# CHAPTER VIII

# ON TRANSCENDING QUANTITIES ARISING FROM THE CIRCLE

126. After logarithms and exponential quantities have been considered, circular arcs and the sines and cosines of these must be considered; not only because they constitute another kind of transcending quantity, but also because of the logarithms and exponentials of these that arise when they are involved with imaginary quantities, which will become clearer below.

Therefore we may put the radius of the circle or the whole sine to be $=1$ and it is clear enough that the periphery of this circle cannot be expressed exactly in rational numbers; but by approximations the semi–circumference of this circle has been found to be

$$\begin{aligned}&=3{,}14159\ 26535\ 89793\ 23846\ 26433\ 83279\ 50288\ 41971\ 69399\ 37510\\ &\quad 58209\ 74944\ 59230\ 78164\ 06286\ 20899\ 86280\ 34825\ 34211\ 70679\\ &\quad 82148\ 08651\ 32823\ 06647\ 09384\ 46\ +,\end{aligned}$$

for which number for the sake of brevity I may write as

$$\pi,$$

thus so that there shall be $\pi =$ semi–circumference of the circle, whose radius $=1$, or $\pi$ will be the length of the arc of 180 degrees.

127. With $z$ denoting the arc of some circle, the radius of which I assume always $=1$, the sines and cosines of this arc $z$ mainly are considered. But the sine of the arc $z$ in the following I will indicate in this manner

$$\sin.\ \mathrm{A}.\ z \text{ or only } \sin. z\ ,$$

truly the cosine in this manner

$$\cos.\ \mathrm{A}.\ z \text{ or only } \cos.\ z\ .$$

Thus, since $\pi\, n$ shall be the $180^0$ arc, there will be

$$\sin.0\pi = 0,\ \ \cos.0\pi = 1$$

and

$$\sin.\tfrac{1}{2}\pi = 1, \qquad \cos.\tfrac{1}{2}\pi = 0,$$
$$\sin.\pi = 0, \qquad \cos.\pi = -1,$$
$$\sin.\tfrac{3}{2}\pi = -1, \qquad \cos.\tfrac{3}{2}\pi = 0,$$
$$\sin.2\pi = 0, \qquad \cos.2\pi = 1.$$

Therefore all the sines and cosines will be contained within the limits $+1$ and $-1$. But again there will be

$$\cos.z = \sin.\left(\tfrac{1}{2}\pi - z\right) \text{ and } \sin.z = \cos.\left(\tfrac{1}{2}\pi - z\right)$$

and

$$\left(\sin.z\right)^2 + \left(\cos.z\right)^2 = 1 .$$

Besides these denominations, these also are well-known :

$$\text{tang}.z ,$$

which denotes the tangent of the arc $z$,

$$\cot.z$$

the cotangent of the arc $z$, and agreed to be

$$\text{tang}.z = \frac{\sin.z}{\cos.z}$$

and

$$\cot.z = \frac{\cos.z}{\sin.z} = \frac{1}{\text{tang}.z} ,$$

which all are well-known from trigonometry.

128. Hence truly also it is agreed, if the two arcs $y$ and $z$ may be had, to become

$$\sin.\left(y+z\right) = \sin.y\cos.z + \cos.y\sin.z$$

and

$$\cos.\left(y+z\right) = \cos.y\cos.z - \sin.y\sin.z$$

and likewise

$$\sin.\left(y-z\right) = \sin.y\cos.z - \cos.y\sin.z$$

and

$$\cos.\left(y-z\right) = \cos.y\cos.z + \sin.y\sin.z$$

Hence in place of $y$ by substituting the arcs $\tfrac{1}{2}\pi,\ \pi,\ \tfrac{3}{2}\pi$ etc. there will be

| | |
|---|---|
| $\sin.\left(\frac{1}{2}\pi+z\right)=+\cos.z$ | $\sin.\left(\frac{1}{2}\pi-z\right)=+\cos.z$ |
| $\cos.\left(\frac{1}{2}\pi+z\right)=-\sin.z$ | $\cos.\left(\frac{1}{2}\pi-z\right)=+\sin.z$ |
| $\sin.\left(\pi+z\right)=-\sin.z$ | $\sin.\left(\pi-z\right)=+\sin.z$ |
| $\cos.\left(\pi+z\right)=-\cos.z$ | $\cos.\left(\pi-z\right)=-\cos.z$ |
| $\sin.\left(\frac{3}{2}\pi+z\right)=-\cos.z$ | $\sin.\left(\frac{3}{2}\pi-z\right)=-\cos.z$ |
| $\cos.\left(\frac{3}{2}\pi+z\right)=+\sin.z$ | $\cos.\left(\frac{3}{2}\pi-z\right)=-\sin.z$ |
| $\sin.\left(2\pi+z\right)=+\sin.z$ | $\sin.\left(2\pi-z\right)=-\sin.z$ |
| $\cos.\left(2\pi+z\right)=+\cos.z$ | $\cos.\left(2\pi-z\right)=+\cos.z$ |

Therefore if $n$ may denote some whole number, there will be

| | |
|---|---|
| $\sin.\left(\frac{4n+1}{2}\pi+z\right)=+\cos.z$ | $\sin.\left(\frac{4n+1}{2}\pi-z\right)=+\cos.z$ |
| $\cos.\left(\frac{4n+1}{2}\pi+z\right)=-\sin.z$ | $\cos.\left(\frac{4n+1}{2}\pi-z\right)=+\sin.z$ |
| $\sin.\left(\frac{4n+2}{2}\pi+z\right)=-\sin.z$ | $\sin.\left(\frac{4n+2}{2}\pi-z\right)=+\sin.z$ |
| $\cos.\left(\frac{4n+2}{2}\pi+z\right)=-\cos.z$ | $\cos.\left(\frac{4n+2}{2}\pi-z\right)=-\cos.z$ |
| $\sin.\left(\frac{4n+3}{2}\pi+z\right)=-\cos.z$ | $\sin.\left(\frac{4n+3}{2}\pi-z\right)=-\cos.z$ |
| $\cos.\left(\frac{4n+3}{2}\pi+z\right)=+\sin.z$ | $\cos.\left(\frac{4n+3}{2}\pi-z\right)=-\sin.z$ |
| $\sin.\left(\frac{4n+4}{2}\pi+z\right)=+\sin.z$ | $\sin.\left(\frac{4n+4}{2}\pi-z\right)=-\sin.z$ |
| $\cos.\left(\frac{4n+4}{2}\pi+z\right)=+\cos.z$ | $\cos.\left(\frac{4n+4}{2}\pi-z\right)=+\cos.z$ |

Which formulas are true, whether $n$ shall be a positive or negative integer.

129. Let

$$\sin.z=p \quad\text{and}\quad \cos.z=q;$$

there will be

$$pp+qq=1;$$

and let

$$\sin.y=m,\quad \cos.y=n,$$

so that also there shall be

$$mm+nn=1;$$

the sine and cosine of the arcs of these thus may themselves be considered:

| | |
|---|---|
| $\sin. z = p$ | $\cos. z = q$ |
| $\sin.(y+z) = mq + np$ | $\cos.(y+z) = nq - mp$ |
| $\sin.(2y+z) = 2mnq + (nn - mm.)p$ | $\cos.(2y+z) = (nn - mm.)q - 2mnp$ |
| $\sin.(3y+z) = (3mn^2 - m^3)q + (n^3 - 3m^2n)p$ | $\cos.(3y+z) = (n^3 - 3m^2n)q - (3mn^2 - m^3)p$ |
| etc. | etc. |

These arcs

$$z, \quad y+z, \quad 2y+z, \quad 3y+z, \quad \text{etc.}$$

proceed in an arithmetic progression, truly both the sine as well as the cosine constitute a recurring progression, such as arises from the denominator

$$1 - 2nx + (mm + nn)xx;$$

for there is

$$\sin.(2y+z) = 2n\sin.(y+z) - (mm+nn)\sin.z$$

or

$$\sin.(2y+z) = 2\cos.y \sin.(y+z) - \sin.z$$

and in a like manner

$$\cos.(2y+z) = 2\cos.y \cos.(y+z) - \cos.z.$$

In the same manner again there will be

$$\sin.(3y+z) = 2\cos.y \sin.(2y+z) - \sin.(y+z)$$

and

$$\cos.(3y+z) = 2\cos.y \cos.(2y+z) - \cos.(y+z)$$

and thus

$$\sin.(4y+z) = 2\cos.y \sin.(3y+z) - \sin.(2y+z)$$

and

$$\cos.(4y+z) = 2\cos.y \cos.(3y+z) - \cos.(2y+z)$$

etc.

The laws of which, with the benefit of the arcs proceeding in an arithmetical progression, both of the sine as well as of the cosine, can be formed in an expedient manner, as far as it pleases.

130. Since there shall be

$$\sin.(y+z)=\sin.y\cos.z+\cos.y\sin.z$$

and

$$\sin.(y-z)=\sin.y\cos.z-\cos.y\sin.z,$$

from these expressions either added or subtracted there will be

$$\sin.y\cos.z=\frac{\sin.(y+z)+\sin.(y-z)}{2},$$
$$\cos.y\sin.z=\frac{\sin.(y+z)-\sin.(y-z)}{2}.$$

Because again there will be

$$\cos.(y+z)=\cos.y\cos.z-\sin.y\sin.z$$

and

$$\cos.(y-z)=\cos.y\cos.z+\sin.y\sin.z,$$

in a equal manner,

$$\cos.y\cos.z=\frac{\cos.(y-z)+\cos.(y+z)}{2},$$
$$\sin.y\sin.z=\frac{\cos.(y-z)-\cos.(y+z)}{2}.$$

Let

$$y=z=\tfrac{1}{2}v;$$

from these last formulas there will be

$$\left(\cos.\tfrac{1}{2}v\right)^2=\frac{1+\cos.v}{2}\text{ and }\cos.\tfrac{1}{2}v=\sqrt{\frac{1+\cos.v}{2}},$$
$$\left(\sin.\tfrac{1}{2}v\right)^2=\frac{1-\cos.v}{2}\text{ and }\sin.\tfrac{1}{2}v=\sqrt{\frac{1-\cos.v}{2}},$$

from which from a given cosine the sine and cosine of each half angle are found.

