ADDITION IB

ELASTIC CURVES cont'd.

34. From these kinds enumerated for any case offered it will be easy to assign, to which kind the curve formed may be related. An elastic lamina (Fig.12) may be fixed into a wall at *G*, truly from the end *A* a weight *P* may be hanging, by which the lamina may be curved into the figure *GA*. The tangent *AT* may be drawn, and the whole judgement will be sought from the angle *TAP*. For if this angle were acute, the curve will refer to the second kind ; if it shall be right, to the third, and it will be the elastic rectangle. But if the angle *TAP* were obtuse, yet less than 130° 41', the curve will be related to the fourth kind; but to the fifth kind, if the angle *TAP* shall be $=130° \, 41'$; moreover if the angle *TAP* were greater, the curve will be contained within the sixth kind. Truly it will belong to the seventh kind, if the angle should become to two right angles, but which cannot happen under any circumstances. Therefore this kind with the two following cannot be produced by a weight attached to the lamina.

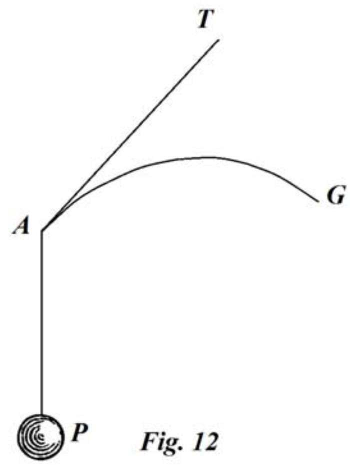

Fig. 12

35. Therefore so that it may be apparent (Fig. 3), how the remaining kinds may be able to produce the curving lamina, for a lamina fixed at *B*, and not at once, but to the end *C* of the lamina a weight *P* may be hung from a rigid rod *AC* firmly connected at *A*, which pulls in the direction *CD*. Let the interval $AC = h$, the absolute elasticity of the lamina *Ekk* and the sine of the angle *MAP*, which the lamina makes at *A* to the horizontal $= m$. With these in place, if the abscissa may be put $AP = t$ and the applied line $PM = y$, this equation will be found for the curve

$$dy = \frac{dt\left(mEkk - Pht - \frac{1}{2}Ptt\right)}{\sqrt{\left(E^2k^4 - \left(mEkk - Pht - \frac{1}{2}Ptt\right)^2\right)}}.$$

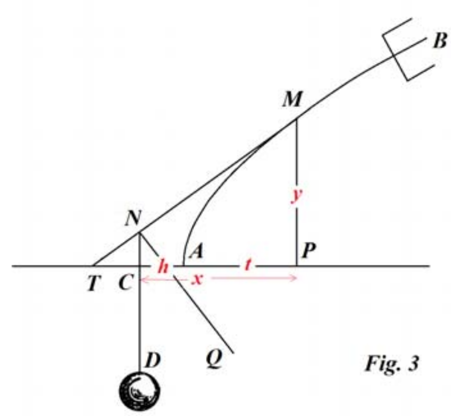

Fig. 3

Now putting $CP = x = h + t$, so that the equation may be reduced to the form we have used in the division of the kinds ; there will be

$$dy = \frac{dx\left(mEkk + \frac{1}{2}Phh - \frac{1}{2}Pxx\right)}{\sqrt{\left(E^2k^4 - \left(mEkk + \frac{1}{2}Phh - \frac{1}{2}Pxx\right)^2\right)}}$$

which compared with the form

$$dy=\frac{dx\left(aa-cc+xx\right)}{\sqrt{\left(cc-xx\right)\left(2aa-cc+xx\right)}}$$

or

$$dy=\frac{dx\left(aa-cc+xx\right)}{\sqrt{\left(a^4-\left(aa-cc+xx\right)^2\right)}}$$

will give $\frac{1}{2}Paa=Ekk$ or

$$aa=\frac{2Ekk}{P} \text{ and } \tfrac{1}{2}Pcc-\tfrac{1}{2}Paa=mEkk+\tfrac{1}{2}Phh;$$

therefore

$$cc=\frac{2\left(1+m\right)Ekk}{P}+hh.$$

36. Therefore the curve will belong to the second kind, if there were

$$\frac{2mEkk}{P}+hh<0 \text{ or } P<-\frac{2mEkk}{hh};$$

therefore unless the angle *PAM* may be negative, the force *P* is negative and the rod must be drawn upwards at *C*. It will belong to the third kind of curvature, if

$$P=-\frac{2mEkk}{hh}.$$

But the fourth kind will be produced, if there were

$$2mEkk+Phh>0, \text{ likewise truly } 2mEkk+Phh<2\alpha Ekk,$$

with $\alpha=0,651868.$ But if there shall be

$$P=\frac{2\left(\alpha-m\right)Ekk}{hh},$$

then the curve will belong to the fifth kind. But if indeed there were

$$Phh>2\left(\alpha-m\right)Ekk, \text{ likewise truly } Phh<2\left(1-m\right)Ekk,$$

the curve is being referred to the sixth kind. And the seventh kind will arise, if

$$Phh = 2(1-m)Ekk .$$

Moreover the eighth kind will be obtained, if

$$Phh > 2(1-m)Ekk ;$$

whereby, if the angle *PAM* were right, on account of $1-m=0$ the curve will always belong to the eighth kind. And finally the ninth kind may arise, if there were $h=\infty$, as I have noted above now.

CONCERNING THE STRENGTH OF COLUMNS

37. The matters which have been observed before about the first kind of curves, can be of assistance in the judgment of columns. For *AB* shall be a column (Fig. 13) placed vertically on the base *A*, bearing a weight *P*. But if now a column may be put in place, so that it may be unable to slide forwards from the load *P*, if it were exceedingly large, nothing other will need to be feared, except for the curving of the column ; therefore in this case the column will be able to be seen as endowed with elasticity. Therefore the absolute elasticity of the column shall be $=Ekk$, and with its height $AB = 2f = a$, and in the above paragraph 25 we have seen the required force for this column or the minimum inclination to be

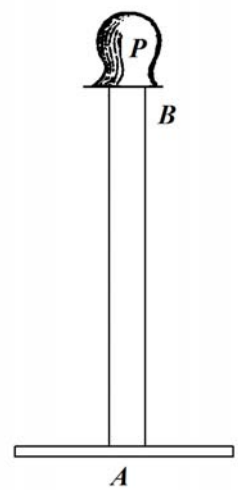

Fig. 13

$$= \frac{\pi\pi Ekk}{4ff} = \frac{\pi\pi}{aa} \cdot Ekk .$$

Therefore unless the load *P* requiring to be borne shall be greater than $E \cdot \frac{\pi\pi kk}{aa}$, at this stage no curvature need be feared; but on the other hand, if the weight *P* were greater, the column will be unable to resist curving. But with the elasticity of the column remaining and thus with its same thickness, the weight *P*, which it is strong enough to bear without danger, will be inversely as the square of the height of the column and with a column twice as high can bear only a quarter of the load. Therefore these especially can be brought to attention in the use of wooden columns, which clearly are liable to curving.

THE DETERMINATION OF ELASTIC CURVATURE BY EXPERIMENT

38. But so that the strength and curvature of each elastic lamina may be able to be determined in the first place, it is necessary, that the absolute elasticity shall be known, which we have expressed up to now by Ekk; that which will be performed conveniently by a single experiment. A uniform elastic lamina FH may be implanted (Fig. 14), of which it is required to find the absolute elasticity, with the other end F in the solid wall GK, thus so that it may maintain a horizontal position FH; for here it may be allowed to ignore the natural weight. To the other end H a weight P may be hung taken arbitrarily, by which the lamina in the state AF will be curved. The length of the lamina shall be $AF = HF = f$, the right horizontal line $AG = g$ and the vertical $GF = h$, which values will all become known by experiment. This curve may be compared now with the general equation :

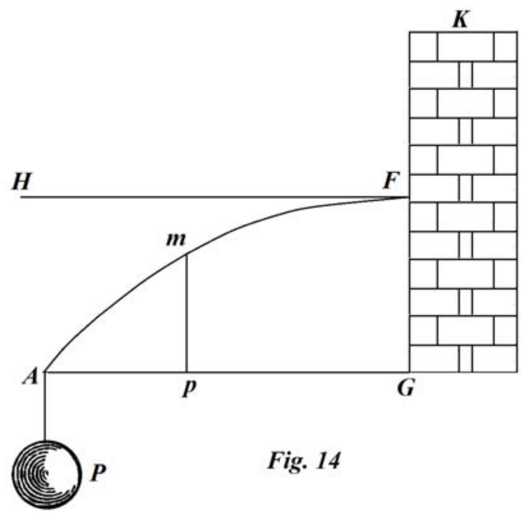

Fig. 14

$$dy = \frac{(cc - aa - xx)\,dx}{\sqrt{(cc - xx)(2aa - cc + xx)}},$$

in which if a and c were defined by f, g, h, the curving force will be $P = \frac{2Ekk}{aa}$ and thus the absolute elasticity $Ekk = \frac{1}{2}Paa$.

39. Now because the tangent at F is horizontal, there will be $\frac{dy}{dx} = 0$ and thus $x = \sqrt{(cc - aa)}$. Hence therefore there will be

$$AG = g = \sqrt{(cc - aa)} \text{ and } aa = cc - gg$$

and thus

$$dy = \frac{(gg - xx)\,dx}{\sqrt{(cc - xx)(cc - 2gg + xx)}},$$

but on putting here $x = g$, there must become $y = GF = h$ or $s = AF = f$; truly there is

$$ds = \frac{(cc-gg)\,dx}{\sqrt{(cc-xx)(cc-2gg+xx)}}.$$

Now if the weight *P* may be taken very small, so that the lamina may be only be depressed by very little, then *c* will be a very large quantity and thus there will be approximately

$$\frac{1}{\sqrt{(cc-xx)(cc-2gg+xx)}} = (c^4-2ccgg+2ggxx-x^4)^{-\frac{1}{2}}$$
$$= \frac{1}{cc}+\frac{gg}{c^4}-\frac{ggxx}{c^4}+\frac{x^4}{2c^6},$$

and thus by integration also approximately :

$$s = \frac{(cc-gg)x}{cc}+\frac{(cc-gg)ggx}{c^4}-\frac{(cc-gg)ggx^3}{3c^4}+\frac{(cc-gg)ggx^5}{10c^4}$$

and

$$y = \frac{ggx}{cc}+\frac{g^4x}{c^4}-\frac{g^4x^3}{3c^4}+\frac{ggx^5}{10c^4}$$
$$-\frac{x^3}{3cc}-\frac{ggx^3}{3c^4}+\frac{ggx^5}{5c^4}-\frac{x^7}{14c^4}.$$

Now let $x = g$, and there becomes

$$f = g-\frac{37g^5}{30c^4} \text{ and } h = \frac{2g^3}{3cc}+\frac{2g^5}{3c^4}.$$

But if therefore the right line $FG = h$ may be called into use, there will be

$$cc = \frac{2g^3}{3h} \text{ and } aa = \frac{g(2gg-3gh)}{3h},$$

from which the absolute elastic constant is elicited

$$Ekk = \frac{Pgg(2g-\ 3h)}{6h};$$

which value will scarcely differ from the true value, provided the curvature of the lamina may not be induced to be exceedingly great.

40. But this absolute value of the elasticity *Ekk* depends in the first place on the nature of the material, from which the lamina has been fabricated, from which other materials are

accustomed to be said more or less springy. In the second place too thus it depends on the width of the lamina, so that the expression *Ekk* everywhere must be proportional to the width of the lamina, if the rest shall be equal. Truly in the third place the thickness of the lamina brings a great deal to determining the value of *Ekk*, which thus may be seen to be prepared thus, so that, with all else being equal, *Ekk* shall be as the square of the thickness. Therefore jointly the expression *Ekk* will hold a ratio composed from an account of the springy material, of the simple width of the lamina, and of the thickness of the lamina squared. Hence by experiments, in which it is allowed for the width and thickness to be measured, the elasticity of all elastic materials may be compared among themselves and can be determined.

THE UNEQUAL CURVATURE OF ELASTIC LAMINAS

41. Therefore just as at this stage for a lamina, of which I have determined the curvature, I have put the absolute elasticity *Ekk* through the whole length to be constant, thus the solution will be able to be resolved by the same method, if the quantity *Ekk* may be put variable in some manner. As it happens, if the absolute elasticity were as some function of a part of the lamina *AM* (Fig. 2), which function shall be $= S$, [*i.e.* the absolute elasticity is in modern terms is $S(x)$] by putting the arc $AM = s$ and with the radius of osculation at $M = R$ the curvature *AM*, which the lamina adopts, thus will be prepared, so that in this, amongst all the others of the same length, $\int \frac{Sds}{RR}$ shall be a minimum. Therefore this case may be solved by a formula of the second kind, [see Ch.IV, sect. 7]. Let $dy = pdx,\ dp = qdx,$ but $dS = Tds$, and between all the curves, in which $\int dx\sqrt{(1+pp)}$ is of the same magnitude, that must be determined, in which $\int \frac{Sqqdx}{(1+pp)^{5:2}}$ must be a minimum, [recall that $R = \frac{(1+pp)^{3:2}}{q}$]. The first formula $\int dx\sqrt{(1+pp)}$ gives for the formula of the differential, $\frac{1}{dx}d.\ \frac{p}{\sqrt{(1+pp)}}$.

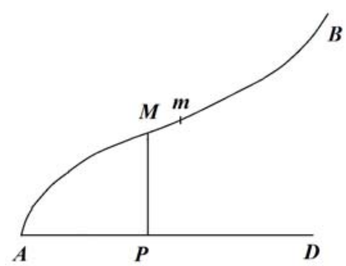

Fig. 2

Truly the other $\int \frac{Sqqdx}{(1+pp)^{5:2}}$ compared with $\int Zdx$ will give

$$Z = \frac{Sqq}{(1+pp)^{5:2}}.$$

Therefore since there may be put

$$dZ = Ld\Pi + Mdx + Ndy + Pdp + Qdq,\ \Pi = \int [Z] dx$$

and

$$d[Z]=[M]dx+[N]dy+[P]dp,$$

there will be

$$Ld\Pi=\frac{qqTds}{(1+pp)^{5:2}},$$

from which

$$L=\frac{qqT}{(1+pp)^{5:2}},\ d\Pi=ds=dx\sqrt{(1+pp)}$$

and thus

$$[Z]=\sqrt{(1+pp)},\ [M]=0,\ [N]=0,\text{ and }[P]=\frac{p}{\sqrt{(1+pp)}}.$$

Then truly there is

$$M=0,\ N=0,\ \ P=-\frac{5Sqqp}{(1+pp)^{7:2}}\ \text{ and }\ Q=\frac{2Sq}{(1+pp)^{5:2}},$$

thus so that there shall be

$$dZ=\frac{qqdS}{(1+pp)^{5:2}}+Pdp+Qdq.$$

42. Now the integral may be taken

$$\int Ldx=\int\frac{qqTdx}{(1+pp)^{5:2}}=\int\frac{qqdS}{(1+pp)^{3}}$$

and its value shall be *H*, if there may be put $x=a$, of which the constant *a* indeed will vanish soon from the calculation. Therefore there will be, [recalling that now we would have the limits (0, *a*) for the definite integral *H*, and (*x*, *a*) for the indefinite integral *V*.]

$$V=H-\int\frac{qqdS}{(1+pp)^{3}}.$$

From which the value of the differential becomes

$$=-\frac{dP}{dx}-\frac{1}{dx}d\cdot[P]V+\frac{ddQ}{dx^2}.$$

On which account from these two differential values this equation arises for the curve sought

$$\frac{\alpha}{dx}d\cdot\frac{p}{\sqrt{(1+pp)}}=+\frac{dP}{dx}+\frac{1}{dx}d\cdot[P]V-\frac{ddQ}{dx^2},$$

which integrated gives

$$\frac{\alpha p}{\sqrt{(1+pp)}}+\beta=P+[P]V-\frac{dQ}{dx}$$

or

$$\frac{\alpha p}{\sqrt{(1+pp)}}+\beta=\frac{Hp}{\sqrt{(1+pp)}}-\frac{p}{\sqrt{(1+pp)}}\int\frac{qqdS}{(1+pp)^3}+P-\frac{dQ}{dx},$$

where the constant H can be taken to be determined by the arbitrary constant α, from which the constant a may depart from the calculation itself. Therefore on this account this equation will be produced

$$\frac{\alpha p}{\sqrt{(1+pp)}}+\beta=P-\frac{dQ}{dx}-\frac{p}{\sqrt{(1+pp)}}\int\frac{qqdS}{(1+pp)^3}.$$

43. This equation may be multiplied by $dp=qdx$ and there will be produced :

$$\frac{\alpha pdp}{\sqrt{(1+pp)}}+\beta dp=Pdp-qdQ-\frac{pdp}{\sqrt{(1+pp)}}\int\frac{qqdS}{(1+pp)^3}.$$

But since there shall be

$$dZ=\frac{qqdS}{(1+pp)^{5:2}}+Pdp+Qdq,$$

it becomes

$$Pdp=dZ-Qdq-\frac{qqdS}{(1+pp)^{5:2}};$$

with which value substituted this integral equation emerges :

$$\frac{\alpha pdp}{\sqrt{(1+pp)}}+\beta dp=dZ-qdQ-Qdq-\frac{qqdS}{(1+pp)^{5:2}}-\frac{pdp}{\sqrt{(1+pp)}}\int\frac{qqdS}{(1+pp)^3},$$

of which the integral is :

$$\alpha\sqrt{(1+pp)}+\beta p+\gamma=Z-Qq-\sqrt{(1+pp)}\int\frac{qqdS}{(1+pp)^3}$$

or

$$\alpha\sqrt{(1+pp)}+\beta p+\gamma=-\frac{Sqq}{(1+pp)^{5:2}}-\sqrt{(1+pp)}\int\frac{qqdS}{(1+pp)^{3}}.$$

So that we may remove the integral sign, with the equation divided by $\sqrt{(1+pp)}$, that may be differentiated again :

$$\frac{\beta dp}{(1+pp)^{3:2}}-\frac{\gamma pdp}{(1+pp)^{3:2}}+\frac{2qqdS}{(1+pp)^{3}}+\frac{2Sqdq}{(1+pp)^{3}}-\frac{6Spqqdp}{(1+pp)^{4}}=0,$$

which multiplied by $\frac{(1+pp)^{3:2}}{2q}$ gives :

$$\frac{\beta dp}{2q}-\frac{\gamma pdp}{2q}+\frac{qdS+Sdq}{(1+pp)^{3:2}}-\frac{3Spqdp}{(1+pp)^{5:2}}=0,$$

of which, on account of $dp=qdx$ and $dy=pdx$, the integral will be

$$\alpha+\tfrac{1}{2}\beta x-\tfrac{1}{2}\gamma y+\frac{Sq}{(1+pp)^{3:2}}=0.$$

But $-\frac{(1+pp)^{3:2}}{q}=$ radius of osculation *R*, from which by doubling the constants β and γ this equation will arise

$$\frac{S}{R}=\alpha+\beta x-\gamma y\ ;$$

which equation is in complete agreement with that, which the other direct method supplies. For $\alpha+\beta x-\gamma y$ expresses the moment of the curving force, with some right line taken for the axis, to which the moment certainly must be equal to the absolute elasticity *S* divided by the radius of osculation *R*. Thus therefore not only has the observed property of elasticity of the celebrated Bernoulli been vindicated most fully, but also the more difficult use of my formulas taken in this example has been made clear.

44. Therefore if a curve were given (Fig. 3), so that a lamina may be formed with unequal elasticity from the force acting $CD = P$, hence the absolute elasticity of the lamina at some place can be known. For with the right line taken *CP*, which is normal to the direction of the force acting, for the axis and putting $CP = x,\ PM = y$, for the arc of the curve $AM = s$ and the radius of osculation at $M = R$, on account of the related moment of the force *P* at the point *M* $= Px$ there will be $\frac{S}{R} = Px$; and thus the absolute elasticity at the point *M*, which is $S, \ = PRx$. Hence, since the given curve may give the radius of osculation *R* at the individual points, the absolute elasticity will be know at any point. But if therefore the material of the lamina were the same everywhere together with a single thickness, but the width shall be variable : because the absolute elasticity is proportional to the width, the width of the lamina will be deduced at individual places from the curve formed.

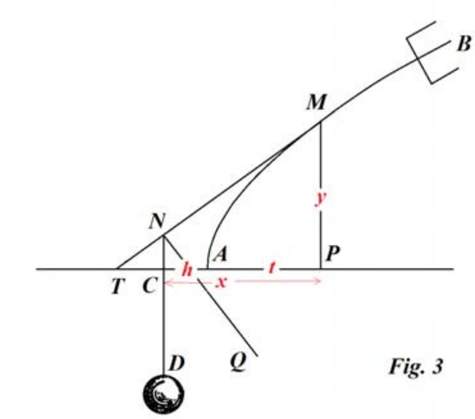

Fig. 3

45. Let a small triangular tongue *fAf* be cut out from an elastic lamina (Fig. 15) of the same thickness everywhere. Therefore because the width *mm* at some place *M* is proportional to the length *AM*, putting $AM = s$ the absolute elasticity at *M* will be as s. That shall be $= Eks$, and the end of the triangle *ff* may be fixed horizontally in the wall and the weight *P* may be suspended from the point *A*, so that the right median line *AF* (Fig. 14) of the lamina may be formed into the curve *FmA*, of which the nature is sought. But with the abscissa put on the horizontal axis $Ap = x$, with the applied line $pm = y$ and with the arc $Am = s$, the moment will be $Px = \frac{Eks}{R}$ with *R* denoting the radius of oscillation at *m*. This equation may be multiplied by *dx* and on account of $R = -\frac{ds^3}{dxddy}$, by putting *dx* constant, there will be

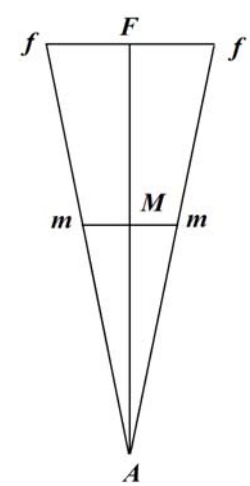

Fig. 15

$$Pxdx = -\frac{Eksdx^2ddy}{ds^3} \quad \text{or} \quad \frac{Pxdx}{Ek} + \frac{sdx^2ddy}{ds^3} = 0.$$

But, since there shall be

$$d \cdot \frac{sdy}{ds} = \frac{sddy}{ds} - \frac{sdydds}{ds^2} + dy = \frac{sdx^2ddy}{ds^3} + dy$$

on account of $dds = \frac{dyddy}{ds}$, there will be

$$\int \frac{sdx^2 ddy}{ds^3} = \frac{sdy}{ds} - y,$$

from which by integration there will be found :

$$\frac{Pxx}{2Ek} + a = -\frac{sdy}{ds} + y\,.$$

46. Let $dy = pdx$, there will be $ds = dx\sqrt{(1+pp)}$ and on putting $\frac{2Ek}{P} = c$ the above equation becomes :

$$a + \frac{xx}{c} = y - \frac{sp}{\sqrt{(1+pp)}}\,;$$

and thus there will be

$$\frac{a\sqrt{(1+pp)}}{p} + \frac{xx\sqrt{(1+pp)}}{cp} = \frac{y\sqrt{(1+pp)}}{p} - s\,;$$

which differentiated gives :

$$-\frac{adp}{pp\sqrt{(1+pp)}} + \frac{2xdx\sqrt{(1+pp)}}{cp} - \frac{xxdp}{cpp\sqrt{(1+pp)}}$$
$$= \frac{dy\sqrt{(1+pp)}}{p} - \frac{ydp}{pp\sqrt{(1+pp)}} - dx\sqrt{(1+pp)} = -\frac{ydp}{pp\sqrt{(1+pp)}}.$$

Hence there arises

$$a - y = \frac{2pxdx(1+pp)}{cdp} - \frac{xx}{c}\,.$$

Putting *dp* constant, and on being differentiated there will be

$$-pdx = \frac{2pxddx(1+pp)}{cdp} + \frac{2pdx^2(1+pp)}{cdp} + \frac{2xdx(1+3pp)}{c} - \frac{2xdx}{c}$$

or

$$0 = cdxdp + 2xddx(1+pp) + 2dx^2(1+pp) + 6pxdxdp,$$

but further resolution of which equation cannot be put in place. Moreover this equation is the most simple for the curve

$$\frac{yds - sdy}{ds} = \frac{Pxx}{2Ek};$$

because indeed on putting $x = 0$, both y and s must vanish, and the constant a must be $= 0$.

CONCERNING THE CURVATURE OF ELASTIC LAMINAS NOT NATURALLY STRAIGHT

47. Therefore the curvature of either equal or unequal elasticity is determined in this manner, if it may be from single force acting, as it is required to be noted particularly, if the lamina naturally were extended along a right line. Because if the lamina in the natural state were now a curve, then by the force acting it may adopt another curvature everywhere ; towards finding which, in addition to the force acting and the elasticity likewise its natural shape is required to be known. Therefore let the elastic lamina naturally be the curve *Bma* (Fig. 16), of which the elasticity shall be the same everywhere $= Ekk$, which may be curved into the figure *BMA* by the force *P* acting. Through *A* the right line *CAP* may be drawn normal to the direction of the force acting, which may be taken for the axis, and the interval *AC* shall be $= c$, the abscissa $AP = x$, the applied line $PM = s$; the moment of the force acting for the point *M* will be $= P(c + x)$.

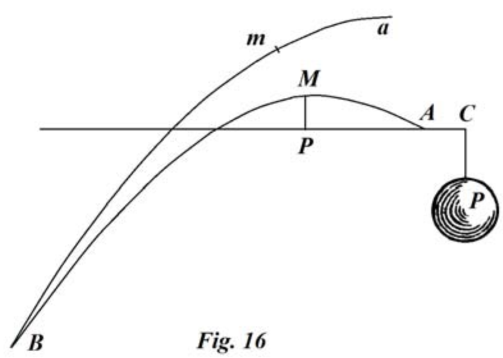

Fig. 16

48. Again the radius of osculation of the curve sought at M shall be $= R$; the arc $am = AM = s$ may be taken in the natural state and at the point *m* the radius of osculation shall be $= r$, which on account of the known curve *amB* will be given by the arc *S*. Therefore at *M*, because the curvature is greater, the radius of osculation *R* is less than *r* and the excess of the elementary angle at *M* above the natural angle will be $= \frac{ds}{R} - \frac{ds}{r}$, which excess will be effect produced by the force acting. On account of which

$$P(c + x) = Ekk\left(\frac{1}{R} - \frac{1}{r}\right),$$

which, since *r* may be give by *s*, will be the equation for the curve sought ; but which thus in general cannot be seen to be reduced further.

49. Therefore we may put the lamina in the natural state *amB* to have a circular figure ; *r* will be the radius of this circle, which shall be $=a$, from which there shall be

$$P(c+x)=Ekk\left(\frac{1}{R}-\frac{1}{a}\right).$$

This equation may be multiplied by *dx* and integrated ; there will arise

$$\frac{P}{Ekk}\left(\tfrac{1}{2}xx+cx+f\right)=-\frac{dy}{ds}-\frac{x}{a};$$

[Recall that :

$$ds=Rd\theta;\frac{dy}{dx}=\tan\theta;\frac{d^2y}{dx^2}=\sec^2\theta\frac{d\theta}{dx}=\left(1+\left(\frac{dy}{dx}\right)^2\right)\frac{d\theta}{dx};$$

$$\therefore\frac{1}{R}=\frac{d\theta}{ds}=\frac{\frac{d^2y}{dx^2}}{\left(1+\left(\frac{dy}{dx}\right)^2\right)^{\frac{3}{2}}}.\text{ Hence }\int\frac{dx}{R}=\int dx\frac{\frac{d^2y}{dx^2}}{\left(1+\left(\frac{dy}{dx}\right)^2\right)^{\frac{3}{2}}}$$

$$=-\frac{\frac{dy}{dx}}{\left(1+\left(\frac{dy}{dx}\right)^2\right)^{\frac{1}{2}}}=-\sin\theta=-\frac{dy}{ds}].$$

which equation, if in place of *c* there may be written $c+\frac{Ekk}{Pa}$, it will change into

$$\frac{P}{Ekk}\left(\tfrac{1}{2}xx+cx+f\right)=-\frac{dy}{ds},$$

which is the same equation, as we have found above for the naturally straight lamina. Therefore the naturally curving lamina of a circle may curve in the same curves, as naturally right laminas adopt ; yet the position of the applied force or the interval *AC*= *c* is known, for each case will have to be following a given law of variation. Therefore the same nine kinds of curves are produced for figures, which a naturally circular lamina can induce, as we have related above. For a circular lamina, if the interval *AC* may be taken infinite, is able first to be extended in a straight line [the ninth kind above] ; then, whatever force applied in addition will put in place the same effect, as if it may be applied to a naturally right elastic lamina alone.

