V = 5,000008$,	$lV = 0,6989707$,	$W = \sqrt{TV}$,
$W = 4,999984$,	$lW = 0,6989687$,	$X = \sqrt{VW}$,
$X = 4,999997$,	$lX = 0,6989697$,	$Y = \sqrt{VX}$,
$Y = 5,000003$,	$1Y = 0,6989702$,	$Z = \sqrt{XY}$,
$Z = 5,000000$,	$1Z = 0,6989700$.	

Therefore with the mean proportions taken thus, finally the calculation arrives at $Z = 5,000000$, from which the logarithm of the number 5 sought is 0,698970 on putting the base of the logarithms $= 10$. Whereby there will be approximately

$$10^{\frac{69897}{100000}} = 5.$$

Moreover, the canon of common logarithms has been computed in this manner by Briggs and Vlaq, after each selected number has been found, the logarithms have been placed together, with the aid of which logarithms can be computed much more readily.

[Complete translations of Briggs's *Arithmetica logarithmica*..... and of Napier's *Mirifici*.... etc., are presented on this website.]

107. Therefore just as many diverse systems of logarithms are given, as there may be numbers taken for the various bases *a*, and thus the number of systems of logarithms will be infinite. But always in two systems the logarithms of the same number maintain the same ratio among themselves. Let the base of one system be $=a$, of the other $=b$ and the logarithm of the number *n* in the former system $=p$, in the latter $=q$; there becomes

$$a^p = n \text{ and } b^q = n,$$

from which

$$a^p = b^q \text{ and thus } a = b^{\frac{q}{p}}.$$

Therefore it is required that the fraction $\frac{q}{p}$ may maintain a constant value, whichever the number were assumed for *n*. But if the logarithms of all the numbers were computed for one system, therefore hence the logarithms for some other system can be found by an easy calculation by the golden rule of logarithms. Thus, since the logarithms may be given for the base 10, hence the logarithms for some other base, for example 2, can be found ; let the logarithm of the number *n* for the base 2 be $=q$, since the logarithm to base 10 of the same number *n* shall be $=p$. Because for the base 10 we have $l2 = 0{,}3010300$ and for the base 2 , $l2 = 1$, the ratio will be $0{,}3010300 : 1 = p : q$ and thus

$$q = \frac{p}{0{,}3010300} = 3{,}3219280 \cdot p;$$

therefore, if all the common logarithms are multiplied by the number 3,3219280 a table of the logarithms to the base 2 will be produced.

108. *Hence it follows that the logarithms of two numbers in any system hold the same ratio.*

Let the two numbers be *M* and *N*, for the base *a* of which the logarithms shall be m and *n;* there will be $M = a^m$ and $N = a^n$; hence we have $a^{mn} = M^n = N^m$ and thus

$$M = N^{\frac{m}{n}};$$

in which an equation with the base *a* shall no longer be present, and it is evident that the fraction $\frac{m}{n}$ has a value independent of the base *a*. Indeed for some other base *b*, the logarithms of the same numbers *M* and *N* shall be μ and ν, in an equal manner the equation may be deduced to be

$$M = N^{\frac{\mu}{\nu}};$$

Therefore it becomes

$$N^{\frac{m}{n}} = N^{\frac{\mu}{\nu}}$$

and hence

$$\frac{m}{n} = \frac{\mu}{\nu} \text{ or } m:n = \mu:\nu.$$

Thus now we see that in every system of diverse logarithms the numbers of the same powers such as y^m and y^n maintain the same ratio of the exponents $m:n$.

109. Therefore for a cannon of logarithms to be put in place for some base *a* only the logarithms of the prime numbers need to be calculated, by the method examined before or by some other more convenient method. For since the logarithms of composite numbers shall be equal to the sum of the logarithms of the individual factors, the logarithms of composite numbers will be found by addition alone. Thus, if the logarithms of the numbers 3 and 5, there will be

$$l15 = l3 + l5, \quad l45 = 2l3 + l5.$$

And, since above for the base $a = 10$ it may be found that $l5 = 0,6989700$, besides moreover there shall be $l10 = 1$, and there will be $l\frac{10}{5} = l2 = l10 - l5$ and thus

$$l2 = 1 - 0,6989700 = 0,3010300 \text{ arises;}$$

moreover from the logarithms found of these prime numbers 2 and 5 the logarithms will be found of all the composite numbers 2 and 5, there are of the kind 4, 8, 16, 32, 64 etc., 20, 40, 80, 25, 50 etc.

110. But the most complete tables of logarithms is used in undertaking numerical calculations, therefore because from tables of this kind not only is the logarithm of any number given, but also the number agreeing with any proposed logarithm can be found. Thus, if *c, d, e, f, g, h* denote any numbers, before multiplication, the value of this expression will be able to be found

$$\frac{ccd\sqrt{e}}{f\sqrt[3]{gh}};$$

for the logarithm of this expression will be

$$= 2lc + ld + \tfrac{1}{2}le - lf - \tfrac{1}{3}lg - \tfrac{1}{3}lh;$$

for which the logarithm, if the corresponding number is sought, will have the value required. Tables of logarithms moreover, are especially worthwhile in serving also to extract the most intricate of roots, of which operations in place of logarithms only multiplication and division is used.

EXAMPLE 1

The value of this power $2^{\frac{7}{12}}$ is required.

Because the logarithm of this is $=\frac{7}{12}l2$, the logarithm of two which is found from the tables, which is 0,3010300, may be multiplies by $\frac{7}{12}$ that is by $\frac{1}{2}+\frac{1}{12};$ there becomes $l2^{\frac{7}{12}}=0{,}1756008$, to which logarithm corresponds the number 1,498307, which therefore shows the approximate value $2^{\frac{7}{12}}$.

EXAMPLE 2

If the number of inhabitants of a certain province should increase by the thirtieth part each year, moreover at start there were 100000 people in the province, the number of inhabitants is sought after 100 years.

For the sake of brevity let the initial number of inhabitants be the number $=n$, thus so that there shall be $n=100000$; in the elapse of one year it will be $=\left(1+\frac{1}{30}\right)n=\frac{31}{30}n,$ the number of inhabitants after two years will be $=\left(\frac{31}{30}\right)^2 n,$ after three years $=\left(\frac{31}{30}\right)^3 n,$ and hence after one hundred years

$$=\left(\tfrac{31}{30}\right)^{100} n=\left(\tfrac{31}{30}\right)^{100} 100000,$$

the logarithm of which is

$$=100l\tfrac{31}{30}+l100000.$$

But

$$100l\tfrac{31}{30}=l31-l30=0{,}014240439,$$

from which

$$100l\tfrac{31}{30}=1{,}4240439;$$

to which if there may be added $l100000=5$, the logarithm of the number of inhabitants sought

$$=6{,}4240439,$$

to which the number corresponds

$$=2654874.$$

Therefore after a century the number of inhabitants shall be more than twenty six and a half times greater.