131. The arc may be put in place

$$y+z=a,\quad\text{and}\quad y-z=b;$$

there will be

$$y=\frac{a+b}{2}\quad\text{et}\quad z=\frac{a-b}{2},$$

with which substituted into the above equations, these equations will be obtained, as just as many theorems :

$$\sin.a + \sin.b = 2\sin.\tfrac{a+b}{2}\cos.\tfrac{a-b}{2},$$
$$\sin.a - \sin.b = 2\cos.\tfrac{a+b}{2}\sin.\tfrac{a-b}{2},$$
$$\cos.a + \cos.b = 2\cos.\tfrac{a+b}{2}\cos.\tfrac{a-b}{2},$$
$$\cos.a - \cos.b = 2\sin.\tfrac{a+b}{2}\sin.\tfrac{a-b}{2}.$$

From these again with the help of division these theorems arise :

$$\frac{\sin.a+\sin.b}{\sin.a-\sin.b} = \text{tang.}\tfrac{a+b}{2}\cot.\tfrac{a-b}{2} = \frac{\text{tang.}\frac{a+b}{2}}{\text{tang.}\frac{a-b}{2}},$$
$$\frac{\sin.a+\sin.b}{\cos.a+\cos.b} = \text{tang.}\tfrac{a+b}{2},$$
$$\frac{\sin.a+\sin.b}{\cos.a-\cos.b} = \cot.\tfrac{a-b}{2},$$
$$\frac{\sin.a-\sin.b}{\cos.a+\cos.b} = \text{tang.}\tfrac{a-b}{2},$$
$$\frac{\sin.a-\sin.b}{\cos.a-\cos.b} = \cot.\tfrac{a+b}{2},$$
$$\frac{\cos.a+\cos.b}{\cos.b-\cos.a} = \cot.\tfrac{a+b}{2}\cot.\tfrac{a-b}{2}.$$

From these finally these theorems are deduced :

$$\frac{\sin.a+\sin.b}{\sin.a-\sin.b}\times\frac{\cos.a+\cos.b}{\cos.b-\cos.a} = \left(\cot.\tfrac{a-b}{2}\right)^2,$$
$$\frac{\sin.a+\sin.b}{\sin.a-\sin.b}\times\frac{\cos.b-\cos.a}{\cos.a+\cos.b} = \left(\text{tang.}\tfrac{a+b}{2}\right)^2.$$

132. Since there shall be

$$(\sin.z)^2 + (\cos.z)^2 = 1,$$

with the factors taken there will be

$$\left(\cos.z + \sqrt{-1}\cdot\sin.z\right)\left(\cos.z - \sqrt{-1}\cdot\sin.z\right) = 1,$$

which factors, even if imaginary, still perform a huge task in the combinations and in the multiplications of arcs. For the product of these factors may be sought :

$$\left(\cos.z + \sqrt{-1}\cdot\sin.z\right)\left(\cos.y + \sqrt{-1}\cdot\sin.y\right)$$

and there will be found :

$$\cos.y\cos.z - \sin.y\sin.z + \sqrt{-1}\cdot(\cos.y\sin.z + \sin.y\cos.z).$$

But since there shall be

$$\cos.y\cos.z - \sin.y\sin.z = \cos.(y+z)$$

and

$$\cos.y\sin.z + \sin.y\cos.z = \sin.(y+z),$$

there will be this product

$$\left(\cos.y + \sqrt{-1}\cdot\sin.y\right)\left(\cos.z + \sqrt{-1}\cdot\sin.z\right) = \cos.(y+z) + \sqrt{-1}\cdot\sin.(y+z)$$

and in a similar manner

$$\left(\cos.y - \sqrt{-1}\cdot\sin.y\right)\left(\cos.z - \sqrt{-1}\cdot\sin.z\right) = \cos.(y+z) - \sqrt{-1}\cdot\sin.(y+z),$$

likewise

$$\left(\cos.x \pm \sqrt{-1}\cdot\sin.x\right)\left(\cos.y \pm \sqrt{-1}\cdot\sin.y\right)\left(\cos.z \pm \sqrt{-1}\cdot\sin.z\right) = \cos.(x+y+z) \pm \sqrt{-1}\cdot\sin.(x+y+z).$$

133. Hence therefore it follows :

$$\left(\cos.z \pm \sqrt{-1}\cdot\sin.z\right)^2 = \cos.2z \pm \sqrt{-1}\cdot\sin.2z$$

$$\left(\cos.z \pm \sqrt{-1}\cdot\sin.z\right)^3 = \cos.3z \pm \sqrt{-1}\cdot\sin.3z$$

and thus generally there will be

$$\left(\cos.z \pm \sqrt{-1}\cdot\sin.z\right)^n = \cos.nz \pm \sqrt{-1}\cdot\sin.nz .$$

From which on account of the ambiguity of the signs there will be

$$\cos.nz = \frac{\left(\cos.z+\sqrt{-1}\cdot\sin.z\right)^n + \left(\cos.z-\sqrt{-1}\cdot\sin.z\right)^n}{2}$$

and

$$\sin.nz = \frac{\left(\cos.z+\sqrt{-1}\cdot\sin.z\right)^n - \left(\cos.z-\sqrt{-1}\cdot\sin.z\right)^n}{2\sqrt{-1}}.$$

Therefore with these binomials developed in series, there will be

$$\cos.nz = (\cos.z)^n - \frac{n(n-1)}{1\cdot2}(\cos.z)^{n-2}(\sin.z)^2 + \frac{n(n-1)(n-2)(n-3)}{1\cdot2\cdot3\cdot4}(\cos.z)^{n-4}(\sin.z)^4$$
$$-\frac{n(n-1)(n-2)(n-3)(n-4)(n-5)}{1\cdot2\cdot3\cdot4\cdot5\cdot6}(\cos.z)^{n-6}(\sin.z)^6 + \text{etc.}$$

and

$$\sin.nz = \tfrac{n}{1}(\cos.z)^{n-1}\sin.z - \tfrac{n(n-1)(n-2)}{1\cdot 2\cdot 3}(\cos.z)^{n-3}(\sin.z)^{3} + \tfrac{n(n-1)(n-2)(n-3)(n-4)}{1\cdot 2\cdot 3\cdot 4\cdot 5}(\cos.z)^{n-5}(\sin.z)^{5} - \text{ etc.}$$

134. Let the arc $z$ be infinitely small ; there will be $\sin.z = z$ and $\cos.z = 1$; moreover $n$ shall be an infinitely great number, so that the arc $nz$ shall be of finite magnitude, for example $nz = v$; on account of $\sin.z = z = \frac{v}{n}$ there becomes

$$\cos.v = 1 - \tfrac{v^2}{1\cdot 2} + \tfrac{v^4}{1\cdot 2\cdot 3\cdot 4} - \tfrac{v^6}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6} + \text{ etc.}$$

and

$$\sin.v = v - \tfrac{v^3}{1\cdot 2\cdot 3} + \tfrac{v^5}{1\cdot 2\cdot 3\cdot 4\cdot 5} - \tfrac{v^7}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7} + \text{ etc.}$$

Therefore from the given arc $v$ with the help of which series the sine and the cosine will be able to be found ; so that the use of which formulas may become more apparent, we may put the arc $v$ to be to the quadrant or 90° as $m$ to $n$ or for there to be $v = \frac{m}{n}\cdot\frac{\pi}{2}$. Because now the value of $\pi$ is agreed upon, if this is substituted everywhere, the equation will be produced

$$\sin.A.\tfrac{m}{n}90^0 =$$
$$+\tfrac{m}{n}\cdot\ 1{,}57079\ 63267\ 94896\ 61923\ 13216\ 916$$
$$-\tfrac{m^3}{n^3}\cdot 0{,}64596\ 40975\ 06246\ 25365\ 57565\ 639$$
$$+\tfrac{m^5}{n^5}\cdot 0{,}07969\ 26262\ 46167\ 04512\ 05055\ 495$$
$$-\tfrac{m^7}{n^7}\cdot 0{,}00468\ 17541\ 35318\ 68810\ 06854\ 639$$
$$+\tfrac{m^9}{n^9}\cdot 0{,}00016\ 04411\ 84787\ 35982\ 18726\ 609$$
$$-\tfrac{m^{11}}{n^{11}}\cdot 0{,}00000\ 35988\ 43235\ 21208\ 53404\ 585$$
$$+\tfrac{m^{13}}{n^{13}}\cdot 0{,}00000\ 00569\ 21729\ 21967\ 92681\ 178$$
$$-\tfrac{m^{15}}{n^{15}}\cdot 0{,}00000\ 00006\ 68803\ 51098\ 11467\ 232$$
$$+\tfrac{m^{17}}{n^{17}}\cdot 0{,}00000\ 00000\ 06066\ 93573\ 11061\ 957$$
$$-\tfrac{m^{19}}{n^{19}}\cdot 0{,}00000\ 00000\ 00043\ 77065\ 46731\ 374$$
$$+\tfrac{m^{21}}{n^{21}}\cdot 0{,}00000\ 00000\ 00000\ 25714\ 22892\ 860$$
$$-\tfrac{m^{23}}{n^{23}}\cdot 0{,}00000\ 00000\ 00000\ 00125\ 38995\ 405$$
$$+\tfrac{m^{25}}{n^{25}}\cdot 0{,}00000\ 00000\ 00000\ 00000\ 51564\ 552$$
$$-\tfrac{m^{27}}{n^{27}}\cdot 0{,}00000\ 00000\ 00000\ 00000\ 00181\ 240$$
$$+\tfrac{m^{29}}{n^{29}}\cdot 0{,}00000\ 00000\ 00000\ 00000\ 00000\ 551$$