50. Moreover we may put, whatever the natural figure of the lamina shall be, the point *C* to be infinitely distant, thus so that the moment of the force acting shall be the same everywhere, which divided by *Ekk* may be put $=\frac{1}{b}$; and there will be

$$\frac{1}{b}=\left(\frac{1}{R}-\frac{1}{r}\right) \text{ and } \frac{1}{R}=\frac{1}{b}+\frac{1}{r}.$$

Hence there becomes

$$\int\frac{ds}{R}=\frac{s}{b}+\int\frac{ds}{r}= \text{ the amplitude of the arc } AM\,,$$

just as $\int\frac{ds}{r}$ expresses the amplitude of the arc *am*; as indeed the most celebrated Johan Bernoulli has been accustomed to call this by the name *amplitude* in the exemplary tract *Motus reptorius* [Lit. *Creeping motion*]. Therefore $\frac{s}{b}+\int\frac{ds}{r}$ shall be a circular arc, of which the radius taken $=1$, which on account of *r* given by some *s*, also *s* will be known. Hence moreover the orthogonal coordinates *x* and *y* will be found, thus so that there shall be

$$x=\int ds\sin\left(\frac{s}{b}+\int\frac{ds}{r}\right) \text{ and } y=\int ds\cos\left(\frac{s}{b}+\int\frac{ds}{r}\right);$$

from which the curve sought will be able to be constructed by quadrature.

51. Hence the figure *amB* (Fig. 17) can be determined, as the lamina may be had in the natural position, so that by the force *P* acting in the direction *AP* it may be unfolded into the right line *AMB*. For with the length taken $AM=s$, the moment of the force acting for the point $M=Ps$; moreover the radius of osculation at *M*, per hypothesis, will be infinite or $\frac{1}{R}=0$. Now with the arc $am=s$ taken in the natural state and by putting the radius of osculation at m to be $=r$, because this curve from its convexity may seen to be the right line *AB*, *r* must be put negative in the preceding calculation. Hence there will be $Ps=\frac{Ekk}{r}$ or $rs=aa$; which is the equation describing the natural of the curve *amB*.

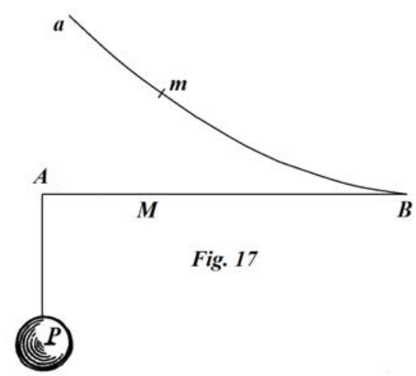

Fig. 17

52 . Therefore since $\frac{1}{r}=\frac{s}{aa}$, there will be $\int\frac{ds}{r}=\frac{ss}{2aa}$, of the amplitude of the arc *am* will be as the square of this arc. Hence the orthogonal coordinates *x* and *y* for this curve *amB* may be defined thus, so that there shall be

$$x=\int ds\sin\frac{ss}{2aa} \text{ and } y=\int ds\cos\frac{ss}{2aa};$$

Evidently in the circle, the radius of which $=1$, the arc $\frac{ss}{2aa}$ must be cut off, of which the sine and cosine must be assumed for determining the coordinates. Moreover from that, because the radius of osculation decreases constantly, so that the greater the arc $am=s$ may be taken, it is evident the curve does not extend to infinity, even if the arc *s* may be taken infinite. Therefore the curve will be generated by a kind of spiral, thus so that from an infinitely many completed turns it may definitely be turning around a certain point as centre, which point may appear difficult to find from this construction. Therefore it will not be a small step required to be taken to consider in the analysis of increments, if some method can be found, with the aid of which, perhaps truly approximately, the value of these integrations

$$\int ds\sin\frac{ss}{2aa} \text{ and } \int ds\cos\frac{ss}{2aa}$$

may be designated in the case in which *s* may be place infinite ; because the problem may not seem unworthy, in which Geometers may exercise their strengths.

[Such problems subsequently emerged in optics with Cornu's spiral and Fresnel's integrals for calculating the vibration curves of near field Fresnel diffraction patterns, usually without a mention of Euler, in the 19th century, and which find a modern application in high speed rail systems in changing from linear to circular motion in a seamless manner ; see Wikipedia for Euler's Spiral.]

53. Let $2aa=bb$, and since there shall be

$$\sin\frac{ss}{bb}=\frac{s^2}{b^2}-\frac{s^6}{1\cdot 2\cdot 3b^6}+\frac{s^{10}}{1\cdot 2\cdot 3\cdot 4\cdot 5b^{10}}-\frac{s^{14}}{1\cdot 2...7b^{14}}+\text{ etc.}$$

$$\cos\frac{ss}{bb}=1-\frac{s^4}{1\cdot 2b^4}+\frac{s^8}{1\cdot 2\cdot 3\cdot 4b^8}-\frac{s^{12}}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6b^{12}}+\text{ etc.},$$

the *x* and *y* coordinates of the curve will be expressed by infinite series conveniently, indeed there will be

$$x = \frac{s^3}{1\cdot 3b^2} - \frac{s^7}{1\cdot 2\cdot 3\cdot 7b^6} + \frac{s^{11}}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 11b^{10}} - \frac{s^{15}}{1\cdot 2 ... 7\cdot 15b^{14}} + \text{ etc.}$$
$$y = s - \frac{s^5}{1\cdot 2\cdot 5b^4} + \frac{s^9}{1\cdot 2\cdot 3\cdot 4\cdot 9b^8} - \frac{s^{13}}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 13b^{12}} + \text{ etc.},$$

from which with the series converging strongly, unless the arc *s* may be assumed very large, the values of the coordinates *x* and *y* are able to be determined readily and near enough. Truly the values *x* and *y* found, if the arc *s* may be put infinitely great, cannot be concluded from these series in any way.

54. Therefore because it produces a great difficulty by making *s* be put into an infinite position, it is indeed possible to bring a remedy to this inconvenience in the following manner.

Putting $\frac{ss}{bb} = v$, thus so that there shall be $s = b\sqrt{v}$, there will be $ds = \frac{bdv}{2\sqrt{v}}$ and there becomes

$$x = \frac{b}{2}\int \frac{dv}{\sqrt{v}} \sin v \text{ and } y = \frac{b}{2}\int \frac{dv}{\sqrt{v}} \cos v.$$

But now I say the values owed for *x* and *y*, if there may be put $s = \infty$, are going to be found from these integral formulas,

$$x = \frac{b}{2}\int dv\left(\frac{1}{\sqrt{v}} - \frac{1}{\sqrt{(\pi+v)}} + \frac{1}{\sqrt{(2\pi+v)}} - \frac{1}{\sqrt{(3\pi+v)}} + \text{ etc.}\right)\sin v$$
$$y = \frac{b}{2}\int dv\left(\frac{1}{\sqrt{v}} - \frac{1}{\sqrt{(\pi+v)}} + \frac{1}{\sqrt{(2\pi+v)}} - \frac{1}{\sqrt{(3\pi+v)}} + \text{ etc.}\right)\cos v,$$

if after integration there may be put $v = \pi$, with π denoting the angle equal to two right angles. Therefore in this manner indeed the position of infinity is avoided, truly on the other hand the infinite series

$$\frac{1}{\sqrt{v}} - \frac{1}{\sqrt{(\pi+v)}} + \frac{1}{\sqrt{(2\pi+v)}} - \text{ etc.}$$

is introduced into the calculation, the sum of which at this stage may still be hidden, a resolution of this kind at this point is liable to the greatest difficulty.

THE CURVATURE OF ELASTIC LAMINAS AT INDIVIDUAL POINTS ARISING FROM WHATEVER FORCES ACTING

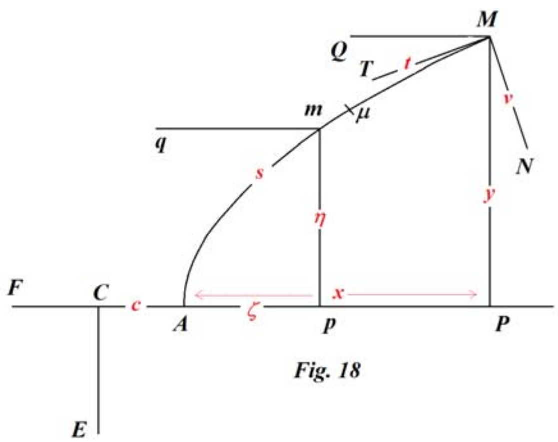

Fig. 18

55. Now with the method treated of investigating the curvature of each elastic lamina, if that may be acted on by one force applied at a given point. It will be agreed also to investigate the curvature of an elastic lamina induced by several, indeed by infinitely many forces. Because truly it is not yet agreed, what kind of expression shall become either a maximum or a minimum from these cases, I shall use only the direct method, so that from that solution perhaps that property which is a maximum or minimum may be able to be deduced. Therefore the elastic lamina (Fig. 18) shall be naturally right, returned in the position *AmM* in the first place by the finite forces *P* and *Q* acting along the directions *CE* and *CF* normal to each other, then truly with the individual elements of the lamina $m\mu$ pulled by infinitely small forces applied both along the directions *mp* and *mq* with these parallel to *CE* and *CF*; with which in place the nature of the induced curve *AmM* is required.

56. The right line *FCA* produced may be taken for the axis, putting $AC = c$ and calling the abscissa $AP = x$, the applied line $PM = y$, the arc of the curve $AM = s$ and the radius of osculation at $M = R$. Let the absolute elastic constant of the lamina $= Ekk$, and the sum of moments arising from all the forces acting with respect to the point *M* must be equal to $\frac{Ekk}{R}$. Indeed in the first place from the finite force *P* acting in the direction *CE* the moment arises $= P(c+x)$ acting in that sense, by which the elastic force may be kept in a state of equilibrium. But the moment arising from the other force *Q*, surely *Qy*, acts in the contrary sense, so that from the finite forces *P* and *Q* jointly the moment $P(c+x)-Qy$ arises. Now some intermediate element of the lamina may be considered $m\mu$, the corresponding abscissa of which may be put $Ap = \zeta$ and with the applied line $pm = \eta$, but the force acting on the element $m\mu$ shall be in the direction *mp* shall be $= dp$ and the force acting in the direction $mq = dq$; the moment arising from these forces for the point *M*

$$=(x-\zeta)dp-(y-\eta)dq.$$

57. Therefore for finding the sum of all these moments the point *M* and hence *x* and *y* for the time being must be had as constants, then the coordinates ζ and η alone with the forces *dp* and *dq* may be considered as variables. Therefore the sum of the moments arising from the arc *Am* acting

$$= xp - \int \zeta dp - yp + \int \eta dq ,$$

where *p* expresses the sum of all the forces of the arc *AM* acting in directions parallel to the applied line *pm* and *q* the sum of all the forces of the arc *Am* acting in directions parallel to the axis *Ap*. But there is

$$\int \zeta dp = \zeta p - \int pd\zeta \text{ and } \int \zeta dq = \eta q - \int qd\eta ;$$

from which the sum of the moments becomes arising from the forces applied to the arc *Am*

$$= (x-\zeta) p + \int pd\zeta - (y-\eta) q - \int qd\eta.$$

Now the point *m* may be moved forwards as to *M* and there becomes $\zeta = x,\ \eta = y$ and $d\zeta = dx$ and $d\eta = dy$; from which the sum of all the moments taken through the whole arc *AM* will be

$$= \int pdx - \int qdy .$$

On account of which this equation will be obtained for the curve sought :

$$\frac{Ekk}{R} = P(c+x) - Qy + \int pdx - \int qdy,$$

where therefore *p* expresses the sum of all the vertical forces or of the forces acting in the directions of the applied line *MP*, and *q* the sum of all the horizontal forces or of the forces acting in the directions *MQ* parallel to the axis *AP*, through the whole arc *AM*.

58. If the formulas *pdx* and *qdy* may not permit integration, then the equation found by differentiation from these formulas will have to be freed from integrations, from which this equation will be found :

$$\frac{EkkdR}{RR} = Pdx - Qdy + pdx - qdy.$$

But if neither *p* nor *q* shall be able to be shown by finite expressions, certainly which now express the sums of an infinitude of infinitely small forces, then by a further differentiation must remove the finite values *p* and *q*, so that at last *dp* and *dq* may be present with the differentiated differentials *ddp* and *ddq*. Moreover after the first differentiation the equation may arise

$$-Ekkd\cdot\frac{dR}{RRdx}=dp-\left(Q+q\right)d\cdot\frac{dy}{dx}-\frac{dy}{dx}dq.$$

Let $\frac{dy}{dx}=\omega,$ and there will be with the equation differentiated anew :

$$-Ekkd\cdot\frac{d\cdot\frac{dR}{RRdx}}{d\omega}=d\cdot\frac{dp}{d\omega}-2dq-\omega d\cdot\frac{dq}{d\omega},$$

which equation rises to differentials of the fourth order.

59. In place of the vertical and horizontal forces *p* and *q*, two forces shall be applied to the lamina at the individual point *M*, the one to the normal $MN=dv$ and the other to the tangent $MT=dt$. Hence there will be

$$dp=\frac{dxdv}{ds}+\frac{dydt}{ds}\ \text{and}\ dq=\frac{dxdt}{ds}-\frac{dydv}{ds}$$

and on account of $dy=\omega dx$ and $ds=dx\sqrt{(1+\omega\omega)}$ there will be found :

$$dp=\frac{dv}{\sqrt{(1+\omega\omega)}}+\frac{\omega dt}{\sqrt{(1+\omega\omega)}}\quad\text{and}\quad dq=\frac{dt}{\sqrt{(1+\omega\omega)}}-\frac{\omega dv}{\sqrt{(1+\omega\omega)}};$$

from which, with the final equation in the preceding paragraph substituted, the following equation will arise :

$$-Ekkd\cdot\frac{d\cdot\frac{dR}{RRdx}}{d\omega}=-\frac{dt}{\sqrt{(1+\omega\omega)}}+\frac{2\omega dv}{\sqrt{(1+\omega\omega)}}+\sqrt{(1+\omega\omega)}d\cdot\frac{dv}{d\omega},$$

which multiplied by $\sqrt{(1+\omega\omega)}$ is made integrable ; for on putting for the sake of brevity, $z=\frac{dR}{RRdx}$ the integral will be found :

$$A-t+\frac{dv\left(1+\omega\omega\right)}{d\omega}=-Ekk\left(\frac{dz\sqrt{(1+\omega\omega)}}{d\omega}-\frac{\omega z}{\sqrt{(1+\omega\omega)}}+\frac{1}{2RR}\right)$$
$$=-Ekk\left(\frac{1+\omega\omega}{d\omega}d\cdot\frac{dR}{RRdx\sqrt{(1+\omega\omega)}}+\frac{1}{2RR}\right).$$

Truly since there shall be

$$R=-\frac{(1+\omega\omega)^{3:2}}{d\omega}$$

there will be

$$d\omega=-\frac{(1+\omega\omega)^{3:2}\,dx}{R};$$

so that with the value substituted in place of $d\omega$ there will be found :

$$A-t-\frac{Rdv}{ds}=-Ekk\left(\frac{1}{2RR}-\frac{R}{ds}d\cdot\frac{dR}{RRds}\right),$$

on account of $dx\sqrt{(1+\omega\omega)}=ds$. On account of which with the equation put in order, this equation will arise for the curve sought

$$t+\frac{Rdv}{ds}-A=Ekk\left(\frac{1}{2RR}-\frac{R}{ds}d\cdot\frac{dR}{RRds}\right).$$

60. Indeed in the first place it is evident, if the elastic force *Ekk* may vanish, the lamina becomes transformed into a perfectly flexible filament ; and hence in all the curves will be contained in these equations, which a perfectly flexible filament can form with some kind of forces acting. Thus if a filament by its own weight is acted on downwards only, there will be $q=0$ and *p* will express the weight of the rope *AM*, and therefore there will be $\frac{pdx}{dy}=Q=\text{constant}$ with $P=0$, which is the general equation for all kinds of catenaries. But if the perfectly flexible filament may be acted on at individual points by forces, the directions of which are normal to the curve itself, thus so that at the point *M* the filament will be acted on along the direction *MN* by a force $=dv$, on account of $t=0$ there will be $\frac{Rdv}{ds}=A=\text{constant}$, which is the general property of curved sails, awnings and of everything, in which actions of this kind have a place.

THE CURVATURE OF ELASTIC LAMINAS ARISING FROM THEIR OWN WEIGHT

61. But I return to elastic laminas, concerning which soon that question before all others worthy of note presents itself: what kind of figure an elastic lamina may take curved by its own weight. Let *AmM* be this curve, which is sought, and because the vertical forces arising from gravity alone act, there becomes $P=0,\ Q=0,\ q=0$ and p will express the weight of the lamina *AM*. Whereby, if *F* shall be the weight of a lamina of length *a*,

because the lamina is assumed to be uniform, there will be $p = \frac{Fs}{a}$; from which the nature of the curve will be expressed by this equation

$$-\frac{EkkdR}{RR} = \frac{Fsdx}{a}.$$

Let the amplitude of the curve $\int \frac{ds}{R} = u$, there will be $R = \frac{ds}{du}$ and $dx = ds \sin u$; from which with the element *ds* constant the equation will be found :

$$sds \sin u + \frac{Eakk}{F} \cdot \frac{ddu}{ds} = 0,$$

but which, as far as it is apparent at first glance, cannot be reduced further.

62. But in the first place the curve deserves to be noted, which a fluid as if of an infinite height induces in an elastic lamina. Let (Fig. 19) *AMB* be this figure, which is sought, and on putting $AP = x$, $PM = y$, $AM = s$ the element *Mm* will be acted on in the direction of the normal *MN* by a force proportional to *ds* itself ; from which there will be $dv = nds$ and $dt = 0$.

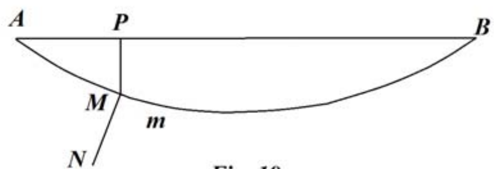

Fig. 19

[*n* is the constant of proportionality.]
Hence the vertical force will arise $dp = ndx$ and the horizontal force $dq = -ndy$; from which at once there becomes $p = nx$ and $q = -ny$; and thus in the first equation there becomes

$$\frac{Ekk}{R} = P(c+x) - Qy + \tfrac{1}{2}nxx + \tfrac{1}{2}nyy.$$

Truly the coordinates *x* and *y* thus can be increased or diminished by constants, so that the equation for a curve of this kind may acquire the form $xx + yy = A + \frac{B}{R}$. But this equation if it may be multiplied by $xdx + ydy$, becomes integrable, [since $\frac{dy}{dx} = \omega$]; indeed

$$\int \frac{xdx + ydy}{R} = -\int \frac{x + y\omega}{(1+\omega\omega)^{3:2}} d\omega = \frac{y - \omega x}{\sqrt{(1+\omega\omega)}} = \frac{ydx - xdy}{ds}.$$

Hence on this account, after integration with the constants changed, the equation will be produced

$$(xx+yy)^2 = A(xx+yy) + \frac{B(ydx-xdy)}{ds} + C.$$

Let $\sqrt{(xx+yy)} = z$ and $y = uz$, there will be $x = z\sqrt{(1-uu)}$; from which

$$ydx - xdy = -\frac{zzdu}{\sqrt{(1-uu)}} \quad \text{and} \quad ds = \sqrt{\left(dz^2 + \frac{zzdu^2}{1-uu}\right)}.$$

Therefore on putting $\frac{du}{\sqrt{(1-uu)}} = dr$, there will be

$$z^4 - Az^2 - C = -\frac{Bzzdr}{\sqrt{(dz^2 + zzdr^2)}} \quad ;$$

and hence

$$dr = \frac{du}{\sqrt{(1-uu)}} = -\frac{dz(z^4 - Az^2 - C)}{z\sqrt{\left(B^2zz - (z^4 - Az^2 - C)^2\right)}}.$$

Therefore this curve, if there were $A = 0$ and $C = 0$, will be algebraic ; for this equation will be had :

$$\frac{du}{\sqrt{(1-uu)}} = \frac{zzdz}{\sqrt{(B^2 - z^6)}} = \frac{3zzdz}{3\sqrt{(a^6 - z^6)}},$$

which integrated gives [Note : Asin is equivalent to arcsin .]

$$\text{Asin}u = \tfrac{1}{3}\text{Asin}\frac{z^3}{a^3} \text{ or } \frac{z^3}{a^3} = 3u - 4u^3 = \frac{3y}{z} - \frac{4y^3}{z^3};$$

from which this equation results $z^6 = 3a^3yzz - 4a^3y^3$, or on account of $zz = xx + yy$, this :

$$x^6 + 3x^4y^2 + 3xxy^4 + y^6 = 3a^3xxy - a^3y^3.$$

CONCERNING THE OSCILLATORY MOTION OF ELASTIC LAMINAS

63. Also from these, any motion of oscillating elastic laminas can be defined according to the motion of the lamina agreed on, which most worthy argument the most celebrated Daniel Bernoulli certainly began to develop first, and had proposed to me now not too many years ago the problem to be determined, concerning the oscillations of elastic laminas with the one end firmly fixed to the wall, the solution of which I have shown in

Comment. Petropol. Book VII. But from this time as it has become easier for me to grasp how to treat this problem, as well also from the exchange of letters with the celebrated Bernoulli, several other questions and considerations have been added, the unraveling of which I have added here, on account of the similarity of the material. But when the motion of the vibration is quick enough, then at the same time a sound is produced by the vibrating lamina, the tone of which and the relation to others will be determined from these principles with the aid of the theory of sound. And because the kinds of sounds may readily be recalled by experiment, from this the agreement of the calculation with the truth can be explored and thus the theory will be able to be confirmed ; with which agreed on our knowledge about the nature of vibrating elastic bodies will be increased not a little.

64. But in the first place it is required to be aware here only a question concerning the smallest oscillations to be put in place and thus the interval, through which the lamina extends, to be as if infinitely small. Truly nor by this limitation is the use and any application to be diminished; for not only the oscillations, if they shall be made through greater spaces, may no longer be isochronous, but also the formation of distinct tones, which we may look for here chiefly, require the smallest oscillations. Therefore here initially I will consider a naturally uniform right elastic lamina, (Fig. 20), one end of which *B* shall be fixed firmly immobile in a pavement, thus so that the remaining lamina itself may maintain the right line *BA*. The length of this lamina shall be $AB = a$ and the absolute elasticity at the individual points shall be $= Ekk$; truly we have considered neither its weight nor the manner of fixture we have put in place, so that its position may not be disturbed by gravity.

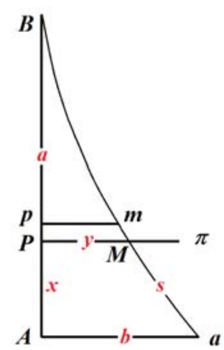

Fig. 20

CONCERNING THE OSCILLATIONS OF AN ELASTIC LAMINA WITH THE OTHER END FIXED TO A WALL

65. Now this lamina from the impulse of some force may perform minimal oscillations about the natural position *BA* in place by extending on each side through the minimum interval *Aa*. And *BMa* shall be some position, which the lamina maintains while oscillating ; which because it shall be only an infinitely small distance from the natural state *BPA*, the right lines *MP*, *Aa* likewise will represent the ways, which the points *M* and *a* extend, or rather these right lines to the true ways will have a ratio an infinitely small difference from the ratio of equality. But towards determining the motion of the oscillations absolutely it is necessary to know the nature of the curve *BMa*, which the lamina adopts while oscillating. Therefore there shall be $AP = x,\ PM = y$, the arc $aM = s$, and the radius of osculation at $M = R$ and the minimum interval $Aa = b$; and from the condition mentioned the arc *s* will be approximately equal to the abscissa *x*, and on this account it will be able to take *dx* for *ds*; for *dx* will vanish before *dy*. And since on putting *dx* constant the radius of osculation generally shall be

$$= \frac{ds^3}{dxddy},$$

in the present case it will become $R = \frac{dx^2}{ddy}$; for the curve *BMa* turns convex towards the *BA* and, because the lamina is fastened firmly to the wall at *B*, the right line *AB* will be a tangent to the curve at the point *B*.

66.With these in place, towards determining the nature of the curve *BMa* as well as the motion of the oscillation itself, the length of the simple isochronous pendulum shall be f; for the minimal oscillations are declared to be isochronous from the nature of the problem, as the calculation being put in place itself will show. Therefore the acceleration, by which the point *M* of the lamina is pushed towards *P*, will be $= \frac{PM}{f} = \frac{y}{f}$.

[This follows from E40, where Euler considers a simple pendulum in which the component of the weight acting horizontally when the mass is extended infinitesimally sideways becomes $\frac{y}{f} \times Mass\ of\ bob$, from similar triangles. There is a continual confusion between mass and weight in this development ; clearly the acceleration of gravity should be present in comparing forces, and simple dimensional analysis indicates where it should be present in such equations.]

Whereby, if the whole mass of the lamina were put $= M$, which is expressed by its weight, the mass of the element $Mm = ds = dx$ will be $= \frac{Mdx}{a}$; from which the moving force acting on the element *Mm* in the direction *MP* will be $= \frac{Mydx}{af}$; and thus the forces, by which the individual particles of the lamina actually are moved, become known both from the curve *BMa* itself, as well as from the length of the isochronous simple pendulum *f*. Because truly the lamina actually is urged to move by the elastic force, with this known both the nature of the curve *BMa* and the length of the simple isochronous pendulum in turn will be determined.

67. Therefore because the lamina may be moved thus, and if to the individual elements *Mm* of that, forces $= \frac{Mydx}{af}$ may be applied in the direction *MP*, it follows, if to the individual elements of the lamina *Mm* , equal forces $\frac{Mydx}{af}$ may be applied in the opposite direction $M\pi$, the lamina will be in the state of equilibrium *BMa*. Hence the lamina while oscillating will approach the same curve which it adopts at rest, if at the individual points *M* with the forces acting $\frac{Mydx}{af}$ in the directions $M\pi$. Therefore by the above rule found from paragraph 56 all these forces applied through the arc *aM* may be

gathered together, and the sum will be produced $\frac{M}{af}\int ydx$ which must be substituted there in place p. Whereby, since the remaining forces P, Q and q may vanish, which may be had there, the nature of the curve will be expressed by the equation

$$\frac{Ekk}{R} = \int pdx\,,$$

from which there will be had

$$\frac{Ekk}{R} = \frac{M}{af}\int dx \int ydx.$$

Truly since there shall be $R = \frac{dx^2}{ddy}$, the equation becomes:

$$\frac{Ekkddy}{dx^2} = \frac{M}{af}\int dx \int ydx$$

and on differentiation

$$\frac{Ekkd^3y}{dx^2} = \frac{Mdx}{af}\int ydx$$

and by differentiating again this equation of the fourth order will be produced :

$$Ekkd^4y = \frac{Mydx^4}{af}.$$

[Compare this with modern theory, and we find that *Ekk* is equivalent to *EI*, the Young's modulus by the moment of inertia, or the stiffness of the beam, while $\frac{M}{a} = \frac{M'}{a}g$ is the weight per unit length, while $\mu = \frac{M'}{a}$ is the mass per unit length, leaving the term $\frac{g}{f}$ relating to the period of a simple pendulum, which on its own is given by $T = 2\pi\sqrt{\frac{f}{g}}$, or the corresponding angular frequency is found from $\omega^2 = \frac{g}{f}$; thus Euler's formula relates directly to $EI\frac{d^4y}{dx^4} = \mu\omega^2 y$.]

68. Therefore from this equation both the nature of the curve *BMa* is expressed as well as from the same, if it may be applied to the present case, the length f will be determined ; with which known the oscillatory motion itself will become known. But before all it is required to integrate this equation ; which since it may belong to that kind of differential equations of higher order, the general integration of which I have shown in *Miscellanea*

Berolinensea Volume VII, [1743] [or see E62, §32, Example 4, *O.O.* vol. 22, p.127.] hence the following equation of the integral may be found by putting therefore for brevity $\frac{Ekk\cdot af}{M}=c^4$; it is known that there will be produced

$$y=Ae^{\frac{x}{c}}+Be^{-\frac{x}{c}}+C\sin\frac{x}{c}+D\cos\frac{x}{c},$$

where *e* denotes the number, the hyperbolic logarithm of which is $=1$, and $\sin\frac{x}{c}$ and $\cos\frac{x}{c}$ denote the sine and cosine of the arc $=x$ in the circle, the radius of which is taken $=1$. Then truly *A*, *B*, *C* and *D* are four arbitrary constants introduced by the four integrations, which it is required to define for the application of the calculation to the present case.

