CHAPTER II

A METHOD OF FINDING ABSOLUTE MAXIMA AND MINIMA FOR CURVED LINES

PROPOSITION I PROBLEM

1. *If in some curve* (Fig. 4) *amz any single applied line Nn may be increased by an infinitely small amount nv, to find the increments or decrements, which the individual determined magnitudes related to the curve hence will take.*

SOLUTION

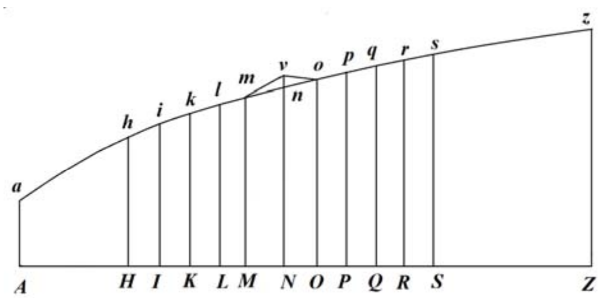

Fig. 4

The determined magnitudes relating to the proposed curve are, besides the abscissa *x*, which is not affected, these *y, p, q, r, s* etc. with their derived values, which are chosen in the following or in the preceding places. So that if now we may put $AM = x$ and $Mm = y$, there will be $Nn = y^{/}$, the value of which by the translation of the point *n* to *v* will be increased by the small amount *nv*, but the remaining applied lines $y^{//}, y^{///}, y^{IV}$ etc. and equally the preceding $y_{/}, y_{//}, y_{///}, y_{IV}$ etc. will not be affected. Therefore since the applied line $y^{/}$ alone will increase by the small amount *nv*, from paragraph 51 of the preceding chapter and the following, how great the increment the remaining magnitudes may be taken from the increment of the applied line $y^{/}$ only. Evidently all the magnitudes, the value of which will depend on $y^{/}$, will undergo a change, truly the rest, which do not depend on $y^{/}$, will remain unchanged. Thus since there shall be $p = \frac{y^{/} - y}{dx}$, this magnitude *p* will increase by the amount $\frac{nv}{dx}$; but since $p^{/} = \frac{y^{//} - y^{/}}{dx}$, this magnitude $p^{/}$ will decrease by the small amount $\frac{nv}{dx}$. In a similar manner the increments or decrements of all the remaining magnitudes will be found by deleting all the values of *y* beyond this $y^{/}$ in these , and by writing *nv* in its place. In this manner the magnitude of all the determined magnitudes, which undergo a certain change, we have collected the increases together in the following table :

Quant.	Increm.	Quant.	Increm.
$y^{/}$	$+nv$	$s_{///}$	$+\frac{nv}{dx^4}$
p	$+\frac{nv}{dx}$	$s_{//}$	$-\frac{4nv}{dx^4}$
$p^{/}$	$-\frac{nv}{dx}$	$s_{/}$	$+\frac{6nv}{dx^4}$
$q_{/}$	$+\frac{nv}{dx^2}$	s	$-\frac{4nv}{dx^4}$
q	$-\frac{2nv}{dx^2}$	$s^{/}$	$+\frac{nv}{dx^4}$
$q^{/}$	$+\frac{nv}{dx^2}$	t_{IV}	$+\frac{nv}{dx^5}$
$r_{//}$	$+\frac{nv}{dx^3}$	$t_{///}$	$-\frac{5nv}{dx^5}$
$r_{/}$	$-\frac{3nv}{dx^3}$	$t_{//}$	$+\frac{10nv}{dx^5}$
r	$+\frac{3nv}{dx^3}$	$t_{/}$	$-\frac{10nv}{dx^5}$
$r^{/}$	$-\frac{nv}{dx^3}$	t	$+\frac{5nv}{dx^5}$
		$t^{/}$	$-\frac{nv}{dx^5}$

And from this table also the increments or decrements of the final magnitude, if which arise, will be able to become known easily. Q.E.I.

COROLLARY I

2. Therefore with the known increments of these primary magnitudes related to the curve, thence the increments of all magnitudes composed from these, which arise from the increased applied line $y^{/}$, will be able to be determined, if the ratio of the composition may be known.

COROLLARY 2

3. Evidently the increments of these magnitudes shown will be able to be considered as the differentials of those. And if some magnitude were proposed composed from these, its agreeing increment arising from the translation of the point from *n* to *v*, will be found by differentiating that magnitude, and by writing these increments in place of the differentials of the individual magnitudes, which have been ascribed to these magnitudes.

COROLLARY 3

4. Therefore if this function $y^{/}\sqrt{(1+pp)}$ may be considered, the increment of which arising from the translation of the point *n* to *v*, shall be required to be determined, at first that function may be differentiated ; from which the function will be produced

$$dy^{/}\sqrt{(1+pp)}+\frac{y^{/}pdp}{\sqrt{(1+pp)}};$$

and here in place of $dy^{/}$ and *dp* the increments agreeing with the magnitudes $dy^{/}$ and p may be written, without doubt $+nv$ and $+\frac{nv}{dx}$; and the increment of the proposed function will be

$$=+nv\sqrt{(1+pp)}+\frac{y^{/}p\cdot nv}{dx\sqrt{(1+pp)}}.$$

COROLLARY 4

5. Therefore by the differentiation of any function, the increment can be assigned readily which arises from the increment *nv* of the applied line $y^{/}$; that which can be done with difficulty and generally minimally from the inspection of the figure.

SCHOLIUM

6. This manner of finding the increments of functions, or of finding the magnitudes from x, y, p, q etc. and from the derivatives $y^{/}$, $y^{//}$, $p^{/}$, $p^{//}$ etc. of these, understood properly, is to be applied to determinate functions only, and truly minimally able to be extended to undetermined ones. For if a function proposed were undetermined or the formula of the integration indefinite, admitting neither an algebraic nor transcending integration, then by differentiation nothing we follow leads to finding its increment. But in the following, where we shall be considering maximum and minimum formulas of this kind $\int Zdx$, in which *Z* shall be an undetermined function of such, we are going to inquire about the increments of functions of this kind. But if *Z* were a determined function, the solution of the proposed problem can suffice for the solutions of related problems requiring to being resolved here.

PROPOSITION II. PROBLEM

7. *If Z were a determined function of x and y only* (Fig. 4), *to find the curve az, in which the value of the formula* $\int Zdx$ *shall be a maximum or a minimum.*

SOLUTION

The abscissa *AZ* may be considered, to which the maximum or minimum of the formula $\int Zdx$ must correspond, to be divided into innumerable equal small elements, with an individual element being denoted by dx ; and by putting the indefinite abscissa $AM = x$ and the applied line $Mm = y$, from the formula $\int Zdx$, Zdx will correspond to the element *MN ;* and following the received manner of notation, $Z^{/}dx$ will correspond to the element *NO* and the values $Z^{//}dx,\ Z^{///}dx$ etc. will correspond to the following elements *OP*, *PQ* etc. ; truly $Z_{/}dx,\ Z_{//}dx,\ Z_{///}dx$ etc will correspond to the preceding elements *LM*, *KL*, *IK* etc. Whereby, if the curve *az* shall be that one itself which is sought, $Zdx + Z^{/}dx + Z^{//}dx + \text{etc.}$, together with $Zdx + Z_{/}dx + Z_{//}dx + \text{etc.}$ must become a maximum or a minimum. So that if therefore a single applied line $Nn = y^{/}$ may be increased by the small amount *nv*, that expression must retain the same value, and thus the value of the differential of the formula $\int Zdx$ or of the sum of the terms

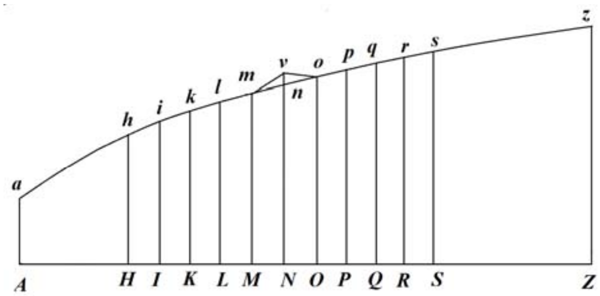

Fig. 4

$Zdx + Z^{/}dx + Z^{//}dx + Z^{///}dx +$ etc . together with $Z_{/}dx + Z_{//}dx + Z_{///}dx + \text{etc}$. must vanish. Therefore the differential values of these terms, which arise from the translation of the point *n* into *v*, will need to be investigated and the sum of these will be the value of the corresponding differential formula $\int Zdx$, which equation put $= 0$ will provide the equation for the curve sought. Moreover because *Z* is put to be a determined function of *x* and *y*, its differential *dZ* will have a form of this kind $Mdx + Ndy$; thus so that there shall be $dZ = Mdx + Ndy$. Therefore the differentials of the values of *Z* thus will be had :

$dZ^{/} = M^{/}dx + N^{/}dy^{/}$.	$dZ_{/} = M_{/}dx + N_{/}dy_{/}$
$dZ^{//} = M^{//}dx + N^{//}dy^{//}$	$dZ_{//} = M_{//}dx + N_{//}dy_{//}$
etc.	etc.

Now since the differential values of the terms $Zdx,\ Z^{/}dx,\ Z^{//}dx$ etc. and likewise of these themselves $Z_{/}dx,\ Z_{//}dx$ etc. may be found, if these terms may be differentiated and *nv* may be written in place of $dy^{/}$ in the differentials, in place of all the remaining differentials truly 0, it is evident the term alone $Z^{/}dx$ shall have a differential value, because $dy^{/}$ occurs in the differential of that only. Therefore with *nv* written everywhere in place of $dy^{/}$ the value of the term $Z^{/}dx$ will be the differential $= N^{/}dx \cdot nv$, which likewise will be the value of the whole differential of the formula $\int Zdx$, because the remaining terms beyond $Z^{/}dx$ undergo no variation. Moreover in place of $N^{/}$ we will be able to put *N*, because there is $N^{/} = N + dN$ and *dN* vanishes besides *N*. Therefore for the curve sought, in which $\int Zdx$ shall be a maximum or a minimum, that equation may be had $Ndx \cdot nv = 0$ or $N = 0$; with $dZ = Mdx + Ndy$. Q. E. I.

COROLLARY 1

8. Therefore if the curve must be defined, in which $\int Zdx$ shall be a maximum or a minimum and *Z* shall be a determined function of x and *y* only, then it is necessary to differentiate the magnitude *Z* ; which may be considered of this form $dZ = Mdx + Ndy$, hence the equation will be formed for the curve sought, which will be $N = 0$.

COROLLARY 2

9. Therefore since *N* shall be a determined function of *x* and *y*, no constant magnitude shall be present in the equation for the curve $N = 0$, which was not in the formula of the maximum of minimum $\int Zdx$; and on this account the curve found will be one of a kind and perfectly determined.

COROLLARY 3

10. Therefore in questions dealt with under this problem the satisfying curve may be determined from the formula of the maximum and minimum alone ; nor will any additional points be allowed to be prescribed, through which the curve sought may pass.

COROLLARY 4

11. But if Z were a function of *x* itself only, thus so that *y* may not be involved, then the determined function $\int Zdx$ equally will be a function of *x* itself only, and thus for that all the curves corresponding to the same abscissa will be satisfied equally. Likewise truly the calculation shows this ; for in this case, in which *y* is not present in *Z*, there becomes $N = 0$ [identically] ; and thus no equation is produced for the curve sought.

COROLLARY 5

12. Also it can be understood at once, whether the curved line may be given in which the formula of this kind $\int Zdx$ shall be a maximum or minimum. For if from differentiation a value of *Z* of this kind may be found for *N*, so that no curve may be expressed by the

equation $N = 0$, then also no curve can be found, in which the proposed formula $\int Zdx$ shall be a maximum or minimum.

COROLLARY 6

13. And then also it is seen that the maximum or minimum property is not to be restricted to any one of the determined abscissas, but, if the curve for one abscissa may return a maximum or minimum formula $\int Zdx$, the same maximum or minimum value will be had equally for any other abscissa.

SCHOLIUM I

14. Therefore we have found an easy method among all the curves corresponding to the same abscissas that can be determined, in which the formula $\int Zdx$ may constitute a maximum or minimum value, if indeed Z is a determined function of x and y only. Likewise truly also it is apparent the satisfying curve will always be algebraic, if indeed Z were an algebraic function of x and y themselves. Therefore a property of a curve found in this manner will be, that, if for the same abscissa some other curved line may be put in place, then for that the value of the formula $\int Zdx$ to be produced certainly will be either lesser or greater than that found, just as that found in the formula $\int Zdx$ were either a maximum or minimum. But since at this stage there may be doubt, whether in the curve found the value of the formula $\int Zdx$ may become a maximum or a minimum, that can be judged easily in any particular case ; but in general nothing at all can be decided. Meanwhile this is certain, if a single equation emerges, then only a maximum or minimum can have a place ; that is, if the curve found shall be for a maximum, then a minimum cannot be given, but the value of the formula $\int Zdx$ can be diminished indefinitely. In a like manner, if a single curve were found and in that the formula $\int Zdx$ shall be a minimum, then the value $\int Zdx$ can be increased indefinitely. But if moreover no solution at all satisfying the curve may be provided, that will indicate a value of the formula $\int Zdx$ for any abscissa , both to increase indefinitely as well as able to decrease indefinitely.

SCHOLIUM 2

15. Also from the same solution, these curves will be able to be found endowed with the property of maxima or minima of another kind mentioned above, which is not come upon through vanishing differentials, but infinitely great ones ; which kind of maxima or minima differs greatly from those. Moreover these curves may be found, if the value of the differential $Ndx \cdot nv$ may be put equal not to zero, but to infinity. Therefore just as often as this equation $N = \infty$ will suggest some curved line, so in that equally the formula

$\int Zdx$ will maintain a maximum or minimum value : Evidently this may come about, when a fraction is produced for *N*, the denominator of which put equal to zero provides the equation for some curved line. And with this agreed upon many curved lines are able to be found, which satisfy the same question, some of which contain maxima, others minima. For it can arise, that many more than two curves satisfying the problem may be found, even if only two equations may be able to arise, evidently $N = 0$ and $N = \infty$. For if *N* were a magnitude composed from factors, then any factor will be able to be put equal to put equal either to zero or infinity by satisfying the equation for the curve ; for it is agreed on many occasions that several more maxima and several more minima can be found. But all this will be made clearer in the following examples contained in this problem.