[The last three places shown here in each power have been corrected from an error in the original work.]

and

$$\cos.A.\tfrac{m}{n}90^0 =$$
$$+ \quad 1{,}00000\ 00000\ 00000\ 00000\ 00000\ 000$$
$$-\frac{m^2}{n^2}\cdot 1{,}23370\ 05501\ 36169\ 82735\ 43113\ 750$$
$$+\frac{m^4}{n^4}\cdot 0{,}25366\ 95079\ 01048\ 01363\ 65633\ 664$$
$$-\frac{m^6}{n^6}\cdot 0{,}02086\ 34807\ 63352\ 96087\ 30516\ 372$$
$$+\frac{m^8}{n^8}\cdot 0{,}00091\ 92602\ 74839\ 42658\ 02417\ 162$$
$$-\frac{m^{10}}{n^{10}}\cdot 0{,}00002\ 52020\ 42373\ 06060\ 54810\ 530$$
$$+\frac{m^{12}}{n^{12}}\cdot 0{,}00000\ 04710\ 87477\ 88181\ 71503\ 670$$
$$-\frac{m^{14}}{n^{14}}\cdot 0{,}00000\ 00063\ 86603\ 08379\ 18522\ 411$$
$$+\frac{m^{16}}{n^{16}}\cdot 0{,}00000\ 00000\ 65659\ 63114\ 97947\ 236$$
$$-\frac{m^{18}}{n^{18}}\cdot 0{,}00000\ 00000\ 00529\ 44002\ 00734\ 624$$
$$+\frac{m^{20}}{n^{20}}\cdot 0{,}00000\ 00000\ 00003\ 43773\ 91790\ 986$$
$$-\frac{m^{22}}{n^{22}}\cdot 0{,}00000\ 00000\ 00000\ 01835\ 99165\ 216$$
$$+\frac{m^{24}}{n^{24}}\cdot 0{,}00000\ 00000\ 00000\ 00008\ 20675\ 330$$
$$-\frac{m^{26}}{n^{26}}\cdot 0{,}00000\ 00000\ 00000\ 00000\ 03115\ 285$$
$$+\frac{m^{28}}{n^{28}}\cdot 0{,}00000\ 00000\ 00000\ 00000\ 00010\ 168$$
$$-\frac{m^{30}}{n^{30}}\cdot 0{,}00000\ 00000\ 00000\ 00000\ 00000\ 029.$$

[The last three places in each power are in error in the original .]

Therefore since it is sufficient to know the sines and cosines of angles to $45^0$, the fraction $\frac{m}{n}$ always will be less than $\frac{1}{2}$ and hence also from the powers of the fraction $\frac{m}{n}$ the series shown will be especially convergent, thus so that only some number of figures more may suffice, particularly if the sine and cosine may not require too many figures.

135. With the sines and cosines found the tangents and cotangents indeed are able to be found by the customary analogies ; but because multiplication and division is such an inconvenience in the generation of numbers of this kind, it is convenient to express these in a particular way. Therefore there will be

$$\text{tang.}v = \frac{\sin.v}{\cos.v} = \frac{v - \frac{v^3}{1\cdot2\cdot3} + \frac{v^5}{1\cdot2\cdot3\cdot4\cdot5} - \frac{v^7}{1\cdot2\cdot3\cdots7} + \text{etc.}}{1 - \frac{v^2}{1\cdot2} + \frac{v^4}{1\cdot2\cdot3\cdot4} - \frac{v^6}{1\cdot2\cdot3\cdots6} + \text{etc.}}$$

and

$$\cot.v = \frac{\cos.v}{\sin.v} = \frac{1 - \frac{v^2}{1\cdot2} + \frac{v^4}{1\cdot2\cdot3\cdot4} - \frac{v^6}{1\cdot2\cdot3\cdots6} + \text{etc.}}{v - \frac{v^3}{1\cdot2\cdot3} + \frac{v^5}{1\cdot2\cdot3\cdot4\cdot5} - \frac{v^7}{1\cdot2\cdot3\cdots7} + \text{etc.}}.$$

Now if the arc shall be $v = \frac{m}{n}90^0$, in the same manner as before there will be

| $\text{tang. A.}\frac{m}{n}90^0$ | $\cot.\text{A.}\frac{m}{n}90^0$ |
|---|---|
| $= +\frac{2mn}{nn-mm}\cdot 0{,}63661\,97723\,676$ | $= +\frac{n}{m}\cdot 0{,}63661\,97723\,676$ |
| $+\frac{m}{n}\cdot 0{,}29755\,67820\,597$ | $-\frac{4mn}{4mn-mm}\cdot 0{,}31830\,98861\,838$ |
| $+\frac{m^3}{n^3}\cdot 0{,}01868\,86502\,773$ | $-\frac{m}{n}\cdot 0{,}20528\,88894\,145$ |
| $+\frac{m^5}{n^5}\cdot 0{,}00184\,24752\,034$ | $-\frac{m^3}{n^3}\cdot 0{,}00655\,10747\,882$ |
| $+\frac{m^7}{n^7}\cdot 0{,}00019\,75800\,715$ | $-\frac{m^5}{n^5}\cdot 0{,}00034\,50292\,554$ |
| $+\frac{m^9}{n^9}\cdot 0{,}00002\,16977\,373$ | $-\frac{m^7}{n^7}\cdot 0{,}00002\,02791\,061$ |
| $+\frac{m^{11}}{n^{11}}\cdot 0{,}00000\,24011\,370$ | $-\frac{m^9}{n^9}\cdot 0{,}00000\,12366\,527$ |
| $+\frac{m^{13}}{n^{13}}\cdot 0{,}00000\,02664\,133$ | $-\frac{m^{11}}{n^{11}}\cdot 0{,}00000\,00764\,959$ |
| $+\frac{m^{15}}{n^{15}}\cdot 0{,}00000\,00295\,865$ | $-\frac{n^{13}}{m^{13}}\cdot 0{,}00000\,00047\,597$ |
| $+\frac{m^{17}}{n^{17}}\cdot 0{,}00000\,00032\,868$ | $-\frac{m^{15}}{n^{15}}\cdot 0{,}00000\,00002\,969$ |
| $+\frac{m^{19}}{n^{19}}\cdot 0{,}00000\,00003\,652$ | $-\frac{m^{17}}{n^{17}}\cdot 0{,}00000\,00000\,185$ |
| $+\frac{m^{21}}{n^{21}}\cdot 0{,}00000\,00000\,406$ | $-\frac{m^{19}}{n^{19}}\cdot 0{,}00000\,00000\,012$ |
| $+\frac{m^{23}}{n^{23}}\cdot 0{,}00000\,00000\,045$ | |
| $+\frac{m^{25}}{n^{25}}\cdot 0{,}00000\,00000\,005$ | |

of which the account of the series will be set out further below.

136. Certainly from the above it is agreed, if the sines and cosines of all the angles less than half a right angle were known, thence likewise the sines and cosines of all the greater angles are to be had. Truly if only the sines and cosines of the angles less than 30° were had, from these by addition and subtraction alone the sines and cosines of all the greater angles are possible to be found. For since there shall be

$$\sin.30^0 = \tfrac{1}{2}$$

on putting $y = 30^0$ from §130

$$\cos.z = \sin.(30+z) + \sin.(30-z)$$

and

$$\sin.z = \cos.(30-z) - \cos.(30+z)$$

and thus from the sines and cosines of the angles $z$ and $30-z$ there are found

$$\sin.(30+z) = \cos.z - \sin.(30-z)$$

and

$$\cos.(30+z) = \cos.(30-z) - \sin.z\,,$$

from which the sine and the cosine of the angles from $30^0$ to $60^0$ and hence all the greater angles are defined.

137. With tangents and cotangents a similar aid in use comes along. For since there shall be

$$\tan.(a+b) = \tfrac{\text{tang.}\,a + \text{tang.}\,b}{1-\text{tang.}a\ \text{tang.}b}\,,$$

there will be

$$\text{tang.}2a = \tfrac{2\text{tang.}a}{1-\text{tang.}a\ \text{tang.}a} \quad \text{and} \quad \cot.2a = \tfrac{\cot.a - \text{tang.}a}{2}\,,$$

from which the tangents and cotangents of the arcs less than $30^0$ are found and the cotangents as far as to $60^0$.

Now let $a = 30-b$; there becomes $2a = 60-2b$ and $\cot.2a = \text{tang.}(30+2b)$; therefore there will be

$$\text{tang.}(30+2b) = \tfrac{\cot.(30-b) - \text{tang.}(30-b)}{2}\,,$$

from which also the tangents of arcs greater than 30° are obtained.

But secants and cosecants are found from tangents by subtraction alone ; indeed there is

$$\text{cosec.}z = \cot.\tfrac{1}{2}z - \cot.z$$

and hence

$$\sec.z = \cot.\left(45^0 - \tfrac{1}{2}z\right) - \text{tang.}z.$$

Therefore it is seen clearly enough from these, how a canon of sines will be constructed.