69. Moreover the determination of the constants may be put in place in the following manner. In the first case putting $x=0$ there must become $y=b$; hence therefore this equation arises $b=A+B+D$, which is the first equation. In the second case, since there shall be

$$\frac{c^4ddy}{dx^2}=\int dx\int ydx$$

by making $x=0$ there must become $\frac{ddy}{dx^2}=0$; but there becomes :

$$\frac{ddy}{dx^2}=\frac{A}{cc}e^{\frac{x}{c}}+\frac{B}{cc}e^{-\frac{x}{c}}-\frac{C}{cc}\sin\frac{x}{c}-\frac{D}{cc}\cos\frac{x}{c},$$

from which this second equation arises : $0=A+\ B-D$.

Thirdly, since there shall be $\frac{c^4d^3y}{dx^3}=\int ydx$, on putting $x=0$ likewise $\frac{d^3y}{dx^3}$ must vanish ; therefore because there will be :

$$\frac{c^3d^3y}{dx^3}=Ae^{\frac{x}{c}}-Be^{-\frac{x}{c}}-C\cos\frac{x}{c}+D\sin\frac{x}{c},$$

the third equation will be produced : $0=A-B-C$.
Moreover in the fourth place, if there is put $x=a$, the applied line *y* vanishes, from which the fourth equation will be obtained :

$$0=Ae^{\frac{a}{c}}+Be^{-\frac{a}{c}}+C\cos\frac{a}{c}+D\sin\frac{a}{c}.$$

In the fifth place, because *AB* is a tangent to the curve at the point *B*, on making $x = a$ there must become $\frac{dy}{dx} = 0$; from which the fifth equation is produced :

$$0 = Ae^{\frac{a}{c}} - Be^{-\frac{a}{c}} + C\cos\frac{a}{c} - D\sin\frac{a}{c}.$$

Therefore from these five equations the first four constants *A*, *B*, *C*, *D* will be defined; then truly, on which the heart of the matter turns, the value of *c* itself will be determined :

$$c = \sqrt[4]{\frac{Ekk\cdot af}{M}};$$

from which the length *f* of the simple isochronous pendulum will be elicited, and from which in turn the times of the oscillations will be known.

70. The constants *C* and *D* thus may be defined from *A* and *B*, so that there shall be $C = A - B$ and $D = A + B$, whereby the values substituted into the fourth and fifth equations will give

$$0 = Ae^{\frac{a}{c}} + Be^{-\frac{a}{c}} + (A - B)\sin\frac{a}{c} + (A + B)\cos\frac{a}{c},$$

$$0 = Ae^{\frac{a}{c}} - Be^{-\frac{a}{c}} - (A + B)\cos\frac{a}{c} - (A + B)\sin\frac{a}{c};$$

and from which there arises,

$$\frac{A}{B} = \frac{-e^{-\frac{a}{c}} + \sin\frac{a}{c} - \cos\frac{a}{c}}{e^{\frac{a}{c}} + \sin\frac{a}{c} + \cos\frac{a}{c}} = \frac{e^{-\frac{a}{c}} + \cos\frac{a}{c} + \sin\frac{a}{c}}{e^{\frac{a}{c}} + \cos\frac{a}{c} - \sin\frac{a}{c}},$$

from which this equation is found :

$$0 = 2 + \left(e^{-\frac{a}{c}} + e^{-\frac{a}{c}}\right)\cos\frac{a}{c}$$

or

$$e^{\frac{2a}{c}}\cos\frac{a}{c} + 2e^{\frac{a}{c}} + \cos\frac{a}{c} = 0,$$

which gives

$$e^{\frac{a}{c}} = \frac{-1 \pm \sin\frac{a}{c}}{\cos\frac{a}{c}}.$$

Therefore since $e^{\frac{a}{c}}$ shall be a positive quantity, the cosine of the angle $\frac{a}{c}$ will be negative, and thus the angle $\frac{a}{c}$ will be more than a right angle.

71. From this equation it is understood an infinite number of angles $\frac{a}{c}$ to be given satisfying the question, from which infinitely diverse modes of oscillation of the same lamina arise. For the curve can cut the axis *AB* at one or more points, before it may touch the axis at *B* ; from which several, indeed an infinite number, of modes of oscillation of the same lamina are equally possible. Therefore since here in especially we will consider the case, so that *B* is the first point, where the lamina is touched by the axis *AB*, the smallest angle $\frac{a}{c}$ will satisfy this case resolving the equation found ; whichever angle since $\frac{a}{c}$ shall be greater than a right angle, may be put to be $\frac{a}{c} = \frac{1}{2}\pi + \varphi$ with the angle φ being less than a right angle. Hence on account of $\sin\frac{a}{c} = \cos\varphi$ and $\cos\frac{a}{c} = -\sin\varphi$, the two-fold equation will be obtained :

$$e^{\frac{a}{c}} = \frac{1 \pm \cos\varphi}{\sin\varphi},$$

which give either $e^{\frac{a}{c}} = \operatorname{tang}\frac{1}{2}\varphi$ or $e^{\frac{a}{c}} = \cot\frac{1}{2}\varphi,$ of which the latter will give the smaller value for the angle φ, which therefore will be appropriate for the case proposed.

72. The following possible modes of oscillation will be found, if for $\frac{a}{c}$ angles may be put in place greater than two right angles, truly less than three. Thus on putting $\frac{a}{c} = \frac{3}{2}\pi + \varphi$ there will be

$$\sin\frac{a}{c} = -\cos\varphi \text{ and } \cos\frac{a}{c} = -\sin\varphi;$$

from which there becomes

$$e^{\frac{a}{c}} = \frac{1 + \cos\varphi}{\sin\varphi}, \text{ either } e^{\frac{a}{c}} = \operatorname{tang}\tfrac{1}{2}\varphi \text{ or } e^{\frac{a}{c}} = \cot\tfrac{1}{2}\varphi.$$

Other modes of oscillation will be found in a similar manner, on putting

$$\frac{a}{c} = \tfrac{5}{2}\pi + \varphi, \ \frac{a}{c} = \tfrac{7}{2}\pi - \varphi \text{ etc.}$$

From all of which, if the hyperbolic logarithms may be taken, the following equations will arise :

I.	$\frac{1}{2}\pi+\varphi=l\cot\frac{1}{2}\varphi$	II.	$\frac{1}{2}\pi+\varphi=l\text{tang}\frac{1}{2}\varphi$
III.	$\frac{3}{2}\pi-\varphi=l\cot\frac{1}{2}\varphi$	IV.	$\frac{3}{2}\pi-\varphi=l\text{tang}\frac{1}{2}\varphi$
V.	$\frac{5}{2}\pi+\varphi=l\cot\frac{1}{2}\varphi$	VI.	$\frac{5}{2}\pi+\varphi=l\text{tang}\frac{1}{2}\varphi$
VII.	$\frac{7}{2}\pi-\varphi=l\cot\frac{1}{2}\varphi$	VIII.	$\frac{7}{2}\pi-\varphi=l\text{tang}\frac{1}{2}\varphi$

etc.

But of these equations the third agrees with the second; for by putting $\frac{1}{2}\varphi=\frac{1}{2}\pi-\frac{1}{2}\theta$, so that there shall be $\cot\frac{1}{2}\varphi=\text{tang}\frac{1}{2}\theta$, the third will change into $\frac{1}{2}\pi+\theta=l\text{tang}\frac{1}{2}\theta$, which is the second itself. In a similar manner the fourth agrees with the first, then the fifth and the eighth agree with each other and the sixth with the seventh. On account of which the following different equations will be produced :

I. $\frac{1}{2}\pi+\varphi=l\cot\frac{1}{2}\varphi$

II. $\frac{1}{2}\pi+\varphi=l\text{tang}\frac{1}{2}\varphi$

III. $\frac{5}{2}\pi+\varphi=l\cot\frac{1}{2}\varphi$

IV. $\frac{5}{2}\pi+\varphi=l\text{tang}\frac{1}{2}\varphi$

V. $\frac{9}{2}\pi+\varphi=l\cot\frac{1}{2}\varphi$

VI. $\frac{9}{2}\pi+\varphi=l\text{tang}\frac{1}{2}\varphi$

etc.

73. But the hyperbolic logarithm of the tangent or cotangent of some angle will be found by taking the tabulated logarithm and then by taking away the logarithm of the whole sine ; [this step is not needed with modern tables, as usually the log of the whole sine is just zero; this was not necessarily the case in the tables used by Euler and his contemporaries, where often the whole sine was a large power of 10 ; now we may simply write $x=e^{\ln x}=10^{\log x}$; thus $lx=\ln 10\log x=2,302585092994\log x$], and multiplying the remainder by 2,302585092994; whereby the labour may be lightened, as may be convenient again by logarithms. Let u be the hyperbolic logarithm of the tangent or of the angle $\frac{1}{2}\varphi$, which is sought , taken from tables of logarithms of the same tangent or cotangent, whereby the logarithm of the whole sine extracted may be put $=v$. Therefor since there shall be $u=2,302585092994\cdot v$, there will be on taking the common logarithms

$$lu=lv+0,3622156886.$$

With this logarithm found, since there shall be $u = \frac{n}{2}\pi + \varphi$, there will be

$$lu = l\left(\frac{n}{2}\pi + \varphi\right).$$

Towards resolving this equation, the angle φ must be expressed in parts of the radius [*i.e.* in radians], and just as π is expressed in the same manner, then $\pi = 3{,}1415926535$ and therefore $\frac{1}{2}\pi = 1{,}57079632679$. Moreover the angle φ is expressed in the same manner, if this may be converted into seconds and with the logarithm of this number 5,3144251332 constantly subtracted [Note that $\log\frac{180\times 60^2}{\pi} = 5{,}3144251332$]; for thus it will produce $l\varphi$, from which number by returning to the number the value of φ is elicited. Moreover there will be for each kind of oscillation :

$$\frac{a}{c} = u = \frac{n}{2}\pi + \varphi.$$

74. With these reminders put in place about the calculation to be put in place, through approximations the value of the angle φ for whatever kind of oscillation will be elicited without difficulty. For by attributing some number to φ as it pleases, and by calculation determining both $\frac{n}{2}\pi + \varphi$ and $l\,{}^{\text{tang}}_{\text{cot}}\,\frac{1}{2}\varphi$, soon the value of φ near the true value will be recognised. So that if moreover the bounds of the angle φ may be found to be distant in some manner, at once closer limits will be found, and from these at last the true value of φ. Thus for the first equation

$$\frac{a}{c} = \frac{n}{2}\pi + \varphi = l\cot\tfrac{1}{2}\varphi$$

the following limits of the angle φ are to be found, 17° 26' and 17° 27', from which by the following calculation the true value of φ will be obtained:

φ =	17° 26' 0"	17° 27' 0"
in min. sec. =	62760"	62820"
log. =	4,797 6S29349	4,79S0979321
subtr.	5,3144251332	5,3144251332
$l\varphi$ =	9, 4832578917	9, 4836727989
φ =	0, 3042690662	0,3045599545
$\frac{1}{2}\pi$ =	1,5707963268	1,5707963268
$\frac{1}{2}\pi+\varphi$ =	1,8750653930	1,8753562813
$\frac{1}{2}\varphi$ =	8° 43' 0"	8° 43' 30"
$l\cot\frac{1}{2}\varphi$ =	10, 8144034109	10,8139819342
v =	0, 8144034109	0 ,8139819342
lv =	9, 9108395839	9,9106147660
add. =	0,3622156886	0 ,3622156886
lu =	0,2730552725	0, 2728304546
u =	1,8752331540	1, 8742626675
diff.	+ 1677610	-10936138

Therefore from these with the errors of each limit there is concluded to be :

$$\varphi = 17°\ 26'\ 7''\frac{98}{100} \text{ and } \frac{1}{2}\pi+\varphi \text{ or } \frac{a}{c} = 107°\ 26'\ 7''\frac{98}{100}.$$

Truly since in seconds there shall be $\varphi = 62767,98$, there will be

$$\begin{aligned} l\varphi &= 4,79773S1525 \\ \text{subtr.} &= \underline{5\ ,3144251332} \\ &\quad 9,4833130193 \\ \therefore\ \varphi &= 0,3043077545 \\ \text{add. } \tfrac{1}{2}\pi &= 1,5707963268 \\ \frac{a}{c} &= 1,8751040813 \end{aligned}$$

with which found there shall be

$$\frac{A}{B} = \text{tang}\,\tfrac{1}{2}\varphi = 0,1533390624\,.$$

Therefore the ratio of the constants *A* and *B* is found, and from which the ratio of the remaining constants *C* et *D* for these will become known.

75. The first equation still remains $b = A + B + D$, which on account of $D = A + B$ will change into $b = 2A + 2B$; and thus $A + B = \frac{1}{2}b$; therefore since there shall be $\frac{A}{B} = \operatorname{tang}\frac{1}{2}\varphi$, it becomes

$$B\left(1 + \operatorname{tang}\tfrac{1}{2}\varphi\right) = \tfrac{1}{2}b \text{ and } B = \frac{b}{2 + 2\operatorname{tang}\frac{1}{2}\varphi}.$$

From which the individual constants of the equation will be determined from $\operatorname{tang}\frac{1}{2}\varphi = 0{,}1533390624$ in the following manner :

$$\frac{A}{b} = \frac{\operatorname{tang}\frac{1}{2}\varphi}{2\left(1 + \operatorname{tang}\frac{1}{2}\varphi\right)} = \frac{0{,}1533390624}{2{,}3066781248}$$

$$\frac{B}{b} = \frac{1}{2\left(1 + \operatorname{tang}\frac{1}{2}\varphi\right)} = \frac{1{,}0000000000}{2{,}3066781248}$$

$$\frac{C}{b} = \frac{-1 + \operatorname{tang}\frac{1}{2}\varphi}{2\left(1 + \operatorname{tang}\frac{1}{2}\varphi\right)} = -\frac{0{,}8466609376}{2{,}3066781248}$$

$$\frac{D}{b} = \frac{1 + \operatorname{tang}\frac{1}{2}\varphi}{2\left(1 + \operatorname{tang}\frac{1}{2}\varphi\right)} = \frac{1{,}1533390624}{2{,}3066781248}$$

with which found the nature of the curve *aMB*, which the lamina adopts during the oscillations, will be expressed by this equation:

$$\frac{y}{b} = \frac{A}{b}e^{\frac{x}{c}} + \frac{B}{b}e^{-\frac{x}{c}} + \frac{C}{b}\sin\frac{x}{c} + \frac{D}{b}\sin\frac{x}{c}.$$

76. But so that it may extend to the velocity of the oscillation, this ratio will be known from the equation $\frac{a}{c} = 1{,}8751040813$. For the sake of brevity there may be put $n = 1{,}8751040813$, so that there shall be $a = nc$.

And since there shall be $c^4 = \frac{Ekk \cdot af}{M}$, where $\frac{M}{a}$ expresses the specific weight of the lamina and *Ekk* the absolute elasticity, in that manner, which I have used so far, there will be

$$a^4 = n^4 \cdot Ekk \cdot \frac{a}{M} \cdot f \text{ and thus } f = \frac{a^4}{n^4} \cdot \frac{1}{Ekk} \cdot \frac{M}{a},$$

from which the length of the simple isochronous pendulum will hold a ratio composed from the fourth power of the length of the lamina, to the simple specific weight [*i.e.* really the mass per unit length, for a given width and thickness], and inversely as the absolute elasticity. Let *g* be the length of simple pendulum performing a single oscillation in one second [*i.e.* a swing from one instantaneous rest point at one side to the corresponding rest point at the other side],thus so that there shall be $g = 3{,}16625$ Rhenish feet; because the durations of the oscillations are in the square root ratio of the lengths of the lengths of the pendulums, the time for our elastic lamina to make one oscillation will be

$$=\frac{\sqrt{f}}{\sqrt{g}}\text{ seconds}=\frac{aa}{nn}\sqrt{\frac{1}{g}\cdot\frac{1}{Ekk}\cdot\frac{M}{a}}\,;$$

from which the number of oscillations produced in one second will be

$$=\frac{nn}{aa}\sqrt{g\cdot Ekk\cdot\frac{a}{M}},$$

which number expresses the tone of the sound, which the lamina excites. Therefore the sounds produced by diverse elastic laminas, with one end fixed in a wall, will be in the ratio composed directly as the square root of the absolute elasticity, inversely as the square root of the specific weight, and inversely as the square of the lengths. Whereby, if two elastic laminas should differ in length only, the sounds will be inversely as the squares of the lengths ; evidently a lamina twice as long will produce a sound lower by two octaves. But a tensed cord twice as long will produce a sound only one octave lower, if the tension may remain the same. From which it is apparent the sounds of elastic laminas and the sounds of tensed cords follow a far different ratio in length.

77. As far as it concerns the nature of the curve *aMB* continued beyond *a* and *B*, indeed in the first place it is apparent it will progress beyond *a* by continually diverging from the axis *BA*. For on putting *x* negative there becomes

$$y = Be^{\frac{x}{c}} + Ae^{-\frac{x}{c}} - C\sin\frac{x}{c} + D\cos\frac{x}{c}.$$

Now here all the terms are positive, because only the coefficient *C* before had obtained a negative value ; so that while *x* increases, *y* also must increase, because the number *B* is greater than *A* and thus the term $Be^{\frac{x}{c}}$ prevails. As at first $\frac{x}{c}$ may have reached perhaps a moderate value, while the term itself $Be^{\frac{x}{c}}$ has increased so much, that the remaining terms besides that may as if vanish. On account of the same reasoning, because the radius of osculation of the curve at *B* is not $=\infty$, indeed it is

$$\frac{Ekk}{R}=\frac{M}{af}\int dx\int ydx,$$

the curve at B will not have a point of contrary inflection and thus it will progress beyond to the same side of the axis AB ; but the abscissa x increased beyond $AB=a$, then the first term $Ae^{\frac{x}{c}}$ soon becomes so large, so that the rest of the terms may be able to be considered as zero before that term.

78. This therefore is the first mode of oscillation among all these innumerable modes, to which the same lamina can be composed. The second mode represented in figure (Fig. 21), so that the lamina fixed at B will cross the axis AB at a single point O, will be deduced from the equation :

$$\frac{a}{c}=\frac{1}{2}\pi+\varphi=l\text{tang}\,\tfrac{1}{2}\varphi \quad \text{or from this} \quad \frac{3}{2}\pi-\varphi=l\cot\tfrac{1}{2}\varphi=\frac{a}{c}.$$

Here by some trials the angle φ is found to be contained between the limits :
1° 2' 40" et 1° 3' 0", from which as before the true value of φ is elicited:

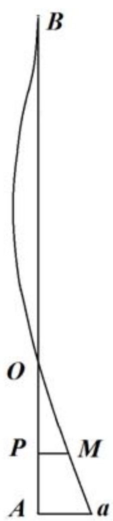

Fig. 21

$\varphi=$	1° 2' 40"	1° 3' 0"
in sec. =	3760"	3780"
log.=	3,5751878450	3,5774917998
subtr. =	5,3144251332	5,3144251332
$l\varphi=$	8,2607627118	8,2630666666
$\varphi=$	0 ,0182289944	0,0183259571
$\frac{3}{2}\pi=$	4,7123889804	4,7123889804
$\frac{a}{c}=$	4,6941599860	4,6940630233
$\frac{1}{2}\varphi=$	31' 20"	31' 30"
$l\cot\frac{1}{2}\varphi=$	2,0402552577	2,0379511745
$lv=$	0,3096845055	0,3091937748
add. =	0,3622156886	0,3622156886
$lu=$	0,6719001941	0,6714094634
$u=$	4,6978613391	4,6925559924
$\frac{a}{c}=$	4,6941599860	4,6940630233
Error	+ 37013531	-15070309

From these errors the true value of the angle is concluded to be

$$\varphi=1°2'54''\frac{213}{1000} \quad \text{and} \quad \frac{a}{c}=268°\ 57'5''\frac{787}{1000}.$$

Therefore since there shall be $\varphi = 3774,213''$, there will be

$$\begin{aligned} l\varphi &= 3,5768264061 \\ \text{subtr.} &= \underline{5,3144251332} \\ & \ 8,2624012729 \\ \varphi &= 0,0182979009 \\ \text{but } \tfrac{3}{2}\pi &= \underline{4,7123889804} \\ \frac{a}{c} &= 4,6940910795 \end{aligned}$$

Therefore the sound of the lamina oscillating in the first mode will be to the sound of the same lamina vibrating in this mode, as the square of the number 1,8751040813 is to the square of the number 4,6940910795, that is as 1 to 6,266891 or in rounded of numbers as 4 to 25 or as 1 to $6\frac{4}{15}$ [Note the error here] . From which the latter sound will be to the former almost as two octaves with a fifth and a semi-tone.

$$[i.e.\ \frac{25}{4} \cong \frac{2}{1}\times\frac{2}{1}\times\frac{3}{2}\times\frac{16}{15}.]$$

79. For the following modes of oscillation of the same elastic lamina, by which the lamina while oscillating may cut the axis *AB* at two or more points, the angle φ shall be made much smaller. Thus for the third mode this equation will be found:

$$\frac{5}{2}\pi + \varphi = l\cot\tfrac{1}{2}\varphi = \frac{a}{c}.$$

Therefore since there shall be

$$e^{\frac{5}{2}\pi+\varphi} = \cot\tfrac{1}{2}\varphi,$$

on account of the very small angle φ there will be

$$e^{\frac{5}{2}\pi+\varphi} = e^{\frac{5}{2}\pi}\left(1+\varphi+\tfrac{1}{2}\varphi^2+\tfrac{1}{6}\varphi^3+\text{etc.}\right)$$

and

$$\cot\tfrac{1}{2}\varphi = \frac{1-\frac{1}{8}\varphi\varphi}{\frac{1}{2}\varphi-\frac{1}{48}\varphi^3} = \frac{2}{\varphi}-\frac{\varphi}{6}.$$

Hence there will be approximately, $e^{\frac{5}{2}\pi} = \frac{2}{\varphi}$, and thus $\varphi = 2e^{-\frac{5}{2}\pi}$ or rather $\varphi = \dfrac{1}{1+\frac{1}{2}e^{\frac{5}{2}\pi}}$;

[On setting $e^{\frac{5}{2}\pi+\varphi} \cong e^{\frac{5}{2}\pi}(1+\varphi) \cong \frac{2}{\varphi}$; to a first approx. $\varphi e^{\frac{5}{2}\pi} \cong 2$; hence $e^{\frac{5}{2}\pi+\varphi} \cong (1+\varphi); \frac{2}{\varphi} \cong e^{\frac{5}{2}\pi}+2; i.e. \varphi = \frac{1}{1+\frac{1}{2}e^{\frac{5}{2}\pi}}$.]

from which there will be

$$\frac{a}{c} = \tfrac{5}{2}\pi + \frac{2}{e^{\frac{5}{2}\pi}+2},$$

which latter term is as the smallest. In a similar manner for the fourth mode of oscillation there will be approximately

$$\frac{a}{c} = \tfrac{7}{2}\pi - 2e^{-\frac{7}{2}\pi}$$

and thus so on; in account of these different terms vanishing the values of $\frac{a}{c}$ will be $\tfrac{7}{2}\pi,\ \tfrac{11}{2}\pi$ etc. which differ less from the truth, where they will progress higher.

THE OSCILLATIONS OF FREE ELASTIC LAMINAS

80. Now we will consider an elastic lamina (Fig. 22) nowhere fixed, but situated free either lying on a highly polished plane or in empty space remote from gravity. Moreover it is readily apparent that a lamina of this kind can undertake an oscillatory motion, provided that the lamina *acb* extends itself by curving alternately this side and then on the other it extends through the state of rest *AB*. Therefore this oscillatory motion can be defined in a similar manner, as in the preceding case, provided the calculation may be applied for this case in the due manner. Therefore *acb* shall be the figure of the curved lamina, which it may obtain while oscillating, but *ACB* shall be the position of the lamina in the state of equilibrium, through which it passes in some oscillation. As before the length of the lamina *AB* may be put $= a$, of which the absolute elasticity $= Ekk$ and the weight of the mass $= M$. Then the abscissa $AP = x$, the applied line $PM = y$, and the arc $aM = s$, which since it may be combined with the abscissa *x*, thus so that it may be able to be put in place $ds = dx$; from which the radius of osculation at *M* will arise $= \frac{dx^2}{ddy} = R$. But again the initial applied line shall be $Aa = b$. With these in place by putting in place the reasoning as before it will arrive at the same equation

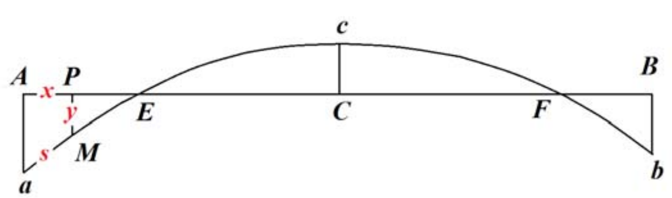

Fig. 22

$$\frac{Ekk}{R}=\frac{M}{a\,f}\int dx\int ydx \;=\frac{Ekkddy}{dx^2}.$$

81. Therefore if we may put $\frac{Ekk\cdot af}{M}=c^4$, where f as before expresses the length of the simple isochronous pendulum, this equation will be found for the curve by integration:

$$y=Ae^{\frac{x}{c}}+Be^{-\frac{x}{c}}+C\sin\frac{x}{c}+D\cos\frac{x}{c},$$

which thus will be to the present case. In the first place, if there may be put $x=0$, there must become $y=b$; from which there comes about

$$b=A+B+D.$$

In the second case, since there shall be

$$\frac{c^4ddy}{dx^2}=\int dx\int ydx,$$

by putting $x=0$ there must become $\frac{ddy}{dx^2}=0,$ from which there is produced

$$0=A+B-D.$$

Thirdly, since there shall be

$$\frac{c^4d^3y}{dx^3}=\int ydx,$$

on putting $x=0$ there must also become $\frac{d^3y}{dx^3}=0,$ from which there arises :

$$0=A-B-C.$$

In the fourth place, if there may be put $x=a$, $\int ydx$ or $\frac{d^3y}{dx^3}$ must vanish, because $\int ydx$ therefore expresses the sum of all the forces pulling the lamina in the direction normal to the axis *AB*, if which sum were not equal to 0, the lamina itself may be advancing by changing locality contrary to what has been set up ; therefore there will be on account of this reason :

$$0=Ae^{\frac{a}{c}}-Be^{-\frac{a}{c}}-C\cos\frac{a}{c}+\sin D\frac{a}{c},$$

In the fifth place, because the lamina is free at the end *B*, there it will have no curvature and thus there will be, on putting $x=a$ also $\frac{ddy}{dx^2}=0,$ from which there will be

$$0 = Ae^{\frac{a}{c}} + Be^{-\frac{a}{c}} - C\sin\frac{a}{c} - D\cos\frac{a}{c}.$$

Therefore from these five conditions introduced into the calculation, not only the four constants *A*, *B*, *C* and *D* will be determined, but also the value of the fraction $\frac{a}{c}$ may be found; from which hence the length of the simple isochronous pendulum *f* will become known.

82. From the second and third of these equations $D = A + B$ and $C = A - B$ is obtained, which substituted into the following will provide

$$0 = Ae^{\frac{a}{c}} - Be^{-\frac{a}{c}} - (A-B)\cos\frac{a}{c} + (A+B)\sin\frac{a}{c},$$

$$0 = Ae^{\frac{a}{c}} + Be^{-\frac{a}{c}} - (A-B)\sin\frac{a}{c} - (A+B)\cos\frac{a}{c},$$

from which there is found :

$$\frac{A}{B} = \frac{e^{-\frac{a}{c}} - \cos\frac{a}{c} - \sin\frac{a}{c}}{e^{\frac{a}{c}} - \cos\frac{a}{c} + \sin\frac{a}{c}} = -\frac{e^{-\frac{a}{c}} + \sin\frac{a}{c} - \cos\frac{a}{c}}{e^{\frac{a}{c}} - \sin\frac{a}{c} - \cos\frac{a}{c}};$$

from which equality this equation is elicited

$$0 = 2 - e^{\frac{a}{c}}\cos\frac{a}{c} - e^{-\frac{a}{c}}\cos\frac{a}{c};\ \text{or}\ e^{\frac{a}{c}} = \frac{1 \pm \sin\frac{a}{c}}{\cos\frac{a}{c}},$$

from which the following equations will be formed

I. $\frac{a}{c} = \frac{1}{2}\pi - \varphi = l\text{tang}\,\frac{1}{2}\varphi,$ which gives

$\frac{a}{c} = 0$ for the natural position of the lamina,

II. $\frac{a}{c} = \frac{1}{2}\pi - \varphi = l\cot\frac{1}{2}\varphi$

III. $\frac{a}{c} = \frac{3}{2}\pi + \varphi = l\cot\frac{1}{2}\varphi$

IV. $\frac{a}{c} = \frac{5}{2}\pi - \varphi = l\cot\frac{1}{2}\varphi$

V. $\frac{a}{c} = \frac{7}{2}\pi + \varphi = l\cot\frac{1}{2}\varphi$

VI. $\frac{a}{c} = \frac{9}{2}\pi - \varphi = l\cot\frac{1}{2}\varphi$

etc.