EXAMPLE I

16. *To find the curve, which among all the curves generally corresponding to the same abscissas,* $\int XYdx$ *may have a maximum or minimum value, with X denoting a function of x and Y a function of y only.*

Therefore in this case there becomes $Z = XY$, and thus $dZ = YdX + XdY = Mdx + Ndy$. Therefore there will be

$$M = \frac{YdX}{dx} \text{ and } N = \frac{XdY}{dy}$$

on account of the function *X* of *x* and *Y* of *y*. Therefore for the curve sought there will be

$$N = \frac{XdY}{dy} = 0;$$

but because *Y* is a function of *y,* there may be put $dY = \Theta dy$; equally Θ will be a function of *y;* and thus for the curve sought, if it may satisfy which, this equation is had $X\Theta = 0$, and thus either $X = 0$ or $\Theta = 0$; of which since neither may give a curved line, it is apparent in general no curve satisfies this question, but the value proposed $\int XYdx$ can be increased or diminished in an infinite number of ways. Moreover from the equation $X = 0$ or $\Theta = 0$, because Θ is a function of *y,* it follows that $y = Const.$, which equation provides a right line parallel to the abscissa *AZ*, of which the distance is so great, that the function *Y* becomes a maxima or minima. For it is apparent, if the quantity *Y* may admit a maximum or minimum value, then also the formula $\int XYdx$ becomes a maximum or minimum. But the other equation $X = 0$, because it provides $x = Const.$, which cannot indeed show a right line satisfying the curve, because it provides a right line normal to the abscissa, which therefore will correspond not to a given abscissa, but only to one point of that.

EXAMPLE II

17. *To find a curve, which amongst all the curves corresponding to the same abscissas may have a maximum or minimum value of the formula* $\int (ax-yy)ydx$.

If this formula may be compared with the general $\int Zdx$, it will become $Z = axy - y^3$, and thus $dZ = aydx + (ax-3yy)dy$; thus so that there may become $M = ay$ and $N = ax - 3yy$; so that this equation will be found for the curve sought $ax - 3yy = 0$ or $yy = \frac{1}{3}ax$, which is the equation for a parabola, the vertex at *A*, the axis AZ and by having the parameter $= \frac{1}{3}a$. Therefore in this parabola the value of the formula $\int (ax-yy)ydx$ will be a maximum or a minimum. But whether it shall be a maximum or a minimum, may be found if we may substitute some other line in place of the parabola and we shall inquire, whether for that the value of the proposed formula shall be greater or less than the parabola. Therefore we may take a right line agreeing with the axis itself, for which there will be $y = 0$. Thus for this the value of the formula $\int (ax-yy)ydx$ equally may become $= 0$, but for the parabola the same value will be positive and thus > 0; from which it follows in the parabola of the proposed formula the value cannot be a minimum but a maximum. Moreover we will be able to indicate algebraically, how great the value of the proposed formula shall become for the parabola ; for since there shall be $yy = \frac{1}{3}ax$, the proposed formula will change into this :

$$\int \tfrac{2}{3}axdx\sqrt{\tfrac{1}{3}ax} = \tfrac{4}{15}ax^2\sqrt{\tfrac{1}{3}ax} .$$

But if we may consider another equation, for example $y = nx$, the proposed formula will change into this

$$\int dx(naxx - n^3x^3) = \tfrac{1}{3}nax^3 - \tfrac{1}{4}n^3x^4 ,$$

which is less than the value of the formula always, which was produced for the parabola found, because that will test anything easily with the defined values substituted in place of x.

EXAMPLE III

18. *To find the curve, in which amongst all the curves in general related to the same abscissa, the value of this formula*

$$\int (15a^2x^2y - 15a^3xy + 5a^2y^3 - 3y^5)dx$$

shall be a maximum or a minimum.

Therefore there will be $Z = 15a^2x^2y - 15a^3xy + 5a^2y^3 - 3y^5$, which if it may be differentiated, on placing *x* constant, will produce

$$N = 15\left(a^2x^2 - a^3x + a^2y^2 - y^4\right);$$

and hence which value put $= 0$ will give the equation for the curve sought ; thus it will be

$$aaxx - a^3x + a^2y^2 - y^4 = 0 = \left(ax - yy\right)(ax + yy - aa).$$

On account of these two factors two satisfying curves will be produced, of which the one will be expressed by this equation : $yy = ax$,the other by this: $yy = aa - ax$; with each for a parabola. Now so that it may become apparent, whether there shall be for a maximum or minimum, we may put the abscissa to be a minimum, and the first equation $yy = ax$ substituted into the formula will give
$\int -10a^3\, x\sqrt{ax}dx.$
[$Z = \sqrt{ax}(15a^2x^2 - 15a^3x + 5a^3x - 3a^2x^2) = \sqrt{ax}(12a^2x^2 - 10a^3x) \cong \sqrt{ax}(-10a^3x)$ for small x; while the other becomes $Z = 15a^5 - 15a^5 + 5a^5 - 3a^5 = 2a^5$.]

Truly the other formula $yy = aa - ax$ or $y = a$ substituted will give $\int 2a^5dx$. But if some other value of y may be attributed, e.g. $y = 0$, then the formula proposed will change into $\int 0dx = 0$. From which it is apparent one of the curves sought shall be a maximum $yy = aa - ax$, but the other $yy = ax$ a minimum, evidently for a negative maximum. But this judgment may be put in place always easily, whether a maximum or minimum may be in place for a curve found, if the abscissa x may be considered infinitely small, then indeed there will be no need for an integration, as that formula itself Zdx will show the value of the formula $\int Zdx$ in this case.

EXAMPLE IV

19. *Among all the curves corresponding to the same abscissas to define that, in which the value of the formula*

$$\int (3ax - 3xx - yy)(ax - xx - \tfrac{4}{3}xy + yy)dx$$

shall be a maximum or a minimum.

Therefore from this formula the following value of Z expanded out will be produced :

$$\begin{aligned} Z = &+ 3a^2x^2 - 4ax^2y + 2axyy + \tfrac{4}{3}xy^3 - y^4 \\ &- 6ax^3 + 4x^3y - 2xxyy + 3x^4, \end{aligned}$$

which differentiated with *x* put constant and divided by *dy*, the following value will be produced for *N*:

$$N = -4ax^2 + 4axy + 4xyy - 4y^3 + 4x^3 - 4xxy,$$

which expression placed equal to zero will give the equation for the curve sought. And thus there will be

$$y^3 - xyy + xxy + axx - axy - x^3 = 0,$$

which has two factors, which provide just as many equations, these

I. $y - x = 0$ for a right line,
II. $yy - ax + xx = 0$ for a circle.

x may be put infinitely small and from the equation $y = x$ the value of *Z* will be $Z = 3a^2x^2$; but from the equation $yy = ax - xx$ or $y = \sqrt{ax}$ there will be $Z = 4aaxx$. But if moreover putting $y = a$, it gives rise to $Z = -a^4$, from which it is apparent each line found to be for a maximum.

SCHOLIUM

20. Also the problems can be resolved by the common method of maxima and minima. For when a curve is sought, for which the value of $\int Zdx$ shall be a maximum or a minimum and that for any abscissa, it is evident, if indeed *Z* shall be a determined function of *x* and *y*, the formula $\int Zdx$ cannot be a maximum or a minimum, unless its element *Zdx* and hence *Z* shall be such. On which account the question will be satisfied, if the quantity *Z* may be differentiated on putting *x* constant and its differential may be put equal to $= 0$. For then *Z* will have a maximum or minimum always, and thus also *Zdx* and the formula $\int Zdx$ itself. But if moreover the function *Z* may be differentiated by considering *x* constant, *Ndy* will emerge, because generally by differentiating we have put $dZ = Mdx + Ndy$; and it is satisfied on putting $N = 0$, which is the same solution, that we have found by the method treated. [Thus it is clear that the differentiation with respect to *y* gives the max. or min. value of the integrand *Zdx* and $\int Zdx$, for all the abscissas *x* in the interval.] But although hence it may be considered these questions able to be resolved in a similar manner, as in the common method of maxima and minima, yet this only arises, if *Z* were a function of *x* and *y* only ; for, if in addition in *Z* magnitudes shall be present from the differentials arising from *p, q, r* etc., then the common method can be of no further use. For even if then the function *Z* may be differentiated on putting *x* constant, the differentials *dp, dq, dr* etc. may be present still in the differential, the relation of which to *dy* since it may not agreed on, a suitable equation for determining the maxima and minima then cannot be deduced. Therefore in these cases the usefulness and necessity of our method will be discerned.

PROPOSITION III. PROBLEM

21. *If* Z (Fig. 4) *were a function of x, y and p to be determined thus so that there shall become*

$$dZ = Mdx + Ndy + Pdp,$$

to find that amongst all the curves corresponding to the same abscissas, in which $\int Zdx$ *shall be a maximum or minimum.*

SOLUTION

Now *amz* shall be the satisfying curve sought and some applied line $Nn = y'$ may be considered to be increased by the small amount *nv*, the value of the differential of the formula $\int Zdx$ or of the magnitude equivalent to this, for example $Zdx + Z'dx + Z''dx + Z'''dx + \text{etc.}$ together with $Z_{/}dx + Z_{//}dx + Z_{///}dx + \text{etc.}$ shall be $= 0$. Therefore the value of the whole differential quantity $\int Zdx$ from the translation of the point *n* into *v* will be found, if the differential values of the individual terms of these, which certainly are affected by this translation, may be sought and may be added into one sum. But from the translation of the point *n* into *v* only these terms will undergo a change, in which the magnitudes y', p and p' are present and thus only the terms $Z\,dx$ and $Z'dx$; for as Z is a function of y and p besides x, thus Z' likewise is a function of y' and p' themselves. On account of which these terms must be differentiated, and in the differentials of these in place of dy', dp and dp' it is required to write the values indicated above $+nv$, $+\frac{nv}{dx}$ and $-\frac{nv}{dx}$. But just as $dZ = Mdx + Ndy + Pdp$, thus there will be $dZ' = M'dx + N'dy' + P'dp'$. And thus hence the value of the differential of Z itself will be $P \cdot \frac{nv}{dx}$ and that of Z' will be $N' \cdot nv - P' \cdot \frac{nv}{dx}$; from which the value of the differential of the formula $\int Zdx$ from each of the terms $Zdx + Z'dx$ thus will be $= nv \cdot (P + N'dx - P')$. But $P' - P = dP$ and in place of N' there can be written *N;* so that the value of the differential will be $= nv \cdot (Ndx - dP)$. Whereby, since the value of the differential of the formula $\int Zdx$ made equal to zero may provide the equation for the curve sought, this equation will become $0 = Ndx - dP$ or $N - \frac{dP}{dx} = 0$, from which equation the nature of the curve sought will be expressed. Q. E. I.
[This last expression is to be much used in examples and is fundamental to the whole theory.]

COROLLARY 1

22. So that therefore if Z were some function of x, y and likewise of their differentials dx and dy or in place of these differentials of p itself, with $dy = pdx$ present the differential of Z of this kind will have the form, so that it shall become $dZ = Mdx + Ndy + Pdp$. And hence the curve may be found, in which $\int Zdx$ shall be a maximum or a minimum, by forming this equation $N - \frac{dP}{dx} = 0$ or $Ndx = dP$.

COROLLARY 2

23. This equation therefore will be always of a differential of the second order, unless p clearly shall not be present in P. For if p shall be present in P then in dP, dp will be present, which on account of $p = \frac{dy}{dx}$ involves a differential the second order.

COROLLARY 3

24. Therefore when the magnitude P at this stage includes p, in the differential itself $dZ = Mdx + Ndy + Pdp$, then on account of the differential equation of the second order for the curve sought, two new arbitrary constants will enter through integration. From which, for the determination of the constants, two points of the curve will be able to be described ; for otherwise not one, but innumerable curves will be found.

COROLLARY 4

25. Thus so that determinate problems of this kind may be proposed, they are to be enunciated thus, so that for a given curve it must be drawn through two given points, which amongst all the other curves drawn through the same two points for the same abscissa x the value $\int Zdx$ may include a maximum or a minimum.

COROLLARY 5

26. But the magnitude p will not be present in P, if Z were a function of x et y, only multiplied by p or by $n + p$, with n denoting some numerical constant. Let V be a function of x and y only, thus so that there shall be $dV = Mdx + Ndy$ and $Z = V(n + p)$, there becomes

$$dZ = (n + p)Mdx + (n + p)Ndy + Vdp .$$

And hence the equation for the curve sought will be

$$0=(n+p)N-\frac{dV}{dx} \text{ or } (n+p)Ndx=dV=Mdx+Ndy.$$

COROLLARY 6

27. Therefore in these cases, in which there is $Z=V(n+p)$ for some function present V of x and y only, it does not result in a differential equation of the second order, because dp clearly is not present in that. Truly nor indeed does it come to a differential equation of the first order, but thus to an algebraic equation. For since there shall be $pdx=dy$, there will be $(n+p)Ndx=nNdx+Ndy$; which put equal to $Mdx+Ndy$ itself will give an equation divisible by dx and thus algebraic, hence $nN=M$, if indeed V were an algebraic function.

COROLLARY 7

28. Moreover as often as this comes about, the formula of the maximum or minimum , which is $\int Zdx$, will be of such a form $\int(Vndx+Vdy)$, or of such, $\int Vdy$, on putting $n=0$. Therefore the maxima or minima formulas equally deduce the determinate equation for a curve sought of this kind, thus so that it may not be required to prescribe one or more points, through which the curve must pass.

COROLLARY 7

29. Therefore with some function V of x and y, that formula of the maxima or minima $\int Vdy$ may be treated in a similar manner, as $\int Vdx$. For on putting $dV=Mdx+Ndy$ for the formula $\int Vdx$, the equation $N=0$ corresponds to this curve, thus for the other formula $\int Vdy$ the equation $M=0$ corresponds. From which it is seen the coordinates x and y can be interchanged between themselves.