EXAMPLE 3

Since the human race shall have propagated from six people after the flood, if we may put the number of people two hundred years after to have increased now to 1000000, it is sought by how great a part the number of people must be increased in a year.

We may put the number of people to have increased in this time by its $\frac{1}{x}$ part per annum and after two centuries by necessity the number of people will be

$$=\left(\frac{1+x}{x}\right)^{200} 6 = 1000000,$$

from which there becomes

$$\frac{1+x}{x} = \left(\frac{1000000}{6}\right)^{\frac{1}{200}}.$$

Therefore there will be

$$l\frac{1+x}{x} = \frac{1}{200} l\frac{1000000}{6} = \frac{1}{200} \cdot 5,2218487 = 0,0261092$$

and thus

$$\frac{1+x}{x} = \frac{1061963}{1000000} \text{ and } 1000000 = 61963x,$$

from which approximately,

$$x = 16.$$

Therefore it may be sufficient [for the population] to multiply to so great a number of people, if the number has increased by its sixteenth part per annum; which multiplication, on account of the long life of the inhabitants, may not be considered especially large. But if moreover the number of people might have gone on to increase in the same ratio through an interval of 400 years, then the number of people must rise to

$$1000000 \cdot \frac{1000000}{6} = 166666666666,$$

with which requiring to be supported, the whole orb of the earth would by no means be equal.

EXAMPLE 4

If the number of people be doubled in each century, the annual increment is sought.

If we may put the annual amount of the number of people to increase by its $\frac{1}{x}$ part, and the initial number of people were $= n$, this will be after a hundred years $= \left(\frac{1+x}{x}\right)^{100} n$; which since it must be equal to $= 2n$, the equation will be

$$\frac{1+x}{x} = 2^{\frac{1}{100}}$$

and

$$l\frac{1+x}{x} = \frac{1}{100} l2 = 0{,}0030103;$$

hence

$$\frac{1+x}{x} = \frac{10069555}{10000000},$$

therefore

$$x = \frac{10000000}{69555} = 144 \text{ approximately.}$$

It suffices therefore, if the number of people may be increased by its $\frac{1}{144}$ part per annum. On account of which reason, the objections of these incredulous men are especially laughable, who deny that so short an interval of time be taken for one man to fill the whole earth with people.

111. But the chief use of logarithms is required for resolving equations of this kind, in which unknown quantities are present in the exponential.
Thus, if an equation of this kind may be arrived at

$$a^x = b,$$

from which the value of the unknown x may be extracted, this is unable to be effected except by logarithms. For since $a^x = b$, there will be

$$la^x = xla = lb$$

and thus

$$x = \frac{lb}{la},$$

where indeed it is the same, with whatever system of logarithms may be used, since in every system the logarithms of the numbers a and b maintain the same ratio between each other.

EXAMPLE 1

If the number of people increased by a hundredth part in one year, it is asked, after how many years will the number be ten times greater.

We may consider this to come about after x and in the beginning the number of people was $=n$; therefore this will be after the elapse of x years $=\left(\frac{101}{100}\right)^{x}n$, which shall be equal to $10n$, making

$$\left(\tfrac{101}{100}\right)^{x}=10$$

and thus

$$xl\tfrac{101}{100}=l10$$

and

$$x=\tfrac{l10}{l101-l100}.$$

And thus there becomes approximately

$$x=\tfrac{10000000}{43214}=231.$$

Therefore after 231 years the number of men [Euler uses the Latin word *homo* for *man*, when he means really *people*, as women were necessary as well !] is ten times greater, the increment of which effects only an annual increment of one hundredth part ; hence after 462 years it becomes a hundred times, and after 693 a thousand times greater.

EXAMPLE 2

Someone owes 400000 florens with this condition, that 5 per cent interest per annum may be held to be paid; but in the individual years he pays 25000 florens. The number of years is sought, after which the debt will be paid off completely.

We may write a for the total debt 400000 fl. and b for the sum repaid per year 25000 fl.; therefore at the end of one year he will owe

$$\tfrac{105}{100}a-b,$$

with two years elapsed

$$\left(\tfrac{105}{100}\right)^{2}a-\tfrac{105}{100}b-b$$

with three years elapsed

$$\left(\tfrac{105}{100}\right)^{3}a-\left(\tfrac{105}{100}\right)^{2}b-\tfrac{105}{100}b-b;$$

hence putting for the sake of brevity : n for $\frac{105}{100}$ with x years elapsed at this stage he will owe

$$n^x a - n^{x-1}b - n^{x-2}b - n^{x-3}b - \cdots - b = n^x a - b\left(1 + n + n^2 + \cdots + n^{x-1}\right).$$

Therefore since there shall be, from the nature of the geometric progression,

$$1 + n + n^2 + \cdots + n^{x-1} = \tfrac{n^x - 1}{n-1},$$

after x years the debtor will owe at this point

$$n^x a - \tfrac{n^x b - b}{n-1} \text{ flor.},$$

which debt put equal to zero will give this equation

$$n^x a = \tfrac{n^x b - b}{n-1}$$

or

$$(n-1)n^x a = n^x b - b \text{ and thus } (b - na + a)n^x = b$$

and

$$n^x = \tfrac{b}{b - na + a}$$

from which arises

$$x = \frac{lb - l\left(b - (n-1)a\right)}{ln}.$$

Since now there shall be

$$a = 400000,\ \ b = 25000,\ \ n = 100,$$

the equation becomes

$$(n-1)a = 20000 \quad \text{and} \quad b - (n-1)a = 5000$$

and the number of years, in which the debt is completely paid off,

$$x = \frac{l25000 - l5000}{l\frac{105}{100}} = \frac{l5}{l\frac{21}{20}} = \frac{6989700}{211893};$$

therefore x will be a little less than 33. It is clear with the elapse of 33 years not only will the debt be paid off, but also the creditor will hold out to return to the debtor

$$\frac{\left(n^{33} - 1\right)b}{n-1} - n^{33}a = \left(\tfrac{21}{20}\right)^{33} \cdot \frac{5000 - 25000}{\frac{1}{20}} = 100000\left(\tfrac{21}{20}\right)^{33} - 500000 \text{ flor.}$$

Because truly

$$l\,\tfrac{21}{20} = 0{,}0211892991,$$

there will be

$$l\left(\tfrac{21}{20}\right)^{33} = 0{,}69924687 \quad \text{and} \quad l100000\left(\tfrac{21}{20}\right)^{33} = 5{,}6992469,$$

to which this number 500318,8 corresponds ; from which the creditor must restore to the debtor after 33 years, $318\tfrac{4}{5}$ florens.