83. These equations again indicate innumerable modes of oscillations, in the second of which they will intersect the axis *AB* only once, in the third twice, in the fourth three times, in the fifth four times, and so on thus. From which it is understood the modes of the second, the fourth, the sixth, etc. are not to be accommodated, according to the present set up . Because in these the number of intersections is odd, the position of the lamina during the oscillations in the second becomes such, as may be shown in Figure 23, in which, although the sum of all the forces acting through the whole lamina may vanish, yet from these the lamina will acquire a rotational motion about the mid-point *C*, because the forces applied on each side to the halves *aC* and *bC* act together to induce a rotational motion of the lamina. On account of this reason, since generally rotational motion must be excluded, the figure of the lamina (Fig. 22), which it adopts during oscillations, thus must be prepared, so that not only the total of the forces acting on the lamina shall be $= 0$, but also so that the sum of the moments shall vanish; which is obtained, if the curve shall be given at the mid point *c* with the diameter *cC*. Which comes about, if the curve may cut the axis *AB* either in two, four, or generally in an even number of points; so that the equations shall be the third, fifth, seventh etc. Only agreeing solutions will be allowed.

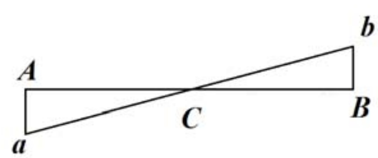

Fig. 23

[This latter assertion of odd numbers of nodes being excluded was shown to be in error by Daniel Bernoulli, who wrote to Euler (Sept. 4, 1743; see Fuss *Correspondence math. et physique*, letter 30) expressing his astonishment at Euler's conclusion. Euler later corrected the mistake, see E84.]

84. This limitation will be found present in the proposition of the problem itself, if we allow only problems of this kind, which may have the right diameter *Cc* or in which the same value of y may be provided, if in place of *x* there may be written $a-x$. Therefore we may put $a-x$ in place of *x* in the general equation, and there will be produced

$$y = Ae^{\frac{a}{c}}e^{-\frac{x}{c}} + Be^{-\frac{a}{c}}e^{\frac{x}{c}} + C\sin\frac{a}{c}\cdot\cos\frac{x}{c} - C\cos\frac{a}{c}\cdot\sin\frac{x}{c}$$
$$+ D\cos\frac{a}{c}\cdot\cos\frac{x}{c} + D\sin\frac{a}{c}\cdot\sin\frac{x}{c},$$

which since it must agree with the equation

$$y = Ae^{\frac{x}{c}} + Be^{-\frac{x}{c}} + C\sin\frac{x}{c} + D\cos\frac{x}{c},$$

there arises

$$Ae^{\frac{a}{c}} = B,\quad C\left(1+\cos\frac{a}{c}\right) = D\sin\frac{a}{c} \text{ and } C\sin\frac{a}{c} = D\left(1-\cos\frac{a}{c}\right),$$

of which the latter two are in agreement. Therefore since there shall be $\frac{A}{B} = e^{-\frac{a}{c}}$, this value compared with the above will produce :

$$e^{-\frac{a}{c}} - \cos\frac{a}{c} - \sin\frac{a}{c} = 1 - e^{-\frac{a}{c}}\cos\frac{a}{c} + e^{-\frac{a}{c}}\sin\frac{a}{c}$$

or

$$e^{-\frac{a}{c}} = \frac{1+\cos\frac{a}{c}+\sin\frac{a}{c}}{1+\cos\frac{a}{c}-\sin\frac{a}{c}} = \frac{1+\sin\frac{a}{c}}{\cos\frac{a}{c}} = \frac{\cos\frac{a}{c}}{1-\sin\frac{a}{c}}.$$

85. Therefore there will be

$$e^{\frac{a}{c}} = \frac{1-\sin\frac{a}{c}}{\cos\frac{a}{c}},$$

and thus in the first equation found :

$$e^{\frac{a}{c}} = \frac{1 \pm \sin\frac{a}{c}}{\cos\frac{a}{c}},$$

only of half the *f* cases shown above, evidently these, which are odd numbers, will solve the present problem. Whereby, since the first equation may contain the natural state of the lamina, all of the oscillations of this kind will be contained in the following equations :

$$\text{I.} \quad \frac{a}{c} = \tfrac{3}{2}\pi + \varphi = l\cot\tfrac{1}{2}\varphi$$

$$\text{II.} \quad \frac{a}{c} = \tfrac{7}{2}\pi + \varphi = l\cot\tfrac{1}{2}\varphi$$

$$\text{III.} \quad \frac{a}{c} = \tfrac{11}{2}\pi + \varphi = l\cot\tfrac{1}{2}\varphi$$

etc.

Therefore the first of these equations will present the first and principle mode of oscillation of that, for which the value of the angle φ will be found by approximation in the same way as above. Moreover the limits of the angle φ soon are deduced to be 1° 0' 40'' and 1° 1' 0'', from which by the following calculation the true value of φ is elicited.

φ =	1° 0' 40''	1° 1' 0''
or	3640''	3660''
log.=	3,5611013836	3,5634810854
subtr. =	5,3144251332	5,3144251332
$l\varphi$ =	8,24667 62504	8,2490559522
φ =	0,0176472180	0,0177441807
$\frac{3}{2}\pi$ =	4,7123889804	4,7123889804
$\frac{a}{c}$ =	4,7300361984	4,7301331611
$\frac{1}{2}\varphi$ =	30' 20''	30' 30''
v =	2,0543424742	2,0519626482
lv =	0,3126728453	0,3121694510
add.	0,3622156886	0,3622156886
lu =	0,67 48885339	0,6743851396
u =	4,7302983543	4,7248186037
Error.	+ 636341	+ 53145574
		636341
		diff. 52509233

Hence it is understood the true value of φ is not held within these limits, but to be a little less than $1^{\text{o}}0'40''$. Truly nothing less will be found from these errors. Indeed let there be $\varphi = 1^{\text{o}}0'40'' - n''$; there will be

$$20'' : 52509233 = n'' : 636341;$$

from which there is found $n = \frac{2423}{10000}$, thus so that there shall be

$$\varphi = 10^{\text{o}}\ 0'\,39''\,\frac{7576}{10000}.$$

Therefore since the angle becomes $\varphi = 3639,7576''$, there will be

$$\begin{aligned} l\varphi &= 3{,}5610724615 \\ \text{subtr. } &\underline{5{,}3144251332} \\ &\ 8{,}2466473283 \\ \varphi &= 0{,}0176460428 \\ \tfrac{3}{2}\pi &= \underline{4{,}7123889804} \\ \frac{a}{c} &= \ 4{,}7300350232 \end{aligned}$$

86. Let this number be $= m$, there will be on account of $c^4 = \frac{Ekk \cdot af}{M}$;

$$a^4 = \frac{m^4 \cdot Ekk \cdot af}{M} \quad \text{and} \quad f = \frac{a^4}{m^4} \cdot \frac{1}{Ekk} \cdot \frac{M}{a}.$$

So that the number of oscillations produced in the same way by the lamina in one second will be

$$= \frac{mm}{aa}\sqrt{g \cdot Ekk \cdot \frac{a}{M}}.$$

where $g = 3{,}16625$ Rhenish feet. But if therefore the same lamina now may have the other end *B* fixed into a wall, now it may be incited to produce sound, the sounds themselves will be as *nn* to *mm*, that is, as the squares of the numbers 1,8751040813 and 4, 7300350232, that is as 1 to 6,363236. Therefore the ratio of these sounds will be approximately as 11 ad 70; the interval of these sounds will be as two octaves with a fifth and a semi-tone.

$$[\textit{i.e.} \ \ \frac{70}{11} \cong \frac{2}{1} \times \frac{2}{1} \times \frac{3}{2} \times \frac{16}{15} = \textit{octave} + \textit{octave} + \textit{fifth} + \textit{semitone}\,]$$

But if the latter free lamina may be taken twice as long as the former, the interval of the sounds becomes a minor sixth [i.e. in the ratio 8:5].

87. With this value of the equation $\frac{a}{c}$ found, the equation for the curve, which the lamina forms while oscillating, until now indeterminate will be able to be determined. For since there shall be

$$e^{\frac{a}{c}} = \frac{1-\sin\frac{a}{c}}{\cos\frac{a}{c}}, \text{ there will be } B = \frac{1-\sin\frac{a}{c}}{\cos\frac{a}{c}}A$$

and

$$C = A - B = A\left(\cos\frac{a}{c} + \sin\frac{a}{c} - 1\right) : \cos\frac{a}{c},$$

and again,

$$D = A + B = A\left(\cos\frac{a}{c} - \sin\frac{a}{c} + 1\right) : \cos\frac{a}{c}.$$

Now there is :

$$b = A + B + D = 2D = 2A\left(\cos\frac{a}{c} - \sin\frac{a}{c} + 1\right) : \cos\frac{a}{c};$$

from which there becomes :

$$A=\frac{b\cos\frac{a}{c}}{2\left(\cos\frac{a}{c}-\sin\frac{a}{c}+1\right)}=\frac{b\left(1+\sin\frac{a}{c}-\cos\frac{a}{c}\right)}{4\sin\frac{a}{c}},$$

$$B=\frac{b\left(1-\sin\frac{a}{c}\right)}{2\left(\cos\frac{a}{c}-\sin\frac{a}{c}+1\right)}=\frac{b\left(-1+\sin\frac{a}{c}+\cos\frac{a}{c}\right)}{4\sin\frac{a}{c}},$$

$$C=\frac{b\left(-1+\sin\frac{a}{c}+\cos\frac{a}{c}\right)}{2\left(\cos\frac{a}{c}-\sin\frac{a}{c}+1\right)}=\frac{b\left(1-\cos\frac{a}{c}\right)}{2\sin\frac{a}{c}},$$

$$D=\frac{b}{2}=\frac{b\sin\frac{a}{c}}{2\sin\frac{a}{c}}.$$

With these in place there arises this equation :

$$\frac{y}{b}=\frac{e^{\frac{x}{c}}\cos\frac{a}{c}+e^{-\frac{x}{c}}\left(1-\sin\frac{a}{c}\right)}{2\left(1-\sin\frac{a}{c}+\cos\frac{a}{c}\right)}+\frac{\left(1-\cos\frac{a}{c}\right)\sin\frac{x}{c}+\sin\frac{a}{c}\cos\frac{x}{c}}{2\sin\frac{a}{c}}.$$

88. But because the right line Cc is the diameter of the curve, the abscissa may be put at the mid-point C, by taking $CP=z$, there will be $x=\frac{1}{2}a-z$. From which there becomes

$$e^{\frac{x}{c}}=e^{\frac{x}{2c}}e^{-\frac{x}{c}}=e^{-\frac{x}{c}}\sqrt{\frac{1-\sin\frac{a}{c}}{\cos\frac{a}{c}}}$$

and

$$e^{-\frac{x}{c}}=e^{\frac{x}{c}}\sqrt{\frac{\cos\frac{a}{c}}{1-\sin\frac{a}{c}}}\ ;$$

from which there will be

$$\frac{Ae^{\frac{x}{c}}+Be^{-\frac{x}{c}}}{b}=\frac{\left(e^{\frac{x}{c}}+e^{-\frac{x}{c}}\right)\sqrt{\cos\frac{a}{c}\left(1-\sin\frac{a}{c}\right)}}{2\left(1-\sin\frac{a}{c}+\cos\frac{a}{c}\right)}=\frac{e^{\frac{x}{c}}+e^{-\frac{x}{c}}}{2\left(e^{\frac{x}{2c}}+e^{-\frac{x}{2c}}\right)}.$$

Then truly there will be

$$\left(1-\cos\frac{a}{c}\right)\sin\frac{x}{c}+\sin\frac{a}{c}\cos\frac{x}{c}=\sin\frac{a}{c}+\sin\frac{(a-x)}{c}$$
$$=\sin\left(\frac{a}{2c}-\frac{z}{c}\right)+\sin\left(\frac{a}{2c}+\frac{z}{c}\right)=2\sin\frac{a}{2c}\cos\frac{z}{c};$$

with which substituted this equation arises :

$$\frac{2y}{b}=\frac{e^{\frac{z}{c}}+e^{-\frac{z}{c}}}{e^{\frac{a}{2c}}+e^{-\frac{a}{2c}}}+\frac{\cos\frac{z}{c}}{\cos\frac{a}{2c}},$$

which is the simplest form, by which the nature of the curve *aMcb* can be expressed ; moreover it is evident, whether *z* may be taken positive or negative, the same value of the applied line *y* is being produced. Truly there is

$$e^{\frac{a}{2c}}+e^{-\frac{a}{2c}}=\frac{2\cos\frac{a}{2c}}{\sqrt{\cos\frac{a}{c}}}.$$

Moreover we have found the angle
$\frac{a}{c}=271^\circ\ 0'39''\frac{3}{4}$; corrected to $271^\circ\ 0'40.94''$ in Oldfather et al.

89. If now there may be put $z=0$, then *y* will give the value of the applied line *Cc*; for there will be

$$\frac{2\cdot Cc}{b}=\frac{2\sqrt{\cos\frac{a}{c}}}{2\cos\frac{a}{2c}}+\frac{1}{\cos\frac{a}{2c}}.$$

or

$$\frac{Cc}{Aa}=\frac{1+\sqrt{\cos\frac{a}{c}}}{2\cos\frac{a}{2c}}=\tfrac{1}{2}\sec\frac{a}{2c}+\tfrac{1}{2}\sec\frac{a}{2c}\sqrt{\cos\frac{a}{c}}.$$

But there is

$\cos\frac{a}{c}=\sin 1^\circ 0'39''\tfrac{3}{4}$ [corr. as above] and $\cos\frac{a}{2c}=\sin 45^\circ 30'19''\tfrac{7}{8}$[corr. to $45^\circ 30'20.47''$].

Hence there is found $\frac{Cc}{Aa}=0,607815$ [corr. to $0,607841$]. Then, if there may be put $y=0$, the points *E* and *F* will be found, at which the curve may intersect the axis. Therefore there will be

$$e^{\frac{z}{c}}+e^{-\frac{z}{c}}=-\frac{\cos\frac{z}{c}}{\cos\frac{a}{2c}}\left(e^{\frac{a}{2c}}+e^{-\frac{a}{2c}}\right)=\frac{2\cos\frac{z}{c}}{\sqrt{\cos\frac{a}{c}}},$$

from which there is found by approximations :

$$\frac{CE}{CA}=0,551685 \quad \text{and} \quad \frac{AE}{AC}=0,448315.$$

Therefore while lamina performs oscillations, these points *E* and *F* will remain fixed ; from which the oscillatory motion of this kind, which previously actually seemed scarcely possible to be produced, can be produced easily. For if the lamina may be fixed at the points *E* and *F* defined in this manner, then it will oscillate henceforth as if it were completely free.

90. If the second of the equations found above may be treated in the same manner

$$\frac{a}{c}=\tfrac{7}{2}\pi+\varphi=l\cot\tfrac{1}{2}\varphi$$

so that in whatever case it will be found that $\varphi=0$ approximately, then the second mode will be produced, so that the free lamina vibrations can be resolved, clearly by cutting the axis *AB* in four points ; and the lamina thus will oscillate, just as if it were fixed at these four points. Therefore in turn, if the lamina may be fixed at these four points or only at two may be fixed in some manner, then it will oscillate in the same manner, as if it were free; but the sound produced will be much higher in tone, evidently which will maintain a ratio according to the manner of the preceding sound almost as 7^2 to 3^2 , that is, the interval will be of two octaves with a quarter and half of a semitone.

$$[i.e.\ \frac{49}{9} \cong \frac{2}{1} \times \frac{2}{1} \times \frac{4}{3} \times \frac{32}{31}.]$$

The third mode of oscillation, from which there is

$$\frac{a}{c} = \tfrac{11}{2}\pi + \varphi = l\cot\tfrac{1}{2}\varphi \ ,$$

will have six intersections of the curve *acb* with the axis *AB* and the sound will be produced higher by one octave sharpened by a minor third ; and hence the lamina will produce this sound, if it may be fixed in two of these six points. Hence it is apparent that various sounds are able to be produced by the same lamina, provided it may be fixed in different ways at two points, and only if the two points, in which it is fixed, may agree with the intersections in the first, second, third mode etc., and thus the oscillations themselves may be composed according to any of the following modes or even to infinitesimals, then the sound becomes sharp to such an extent that generally it cannot be perceived or, what amounts to the same thing, the lamina in short will not be able to receive the motion of the oscillations; or perhaps in the case of a vibrating cord, to which a little bridge thus may be put in place below, so that no parts among themselves may hold a whole ratio, the sound will be produced less distinct.

CONCERNING THE OSCILLATIONS OF AN ELASTIC LAMINA FIXED AT EACH END

90. Now the elastic lamina shall be fixed at each end (Fig. 24) *A* and *B*, thus so that the tangents to the curve at these points will not be determined. Clearly for this case in trials at each end of the lamina the sharpest most tenuous needle points $A\alpha,\ B\beta$ are extended fixed to the wall, which render the extreme ends *A* and *B* immobile. Towards investigating the motion of the oscillations of this elastic lamina, as before the absolute elastic constant may be put $= Ekk$, with the length *AB* and the weight $= M$ and the length of the simple isochronous pendulum $= f$. *AMB* shall be the curved figure, which the lamina adopts while oscillating, and the abscissa may be put $AP = AM = x$, the applied line $PM = y$ and the radius of osculation at $M = R$. Again P shall be the force, which the sharp point $A\alpha$ sustains in the direction $A\alpha$, and because the force, by which the element *Mm* must be urged in the direction $M\mu$, by which the lamina will be preserved in its state, is equal to $= \dfrac{Mydx}{af}$,

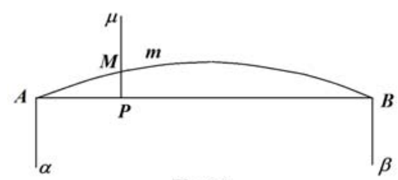

Fig. 24

[recall that *M* is the weight here and so is a force ;] by the rules described above, the equation for this curve :

$$\frac{Ekk}{R} = Px - \frac{M}{af}\int dx\int ydx.$$

Truly there is $R=-\frac{dx^2}{ddy}$, because the curve is concave towards the axis ; from which there becomes :

$$\frac{Ekk}{dx^2}ddy=\frac{M}{af}\int dx\int ydx=-Px.$$

Therefore on making $x=0$ the radius of osculation R at A will be infinite and thus $ddy=0$.

91. If this equation may be differentiated twice, it will produce the same equation, which we found in the previous case,

$$Ekkd^4y=\frac{M}{af}ydx^4.$$

So that therefore if there may be put $\frac{Ekk\cdot af}{M}=c^4$, the equation of the integral will be

$$y=Ae^{\frac{x}{c}}+Be^{-\frac{x}{c}}+C\sin\frac{x}{c}+D\cos\frac{x}{c}.$$

Towards determining which there may be put $x=0$, and because likewise y must vanish, there will be $0=A+B+D$.

In the second case putting $x=a$, and because equally there must become $y=0$, there will be

$$0=Ae^{\frac{a}{c}}+Be^{-\frac{a}{c}}+C\sin\frac{a}{c}+D\cos\frac{a}{c}.$$

Thirdly, because $\frac{ddy}{dx^2}$ must vanish, on putting both $x=0$ and $x=a$ there becomes

$$0=A+B-D \text{ and } 0=Ae^{\frac{a}{c}}+Be^{-\frac{a}{c}}-C\sin\frac{a}{c}-D\cos\frac{a}{c}.$$

Now the equations $0=A+B-D$ and $0=A+B+D$ give $D=0$ and $B=-A$; which values substituted into the remaining two equations give :

$$0=A\left(e^{\frac{a}{c}}-e^{-\frac{a}{c}}\right)+C\sin\frac{a}{c}$$

and

$$0=A\left(e^{\frac{a}{c}}-e^{-\frac{a}{c}}\right)-C\sin\frac{a}{c};$$

which cannot be satisfied, unless there shall be $A = 0$, because there cannot become $e^{\frac{a}{c}} = e^{-\frac{a}{c}}$ besides the case $\frac{a}{c} = 0$, then truly there must become $C \sin \frac{a}{c} = 0$. Since here there cannot be put $C = 0$, because the motion of the oscillation would become zero, therefore there will be $\sin \frac{a}{c} = 0$ and thus either $\frac{a}{c} = \pi$ or $\frac{a}{c} = 2\pi$ etc., so that again infinitely many different modes of oscillation arise, just as the curve *AMB* either may never may cut the axis besides the terms *A* and *B*, or at a single or in two or in several points, as is deduced from the equation $y = C \sin \frac{x}{c}$. But the points of intersection, however many there should be, will stand apart from each other by equal intervals.

93. Therefore for the first and principal mode of oscillation there shall be $\frac{a}{c} = \pi$, there will be

$$a^4 = \pi^4 c^4 = \pi^4 \times Ekk \times \frac{a}{M} \times f,$$

from which there becomes :

$$f = \frac{a^4}{\pi^4} \times \frac{1}{Ekk} \times \frac{M}{a}.$$

Whereby on account of the length of the lamina, the sounds again will maintain the reciprocal square ratio of the lengths. But the sound of this lamina produced in this manner will itself be had to the sound of the same lamina, if the other end *B* were fastened to the wall, as $\pi\pi$ to the square of the number 1,8751040813, that is as 2,807041 to 1 or in rounded numbers as 57 to 160, which interval is an octave with a tritone almost [The ratio of the frequencies required is 2.8, giving $2.8:1 = \frac{2}{1} \times \frac{14}{10} \cong \frac{2}{1} \times \frac{14.25}{10}$]. If the oscillations may be prepared themselves according to this manner, so that there shall be $\frac{a}{c} = 2\pi$, the [musical] sound becomes two octaves higher; but if there shall be $\frac{a}{c} = 3\pi$, the sound becomes sharper by three octaves in tone than with the case, where $\frac{a}{c} = \pi$, and thus so forth. Which so that they may be found more readily from experiment, it is to be noted here that the oscillations are to be made minimally, thus so that there shall be no need for an extension of the lamina. Whereby, lest the holding strength of the lamina may offer resistance, by which even the smallest extensions may be prevented, without which these oscillations will be unable to be performed, here an alteration may be offered, these sharp points [cusps] must be constituted, so that the smallest extension may not be impeded ; which comes about, if they may rest on a highly polished plane. Thus the elastic lamina *AB* provided with the cusps $A\alpha$ and $B\beta$ at *A* and *B*, if the cusps may be put in place on a mirror, will produce sound agreeing with the calculation.

CONCERNING THE ELASTIC OSCILLATIONS OF A LAMINA WITH EACH END FIXED TO A WALL

94. The oscillatory motion of elastic laminas may be completed with this case set out, with each end *A* and *B* fixed in a wall (Fig. 25), thus so that while oscillating the points *A* and *B* not only remain fixed, but also the right line *AB* will always be a tangent to the curve *AMB* at the points *A* and *B*. Therefore here again there is a warning, so that the retaining end bolts *A* and *B* shall not be firm to such an extent, but that they may allow a little extension, as much as is required for the extension. Therefore whatever forces shall be required to hold the lamina in place at the ends *A* and *B*, it arrives at the following differential equation of the fourth order :

Fig. 25

$$Ekkd^4y = \frac{M}{af}dx^4;$$

of which, if there may be put $\frac{Ekk \cdot af}{M} = c^4$, the integral will be as above :

$$y = Ae^{\frac{x}{c}} + Be^{-\frac{x}{c}} + C\sin\frac{x}{c} + D\cos\frac{x}{c}.$$

95. But the constants *A*, *B*, *C* and *D* have to be prepared thus, so that on putting $x = 0$ not only *y* shall vanish, but also there becomes $dy = 0$, because at *A* the curve is a tangent with the axis *AB*. Truly with this, the same must happen at each end, if there may be put $x = a$; from which these four equations arise :

I. $0 = A + B + D$

II. $0 = A - B + C$

III. $0 = Ae^{\frac{a}{c}} + Be^{-\frac{a}{c}} + C\sin\frac{a}{c} + D\cos\frac{a}{c}$

IV. $0 = Ae^{\frac{a}{c}} - Be^{-\frac{a}{c}} + C\cos\frac{a}{c} - D\sin\frac{a}{c}.$

From the first and second of these equations arises $C = -A + B$ and $D = -A - B$, which values substituted into the remaining two equations will give :

$$0 = Ae^{\frac{a}{c}} + Be^{-\frac{a}{c}} - (A-B)\sin\frac{a}{c} - (A+B)\cos\frac{a}{c}$$
$$0 = Ae^{\frac{a}{c}} - Be^{-\frac{a}{c}} - (A-B)\cos\frac{a}{c} + (A+B)\sin\frac{a}{c}.$$

the sum and difference of which is

$$0 = Ae^{\frac{a}{c}} + B\sin\frac{a}{c} - A\cos\frac{a}{c} \quad \text{or} \quad \frac{A}{B} = \frac{\sin\frac{a}{c}}{\cos\frac{a}{c} - e^{\frac{a}{c}}}$$

$$0 = Be^{-\frac{a}{c}} - A\sin\frac{a}{c} - B\cos\frac{a}{c} \quad \text{or} \quad \frac{A}{B} = \frac{e^{-\frac{a}{c}} - \cos\frac{a}{c}}{\sin\frac{a}{c}},$$

from which there becomes :

$$2 = \left(e^{\frac{a}{c}} + e^{-\frac{a}{c}}\right)\cos\frac{a}{c} \quad \text{or} \quad e^{\frac{a}{c}} = \frac{1 \pm \sin\frac{a}{c}}{\cos\frac{a}{c}}.$$

Which equation because it agrees with that, which we have found in paragraph 81, the following solutions will satisfy, from an infinite number of such :

$$\text{I.} \quad \frac{a}{c} = \tfrac{1}{2}\pi - \varphi = l\cot\tfrac{1}{2}\varphi$$
$$\text{II.} \quad \frac{a}{c} = \tfrac{3}{2}\pi + \varphi = l\cot\tfrac{1}{2}\varphi$$
$$\text{III.} \quad \frac{a}{c} = \tfrac{5}{2}\pi - \varphi = l\cot\tfrac{1}{2}\varphi$$

etc.

96. Of these equations the first is unable to be satisfied, unless there shall be $\varphi = 90°$ and thus $\frac{a}{c} = 0$; from which the first mode of oscillation will arise from the equation

$$\frac{a}{c} = \tfrac{3}{2}\pi + \varphi = l\cot\tfrac{1}{2}\varphi;$$

which since now it shall have been treated, there will be $\frac{a}{c} = 4,7300350232.$ On account of which the elastic lamina, each end of which may be retained fixed by the wall, will perform its vibrations in the same way, just as if it were completely free. But this agreement is considered only for the first mode of oscillation ; indeed the second mode of oscillation, when there is

$$\frac{a}{c}=\tfrac{5}{2}\pi-\varphi=l\cot\tfrac{1}{2}\varphi$$

and the lamina may cut the axis *AB* while oscillating at a single point, does not have an equal oscillation in the free lamina case ; but the third mode of the lamina fixed at each end will agree with the second mode of the free lamina, and so on thus.

97. These two last kinds of oscillation [*i.e.* § 91 & § 94] on account of the reason established are unable to be investigated by experiment : but the first [*i.e.* § 65] not only is especially suitable for experimental investigation, but also can be used for investigating the absolute elasticity of each lamina proposed, as we have indicated by *Ekk*. So that indeed if the sound may be noted, which a lamina of this kind produces with the other end fixed in a wall, and a tone agreeing with that may be produced in a chord, likewise the number of oscillations in a second produced will be known. Which if it may be put equal to the expression

$$\frac{nn}{aa}\sqrt{g}\cdot Ekk\cdot\frac{a}{M},$$

on account of the known number *n* and the magnitudes *g*, *a* and *M* found from the dimensions measured, the value of the expression *Ekk* and thus the absolute elasticity will be known; which can be compared with that, which we have shown above how to find from the curvature[§ 38] .