SCHOLIUM 1

30. And thus it is evident in the solution of problems of this kind, in which the curve is sought having a maximum or minimum value of the formula $\int Zdx$ for some function Z of x, y and p arising, to arrive at a differential equation of the second order, unless in Z the quantity p may have a single dimension only. But for a number this equation of the second order often allows an integration, which will be required to be seen in individual cases. Meanwhile to have noted here generally it will help the integration to succeed, if in the function Z certainly x shall not be present, that is, if in the differential of that, $dZ=Mdx+Ndy+Pdp$ the value M may vanish, thus so that there shall be only $dZ=Ndy+Pdp$. Since indeed for the curve found there shall be this equation $N-\frac{dP}{dx}=0$, that may be multiplied by dy, and because $dy=pdx$, that will change into

this $Ndy - pdP = 0$, to which this itself will be equivalent $Ndy + Pdp = Pdp + pdP = dZ$, the integral of which is $Z + C = Pp$, which equation now is a differential of the first order only. Therefore as often among all the curves of the same corresponding abscissas that is sought, in which the value of the formula $\int Zdx$ shall be a maximum or a minimum, and Z shall be a function of y and p only, thus so that there shall be $dZ = Ndy + Pdp$, then for the satisfying curve this differential equation of the first order will be able to be shown at once $Z + c = Pp$. Then truly also, if Z were a function of x and p only and $dZ = Mdx + Pdp$ with the term Ndy vanishing, then for the curve a differential equation of the first order will be produced. For on account of $dP = 0$ there will be $P = C$, which will give a differential equation of the first order only. But if in addition M shall vanish or Z shall be a function of p itself only and $dZ = Pdp$, the equation $P = C$ will be found and it will be changed into that $Pdp = Cdp = dZ$, which again integrated gives $Z + D = Cp$. But in this case, because Z and P are functions of p itself only, each equation $P = C$ and $Z + D = Cp$ will provide a constant value for p and thus an equation of this form $dy = ndx$, which indicates that right lines are sufficient to satisfy problems of this kind and indeed you can draw such as it pleases. For in the equation $P = C$, since C shall be an arbitrary constant, the value of p not only is constant, but also emerges arbitrary ; from which any right line will result. On account of which, if a curve must be drawn through two given points, in which $\int Zdx$ shall be a maximum or minimum, and Z shall be a function of p only, then the right line through those two given points will be satisfactory.

SCHOLIUM 2

31. Because now above we have seen the coordinates x and y are interchanged in problems of this kind and, if it may considered convenient, the applied line may be treated as the abscissa, it will help for this case to be confirmed also. Therefore a curve shall be required to be found, in which $\int Zdy$ shall be a maximum or minimum with the function Z of x, y and p present, and

$$dZ = Mdx + Ndy + Pdp .$$

But this formula $\int Zdy$ reduced to our form will become $\int Zpdx$, in which there shall be

$$d \cdot Zp = Mpdx + Npdy + (Z + Pp)dp ,$$

from which the corresponding value of the differential of the proposed formula will be

$$(Npdx - dZ - Pdp - pdP)nv = (-Mdx - 2Pdp - pdP)nv$$

[Note that *M* and *N* previously are now *Mp* and *Np*, while *P* has become $Z+Pp$, to which the condition $N-\frac{dP}{dx}=0$ applies for the new functions.]
and the equation for the proposed curve will be

$$0=-Mdx-2Pdp-pdP \text{ or } 0=Mdy-d\cdot Pp^2.$$

Because now for the similarity to be shown, because here we may consider *y* as the abscissa, we may put $dx=\pi dy$, there will be

$$p=\frac{1}{\pi} \text{ and } dp=-\frac{d\pi}{\pi\pi}=-ppd\pi\,;$$

there will be

$$dZ=Mdx+Ndy-Pppd\pi=Mdx+Ndy+\Pi d\pi$$

on putting $\Pi=-Ppp$, so that the similarity of the terms will be preserved. From which therefore the equation for the curve will be $0=-Mdy+d\Pi$; which may be produced by the same equation, if in the formula $\int Zdx$ the applied line *y* will be changed into the abscissa and in turn the abscissa into the applied line. Therefore any indeterminate formula proposed from *x* and *y* and composed from the differentials of those, which must be maxima or minima, of the coordinates *x* and *y* whichever will be allowed to be treated as the abscissa and to be adapted to the same maximum or minimum.

EXAMPLE I

32. *Among all the curves related to the same abscissas to determine that, in which* $\int(Zdx+[Z]dy)$ *shall be a maximum or a minimum ; with some functions Z and* $[Z]$ *of* x *and y arising, thus so that there shall be*

$$dZ=Mdx+Ndy \text{ and } d[Z]=[M]dx+[N]dy\,.$$

So that this formula $\int(Zdx+[Z]dy)$ may be reduced to the customary form, *pdx* may be put in place of *dy* and this formula will be had $\int(Z+[Z]p)dx$ effecting the maximum or minimum. Therefore the value $Z+[Z]p$ will be differentiated and its differential will be

$$=+Mdx+Ndy+[M]pdx+[N]pdy+[Z]dp\,.$$

Now by the rule found hence this equation will be produced for the curve sought

$$0=(N+[N]p)dx-d[Z]=(N+[N]p)dx-[M]dx-[N]dy,$$

which, on account of $[N]pdx = [N]dy$, divided by *dx* will give this equation for the curve sought, algebraic or finite $N - [M] = 0$ or $N = [M]$. Hence it is understood, if the proposed formula $\int (Zdx + [Z]dy)$ were determinate or with the differential $Zdx + [Z]dy$ prepared thus, so that it may admit integration, then no [particular] line sought will be satisfactory, or rather all lines will be equally satisfactory. For if $Zdx + [Z]dy$ can be integrated, it follows that $N = [M]$, as we have shown elsewhere concerning the differentiation of formulas of two determinate variables ; and thus from these cases the identical equation $0 = 0$ will be produced. And hence it is understood more clearly, as we have observed now before, a formula of the maximum or minimum is required to be an indeterminate one ; for otherwise all the curved lines will be equally satisfying.

EXAMPLE II

33. *Among all the curved lines related to the same abscissa, to determine that, the length of which shall be a minimum or in which* $\int dx\sqrt{(1+pp)}$ *shall be a minimum.*

It is apparent indeed in the first place in this question that a maximum cannot be given, since the length of the lines may be increased indefinitely, with the same abscissa maintained. Thus only a minimum will have a place, since that agrees with the elements of geometry, in which it may be demonstrated a right line among all other lines situated between the same ends to be the shortest. Therefore this example introduced thus has shown, since in order that the agreement of our method may be understood with the truth now recognised from other places, then also, so that the circumstances with two arbitrary points, which must be added to questions of this kind, may be understood better. Therefore the formula $\int dx\sqrt{(1+pp)}$ will be compared with the general one $\int Zdx$,

$$Z = \sqrt{(1+pp)} \text{ and } dZ = \frac{pdp}{\sqrt{(1+pp)}};$$

from which $M = 0, N = 0$ and $P = \dfrac{p}{\sqrt{(1+pp)}}$. Whereby, since in general the equation for the line sought shall be $N - \dfrac{dP}{dx} = 0$, we will have in this case $dP = 0$ and thus

$$P = \frac{p}{\sqrt{(1+pp)}} = Const.$$

from which equation $p = Const. = n$ or $dy = ndx$ arises, which integrated again gives $y = a + nx$. Therefore not only is it apparent that the line sought is right, but also, on account of the two arbitrary constants *a* and *n*, any straight line. Whereby, if it may be commanded to draw the shortest line between two points, that will be a right line. But likewise it is understood, if a line must be found, in which $\int Zdx$ shall be a maximum or minimum, where Z is a function of p only, then only a right line is satisfactory ; as we have now observed before.

EXAMPLE III

34. *Among all the curved lines related to the same abscissa, to determine that, in which*

$$\int \frac{dx\sqrt{(1+pp)}}{\sqrt{x}}$$

shall be a maximum or minimum.

This formula arises, if the line of quickest descent may be sought under the hypothesis of uniform gravity, by putting the axis on which the abscissas are taken vertical. Therefore there will be

$$Z = \frac{\sqrt{(1+pp)}}{\sqrt{x}} \text{ and } dZ = -\frac{dx\sqrt{(1+pp)}}{2x\sqrt{x}} + \frac{pdp}{\sqrt{x(1+pp)}};$$

from which becomes

$$M = -\frac{\sqrt{(1+pp)}}{2x\sqrt{x}}, \ N = 0 \text{ and } P = \frac{p}{\sqrt{x(1+pp)}}.$$

Now since the curve sought may be expressed by the equation $N - \frac{dP}{dx} = 0$ there will be $dP = 0$ and

$$P = \frac{p}{\sqrt{x(1+pp)}} = Const. = \frac{1}{\sqrt{a}};$$

whereby multiplication gives

$$ap^2 = x + p^2x \text{ and } p = \frac{dy}{dx} = \sqrt{\frac{x}{a-x}} \text{ or } y = \int dx\sqrt{\frac{x}{a-x}},$$

which equation indicates the curve sought to be a cycloid produced on a horizontal base and having a cusp in the upper region of the axis, which thus will be able to be drawn through any two points. [See Goldstine p.72 *History C.ofV.*]

EXAMPLE IV

35. *Among all the curved lines related to the same abscissa, to determine that, in which* $\int y^n dx\sqrt{(1+pp)}$ *shall be a maximum or minimum.*

Hence for this proposed formula there will be

$$Z = y^n\sqrt{(1+pp)} \text{ and } dZ = ny^{n-1}dy\sqrt{(1+pp)} + \frac{y^n pdp}{\sqrt{(1+pp)}};$$

thus so that it becomes

$$M = 0, \quad N = ny^{n-1}\sqrt{(1+pp)} \text{ and } P = \frac{y^n p}{\sqrt{(1+pp)}}.$$

Therefore because $M = 0$, this equation will be had immediately for the curve sought now once integrated $Z + C = Pp$ (§ 30), which in our case becomes

$$y^n\sqrt{(1+pp)} + ma^n = \frac{y^n pp}{\sqrt{(1+pp)}}.$$

But if the constant may be put as $a = 0$, it will produce $1+pp = pp$ or $p = \infty$ and it will be satisfied by a right line normal to the axis. Generally truly the satisfying lines will be found from the equation, which will change into

$$y^n + ma^n\sqrt{(1+pp)} = 0 \text{ or } y^{2n} = m^2a^{2n} + m^2a^{2n}p^2;$$

which gives

$$p\ (=\frac{dy}{dx}) = \frac{\sqrt{(y^{2n} - m^2a^{2n})}}{ma^n} \text{ and } x = \int\frac{ma^n dy}{\sqrt{(y^{2n} - m^2a^{2n})}};$$

which line can be drawn through two given points. If there were $n = -\frac{1}{2}$, thus so that $\int\frac{dx\sqrt{(1+pp)}}{\sqrt{y}}$ must become a maximum or minimum, equally it must produce brachystochrone lines relative to the horizontal axis and for that there will be $x = \int dy\sqrt{\frac{y}{a-y}}$; which since it agrees entirely with the preceding, provided *x* and *y* may commute between themselves. Clearly it will be as before a curve satisfying a cycloid generated by rotating on a horizontal base, such that it is allowed to be drawn through any two points.

EXAMPLE V

36. *Among all the curved lines related to the same abscissa, to determine that, in which* $\int \frac{ydy^3}{dx^2+dy^2}$ *shall be a maximum or minimum.*

This formula will become reduced to the accustomed form with the aid of the substitution $dy = pdx$ into this :

$$\int \frac{yp^2dx}{1+pp};$$

and that is accustomed to be found, if the solid of rotation arising may be sought from the rotation of the curve about the axis, so that the motion in a fluid along the direction of the axis may suffer the minimum resistance ; in as much as the resistance in this case is considered proportional to the formula

$$\int \frac{ydy^3}{dx^2+dy^2} \text{ or } \int \frac{yp^2dx}{1+pp}.$$

Therefore there will be

$$Z = \frac{yp^3dx}{1+pp} \text{ and } dZ = \frac{p^3dy}{1+pp} + \frac{ydp(3pp+p^4)}{(1+pp)^2};$$

thus so that there becomes

$$M = 0, \quad N = \frac{p^3}{1+pp} \text{ and } P = \frac{p^2y(3+pp)}{(1+pp)^2};$$

Therefore since there shall be $M = 0$, as single integration generally will succeed and the equation for the curve sought will be $Z + C = Pp$ [recall that $Ndy - pdP = 0$], or

$$\frac{yp^3}{1+pp} + a = \frac{p^3y(3+pp)}{(1+pp)^2},$$

which will change into this $a(1+pp)^2 = 2p^3y$. But the expansion of this equation cannot thus be put in place, so that p may be eliminated ; whereby it will be appropriate to define each coordinate y and x by the same variable p. And indeed in the first place there is

$$y = \frac{a(1+pp)^2}{2p^3}.$$

Then on account of $dy = pdx$ there will be

$$dx = \frac{dy}{p} \text{ and } x = \int \frac{dy}{p} = \frac{y}{p} + \int \frac{ydp}{pp}.$$

But if therefore in place of *y* the value found may be substituted, it will produce

$$x = \frac{a(1+pp)^2}{2p^4} + a\int \frac{dp(1+pp)^2}{2p^5} = \frac{a}{2}\left(\frac{3}{4p^4} + \frac{1}{pp} + 1 + lp\right),$$

from which the construction of the curve will be made, with the aid of logarithms.