112. But common logarithms extracted on the base $=10$ besides this use, at which logarithms in general are outstanding, suitably enjoy the advantages of decimal arithmetic and for that reason bring a significant usefulness before other systems of logarithms. For since the logarithms of all numbers besides the powers of ten may be shown as decimal fractions, the logarithms of numbers contained between 1 and 10 will be contained within the limits 0 and 1, also the logarithms of the numbers between 10 and 100 will be contained between the limits 1 and 2, and thus henceforth. Therefore any logarithm may be constructed from a whole number and a decimal fraction, and that whole number is accustomed to be called the *characteristic,* moreover the decimal fraction is called the *mantissa* [meaning *the extra part*]. And thus the characteristic falls short by one from a known number, with which the number agrees ; thus the characteristic of the logarithm of the number 78509 will be 4, because this agrees with five known places or figures. Hence from the logarithm of any number it is understood at once, from how many figures the number shall be composed. Thus the number corresponding to the logarithm 7,5804631 will be constructed from 8 figures.

113. Therefore if the mantissas of two logarithms may agree, truly the characteristics only may differ, then the numbers corresponding to these logarithms will be in the ratio of powers of ten to one and thus agree in the ratio of the figures, from which they are constructed. Thus the logarithms 4,9130187 and 6,9130187 correspond to these numbers 81850 and 8185000 ; but the logarithm 3,9130187 agrees with 8185 and to this logarithm 0,9130187 , 8,185 is the number in agreement. Therefore the mantissa alone will indicate the figures composing the number ; with which found it will be apparent from the characteristic, how many figures must be referred from the left to the whole number, the remaining figures to the right truly will give the decimal fraction. Thus, if this logarithm were found 2,7603429, the mantissa will indicate these figures 5758945, but the characteristic 2 determines the number for that logarithm, so that it shall be 575,8945; if the characteristic were 0, the number becomes 5,758945; but if it may be diminished by one anew, so that it shall be –1, the corresponding number will be ten times less, surely 0,5758945, and the characteristic –2 will correspond to 0,05758945 etc. But in place of the negative characteristics of this kind –1, –2, –3 etc. they are accustomed to be written as 9, 8, 7 etc. and it is understood that these logarithms must be taken from ten. Truly these are accustomed to be explained further in the introductions to tables of logarithms.

EXAMPLE

If this progression 2, 4, 16, 256 etc., of which any term is the square of the preceding, may be continued as far as to the twenty fifth term, the magnitude of this final term is sought.

The terms of this progression may be expressed thus more conveniently $2^1, \quad 2^2, \quad 2^4, \quad 2^8$ etc.,where it is apparent that the exponents constitute a geometric progression and the twenty fifth term becomes the exponent

$$2^{24} = 16777216,$$

thus so that term sought itself shall be

$$= \ 2^{16777216};$$

the logarithm of this will be

$$= 16777216 l2.$$

Therefore since there shall be

$$l2 = 0{,}30102\ 99956\ 639811952$$

the logarithm of the number sought will be

$$= 5050445{,}25973367,$$

from the characteristic of which it is apparent that the number sought may be constructed in the usual manner from

$$5050446$$

figures. Moreover the mantissa 259733675932 found in tables of logarithms will give the initial figures of the number, which will be 181858. Therefore although this number cannot be shown in any way, yet it is possible to affirm generally that it is constructed from 5050446 figures and the first six are 181858, which to this 5050440 figures to the right may follow, of which some may be defined above from a larger canon of logarithms; evidently the first eleven figures will be 18185852986.

CAPUT VI

DE QUANTITATIBUS EXPONENTIALIBUS AC LOGARITHMIS

96. Quanquam notio functionum transcendentium in Calculo integrali demum perpendetur, tamen, antequam eo perveniamus, quasdam species magis obvias atque ad plures investigationes aditum aperientes evolvere conveniet. Primum ergo considerandae sunt quantitates exponentiales seu potestates, quarum exponens ipse est quantitas variabilis. Perspicuum enim est huiusmodi quantitates ad functiones algebraicas referri non posse, cum in his exponentes non nisi constantes locum habeant. Multiplices autem sunt quantitates exponentiales, prout vel solus exponens est quantitas variabilis vel praeterea etiam ipsa quantitas elevata. Prioris generis est a^z, huius vero y^z; quin etiam ipse exponens potest esse quantitas exponentialis, uti in his formis a^{a^z}, a^{y^z}, y^{a^z}, x^{y^z}. Huiusmodi autem quantitatum non plura constituemus genera, eum earum natura satis clare intelligi queat, si primam tantum speciem a^z evolverimus.

97. Sit igitur proposita huiusmodi quantitas exponentialis a^z, quae est potestas quantitatis constantis *a* exponentem habens variabilem *z*. Cum igitur iste exponens *z* omnes numeros determinatos in se complectatur, primum patet, si loco *z* omnes numeri integri affirmativi successive substituantur, loco a^z hos prodituros esse valores determinatos

$$a^1,\quad a^2,\quad a^3,\quad a^4,\quad a^5,\quad a^6\quad \text{etc.}$$

Sin autem pro *z* ponantur successive numeri negativi $-1,\ -2,\ -3,\ -4$ etc., prodibunt

$$\tfrac{1}{a},\quad \tfrac{1}{a^2},\quad \tfrac{1}{a^3},\quad \tfrac{1}{a^4}\quad \text{etc.}$$

ac, si fuerit $z=0$, habebitur semper $a^0=1$.