ADDITAMENTUM IB

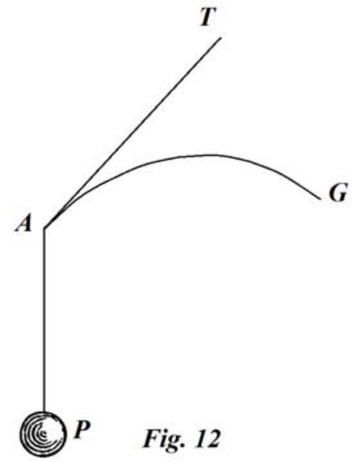

Fig. 12

34. His enumeratis speciebus facile erit pro quovis casu oblato assignare, ad quamnam speciem curva formata pertineat. Sit lamina Elastica (Fig.12) in *G* muro infixa, termino vero *A* appendatur pondus *P*, quo lamina in figuram *GA* incurvetur. Ducatur tangens *AT*, atque ex angulo *TAP* totum iudicium erit petendum. Si enim hic angulus fuerit acutus, referetur curva ad speciem secundam; sin sit rectus, ad tertiam, eritque Elastica rectangula. Quodsi angulus *TAP* fuerit obtusus, minor tamen quam 130° 41', curva ad speciem quartam pertinebit; ad quintam autem, si angulus *TAP* sit $=130° \ 41'$; sin autem angulus *TAP* maior fuerit, curva sub specie sexta continebitur. Ad septimam vero pertineret, si iste angulus fieret duobus rectis aequalis, quod autem nunquam fieri potest. Haec igitur species *T* cum duabus sequentibus produci nequit laminae immediate pondus appendendo.

35. Ut igitur pateat (Fig. 3), quomodo reliquae species laminam incurvando produci queant, laminae in *B* fixae, non immediate, sed virgae rigidae *AG* cum laminae termino *A* firmissime connexae in *C* appendatur pondus *P*, quod trahat in directione *GD*. Sit intervallum *AGh*, elasticitas laminae absoluta *Ekk* et anguli *MAP*, quem lamina in *A* cum horizontali constituit, sinus $= m$. His positis, si ponatur abscissa $AP = t$ et applicata $PM = y$, reperietur pro curva ista aequatio

$$dy = \frac{dt\left(mEkk - Pht \ -\tfrac{1}{2}Ptt\right)}{\sqrt{\left(E^2k^4 - \left(mEkk - \ Pht - \tfrac{1}{2}Ptt\right)^2\right)}}.$$

Ponatur iam $CP = x = h + t$, quo aequatio ad formam, qua in divisione specierum usi sumus, reducatur; erit

$$dy = \frac{dx\left(mEkk \ + \tfrac{1}{2}Phh - \tfrac{1}{2}Pxx\right)}{\sqrt{\left(E^2k^4 - \left(mEkk + \tfrac{1}{2}Phh - \tfrac{1}{2}Pxx\right)^2\right)}}$$

quae comparata cum forma

$$dy = \frac{dx\left(aa - cc + xx\right)}{\sqrt{\left(cc - xx\right)\left(2aa - cc + xx\right)}}$$

seu

$$dy = \frac{dx\left(aa - cc + xx\right)}{\sqrt{\left(a^4 - \left(aa - cc + xx\right)^2\right)}}$$

dabit $\frac{1}{2}Paa = Ekk$ seu

$$aa = \frac{2Ekk}{P} \text{ et } \tfrac{1}{2}Pcc - \tfrac{1}{2}Paa = mEkk + \tfrac{1}{2}Phh;$$

ergo

$$cc = \frac{2\left(1+m\right)Ekk}{P} + hh.$$

36. Curva ergo ad speciem secundam pertinebit, si fuerit

$$\frac{2mEkk}{P} + hh < 0 \text{ seu } P < -\frac{2mEkk}{hh};$$

nisi ergo angulus *PAM* sit negativus, vis *P* negativa esse atque virga in *C* sursum trahi debet. Ad speciem tertiam curvatura pertinebit, si

$$P = -\frac{2mEkk}{hh}.$$

Quarta autem species prodibit, si fuerit

$$2mEkk + Phh > 0, \text{ simul vero } 2mEkk + Phh < 2\alpha Ekk,$$

existente $\alpha = 0,651868.$ Sin autem sit

$$P = \frac{2\left(\alpha - m\right)Ekk}{hh},$$

tum curva ad speciem quintam pertinebit. Quodsi vero fuerit

$$Phh > 2\left(\alpha - m\right)Ekk, \text{ simul vero } Phh < 2\left(1-m\right)Ekk,$$

curva ad speciem sextam est referenda. Septimaqua species proveniet, si

$$Phh = 2\left(1-m\right)Ekk.$$

Octava autem species obtinebitur, si

$$Phh > 2\left(1-m\right)Ekk;$$

quare, si angulus *PAM* fuerit rectus, ob $1-m=0$ curva semper ad speciem octavam pertinebit. Species denique nona orietur, si fuerit $h=\infty$, uti iam supra annotavi.

DE VI COLLUMNARUM

37. Quae ante de specie prima sunt annotata, inservire possunt viribus columnarum diiudicandis. Sit enim (Fig. 13) *AB* columna super basi *A* verticaliter posita, gestans pondus *P*. Quodsi iam columna ita sit constituta, ut prolabi nequeat, ab onere *P*, si fuerit nimis magnum, nil aliud erit metuendum, nisi columnae incurvatio; hoc ergo casu columna spectari poterit tanquam elasticitate praedita. Sit igitur elasticitas absoluta columnae $= Ekk$ eiusque altitudo $AB = 2f = a$, atque supra paragrapho 25 vidimus vim requisitam ad hanc columnam vel minimum inclinandam esse

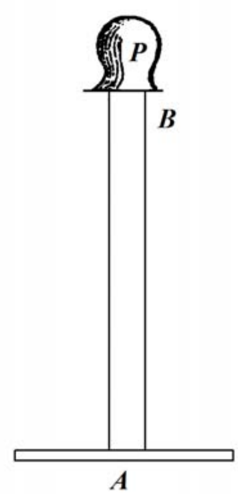

Fig. 13

$$= \frac{\pi\pi Ekk}{4ff} = \frac{\pi\pi}{aa} \cdot Ekk$$

Nisi ergo onus gestandum *P* maius sit quam $E \cdot \frac{\pi\pi kk}{aa}$, nulla prorsus incurvatio erit metuenda; contra vero, si pondus *PA* fuerit maius, columna incurvationi resistere non poterit. Manente autem elasticitate columnae atque adeo eius crassitie eadem, pondus *P*, quod sine periculo gestare valet, erit reciproce ut quadratum altitudinis columnae columnaque duplo altior quartam tantum oneris partem gestare poterit. Haec igitur praecipue in usum vocari possunt circa columnas ligneas, quippe quae incurvationi sunt obnoxiae.

ELASTICITATIS ABSOLUTAE DETERMINATIO PER EXPERIMENTA

38. Quo autem vis atque incurvatio cuiusque laminae Elasticae a priori determinari queat, necesse est, ut elasticitas absoluta, quam hactenus per *Ekk* expressimus, sit cognita; id quod unico experimento commode praestabitur. Infigatur lamina Elastica (Fig. 14) uniformis *FH*, cuius elasticitatem absolutam investigari oportet, altero termino F parieti firmo *GK*, ita ut situm teneat horizontalem *FH*; hic enim gravitatem naturalem negligere liceat. Alteri termino *H* appendatur pondus pro arbitrio sumptum *P*, quo lamina in statum *AF* incurvetur. Sit longitudo laminae $AF = HF = f$, recta horizontalis $AG = g$ et verticalis $GF = h$, qui valores omnes per experimentum erunt cogniti. Comparetur iam haec curva cum aequatione generali

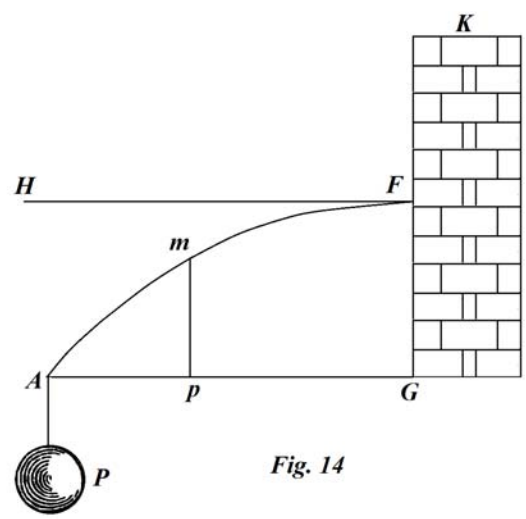

Fig. 14

$$dy = \frac{(cc-aa-xx)dx}{\sqrt{(cc-xx)(2aa-cc+xx)}},$$

in qua si fuerit *a* et *c* per *f, g, h* definite, erit vis incurvans $P = \frac{2Ekk}{aa}$ ideoque elasticitas absoluta $Ekk = \frac{1}{2}Paa$.

39. Quia iam tangens in *F* est horizontalis, erit hic $\frac{dy}{dx} = 0$ ideoque $x = \sqrt{(cc-aa)}$. Hinc ergo erit

$$AG = g = \sqrt{(cc-aa)} \text{ et } aa = cc - gg$$

ideoque

$$dy = \frac{(gg-xx)dx}{\sqrt{(cc-xx)(cc-2gg+xx)}},$$

posito autem hic $x = g$, fieri debebit $y = GF = h$ seu $s = AF = f$; est vero

$$ds = \frac{(cc-gg)dx}{\sqrt{(cc-xx)(cc-2gg+xx)}}.$$

Iam si pondus *P* sumatur valde parvum, ut lamina paulisper tantum deprimatur, tum erit *c* quantitas valde magna ideoque erit proxime

$$\frac{1}{\sqrt{(cc-xx)(cc-2gg+xx)}} = (c^4 - 2ccgg + 2ggxx - x^4)^{-\frac{1}{2}}$$
$$= \frac{1}{cc} + \frac{gg}{c^4} - \frac{ggxx}{c^4} + \frac{x^4}{2c^6},$$

ideoque integrando quoque proxime:

$$s = \frac{(cc-gg)x}{cc} + \frac{(cc-gg)ggx}{c^4} - \frac{(cc-gg)ggx^3}{3c^4} + \frac{(cc-gg)ggx^5}{10c^4}$$

et

$$y = \frac{ggx}{cc} + \frac{g^4x}{c^4} - \frac{g^4x^3}{3c^4} + \frac{ggx^5}{10c^4}$$
$$- \frac{x^3}{3cc} - \frac{ggx^3}{3c^4} + \frac{ggx^5}{5c^4} - \frac{x^7}{14c^4}.$$

Sit nunc $x = g$, fietque

$$f = g - \frac{37g^5}{30c^4} \text{ et } h = \frac{2g^3}{3cc} + \frac{2g^5}{3c^4}.$$

Quodsi ergo recta $FG = h$ in usum vocetur, erit

$$cc = \frac{2g^3}{3h} \quad \text{et} \quad aa = \frac{g(2gg - 3gh)}{3h},$$

unde elicitur elasticitas absoluta

$$Ekk = \frac{Pgg(2g - 3h)}{6h};$$

qui valor a vero vix sensibiliter discrepabit, dummodo laminae curvatura non nimis magna inducatur.

40. Haec autem elasticitas absoluta *Ekk* primum pendet ab natura materiae, ex qua lamina est fabrefacta, unde alia materia magis, alia minus elatere praedita dici solet. Secundo quoque ita pendet a laminae latitudine, ut expressio *Ekk* ubique latitudini laminae debeat esse proportionalis, si cetera sint paria. Tertio verum crassities laminae plurimum confert ad valorem ipsius *Ekk* determinandum, quae ita comparata esse videtur, ut, ceteris paribus, *Ekk* sit ut crassitiei quadratum. Coniunctim ergo tenebit expressio *Ekk* rationem compositam ex ratione elateris materiae, latitudinis laminae simplici ac duplicata crassitiei laminae. Hinc per experimenta, quibus latitudinem et crassitiem metiri licet, omnium materiarum elasticitates inter se comparari ac determinari poterunt.

De curvatura laminae elasticae inaequabiles

41. Quemadmodum igitur hactenus laminae, cuius curvaturam determinavi, elasticitatem absolutam *Ekk* per totam longitudinem constautem posui, ita solutio eadem methodo poterit absolvi, si quantitas *Ekk* utcunque ponatur variabilis. Scilicet, si elasticitas absoluta fuerit ut functio quaecunque portionis laminae *AM* (Fig. 2), quae functio sit $= S$, posito arcu $AM = s$ atque existente radio osculi in $M = R$ curva *AM*, quam lamina induit, ita erit comparata, ut in ea inter omnes alias eiusdem longitudinis sit $\int \frac{Sds}{RR}$ minimum. Solvetur ergo iste casus per formulam secundam generalem. Sit $dy = pdx,\ dp = qdx,$ et $dS = Tds$, atque inter omnes curvas, in quibus est $\int dx\sqrt{(1+pp)}$ eiusdem magnitudinis, ea determinari debebit, in qua sit $\int \frac{Sqqdx}{(1+pp)^{5:2}}$ minimum. Prior formula $\int dx\sqrt{(1+pp)}$ dat pro formula differentiali $\frac{1}{dx}d.\ \frac{p}{\sqrt{(1+pp)}}$.

Altera vero $\int \frac{Sqqdx}{(1+pp)^{5:2}}$ cum $\int Zdx$ comparata dabit

$$Z = \frac{Sqq}{(1+pp)^{5:2}}.$$

Cum igitur positum sit

$$dZ = Ld\Pi + Mdx + Ndy + Pdp + Qdq,\ \Pi = \int [Z]dx$$

et

$$d[Z] = [M]dx + [N]dy + [P]dp,$$

erit

$$Ld\Pi = \frac{qqTds}{(1+pp)^{5:2}},$$

unde

$$L = \frac{qqT}{(1+pp)^{5:2}},\ d\Pi = ds = dx\sqrt{(1+pp)}$$

ideoque

$$[Z] = \sqrt{(1+pp)},\ [M] = 0,\ [N] = 0,\ \text{et}\ [P] = \frac{p}{\sqrt{(1+pp)}}.$$

Deinde vero est

$$M = 0,\ N = 0,\ \ P = -\frac{5Sqqp}{(1+pp)^{7:2}}\ \ \text{et}\ \ Q = \frac{2Sq}{(1+pp)^{5:2}},$$

ita ut sit

$$dZ = \frac{qqdS}{(1+pp)^{5:2}} + Pdp + Qdq.$$

42. Iam sumatur integrale

$$\int Ldx = \int \frac{qqTdx}{(1+pp)^{5:2}} = \int \frac{qqdS}{(1+pp)^{3}}$$

sitque H eius valor, si ponatur $x = a$, cuius quidem constantis a consideratio mox ex calculo rursus evanescet. Erit ergo

$$V = H - \int \frac{qqdS}{(1+pp)^{3}}.$$

Unde valor differentialis fiet

$$= -\frac{dP}{dx} - \frac{1}{dx}d\cdot[P]V + \frac{ddQ}{dx^{2}}.$$

Quamobrem ex his duobus valoribus differentialibus nascetur haec aequatio pro curva quaesita

$$\frac{\alpha}{dx} d \cdot \frac{p}{\sqrt{(1+pp)}} = +\frac{dP}{dx} + \frac{1}{dx} d \cdot [P]V - \frac{ddQ}{dx^2},$$

quae integrata dat

$$\frac{\alpha p}{\sqrt{(1+pp)}} + \beta = P + [P]V - \frac{dQ}{dx}$$

sive

$$\frac{\alpha p}{\sqrt{(1+pp)}} + \beta = \frac{Hp}{\sqrt{(1+pp)}} - \frac{p}{\sqrt{(1+pp)}} \int \frac{qqdS}{(1+pp)^3} + P - \frac{dQ}{dx},$$

ubi constans *H* alias determinata in constante arbitraria α comprehendi potest, quo ipso constans *a* ex calculo egreditur. Idcirco ergo prodibit haec aequatio

$$\frac{\alpha p}{\sqrt{(1+pp)}} + \beta = P - \frac{dQ}{dx} - \frac{p}{\sqrt{(1+pp)}} \int \frac{qqdS}{(1+pp)^3}.$$

43. Multiplicetur haec aequatio per $dp = qdx$ atque prodibit:

$$\frac{\alpha pdp}{\sqrt{(1+pp)}} + \beta dp = Pdp - qdQ - \frac{pdp}{\sqrt{(1+pp)}} \int \frac{qqdS}{(1+pp)^3}.$$

Cum autem sit

$$dZ = \frac{qqdS}{(1+pp)^{5:2}} + Pdp + Qdq.$$

erit

$$Pdp = dZ - Qdq - \frac{qqdS}{(1+pp)^{5:2}};$$

quo valore substituto emerget aequatio integrabilis haec:

$$\frac{\alpha pdp}{\sqrt{(1+pp)}} + \beta dp = dZ - Qdq - qdQ - \frac{qqdS}{(1+pp)^{5:2}} - \frac{pdp}{\sqrt{(1+pp)}} \int \frac{qqdS}{(1+pp)^3},$$

cuius integralis est:

$$\alpha\sqrt{(1+pp)}+\beta p+\gamma=Z-Qq-\sqrt{(1+pp)}\int\frac{qqdS}{(1+pp)^3}$$

seu

$$\alpha\sqrt{(1+pp)}+\beta p+\gamma=-\frac{Sqq}{(1+pp)^{5:2}}-\sqrt{(1+pp)}\int\frac{qqdS}{(1+pp)^3}.$$

Quo signum integrale tollamus, divisa aequatione per $\sqrt{(1+pp)}$ ea denuo differentietur:

$$\frac{\beta dp}{(1+pp)^{3:2}}-\frac{\gamma pdp}{(1+pp)^{3:2}}+\frac{2qqdS}{(1+pp)^3}+\frac{2Sqdq}{(1+pp)^3}-\frac{6Spqqdp}{(1+pp)^4}=0,$$

quae per $\frac{(1+pp)^{3:2}}{2q}$ multiplicata praebet:

$$\frac{\beta dp}{2q}-\frac{\gamma pdp}{2q}+\frac{qdS+Sdq}{(1+pp)^{3:2}}-\frac{3Spqdp}{(1+pp)^{5:2}}=0,$$

cuius ob $dp=qdx$ et $dy=pdx$ integrale erit

$$\alpha+\tfrac{1}{2}\beta x-\tfrac{1}{2}\gamma y+\frac{Sq}{(1+pp)^{3:2}}=0.$$

At est $-\frac{(1+pp)^{3:2}}{q}=$ radio osculi *R*, unde constantes β et γ duplicando orietur haec aequatio

$$\frac{S}{R}=\alpha+\beta x-\gamma y\ ;$$

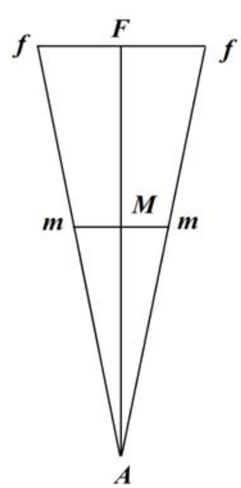

Fig. 15

quae aequatio apprime congruit cum ea, quam altera Methodus directa suppeditat. Exprimet enim $\alpha+\beta x-\gamma y$ momentum potentiae incurvantis, recta quacunque pro axe assumpta, cui momento utique aequalis esse debet elasticitas absoluta *S* per radium osculi *R* divisa. Sic igitur non solum Celeberrimi BERNOULLII observata proprietas Elasticae plenissime est evicta, sed etiam formularum mearum difficiliorum usus summus in hoc Exemplo est declaratus.

44. Si ergo curva (Fig. 3) fuerit data, quam lamina inaequabiliter Elastica a potentia $CD=P$ sollicitata format, hinc elasticitas absoluta laminae in quovis loco poterit cognosci. Sumpta enim recta *CP*, quae ad directionem vis sollicitantis est normalis, pro axe

ac posita $CP = x,\ PM = y$, arcu curvae $AM = s$ et radio osculi in $M = R$, ob momentum potentiae *P* ad punctum *M* relatum $= Px$ erit $\frac{S}{R} = Px$; ideoque elasticitas absoluta in puncto *M*, quae est $S, = PRx$. Hinc, cum data curva in singulis punctis detur radius osculi *R*, elasticitas absoluta in quovis loco innotescit. Quodsi ergo materia laminae una cum crassitie ubique fuerit eadem, latitudo autem sit variabilis: quia elasticitas absoluta latitudini est proportionalis, ex curva formata latitudo laminae in singulis locis colligitur.

45. Sit ex lamina Elastica (Fig. 15) excissa lingula triangularis *fAf* ubique eiusdem crassitiei. Quoniam ergo latitudo *mm* in quovis loco *M* est longitudini *AM* proportionalis, posita $AM = s$ erit elasticitas absoluta in *M* ut *s*. Sit ea $= Eks$ atque laminae termino *ff* muro horizontaliter infixo appendatur cuspidi *A* pondus *P*, quo laminae recta media *AF* (Fig. 14) in curvam *FmA* incurvetur, cuius natura quaeritur. Positis autem in axe horizontali abscissa $Ap = x$, applicata $pm = y$ et arcu $Am = s$, erit $Px = \frac{Eks}{R}$ denotante *R* radium osculi in *m*. Multiplicetur haec aequatio per *dx* et ob $R = -\frac{ds^3}{dxddy}$, posito *dx* constante, erit

$$Pxdx = -\frac{Eksdx^2ddy}{ds^3} \text{ seu } \frac{Pxdx}{Ek} + \frac{sdx^2ddy}{ds^3} = 0.$$

At, cum sit

$$d\cdot\frac{sdy}{ds} = \frac{sddy}{ds} - \frac{sdydds}{ds^2} + dy = \frac{sdx^2ddy}{ds^3} + dy$$

ob $dds = \frac{dyddy}{ds}$, erit

$$\int\frac{sdx^2ddy}{ds^3} = \frac{sdy}{ds} - y,$$

unde integrando habebitur

$$\frac{Pxx}{2Ek} + a = -\frac{sdy}{ds} + y.$$

46. Sit $dy = pdx$, erit $ds = dx\sqrt{(1+pp)}$ et posito $\frac{2Ek}{P} = c$ fiet

$$a + \frac{xx}{c} = y - \frac{sp}{\sqrt{(1+pp)}};$$

ideoque erit

$$\frac{a\sqrt{(1+pp)}}{p} + \frac{xx\sqrt{(1+pp)}}{cp} = \frac{y\sqrt{(1+pp)}}{p} - s;$$

quae differentiata dat

$$-\frac{adp}{pp\sqrt{(1+pp)}}+\frac{2xdx\sqrt{(1+pp)}}{cp}-\frac{xxdp}{cpp\sqrt{(1+pp)}}$$
$$=\frac{dy\sqrt{(1+pp)}}{p}-\frac{ydp}{pp\sqrt{(1+pp)}}-dx\sqrt{(1+pp)}=-\frac{ydp}{pp\sqrt{(1+pp)}}.$$

Hinc oritur

$$a-y=\frac{2pxdx(1+pp)}{cdp}-\frac{xx}{c}.$$

Ponatur *dp* constans, ac differentiando erit

$$-pdx=\frac{2pxddx(1+pp)}{cdp}+\frac{2pdx^2(1+pp)}{cdp}+\frac{2xdx(1+3pp)}{c}-\frac{2xdx}{c}$$

seu

$$0=cdxdp+2xddx(1+pp)+2dx^2(1+pp)+6pxdxdp,$$

cuius aequationis autem resolutio ulterior non constat. Simplicissima autem pro curva est aequatio haec

$$\frac{yds-sdy}{ds}=\frac{Pxx}{2Ek};$$

quia enim posito $x=0$ et *y* et *s* evanescere debent, constans *a* debet esse = 0.

DE INCURVATIONE LAMINARUM
ELASTICARUM NATURALITER NON RECTARUM

47. Hoc igitur modo curvatura laminae sive aequaliter sive inaequaliter Elasticae determinatur, si ab una potentia sollicitetur atque, quod praecipue est notandum, si lamina naturaliter fuerit in directum extensa. Quodsi enim lamina in statu naturali iam fuerit curva, tum utique a vi sollicitante aliam curvaturam induet; ad quam inveniendam, praeter sollicitationem atque elasticitatem simul figuram eius naturalem nosse oportet. Sit igitur lamina Elastica (Fig. 16) naturaliter curva *Bma*, cuius quidem elasticitas sit ubique eadem $=Ekk$, quae a vi sollicitante *P* in figuram

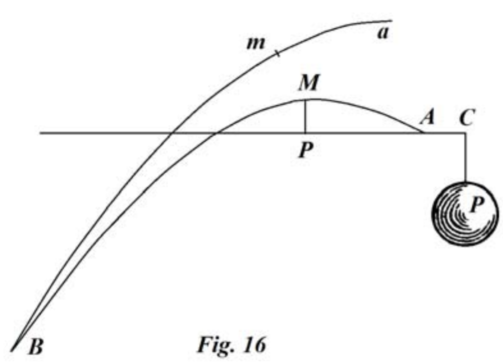

Fig. 16

BMA incurvetur. Per *A* ducatur recta *CAP* ad directionem vis sollicitantis normalis, quae habeatur pro axe, sitque intervallum $AC = c$, abscissa $AP = x$, applicata $PM = s$; erit momentum vis sollicitantis pro puncto $M = P(c+x)$.

48. Sit porro radius osculi curvae quaesitae in $M = R$; sumatur in statu naturali arcus $am = AM = s$ sitque in puncto *m* radius osculi $= r$, qui ob curvam *amB* cognitam dabitur per arcum *S*. In *M* ergo, quia curvatura maior est, radius osculi *R* minor est quam *r* atque excessus anguli elementaris in *M* supra angulum in statu naturali erit $= \frac{ds}{R} - \frac{ds}{r}$, qui excessus erit effectus a potentia sollicitante productus. Quamobrem erit $P(c+x) = Ekk\left(\frac{1}{R} - \frac{1}{r}\right)$, quae, cum *r* per *s* detur, erit aequatio pro curva quaesita ; quae autem sic in genere spectata ulterius reduci non potest.

49. Ponamus ergo laminam in statu naturali *amB* habere figuram circularem; erit radius eius circuli, qui sit $= a$, unde fit

$$P(c+x) = Ekk\left(\frac{1}{R} - \frac{1}{a}\right).$$

Multiplicetur haec aequatio per *dx* et integretur; orietur

$$\frac{P}{Ekk}\left(\tfrac{1}{2}xx + cx + f\right) = -\frac{dy}{ds} - \frac{x}{a};$$

quae aequatio, si loco *c* scribatur $c + \frac{Ekk}{Pa}$, abibit in

$$\frac{P}{Ekk}\left(\tfrac{1}{2}xx + cx + f\right) = -\frac{dy}{ds},$$

quae est eadem aequatio, quam supra pro lamina naturaliter recta invenimus. Lamina ergo naturaliter circularis in easdem curvas incurvatur, quae laminae naturaliter rectae inducuntur; tantum scilicet locus applicationis potentiae seu intervallum *AC*= *c* pro utroque casu secundum datam legem variari debebit. Eaedem ergo novem species curvarum prodibunt pro figuris, quas lamina naturaliter circularis inducere potest, quas supra numeravimus. Lamina enim circularis, si intervallum *AC* capiatur infinitum, primum in lineam rectam extendi potest; tum, quaecunque potentia insuper applicata eundem praestabit effectum, ac si sola laminae Elasticae naturaliter rectae applicaretur.

50. Ponamus autem, quaecunque sit laminae figura naturalis, punctum *C* infinite distare, ita ut momentum vis sollicitantis ubique sit idem, quod per *Ekk* divisum ponatur $=\frac{1}{b}$; eritque

$$\frac{1}{b}=\left(\frac{1}{R}-\frac{1}{r}\right) \text{ et } \frac{1}{R}=\frac{1}{b}+\frac{1}{r}.$$

Hinc fiet

$$\int\frac{ds}{R}=\frac{s}{b}+\int\frac{ds}{r}= \text{ amplitudini arcus AM },$$

sicuti $\int\frac{ds}{r}$ exprimit amplitudinem arcus *am*; quemadmodum quidem Celeberrimus JOHANNES BERNOULLI hoc *amplitudinis* nomine in eximio Tractatu *De motu reptorio* uti est solitus. Sit igitur $\frac{s}{b}+\int\frac{ds}{r}$ arcus in circulo, cuius radius $=1$ sumptus, qui ob *r* per *s* datum quoque in *s* erit cognitus. Hinc autem reperientur coordinatae orthogonales *x* et *y*, ita ut sit

$$x=\int ds\sin\left(\frac{s}{b}+\int\frac{ds}{r}\right) \text{ et } y=\int ds\cos\left(\frac{s}{b}+\int\frac{ds}{r}\right);$$

unde curva quaesita per quadraturas construi poterit.