EXAMPLE VI

37. *To find the curve, in which this formula* $\int yxdx\sqrt{(1+pp)}$ *shall be a maximum or minimum.*

Therefore there will be $Z = yx\sqrt{(1+pp)}$ and

$$dZ = ydx\sqrt{(1+pp)} + xdy\sqrt{(1+pp)} + \frac{yxpdp}{\sqrt{(1+pp)}}.$$

Hence on this account there will be had

$$M = ydx\sqrt{(1+pp)},\ N = x\sqrt{(1+pp)}, \text{ and } P = \frac{yxp}{\sqrt{(1+pp)}};$$

from which this equation will be formed for the curve $Ndx = dP$, which suggests

$$xdx\sqrt{(1+pp)} = \frac{p^2xdx + yxpdp}{\sqrt{(1+pp)}} + \frac{yxdp}{(1+pp)^{\frac{3}{2}}}$$

or

$$xdx - ydy = \frac{yxdp}{1+pp},$$

on account of $dy = pdx$. This is a differential equation of the second order and, nevertheless with the help of a suitable substitution that may be reduced to a simpler differential, so that from that the variables *x* and *y* everywhere constitute a number of the same dimensions, yet this differential equation thus has been prepared, so that neither shall it be able to be integrated nor separated ; evidently it can be reduced to an equation of this form

$$\frac{du}{u^3} + \frac{dv}{v^3} = \frac{vdv(1+u^3)}{u^3}.$$

Since it shall be so, neither can the equation found $xdx - ydy = \frac{yxdp}{1+pp}$ be able to be reduced to a simple or more convenient form ; and hence nothing certainly can be judged about the nature of this curve. Yet meanwhile that potential equation will involve two arbitrary constants from which the satisfying curve can be drawn through two given points.

EXAMPLE VII

38. *To find the curve in which* $\int (xx+yy)^n dx\sqrt{(1+pp)}$ *shall be a maximum or minimum.*

Here since there shall be $Z = (xx+yy)^n dx\sqrt{(1+pp)}$, there will be

$$dZ = 2n(xx+yy)^{n-1}(xdx+ydy)\sqrt{(1+pp)} + \frac{(xx+yy)^n pdp}{\sqrt{(1+pp)}},$$

therefore

$$N = 2n(xx+yy)^{n-1} y\sqrt{(1+pp)} \text{ and } P = \frac{(xx+yy)^n p}{\sqrt{(1+pp)}};$$

from which the curve sought shall have the equation:

$$2n(xx+yy)^{n-1} ydx\sqrt{(1+pp)} = d\frac{(xx+yy)^n p}{\sqrt{(1+pp)}} = \frac{2n(xx+yy)^{n-1} p(xdx+ydy)}{\sqrt{(1+pp)}} + \frac{dp(xx+yy)^n}{(1+pp)^{\frac{3}{2}}},$$

which divided by $(xx+yy)^{n-1}$ and multiplied by $\sqrt{(1+pp)}$ will change into

$$2nydx = 2nxdy + \frac{(xx+yy)dp}{(xx+yy)} \text{ or } \frac{2n(ydx-xdy)}{xx+yy} = \frac{dp}{1+pp}.$$

Each member of this equation is integrable by the quadrature of the circle, and the integral becomes

$$2n\text{Atang}\frac{x}{y} = \text{Atang } p + \text{Atang } k = \text{Atang } \frac{p+k}{1-pk},$$

from which there becomes

$$\frac{x}{y} = \text{tang } \frac{1}{2n}\text{Atang } \frac{k+p}{1-kp} = T;$$

and *T* will be an algebraic function of *p*, provided *2n* shall be a rational number. Therefore since $x = Ty$ or $dy = pdx = \frac{dx}{T} - \frac{xdT}{TT}$ or

$$xdT = Tdx - pTTdx;$$

therefore

$$\frac{dx}{x} = \frac{dT}{T - pTT} + \frac{Tdp}{1 - pT} - \frac{Tdp}{1 - pT};$$

from which there arises

$$lx = l\frac{T}{1 - pT} - \int \frac{Tdp}{1 - pT},$$

which indeed will be satisfactory enough for the constructing the curve. Truly so that the nature of these curves may be better understood, which arise for definite values of the exponent *n*, we will consider some cases.

I. Let $n = \frac{1}{2}$ and $2n = 1$; there will be $\text{Atang}\frac{x}{y} = \text{Atang}\frac{k+p}{1-kp}$, and thus

$$\frac{x}{y} = \frac{k+p}{1-kp} = \frac{kdx + dy}{dx - kdy},$$

or

$$xdx - kxdy = kydx + ydy;$$

which integrated provides

$$x^2 - y^2 = 2kxy + C;$$

which is the equation for an equilateral hyperbola.

II. Let $n = 1$ and $2n = 2$; there will be $2\text{Atang}\frac{x}{y} = \text{Atang}\frac{k+p}{1-kp}$; or

$$\text{Atang}\frac{2xy}{yy - xx} = \text{Atang}\frac{k+p}{1-kp};$$

from which there becomes

$$\frac{2xy}{yy - xx} = \frac{kdx + dy}{dx - kdy}$$

or

$$2xydx - 2kxydy = kyydx - kxxdx + yydy - xxdy\ ;$$

which integrated gives

$$yx^2 = ky^2x - \tfrac{1}{3}kx^3 + \tfrac{1}{3}y^3 + C \quad \text{or} \quad y^3 + 3ky^2x - 3yx^2 - kx^3 = C.$$

III. Let $n = \tfrac{3}{2}$ or $2n = 3$; erit

$$3\text{Atang}\frac{x}{y} = \text{Atang}\frac{3y^2x - x^3}{y^3 - 3yx^2} = \text{Atang}\frac{kdx + dy}{dx - kdy}\ ;$$

and hence

$$3y^2xdx - 3ky^2xdy - x^3dx + kx^3dy = ky^3dx + y^3dy - 3kyx^2dx - 3yx^2dy\ ;$$

which integrated gives

$$\tfrac{3}{2}y^2x^2 - ky^3x - \tfrac{1}{4}x^4 + kyx^3 - \tfrac{1}{4}y^4 = C$$

or

$$y^4 + 4y^3x - 6y^2x^2 - 4kyx^3 + x^4 = C.$$

From these cases the equation of the integral can be deduced for any value of *n*. For since there shall be

$$2n\text{Atang}\frac{x}{y} = \text{Atang}\frac{2ny^{2n-1}x - \frac{2n(2n-1)(2n-2)}{1\cdot 2\cdot 3}y^{2n-3}x^3 + \text{ etc.}}{y^{2n} - \frac{2n(2n-1)}{1\cdot 2}y^{2n-2}x^2 + \frac{2n(2n-1)(2n-2)(2n-3)}{1\cdot 2\cdot 3\cdot 4}y^{2n-4}x^4 - \text{etc}}$$

$$= \text{Atang}\frac{(y + x\sqrt{-1})^{2n} - (y - x\sqrt{-1})^{2n}}{(y + x\sqrt{-1})^{2n}\sqrt{-1} + (y - x\sqrt{-1})^{2n}\sqrt{-1}},$$

there becomes

$$\frac{kdx + dy}{dx - kdy} = \frac{(y + x\sqrt{-1})^{2n} - (y - x\sqrt{-1})^{2n}}{(y + x\sqrt{-1})^{2n}\sqrt{-1} + (y - x\sqrt{-1})^{2n}\sqrt{-1}}\ ;$$

which reduced provides

$$\begin{aligned}&kdx(y+x\sqrt{-1})^{2n}\sqrt{-1}+kdx(y-x\sqrt{-1})^{2n}\sqrt{-1}\\&+kdy(y+x\sqrt{-1})^{2n}-kdy(y-x\sqrt{-1})^{2n}\\&=-dy(y+x\sqrt{-1})^{2n}\sqrt{-1}-dy(y-x\sqrt{-1})^{2n}\sqrt{-1}\\&+dx(y+x\sqrt{-1})^{2n}-dx(y-x\sqrt{-1})^{2n},\end{aligned}$$

of which the integral is

$$k(y+x\sqrt{-1})^{2n+1}-k(y-x\sqrt{-1})^{2n+1}=-\frac{1}{\sqrt{-1}}(y+x\sqrt{-1})^{2n+1}-\frac{1}{\sqrt{-1}}(y-x\sqrt{-1})^{2n+1}+C$$

or

$$C=(y+x\sqrt{-1})^{2n+1}(k\sqrt{-1}+1)+(y-x\sqrt{-1})^{2n+1}(1-k\sqrt{-1}).$$

But generally there is:

$$\left(y+x\sqrt{-1}\right)^{2n+1}+(y-x\sqrt{-1})^{2n+1}=2(yy+xx)^{(2n+1):2}\cos(2n+1)\,\text{Atang}\,\frac{x}{y}$$

and

$$\frac{\left(y+x\sqrt{-1}\right)^{2n+1}-(y-x\sqrt{-1})^{2n+1}}{\sqrt{-1}}=2(yy+xx)^{(2n+1):2}\sin(2n+1)\,\text{Atang}\,\frac{x}{y}.$$

With which values substituted this equation of the integral will be produced free from imaginary numbers

$$2k(yy+xx)^{(2n+1):2}\ \sin(2n+1)\,\text{Atang}\,\frac{x}{y}=2(yy+xx)^{(2n+1):2}\cos(2n+1)\ \text{Atang}\,\frac{x}{y}-C$$

or, on account of the arbitrary constants *k* and *C*, this :

$$C=(yy+xx)^{(2n+1):2}\ \left(k\sin(2n+1)\,\text{Atang}\,\frac{x}{y}+h\cos(2n+1)\ \text{Atang}\,\frac{x}{y}\right),$$

which equation always is algebraic, as long as *n* should be a rational number. Or if a certain circular arc may be put $=g$, the curve sought from an equation of this kind

$$C=(yy+xx)^{(2n+1):2}\ \sin\left(g+(2n+1)\,\text{Atang}\,\frac{x}{y}\right)$$

can be expressed from the position of the radius of the circle, that here we consider, $=1$.

SCHOLIUM 3

39. Therefore if among all the curves corresponding to the same abscissa that must be found, in which $\int Zdx$ shall be a maximum or minimum, with the function Z present of x, y and p, thus so that there shall be $dZ = Mdx + Ndy + Pdp$, this equation $N - \frac{dP}{dx} = 0$ will be had for the curve sought. Moreover because in the preceding problem we have observed, if Z were a function of x and y only, then the solution can be absolved by a common method : for in order that $\int Zdx$ shall be a maximum or minimum, also Zdx and hence Z is required to be such, with respect to x ; and on this account with its differential dZ, by taking x constant, placed equal to zero will give the equation for the curve sought. A similar method may succeed in the present problem, but only if in the differential of Z, which arises on putting x constant and is $Ndy + Pdp$, a relation between the differentials dy and dp may be apparent, so that division by dy put in place, and the finite value elicited may be put equal to zero. But since that relation between dy and dp, without which the common method of maxima or minima cannot be used, in the first place that even now may not be allowed, we will be able to assign later : For because this equation $N - \frac{dP}{dx} = 0$ has been found for the curve sought, this may be understood to arise from that $Ndy + Pdp$ or $N + \frac{Pdp}{dy}$, if there may be put in place $-\frac{dP}{dx} = \frac{Pdp}{dy}$ or $0 = dP + \frac{Pdp}{p}$ on account of $dy = pdx$. Concerning which the relation between these differentials dy and dp thus will be prepared, so that it may be included in the equation $pdP + Pdp = 0$; which property corresponds to this, so that Pp must be considered as constant. Hence for problems being resolved, in which a curve is sought having a maximum or minimum value of the formula $\int Zdx$, with $dZ = Mdx + Ndy + Pdp$ satisfied, the value of Z itself has to be differentiated and in the differential $Mdx + Ndy + Pdp$ in place of Mdx there must be put 0, Ndy remains unchanged, then truly in place of Pdp write $-pdP$ and that which emerges to be placed equal to zero. With this agreed upon there will be obtained $Ndy - pdP = 0$; which equation, on account of $dy = pdx$, will change into this $N - \frac{dP}{dx} = 0$, which is that very equation which we have found. And thus the method is desired free from lines and a geometric resolution, for which it may be shown in such an investigation of maxima and minima $-pdP$ must be written in place of Pdp.

CAPUT II

DE METHODO MAXIMORUM AC MINIMORUM AD LINEAS CURVAS INVENIENDAS ABSOLUTA

PROPOSITIO I PROBLEMA

1. *Si in curva* (Fig. 4) *quacunque amz una applicata quaevis Nn augeatur particula infinite parva nv, invenire incrementa vel decrementa, quae singulae quantitates determinatae ad curvam pertinentes hinc accipient.*

SOLUTIO

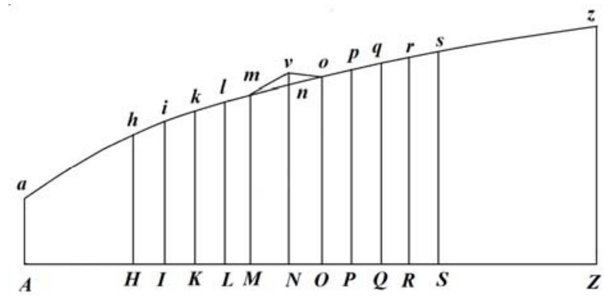

Fig. 4

Quantitates determinatae ad curvam propositam pertinentes sunt, praeter abscissam *x*, quae non afficitur, hae *y*, *p*, *q*, *r*, *s* etc. cum suis derivatis valoribus, quos in locis vel sequentibus vel antecedentibus sortiuntur. Quodsi nunc ponamus $AM = x$ et $Mm = y$, erit $Nn = y^{/}$, huiusque valor per translationem puncti *n* in *v* augebitur particula *nv*, reliquae autem applicatae $y^{//}$, $y^{///}$, y^{IV} etc. pariter ac praecedentes $y_{/}$, $y_{//}$, $y_{///}$, y_{IV} etc. non afficientur. Cum igitur sola applicata $y^{/}$ crescat particula *nv*, ex Capitis praecedentis paragraphis 51 et seqq. colligetur, quantum incrementum reliquae quantitates omnes capiant ex incremento solius applicatae $y^{/}$. Omnes scilicet quantitates, quarum valor pendet ab $y^{/}$, mutationem subibunt, reliquae vero, *quae* ab $y^{/}$ non pendent, manebunt invariatae. Ita cum sit $p = \frac{y^{/} - y}{dx}$, haec quantitas *p* crescet particula $\frac{nv}{dx}$; at cum sit $p^{/} = \frac{y^{//} - y^{/}}{dx}$, haec quantitas $p^{/}$ decrescet particula $\frac{nv}{dx}$. Similique modo reliquarum quantitatum incrementa vel decrementa reperientur delendo in earum valoribus supra exhibitis omnes valores ipsius *y*, praeter hunc $y^{/}$, huiusque loco scribendo *nv*. Hoc modo omnium quantitatum determinatarum, quae quidem mutationem patiuntur, incrementa in sequenti Tabella congessimus:

Quant.	*Increm.*	*Quant.*	*Increm.*
$y^{/}$	$+nv$	$s_{///}$	$+\frac{nv}{dx^4}$
p	$+\frac{nv}{dx}$	$s_{//}$	$-\frac{4nv}{dx^4}$
$p^{/}$	$-\frac{nv}{dx}$	$s_{/}$	$+\frac{6nv}{dx^4}$
$q_{/}$	$+\frac{nv}{dx^2}$	s	$-\frac{4nv}{dx^4}$
q	$-\frac{2nv}{dx^2}$	$s^{/}$	$+\frac{nv}{dx^4}$
$q^{/}$	$+\frac{nv}{dx^2}$	t_{IV}	$+\frac{nv}{dx^5}$
$r_{//}$	$+\frac{nv}{dx^3}$	$t_{///}$	$-\frac{5nv}{dx^5}$
$r_{/}$	$-\frac{3nv}{dx^3}$	$t_{//}$	$+\frac{10nv}{dx^5}$
r	$+\frac{3nv}{dx^3}$	$t_{/}$	$-\frac{10nv}{dx^5}$
$r^{/}$	$-\frac{nv}{dx^3}$	t	$+\frac{5nv}{dx^5}$
		$t^{/}$	$-\frac{nv}{dx^5}$

Atque ex hac Tabella etiam ulteriorum quantitatum, si quae occurrunt, incrementa vel decrementa facile cognosci poterunt. Q.E.I.