Quodsi 1oco *z* numeri fracti ponantur, ut $\tfrac{1}{2},\ \tfrac{1}{3},\ \tfrac{2}{3},\ \tfrac{1}{4},\ \tfrac{3}{4}$ etc., orientur isti valores

$$\sqrt{a},\quad \sqrt[3]{a},\quad \sqrt[3]{aa},\quad \sqrt[4]{a},\quad \sqrt[4]{a^3}\quad \text{etc.,}$$

qui in se spectati geminos pluresve induunt valores, cum radicum extractio semper valores multiformes producat. Interim tamen hoc loco valores tantum primarii, reales scilicet atque affirmativi, admitti solent, quia quantitas a^z tanquam functio uniformis ipsius *z* spectatur. Sic $a^{\frac{5}{2}}$ medium quendam tenebit locum inter a^2 et a^3 eritque ideo quantitas eiusdem generis; et quamvis valor $a^{\frac{5}{2}}$ sit aeque $=-aa\sqrt{a}$ ac $=+aa\sqrt{a}$, tamen posterior tantum in censum venit. Eodem modo res se habet, si exponens *z* valores irrationales accipiat, quibus casibus, cum difficile sit numerum valorum involutorum concipere, unicus tantum realis consideratur. Sic $a^{\sqrt{7}}$ erit valor determinatus intra limites a^2 et a^3 comprehensus.

98. Maxime autem valores quantitatis exponentialis a^z a magnitudine numeri constantis *a* pendebunt. Si enim fuerit $a=1$, semper erit $a^z=1$, quicunque valores exponenti *z* tribuantur. Sin autem fuerit $a>1$, tum valores ipsius a^z eo erunt maiores, quo maior numerus loco *z* substituatur, atque adeo posito $z=\infty$ in infinitum excrescunt; si fuerit $z=0$, fiet $a^z=1$, et si sit $z<0$, valores a^z fient unitate minores, quoad posito $z=-\infty$ fiat $a^z=0$. Contrarium evenit, si sit $a<1$, verum tamen quantitas affirmativa; tum enim valores ipsius a^z decrescent crescente *z* supra 0; crescent vero, si pro *z* numeri negativi substituantur. Cum enim sit $a<1$, erit $\frac{1}{a}>1$; posito ergo $\frac{1}{a}=b$ erit $a^z=b^{-z}$, unde posterior casus ex priori diiudicari poterit.

99. Si sit $a=0$, ingens saltus in valoribus ipsius a^z deprehenditur. Quamdiu enim fuerit *z* numerus affirmativus seu maior nihilo, erit perpetuo $a^z=0$; si sit $z=0$, erit $a^0=1$; sin autem fuerit *z* numerus negativus, tum a^z obtinebit valorem infinite magnum. Sit enim $z=-3$; erit

$$a^z=0^{-3}=\frac{1}{0^3}=\frac{1}{0}$$

ideoque infinitum.

Multo maiores autem saltus occurrent, si quantitas constans *a* habeat valorem negativum, puta –2. Tum enim ponendis loco *z* numeris integris valores ipsius a^z alternatim erunt affirmativi et negativi, ut ex hac serie intelligitur

$$\begin{array}{cccccccccc} a^{-4}, & a^{-3}, & a^{-2}, & a^{-1}, & a^0, & a^1, & a^2, & a^3, & a^4 & \text{etc.} \\ +\frac{1}{16}, & -\frac{1}{8}, & +\frac{1}{4}, & -\frac{1}{2}, & 1, & -2, & +4, & -8, & +16 & \text{etc.} \end{array}$$

Praeterea vero, si exponenti *z* valores tribuantur fracti, potestas $a^z=(-2)^z$ mox reales mox imaginarios induet valores; erit enim $a^{\frac{1}{2}}=\sqrt{-2}$ imaginarium, at erit

$a^{\frac{1}{3}} = \sqrt[3]{-2} = -\sqrt[3]{2}$ reale; utrum autem, si exponenti z tribuantur valores irrationales, potestas a^z exhibeat quantitates reales an imaginarias, ne quidem definiri licet.

100. His igitur incommodis numerorum negativorum loco a substituendorum commemoratis statuamus a esse numerum affirmativum et unitate quidem maiorem, quia huc quoque illi casus, quibus a est numerus affirmativus unitate minor, facile reducuntur. Si ergo ponatur $a^z = y$, loco z substituendo omnes numeros reales, qui intra limites $+\infty$ et $-\infty$ continentur, y adipiscetur omnes valores affirmativos intra limites $+\infty$ et 0 contentos. Si enim sit $z = \infty$, erit $y = \infty$; si $z = 0$,erit $y = 1$, et si $z = -\infty$, fiet $y = 0$. Vicissim ergo quicunque valor affirmativus pro y accipiatur, dabitur quoque valor realis respondens pro z, ita ut sit $a^z = y$; sin autem ipsi y tribueretur valor negativus, exponens z valorem realem habere non poterit.

101. Si igitur fuerit $a^z = y$, erit y functio quaedam ipsius z; et quemadmodum y a z pendeat, ex natura potestatum facile intelligitur; hinc enim, quicunque valor ipsi z tribuatur, valor ipsius y determinatur. Erit autem

$$yy = a^{2z}, \quad y^3 = a^{3z}$$

et generaliter erit

$$y^n = a^{nz};$$

unde sequitur fore

$$\sqrt{y} = a^{\frac{1}{2}z}, \quad \sqrt[3]{y} = a^{\frac{1}{3}z}, \quad \text{et} \quad \tfrac{1}{y} = a^{-z}, \quad \tfrac{1}{yy} = a^{-2z}, \quad \text{et} \quad \tfrac{1}{\sqrt{y}} = a^{-\frac{1}{2}z}$$

et ita porro. Praeterea, si fuerit $v = a^x$, erit

$$vy = a^{x+z} \quad \text{et} \quad \tfrac{v}{y} = a^{x-z},$$

quorum subsidiorum beneficio eo facilius valor ipsius y ex dato valore ipsius z inveniri potest.