51. Hinc determinari potest figura *amB* (Fig. 17), quam lamina in situ naturali habere debet, ut a potentia *P* in directione *AP* sollicitante in lineam rectam *AMB* explicetur. Sumpta enim longitudine $AM = s$ erit momentum potentiae sollicitantis pro puncto $M = Ps$; radius osculi autem in M , per hypothesin, erit infinitus seu $\frac{1}{R}=0$. Sumpto iam in statu naturali arcu $am=s$ positoque radio osculi in $m=r$, quia haec curva convexitate sua rectam *AB* spectat, in calculo praecedente poni debet *r* negativum. Hinc erit

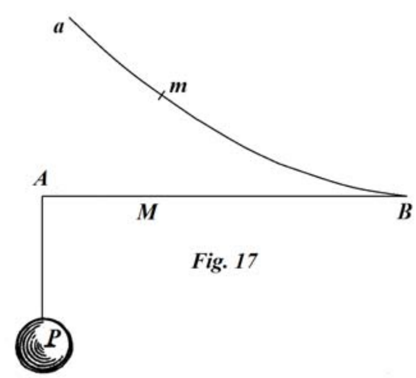

Fig. 17

$$Ps=\frac{Ekk}{r} \text{ seu } rs=aa;$$ quae est aequatio naturam curvae *amB* complectens.

52 . Cum igitur $\frac{1}{r}=\frac{s}{aa}$, erit $\int\frac{ds}{r}=\frac{ss}{2aa}$, seu erit amplitudo arcus *am* ut quadratum ipsius arcus. Hinc coordinatae orthogonales *x* et *y* pro hac curva *amB* ita definientur, ut sit

$$x=\int ds\sin\frac{ss}{2aa} \text{ et } y=\int ds\cos\frac{ss}{2aa};$$

Scilicet in circulo, cuius radius $=1$, abscindi debet arcus $\frac{ss}{2aa}$, cuius sinus et cosinus ad coordinatas determinandas assumi debent. Ex eo autem, quod radius osculi continuo decrescit, quo maior capiatur arcus $am=s$, manifestum est curvam in infinitum non protendi, etiamsi arcus *s* capiatur infinitus. Curva ergo erit ex spiralium genere, ita ut infinitis peractis spiris in certo quodam puncto tanquam centro convolvatur, quod punctum ex hac constructione invenire difficillimum videtur. Non exiguum ergo analysis incrementum capere existimanda erit, si quis methodum inveniret, cuius ope, saltem vero proxime, valor horum integralium

$$\int ds\sin\frac{ss}{2aa} \text{ et } \int ds\cos\frac{ss}{2aa}$$

assignari posset casu, quo *s* ponitur infinitum ; quod Problema non indignum videtur, in quo Geometrae vires suas excerceant.

53. Sit $2aa=bb$, et cum sit

$$\sin\frac{ss}{bb}=\frac{s^2}{b^2}-\frac{s^6}{1\cdot 2\cdot 3b^6}+\frac{s^{10}}{1\cdot 2\cdot 3\cdot 4\cdot 5b^{10}}-\frac{s^{14}}{1\cdot 2...7b^{14}}+\text{ etc.}$$
$$\cos\frac{ss}{bb}=1-\frac{s^4}{1\cdot 2b^4}+\frac{s^8}{1\cdot 2\cdot 3\cdot 4b^8}-\frac{s^{12}}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6b^{12}}+\text{ etc.},$$

coordinatae *x* et *y* curvae quaesitae commode per series infinitas exprimi poterunt, erit enim

$$x=\frac{s^3}{1\cdot 3b^2}-\frac{s^7}{1\cdot 2\cdot 3\cdot 7b^6}+\frac{s^{11}}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 11b^{10}}-\frac{s^{15}}{1\cdot 2...7\cdot 15b^{14}}+\text{ etc.}$$
$$y=s-\frac{s^5}{1\cdot 2\cdot 5b^4}+\frac{s^9}{1\cdot 2\cdot 3\cdot 4\cdot 9b^8}-\frac{s^{13}}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 13b^{12}}+\text{ etc.},$$

ex quibus seriebus vehementer convergentibus, nisi arcus *s* assumatur valde magnus, valores coordinatarum *x* et *y* vero proxime satis expedite determinari possunt. Verum cuiusmodi valores *x* et *y* acquirant, si ponatur arcus *s* infinite magnus, ex his seriebus nullo modo concludi potest.

54. Quoniam igitur positio infiniti loco *s* facienda maximam parit difficultatem, huic quidem incommodo sequenti modo remedium afferri potest.
Ponatur $\frac{ss}{bb}=v$, ut sit $s=b\sqrt{v}$, erit $ds=\frac{bdv}{2\sqrt{v}}$ fietque

$$x=\frac{b}{2}\int\frac{dv}{\sqrt{v}}\sin v \text{ et } y=\frac{b}{2}\int\frac{dv}{\sqrt{v}}\cos v.$$

Nunc autem dico valores debitos pro *x* et *y*, si ponatur $s=\infty$, inventum iri ex his formulis integralibus,

$$x=\frac{b}{2}\int dv\left(\frac{1}{\sqrt{v}}-\frac{1}{\sqrt{(\pi+v)}}+\frac{1}{\sqrt{(2\pi+v)}}-\frac{1}{\sqrt{(3\pi+v)}}+\text{ etc.}\right)\sin v$$
$$y=\frac{b}{2}\int dv\left(\frac{1}{\sqrt{v}}-\frac{1}{\sqrt{(\pi+v)}}+\frac{1}{\sqrt{(2\pi+v)}}-\frac{1}{\sqrt{(3\pi+v)}}+\text{ etc.}\right)\cos v,$$

si post integrationem ponatur $v=\pi$, denotante π angulum duobus rectis aequalem. Hoc ergo modo positio infiniti quidem evitatur, contra vero series infinita

$$\frac{1}{\sqrt{v}}-\frac{1}{\sqrt{(\pi+v)}}+\frac{1}{\sqrt{(2\pi+v)}}-\text{ etc.}$$

in calculum introducitur, cuius summa cum adhuc lateat, resolutio huius modi maximae adhuc difficultati est obnoxia.

DE INCURVATIONE LAMINAE ELASTICAE IN SINGULIS PUNCTIS A VIRIBUS QUIBUSCUNQUE SOLLICITATAE

55. Tradita iam methodo investigandi curvaturam cuiusque laminae Elasticae, si ea ab una vi in dato loco applicata sollicitetur, conveniet quoque curvaturam a pluribus, imo infinitis, potentiis laminae Elasticae inductam indagare. Quoniam vero nondum constat, cuiusmodi expressio his casibus futura sit vel maxima vel minima, methodo utar tantum directa, quo ex ipsa solutione fortasse proprietas ea, quae est maxima vel minima, erui queat. Sit igitur

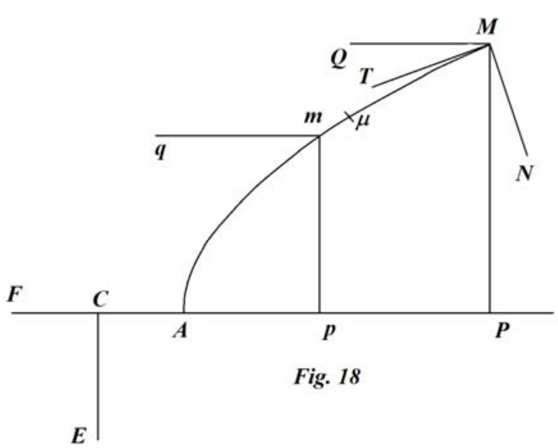

Fig. 18

lamina Elastica (Fig. 18) naturaliter recta, in statum *AmM* redacta primum a viribus finitis *P* et *Q* secundum directiones *CE* et *CF* inter se normales sollicitantibus, tum vero a viribus infinite parvis singulis laminae elementis $m\mu$ applicatis et secundum directiones *mp* et *mq* illis *CE* et *CF* parallelas trahentibus; quibus positis requiritur natura curvae *AmM* laminae inductae.

56. Sumatur recta *FCA* producta pro axe, ponatur $AC = c$ et vocetur abscissa $AP = x$, applicata $PM = y$, arcus curvae $AM = s$ et radius osculi in $M = R$. Sit elasticitas laminae absoluta constans $= Ekk$, atque summa momentorum ex omnibus viribus sollicitantibus respectu puncti *M* ortorum aequalis esse debet $\frac{Ekk}{R}$. Primum quidem a vi finita *P* in directione *CE* trahente oritur momentum $= P(c+x)$ in eam plagam agens, qua vis Elastica aequilibratur. Momentum autem ex altera vi *Q* ortum, nempe *Qy*, in contrariam plagam tendit, ex quo ex viribus finitis *P* et *Q* coniunctim oritur momentum $P(c+x)-Qy$. Iam consideretur quodvis elementum laminae intermedium $m\mu$, cuius respondens abscissa *Ap* ponatur $=\zeta$ et applicata $pm=\eta$, sit autem vis elementum $m\mu$ in directione *mp* urgens $= dp$ et vis urgens in directione $mq = dq$; erit momentum ex his viribus pro puncto *M* ortum

$$=(x-\zeta)\,dp-(y-\eta)\,dq.$$

57. Ad summam ergo omnium horum momentorum inveniendam punctum *M* ac proinde *x* et *y* tantisper pro constantibus haberi debent, dum solae coordinatae ζ et η cum viribus *dp* et *dq* tanquam variabiles spectantur. Erit ergo summa momentorum a viribus arcum *Am* sollicitantibus ortorum

$$= xp-\int\zeta dp-yp+\int\eta dq\,,$$

ubi *p* exprimit summam omnium virium arcum *AM* in directionibus applicatis *pm* parallelis sollicitantium et *q* summam omnium virium arcum *Am* in directionibus axi *Ap* parallelis sollicitantium. At est

$$\int\zeta dp=\zeta p-\int pd\zeta \quad \text{et} \quad \int\zeta dq=\eta q-\int qd\eta\,;$$

unde fit summa momentorum ex viribus arcui *Am* applicatis ortorum

$$=(x-\zeta)\,p+\int pd\zeta-(y-\eta)\,q-\int qd\eta.$$

Promoveatur iam punctum *m* in *M* usque fietque $\zeta = x,\ \eta = y$ et $d\zeta = dx$ atque $d\eta = dy$; unde summa omnium momentorum per totum arcum *AM* sumptorum erit

$$=\int pdx-\int qdy .$$

Quocirca obtinebitur pro curva quaesita haec aequatio

$$\frac{Ekk}{R}=P(c+x)-Qy+\int pdx-\int qdy,$$

ubi ergo *p* exprimit summam omnium virium verticalium seu in directionibus applicatarum *MP* agentium et *q* summam omnium virium horizontalium seu in directionibus *MQ* axi *AP* parallelis agentium per totum arcum *AM*.

58. Si formulae *pdx* et *qdy* integrationem non admittant, tum aequatio inventa per differentiationem ab his formulis integralibus liberari debebit, unde habebitur ista aequatio :

$$\frac{EkkdR}{RR}=Pdx-Qdy+pdx-qdy.$$

Sin autem nec *p* nec *q* per expressiones finitas exhiberi possint, quippe quae iam exprimunt summas infinitarum virium infinite parvarum, tum per ulteriorem differentiationem valores finiti *p* et *q* exterminari debebunt, ut tantum insint *dp* et *dq* cum differentio-differentialibus *ddp* et *ddq*. Orietur autem post primam differentiationem

$$-Ekkd\cdot\frac{dR}{RRdx}=dp-(Q+q)d\cdot\frac{dy}{dx}-\frac{dy}{dx}dq.$$

Sit $\frac{dy}{dx}=\omega,$ eritque denuo aequatione differentiata:

$$-Ekkd\cdot\frac{d\cdot\frac{dR}{RRdx}}{d\omega}=d\cdot\frac{dp}{d\omega}-2dq-\omega d\cdot\frac{dq}{d\omega},$$

quae aequatio ad differentialia quarti ordinis ascendit.

59. Sint laminae, loco potentiarum verticalium et horizontalium *p* et *q*, in singulis punctis *M* applicatae duae potentiae, altera normalis $MN=dv$ et altera tangentialis $MT=dt$. Hinc erit

$$dp=\frac{dxdv}{ds}+\frac{dydt}{ds}\ \text{ et }\ dq=\frac{dxdt}{ds}-\frac{dydv}{ds}$$

et ob $dy=\omega dx$ et $ds=dx\sqrt{(1+\omega\omega)}$ habebitur

$$dp=\frac{dv}{\sqrt{(1+\omega\omega)}}+\frac{\omega dt}{\sqrt{(1+\omega\omega)}}\quad\text{et}\quad dq=\frac{dt}{\sqrt{(1+\omega\omega)}}-\frac{\omega dv}{\sqrt{(1+\omega\omega)}};$$

quibus in praecedentis paragraphi aequatione ultima substitutis proveniet sequens aequatio

$$-Ekkd\cdot\frac{d\cdot\dfrac{dR}{RRdx}}{d\omega}=-\frac{dt}{\sqrt{(1+\omega\omega)}}+\frac{2\omega dv}{\sqrt{(1+\omega\omega)}}+\sqrt{(1+\omega\omega)}d\cdot\frac{dv}{d\omega},$$

quae multiplicata per $\sqrt{(1+\omega\omega)}$ fit integrabilis; posito enim brevitatis gratia $z=\frac{dR}{RRdx}$ reperietur integrale

$$A-t+\frac{dv(1+\omega\omega)}{d\omega}=-Ekk\left(\frac{dz\sqrt{(1+\omega\omega)}}{d\omega}-\frac{\omega z}{\sqrt{(1+\omega\omega)}}+\frac{1}{2RR}\right)$$
$$=-Ekk\left(\frac{1+\omega\omega}{d\omega}d\cdot\frac{dR}{RRdx\sqrt{(1+\omega\omega)}}+\frac{1}{2RR}\right).$$

Cum vero sit

$$R=-\frac{(1+\omega\omega)^{3:2}}{d\omega}$$

erit

$$d\omega=-\frac{(1+\omega\omega)^{3:2}dx}{R};$$

quo loco $d\omega$ valore substituto habebitur:

$$A-t-\frac{Rdv}{ds}=-Ekk\left(\frac{1}{2RR}-\frac{R}{ds}d\cdot\frac{dR}{RRds}\right),$$

ob $dx\sqrt{(1+\omega\omega)}=ds$. Quacirca aequatione ordinata, pro curva quaesita orietur haec aequatio

$$t+\frac{Rdv}{ds}-A=Ekk\left(\frac{1}{2RR}-\frac{R}{ds}d\cdot\frac{dR}{RRds}\right).$$

60. Primum quidem manifestum est, si vis Elastica *Ekk* evanescat, laminam transmutari in filum perfecte flexile; atque hinc in his aequationibus continentur omnes curvae, quas filum perfecte flexile a viribus quibuscunque sollicitatum formare potest. Sic si filum a propria gravitate tantum deorsum sollicitatur, erit $q=0$ et *p* exprimet pondus funis *AM*,

eritque ergo $P\frac{dx}{dy} = Q =$ constanti facto $P = 0$, quae est aequatio generalis pro omnis generis Catenariis. Sin autem filum perfecte flexile in singulis punctis a viribus, quarum directiones sunt normales ad ipsam curvam, sollicitetur, ita ut in puncto *M* filum sollicitetur secundum directionem *MN* vi $= dv$, ob $t = 0$ erit $\frac{Rdv}{ds} = A =$ constanti, quae est proprietas generalis curvarum Velariarum, Lintheariarum omniumque, in quibus huiusmodi sollicitationes locum habent.

DE CURVATURA LAMINAE ELASTICAE A PROPRIO PONDERE ORTA

61. Ad laminas Elasticas autem revertor, de quibus mox ista quaestio prae ceteris notatu digna se offert, cuiusmodi figuram accipiat lamina Elastica proprio pondere incurvata. Sit *AmM* haec curva, quae quaeritur, et quia solae vires verticales a gravitate ortae urgent, fiet $P = 0,\ Q = 0,\ q = 0$ et *p* exprimet pondus laminae *AM*. Quare, si *F* sit pondus laminae longitudinis *a*, quia lamina uniformis assumitur, erit $p = \frac{Fs}{a}$; unde curvae natura hac exprimetur aequatione

$$-\frac{EkkdR}{RR} = \frac{Fsdx}{a}.$$

Sit amplitudo curvae $\int \frac{ds}{R} = u$, erit $R = \frac{ds}{du}$ et $dx = ds \sin u$; unde sumpto elemento *ds* constante reperietur aequatio

$$sds \sin u + \frac{Eakk}{F} \cdot \frac{ddu}{ds} = 0,$$

quae autem, quantum primo intuitu patet, ulterius reduci nequit.

62. In primis autem notari meretur curva, quam fluidum altitudinis quasi infinitae laminae Elasticae inducit. Sit (Fig. 19) *AMB* figura haec, quae quaeritur, et posito $AP = x$, $PM = y$, $AM = s$ elementum *Mm* in directione normali *MN* urgebitur vi ipsi *ds* proportionali; unde erit $dv = nds$ et $dt = 0$. Hinc orietur vis verticalis $dp = ndx$ et horizontalis $dq = -ndy$; ex quibus statim fit $p = nx$ et $q = -ny$; ideoque in aequatione prima fiet

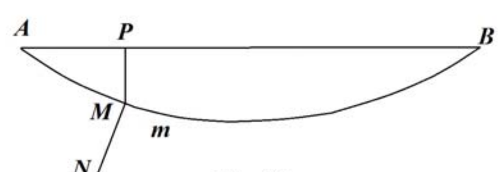

Fig. 19

$$\frac{Ekk}{R} = P(c + x) - Qy + \tfrac{1}{2}nxx + \tfrac{1}{2}nyy.$$

Coordinatae vero *x* et *y* ita quantitatibus constantibus augeri diminuive possunt, ut aequatio pro curva huiusmodi faciem acquirat $xx+yy=A+\frac{B}{R}$. Haec autem aequatio si multiplicetur per $xdx+ydy$, fiet integrabilis; est enim

$$\int\frac{xdx+ydy}{R}=-\int\frac{x+y\omega}{(1+\omega\omega)^{3:2}}d\omega=\frac{y-\ \omega x}{\sqrt{(1+\omega\omega)}}=\frac{ydx-xdy}{ds}.$$

Hanc ob rem, post integrationem constantibus mutatis, prodibit

$$(xx+yy)^2=A(xx+yy)+\frac{B(ydx-xdy)}{ds}+C.$$

Sit $\sqrt{(xx+yy)}=z$ et $y=uz$, erit $x=z\sqrt{(1-uu)}$; unde

$$ydx-xdy=-\frac{zzdu}{\sqrt{(1-uu)}}\quad \text{et}\quad ds=\sqrt{\left(dz^2+\frac{zzdu^2}{1-uu}\right)}.$$

Ergo posito $\frac{du}{\sqrt{(1-uu)}}=dr$ erit

$$z^4-Az^2-C=-\frac{Bzzdr}{\sqrt{(dz^2+zzdr^2)}}\quad ;$$

hincque

$$dr=\frac{du}{\sqrt{(1-uu)}}=-\frac{dz\left(z^4-Az^2-C\right)}{z\sqrt{\left(B^2zz-\left(z^4-Az^2-C\right)^2\right)}}.$$

Curva haec ergo, si fuerit $A=0$ et $C=0$, erit algebraica; habebitur enim haec aequatio

$$\frac{du}{\sqrt{(1-uu)}}=\frac{zzdz}{\sqrt{(B^2-z^6)}}=\frac{3zzdz}{3\sqrt{(a^6-z^6)}},$$

quae integrata dat

$$\mathrm{A}\sin u=\tfrac{1}{3}\mathrm{A}\sin\frac{z^3}{a^3}\ \text{ or }\ \frac{z^3}{a^3}=3u-4u^3=\frac{3y}{z}-\frac{4y^3}{z^3};$$

unde haec resultat aequatio $z^6=3a^3yzz-4a^3y^3$ seu ob $zz=xx+yy$ haec

$$x^6 + 3x^4y^2 + 3xxy^4 + y^6 = 3a^3xxy - a^3y^3.$$

DE MOTU OSCILLATORIO LAMINARUM ELASTICARUM

63. Ex his etiam motus oscillatorius laminarum Elasticarum utcunque ad motum comparatarum definiri potest, quod argumentum profecto dignissimum primum excolere coepit Vir Celeberrimus DANIEL BERNOULLI mihique iam ante complures annos Problema de oscillationibus laminae Elasticae altero termino parieti firmo infixae determinandis proposuit, cuius Solutionem exhibui in *Comment. Petropo*l. Tomo VII. Ex hoc autem tempore cum mihi commodius hoc Problema tractare contigit, tum etiam per commercium cum Celeberrimo BERNOULLIO plures accesserunt aliae quaestiones et considerationes, quarum enodationem ob materiae affinitatem hic adiungam. Quando autem motus vibratorius est satis promtus, tum simul a lamina vibrante sonus editur, cuius tenor ac relatio ad alios ope doctrinae de sonis ex his principüs determinabitur. Et quoniam sonorum indoles facillime ad experimenta revocatur, hoc ipso consensus calculi cum veritate explorari atque adeo Theoria confirmari poterit; quo pacto cognitio nostra circa naturam corporum elasticorum non parum amplificabitur.

64. Primum autem monendum est hic tantum circa oscillationes minimas quaestionem institui atque adeo intervallum, per quod lamina inter oscillandum excurrit, esse quasi infinite parvum. Neque vero hac limitatione usus et applicatio quicquam diminuitur; non solum enim oscillationes, si per maiora spatia fierent, isochronismo destituerentur, sed etiam sonorum distinctorum formatio, ad quam hic potissimum spectamus, minimas oscillationes requirit. Considero igitur hic primum laminam Elasticam uniformem naturaliter, rectam (Fig. 20), cuius alter terminus *B* pavimento immobili firmiter sit infixus, ita ut lamina sibi relicta situm teneat rectum *BA*. Sit huius laminae longitudo $AB = a$ eiusque elasticitas absoluta in singulis locis $= Ekk$; ab eius vero pondere vel mentem revocamus vel infixionem eiusmodi statuimus, ut eius status a gravitate turbari nequeat.

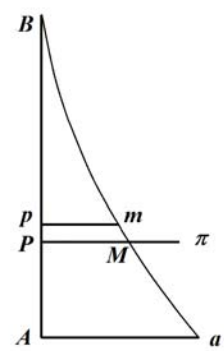

Fig. 20

DE OSCILLATIONIBUS LAMINAE ELASTICAE ALTERO TERMINO MURO INFIXAE

65. Iam lamina haec a vi quacunque impulsa vibrationes peragat minimas circa statum naturalem *BA* utrinque excurrendo per minima intervalla *Aa*. Sitque *BMa* status quispiam, quem lamina inter oscillandum tenet; qui quoniam infinite parum tantum distat a statu naturali *BPA*, rectae *MP*, *Aa* simul repraesentabunt vias, quas laminae puncta *M* et *a* percurrunt, vel potius hae rectae ad vias veras rationem habebunt a ratione aequalitatis infinite parum discrepantem. Ad motum autem oscillatorium determinandum absolute necesse est naturam curvae *BMa*, quam lamina inter oscillandum induit, nosse. Sit

igitur $AP = x,\ PM = y$, arcus $aM = s$ et radius osculi in $M = R$ et intervallum minimum $Aa = b$; atque ex conditione memorata erit arcus *s* proxime aequalis abscissae *x*, ac proinde pro *ds* sumi poterit *dx*; prae dx enim evanescet *dy*. Et cum posito *dx* constante sit generatim radius osculi

$$= \frac{ds^3}{dxddy},$$

erit praesenti casu $R = \frac{dx^2}{ddy}$; nam curva *BMa* convexitatem axi *BA* obvertit et, quia lamina in *B* muro firmiter est infixa, erit recta *AB* tangens curvae in puncto *B*.

66. His positis tam ad naturam curvae *BMa* quam ad ipsum motum oscillatorium determinandum sit *f* longitudo penduli simplicis isochroni; oscillationes enim minimas esse isoohronas cum natura rei declarat, tum ipse calculus instituendus monstrabit. Acceleratio ergo, qua laminae punctum *M* versus *P* urgetur, erit $= \frac{PM}{f} = \frac{y}{f}$. Quare, si massa totius laminae ponatur $= M$, quae per eius pondus exprimitur, erit elementi $Mm = ds = dx$ massa $= \frac{Mdx}{a}$; unde vis motrix elementum *Mm* in directione *MP* sollicitans erit $= \frac{Mydx}{af}$; sicque vires, quibus singulae laminae particulae ad motum actu cientur, innotescunt cum ex ipsa curva *BMa*, tum ex longitudine penduli simplicis isochroni *f*. Quoniam vero lamina a vi Elastica revera ad motum incitatur, ex hac cognita vicissim et natura curvae *BMa* et longitudo penduli simplicis isochroni determinabitur.

67. Quoniam ergo lamina perinde movetur, ac si singulis ipsius elementis *Mm* in directione *MP* vires essent applicatae $= \frac{Mydx}{af}$, sequitur, si laminae singulis elementis *Mm* in directionibus contrariis $M\pi$ aequales vires $\frac{Mydx}{af}$ applicarentur, laminam in statu *BMa* aequilibrari. Hinc lamina inter oscillandum eandem curvaturam subibit, quam indueret quieta, si in singulis punctis *M* sollicitaretur viribus $\frac{Mydx}{af}$ in directionibus $M\pi$. Per regulam ergo supra paragrapho 56 inventam colligantur omnes hae vires per arcum *aM* applicatae, atque prodibit summa $\frac{M}{af}\int ydx$ quae ibi in locum ipsius *p* substitui debet. Quare, cum vires reliquae *P*, *Q* et *q*, quae ibi habebantur, evanescant, natura curvae exprimetur aequatione

$$\frac{Ekk}{R} = \int pdx,$$

unde habebitur

$$\frac{Ekk}{R}=\frac{M}{af}\int dx\int ydx.$$

Cum vero sit $R=\frac{dx^2}{ddy}$, erit

$$\frac{Ekkddy}{dx^2}=\frac{M}{af}\int dx\int ydx$$

et differentiando

$$\frac{Ekkd^3y}{dx^2}=\frac{Mdx}{af}\int ydx$$

denuoque differentiando prodibit ista aequatio differentialis quarti ordinis

$$Ekkd^4y=\frac{Mydx^4}{af}.$$

68. Hac ergo aequatione et natura curvae *BMa* exprimitur et ex eadem, si ad casum oblatum accommodetur, longitudo *f* determinabitur; qua cognita ipse motus oscillatorius innotescet. Ante omnia autem hanc aequationem integrari oportet; quae cum pertineat ad id aequationum differentialium altiorum graduum genus, cuius integrationem generalem exhibui in *Miscell. Berol.* Volumine VII, hinc sequens aequatio integralis reperietur ponendo brevitatis ergo $\frac{Ekk\cdot af}{M}=c^4$; prodibit scilicet

$$y=Ae^{\frac{x}{c}}+Be^{-\frac{x}{c}}+C\sin\frac{x}{c}+D\cos\frac{x}{c},$$

ubi *e* denotat numerum, cuius logarithmus hyperbolicus est $=1$, et $\sin\frac{x}{c}$ et $\cos\frac{x}{c}$ et denotant sinum et cosinum arcus $=x$ in circulo, cuius radius $=1$, assumpti. Tum vero *A*, *B*, *C* et *D* sunt quatuor constantes arbitrariae per quadruplicem integrationem introductae, quas ex accommodatione calculi ad praesentem casum definire oportet.

69. Determinatio autem constantium sequenti modo instituetur. Primum posito $x=0$ fieri debet $y=b$; hinc ergo oritur ista aequatio $b=A+B+D$. quae est prima. Secundo, cum sit

$$\frac{c^4ddy}{dx^2}=\int dx\int ydx$$

facto $x=0$ fieri debet $\frac{ddy}{dx^2}=0$; at est

$$\frac{ddy}{dx^2}=\frac{A}{cc}e^{\frac{x}{c}}+\frac{B}{cc}e^{-\frac{x}{c}}-\frac{C}{cc}\sin\frac{x}{c}-\frac{D}{cc}\cos\frac{x}{c},$$

unde oritur haec aequatio secunda $0=A+B-D$.