COROLLARIUM I

2. Cognitis igitur incrementis harum quantitatum primariarum ad curvam pertinentium, inde omnium quantitatum ex iis compositarum incrementa, quae oriuntur ex aucta applicata $y^{/}$, determinari poterunt, si ratio compositionis spectetur.

COROLLARIUM 2

3. Harum scilicet quantitatum incrementa exhibita, considerari poterunt tanquam earum differentialia. Atque si proposita fuerit quantitas quaecunque ex illis composita, eius conveniens incrementum ex translatione puncti *n* in *v* ortum invenietur differentiando illam quantitatem et loco differentialium singularum quantitatum scribendo ea incrementa, quae his quantitatibus sunt adscripta.

COROLLARIUM 3

4. Si igitur habeatur haec functio $y^{/}\sqrt{(1+pp)}$, cuius incrementum, quod ex translatione puncti *n* in *v* oritur, sit determinandum, ea functio primum differentietur; unde prodibit

$$dy^{/}\sqrt{(1+pp)}+\frac{y^{/}pdp}{\sqrt{(1+pp)}};$$

hicque loco $dy^{/}$ et dp scribantur incrementa quantitatibus $dy^{/}$ et p convenientia, nempe $+nv$ et $+\frac{nv}{dx}$; eritque functionis propositae incrementum

$$=+nv\sqrt{(1+pp)}+\frac{y^{/}p\cdot nv}{dx\sqrt{(1+pp)}}.$$

COROLLARIUM 4

5. Expedite igitur per differentiationem functionis cuiuscunque incrementum, quod ex incremento *nv* applicatae $y^{/}$ oritur, assignari potest; id quod ex inspectione figurae difficulter et minime generaliter fieri potest.

SCHOLION

6. Probe notandum est hunc modum incrementa functionum seu quantitatum ex *x, y, p, q* etc. harumque derivatis $y^{/}$, $y^{//}$, $p^{/}$, $p^{//}$ etc. datarum incrementa inveniendi tantum ad functiones determinatas patere, minime vero ad indeterminatas extendi posse. Quodsi enim functio proposita fuerit indeterminata seu formula integralis indefinita, integrationem neque algebraice neque transcendenter admittens, tum differentiatione nihil consequimur ad eius incrementum inveniendum. In sequentibus autem, ubi eiusmodi maximi minimive formulas $\int Zdx$ sumus contemplaturi, in quibus *Z* sit functio talis indeterminata, in huiusmodi functionum incrementa sumus inquisituri. Sin autem *Z* fuerit functio determinata, propositi Problematis solutio sufficere potest ad solutiones Problematum huc pertinentium absolvendas.

PROPOSITIO II. PROBLEMA

7. *Si fuerit* (Fig. 4) *Z functio determinata ipsarum x et y tantum, invenire curvam az, in qua valor formulae* $\int Zdx$ *sit maximus vel minimus.*

SOLUTIO

Concipiatur abscissa *AZ*, cui maximum minimumve formulae $\int Zdx$ respondere debet, divisa in innumerabilia elementa aequalia, singula per *dx* denotanda; positaque abscissa indefinita $AM = x$ et applicata $Mm = y$, ex formula $\int Zdx$ elemento *MN* respondebit *Zdx;* atque secundum receptum notandi modum elemento sequenti *NO* respondebit $Z^{/}dx$ et sequentibus elementis *OP*, *PQ* etc. respondebunt valores

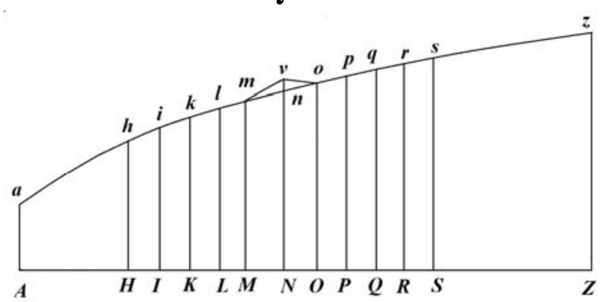

Fig. 4

$Z^{//}dx,\ Z^{///}dx$ etc., antecedentibus vero elementis *LM, KL, IK* etc. respondebunt $Z_{/}dx,\ Z_{//}dx,\ Z_{///}dx$ etc. Quare, si curva *az* sit ea ipsa, quae quaeritur, debebit esse $Zdx + Z^{/}dx + Z^{//}dx +$ etc., una cum $Zdx + Z_{/}dx + Z_{//}dx +$ etc. maximum vel minimum. Quodsi igitur una applicata $Nn = y^{/}$ augeatur particula *nv*, illa expressio eundem valorem retinere atque adeo valor differentialis formulae $\int Zdx$ seu summae terminorum $Zdx + Z^{/}dx + Z^{//}dx + Z^{///}dx +$ etc . una cum $Z_{/}dx + Z_{//}dx + Z_{///}dx +$ etc . evanescere debet. Singulorum igitur horum terminorum valores differentiales, qui oriuntur ex translatione puncti *n* in *v*, investigari debebunt eorumque aggregatum erit valor differentialis formulae $\int Zdx$ respondens, qui positus $= 0$ aequationem pro curva quaesita praebebit. Quoniam autem *Z* ponitur functio determinata ipsarum *x* et *y*, habebit ipsius differentiale *dZ* huiusmodi formam $Mdx + Ndy$; ita ut sit $dZ = Mdx + Ndy$. Valorum igitur derivatorum ipsius *Z* differentialia ita se habebunt:

$dZ^{/} = M^{/}dx + N^{/}dy^{/}.$	$dZ_{/} = M_{/}dx + N_{/}dy_{/}$
$dZ^{//} = M^{//}dx + N^{//}dy^{//}$	$dZ_{//} = M_{//}dx + N_{//}dy_{//}$
etc.	etc.

Cum nunc valores differentiales terminorum $Zdx,\ Z^{/}dx,\ Z^{//}dx$ etc. itemque ipsorum $Z_{/}dx,\ Z_{//}dx$ etc. inveniantur, si hi termini differentientur atque loco $dy^{/}$ in differentialibus scribatur *nv*, loco omnium reliquorum differentialium vero 0, manifestum est solum terminum $Z^{/}dx$ habiturum esse valorem differentialem, quoniam in eius solius differentiali occurrit $dy^{/}$. Scripta itaque *nv* loco $dy^{/}$ erit termini $Z^{/}dx$ valor differentialis $= N' dx \cdot nv$, qui simul erit valor differentialis totius formulae $\int Zdx$, quia reliqui termini praeter $Z' dx$ nullam variationem patiuntur. Loco N' autem ponere poterimus *N*, quia est $N' = N + dN$ et *dN* prae *N* evanescit. Pro curva igitur quaesita, in qua sit $\int Zdx$ maximum vel minimum, ista habetur aequatio $Ndx \cdot nv = 0$ seu $N = 0$; existente $dZ = Mdx + Ndy$. Q. E. I.

COROLLARIUM 1

8. Si igitur curva debeat definiri, in qua sit $\int Zdx$ maximum vel minimum atque *Z* sit functio determinata ipsarum *x* et *y* tantum, tum quantitatem *Z* differentiari oportet; quod cum habiturum sit huiusmodi formam $dZ = Mdx + Ndy$, hinc formabitur aequatio pro curva quaesita, quae erit $N = 0$.

COROLLARIUM 2

9. Cum ergo *N* sit functio ipsarum *x* et *y* determinata, in aequatione pro curva $N = 0$ nulla inerit quantitas constans, quae non fuit in formula maximi minimive $\int Zdx$; et hanc ob rem curva inventa erit unica et perfecte determinata.

COROLLARIUM 3

10. In quaestionibus igitur sub hoc Problemate comprehensis curva satisfaciens ex sola maximi minimive formula determinatur; neque licebit insuper puncta aliqua praescribere, per quae curva quaesita transeat.

COROLLARIUM 4

11. Quodsi *Z* fuerit functio tantum ipsius *x*, ita ut *y* non involvat, erit tum $\int Zdx$ functio determinata pariter ipsius *x* tantum eique adeo omnes curvae eidem abscissae respondentes aeque satisfacient. Idem vero hoc monstrat calculus; hoc enim casu, quo in *Z* non inest *y*, fiet $N = 0$; ideoque nulla prodit aequatio pro curva quaesita.

COROLLARIUM 5

12. Statim etiam intelligi potest, utrum detur linea curva, in qua huiusmodi formula $\int Zdx$ sit maximum vel minimum. Si enim ex differentiatione ipsius *Z* eiusmodi valor pro *N* reperiatur, ut per aequationem $N = 0$ nulla curva exprimatur, tum etiam nulla curva extat, in qua proposita formula $\int Zdx$ sit maximum vel minimum.

COROLLARIUM 6

13. Denique etiam perspicitur hanc maximi minimive proprietatem non uni alicui determinatae abscissae esse adstrictam, sed, si curva pro una abscissa reddat formulam $\int Zdx$ maximum vel minimum, eandem pro quacunque alia abscissa pariter maximum minimumve valorem esse habiturum.

SCHOLION I

14. Nacti ergo sumus methodum facilem inter omnes curvas eidem abscissae respondentes eam determinandi, in qua constituat formula $\int Zdx$ valorem maximum vel minimum, siquidem *Z* est functio determinata ipsarum *x* et *y* tantum. Simul vero etiam patet curvam satisfacientem semper fore algebraicam, siquidem *Z* fuerit functio algebraica ipsarum *x* et *y*. Curvae igitur hoc modo inventae ista erit proprietas, ut, si ad eandem abscissam alia quaecunque constituatur linea curva, tum pro ea valor formulae $\int Zdx$ certo vel minor vel maior sit prodituras quam pro inventa, prout in inventa formula $\int Zdx$ vel fuerit maxima vel minima. Cum autem adhuc dubium sit, utrum in curva inventa valor formulae $\int Zdx$ futurus sit maximus an minimus, de eo in quovis casu particulari facile fiet diiudicatio; in genere autem nihil omnino decidi potest. Interim hoc certum est, si unica prodit aequatio, tum tantum vel maximum vel minimum locum habere posse; hoc est, si curva inventa sit pro maximo, tum minimum non dari, sed valorem formulae $\int Zdx$ in infinitum diminui posse. Pari modo, si unica inventa fuerit curva in eaque formula $\int Zdx$ sit minima, tum valorem $\int Zdx$ in infinitum augeri posse. Quod si autem solutio nullam prorsus praebeat curvam satisfacientem, id indicio erit valorem formulae $\int Zdx$ pro quacunque abscissa tam in infinitum crescere quam decrescere posse.

SCHOLION 2

15. Ex eadem etiam solutione reperiri poterunt illae curvae maximi minimive proprietate praeditae alterius generis supra memoratae, ad quas non pervenitur per valores differentiales evanescentes, sed infinite magnos; quod maximorum et minimorum genus ab illo maxime discrepat. Reperientur autem istae curvae, si valor differentialis $Ndx \cdot nv$ non nihilo, sed infinito aequalis ponatur. Quoties igitur haec aequatio $N = \infty$ lineam aliquam curvam suggerit, tum in ea pariter formula $\int Zdx$ maximum vel minimum ohtinebit valorem: Hoc scilicet eveniet, quando pro *N* prodit fractio, cuius denominator nihilo aequalis positus praebet aequationem pro aliqua linea curva. Hoc itaque pacto plures curvae reperiri possunt, quae eidem quaestioni satisfaciant, quarum aliae maxima continebunt, aliae minima. Fieri etiam potest, ut plures quam duae curvae Problemati satisfacientes reperiantur, etiamsi binae tantum oriri queant aequationes, scilicet $N = 0$ et

$N = \infty$. Si enim *N* fuerit quantitas ex factoribus composita, tum quilibet factor vel nihilo vel infinito aequalis positus dabit aequationem pro curva satisfaciente; constat enim saepenumero plura maxima pluraque minima locum habere posse. Haec autem omnia clarius enodabuntur in sequentibus Exemplis in hoc Problemate contentis.