EXEMPLUM

Si fuerit $a = 10$, ex Arithmetica, qua utimur, denaria in promptu erit valores ipsius y exhibere, si quidem pro z numeri integri ponantur. Erit enim

$$10^1 = 10,\ 10^2 = 100,\ \ 10^3 = 1000,\ 10^4 = 10000 \ \text{ et } \ 10^0 = 1;$$

item

$$10^{-1} = \tfrac{1}{10} = 0{,}1,\ \ 10^{-2} = \tfrac{1}{100} = 0{,}01,\ \ 10^{-3} = \tfrac{1}{1000} = 0{,}001;$$

sin autem pro z fractiones ponantur, ope radicum extractionis valores ipsius y indicari possunt; sic erit

$$10^{\frac{1}{2}} = \sqrt{10} \ = 3{,}162277 \ \text{ etc.}$$

102. Quemadmodum autem dato numero a ex quovis valore ipsius z , reperiri potest valor ipsius y, ita vicissim dato valore quocunque affirmativo ipsius y conveniens dabitur valor ipsius z, ut sit $a^z = y$; iste autem valor ipsius z, quatenus tanquam functio ipsius y spectatur, vocari solet *Logarithmus* ipsius y. Supponit ergo doctrina logarithmorum numerum certum constantem loco a substituendum, qui propterea vocatur *basis* logarithmorum; qua assumpta erit logarithmus cuiusque numeri y exponens potestatis a^z , ita ut ipsa potestas a^z aequalis sit numero illi y; indicari autem logarithmus numeri y solet hoc modo ly. Quodsi ergo fuerit $a^z = y$, erit $z = ly$, ex quo intelligitur basin logarithmorum, etiamsi ab arbitrio nostro pendeat, tamen esse debere numerum unitate maiorem hincque nonnisi numerorum affirmativorum logarithinos realiter exhiberi posse.

103. Quicunque ergo numerus pro basi logarithmica a accipiatur, erit semper

$$l1 = 0\,;$$

si enim in aequatione $a^z = y$, quae convenit cum hac $z = ly$, ponatur $y = 1$, erit $z = 0$.

Deinde numerorum unitate maiorum logarithmi erunt affirmativi, pendentes a valore basis a; sic erit

$$la = 1, \quad laa = 2, \quad la^3 = 3, \quad la^4 = 4 \quad \text{etc.,}$$

unde a posteriori intelligi potest, quantus numerus pro basi sit assumptus; scilicet ille numerus, cuius logarithmus est $= 1$, erit basis logarithmica.

Numerorum autem unitate minorum, affirmativorum tamen, logarithmi erunt negativi; erit enim

$$l\tfrac{1}{a} = -1, \quad l\tfrac{1}{aa} = -2, \quad l\tfrac{1}{a^3} = -3 \quad \text{etc.}$$

Numerorum autem negativorum logarithmi non erunt reales, sed imaginarii, uti iam notavimus.

104. Simili modo si fuerit $ly = z$, erit $lyy = 2z$, $ly^3 = 3z$ et generaliter $ly^n = nz$ seu $ly^n = nly$ ob $z = ly$. Logarithmus igitur cuiusque potestatis ipsius *y* aequatur logarithmo ipsius *y* per exponentem potestatis multiplicato; sic erit

$$l\sqrt{y} = \tfrac{1}{2}z = \tfrac{1}{2}ly,\ \ l\frac{1}{\sqrt{y}} = ly^{-\frac{1}{2}} = -\tfrac{1}{2}ly$$

et ita porro; unde ex dato logarithmo cuiusque numeri inveniri possunt logarithmi quarumcunque ipsius potestatum.

Sin autem iam inventi sint duo logarithmi, nempe

$$ly = z \ \text{ et } \ lv = x,$$

cum sit $y = a^z$ et $v = a^x$, erit

$$lvy = x + z = lv + ly;$$

hinc logarithmus producti duorum numerorum aequatur summae logarithmorum factorum; simili vero modo erit

$$l\frac{y}{v} = z - x = ly - lv$$

hincque logarithmus fractionis aequatur logarithmo numeratoris dempto logarithmo denominatoris; quae regulae inserviunt logarithmis plurium numerorum inveniendis ex cognitis iam aliquot logarithmis.

105. Ex his autem patet aliorum numerorum non dari logarithmos rationales nisi potestatum basis *a;* nisi enim numerus alius *b* fuerit potestas basis *a,* eius logarithmus numero rationali exprimi non poterit. Neque vero etiam logarithmus ipsius *b* erit numerus irrationalis; Si enim foret $lb = \sqrt{n}$, tum esset $a^{\sqrt{n}} = b$; id quod fieri nequit, si quidem numeri *a* et *b* rationales statuantur. Solent autem imprimis numerorum rationalium et integrorum logarithmi desiderari, quia ex his logarithmi fractionum ac numerorum surdorum inveniri possunt. Cum igitur logarithmi numerorum, qui non sunt potestates basis *a,* neque rationaliter neque irrationaliter exhiberi queant, merito ad quantitates transcendentes referuntur hincque logarithmi quantitatibus transcendentibus annumerari solent.

106. Hanc ob rem logarithmi numerorum vero tantum proxime per fractiones decimales exprimi solent, qui eo minus a veritate discrepabunt, ad quo plures figuras fuerint exacti. Atque hoc modo per solam radicis quadratae extractionem cuiusque numeri logarithmus vero proxime determinari poterit.
Cum enim posito

$$ly = z \quad \text{et} \quad lv = x$$

sit

$$l\sqrt{vy} = \frac{x+z}{2}$$

si numerus propositus b contineatur intra limites a^2 et a^3, quorum logarithmi sunt 2 et 3, quaeratur valor ipsius $a^{2\frac{1}{2}}$ seu $a^2\sqrt{a}$ atque b vel intra limites a et $a^{2\frac{1}{2}}$ vel $a^{2\frac{1}{2}}$ et a^3 continebitur; utrumvis accidat, sumendo medio proportionali denuo limites propiores prodibunt hocque modo ad limites pervenire licebit, quorum intervallum data quantitate minus evadat et quibuscum numerus propositus b sine errore confundi possit. Quoniam vero horum singulorum limitum logarithmi dantur, tandem logarithmus numeri b reperietur.

EXEMPLUM

Ponatur basis logarithmica $a = 10$, quod in tabulis usu receptis fieri solet, et quaeratur vero tantum proxime logarithmus numeri 5; quia hic continetur intra limites 1 et 10, quorum logarithmi sunt 0 et 1, sequenti modo radicum extractio continua instituatur, quoad ad limites a numero proposito 5 non amplius discrepantes perveniatur.