Tertio, cum sit $\frac{c^4d^3y}{dx^3}=\int ydx$, posito $x=0$ simul $\frac{d^3y}{dx^3}$ evanescere debet ; quia ergo erit

$$\frac{c^3d^3y}{dx^3}=Ae^{\frac{x}{c}}-Be^{-\frac{x}{c}}-C\cos\frac{x}{c}+D\sin\frac{x}{c},$$

prodit aequatio *tertia* $0=A-B-C$.

Quarto autem, si ponatur $x=a$, applicata *y* evanescit, unde obtinebitur aequatio *quarta*

$$0=Ae^{\frac{a}{c}}+Be^{-\frac{a}{c}}+C\cos\frac{a}{c}+D\sin\frac{a}{c}.$$

Quinto, quia *AB* est tangens curvae in puncto *B*, facto $x=a$ fieri debet $\frac{dy}{dx}=0$; unde prodit aequatio *quinta*

$$0=Ae^{\frac{a}{c}}-Be^{-\frac{a}{c}}+C\cos\frac{a}{c}-D\sin\frac{a}{c}.$$

Ex his ergo quinque aequationibus primum quatuor constantes *A*, *B*, *C*, *D* definientur; tum vero, in quo cardo rei versatur, determinabitur valor ipsius

$$c=\sqrt[4]{\frac{Ekk\cdot af}{M}};$$

ex quo longitudo penduli simplicis isochroni *f* elicietur, quo ipso durationes oscillationum cognoscentur.

70. Ex aequationibus secunda et tertia constantes *C* et *D* ex *A* et *B* ita definientur, ut sit $C=A-B$ et $D=A+B$, qui valores in aequationibus quarta et quinta substituti dabunt

$$0=Ae^{\frac{a}{c}}+Be^{-\frac{a}{c}}+(A-B)\sin\frac{a}{c}+(A+B)\cos\frac{a}{c},$$

$$0=Ae^{\frac{a}{c}}-Be^{-\frac{a}{c}}-(A+B)\cos\frac{a}{c}-(A+B)\sin\frac{a}{c};$$

ex quibus eruitur,

$$\frac{A}{B}=\frac{-e^{-\frac{a}{c}}+\sin\frac{a}{c}-\cos\frac{a}{c}}{e^{\frac{a}{c}}+\sin\frac{a}{c}+\cos\frac{a}{c}}=\frac{e^{-\frac{a}{c}}+\cos\frac{a}{c}+\sin\frac{a}{c}}{e^{\frac{a}{c}}+\cos\frac{a}{c}-\sin\frac{a}{c}},$$

unde obtinebitur haec aequatio

$$0=2+\left(e^{-\frac{a}{c}}+e^{-\frac{a}{c}}\right)\cos\frac{a}{c}$$

seu

$$e^{\frac{2a}{c}}\cos\frac{a}{c}+2e^{\frac{a}{c}}+\cos\frac{a}{c}=0,$$

quae dat

$$e^{\frac{a}{c}}=\frac{-1\pm\sin\frac{a}{c}}{\cos\frac{a}{c}}.$$

Cum igitur $e^{\frac{a}{c}}$ sit quantitas affirmativa, cosinus anguli $\frac{a}{c}$ erit negativus ideoque angulus $\frac{a}{c}$ recto maior.

71. Ex hac aequatione intelligitur dari infinitos angulos $\frac{a}{c}$ quaesito satisfacientes, ex quibus infiniti diversi modi oscillationum eiusdem laminae oriuntur. Curva enim in uno pluribusve punctis axem *AB* secare potest, antequam in *B* axem tangat; ex quo eiusdem laminae plures, imo infiniti, oscillandi modi aeque sunt possibiles. Cum igitur hic inprimis contemplemur casum, quo *B* primum est punctum, ubi lamina ab axe *AB* tangitur, huic casui satisfaciet minimus angulus $\frac{a}{c}$ aequationem inventam resolvens ; qui angulus cum sit $\frac{a}{c}$ recto maior, ponatur $\frac{a}{c}=\frac{1}{2}\pi+\varphi$ existente φ angulo recto minore. Hinc ob $\sin\frac{a}{c}=\cos\varphi$ et $\cos\frac{a}{c}=-\sin\varphi$ obtinebitur duplex aequatio

$$e^{\frac{a}{c}}=\frac{1\pm\cos\varphi}{\sin\varphi},$$

quae praebet vel $e^{\frac{a}{c}} = \text{tang}\,\frac{1}{2}\varphi$ vel $e^{\frac{a}{c}} = \cot\frac{1}{2}\varphi$, quarum posterior minorem dabit valorem pro angulo φ, quae ergo ad casum propositum erit accommodata.

72. Sequentes possibiles oscillationum modi reperientur, si pro $\frac{a}{c}$ ponantur anguli duobis rectis maiores, tribus vero minores. Sic posito $\frac{a}{c} = \frac{3}{2}\pi + \varphi$ erit

$$\sin\frac{a}{c} = -\cos\varphi \text{ et } \cos\frac{a}{c} = -\sin\varphi;$$

unde fit

$$e^{\frac{a}{c}} = \frac{1+\cos\varphi}{\sin\varphi}, \text{ seu vel } e^{\frac{a}{c}} = \text{tang}\,\tfrac{1}{2}\varphi \text{ vel } e^{\frac{a}{c}} = \cot\tfrac{1}{2}\varphi.$$

Simili modo alii oscillationum modi reperientur, ponendo

$$\frac{a}{c} = \tfrac{5}{2}\pi + \varphi, \ \frac{a}{c} = \tfrac{7}{2}\pi - \varphi \text{ etc. etc.}$$

Ex quibus omnibus, si sumantur logarithmi hyperbolici, orientur sequentes aequationes:

I.	$\frac{1}{2}\pi + \varphi = l\cot\frac{1}{2}\varphi$	II.	$\frac{1}{2}\pi + \varphi = l\text{tang}\,\frac{1}{2}\varphi$
III.	$\frac{3}{2}\pi - \varphi = l\cot\frac{1}{2}\varphi$	IV.	$\frac{3}{2}\pi - \varphi = l\text{tang}\,\frac{1}{2}\varphi$
V.	$\frac{5}{2}\pi + \varphi = l\cot\frac{1}{2}\varphi$	VI.	$\frac{5}{2}\pi + \varphi = l\text{tang}\,\frac{1}{2}\varphi$
VII.	$\frac{7}{2}\pi - \varphi = l\cot\frac{1}{2}\varphi$	VIII.	$\frac{7}{2}\pi - \varphi = l\text{tang}\,\frac{1}{2}\varphi$

etc.

Harum autem aequationum tertia congruit cum secunda; posito enim $\frac{1}{2}\varphi = \frac{1}{2}\pi - \frac{1}{2}\theta$, ut sit $\cot\frac{1}{2}\varphi = \text{tang}\,\frac{1}{2}\theta$, tertia transit in $\frac{1}{2}\pi + \theta = l\text{tang}\,\frac{1}{2}\theta$, quae est ipsa secunda. Simili modo quarta congruit cum prima, tum quinta et octava inter se congruunt atque sexta cum septima. Quamobrem sequentes tantum prodibunt aequationes diversae:

I. $\frac{1}{2}\pi+\varphi = l\cot\frac{1}{2}\varphi$

II. $\frac{1}{2}\pi+\varphi = l\operatorname{tang}\frac{1}{2}\varphi$

III. $\frac{5}{2}\pi+\varphi = l\cot\frac{1}{2}\varphi$

IV. $\frac{5}{2}\pi+\varphi = l\operatorname{tang}\frac{1}{2}\varphi$

V. $\frac{9}{2}\pi+\varphi = l\cot\frac{1}{2}\varphi$

VI. $\frac{9}{2}\pi+\varphi = l\operatorname{tang}\frac{1}{2}\varphi$

etc.

73. Logarithmus autem hyperbolicus tangentis vel cotangentis cuiuspiam anguli reperitur sumendo logarithmum Tabularem indeque auferendo logarithmum sinustotius atque residuum multiplicando per 2,302585092994; qui labor ut sublevetur, denuo logarithmis uti conveniet. Sit u logarithmus hyperbolicus tangentis seu cotangentis anguli $\frac{1}{2}\varphi$, qui quaeritur; sumatur exTabulis Logarithmus eiusdem tangentis cotangentisve, qui logarithmo sinus totius multatus ponatur $=v$. Cum ergo sit $u=2,302585092994\cdot v$, erit sumendis logarithmis vulgaribus

$$lu = lv + 0,3622156886..$$

Hoc logarithmo invento, cum sit $u=\frac{n}{2}\pi+\varphi$, erit

$$lu = l\left(\frac{n}{2}\pi+\varphi\right).$$

Ad hoc evolvendum angulus φ in partibus radii exprimi debet, quemadmodum et π eodem modo exprimitur, dum est $\pi=3,1415926535$ ac propterea $\frac{1}{2}\pi=1,57079632679$. Angulus autem φ eodem modo exprimetur, si is in minuta secunda convertatur atque ab huius numeri logarithmo subtrahatur constanter 5,3144251332; sic enim prodibit $l\varphi$, ex quo ad numeros regrediendo valor ipsius φ eruitur. Erit autem constanter pro unoquoque oscillationum genere

$$\frac{a}{c} = u = \frac{n}{2}\pi+\varphi.$$

74. His circa calculum instituendum monitis, per approximationes valor anguli φ pro quovis oscillationum genere non difficulter eruetur. Tribuendo enim pro lubitu ipsi φ valores aliquot et per calculum determinando et $\frac{n}{2}\pi+\varphi$ et $l\begin{matrix}\text{tang}\\\text{cot}\end{matrix}\frac{1}{2}\varphi$, mox valor ipsius φ prope verus cognoscetur. Quodsi autem habeantur limites anguli φ utcunque remoti, statim invenientur limites propiores ex hisque tandem verus valor ipsius φ. Sic pro aequatione prima

$$\frac{a}{c} = \frac{n}{2}\pi + \varphi = l\cot\tfrac{1}{2}\varphi$$

sequentes limites anguli φ erui 17° 26' et 17° 27', ex quibus per sequentem calculum verus valor ipsius φ obtinebitur:

φ =	17° 26' 0"	17° 27' 0"
in min. sec. =	62760"	62820"
log. =	4,797 6S29349	4,79S0979321
subtr.	5,3144251332	5,3144251332
$l\varphi$ =	9, 4832578917	9, 4836727989
φ =	0, 3042690662	0,3045599545
$\frac{1}{2}\pi$ =	1,5707963268	1,5707963268
$\frac{1}{2}\pi+\varphi$ =	1,8750653930	1,8753562813
$\frac{1}{2}\varphi$ =	8° 43' 0"	8° 43' 30"
$l\cot\frac{1}{2}\varphi$ =	10, 8144034109	10,8139819342
v =	0, 8144034109	0 ,8139819342
lv =	9, 9108395839	9,9106147660
add. =	0,3622156886	0 ,3622156886
lu =	0,2730552725	0, 2728304546
u =	1,8752331540	1, 8742626675
diff.	+ 1677610	-10936138

Ex his ergo utriusque limitis erroribus concluditur fore

$$\varphi = 17°\ 26'\ 7''\frac{98}{100} \text{ et } \frac{1}{2}\pi + \varphi \text{ seu } \frac{a}{c} = 107°\ 26'\ 7''\frac{98}{100}.$$

Cum vero in minutis secundis sit $\varphi = 62767,98$, erit

$l\varphi$ =	4,79773S1525
subtr. =	5 ,3144251332
	9,4833130193
ergo φ =	0,3043077545
add. $\frac{1}{2}\pi$ =	1, 5707963268
$\frac{a}{c}$ =	1,8751040813

quo invento erit

$$\frac{A}{B} = \text{tang}\,\tfrac{1}{2}\varphi = 0{,}1533390624\,.$$

Reperitur ergo ratio constantium A et B , ex quibus et ratio reliquarum constantium C et D ad illas cognoscetur.

75. Restat adhuc prima aequatio $b = A + B + D$, quae ob $D = A + B$ abit in $b = 2A + 2B$; ideoque $A + B = \tfrac{1}{2}b$; cum ergo sit $\frac{A}{B} = \text{tang}\,\tfrac{1}{2}\varphi$,
fiet

$$B\left(1 + \text{tang}\,\tfrac{1}{2}\varphi\right) = \tfrac{1}{2}b \text{ et } B = \frac{b}{2 + 2\text{tang}\,\tfrac{1}{2}\varphi}\,.$$

Unde ex $\text{tang}\,\tfrac{1}{2}\varphi = 0{,}1533390624$ singulae aequationis constantes sequenti modo determinabuntur:

$$\frac{A}{b} = \frac{\text{tang}\,\tfrac{1}{2}\varphi}{2\left(1 + \text{tang}\,\tfrac{1}{2}\varphi\right)} = \frac{0{,}1533390624}{2{,}3066781248}$$

$$\frac{B}{b} = \frac{1}{2\left(1 + \text{tang}\,\tfrac{1}{2}\varphi\right)} = \frac{1{,}0000000000}{2{,}3066781248}$$

$$\frac{C}{b} = \frac{-1 + \text{tang}\,\tfrac{1}{2}\varphi}{2\left(1 + \text{tang}\,\tfrac{1}{2}\varphi\right)} = -\frac{0{,}8466609376}{2{,}3066781248}$$

$$\frac{D}{b} = \frac{1 + \text{tang}\,\tfrac{1}{2}\varphi}{2\left(1 + \text{tang}\,\tfrac{1}{2}\varphi\right)} = \frac{1{,}1533390624}{2{,}3066781248}$$

quibus inventis natura curvae *aMB*, quam lamina inter oscillandum induit, hac exprimetur aequatione

$$\frac{y}{b} = \frac{A}{b}e^{\frac{x}{c}} + \frac{B}{b}e^{-\frac{x}{c}} + \frac{C}{b}\sin\frac{x}{c} + \frac{D}{b}\sin\frac{x}{c}.$$

76. Quod autem ad oscillationum velocitatem attinet, ea ex aequatione $\frac{a}{c} = 1{,}8751040813$ cognoscetur. Ponatur brevitatis gratia $n = 1{,}8751040813$, ut sit $a = nc$.

Et cum sit $c^4 = \frac{Ekk \cdot af}{M}$, ubi $\frac{M}{a}$ exprimit gravitatem specificam laminae et *Ekk* elasticitatem absolutam, eo modo, quo hactenus sum usus, erit

$$a^4 = n^4 \cdot Ekk \cdot \frac{a}{M} \cdot f \text{ ideoque } f = \frac{a^4}{n^4} \cdot \frac{1}{Ekk} \cdot \frac{M}{a},$$

ex quo longitudo penduli simplicis isochroni tenebit rationem compositam ex quadruplicata longitudinis laminae, simplici gravitatis specificae et inversa elasticitatis absolutae. Sit *g* longitudo penduli simplicis singulis minutis secundis oscillantis, ita ut sit $g = 3{,}16625$ ped. Rhenani; quia durationes oscillationum sunt in subduplicata ratione pendulorum, tempus unius oscillationis a lamina nostra Elastica factae erit

$$= \frac{\sqrt{f}}{\sqrt{g}} \text{ secund.} = \frac{aa}{nn}\sqrt{\frac{1}{g}\cdot\frac{1}{Ekk}\cdot\frac{M}{a}}\ ;$$

unde numerus oscillationum uno minuto secundo editarum erit

$$= \frac{nn}{aa}\sqrt{g\cdot Ekk\cdot\frac{a}{M}},$$

qui numerus exprimit soni, quem lamina excitat, tenorem. Soni ergo a diversis laminis elasticis uno termino muro infixis editi erunt in ratione composita subduplicata elasticitatum absolutarum directe, inversa subduplicata gravitatum specificarum et inversa duplicata longitudinum. Quare, si duae laminae Elasticae tantum longitudine differant, erunt soni reciproce ut quadrata longitudinum; scilicet lamina duplo longior edet sonum duabus octavis graviorem. Corda autem tensa duplo longior sonum una tantum octava graviorem edit, si tensio maneat eadem. Ex quo patet sonos laminarum elasticarum longe aliam sequi rationem, atque sonos cordarum tensarum.

77. Quod ad naturam curvae *aMB* ultra terminos *a* et *B* continuatae attinet, primum quidem patet curvam ultra *a* divergendo ab axe *BA* continuato progredi. Posito enim *x* negativo fiet

$$y = Be^{\frac{x}{c}} + Ae^{-\frac{x}{c}} - C\sin\frac{x}{c} + D\cos\frac{x}{c}.$$

Hic iam omnes termini sunt affirmativi, quia solus coefficiens *C* ante obtinuerat valorem negativum; unde dum crescit *x*, etiam *y* crescere debet, quia *x* numerus *B* maior est quam *A* atque adeo terminus $Be^{\frac{x}{c}}$ praevalet. Quam primum autem $\frac{x}{c}$ valorem saltem mediocrem est adeptum, tum iste terminus $Be^{\frac{x}{c}}$ tantopere crescit, ut reliqui termini prae eo quasi evanescant. Ob eandem rationem, quia curvae in *B* radius osculi non est $= \infty$, est enim

$$\frac{Ekk}{R} = \frac{M}{af}\int dx\int ydx\,,$$

curva in B non habebit punctum flexus contrarii ideoque ad eandem axis *AB* partem ulterius progredietur; aucta autem abscissa *x* ultra $AB = a$, tum primus terminus $Ae^{\frac{x}{c}}$ mox tam fit magnus, ut reliqui prae eo pro nihilo reputari queant.

78. Hic igitur est primus oscillationum modus inter illos innumerabiles, ad quos eadem lamina se componere potest. Secundus modus in figura (Fig. 21) repraesentatus, quo lamina in *B* fixa axem *AB* in uno puncto *O* traiicit, deducetur ex aequatione

$$\frac{a}{c} = \frac{1}{2}\pi + \varphi = l\mathrm{tang}\,\tfrac{1}{2}\varphi \quad \text{seu hac} \quad \frac{3}{2}\pi - \varphi = l\cot\tfrac{1}{2}\varphi = \frac{a}{c}.$$

Hic per nonnulla tentamina inveni angulum p contineri intra hos limites:
1° 2' 40'' et 1° 3' 0'', ex quibus ut ante verus valor ipsius φ eruetur:

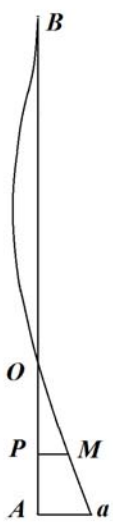

Fig. 21

$\varphi =$ 1° 2' 40''	1° 3' 0''
in min.sec. =3760''	3780''
log.= 3,5751878450	3,5774917998
subtr. = 5,3144251332	5,3144251332
$l\varphi =$ 8,2607627118	8,2630666666
$\varphi =$ 0 ,0182289944	0,0183259571
$\frac{3}{2}\pi =$ 4,7123889804	4,7123889804
$\frac{a}{c} =$ 4,6941599860	4,6940630233
$\frac{1}{2}\varphi =$ 31' 20''	31' 30''
$l\cot\frac{1}{2}\varphi =$ 2,0402552577	2,0379511745
$lv =$ 0,3096845055	0,3091937748
add. = 0,3622156886	0,3622156886
$lu =$ 0,6719001941	0,6714094634
$u =$ 4,6978613391	4,6925559924
$\frac{a}{c} =$ 4,6941599860	4,6940630233
Error + 37013531	-15070309

Ex his erroribus concluditur verus valor anguli

$$\varphi = 1°2'54''\frac{213}{1000} \quad \text{et} \quad \frac{a}{c} = 268°\ 57'5''\frac{787}{1000}.$$

Cum igitur sit $\varphi = 3774,213''$, erit

$$
\begin{aligned}
l\varphi &= 3{,}5768264061\\
\text{subtr.} &= \underline{5{,}3144251332}\\
&\ 8{,}2624012729\\
\varphi &= 0{,}0182979009\\
\text{a } \tfrac{3}{2}\pi &= \underline{4{,}\ 7123889804}\\
\frac{a}{c} &= 4{,}6940910795
\end{aligned}
$$

Sonus ergo laminae priori modo oscillantis erit ad sonum eiusdem laminae hoc modo vibrantis, uti est quadratum numeri 1,8751040813 ad quadratum numeri 4,6940910795, hoc est ut 1 ad 6,266891 seu in minimis numeris ut 4 ad 25 seu ut 1 ad $6\tfrac{4}{15}$. Unde sonus posterior erit ad priorem duplex octava cum quinta et cum hemitonio fere.

79. Pro sequentibus oscillationum modis eiusdem laminae elasticae, quibus lamina inter oscillandum axem *AB* in duobus pluribusve punctis intersecat, fit angulus φ multo minor. Sic pro tertio modo habetur haec aequatio

$$\frac{5}{2}\pi+\varphi = l\cot\tfrac{1}{2}\varphi = \frac{a}{c}.$$

Cum ergo sit

$$e^{\frac{5}{2}\pi+\varphi} = \cot\tfrac{1}{2}\varphi,$$

ob φ angulum vehementer parvum erit

$$e^{\frac{5}{2}\pi+\varphi} = e^{\frac{5}{2}\pi}\left(1+\varphi+\tfrac{1}{2}\varphi^2+\tfrac{1}{6}\varphi^3+\text{etc.}\right)$$

et

$$\cot\tfrac{1}{2}\varphi = \frac{1-\frac{1}{8}\varphi\varphi}{\frac{1}{2}\varphi-\frac{1}{48}\varphi^3} = \frac{2}{\varphi}-\frac{\varphi}{6}.$$

Hinc erit proxime $e^{\frac{5}{2}\pi} = \dfrac{2}{\varphi}$ ideoque $\varphi = 2e^{-\frac{5}{2}\pi}$ et propius $\varphi = \dfrac{1}{1+\frac{1}{2}e^{\frac{5}{2}\pi}}$;

unde erit

$$\frac{a}{c} = \tfrac{5}{2}\pi + \frac{2}{e^{\frac{5}{2}\pi}+2},$$

qui posterior terminus est quam minimus. Simili modo pro quarto oscillationum modo erit proxime

$$\frac{a}{c} = \tfrac{7}{2}\pi - 2e^{-\frac{7}{2}\pi}$$

et ita porro; ob hos alteros terminos evanescentes ipsius $\frac{a}{c}$ valores erunt $\tfrac{7}{2}\pi$, $\tfrac{11}{2}\pi$ etc. qui eo minus a veritate aberrabunt, quo ulterius progredientur.

DE OSCILLATIONS LAMINAE ELASTICAE LIBERAE

80. Consideremus iam laminam Elasticam (Fig. 22) nusquam fixam, sed liberam vel plano politissimo incumbentem vel remota gravitate in spatio vacuo versantem. Facile autem patet huiusmodi laminam motum oscillatorium recipere posse, dum lamina *acb* sese incurvando alternatim cis et ultra statum quietis *AB* excurrit. Motus igitur iste oscillatorius simili modo, quo in casu praecedente, definiri poterit, dummodo calculus debito modo ad hunc casum accommodetur. Sit igitur *acb* figura laminae incurvata, quam inter oscillandum obtinet, at *ACB* sit situs eiusdem laminae in statu aequilibrii, per quem in quavis oscillatione transit. Ponatur ut ante longitudo laminae $AB = a$, eius elasticitas absoluta $= Ekk$ atque pondus seu massa $= M$. Deinde sit abscissa $AP = x$, applicata $PM = y$, arcus $aM = s$, qui cum abscissa x confundetur, ita ut statui queat $ds = dx$; ex quo radius osculi in M orietur $= \frac{dx^2}{ddy} = R$. Sit autem porro applicata prima $Aa = b$. His positis ratiocinium ut ante instituendo ad eandem pervenietur aequationem

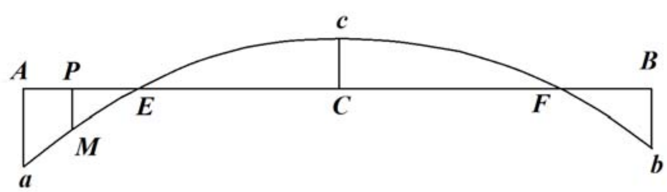

Fig. 22

$$\frac{Ekk}{R} = \frac{M}{af}\int dx \int y dx = \frac{Ekkddy}{dx^2}.$$

81. Si igitur ponamus $\frac{Ekk \cdot af}{M} = c^4$, ubi f ut ante exprimit longitudinem penduli simplicis isochroni, habebitur integrando pro curva haec aequatio

$$y = Ae^{\frac{x}{c}} + Be^{-\frac{x}{c}} + C\sin\frac{x}{c} + D\cos\frac{x}{c},$$

quae ad praesentem casum ita accommodabitur. Primo, si ponatur $x = 0$, fieri debet $y = b$; unde fit

$$b = A + B + D.$$

Secundo, cum sit

$$\frac{c^4 ddy}{dx^2} = \int dx \int ydx,$$

posito $x = 0$ fieri debet $\frac{ddy}{dx^2} = 0,$ unde prodit

$$0 = A + B - D.$$

Tertio, cum sit

$$\frac{c^4 d^3 y}{dx^3} = \int ydx,$$

posito $x = 0$ fieri quoque debet $\frac{d^3 y}{dx^3} = 0,$ unde nascitur:

$$0 = A - B - C.$$

Quarto, si ponatur $x = a$, evanescere debet $\int ydx$ seu $\frac{d^3 y}{dx^3}$, propterea quod $\int ydx$ exprimit summam omnium virium laminam in directione ad axem *AB* normali trahentium, quae summa si non esset 0, ipsa lamina motu locali promoveretur contra institutum; erit ergo ob hanc rationem

$$0 = Ae^{\frac{a}{c}} - Be^{-\frac{a}{c}} - C\cos\frac{a}{c} + \sin D\frac{a}{c},$$

Quinto, quia lamina in extremitate *B* est libera, ibi curvaturam nullam habere poterit eritque ideo posito $x = a$ quoque $\frac{ddy}{dx^2} = 0,$ unde erit

$$0 = Ae^{\frac{a}{c}} + Be^{-\frac{a}{c}} - C\sin\frac{a}{c} - D\cos\frac{a}{c}.$$

His igitur quinque conditionibus in computum ductis non solum quatuor constantes *A*, *B*, *C* et *D* determinabuntur, sed etiam fractionis $\frac{a}{c}$ valor reperietur; ex quo proinde longitudo penduli simplicis isochroni *f* innotescet.

82. Ex harum aequationum secunda et tertia obtinetur $D = A + B$ et $C = A - B$, qui in sequentibus substituti praebebunt

$$0 = Ae^{\frac{a}{c}} - Be^{-\frac{a}{c}} - (A - B)\cos\frac{a}{c} + (A + B)\sin\frac{a}{c},$$

$$0 = Ae^{\frac{a}{c}} + Be^{-\frac{a}{c}} - (A - B)\sin\frac{a}{c} - (A + B)\cos\frac{a}{c},$$

ex quibus reperitur:

$$\frac{A}{B}=\frac{e^{-\frac{a}{c}}-\cos\frac{a}{c}-\sin\frac{a}{c}}{e^{\frac{a}{c}}-\cos\frac{a}{c}+\sin\frac{a}{c}}=-\frac{e^{-\frac{a}{c}}+\sin\frac{a}{c}-\cos\frac{a}{c}}{e^{\frac{a}{c}}-\sin\frac{a}{c}-\cos\frac{a}{c}};$$

ex qua aequalitate elicitur ista aequatio

$$0=2-e^{\frac{a}{c}}\cos\frac{a}{c}-e^{-\frac{a}{c}}\cos\frac{a}{c};\ \text{seu}\ \ e^{\frac{a}{c}}=\frac{1\pm\sin\frac{a}{c}}{\cos\frac{a}{c}},$$

unde sequentes formabuntur aequationes

I. $\frac{a}{c}=\frac{1}{2}\pi-\varphi=l\text{tang}\,\frac{1}{2}\varphi,$ qui dat

$\frac{a}{c}=0$ pro situ laminae naturali,

II. $\frac{a}{c}=\frac{1}{2}\pi-\varphi=l\cot\frac{1}{2}\varphi$

III. $\frac{a}{c}=\frac{3}{2}\pi+\varphi=l\cot\frac{1}{2}\varphi$

IV. $\frac{a}{c}=\frac{5}{2}\pi-\varphi=l\cot\frac{1}{2}\varphi$

V. $\frac{a}{c}=\frac{7}{2}\pi+\varphi=l\cot\frac{1}{2}\varphi$

VI. $\frac{a}{c}=\frac{9}{2}\pi-\varphi=l\cot\frac{1}{2}\varphi$

etc.