138. An infinitely small arc $z$ may be put anew in the formulas §133 and $n$ shall be an infinitely large number $i$, so that $iz$ may maintain a finite value $v$. Therefore there will be

$nz = v$ and $z = \frac{v}{i}$, from which $\sin.z = \frac{v}{i}$ and $\cos.z = 1$; with these put in place there becomes

$$\cos.v = \frac{\left(1+\frac{v\sqrt{-1}}{i}\right)^i + \left(1-\frac{v\sqrt{-1}}{i}\right)^i}{2}$$

and

$$\sin.v = \frac{\left(1+\frac{v\sqrt{-1}}{i}\right)^i - \left(1-\frac{v\sqrt{-1}}{i}\right)^i}{2\sqrt{-1}}$$

But in the preceding chapter we have seen that

$$\left(1+\frac{z\sqrt{-1}}{i}\right)^i = e^z$$

with $e$ denoting the base of hyperbolic logarithms ; therefore for $z$ in one part I write $+v\sqrt{-1}$ ,and for the other part $-v\sqrt{-1}$ , and there becomes

$$\cos.v = \frac{e^{+v\sqrt{-1}} + e^{-v\sqrt{-1}}}{2}$$

and

$$\sin.v = \frac{e^{+v\sqrt{-1}} - e^{-v\sqrt{-1}}}{2\sqrt{-1}}$$

From which it is understood, how imaginary exponential quantities may be reduced to the sine and cosine of real arcs. Truly there will be

$$e^{+v\sqrt{-1}} = \cos.v + \sqrt{-1}\cdot\sin.v$$

and

$$e^{+v\sqrt{-1}} = \cos.v - \sqrt{-1}\cdot\sin.v$$

139. Now in the same formulas § 133 $n$ shall be an infinitely small number or $n=\frac{1}{i}$ with $i$ being an infinitely great number ; the equations become

$$\cos.nz=\cos.\tfrac{z}{i}=1 \text{ and } \sin.nz=\sin.\tfrac{z}{i}=\tfrac{z}{i};$$

for with the arc vanishing $\frac{z}{i}$ is equal to the sine itself, truly the cosine $=1$. With these in place there is had

$$1=\frac{\left(\cos.z+\sqrt{-1}\cdot\sin.z\right)^{\frac{1}{i}}+\left(\cos.z-\sqrt{-1}\cdot\sin.z\right)^{\frac{1}{i}}}{2}$$

and

$$\frac{z}{i}=\frac{\left(\cos.z+\sqrt{-1}\cdot\sin.z\right)^{\frac{1}{i}}-\left(\cos.z-\sqrt{-1}\cdot\sin.z\right)^{\frac{1}{i}}}{2\sqrt{-1}}.$$

Moreover with the hyperbolic logarithms obtained above (§125) we have shown that

$$l(1+x)=i(1+x)^{\frac{1}{i}}-1 \text{ or } y^{\frac{1}{i}}=1+\tfrac{1}{i}ly$$

on putting $y$ in place of $1+x$. Now therefore on putting $y$ in place of $\cos.z+\sqrt{-1}\cdot\sin.z$ for the one part and $\cos.z-\sqrt{-1}\cdot\sin.z$ for the other part, it will produce

$$1=\frac{1+\frac{1}{i}l\left(\cos.z+\sqrt{-1}\cdot\sin.z\right)1+\frac{1}{i}l\left(\cos.z-\sqrt{-1}\cdot\sin.z\right)}{2}=1$$

on account of the vanishing logarithms, thus so that nothing can be concluded. Truly the other equation for the sine may be put in place

$$\frac{z}{i}=\frac{\frac{1}{i}l\left(\cos.z+\sqrt{-1}\cdot\sin.z\right)-\frac{1}{i}l\left(\cos.z-\sqrt{-1}\cdot\sin.z\right)}{2\sqrt{-1}}$$

and thus

$$z=\frac{1}{2\sqrt{-1}}\,l\,\frac{\cos.z+\sqrt{-1}\cdot\sin.z}{\cos.z-\sqrt{-1}\cdot\sin.z},$$

from which it is apparent, in what way imaginary logarithms relate back to circular arcs.

140. Since there shall be $\frac{\sin.z}{\cos.z} = \text{tang}.z$ ,the arc $z$ may be expressed by its tangent, so that there shall be

$$z = \frac{1}{2\sqrt{-1}} l \frac{1+\sqrt{-1}\cdot tang.z}{1-\sqrt{-1}\cdot tang.z}.$$

Truly above (§ 123) we have seen that

$$l\frac{1+x}{1-x} = \frac{2x}{1} + \frac{2x^2}{3} + \frac{2x^5}{5} + \frac{2x^7}{7} + \text{etc.}$$

Therefore on putting $z = \sqrt{-1} \cdot \text{tang}.z$ there becomes

$$z = \frac{\text{tang}.z}{1} - \frac{(\text{tang}.z)^3}{3} + \frac{(\text{tang}.z)^5}{5} - \frac{(\text{tang}.z)^7}{7} + \text{etc.}$$

Therefore if we may put $\text{tang}.z = t$ , so that the arc shall be $z$, of which the tangent is $t$, that we will indicate thus A.tang.$t$ , and thus there will be

$$z = \text{A.tang}.t.$$

Therefore with the tangent $t$ known the corresponding arc

$$z = \frac{t}{1} - \frac{t^3}{3} + \frac{t^5}{5} - \frac{t^7}{7} + \frac{t^9}{9} - \text{etc.}$$

Since therefore,

if the tangent $t$ is equal to the radius 1, the arc becomes $z =$ to the arc $45^0$ or $z = \frac{\pi}{4}$, there will be

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \text{etc.},,$$

which is the series first produced by Leibnitz for the value being expressed for the periphery of the circle.
[Usually called Gregory's series, which predates the derivation of Leibnitz.]

141. But so that the length of a circular arc can be defined readily from a series of this kind, it is evident that for the tangent $t$ a small enough fraction must be substituted. Thus with the aid of this series the length of the arc $z$ will be found, the tangent $t$ of which is equal to $\frac{1}{10}$ ; for this arc becomes

$$z = \frac{1}{10} - \frac{1}{3000} + \frac{1}{500000} - \text{etc.},$$

the value of which series may be shown as a decimal fraction approximately without difficulty. But truly from such a known arc nothing will be able to be concluded about the length of the whole periphery, since the ratio shall not be assignable, that the arc of which the tangent is $=\frac{1}{10}$ , may hold to the whole periphery. Hence on this account towards finding the periphery of this kind an arc must be sought, which shall be at the same time some part of the periphery and the tangent small enough to be able to be expressed conveniently. Therefore according to this it is customary to take the arc of 30°, the tangent of which is $=\frac{1}{\sqrt{3}}$ because the tangents of smaller arcs commensurable with the periphery become exceedingly irrational. Whereby on account of the arc of $30°=\frac{\pi}{6}$ there will be

$$\frac{\pi}{6}=\frac{1}{\sqrt{3}}-\frac{1}{3\cdot 3\sqrt{3}}+\frac{1}{5\cdot 3^2\sqrt{3}}-\text{etc.}$$

and

$$\pi=\frac{2\sqrt{3}}{1}-\frac{2\sqrt{3}}{3\cdot 3}+\frac{2\sqrt{3}}{5\cdot 3^2}-\frac{2\sqrt{3}}{7\cdot 3^3}+\text{etc.},$$

with the aid of which series, the value of $\pi$ would be determined by incredible labour.

142. But here the labour with that is greater, because in the first place the individual terms shall be irrational, then truly any term is only about a third less than the preceding. A remedy to this inconvenience can arise. The arc of $45^0$ or $\frac{\pi}{4}$ may be taken; the value of which itself can be expressed by the scarcely converging series

$$1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\text{etc.},$$

yet this may be retained and separated into two arcs *a* and *b*, so that there shall be $a+b=\frac{\pi}{4}=45°$ . Therefore since there shall be

$$\text{tan.}(a+b)=1=\frac{\text{tang.}a+\text{tang.}b}{1-\text{tang.}a\ \text{tang.}b},$$

the equation becomes

$$1-\text{tang.}a\ \text{tang.}b=\text{tang.}a+\text{tang.}b$$

and

$$\text{tang.}b=\frac{1-\text{tang.}a}{1+\text{tang.}a}.$$

Now let $\text{tang.}a=\frac{1}{2}$; then $\text{tang.}b=\frac{1}{3}$; hence each of the arcs *a* and *b* may be expressed by rational series converging much more quickly than above, and the sum of these will give the value of the arc $\frac{\pi}{4}$ ; and hence thus it will be

$$\pi = 4\cdot\left(\begin{array}{c}\frac{1}{1\cdot 2}-\frac{1}{3\cdot 2^3}+\frac{1}{5\cdot 2^5}-\frac{1}{7\cdot 2^7}+\frac{1}{9\cdot 2^9}-\text{ etc.}\\ +\\ \frac{1}{1\cdot 3}-\frac{1}{3\cdot 3^3}+\frac{1}{5\cdot 3^5}-\frac{1}{7\cdot 3^7}+\frac{1}{9\cdot 3^9}-\text{ etc.}\end{array}\right)$$

Therefore in this manner the length of the semi–circumference $\pi$ would be able to be found, as indeed it has been made with the aid of the series mentioned before.

## CAPUT VIII

# DE QUANTITATIBUS TRANSCENDENTIBUS EX CIRCULO ORTIS

126. Post logarithmos et quantitates exponentiales considerari debent arcus circulares eorumque sinus et cosinus, quia non solum aliud quantitatum transcendentium genus constituunt, sed etiam ex ipsis logarithmis et exponentialibus, quando imaginariis quantitatibus involvuntur, proveniunt, id quod infra clarius patebit.