A= 1, 000000,	lA = 0,0000000,	sit
B =10,000000,	lB = 1,0000000,	$C=\sqrt{AB}$,
C= 3,162277,	lC= 0,5000000,	$D=\sqrt{BC}$,
D = 5,623413,	lD = 0,7500000,	$E=\sqrt{CD}$,
E = 4,216964,	lE = 0,6250000,	$F=\sqrt{DE}$,
F = 4,869674,	lF = 0,6875000,	$G=\sqrt{DF}$,
G= 5,232991,	lG = 0,7187500,	$H=\sqrt{FG}$,
H =5,048065,	lH = 0,7031250,	$I=\sqrt{FH}$,
1 = 4,958069,	lI = 0,6953125,	$K=\sqrt{HI}$,
K =5,002865,	lK= 0,6992187,	$L=\sqrt{IK}$,
L =4,980416,	lL = 0,6972656,	$M=\sqrt{KL}$,
M=4,991627,	lM =0,6982421,	$N=\sqrt{KM}$,
N= 4,997242,	lN = 0,6987304,	$O=\sqrt{KN}$,
O= 5,000052,	lO = 0,6989745,	$P=\sqrt{NO}$,
P= 4,998647,	lP = 0,6988525,	$Q=\sqrt{OP}$,
Q= 4,999350,	lQ= 0,6989135,	$R=\sqrt{OQ}$,
R = 4,999701,	lR = 0,6989440,	$S=\sqrt{OR}$,
S = 4,999876,	lS = 0,6989592,	$T=\sqrt{OS}$,
T = 4,999963,	lT = 0,6989668,	$V=\sqrt{OT}$,
V = 5,000008,	lV = 0,6989707,	$W=\sqrt{TV}$,
W =4,999984,	lW= 0,6989687,	$X=\sqrt{VW}$,
X = 4,999997,	lX = 0,6989697,	$Y=\sqrt{VX}$,
Y = 5,000003,	*1Y* = 0,6989702,	$Z=\sqrt{XY}$,
Z = 5,000000,	*1Z* = 0,6989700.	

Sic ergo mediis proportionalibus sumendis tandem perventum est ad $Z = 5{,}000000$, ex quo logarithmus numeri 5 quaesitus est 0,698970 posita basi logarithmica $=10$. Quare erit proxime

$$10^{\frac{69897}{100000}} = 5.$$

Hoc autem modo computatus est canon logarithmorum vulgaris a BRIGGIO et VLACQUIO, quamquam postea eximia inventa sunt compendia, quorum ope multo expeditius logarithmi supputari possunt.

107. Dantur ergo tot diversa logarithmorum systemata, quot varii numeri pro basi a accipi possunt, atque ideo numerus systematum logarithmicorum erit infinitus. Perpetuo autem in duobus systematis logarithmi eiusdem numeri eandem inter se servant rationem. Sit basis unius systematis $= a$, alterius $= b$ atque numeri n logarithmus in priori systemate $= p$, in posteriori $= q$; erit

$$a^p = n \text{ et } b^q = n,$$

unde

$$a^p = b^q \quad \text{ideoque} \quad a = b^{\frac{q}{p}}.$$

Oportet ergo, ut fractio $\frac{q}{p}$ constantem obtineat valorem, quicunque numerus pro n fuerit assumptus. Quodsi ergo pro uno systemate logarithmi omnium numerorum fuerint computati, hinc facile negotio per regulam auream logarithmi pro quovis alio systemate reperiri possunt. Sic, cum dentur logarithmi pro basi 10, hinc logarithmi pro quavis alia basi, puta 2, inveniri possunt; quaeratur enim logarithmus numeri n pro basi 2, qui sit $= q$, cum eiusdem numeri n logarithmus sit $= p$ pro basi 10. Quoniam pro basi 10 est $l2 = 0{,}3010300$ et pro basi 2 est $l2 = 1$, erit $0{,}3010300 : 1 = p : q$ ideoque

$$q = \frac{p}{0{,}3010300} = 3{,}3219280 \cdot p;$$

si ergo omnes logarithmi communes multiplicuntur per numerum 3,3219280 prodibit tabula logarithmorum pro basi 2.

108. *Hinc sequitur duorum numerorum logarithmos in quocunque systemate eandem tenere rationem.*

Sint enim duo numeri M et N, quorum pro basi a logarithmi sint m et n; erit $M = a^m$ et $N = a^n$; hinc fiet $a^{mn} = M^n = N^m$ ideoque

$$M = N^{\frac{m}{n}};$$

in qua aequatione cum basis a non amplius insit, perspicuum est fractionem $\frac{m}{n}$ habere valorem a basi a non pendentem. Sint enim pro alia basi b numerorum eorundem M et N logarithmi μ et ν, pari modo colligetur fore

$$M = N^{\frac{\mu}{\nu}};$$

Erit ergo

$$N^{\frac{m}{n}} = N^{\frac{\mu}{\nu}}$$

hincque

$$\frac{m}{n} = \frac{\mu}{\nu} \quad \text{seu} \quad m : n = \mu : \nu.$$

Ita iam vidimus in omni logarithmorum systemate logarithmos diversarum eiusdem numeri potestatum ut y^m et y^n tenere rationem exponentium $m:n$.

109. Ad canonem ergo logarithmorum pro basi quacunque *a* condendum sufficit numerorum tantum primorum logarithmos methodo ante tradita vel alia commodiori supputasse. Cum enim logarithmi numerorum compositorum sint aequales summis logarithmorum singulorum factorum, logarithmi numerorum compositorum per solam additionem reperientur. Sic, si habeantur logarithmi numerorum 3 et 5, erit

$$l15 = l3 + l5, \quad l45 = 2l3 + l5.$$

Atque, cum supra pro basi $a = 10$ inventus sit $l5 = 0{,}6989700$, praeterea autem sit $l10 = 1$, erit $l\frac{10}{5} = l2 = l10 - l5$ ideoque orietur

$$l2 = 1 - 0{,}6989700 = 0{,}3010300;$$

ex his autem numerorum primorum 2 et 5 logarithmis inventis reperientur logarithmi omnium numerorum ex his 2 et 5 compositorum, cuiusmodi sunt isti 4, 8, 16, 32, 64 etc., 20, 40, 80, 25, 50 etc.

110. Tabularum autem logarithmicarum amplissimus est usus in contrahendis calculis numericis, propterea quod ex eiusmodi tabulis non solum dati cuiusque numeri logarithmus, sed etiam cuiusque logarithmi propositi numerus conveniens reperiri potest. Sic, si *c, d, e, f, g, h* denotent numeros quoscunque, citra multiplicationem reperiri poterit valor istius expressionis

$$\frac{ccd\sqrt{e}}{f\sqrt[3]{gh}};$$

erit enim huius expressionis logarithmus

$$= 2lc + ld + \tfrac{1}{2}le - lf - \tfrac{1}{3}lg - \tfrac{1}{3}lh;$$

cui logarithmo si quaeratur numerus respondens, habebitur valor quaesitus. Inprimis autem inserviunt tabulae logarithmicae dignitatibus atque radicibus intricatissimis inveniendis, quarum operationum loco in logarithmis tantum multiplicatio et divisio adhibetur.