EXEMPLUM I

16. *Invenire curvam, quae inter omnes omnino curvas eidem abscissae respondentes habeat* $\int XYdx$ *maximum vel minimum, denotante X functionem ipsius x et Y ipsius y tantum.*

In hoc igitur casu fiet $Z = XY$, ideoque $dZ = YdX + XdY = Mdx + Ndy$.
Erit ergo

$$M = \frac{YdX}{dx} \text{ et } N = \frac{XdY}{dy}$$

ob *X* ipsius *x* et *Y* ipsius *y* functionem. Pro curva igitur quaesita erit

$$N = \frac{XdY}{dy} = 0\,;$$

quoniam autem *Y* est functio ipsius *y*, ponatur $dY = \Theta dy$; erit Θ pariter functio ipsius *y;* ideoque pro curva quaesita, si quae satisfacit, habetur haec aquatio $X\Theta = 0$, ideoque vel $X = 0$ vel $\Theta = 0$; quarum cum neutra lineam curvam praebeat, apparet huic quaestioni nullam omnino curvam satisfacere, sed valorem propositum $\int XYdx$ in infinitum cum augeri tum diminui posse. Ex aequatione autem $\Theta = 0$, quia Θ est functio ipsius *y*, sequitur $y = Const.$, quae aequatio praebet lineam rectam parallelam abscissae *AZ*, cuius distantia tanta est, ut fiat functio *Y* maxima vel minima. Patet enim, si quantitas *Y* maximum minimumve valorem admittat, tum etiam formulam $\int XYdx$ fieri maximum vel minimum. Altera autem aequatio $X = 0$, quia praebet $x = Const.$, nequidem lineam rectam quaestioni satisfacientem exhibet, quia praebet lineam rectam normalem ad abscissam, quae propterea non datae abscissae cuipiam, sed tantum eius uni puncto respondebit.

EXEMPLUM II

17. *Invenire curvam, quae inter omnes eidem abscissae respondentes curvas habeat valorem formulae* $\int (ax - yy)ydx$ *maximum vel minimum.*

Si haec formula cum generali $\int Zdx$ comparetur, fiet $Z = axy - y^3$, ideoque $dZ = aydx + (ax - 3yy)dy$; ita ut fiat $M = ay$ et $N = ax - 3yy$; unde pro curva quaesita habebitur ista aequatio $ax - 3yy = 0$ seu $yy = \frac{1}{3}ax$, quae est pro Parabola verticem in *A*, axem *AZ* et parametrum $= \frac{1}{3}a$ habente. In hac igitur Parabola erit valor formulae

$\int (ax-yy)ydx$ maximus vel minimus. Utrum autem sit maximus an minimus, reperietur, si aliam quamcunque lineam loco Parabolae substituamus atque inquiramus, utrum pro ea valor formulae propositae maior sit an minor quam pro Parabola. Sumamus igitur lineam rectam cum ipso axe congruentem, pro qua erit $y=0$. Pro hac itaque valor formulae $\int (ax-yy)ydx$ fiet pariter $=0$, pro Parabola autem idem valor erit affirmativus ideoque > 0; ex quo sequitur in Parabola formulae propositae valorem non esse minimum, sed maximum. Poterimus autem algebraice indicare, quantus futurus sit valor formulae propositae pro Parabola; cum enim sit $yy=\frac{1}{3}ax$, abibit formula proposita in hanc

$$\int \tfrac{2}{3}axdx\sqrt{\tfrac{1}{3}ax} = \tfrac{4}{15}ax^2\sqrt{\tfrac{1}{3}ax}.$$

Quodsi autem ponamus aliam aequationem, puta $y=nx$, abibit formula proposita in hanc

$$\int dx(naxx-n^3x^3) = \tfrac{1}{3}nax^3 - \tfrac{1}{4}n^3x^4,$$

quae semper est minor quam valor formulae, qui pro Parabola inventa prodiit, id quod quilibet facile substituendis loco x definitis valoribus experietur.

EXEMPLUM III

18. *Invenire curvam, in qua sit inter omnes omnino curvas ad eandem abscissam relatas valor huius formulae*

$$\int (15a^2x^2y-15a^3xy+5a^2y^3-3y^5)dx$$

maximus vel minimus.

Erit igitur $Z=15a^2x^2y-15a^3xy+5a^2y^3-3y^5$, qui si differentietur, posito x constante, prodibit

$$N=15\left(a^2x^2-a^3x+a^2y^2-y^4\right);$$

hincque qui valor positus $=0$ dabit aequationem pro curva quaesita; erit itaque

$$aaxx-a^3x+a^2y^2-y^4=0=\left(ax-yy\right)(ax+yy-aa).$$

Ob binos hos factores prodeunt binae curvae satisfacientes, quarum altera exprimetur hac aequatione $yy=ax$, altera hac $yy=aa-ax$; utraque pro Parabola. Ut nunc appareat, utra sit pro maximo vel minimo, ponamus abscissam esse minimam, ac prior aequatio $yy=ax$ in formula substituta dabit

$$\int -10a^3\, x\sqrt{ax}dx.$$

Altera vero formula $yy = aa - ax$ seu $y = a$ substituta dabit $\int 2a^5 dx$. Quodsi autem ipsi y alius quicunque valor tribuatur, puta $y = 0$, tum formula proposita abit in $\int 0dx = 0$. Ex quo patet curvarum inventarum alteram $yy = aa - ax$ esse pro maximo, alteram autem $yy = ax$ pro minimo, scilicet pro maximo negativo. Facillime autem perpetuo haec diiudicatio, utrum maximum an minimum in curva inventa locum habeat, instituetur, si abscissa x ponatur infinite parva; tum enim integratione non erit opus, sed ipsa formula Zdx monstrabit valorem formulae $\int Zdx$ hoc casu.

EXEMPLUM IV

19. *Inter omnes curvas eidem abscissae respondentes definire eam, in qua sit formulae*

$$\int (3ax - 3xx - yy)(ax - xx - \tfrac{4}{3}xy + yy)dx$$

valor maximus vel minimus.

Ex hac igitur formula prodibit sequens ipsius Z valor evolutus:

$$\begin{aligned} Z = &+ 3a^2x^2 - 4ax^2y + 2axyy + \tfrac{4}{3}xy^3 - y^4 \\ &- 6ax^3 + 4x^3y - 2xxyy + 3x^4 \end{aligned}$$

quae differentiata posito x constante ac divisa per dy sequentem praebebit valorem pro N

$$N = -4ax^2 + 4axy + 4xyy - 4y^3 + 4x^3 - 4xxy,$$

quae expressio nihilo aequalis posita dabit aequationem pro curva quaesita.
Erit itaque

$$y^3 - xyy + xxy + axx - axy - x^3 = 0,$$

quae duos habet factores, qui totidem praebent aequationes, hasce

I. $y - x = 0$ pro linea recta,
II. $yy - ax + xx = 0$ pro circulo.

Ponatur x infinite parva eritque ex aequatione $y = x$ valor ipsius $Z = 3a^2x^2$; at ex aequatione $yy = ax - xx$ seu $y = \sqrt{ax}$ erit $Z = 4aaxx$. Quodsi autem ponatur $y = a$, prodit $Z = -a^4$, unde apparet utramque lineam inventam esse pro maximo.

SCHOLION

20. Problemata etiam resolvi possunt per Methodum maximorum et minimorum vulgarem. Quando enim curva quaeritur, pro qua valor ipsius $\int Zdx$ sit maximus vel minimus idque pro qualibet abscissa, manifestum est, siquidem *Z* sit functio determinata ipsarum *x* et *y*, formulam $\int Zdx$ maximum minimumve esse non posse, nisi elementum eius *Z dx* ac proinde ipsum *Z* tale sit. Quamobrem quaestioni satisfiet, si quantitas *Z* differentietur posito *x* constante eiusque differentiale ponatur $=0$. Tum enim perpetuo *Z* habebit valorem maximum vel minimum, ac proinde etiam *Zdx* et ipsa formula $\int Zdx$. Quodsi autem functio *Z* differentietur posito *x* constante, prodibit *Ndy*, quoniam generaliter differentiando posuimus $dZ = Mdx + Ndy$; satisfietque ponendo $N = 0$, quae est eadem solutio, quam per Methodum traditam invenimus. Quamvis autem hinc videantur istae quaestiones simili modo resolvi posse, quo in Methodo maximorum et minimorum vulgari, tamen hoc tantum evenit, si *Z* fuerit functio ipsarum *x* et *y* tantum; namque, si in *Z* praeterea insint quantitates ex differentialibus ortae *p, q, r* etc., tum vulgaris Methodus nullius amplius usus esse potest. Etsi enim tum differentietur functio *Z* posito *x* constante, tamen in differentiale etiam ingrederentur differentialia *dp, dq, dr* etc., quorum relatio ad *dy* cum non constet, aequatio inde ad maximum minimumve determinandum apta deduci non poterit. His igitur casibus utilitas et necessitas nostrae Methodi maxime cernetur.

PROPOSITIO III. PROBLEMA

21. *Si Z* (Fig. 4) *fuerit functio ipsarum x, y et p determinata. ita ut sit*

$$dZ = Mdx + Ndy + Pdp,$$

invenire inter omnes curvas eidem abscissae respondentes eam, in qua sit $\int Zdx$ *maximum vel minimum.*

SOLUTIO

Sit *amz* curva quaesito satisfaciens atque concipiatur applicata quaecunque $Nn = y^{\prime}$ augeri particula *nv*, debebit valor differentialis formulae $\int Zdx$ seu quantitatis huic aequivalentis, puta $Zdx + Z^{\prime}dx + Z^{\prime\prime}dx + Z^{\prime\prime\prime}dx + \text{etc.}$ una cum $Z_{\prime}dx + Z_{\prime\prime}dx + Z_{\prime\prime\prime}dx + \text{etc.}$, esse $=0$. Totius igitur quantitatis $\int Zdx$ valor differentialis ex translatione puncti *n* in *v* habebitur, si singulorum illorum terminorum, qui quidem hac translatione afficiuntur, valores differentiales quaerantur et in unam summam addantur. Ex translatione autem puncti *n* in *v* illi tantum termini mutationem subeunt, in quibus insunt quantitates $y^{\prime}$, p et $p^{\prime}$ ideoque tantum termini *Z dx* et $Z^{\prime}dx$; nam uti *Z* est functio ipsarum *y* et *p* praeter *x*, ita $Z^{\prime}$ similis est functio ipsarum $y^{\prime}$ et $p^{\prime}$. Quamobrem hi termini debebunt differentiari, atque in eorum differentialibus loco dy', dp et dp' scribi oportet valores supra indicatos $+nv$, $+\frac{nv}{dx}$ et $-\frac{nv}{dx}$. Sicut autem est $dZ = Mdx + Ndy + Pdp$, ita erit $dZ' = M'dx + N'dy' + P'dp'$. Hinc itaque valor differentialis ipsius Z erit $P \cdot \frac{nv}{dx}$ et ipsius Z' erit $N' \cdot nv - P' \cdot \frac{nv}{dx}$; ex quo utriusque termini $Zdx + Z'dx$ ideoque integrae formulae $\int Zdx$ valor differentialis erit $= nv \cdot (P + N'dx - P')$. At est $P' - P = dP$ et loco N' scribi potest N; unde valor differentialis erit $= nv \cdot (Ndx - dP)$. Quare, cum formulae $\int Zdx$ valor differentialis nihilo aequalis factus praebeat aequationem pro curva quaesita, haec erit $0 = Ndx - dP$ vel $N - \frac{dP}{dx} = 0$, qua aequatione natura curvae quaesitae exprimetur. Q. E. I.

COROLLARIUM 1

22. Quodsi ergo fuerit Z functio quaecunque ipsarum x, y itemque earum differentialium dx et dy seu loco horum differentialium ipsius p, existente $dy = pdx$, differentiale ipsius Z huiusmodi habebit formam, ut sit $dZ = Mdx + Ndy + Pdp$. Atque hinc reperietur curva, in qua sit $\int Zdx$ maximum vel minimum, formando hanc aequationem $N - \frac{dP}{dx} = 0$ seu $Ndx = dP$.

COROLLARIUM 2

23. Aequatio haec igitur semper erit differentialis secundi gradus, nisi in P plane non insit p. Nam si p continetur in P, tum in dP inerit dp, quod ob $p = \frac{dy}{dx}$ differentialia secundi gradus involvet.

COROLLARIUM 3

24. Quando ergo in differentiali ipsius $dZ = Mdx + Ndy + Pdp$ quantitas P adhuc in se complectitur p, tum ob aequationem pro curva quaesita differentialem secundi gradus duae novae constantes arbitrariae per integrationem ingredientur. Ex quo ad harum constantium determinationem duo curvae puncta praescribi poterunt; alias enim non una, sed innumerabiles curvae reperirentur.

COROLLARIUM 4

25. Ut itaque huiusmodi Problemata determinate proponantur, ita sunt enuncianda, ut per data duo puncta curva duci debeat, quae inter omnes alias curvas per eadem puncta ductas pro eadem abscissa x valorem $\int Zdx$ maximum minimumve complectatur.

COROLLARIUM 5

26. In P autem quantitas p non inerit, si Z fuerit functio ipsarum x et y tantum per p vel per $n + p$, denotante n numerum constantem, multiplicata. Sit enim V functio ipsarum x et y tantum, ita ut sit $dV = Mdx + Ndy$ atque $Z = V(n + p)$, erit

$$dZ = (n+p)Mdx + (n+p)Ndy + Vdp.$$

Hincque aequatio pro curva quaesita erit

$$0 = \left(n+p\right)N - \frac{dV}{dx} \quad \text{seu} \quad \left(n+p\right)Ndx = dV = Mdx + Ndy.$$

COROLLARIUM 6

27. His igitur casibus, quibus est $Z = V\left(n+p\right)$ existente V functione ipsarum x et y tantum, non pervenitur ad aequationem differentialem secundi gradus, quia dp in ea prorsus non inest. Verum nequidem ad differentialem aequationem primi gradus pervenitur, sed adeo ad algebraicam. Nam cum sit $pdx = dy$, erit $(n+p)Ndx = nNdx + Ndy$; quod ipsi $Mdx + Ndy$ aequale positum dabit aequationem per dx divisibilem adeoque algebraicam hanc $nN = M$, siquidem V fuerit functio algebraica.