Ponamus ergo radium circuli seu sinum totum esse $=1$ atque satis liquet peripheriam huius circuli in numeris rationalibus exacte exprimi non posse; per approximationes autem inventa est semicircumferentia huius circuli esse

$$\begin{aligned}=3{,}14159\,26535\,89793\,23846\,26433\,83279\,50288\,41971\,69399\,37510\\ 58209\,74944\,59230\,78164\,06286\,20899\,86280\,34825\,34211\,70679\\ 82148\,08651\,32823\,06647\,09384\,46+,\end{aligned}$$

pro quo numero brevitatis ergo scribam

$$\pi,$$

ita ut sit $\pi =$ semicircumferentiae circuli, cuius radius $=1$, seu $\pi$ erit longitudo arcus 180 graduum.

127. Denotante $z$ arcum huius circuli quemcunque, cuius radium perpetuo assumo $=1$, huius arcus $z$ considerari potissimum solent sinus et cosinus. Sinum autem arcus $z$ in posterum hoc modo indicabo

$$\sin. \text{A}. z \text{ seu tantum } \sin.z,$$

cosinum vero hoc modo

$$\cos. \text{A}. z \text{ seu tantum } \cos. z.$$

Ita, cum $n$ sit arcus 180°, erit

$$\sin.0\pi = 0,\quad \cos.0\pi = 1$$

et

$$\begin{aligned}&\sin.\tfrac{1}{2}\pi = 1, &&\cos.\tfrac{1}{2}\pi = 0,\\ &\sin.\pi = 0, &&\cos.\pi = -1,\\ &\sin.\tfrac{3}{2}\pi = -1, &&\cos.\tfrac{3}{2}\pi = 0,\\ &\sin.2\pi = 0, &&\cos.2\pi = 1.\end{aligned}$$

Omnes ergo sinus et cosinus intra limites $+1$ et $-1$ continentur. Erit autem porro

$$\cos.z = \sin.\left(\tfrac{1}{2}\pi - z\right) \text{ et } \sin.z = \cos.\left(\tfrac{1}{2}\pi - z\right)$$

atque

$$(\sin.z)^2 + (\cos.z)^2 = 1 .$$

Praeter has denominationes notandae sunt quoque hae:

$$\text{tang}.z ,$$

quae denotat tangentem arcus $z$,

$$\cot.z$$

cotangentem arcus $z$, constatque esse

$$\text{tang}.z = \frac{\sin.z}{\cos.z}$$

et

$$\cot.z = \frac{\cos.z}{\sin.z} = \frac{1}{\text{tang}.z} ,$$

quae omnia ex trigonometria sunt nota.

128. Hinc vero etiam constat, si habeantur duo arcus $y$ et $z$, fore

$$\sin.(y+z) = \sin.y \cos.z + \cos.y \sin.z$$

et

$$\cos.(y+z) = \cos.y \cos.z - \sin.y \sin.z$$

itemque

$$\sin.(y-z) = \sin.y \cos.z - \cos.y \sin.z$$

et

$$\cos.(y-z) = \cos.y \cos.z + \sin.y \sin.z$$

Hinc loco $y$ substituendo arcus $\frac{1}{2}\pi$, $\pi$, $\frac{3}{2}\pi$ etc. erit

| | |
|---|---|
| $\sin.\left(\tfrac{1}{2}\pi + z\right) = +\cos.z$ | $\sin.\left(\tfrac{1}{2}\pi - z\right) = +\cos.z$ |
| $\cos.\left(\tfrac{1}{2}\pi + z\right) = -\sin.z$ | $\cos.\left(\tfrac{1}{2}\pi - z\right) = +\sin.z$ |
| $\sin.(\pi + z) = -\sin.z$ | $\sin.(\pi - z) = +\sin.z$ |
| $\cos.(\pi + z) = -\cos.z$ | $\cos.(\pi - z) = -\cos.z$ |
| $\sin.\left(\tfrac{3}{2}\pi + z\right) = -\cos.z$ | $\sin.\left(\tfrac{3}{2}\pi - z\right) = -\cos.z$ |
| $\cos.\left(\tfrac{3}{2}\pi + z\right) = +\sin.z$ | $\cos.\left(\tfrac{3}{2}\pi - z\right) = -\sin.z$ |
| $\sin.(2\pi + z) = +\sin.z$ | $\sin.(2\pi - z) = -\sin.z$ |

| | |
|---|---|
| $\cos.(2\pi+z)=+\cos.z$ | $\cos.(2\pi-z)=+\cos.z$ |

Si ergo $n$ denotet numerum integrum quemcumque, erit

| | |
|---|---|
| $\sin.\left(\frac{4n+1}{2}\pi+z\right)=+\cos.z$ | $\sin.\left(\frac{4n+1}{2}\pi-z\right)=+\cos.z$ |
| $\cos.\left(\frac{4n+1}{2}\pi+z\right)=-\sin.z$ | $\cos.\left(\frac{4n+1}{2}\pi-z\right)=+\sin.z$ |
| $\sin.\left(\frac{4n+2}{2}\pi+z\right)=-\sin.z$ | $\sin.\left(\frac{4n+2}{2}\pi-z\right)=+\sin.z$ |
| $\cos.\left(\frac{4n+2}{2}\pi+z\right)=-\cos.z$ | $\cos.\left(\frac{4n+2}{2}\pi-z\right)=-\cos.z$ |
| $\sin.\left(\frac{4n+3}{2}\pi+z\right)=-\cos.z$ | $\sin.\left(\frac{4n+3}{2}\pi-z\right)=-\cos.z$ |
| $\cos.\left(\frac{4n+3}{2}\pi+z\right)=+\sin.z$ | $\cos.\left(\frac{4n+3}{2}\pi-z\right)=-\sin.z$ |
| $\sin.\left(\frac{4n+4}{2}\pi+z\right)=+\sin.z$ | $\sin.\left(\frac{4n+4}{2}\pi-z\right)=-\sin.z$ |
| $\cos.\left(\frac{4n+4}{2}\pi+z\right)=+\cos.z$ | $\cos.\left(\frac{4n+4}{2}\pi-z\right)=+\cos.z$ |

Quae formulae verae sunt, sive $n$ sit numerus affirmativus sive negativus integer.

129. Sit

$$\sin.z=p \text{ et } \cos.z=q;$$

erit

$$pp+qq=1;$$

et

$$\sin.y=m, \quad \cos.y=n,$$

ut sit quoque

$$mm+nn=1;$$

arcuum ex his compositorum sinus et cosinus ita se habebunt:

| | |
|---|---|
| $\sin.z=p$ | $\cos.z=q$ |
| $\sin.(y+z)=mq+np$ | $\cos.(y+z)=nq-mp$ |
| $\sin.(2y+z)=2mnq+(nn-mm.)p$ | $\cos.(2y+z)=(nn-mm.)q-2mnp$ |
| $\sin.(3y+z)=(3mn^2-m^3)q+(n^3-3m^2n)p$ | $\cos.(3y+z)=(n^3-3m^2n)q-(3mn^2-m^3)p$ |
| etc. | etc. |

Arcus isti

$$z, \quad y+z, \quad 2y+z, \quad 3y+z, \quad \text{etc.}$$

in arithmetica progressione progrediuntur, eorum vero tam sinus quam cosinus progressionem recurrentem constituunt, qualis ex denominatore

$$1-2nx+(mm+nn)xx$$

oritur; est enim

$$\sin.(2y+z)=2n\sin.(y+z)-(mm+nn)\sin.z$$

sive

$$\sin.(2y+z)=2\cos.y\sin.(y+z)-\sin.z$$

atque simili modo

$$\cos.(2y+z)=2\cos.y\cos.(y+z)-\cos.z.$$

Eodem modo erit porro

$$\sin.(3y+z)=2\cos.y\sin.(2y+z)-\sin.(y+z)$$

et

$$\cos.(3y+z)=2\cos.y\cos.(2y+z)-\cos.(y+z)$$

itemque

$$\sin.(4y+z)=2\cos.y\sin.(3y+z)-\sin.(2y+z)$$

et

$$\cos.(4y+z)=2\cos.y\cos.(3y+z)-\cos.(2y+z)$$

etc.

Cuius legis beneficio arcuum in progressione arithmetica progredientium tam sinus quam cosinus, quousque libuerit, expedite formari possunt.

130. Cum sit

$$\sin.(y+z)=\sin.y\cos.z+\cos.y\sin.z$$

atque

$$\sin.(y-z)=\sin.y\cos.z-\cos.y\sin.z,$$

erit his expressionibus vel addendis vel subtrahendis

$$\sin.y\cos.z=\frac{\sin.(y+z)+\sin.(y-z)}{2},$$

$$\cos.y\sin.z=\frac{\sin.(y+z)-\sin.(y-z)}{2}.$$

Quia porro est

$$\cos.(y+z)=\cos.y\cos.z-\sin.y\sin.z$$

atque

$$\cos.(y-z) = \cos.y\cos.z + \sin.y\sin.z\,,$$

erit pari modo

$$\cos.y\cos.z = \frac{\cos.(y-z)+\cos.(y+z)}{2}\,,$$

$$\sin.y\sin.z = \frac{\cos.(y-z)-\cos.(y+z)}{2}\,.$$

Sit

$$y = z = \tfrac{1}{2}v\,;$$

erit ex his postremis formulis

$$\left(\cos.\tfrac{1}{2}v\right)^2 = \frac{1+\cos.v}{2} \text{ et } \cos.\tfrac{1}{2}v = \sqrt{\frac{1+\cos.v}{2}}\,,$$

$$\left(\sin.\tfrac{1}{2}v\right)^2 = \frac{1-\cos.v}{2} \text{ et } \sin.\tfrac{1}{2}v = \sqrt{\frac{1-\cos.v}{2}}\,,$$

unde ex dato cosinu cuiusque anguli reperiuntur eius semissis sinus et cosinus.