EXEMPLUM 1

Quaeratur valor huius potestatis $2^{\frac{7}{12}}$.

Quoniam eius logarithmus est $= \frac{7}{12} l2$, multiplicetur logarithmus binarii ex tabulis, qui est 0,3010300, per $\frac{7}{12}$ hoc est per $\frac{1}{2}+\frac{1}{12}$; erit $l2^{\frac{7}{12}} = 0{,}1756008$, cui logarithmo respondet numerus 1,498307, qui ergo proxime exhibet valorem $2^{\frac{7}{12}}$

EXEMPLUM 2

Si numerus incolarum cuiuspiam provinciae quotannis sui parte trigesima augeatur, initio autem in provincia habitaverint 100000 hominum, quaeritur post 100 annos incolarum numerus.

Sit brevitatis gratia initio incolarum numerus $= n$, ita ut sit $n = 100000$; anno elapso uno erit $= \left(1+\frac{1}{30}\right)n = \frac{31}{30}n$, incolarum numerus post duos annos $= \left(\frac{31}{30}\right)^2 n$, post tres annos $= \left(\frac{31}{30}\right)^3 n$, hincque post centum annos

$$= \left(\frac{31}{30}\right)^{100} n = \left(\frac{31}{30}\right)^{100} 100000,$$

cuius logarithmus est

$$= 100 l \tfrac{31}{30} + l100000.$$

At est

$$100 l \tfrac{31}{30} = l31 - l30 = 0{,}014240439,$$

unde

$$100 l \tfrac{31}{30} = 1{,}4240439;$$

ad quem si addatur $l100000 = 5$, erit logarithmus numeri incolarum quaesiti

$$= 6{,}4240439,$$

cui respondet numerus

$$= 2654874.$$

Post centum ergo annos numerus incolarum fit plus quam vicies sexies cum semisse maior.

EXEMPLUM 3

Cum post diluvium a sex hominibus genus humanum sit propagatum, si ponamus ducentis annis post numerum hominum iam ad 1000000 excrevisse, quaeritur, quanta sui parte numerus hominum quotannis augeri debuerit.

Ponamus hoc tempore numerum hominum parte sua $\frac{1}{x}$ quotannis increvisse atque post ducentos annos prodierit necesse est numerus hominum

$$=\left(\frac{1+x}{x}\right)^{200} 6 = 1000000,$$

unde fit

$$\frac{1+x}{x} = \left(\frac{1000000}{6}\right)^{\frac{1}{200}}.$$

Erit ergo

$$l\,\frac{1+x}{x} = \frac{1}{200}\,l\,\frac{1000000}{6} = \frac{1}{200}\cdot 5{,}2218487 = 0{,}0261092$$

ideoque

$$\frac{1+x}{x} = \frac{1061963}{1000000} \text{ et } 1000000 = 61963x,$$

unde fit

$$x = 16 \text{ circiter.}$$

Ad tantam ergo hominum multiplicationem suffecisset, si quotannis decima sexta sui parte increverint; quae multiplicatio ob longaevam vitam non nimis magna censeri potest. Quodsi autem eadem ratione per intervallum 400 annorum numerus hominum crescere perrexisset, tum numerus hominum ad

$$1000000\cdot\frac{1000000}{6} = 166666666666$$

ascendere debuisset, quibus sustentandis universus orbis terrarum nequaquam par fuisset.

EXEMPLUM 4

Si singulis seculis numerus hominum duplicetur, quaeritur incrementum annuum.

Si quotannis hominum numerum parte sua $\frac{1}{x}$ crescere ponamus et initio numerus hominum fuert $= n$, erit is post centum annos $= \left(\frac{1+x}{x}\right)^{100} n$; ; qui cum esse debeat $= 2n$, erit

$$\frac{1+x}{x} = 2^{\frac{1}{100}}$$

et

$$l\frac{1+x}{x} = \frac{1}{100}l2 = 0{,}0030103;$$

hinc

$$\frac{1+x}{x} = \frac{10069555}{10000000},$$

ergo

$$x = \frac{10000000}{69555} = 144 \text{ circiter.}$$

Sufficit ergo, si numerus hominum quotannis parte sua $\frac{1}{144}$ augeatur. Quam ob causam maxime ridiculae sunt eorum incredulorum hominum obiectiones, qui negant tam brevi temporis spatia ab uno homine universam terram incolis impleri potuisse.

111. Potissimum autem logarithmorum usus requiritur ad eiusmodi aequationes resolvendas, in quibus quantitas incognita in exponentem ingreditur.
Sic, si ad huiusmodi perveniatur'aequationem

$$a^x = b,$$

ex qua incognitae x valorem erui oporteat, hoc non nisi per logarithmos effici poterit.
Cum enim sit $a^x = b$, erit

$$la^x = xla = lb$$

ideoque

$$x = \frac{lb}{la},$$

ubi quidem perinde est, quonam systemate logarithmico utatur, cum in omni systemate logarithmi numerorum a et b eandem inter se teneant rationem.

EXEMPLUM 1

Si numerus hominum quotannis centesima sui parte augeatur, quaeritur, post quot annos numerus hominum fiat decuplo maior.

Ponamus hoc evenire post x annos et initio hominum numerum fuisse $= n$; erit is ergo elapsis x annis $= \left(\frac{101}{100}\right)^x n$, qui cum aequalis sit $10n$, fiet

$$\left(\frac{101}{100}\right)^x = 10$$

ideoque

$$xl\frac{101}{100} = l10$$

et

$$x = \frac{l10}{l101 - l100}.$$

Prodibit itaque

$$x = \frac{10000000}{43214} = 231 \text{ circiter}.$$

Post annos ergo 231 fiet hominum numerus, quorum incrementum annuum tantum centesimam partem efficit, decuplo maior; hinc post 462 annos fiet centies et post 693 annos millies maior.

EXEMPLUM 2

Quidam debet 400000 florenos hac conditione, ut quotannis usuram 5 de centenis solvere teneatur; exsolvit autem singulis annis 25000 florenos. Quaeritur, post quot annos debitum penitus extinguatur.