83. Hae aequationes iterum indicant innumerabiles oscillationum modos, in quorum secundo lamina semel tantum axem *AB* intersecabit, in tertio bis, in quarto ter, in quinto quater et ita porro. Ex quibus intelligitur modos secundum, quartum, sextum etc. ad praesens institutum non esse accommodatos. Quoniam enim in his numerus intersectionum est impar, laminae situs inter oscillandum in secundo foret talis, qualem Figura 23 repraesentat, in quo, quamvis summa virium sollicitantium per totam laminam evanescat, tamen ab iis lamina circa punctum medium *C* motum rotatorium acquireret, quia vires utrique semissi *aC* et *bC* applicatae ad eundem laminae motum rotatorium inducendum conspirarent. Quam ob causam, cum omnino motus

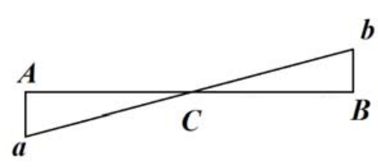

Fig. 23

rotatorius excludi debeat, figura laminae (Fig. 22), quam inter oscillandum induit, ita debet esse comparata, ut non solum virium sollicitantium toti laminae applicatarum sit $=0$, sed etiam ut earum summa momentorum evanescat; quod obtinetur, si curva in puncto medio c diametro *cC* sit praedita. Quod evenit, si curva axem *AB* vel in duobus vel in quatuor vel generatim in punctorum numero pari secet; ex quo aequationes tertia, quinta, septima etc. Solutiones tantum convenientes praebebunt.

84. Haec ipsa limitatio in ipsa Problematis propositione contenta reperietur, si eiusmodi tantum curvas admittamus, quae rectam *Cc* habeant diametrum seu in quibus valor ipsius *y* prodeat idem, si loco *x* scribatur $a-x$. Ponamus ergo in aequatione generali $a-x$ loco *x*, atque prodibit

$$y = Ae^{\frac{a}{c}}e^{-\frac{x}{c}} + Be^{-\frac{a}{c}}e^{\frac{x}{c}} + C\sin\frac{a}{c}\cdot\cos\frac{x}{c} - C\cos\frac{a}{c}\cdot\sin\frac{x}{c}$$
$$+ D\cos\frac{a}{c}\cdot\cos\frac{x}{c} + D\sin\frac{a}{c}\cdot\sin\frac{x}{c},$$

quae cum congruere debeat cum aequatione

$$y = Ae^{\frac{x}{c}} + Be^{-\frac{x}{c}} + C\sin\frac{x}{c} + D\cos\frac{x}{c},$$

fiet

$$Ae^{\frac{a}{c}} = B,\quad C\left(1+\cos\frac{a}{c}\right) = D\sin\frac{a}{c} \text{ et } C\sin\frac{a}{c} = D\left(1-\cos\frac{a}{c}\right),$$

quarum duae posterioras congruunt. Cum ergo sit $\frac{A}{B} = e^{-\frac{a}{c}}$, hoc valore cum superioribus comparato prodibit:

$$e^{-\frac{a}{c}} - \cos\frac{a}{c} - \sin\frac{a}{c} = 1 - e^{-\frac{a}{c}}\cos\frac{a}{c} + e^{-\frac{a}{c}}\sin\frac{a}{c}$$

seu

$$e^{-\frac{a}{c}} = \frac{1+\cos\frac{a}{c}+\sin\frac{a}{c}}{1+\cos\frac{a}{c}-\sin\frac{a}{c}} = \frac{1+\sin\frac{a}{c}}{\cos\frac{a}{c}} = \frac{\cos\frac{a}{c}}{1-\sin\frac{a}{c}}.$$

85. Erit ergo

$$e^{\frac{a}{c}} = \frac{1-\sin\frac{a}{c}}{\cos\frac{a}{c}},$$

sicque in aequatione prius inventa

$$e^{\frac{a}{c}} = \frac{1 \pm \sin \frac{a}{c}}{\cos \frac{a}{c}},$$

semissis tantum *f* casuum supra exhibitorum, scilicet ii, qui sunt numeris imparibus, praesens Problema resolvent. Quare, cum prima aequatio contineat laminae statum naturalem, omnes oscillationum modi in sequentibus aequationibus continebuntur:

$$\text{I.} \quad \frac{a}{c} = \tfrac{3}{2}\pi + \varphi = l\cot \tfrac{1}{2}\varphi$$

$$\text{II.} \quad \frac{a}{c} = \tfrac{7}{2}\pi + \varphi = l\cot \tfrac{1}{2}\varphi$$

$$\text{III.} \quad \frac{a}{c} = \tfrac{11}{2}\pi + \varphi = l\cot \tfrac{1}{2}\varphi$$

etc.

Aequationum ergo harum prima praebebit primum eumque principalem oscillandi modum, pro quo valor anguli φ; simili modo, quo supra, per approximationem reperietur. Limites autem anguli φ mox colliguntur esse 1° 0' 40" et 1° 1' 0", ex quibus per sequentem calculum verus ipsius φ valor eruitur.

φ =	1° 0' 40"	1° 1' 0"
seu	3640"	3660"
log.=	3,5611013836	3,5634810854
subtr. =	5,3144251332	5,3144251332
$l\varphi$ =	8,24667 62504	8,2490559522
φ =	0,0176472180	0,0177441807
$\frac{3}{2}\pi$ =	4,7123889804	4,7123889804
$\frac{a}{c}$ =	4,7300361984	4,7301331611
$\frac{1}{2}\varphi$ =	30' 20"	30' 30"
v =	2,0543424742	2,0519626482
lv =	0,3126728453	0,3121694510
add.	0,3622156886	0,3622156886
lu =	0,67 48885339	0,6743851396
u =	4,7302983543	4,7248186037
Error.	+ 636341	+ 53145574

636341
diff. 52509233

Hinc intelligitur verum valorem ipsius φ non intra istos limites contineri, sed aliquantulum esse minorem quam $1^{o}0'40''$. Nihilo vero minus is ex his erroribus reperietur. Sit enim $\varphi = 1^{o}0'40'' - n''$; erit
$20'' : 52509233 = n'' : 636341$;
unde repentur $n = \frac{2423}{10000}$, ita ut sit

$$\varphi = 10^{o}\ 0'\ 39''\frac{7576}{10000}.$$

Cum ergo sit $\varphi = 3639{,}7576''$, erit

$$\begin{aligned} l\varphi &= 3{,}5610724615 \\ \text{subtr. } & \underline{5{,}3144251332} \\ & 8{,}2466473283 \\ \varphi &= 0{,}0176460428 \\ \tfrac{3}{2}\pi &= \underline{4{,}7123889804} \\ \frac{a}{c} &= 4{,}7300350232 \end{aligned}$$

86. Sit hic numerus $= m$, erit, ob $c^{4} = \frac{Ekk \cdot af}{M}$;

$$a^{4} = \frac{m^{4} \cdot Ekk \cdot af}{M} \quad \text{et} \quad f = \frac{a^{4}}{m^{4}} \cdot \frac{1}{Ekk} \cdot \frac{M}{a}.$$

Unde pari modo numerus oscillationum ab hac lamina uno minuto secundo editarum erit

$$= \frac{mm}{aa}\sqrt{g \cdot Ekk \cdot \frac{a}{M}}.$$

existente $g = 3{,}16625$ ped. Rhen. Quodsi ergo eadem lamina nunc altero termino *B* muro infixo, nunc libera ad sonum edendum incitetur, erunt soni inter se ut *nn* ad *mm*, hoc est ut quadrata numerorum 1,8751040813 et 4, 7300350232, hoc est ut 1 ad 6,363236. Ratio ergo horum sonorum erit proxime ut 11 ad 70; horum ergo sonorum intervallum constituet duas octavas cum quinta et hemitonio. Sin autem posterior lamina libera duplo longior capiatur quam prior fixa, intervallum sonorum erit fere sexta minor.

87. Invento hoc valore fractionis a aequatio pro curva, quam lamina $\frac{a}{c}$ inter oscillandum format, hactenus indeterminata poterit determinari. Cum enim sit

$$e^{\frac{a}{c}} = \frac{1-\sin\frac{a}{c}}{\cos\frac{a}{c}}, \text{ erit } B = \frac{1-\sin\frac{a}{c}}{\cos\frac{a}{c}}A$$

et

$$C = A - B = A\left(\cos\frac{a}{c} + \sin\frac{a}{c} - 1\right) : \cos\frac{a}{c}$$

et

$$D = A + B = A\left(\cos\frac{a}{c} - \sin\frac{a}{c} + 1\right) : \cos\frac{a}{c}.$$

Iam est

$$b = A + B + D = 2D = 2A\left(\cos\frac{a}{c} - \sin\frac{a}{c} + 1\right) : \cos\frac{a}{c};$$

unde fit

$$A = \frac{b\cos\frac{a}{c}}{2\left(\cos\frac{a}{c} - \sin\frac{a}{c} + 1\right)} = \frac{b\left(1 + \sin\frac{a}{c} - \cos\frac{a}{c}\right)}{4\sin\frac{a}{c}},$$

$$B = \frac{b\left(1 - \sin\frac{a}{c}\right)}{2\left(\cos\frac{a}{c} - \sin\frac{a}{c} + 1\right)} = \frac{b\left(-1 + \sin\frac{a}{c} + \cos\frac{a}{c}\right)}{4\sin\frac{a}{c}},$$

$$C = \frac{b\left(-1 + \sin\frac{a}{c} + \cos\frac{a}{c}\right)}{2\left(\cos\frac{a}{c} - \sin\frac{a}{c} + 1\right)} = \frac{b\left(1 - \cos\frac{a}{c}\right)}{2\sin\frac{a}{c}},$$

$$D = \frac{b}{2} = \frac{b\sin\frac{a}{c}}{2\sin\frac{a}{c}}.$$

His substitutis oritur haec aequatio:

$$\frac{y}{b}=\frac{e^{\frac{x}{c}}\cos\frac{a}{c}+e^{-\frac{x}{c}}\left(1-\sin\frac{a}{c}\right)}{2\left(1-\sin\frac{a}{c}+\cos\frac{a}{c}\right)}+\frac{\left(1-\cos\frac{a}{c}\right)\sin\frac{x}{c}+\sin\frac{a}{c}\cos\frac{x}{c}}{2\sin\frac{a}{c}}.$$

88. Quia autem recta *Cc* est curvae diameter, ponatur abscissa a puncto medio *C* sumpta $CP=z$, erit $x=\frac{1}{2}a-z$. Unde fit

$$e^{\frac{x}{c}}=e^{\frac{x}{2c}}e^{-\frac{x}{c}}=e^{-\frac{x}{c}}\sqrt{\frac{\left(1-\sin\frac{a}{c}\right)}{\cos\frac{a}{c}}}$$

et

$$e^{-\frac{x}{c}}=e^{\frac{x}{c}}\sqrt{\frac{\cos\frac{a}{c}}{\left(1-\sin\frac{a}{c}\right)}};$$

ex quo erit

$$\frac{Ae^{\frac{x}{c}}+Be^{-\frac{x}{c}}}{b}=\frac{\left(e^{\frac{x}{c}}+e^{-\frac{x}{c}}\right)\sqrt{\cos\frac{a}{c}\left(1-\sin\frac{a}{c}\right)}}{2\left(1-\sin\frac{a}{c}+\cos\frac{a}{c}\right)}=\frac{e^{\frac{x}{c}}+e^{-\frac{x}{c}}}{2\left(e^{\frac{x}{2c}}+e^{-\frac{x}{2c}}\right)}.$$

Tum vero erit

$$\left(1-\cos\frac{a}{c}\right)\sin\frac{x}{c}+\sin\frac{a}{c}\cos\frac{x}{c}=\sin\frac{a}{c}+\sin\frac{(a-x)}{c}$$
$$=\sin\left(\frac{a}{2c}-\frac{z}{c}\right)+\sin\left(\frac{a}{2c}+\frac{z}{c}\right)=2\sin\frac{a}{2c}\cos\frac{z}{c};$$

quibus substitutis oritur haec aequatio:

$$\frac{2y}{b}=\frac{e^{\frac{z}{c}}+e^{-\frac{z}{c}}}{e^{\frac{a}{2c}}+e^{-\frac{a}{2c}}}=\frac{\cos\frac{z}{c}}{\cos\frac{a}{2c}},$$

quae est forma simplicissima, qua natura curvae *aMcb* exprimi potest; manifestum autem est, sive *z* sumatur affirmative sive negative, eundem esse prodituram valorem applicatae *y*. Est vero

$$e^{\frac{a}{2c}}+e^{-\frac{a}{2c}}=\frac{2\cos\frac{a}{2c}}{\sqrt{\cos\frac{a}{c}}}.$$

Invenimus autem angulum $\frac{a}{c}=271^\circ\ 0'39''\tfrac{3}{4}$.

89. Si iam ponatur $z=0$, praebebit *y* valorem applicatae *Cc*; erit ergo

$$\frac{2\cdot Cc}{b}=\frac{2\sqrt{\cos\frac{a}{c}}}{2\cos\frac{a}{2c}}+\frac{1}{\cos\frac{a}{2c}}.$$

seu

$$\frac{Cc}{Aa}=\frac{1+\sqrt{\cos\frac{a}{c}}}{2\cos\frac{a}{2c}}=\tfrac{1}{2}\sec\frac{a}{2c}+\tfrac{1}{2}\sec\frac{a}{2c}\sqrt{\cos\frac{a}{c}}.$$

At est

$$\cos\frac{a}{c}=\sin 1^\circ 0'39''\tfrac{3}{4}\ \text{ et }\ \cos\frac{a}{2c}=\sin 45^\circ 30'19''\tfrac{7}{8}.$$

Hinc reperitur $\frac{Cc}{Aa}=0,607815$. Deinde, si ponatur $y=0$, reperientur puncta *E* et *F*, quibus curva axem intersecat. Erit ergo

$$e^{\frac{z}{c}}+e^{-\frac{z}{c}}=-\frac{\cos\frac{z}{c}}{\cos\frac{a}{2c}}\left(e^{\frac{a}{2c}}+e^{-\frac{a}{2c}}\right)=\frac{2\cos\frac{z}{c}}{\sqrt{\cos\frac{a}{c}}},$$

ex qua per approximationes reperitur

$$\frac{CE}{CA} = 0,551685 \quad \text{et} \quad \frac{AE}{AC} = 0,448315.$$

Dum ergo lamina oscillationes peragit, haec puncta *E* et *F* restabunt immobilia; ex quo huiusmodi motus oscillatorius, qui alias vix actu produci posse videatur, facile produci poterit. Si enim lamina in punctis *E* et *F* hoc modo definitis figatur, tum perinde oscillabitur, ac si penitus esset libera.

90. Si eodem modo tractetur aequationum supra inventarum secunda

$$\frac{a}{c} = \tfrac{7}{2}\pi + \varphi = l\cot\tfrac{1}{2}\varphi$$

quo quidem casu reperietur proxime $\varphi = 0$, tum prodibit secundus modus, quo lamina libera vibrationes absolvere potest, secando scilicet axem *AB* in quatuor punctis; ideoque lamina perinde oscillabitur, ac si in his quatuor punctis esset fixa. Vicissim ergo, si lamina in his quatuor punctis vel eorum duobus tantum quibusvis figatur, tum eodem modo oscillabitur, ac si esset libera; sonum autem edet multo auctiorem, quippe qui ad sonum praecedentem modo editum rationem tenebit fere ut 7^2 ad 3^2, hoc est, intervallum erit duarum octavarum cum quarta et hemitonii semisse. Tertius oscillandi modus, quo est

$$\frac{a}{c} = \tfrac{11}{2}\pi + \varphi = l\cot\tfrac{1}{2}\varphi \ ,$$

habebit sex curvae *acb* intersectiones cum axe *AB* sonusque edetur plus una octava cum tertia minore acutior; huncque sonum lamina edet, si in duobus illorum sex punctorum figatur. Hinc patet quam varii soni ab eadem lamina, prout in duobus punctis diversimode figitur, edi queant, et nisi puncta bina, quibus infigitur, congruant cum intersectionibus in modo primo vel secundo vel tertio, atque adeo oscillationes sese ad modorum aliquem sequentium vel etiam ad infinitesimum componant, tum sonum fore tantopere acutum, ut percipi omnino nequeat seu, quod eodem redit, lamina motum oscillatorium prorsus recipere non poterit; vel saltem instar cordae vibrantis, cui ponticulus ita subiicitur, ut partes nullam inter se teneant rationem rationalem, sonus minus distinctus producetur.

DE OSCILLATIONIBUS LAMINAE ELASTICAE UTROQUE TERMINO FIXAE

90. Infixa nunc sit lamina Elastica (Fig. 24) in utroque termino *A* et *B*, ita tamen, ut tangentes curvae in his punctis non determinentur. Ad hunc scilicet casum in experimentis producendum laminae in utroque termino infigantur tenuissimi aculei

$A\alpha,\ B\beta$, qui parieti infixi reddant laminae extremos terminos A et B immobiles. Ad motum oscillatorium huius laminae Elasticae investigandum ponatur ut ante elasticitas absoluta laminae $=Ekk$, longitudo AB et pondus $=M$ atque longitudo penduli simplicis isochroni $=f$. Sit AMB figura curvilinea, quam lamina inter oscillandum induit, ac ponatur abscissa $AP=AM=x$, applicata $PM=y$ et radius osculi in $M=R$. Sit porro P vis, quam aculeus $A\alpha$ sustinet in directione $A\alpha$, et quia vis, qua elementum Mm in directione $M\mu$ urgeri debet,

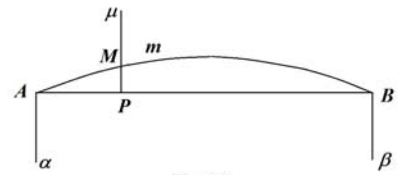

Fig. 24

quo lamina in hoc statu conservetur, est $=\dfrac{Mydx}{af}$, per Regulas supra descriptas aequatio pro curva haec

$$\frac{Ekk}{R}=Px-\frac{M}{af}\int dx\int ydx.$$

Est vero $R=-\dfrac{dx^2}{ddy}$, quia curva versus axem est concava; unde fit

$$\frac{Ekk}{dx^2}ddy=\frac{M}{af}\int dx\int ydx=-Px..$$

Facto ergo $x=0$ erit radius osculi R in A infinitus ideoque $ddy=0$.

91. Si haec aequatio bis differentietur, prodibit eadem aequatio, quam pro casibus praecedentibus invenimus,

$$Ekkd^4y=\frac{M}{af}ydx^4.$$

Quodsi ergo ponatur $\dfrac{Ekk\cdot af}{M}=c^4$, erit aequatio integralis

$$y=Ae^{\frac{x}{c}}+Be^{-\frac{x}{c}}+C\sin\frac{x}{c}+D\cos\frac{x}{c}.$$

Ad quam determinandam ponatur $x=0$, et quia simul y evanescere debet, erit $0=A+B+D$.
Secundo ponatur $x=a$, et quia pariter fieri debet $y=0$, erit

$$0=Ae^{\frac{a}{c}}+Be^{-\frac{a}{c}}+C\sin\frac{a}{c}+D\cos\frac{a}{c}.$$

Tertio, quia $\dfrac{ddy}{dx^2}$ evanescere debet, posito et $x=0$ et $x=a$ fiet

$$0 = A + B - D \text{ et } 0 = Ae^{\frac{a}{c}} + Be^{-\frac{a}{c}} - C\sin\frac{a}{c} - D\cos\frac{a}{c}.$$

Iam aequationes $0 = A + B - D$ et $0 = A + B + D$ dant $D = 0$ et $B = -A$; qui valores in reliquis duabus aequationibus substituti praebent

$$0 = A\left(e^{\frac{a}{c}} - e^{-\frac{a}{c}}\right) + C\sin\frac{a}{c}$$

et

$$0 = A\left(e^{\frac{a}{c}} - e^{-\frac{a}{c}}\right) - C\sin\frac{a}{c};$$

quibus satisfieri nequit, nisi sit $A = 0$, quia non potest esse $e^{\frac{a}{c}} = e^{-\frac{a}{c}}$ praeter casum $\frac{a}{c} = 0$, tum vero esse debet $C\sin\frac{a}{c} = 0$. Hic cum nequeat poni $C = 0$, quia motus oscillatorius foret nullus, erit $\sin\frac{a}{c} = 0$ ideoque vel $\frac{a}{c} = \pi$ vel $\frac{a}{c} = 2\pi$ etc., unde iterum infiniti diversi oscillationum oriuntur modi, prout curva *AMB* axem vel nusquam praeter terminos *A* et *B* secat vel in uno vel in duobus vel in pluribus punctis, uti colligitur ex aequatione $y = C\sin\frac{x}{c}$. Puncta intersectionum autem, quotcunque fuerint, aequalibus intervallis inter se distabunt.

93. Cum igitur pro primo ac principali oscillandi modo sit $\frac{a}{c} = \pi$, erit

$$a^4 = \pi^4 c^4 = \pi^4 \times Ekk \times \frac{a}{M} \times f,$$

unde fit

$$f = \frac{a^4}{\pi^4} \times \frac{1}{Ekk} \times \frac{M}{a}.$$

Quare ratione longitudinis laminae soni iterum tenebunt rationem reciprocam duplicatam longitudinum. Sonus autem huius laminae hoc modo editus se habebit ad sonum eiusdem laminae, si altero termino *B* muro esset infixa, uti $\pi\pi$ ad quadratum numeri 1,8751040813, hoc est ut 2,807041 ad 1 seu in numeris minimis ut 57 ad 160, quod intervallum est octava cum tritono fere. Si oscillationes se ad secundum modum, quo est $\frac{a}{c} = 2\pi$, componant, sonus fiet duplici octava acutior; sin sit $\frac{a}{c} = 3\pi$, sonus acutior fiet tribus octavis cum tono maiore quam casu, quo $\frac{a}{c} = \pi$, et ita porro. Quae quo facilius ad experimenta revocari queant, notandum est oscillationes hic quam-minimas poni, ita ut nulla laminae elongatione sit opus. Quare, ne tenacitas laminae, qua etiam minimae extensioni, sine qua oscillationes istae peragi nequeunt, reluctatur, hic alterationem afferat, cuspides illae ita debent constitui, ut tantilla extensio non impediatur; quod evenit, si plano politissimo incumbant. Sic lamina Elastica *AB* in *A* et *B* cuspidibus $A\alpha$ et $B\beta$ munita, si cuspides speculo imponantur, sonum calculo conformem edet.

DE OSCILLATIONIBUS LAMINAE ELASTICAE UTROQUE TERMINO P ARIETI INFIXAE

94. Hoc casu expedito istam de laminis elasticis tractationem claudat motus oscillatorius laminae Elasticae, utroque termino *A* et *B* muro infixae (Fig. 25), ita ut inter oscillandum puncta *A* et *B* non solum maneant immota, sed etiam recta *AB* perpetuo sit tangens curvae *AMB* in punctis *A* et *B*. Hic ergo iterum cavendum est, ut obices terminos *A* et *B* comprehendentes non sint adeo firmi, sed tantillam extensionem, quanta ad curvaturam requiritur, permittant. Quaecunque ergo sint vires in terminis *A* et *B* ad laminam continendam requisitae, ad sequentem pervenietur aequationem differentialem quarti ordinis

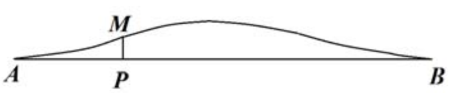

Fig. 25

$$Ekkd^4 y = \frac{M}{af} dx^4 ;$$

cuius, si ponatur $\frac{Ekk \cdot af}{M} = c^4$, integral erit ut supra

$$y = Ae^{\frac{x}{c}} + Be^{-\frac{x}{c}} + C\sin\frac{x}{c} + D\cos\frac{x}{c}.$$

95. Constantes *A*, *B*, *C* et *D* autem ita debent esse comparatae, ut posito $x = 0$ non solum *y* evanescat, sed etiam fiat $dy = 0$, quia in *A* curva ab axe *AB* tangitur. Hoc idem utrumque vero evenire debet, si ponatur $x = a$; unde istae quatuor aequationes nascentur

$$\text{I.}\quad 0 = A + B + D$$

$$\text{II.}\quad 0 = A - B + C$$

$$\text{III.}\quad 0 = Ae^{\frac{a}{c}} + Be^{-\frac{a}{c}} + C\sin\frac{a}{c} + D\cos\frac{a}{c}$$

$$\text{IV.}\quad 0 = Ae^{\frac{a}{c}} - Be^{-\frac{a}{c}} + C\cos\frac{a}{c} - D\sin\frac{a}{c}.$$

Ex harum aequationum prima et secunda oritur $C = -A + B$ et $D = -A - B$, qui valores in reliquis duabus substituti dabunt

$$0 = Ae^{\frac{a}{c}} + Be^{-\frac{a}{c}} - (A - B)\sin\frac{a}{c} - (A + B)\cos\frac{a}{c}$$
$$0 = Ae^{\frac{a}{c}} - Be^{-\frac{a}{c}} - (A - B)\cos\frac{a}{c} + (A + B)\sin\frac{a}{c}.$$

quarum summa ac differentia est

$$0 = Ae^{\frac{a}{c}} + B\sin\frac{a}{c} - A\cos\frac{a}{c} \quad \text{seu} \quad \frac{A}{B} = \frac{\sin\frac{a}{c}}{\cos\frac{a}{c} - e^{\frac{a}{c}}}$$

$$0 = Be^{-\frac{a}{c}} - A\sin\frac{a}{c} - B\cos\frac{a}{c} \quad \text{seu} \quad \frac{A}{B} = \frac{e^{-\frac{a}{c}} - \cos\frac{a}{c}}{\sin\frac{a}{c}},$$

unde fit

$$2 = \left(e^{\frac{a}{c}} + e^{-\frac{a}{c}}\right)\cos\frac{a}{c} \quad \text{seu} \quad e^{\frac{a}{c}} = \frac{1 \pm \sin\frac{a}{c}}{\cos\frac{a}{c}}.$$

Quae aequatio quia congruit cum ea, quam paragrapho 81 invenimus, sequentes Solutiones numero infinitae satisfacient:

$$\text{I.} \quad \frac{a}{c} = \tfrac{1}{2}\pi - \varphi = l\cot\tfrac{1}{2}\varphi$$
$$\text{II.} \quad \frac{a}{c} = \tfrac{3}{2}\pi + \varphi = l\cot\tfrac{1}{2}\varphi$$
$$\text{III.} \quad \frac{a}{c} = \tfrac{5}{2}\pi - \varphi = l\cot\tfrac{1}{2}\varphi$$

etc.

96. Harum aequationum primae satisfieri nequit, nisi sit $\varphi = 90°$ ideoque $\frac{a}{c} = 0$; unde primus oscillandi modus oritur ex aequatione

$$\frac{a}{c}=\tfrac{3}{2}\pi+\varphi=l\cot\tfrac{1}{2}\varphi\,;$$

quae cum iam supra sit tractata, erit $\frac{a}{c}=4,7300350232.$ Quamobrem lamina elastica, cuius uterque terminus parieti infixus tenetur, perinde vibrationes suas peraget, ac si esset omnino libera. Haec autem convenientia tantum ad primum oscillandi modum spectat ; secundus enim oscillandi modus, quo est

$$\frac{a}{c}=\tfrac{5}{2}\pi-\varphi=l\cot\tfrac{1}{2}\varphi$$

atque lamina axem *AB* inter oscillandum in uno puncto intersecat, in lamina libera sui parem non habet; tertius autem modus laminae utrinque infixae congruet cum modo secundo laminae liberae, atque ita porro.

97. Haec duo postrema oscillationum genera ob causam allatam non congrue per experimenta explorari possunt: primum autem non solum ad experimenta instituenda maxime est aptum, sed etiam adhiberi potest ad elasticitatem absolutam cuiusque laminae propositae, quam per *Ekk* indicavimus, investigandam. Quodsi enim sonus notetur, quem huiusmodi lamina altero termino muro infixa edit, eique in corda consonus efficiatur, simul numerus oscillationum uno minuto secundarum editarum cognoscetur. Qui si aequalis ponatur expressioni

$$\frac{nn}{aa}\sqrt{g}\cdot Ekk\cdot\frac{a}{M},$$

ob numerum *n* cognitum et quantitates *g, a* et *M* per dimensiones inventas, reperietur valor expressionis *Ekk* sicque elasticitas absoluta innotescit; quae cum ea, quam supra ex incurvatione reperire docuimus, comparari potest.