131. Ponatur arcus

$$y+z=a \text{ et } y-z=b\,;$$

erit

$$y = \frac{a+b}{2} \text{ et } z = \frac{a-b}{2}\,,$$

quibus in superioribus formulis substitutis habebuntur hae aequationes, tanquam totidem theoremata:

$$\sin.a+\sin.b = 2\sin.\frac{a+b}{2}\cos.\frac{a-b}{2}\,,$$

$$\sin.a-\sin.b = 2\cos.\frac{a+b}{2}\sin.\frac{a-b}{2}\,,$$

$$\cos.a+\cos.b = 2\cos.\frac{a+b}{2}\cos.\frac{a-b}{2}\,,$$

$$\cos.a-\cos.b = 2\sin.\frac{a+b}{2}\sin.\frac{a-b}{2}\,.$$

Ex his porro nascuntur ope divisionis haec theoremata

$$\frac{\sin.a+\sin.b}{\sin.a-\sin.b}=\text{tang.}\frac{a+b}{2}\cot.\frac{a-b}{2}=\frac{\text{tang.}\frac{a+b}{2}}{\text{tang.}\frac{a-b}{2}},$$
$$\frac{\sin.a+\sin.b}{\cos.a+\cos.b}=\text{tang.}\frac{a+b}{2},$$
$$\frac{\sin.a+\sin.b}{\cos.a-\cos.b}=\cot.\frac{a-b}{2},$$
$$\frac{\sin.a-\sin.b}{\cos.a+\cos.b}=\text{tang.}\frac{a-b}{2},$$
$$\frac{\sin.a-\sin.b}{\cos.a-\cos.b}=\cot.\frac{a+b}{2},$$
$$\frac{\cos.a+\cos.b}{\cos.b-\cos.a}=\cot.\frac{a+b}{2}\cot.\frac{a-b}{2}.$$

Ex his denique deducuntur ista theoremata

$$\frac{\sin.a+\sin.b}{\sin.a-\sin.b}\times\frac{\cos.a+\cos.b}{\cos.b-\cos.a}=\left(\cot.\frac{a-b}{2}\right)^2,$$
$$\frac{\sin.a+\sin.b}{\sin.a-\sin.b}\times\frac{\cos.b-\cos.a}{\cos.a+\cos.b}=\left(\text{tang.}\frac{a+b}{2}\right)^2.$$

132. Cum sit

$$(\sin.z)^2+(\cos.z)^2=1,$$

erit factoribus sumendis

$$\left(\cos.z+\sqrt{-1}\cdot\sin.z\right)\left(\cos.z-\sqrt{-1}\cdot\sin.z\right)=1,$$

qui factores, etsi imaginarii, tamen ingentem praestant usum in arcubus combinandis et multiplicandis. Quaeratur enim productum horum factorum

$$\left(\cos.z+\sqrt{-1}\cdot\sin.z\right)\left(\cos.y+\sqrt{-1}\cdot\sin.y\right)$$

ac reperietur

$$\cos.y\cos.z-\sin.y\sin.z+\sqrt{-1}\cdot(\cos.y\sin.z+\sin.y\cos.z).$$

Cum autem sit

$$\cos.y\cos.z-\sin.y\sin.z=\cos.(y+z)$$

et

$$\cos.y\sin.z+\sin.y\cos.z=\sin.(y+z),$$

erit hoc productum

$$\left(\cos.y+\sqrt{-1}\cdot\sin.y\right)\left(\cos.z+\sqrt{-1}\cdot\sin.z\right)=\cos.(y+z)+\sqrt{-1}\cdot\sin.(y+z)$$

et simili modo

$$\left(\cos.y-\sqrt{-1}\cdot\sin.y\right)\left(\cos.z-\sqrt{-1}\cdot\sin.z\right)=\cos.(y+z)-\sqrt{-1}\cdot\sin.(y+z),$$

item

$$\left(\cos.x\pm\sqrt{-1}\cdot\sin.x\right)\left(\cos.y\pm\sqrt{-1}\cdot\sin.y\right)\left(\cos.z\pm\sqrt{-1}\cdot\sin.z\right)=\cos.(x+y+z)\pm\sqrt{-1}\cdot\sin.(x+y+z).$$

133. Hinc itaque sequitur fore

$$\left(\cos.z\pm\sqrt{-1}\cdot\sin.z\right)^2=\cos.2z\pm\sqrt{-1}\cdot\sin.2z$$

$$\left(\cos.z\pm\sqrt{-1}\cdot\sin.z\right)^3=\cos.3z\pm\sqrt{-1}\cdot\sin.3z$$

ideoque generaliter erit

$$\left(\cos.z\pm\sqrt{-1}\cdot\sin.z\right)^n=\cos.nz\pm\sqrt{-1}\cdot\sin.nz\ .$$

.
Unde ob signorum ambiguitatem erit

$$\cos.nz=\frac{\left(\cos.z+\sqrt{-1}\cdot\sin.z\right)^n+\left(\cos.z-\sqrt{-1}\cdot\sin.z\right)^n}{2}$$

et

$$\sin.nz=\frac{\left(\cos.z+\sqrt{-1}\cdot\sin.z\right)^n-\left(\cos.z-\sqrt{-1}\cdot\sin.z\right)^n}{2\sqrt{-1}}.$$

Evolutis ergo binomiis hisce erit per series

$$\cos.nz=(\cos.z)^n-\frac{n(n-1)}{1\cdot2}(\cos.z)^{n-2}(\sin.z)^2+\frac{n(n-1)(n-2)(n-3)}{1\cdot2\cdot3\cdot4}(\cos.z)^{n-4}(\sin.z)^4$$
$$-\frac{n(n-1)(n-2)(n-3)(n-4)(n-5)}{1\cdot2\cdot3\cdot4\cdot5\cdot6}(\cos.z)^{n-6}(\sin.z)^6+\text{etc.}$$

et

$$\sin.nz=\frac{n}{1}(\cos.z)^{n-1}\sin.z-\frac{n(n-1)(n-2)}{1\cdot2\cdot3}(\cos.z)^{n-3}(\sin.z)^3+\frac{n(n-1)(n-2)(n-3)(n-4)}{1\cdot2\cdot3\cdot4\cdot5}(\cos.z)^{n-5}(\sin.z)^5$$
$$-\ \text{etc.}$$

134. Sit arcus $z$ infinite parvus; erit $\sin.z = z$ et $\cos.z = 1$; sit autem $n$ numerus infinite magnus, ut sit arcus $nz$ finitae magnitudinis, puta $nz = v$; ob $\sin.z = z = \frac{v}{n}$ erit

$$\cos.v = 1 - \frac{v^2}{1\cdot 2} + \frac{v^4}{1\cdot 2\cdot 3\cdot 4} - \frac{v^6}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6} + \text{etc.}$$

et

$$\sin.v = v - \frac{v^3}{1\cdot 2\cdot 3} + \frac{v^5}{1\cdot 2\cdot 3\cdot 4\cdot 5} - \frac{v^7}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7} + \text{etc.}$$

Dato ergo arcu $v$ ope harum serierum eius sinus et cosinus inveniri poterunt; quarum formularum usus quo magis pateat, ponamus arcum $v$ esse ad quadrantem seu 90° ut $m$ ad $n$ seu esse $v = \frac{m}{n}\cdot\frac{\pi}{2}$. Quia nunc valor ipsius $\pi$ constat, si is ubique substituatur, prodibit

$$\sin.A.\tfrac{m}{n}90^0 =$$

$$+\frac{m}{n}\cdot\ 1{,}57079\ 63267\ 94896\ 61923\ 13216\ 916$$

$$-\frac{m^3}{n^3}\cdot 0{,}64596\ 40975\ 06246\ 25365\ 57565\ 639$$

$$+\frac{m^5}{n^5}\cdot 0{,}07969\ 26262\ 46167\ 04512\ 05055\ 495$$

$$-\frac{m^7}{n^7}\cdot 0{,}00468\ 17541\ 35318\ 68810\ 06854\ 639$$

$$+\frac{m^9}{n^9}\cdot 0{,}00016\ 04411\ 84787\ 35982\ 18726\ 609$$

$$-\frac{m^{11}}{n^{11}}\cdot 0{,}00000\ 35988\ 43235\ 21208\ 53404\ 585$$

$$+\frac{m^{13}}{n^{13}}\cdot 0{,}00000\ 00569\ 21729\ 21967\ 92681\ 178$$

$$-\frac{m^{15}}{n^{15}}\cdot 0{,}00000\ 00006\ 68803\ 51098\ 11467\ 232$$

$$+\frac{m^{17}}{n^{17}}\cdot 0{,}00000\ 00000\ 06066\ 93573\ 11061\ 957$$

$$-\frac{m^{19}}{n^{19}}\cdot 0{,}00000\ 00000\ 00043\ 77065\ 46731\ 374$$

$$+\frac{m^{21}}{n^{21}}\cdot 0{,}00000\ 00000\ 00000\ 25714\ 22892\ 860$$

$$-\frac{m^{23}}{n^{23}}\cdot 0{,}00000\ 00000\ 00000\ 00125\ 38995\ 405$$

$$+\frac{m^{25}}{n^{25}}\cdot 0{,}00000\ 00000\ 00000\ 00000\ 51564\ 552$$

$$-\frac{m^{27}}{n^{27}}\cdot 0{,}00000\ 00000\ 00000\ 00000\ 00181\ 240$$

$$+\frac{m^{29}}{n^{29}}\cdot 0{,}00000\ 00000\ 00000\ 00000\ 00000\ 551$$

[The last three places in each power are in error in the original.]