COROLLARIUM 7

28. Quoties autem hoc evenit, maximi minimive formula, quae est $\int Zdx$, erit talis formae $\int (Vndx + Vdy)$ vel posito $n = 0$ talis $\int Vdy$. Huiusmodi igitur maximi minimive formulae pariter aequationem determinatam pro curva quaesita deducunt, ita ut non liceat unum plurave puncta praescribere, per quae curva transire debeat.

COROLLARIUM 8

29. Posita igitur V functione ipsarum x et y, ista maximi minimive formula $\int Vdy$ pari modo tractatur, quo $\int Vdx$. Nam posito $dV = Mdx + Ndy$ formulae $\int Vdx$ respondet

aequatio pro curva haec $N=0$, ita alteri formulae $\int Vdy$ respondet aequatio $M=0$. Ex quo perspicuum est coordinatas *x* et *y* inter se commutari posse.

SCHOLION 1

30. Apparet itaque in solutione huiusmodi Problematum, quibus quaeritur curva valorem formulae $\int Zdx$ maximum minimumve habens existente *Z* functione ipsarum *x*, y et *p*, perveniri ad aequationem differentialem secundi gradus, nisi in *Z* quantitas *p* unicam tantum habeat dimensionem. Saepe numero autem ista aequatio differentialis secundi gradus integrationem admittit, de quo in singulis casibus erit videndum. Interim hic annotasse iuvabit generaliter integrationem succedere, si in functione *Z* omnino non insit *x*, hoc est, si in eius differentiali $dZ=Mdx+Ndy+Pdp$ valor *M* evanescat, ita ut sit tantum $dZ=Ndy+Pdp$. Cum enim pro curva inventa sit haec aequatio $N-\frac{dP}{dx}=0$, multiplicetur ea per *dy*, et quia est $dy=pdx$, ea abibit in hanc $Ndy-pdP=0$, cui aequivalet ista $Ndy+Pdp=Pdp+pdP=dZ$, cuius integrale est $Z+C=Pp$, quae aequatio iam tantum est differentialis primi gradus. Quoties ergo inter omnes curvas eidem abscissae respondentes ea quaeritur, in qua sit valor formulae $\int Zdx$ maximus vel minimus atque *Z* tantum sit functio ipsarum *y* et *p*, ita ut sit $dZ=Ndy+Pdp$, tum pro curva satisfaciente statim exhiberi poterit aequatio differentialis primi gradus ista $Z+c=Pp$. Deinde vero etiam, si *Z* fuerit functio ipsarum *x* et *p* tantum atque $dZ=Mdx+Pdp$ evanescente termino *Ndy*, tum pro curva prodibit aequatio differentialis primi gradus. Nam ob $dP=0$ erit $P=C$, quae pro curva quaesita dabit aequationem differentialem primi gradus tantum. Quodsi autem insuper *M* evanescat seu *Z* functio sit ipsius *p* tantum et $dZ=Pdp$, aequatio inventa $P=C$ transmutabitur in istam $Pdp=Cdp=dZ$, quae denuo integrata dat $Z+D=Cp$. Hoc autem casu, quia *Z* et *P* sunt functiones ipsius *p* tantum, utraque aequatio $P=C$ et $Z+D=Cp$ praebebit pro *p* valorem constantem ideoque aequationem huius formae $dy=ndx$, quae indicat huiusmodi Problematis satisfacere lineas rectas et quidem quascunque utlibet ductas. Nam in aequatione $P=C$, cum *C* sit constans arbitraria, valor ipsius *p* non solum constans, sed etiam arbitrarius evadet; ex quo linea recta quaecunque resultabit. Quamobrem, si per data duo puncta curva duci debeat, in qua sit $\int Zdx$ maximum vel minimum, ac *Z* sit functio ipsius *p* tantum, tum satisfaciet linea recta per illa data duo puncta ducta.

SCHOLION 2

31. Quoniam supra iam vidimus in huiusmodi Problematis coordinatas *x* et *y* inter se commutari atque, si commodum videatur, applicatam *y* tanquam abscissam tractari posse, idem hoc quoque casu confirmari iuvabit. Sit igitur curva investiganda, in qua sit $\int Zdy$ maximum vel minimum existente *Z* functione ipsarum *x*, *y* et *p* et

$$dZ=Mdx+Ndy+Pdp.$$

Haec autem formula $\int Zdy$ ad nostram formam reducta abit in $\int Zpdx$, in qua erit

$$d\cdot Zp = Mpdx + Npdy + (Z+Pp)dp\,,$$

ex qua formulae propositae valor differentialis respondens erit

$$(Npdx - dZ - Pdp - pdP)nv = (-Mdx - 2Pdp - pdP)nv$$

et aequatio pro curva quaesita erit

$$0 = -Mdx - 2Pdp - pdP \text{ seu } 0 = Mdy - d\cdot Pp^2.$$

Quodsi nunc ad similitudinem ostendendam, quia hic y tanquam abscissam consideramus, ponamus $dx = \pi dy$, erit

$$p = \frac{1}{\pi} \text{ et } dp = -\frac{d\pi}{\pi\pi} = -ppd\pi\,;$$

erit

$$dZ = Mdx + Ndy - Pppd\pi = Mdx + Ndy + \Pi d\pi$$

ponendo $\Pi = -Ppp$, ut similitudo terminorum conservetur. Quapropter aequatio pro curva erit $0 = -Mdy + d\Pi$; quae eadem aequatio prodiisset, si in formula $\int Zdx$ applicata y in abscissam et vicissim abscissa in applicatam transmutetur. Proposita igitur quacunque formula indeterminata ex x et y horumque differentialibus composita, quae debeat esse maxima vel minima, coordinatarum x et y utramlibet licebit tanquam abscissam tractare ad eamque maximum minimumve accommodare.

EXEMPLUM I

32. *Inter omnes curvas ad eandem abscissam relatas eam determinare, in qua sit* $\int (Zdx + [Z]dy)$ *maximum vel minimum; existentibus Z et* $[Z]$ *functionibus quibuscunque ipsarum x et y, ita ut sit* $dZ = Mdx + Ndy$ et $d[Z] = [M]dx + [N]dy$.

Ut formula haec $\int (Zdx + [Z]dy)$ ad formam receptam reducatur, ponatur pdx loco dy habebiturque haec formula $\int (Z + [Z]p)dx$ dx maxima minimave efficienda. Differentietur ergo valor $Z + [Z]p$ eritque eius differentiale

$$= +Mdx + Ndy + [M]pdx + [N]pdy + [Z]dp\,.$$

Iam per regulam inventam hinc pro curva quaesita ista prodibit aequatio

$$0=(N+[N]p)dx-d[Z]=(N+[N]p)dx-[M]dx-[N]dy,$$

quae, ob $[N]pdx=[N]dy$, per *dx* divisa dabit hanc aequationem pro curva quaesita algebraicam seu finitam $N-[M]=0$ seu $N=[M]$. Hinc intelligitur, si formula proposita $\int(Zdx+[Z]dy)$ fuerit determinata seu differentiale $Zdx+[Z]dy$ ita comparatum, ut integrationem admittat, tum nullam lineam quaesito esse satisfacturam, seu potius omnes lineas aeque satisfacere. Nam si $Zdx+[Z]dy$ integrationem admittit, per se erit $N=[M]$, uti alibi de formulis differentialibus duarum variabilium determinatis demonstravimus; ideoque his casibus prodit aequatio identica $0=0$. Hincque luculenter intelligitur, quod iam ante notavimus, maximi minimive formulam oportere esse formulam indeterminatam; alioquin enim omnes lineae curvae aeque satisfacerent.

EXEMPLUM II

33. *Inter omnes lineas ad eandem abscissam relatas determinare eam, cuius longitudo sit minima seu in qua sit* $\int dx\sqrt{(1+pp)}$ *minimum.*

Primum quidem apparet in hac quaestione maximum non dari, cum linearum longitudo in infinitum augeri queat, manente abscissa eadem. Ita minimum tantum habebit locum, id quod ex ipsa Geometria elementari constat, in qua demonstratur lineam rectam inter omnes alias lineas intra eosdem terminos sitas esse brevissimam. Hoc igitur Exemplum ideo attulisse visum est, cum ut consensus nostrae Methodi cum veritate aliunde iam cognita intelligatur, tum etiam, ut circumstantia de duobus punctis arbitrariis, quae ad huius generis quaestiones addi debet, melius percipiatur. Erit igitur, formula $\int dx\sqrt{(1+pp)}$ cum generali $\int Zdx$ comparata,

$$Z=\sqrt{(1+pp)} \text{ et } dZ=\frac{pdp}{\sqrt{(1+pp)}};$$

unde fit $M=0, N=0$ et $P=\frac{p}{\sqrt{(1+pp)}}$. Quare, cum in genera aequatio pro linea quaesita sit $N-\frac{dP}{dx}=0$, habebimus hoc casu $dP=0$ ideoque

$$P=\frac{p}{\sqrt{(1+pp)}}=Const.,$$

ex qua aequatione oritur $p=Const.=n$ seu $dy=ndx$, quae denuo integrata dat $y=a+nx$. Non solum ergo patet lineam quaesitam esse rectam, sed etiam, ob duas arbitrarias constantes *a* et *n*, rectam utcunque ductam. Quare, si per data duo puncta linea duci iubeatur brevissima, erit illa recta. Similiter autem intelligitur, si linea debeat inveniri, in qua sit $\int Zdx$, ubi Z est functio ipsius p tantum, maximum vel minimum, tum lineam rectam tantum satisfacere; uti ante iam notavimus.

EXEMPLUM III

34. *Inter omnes curvas ad eandem abscissam relatas determinare eam, in qua sit*

$$\int \frac{dx\sqrt{(1+pp)}}{\sqrt{x}}$$

maximum vel minimum.

Haec formula oritur, si quaeratur linea celerrimi descensus in hypothesi gravitatis uniformis, ponendo axem in quo abscissae capiuntur verticalem. Erit igitur

$$Z = \frac{\sqrt{(1+pp)}}{\sqrt{x}} \text{ et } dZ = -\frac{dx\sqrt{(1+pp)}}{2x\sqrt{x}} + \frac{pdp}{\sqrt{x(1+pp)}};$$

unde fit

$$M = -\frac{\sqrt{(1+pp)}}{2x\sqrt{x}},\ N = 0 \text{ et } P = \frac{p}{\sqrt{x(1+pp)}}.$$

Cum iam curva quaesta exprimatur aequatione $N - \frac{dP}{dx} = 0$ erit $dP = 0$ et

$$P = \frac{p}{\sqrt{x(1+pp)}} = Const. = \frac{1}{\sqrt{a}};$$

quaere ducta praebet $ap^2 = x + p^2x$ et $p = \frac{dy}{dx} = \sqrt{\frac{x}{a-x}}$ seu $y = \int dx\sqrt{\frac{x}{a-x}}$,

quae aequatio indicat curvam quaesitam esse Cycloidem super basi horizontali natam et cuspidem in suprema axis regione habentem, quae adeo per data duo quaecunque puncta duci poterit.

EXEMPLUM IV

35. *Inter omnes curvas eidem abscissae respondentes eam determinare, in qua sit* $\int y^n dx\sqrt{(1+pp)}$ *maximum vel minimum.*

Pro hac ergo formula proposita erit

$$Z = y^n\sqrt{(1+pp)} \text{ et } dZ = ny^{n-1}dy\sqrt{(1+pp)} + \frac{y^n pdp}{\sqrt{(1+pp)}};$$

ita ut fiat

$$M=0 \text{ et } N=ny^{n-1}\sqrt{(1+pp)} \text{ atque } P=+\frac{y^n p}{\sqrt{(1+pp)}}.$$

Quoniam igitur est $M=0$, statim pro curva quaesita habebitur ista aequatio semel iam integrata $Z+C=Pp$ (§ 30), quae nostro casu fit

$$y^n\sqrt{(1+pp)} \ +ma^n=\frac{y^n pp}{\sqrt{(1+pp)}}.$$

Quodsi ponatur constans $a=0$, prodibit $1+pp=pp$ seu $p=\infty$ satisfacietque linea recta normalis ad axem. Generatim vero lineae satisfacientes reperientur ex aequatione, quae abit in

$$y^n+ma^n\sqrt{(1+pp)}=0 \text{ seu } y^{2n}=m^2a^{2n}+m^2a^{2n}p^2\ ;$$

quae dat

$$p\ (=\frac{dy}{dx})=\frac{\sqrt{(y^{2n}-\ m^2a^{2n})}}{ma^n} \text{ et } x=\int\frac{ma^n dy}{\sqrt{(y^{2n}-\ m^2a^{2n})}}\ ;$$

quae linea per data duo puncta duci potest. Si fuerit $n=-\frac{1}{2}$, ita ut $\int\frac{dx\sqrt{(1+pp)}}{\sqrt{y}}$ debeat esse maximum vel minimum, pariter prodire debet linea brachystochrona ad axem horizontalem relata eritque pro ea $x=\int dy\sqrt{\frac{y}{a-y}}$; quae cum praecedente omnino congruit, dummodo coordinatae *x* et *y* inter se commutentur. Erit scilicet, ut ante, curva satisfaciens Cyclois super basi horizontali rotando generata, qualem per data duo quaecunque puncta ducere licet.