Scribamus a pro debita summa 400000 fl. et b pro summa 25000 fl. quotannis soluta; debebit ergo elapso uno anno

$$\frac{105}{100}a - b,$$

elapsis duobus annus

$$\left(\frac{105}{100}\right)^2 a - \frac{105}{100}b - b$$

elapsia tribua annia

$$\left(\frac{105}{100}\right)^3 a - \left(\frac{105}{100}\right)^2 b - \frac{105}{100}b - b;$$

hinc posito brevitatis causa n pro $\frac{105}{100}$ elapsis x annis adhuc debebit

$$n^x a - n^{x-1}b - n^{x-2}b - n^{x-3}b - \cdots - b = n^x a - b\left(1 + n + n^2 + \cdots + n^{x-1}\right).$$

Cum igitur sit ex natura progressionum geometricarum

$$1+n+n^2+\cdots+n^{x-1}=\tfrac{n^x-1}{n-1},$$

post x annos debitor adhuc debebit

$$n^x a-\tfrac{n^x b-b}{n-1}\text{ flor.},$$

quod debitum nihilo aequale positum dabit hanc aequationem

$$n^x a=\tfrac{n^x b-b}{n-1}$$

seu

$$(n-1)n^x a=n^x b-b \text{ ideoque } (b-na+a)n^x=b$$

et

$$n^x=\tfrac{b}{b-na+a}$$

unde fit

$$x=\frac{lb-l\big(b-(n-1)a\big)}{ln}.$$

Cum iam sit

$$a=400000,\ \ b=25000,\ \ n=100,$$

erit

$$(n-1)a=20000 \text{ et } b-(n-1)a=5000$$

atque annorum, quibus debitum penitus extinguitur, numerus

$$x=\tfrac{l25000-l5000}{l\frac{105}{100}}=\tfrac{l5}{l\frac{21}{20}}=\tfrac{6989700}{211893};$$

erit ergo x aliquanto minor quam 33. Scilicet elapsis annis 33 non solum debitum extinguetur, sed creditor debitori reddere tenebitur

$$\frac{(n^{33}-1)b}{n-1}-n^{33}a=\left(\tfrac{21}{20}\right)^{33}\cdot\tfrac{5000-25000}{\frac{1}{20}}=100000\left(\tfrac{21}{20}\right)^{33}-500000\text{ flor.}$$

Quia vero est

$$l\,\tfrac{21}{20}=0{,}0211892991,$$

erit

$$l\left(\tfrac{21}{20}\right)^{33}=0{,}69924687 \text{ et } l100000\left(\tfrac{21}{20}\right)^{33}=5{,}6992469,$$

cui respondet hic numerus 500318,8 ; unde creditor debitori post 33 annos restituere debet $318\frac{4}{5}$ florenos.

112. Logarithmi autem vulgares super basi $=10$ extracti praeter hunc usum, quem logarithmi in genere praestant, in Arithmetica decimali usu recepta singulari gaudent commodo atque ob hanc causam prae aliis systematibus insignem afferunt utilitatem. Cum enim logarithmi omnium numerorum praeter denarii potestates in fractionibus decimalibus exhibeantur, numerorum inter 1 et 10 contentorum logarithmi intra limites 0 et 1, numerorum autem inter 10 et 100 contentorum logarithmi inter limites 1 et 2, et ita porro, continebuntur. Constat ergo logarithmus quisque ex numero integro et fractione decimali et ille numerus integer vocari solet *characteristica,* fractio decimalis autem *mantissa.* Characteristica itaque unitate deficiet a numero notarum, quibus numerus constat; ita logarithmi numeri 78509 characteristica erit 4, quia is ex quinque notis seu figuris constat. Hinc ex logarithmo cuiusvis numeri statim intelligitur, ex quot figuris numerus sit compositus. Sic numerus logarithmo 7,5804631 respondens ex 8 figuris constabit.

113. Si ergo duorum logarithmorum mantissae conveniant, characteristicae vero tantum discrepent, tum numeri his logarithmis respondentes rationem habebunt ut potestas denarii ad unitatem ideoque ratione figurarum, quibus constant, convenient. Ita horum logarithmorum 4,9130187 et 6,9130187 numeri erunt 81850 et 8185000; logarithmo autem 3,9130187 conveniet 8185 et logarithmo huic 0,9130187 convenit 8,185. Sola ergo mantissa indicabit figuras numerum componentes; quibus inventis ex characteristica patebit, quot figurae a sinistra ad integra referri debeant, reliquae ad dextram vero dabunt fractiones decimales. Sic, si hic logarithmus fuerit inventus 2,7603429, mantissa indicabit has figuras 5758945, characteristica 2 autem numerum illi logarithmo determinat, ut sit 575,8945; si characteristica esset 0, foret numerus 5,758945; sin denuo unitate minuatur, ut sit –1, erit numerus respondens decies minor, nempe 0,5758945, et characteristicae –2 respondebit 0,05758945 etc. Loco characteristicarum autem huiusmodi negativarum –1, –2, –3 etc. scribi solent 9, 8, 7 etc. atque subintelligitur hos logarithmos denario minui debere. Haec vero in manductionibus ad tabulas logarithmorum fusius exponi solent.

EXEMPLUM

Si haec progressio 2, 4, 16, 256 etc., cuius quisque terminus est quadratum praecedentis, continuetur usque ad terminum vigesimum quintum, quaeritur magnitudo huius termini ultimi.

Termini huius progressionis per exponentes ita commodius exprimuntur $2^1, \quad 2^2, \quad 2^4, \quad 2^8$ etc.,ubi patet exponentes progressionem geometricam constituere atque termini vigesimi quinti exponentem fore

$$2^{24}=16777216,$$

ita ut ipse terminus quaesitus sit

$$= 2^{16777216};$$

huius ergo logarithmus erit

$$= 16777216 l2.$$

Cum ergo sit

$$l2 = 0{,}30102\ 99956\ 639811952$$

erit, numeri quaesiti logarithmus

$$= 5050445{,}25973367,$$

ex cuius characteristica patet numerum quaesitum more solito expressum constare ex

5050446

figuris. Mantissa autem 259733675932 in tabula logarithmorum quaesita dabit figuras initiales numeri quaesiti, quae erunt 181858. Quanquam ergo iste numerus nullo modo exhiberi potest, tamen affirmari potest eum omnino ex 5050446 figuris constare atque figuras initiales sex esse 181858, quas dextrorsum adhuc 5050440 figurae sequantur, quarum insuper nonnullae ex maiori logarithmorum canone definiri possent; undecim scilicet figurae initiales erunt 18185852986.