atque

$$\begin{aligned}
&\cos.\mathrm{A}.\tfrac{m}{n}90^\circ = \\
&+\quad 1{,}00000\ 00000\ 00000\ 00000\ 00000\ 000 \\
&-\tfrac{m^2}{n^2}\cdot 1{,}23370\ 05501\ 36169\ 82735\ 43113\ 750 \\
&+\tfrac{m^4}{n^4}\cdot 0{,}25366\ 95079\ 01048\ 01363\ 65633\ 664 \\
&-\tfrac{m^6}{n^6}\cdot 0{,}02086\ 34807\ 63352\ 96087\ 30516\ 372 \\
&+\tfrac{m^8}{n^8}\cdot 0{,}00091\ 92602\ 74839\ 42658\ 02417\ 162 \\
&-\tfrac{m^{10}}{n^{10}}\cdot 0{,}00002\ 52020\ 42373\ 06060\ 54810\ 530 \\
&+\tfrac{m^{12}}{n^{12}}\cdot 0{,}00000\ 04710\ 87477\ 88181\ 71503\ 670 \\
&-\tfrac{m^{14}}{n^{14}}\cdot 0{,}00000\ 00063\ 86603\ 08379\ 18522\ 411 \\
&+\tfrac{m^{16}}{n^{16}}\cdot 0{,}00000\ 00000\ 65659\ 63114\ 97947\ 236 \\
&-\tfrac{m^{18}}{n^{18}}\cdot 0{,}00000\ 00000\ 00529\ 44002\ 00734\ 624 \\
&+\tfrac{m^{20}}{n^{20}}\cdot 0{,}00000\ 00000\ 00003\ 43773\ 91790\ 986 \\
&-\tfrac{m^{22}}{n^{22}}\cdot 0{,}00000\ 00000\ 00000\ 01835\ 99165\ 216 \\
&+\tfrac{m^{24}}{n^{24}}\cdot 0{,}00000\ 00000\ 00000\ 00008\ 20675\ 330 \\
&-\tfrac{m^{26}}{n^{26}}\cdot 0{,}00000\ 00000\ 00000\ 00000\ 03115\ 285 \\
&+\tfrac{m^{28}}{n^{28}}\cdot 0{,}00000\ 00000\ 00000\ 00000\ 00010\ 168 \\
&-\tfrac{m^{30}}{n^{30}}\cdot 0{,}00000\ 00000\ 00000\ 00000\ 00000\ 029.
\end{aligned}$$

[The last three places in each power are in error in the original.]

Cum igitur sufficiat sinus et cosinus angulorum ad $45^0$ nosse, fractio $\frac{m}{n}$ semper minor erit quam $\frac{1}{2}$ hincque etiam ob potestates fractionis $\frac{m}{n}$ series exhibitae maxime convergent, ita ut plerumque aliquot tantum termini sufficiant, praecipue si sinus et cosinus non ad tot figuras desiderentur.

135. Inventis sinibus et cosinibus inveniri quidem possunt tangentes et cotangentes per analogias consuetas; at quia in huiusmodi ingentibus numeris multiplicatio et divisio vehementer est molesta, peculiari modo eas exprimere convenit. Erit ergo

$$\text{tang.}v = \frac{\sin.v}{\cos.v} = \frac{v - \frac{v^3}{1\cdot2\cdot3} + \frac{v^5}{1\cdot2\cdot3\cdot4\cdot5} - \frac{v^7}{1\cdot2\cdot3\cdots7} + \text{etc.}}{1 - \frac{v^2}{1\cdot2} + \frac{v^4}{1\cdot2\cdot3\cdot4} - \frac{v^6}{1\cdot2\cdot3\cdots6} + \text{etc.}}$$

et

$$\cot.v = \frac{\cos.v}{\sin.v} = \frac{1 - \frac{v^2}{1\cdot2} + \frac{v^4}{1\cdot2\cdot3\cdot4} - \frac{v^6}{1\cdot2\cdot3\cdots6} + \text{etc.}}{v - \frac{v^3}{1\cdot2\cdot3} + \frac{v^5}{1\cdot2\cdot3\cdot4\cdot5} - \frac{v^7}{1\cdot2\cdot3\cdots7} + \text{etc.}}.$$

Si iam sit arcus $v = \frac{m}{n}90^0$, erit eodem modo quo ante

| $\text{tang. A.}\frac{m}{n}90^0$ | $\cot.\text{A.}\frac{m}{n}90^0$ |
|---|---|
| $= +\frac{2mn}{nn-mm}\cdot 0{,}63661\,97723\,676$ | $= +\frac{n}{m}\cdot 0{,}63661\,97723\,676$ |
| $+\frac{m}{n}\cdot 0{,}29755\,67820\,597$ | $-\frac{4mn}{4mn-mm}\cdot 0{,}31830\,98861\,838$ |
| $+\frac{m^3}{n^3}\cdot 0{,}01868\,86502\,773$ | $-\frac{m}{n}\cdot 0{,}20528\,88894\,145$ |
| $+\frac{m^5}{n^5}\cdot 0{,}00184\,24752\,034$ | $-\frac{m^3}{n^3}\cdot 0{,}00655\,10747\,882$ |
| $+\frac{m^7}{n^7}\cdot 0{,}00019\,75800\,715$ | $-\frac{m^5}{n^5}\cdot 0{,}00034\,50292\,554$ |
| $+\frac{m^9}{n^9}\cdot 0{,}00002\,16977\,373$ | $-\frac{m^7}{n^7}\cdot 0{,}00002\,02791\,061$ |
| $+\frac{m^{11}}{n^{11}}\cdot 0{,}00000\,24011\,370$ | $-\frac{m^9}{n^9}\cdot 0{,}00000\,12366\,527$ |
| $+\frac{m^{13}}{n^{13}}\cdot 0{,}00000\,02664\,133$ | $-\frac{m^{11}}{n^{11}}\cdot 0{,}00000\,00764\,959$ |
| $+\frac{m^{15}}{n^{15}}\cdot 0{,}00000\,00295\,865$ | $-\frac{n^{13}}{m^{13}}\cdot 0{,}00000\,00047\,597$ |
| $+\frac{m^{17}}{n^{17}}\cdot 0{,}00000\,00032\,868$ | $-\frac{m^{15}}{n^{15}}\cdot 0{,}00000\,00002\,969$ |
| $+\frac{m^{19}}{n^{19}}\cdot 0{,}00000\,00003\,652$ | $-\frac{m^{17}}{n^{17}}\cdot 0{,}00000\,00000\,185$ |
| $+\frac{m^{21}}{n^{21}}\cdot 0{,}00000\,00000\,406$ | $-\frac{m^{19}}{n^{19}}\cdot 0{,}00000\,00000\,012$ |
| $+\frac{m^{23}}{n^{23}}\cdot 0{,}00000\,00000\,045$ | |
| $+\frac{m^{25}}{n^{25}}\cdot 0{,}00000\,00000\,005$ | |

quarum serierum ratio infra fusius exponetur.

136. Ex superioribus quidem constat, si cogniti fuerint omnium angulorum semirecto minorum sinus et cosinus, inde simul omnium angulorum maiorum sinus et cosinus haberi. Verum si tantum angulorum 30° minorum habeantur sinus et cosinus, ex iis per solam additionem et subtractionem omnium angulorum maiorum sinus et cosinus inveniri possunt. Cum enim sit

$$\sin.30^0 = \tfrac{1}{2}$$

erit posito $y = 30^0$ ex §130

$$\cos.z = \sin.(30+z) + \sin.(30-z)$$

et

$$\sin.z = \cos.(30-z) - \cos.(30+z)$$

ideoque ex sinibus et cosinibus angulorum $z$ et $30-z$ s reperiuntur

$$\sin.(30+z) = \cos.z - \sin.(30-z)$$

et

$$\cos.(30+z) = \cos.(30-z) - \sin.z\,,$$

unde sinus et cosinus angulorum a 30° ad 60° hincque omnes maiores definiuntur.

137. In tangentibus et cotangentibus simile subsidium usu venit. Cum enim sit

$$\tan.(a+b) = \frac{\text{tang.}\,a + \text{tang.}\,b}{1 - \text{tang.}a\ \text{tang.}b},$$

erit

$$\text{tang.}2a = \frac{2\text{tang.}a}{1-\text{tang.}a\ \text{tang.}a} \quad \text{et} \quad \cot.2a = \frac{\cot.a - \text{tang.}a}{2},$$

unde ex tangentibus et cotangentibus arcuum 30° minorum inveniuntur cotangentes usque ad 60°.

Sit iam $a = 30 - b$; erit $2a = 60 - 2b$ et $\cot.2a = \text{tang.}(30+2b)$; erit ergo

$$\text{tang.}(30+2b) = \frac{\cot.(30-b) - \text{tang.}(30-b)}{2},$$

unde etiam tangentes arcuum 30° maiorum obtinentur.

Secantes autem et cosecantes ex tangentibus per solam subtractionem inveniuntur; est enim

$$\text{cosec.}z = \cot.\tfrac{1}{2}z - \cot.z$$

et hinc

$$\sec.z = \cot.\left(45^0 - \tfrac{1}{2}z\right) - \tang.z.$$

Ex his ergo luculenter perspicitur, quomodo canones sinuum construi potuerint.

138. Ponatur denuo in formulis §133 arcus $z$ infinite parvus et sit $n$ numerus infinite magnus $i$, ut $iz$ obtineat valorem finitum $v$. Erit ergo

$nz = v$ et $z = \frac{v}{i}$, unde $\sin.z = \frac{v}{i}$ et $\cos.z = 1$; his substitutis fit

$$\cos.v = \frac{\left(1+\frac{v\sqrt{-1}}{i}\right)^i + \left(1-\frac{v\sqrt{-1}}{i}\right)^i}{2}$$

atque

$$\sin.v = \frac{\left(1+\frac{v\sqrt{-1}}{i}\right)^i - \left(1-\frac{v\sqrt{-1}}{i}\right)^i}{2\sqrt{-1}}$$

In capite autem praecedente vidimus esse

$$\left(1+\frac{z\sqrt{-1}}{i}\right)^i = e^z$$

denotante $e$ basin logarithmorum hyperbolicorum; scripto ergo pro $z$ partim $+v\sqrt{-1}$ partim $-v\sqrt{-1}$ erit

$$\cos.v = \frac{e^{+v\sqrt{-1}} + e^{-v\sqrt{-1}}}{2}$$

et

$$\sin.v = \frac{e^{+v\sqrt{-1}} - e^{-v\sqrt{-1}}}{2\sqrt{-1}}$$

Ex quibus intelligitur, quomodo quantitates exponentiales imaginariae ad sinus et cosinus arcuum realium reducantur. Erit vero

$$e^{+v\sqrt{-1}} = \cos.v + \sqrt{-1}\cdot\sin.v$$

et

$$e^{+v\sqrt{-1}} = \cos.v - \sqrt{-1}\cdot\sin.v$$

139. Sit iam in iisdem formulis § 133 $n$ numerus infinite parvus seu $n = \frac{1}{i}$ existente $i$ numero infinite magno; erit

$$\cos.nz = \cos.\tfrac{z}{i} = 1 \text{ et } \sin.nz = \sin.\tfrac{z}{i} = \tfrac{z}{i};$$

arcus enim evanescentis $\frac{z}{i}$ sinus est ipsi aequalis, cosinus vero $=1$. His positis habebitur

$$1 = \frac{\left(\cos.z+\sqrt{-1}\cdot\sin.z\right)^{\frac{1}{i}}+\left(\cos.z-\sqrt{-1}\cdot\sin.z\right)^{\frac{1}{i}}}{2}$$

et

$$\frac{z}{i} = \frac{\left(\cos.z+\sqrt{-1}\cdot\sin.z\right)^{\frac{1}{i}}-\left(\cos.z-\sqrt{-1}\cdot\sin.z\right)^{\frac{1}{i}}}{2\sqrt{-1}}.$$

Sumendis autem logarithmis hyperbolicis supra (§125) ostendimus esse

$$l(1+x) = i(1+x)^{\frac{1}{i}} - 1 \text{ seu } y^{\frac{1}{i}} = 1+\tfrac{1}{i}ly$$

posito y loco $1+x$. Nunc igitur posito loco $y$ partim $\cos.z+\sqrt{-1}\cdot\sin.z$ partim $\cos.z-\sqrt{-1}\cdot\sin.z$ prodibit

$$1 = \frac{1+\frac{1}{i}l\left(\cos.z+\sqrt{-1}\cdot\sin.z\right)1+\frac{1}{i}l\left(\cos.z-\sqrt{-1}\cdot\sin.z\right)}{2} = 1$$

ob logarithmos evanescentes, ita ut hinc nil sequatur. Altera vero aequatio pro sinu suppeditat

$$\frac{z}{i} = \frac{\frac{1}{i}l\left(\cos.z+\sqrt{-1}\cdot\sin.z\right)-\frac{1}{i}l\left(\cos.z-\sqrt{-1}\cdot\sin.z\right)}{2\sqrt{-1}}$$

ideoque

$$z = \frac{1}{2\sqrt{-1}}\,l\,\frac{\cos.z+\sqrt{-1}\cdot\sin.z}{\cos.z-\sqrt{-1}\cdot\sin.z},$$

unde patet, quemadmodum logarithmi imaginarii ad arcus circulares revocentur.

140. Cum sit $\frac{\sin.z}{\cos.z} = \text{tang}.z$, arcus $z$ per suam tangentem ita exprimetur, ut sit

$$z = \frac{1}{2\sqrt{-1}}\,l\,\frac{1+\sqrt{-1}\cdot\text{tang}.z}{1-\sqrt{-1}\cdot\text{tang}.z}.$$

Supra vero (§ 123) vidimus esse

$$l\frac{1+x}{1-x} = \frac{2x}{1} + \frac{2x^2}{3} + \frac{2x^5}{5} + \frac{2x^7}{7} + \text{etc.}$$

Posito ergo $z = \sqrt{-1} \cdot \text{tang.}z$ fiet

$$z = \frac{\text{tang.}z}{1} - \frac{(\text{tang.}z)^3}{3} + \frac{(\text{tang.}z)^5}{5} - \frac{(\text{tang.}z)^7}{7} + \text{etc.}$$

Si ergo ponamus $\text{tang.}z = t$, ut sit $z$ arcus, cuius tangens est $t$, quem ita indicabimus $\text{A.tang.}t$, ideoque erit

$$z = \text{A.tang.}t.$$

Cognita ergo tangente $t$ erit arcus respondens

$$z = \frac{t}{1} - \frac{t^3}{3} + \frac{t^5}{5} - \frac{t^7}{7} + \frac{t^9}{9} - \text{etc.}$$

Cum igitur,

si tangens $t$ aequetur radio 1, fiat arcus $z =$ arcui 45° seu $z = \frac{\pi}{4}$, erit

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \text{etc.},,$$

quae est series a LEIBNITZIO primum producta ad valorem peripheriae circuli exptimendum.

141. Quo autem ex hiusmodi serie longitudo arcus circuli expedite definiri possit, perspicuum est pro tangente $t$ fractionem satis parvam substitui debere. Sic ope huius seriei facile reperietur longitudo arcus $z$, cuius tangens $t$ aequetur $\frac{1}{10}$ ; foret enim iste arcus

$$z = \frac{1}{10} - \frac{1}{3000} + \frac{1}{500000} - \text{etc.},$$

cuius seriei valor per approximationem non difficulter in fractione decimali exhiberetur. At vero ex tali arcu cognito nihil pro longitudine totius circumferentiae concludere licebit, cum ratio, quam arcus, cuius tangens est $= \frac{1}{10}$ ad totam peripheriam tenet, non sit assignabilis. Hanc ob rem ad peripheriam indagandam eiusmodi arcus quaeri debet, qui sit simul pars aliquota peripheriae et cuius tangens satis exigua commode exprimi queat. Ad hoc ergo institutum sumi solet arcus 30°, cuius tangens est $= \frac{1}{\sqrt{3}}$ quia minorum arcuum cum peripheria commensurabilium tangentes nimis fiunt irrationales. Quare ob arcum $30° = \frac{\pi}{6}$ erit

$$\frac{\pi}{6} = \frac{1}{\sqrt{3}} - \frac{1}{3 \cdot 3\sqrt{3}} + \frac{1}{5 \cdot 3^2\sqrt{3}} - \text{etc.}$$

et

$$\pi = \frac{2\sqrt{3}}{1} - \frac{2\sqrt{3}}{3\cdot 3} + \frac{2\sqrt{3}}{5\cdot 3^2} - \frac{2\sqrt{3}}{7\cdot 3^3} + \text{etc.},$$

cuius series ope valor ipsius $\pi$ ante exhibitus incredibili labore fuit determinatus.

142. Hic autem labor eo maior est, quod primum singuli termini sint irrationales, tum vero quisque tantum circiter triplo sit minor quam praecedens. Huic itaque incommodo ita occurri poterit. Sumatur arcus 45° seu $\frac{\pi}{4}$; cuius valor etsi per seriem vix convergentem

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \text{etc.}$$

exprimitur, tamen is retineatur atque in duos arcus $a$ et $b$ dispertiatur, ut sit $a + b = \frac{\pi}{4} = 45°$. Cum igitur sit

$$\tan.(a+b) = 1 = \frac{\text{tang.}a + \text{tang.}b}{1 - \text{tang.}a\ \text{tang.}b},$$

erit

$$1 - \text{tang.}a\ \text{tang.}b = \text{tang.}a + \text{tang.}b$$

et

$$\text{tang.}b = \frac{1 - \text{tang.}a}{1 + \text{tang.}a}.$$

Sit nunc $\text{tang.}a = \frac{1}{2}$; erit $\text{tang.}b = \frac{1}{3}$; hinc uterque arcus $a$ et $b$ per seriem rationalem multo magis quam superior convergentem exprimetur eorumque summa dabit valorem arcus $\frac{\pi}{4}$ hinc itaque erit

$$\pi = 4 \cdot \left( \begin{array}{l} \frac{1}{1\cdot 2} - \frac{1}{3\cdot 2^3} + \frac{1}{5\cdot 2^5} - \frac{1}{7\cdot 2^7} + \frac{1}{9\cdot 2^9} - \text{ etc.} \\ \frac{1}{1\cdot 3} - \frac{1}{3\cdot 3^3} + \frac{1}{5\cdot 3^5} - \frac{1}{7\cdot 3^7} + \frac{1}{9\cdot 3^9} - \text{ etc.} \end{array} \right)$$

Hoc ergo modo multo expeditius longitudo semicircumferentiae $\pi$ inveniri potuisset, quam quidem factum est ope seriei ante commemoratae.