EXEMPLUM V

36. *Inter omnes curvas eidem abscissae respondentes eam determinare, in qua sit* $\int\frac{ydy^3}{dx^2+dy^2}$ *maximum vel minimum.*

Formula haec ad formam consuetam ope substitutionis $dy=pdx$ reducta abit in hanc

$$\int\frac{yp^2dx}{1+pp};$$

eaque reperiri solet, si quaeratur solidum rotundum rotatione curvae circa axem ortum, quod secundum axis directionem in fluido motum minimam patiatur resistentiam; resistentia namque hoc casu proportionalis censetur formulae

$$\int \frac{ydy^3}{dx^2+dy^2} \quad \text{seu} \quad \int \frac{yp^2dx}{1+pp}.$$

Erit ergo

$$Z = \frac{yp^3dx}{1+pp} \ \text{ et } \ dZ = \frac{p^3dy}{1+pp} + \frac{ydp(3pp+p^4)}{(1+pp)^2};$$

ita ut fiat

$$M = 0, \quad N = \frac{p^3}{1+pp} \ \text{ et } \ P = \frac{p^2y(3+pp)}{(1+pp)^2} \ ;$$

Cum igitur sit $M = 0$, una integratio generaliter succedit eritque aequatio pro curva quaesita $Z + C = Pp$ seu

$$\frac{yp^3}{1+pp} + a = \frac{p^3y(3+pp)}{(1+pp)^2},$$

quae abit in hanc $a(1+pp)^2 = 2p^3y$. Huius aequationis autem evolutio non ita potest institui, ut eliminetur *p;* quare conveniet utramque coordinatam *y* et *x* per eandem variabilem *p* definiri. Ac primo quidem est

$$y = \frac{a(1+pp)^2}{2p^3}.$$

Deinde ob $dy = pdx$ erit

$$dx = \frac{dy}{p} \ \text{ et } \ x = \int \frac{dy}{p} = \frac{dy}{p} + \int \frac{ydp}{pp}.$$

Quodsi ergo loco *y* valor inventus substituatur, prodibit

$$x = \frac{a(1+pp)^2}{2p^4} + a\int \frac{dp(1+pp)^2}{2p^5} = \frac{a}{2}\left(\frac{3}{4p^4} + \frac{1}{pp} + 1 + lp\right),$$

ex quibus curvae constructio poterit confici, logarithmis in subsidium vocandis.

EXEMPLUM VI

37. *Invenire curvam, in qua ista formula* $\int yxdx\sqrt{(1+pp)}$ *sit maximum minimumve.*

Erit ergo $Z = yx\sqrt{(1+pp)}$ atque

$$dZ = ydx\sqrt{(1+pp)} + xdy\sqrt{(1+pp)} + \frac{yxpdp}{\sqrt{(1+pp)}}.$$

Hanc ob rem habebitur

$$M = ydx\sqrt{(1+pp)},\; N = x\sqrt{(1+pp)},\; \text{et } P = \frac{yxp}{\sqrt{(1+pp)}};$$

unde aequatio pro curva formabitur haec $Ndx = dP$, quae suggerit

$$xdx\sqrt{(1+pp)} = \frac{p^2xdx + yxpdp}{\sqrt{(1+pp)}} + \frac{yxdp}{(1+pp)^{\frac{3}{2}}}$$

seu

$$xdx - ydy = \frac{yxdp}{1+pp},$$

ob $dy = pdx$. Haec est aequatio differentialis secundi gradus et, quanquam ope idonearum substitutionum ea ad formam simpliciter differentialem reduci potest, eo quod variabiles x et y ubique eundem dimensionum numerum constituunt, tamen aequatio ista differentialis ita est comparata, ut neque integrari neque separari possit; deduci scilicet potest ad aequationem huius formae

$$\frac{du}{u^3} + \frac{dv}{v^3} = \frac{vdv(1+u^3)}{u^3}.$$

Quod cum ita sit, neque aequatio inventa $xdx - ydy = \dfrac{yxdp}{1+pp}$ ad formam vel simpliorem vel commodiorem revocari potest; hincque nihil admodum de natura curvae inventae iudicare licet. Interim tamen illa aequatio potentia duas arbitrarias constantes involvit, ex quo curva satisfaciens per bina puncta data duci potest.

EXEMPLUM VII

38. *Invenire curvam, in qua sit* $\int (xx+yy)^n dx\sqrt{(1+pp)}$ *maximum vel minimum.*

Cum hic sit $Z=(xx+yy)^n dx\sqrt{(1+pp)}$, erit

$$dZ=2n(xx+yy)^{n-1}(xdx+ydy)\sqrt{(1+pp)}+\frac{(xx+yy)^n pdp}{\sqrt{(1+pp)}},$$

ergo

$$N=2n(xx+yy)^{n-1}y\sqrt{(1+pp)} \text{ et } P=\frac{(xx+yy)^n p}{\sqrt{(1+pp)}};$$

ex quo pro curva quaesita ista habebitur aequatio

$$2n(xx+yy)^{n-1}ydx\sqrt{(1+pp)}=d\frac{(xx+yy)^n p}{\sqrt{(1+pp)}}=\frac{2n(xx+yy)^{n-1}p(xdx+ydy)}{\sqrt{(1+pp)}}+\frac{dp(xx+yy)^n}{(1+pp)^{\frac{3}{2}}},$$

quae per $(xx+yy)^{n-1}$ divisa ac per $\sqrt{(1+pp)}$ multiplicata abit in

$$2nydx=2nxdy+\frac{(xx+yy)dp}{(xx+yy)} \text{ seu } \frac{2n(ydx-xdy)}{xx+yy}=\frac{dp}{1+pp}.$$

Huius aequationis utrumque membrum integrabile est per quadraturam circuli, fitque integrale

$$2n\text{Atang}\frac{x}{y}=\text{Atang } p+\text{Atang } k=\text{Atang } \frac{p+k}{1-pk},$$

unde fiet

$$\frac{x}{y}=\text{tang }\frac{1}{2n}\text{Atang }\frac{k+p}{1-kp}=T;$$

eritque T functio algebraica ipsius p, dummodo sit $2n$ numerus rationalis.
Cum ergo $x=Ty$ seu $dy=pdx=\frac{dx}{T}-\frac{xdT}{TT}$ sive

$$xdT=Tdx-pTTdx;$$

ideoque

$$\frac{dx}{x}=\frac{dT}{T-pTT}+\frac{Tdp}{1-pT}-\frac{Tdp}{1-pT};$$

unde prodit

$$lx = l\frac{T}{1-pT} - \int\frac{Tdp}{1-pT},$$

quae quidem ad construendam curvam abunde satisfaciunt. Verum ut harum curvarum, quae pro definitis exponentis n valoribus prodeunt, natura melius cognoscatur, Casus nonnullos contemplabimur.

I. Sit $n = \frac{1}{2}$ et $2n = 1$; erit $\text{Atang}\frac{x}{y} = \text{Atang}\frac{k+p}{1-kp}$, ideoque

$$\frac{x}{y} = \frac{k+p}{1-kp} = \frac{kdx+dy}{dx-kdy},$$

seu

$$xdx - kxdy = kydx + ydy;$$

quae integrata praebet

$$x^2 - y^2 = 2kxy + C;$$

quae est aequatio pro Hyperbola aequilatera.

II. Sit $n = 1$ et $2n = 2$; erit $2\text{Atang}\frac{x}{y} = \text{Atang}\frac{k+p}{1-kp}$; seu

$$\text{Atang}\frac{2xy}{yy-xx} = \text{Atang}\frac{k+p}{1-kp};$$

unde fit

$$\frac{2xy}{yy-xx} = \frac{kdx+dy}{dx-kdy}$$

seu

$$2xydx - 2kxydy = kyydx - kxxdx + yydy - xxdy;$$

quae integrata dat

$$yx^2 = ky^2x - \tfrac{1}{3}kx^3 + \tfrac{1}{3}y^3 + C \quad \text{sive} \quad y^3 + 3ky^2x - 3yx^2 - kx^3 = C.$$

III. Sit $n = \frac{3}{2}$ seu $2n = 3$; erit

$$3\text{Atang}\frac{x}{y} = \text{Atang}\frac{3y^2x - x^3}{y^3 - 3yx^2} = \text{Atang}\frac{kdx + dy}{dx - kdy};$$

hincque

$$3y^2xdx - 3ky^2xdy - x^3dx + kx^3dy = ky^3dx + y^3dy - 3kyx^2dx - 3yx^2dy;$$

quae integrata dat

$$\tfrac{3}{2}y^2x^2 - ky^3x - \tfrac{1}{4}x^4 + kyx^3 - \tfrac{1}{4}y^4 = C$$

seu

$$y^4 + 4y^3x - 6y^2x^2 - 4kyx^3 + x^4 = C.$$

Ex his iam casibus colligi poterit aequatio integralis pro valore quocunque ipsius *n*. Cum enim sit

$$2n\text{Atang}\frac{x}{y} = \text{Atang}\frac{2ny^{2n-1}x - \frac{2n(2n-1)(2n-2)}{1\cdot 2\cdot 3}y^{2n-3}x^3 + \text{ etc.}}{y^{2n} - \frac{2n(2n-1)}{1\cdot 2}y^{2n-2}x^2 + \frac{2n(2n-1)(2n-2)(2n-3)}{1\cdot 2\cdot 3\cdot 4}y^{2n-4}x^4 - \text{etc}}$$

$$= \text{Atang}\frac{(y + x\sqrt{-1})^{2n} - (y - x\sqrt{-1})^{2n}}{(y + x\sqrt{-1})^{2n}\sqrt{-1} + (y - x\sqrt{-1})^{2n}\sqrt{-1}},$$

fiet

$$\frac{kdx + dy}{dx - kdy} = \frac{(y + x\sqrt{-1})^{2n} - (y - x\sqrt{-1})^{2n}}{(y + x\sqrt{-1})^{2n}\sqrt{-1} + (y - x\sqrt{-1})^{2n}\sqrt{-1}};$$

quae reducta praebet

$$\begin{aligned}&kdx(y + x\sqrt{-1})^{2n}\sqrt{-1} + kdx(y - x\sqrt{-1})^{2n}\sqrt{-1}\\&+kdy(y + x\sqrt{-1})^{2n} - kdy(y - x\sqrt{-1})^{2n}\\&= -dy(y + x\sqrt{-1})^{2n}\sqrt{-1} - dy(y - x\sqrt{-1})^{2n}\sqrt{-1}\\&+dx(y + x\sqrt{-1})^{2n} - dx(y - x\sqrt{-1})^{2n},\end{aligned}$$

cuius integrale est

$$k(y + x\sqrt{-1})^{2n+1} - k(y - x\sqrt{-1})^{2n+1} = -\frac{1}{\sqrt{-1}}(y + x\sqrt{-1})^{2n+1} - \frac{1}{\sqrt{-1}}(y - x\sqrt{-1})^{2n+1} + C$$

seu

$$C=(y+x\sqrt{-1})^{2n+1}(k\sqrt{-1}+1)+(y-x\sqrt{-1})^{2n+1}(1-k\sqrt{-1}).$$

At est generaliter

$$\left(y+x\sqrt{-1}\right)^{2n+1}+(y-x\sqrt{-1})^{2n+1}=2(yy+xx)^{(2n+1):2}\cos(2n+1)\ A\ \mathrm{tang}\frac{x}{y}$$

atque

$$\frac{\left(y+x\sqrt{-1}\right)^{2n+1}-(y-x\sqrt{-1})^{2n+1}}{\sqrt{-1}}=2(yy+xx)^{(2n+1):2}\sin(2n+1)\ A\ \mathrm{tang}\frac{x}{y}.$$

Quibus valoribus substitutis prodibit aequatio integralis ab imaginariis libera haec

$$2k(yy+xx)^{(2n+1):2}\ \sin(2n+1)\,\mathrm{A}\ \mathrm{tang}\frac{x}{y}=2(yy+xx)^{(2n+1):2}\cos(2n+1)\ A\ \mathrm{tang}\frac{x}{y}-C$$

vel, ob constantes arbitrarias *k* et *C*, ista

$$C=(yy+xx)^{(2n+1):2}\ \left(k\sin(2n+1)\,\mathrm{A}\ \mathrm{tang}\frac{x}{y}+h\cos(2n+1)\ A\,\mathrm{tang}\frac{x}{y}\right),$$

quae aequatio semper est algebraica, dummodo fuerit *n* numerus rationalis. Vel si arcus quidam circularis arbitrarius ponatur $=g$, curva quaesita huiusmodi aequatione

$$C=(yy+xx)^{(2n+1):2}\ \sin\left(g+(2n+1)\,\mathrm{A}\ \mathrm{tang}\frac{x}{y}\right)$$

exprimi potest posito radio circuli, quem hic contemplamur, $=1$.

SCHOLION 3

39. Si ergo inter omnes curvas eidem abscissae respondentes ea debeat inveniri, in qua sit $\int Zdx$ maximum vel minimum, existente *Z* functione ipsarum *x, y* et *p*, ita ut sit $dZ=Mdx+Ndy+Pdp$, pro curva quaesita ista habebitur aequatio $N-\frac{dP}{dx}=0$. Quonian autem in Problemate praecedente annotavimus, si *Z* tantum fuerit functio ipsarum *x* et *y*, tum Methodo vulgari solutionem absolvi posse: nam ut $\int Zdx$ sit maximum minimumve, etiam *Zdx* ac proinde *Z* tale esse oportet, respectu ad *x* habito; et hanc ob rem differentiale ipsius *dZ*, sumto *x* constante, nihilo aequale positum dabit aequationem pro curva quaesita. Similis Methodus succederet in praesente Problemate, si

modo in differentiali ipsius *Z*, quod oritur posito *x* constante atque est $Ndy + Pdp$, relatio inter differentialia *dy* et *dp* pateret, ut per *dy* divisio institui atque valor finitus nihilo aequandus erui posset. Cum autem istam relationem inter *dy* et *dp*, sine qua Methodus maximorum et minimorum vulgaris adhiberi nequit, a priori definire etiamnum non liceat, poterimus eam a posteriori assignare: Quia enim inventa est aequatio pro curva quaesita haec $N - \frac{dP}{dx} = 0$, intelligitur hanc ex illa $Ndy + Pdp$ seu $N + \frac{Pdp}{dy}$ oriri potuisse, si constitisset esse $-\frac{dP}{dx} = \frac{Pdp}{dy}$ seu $0 = dP + \frac{Pdp}{p}$ ob $dy = pdx$. Quocirca relatio illa inter differentialia *dy* et *dp* ita erit comparata, ut contineatur aequatione $pdP + Pdp = 0$; quae proprietas ad hanc redit, ut considerari debeat *Pp* tanquam constans. Hinc ad Problemata resolvenda, in quibus curva quaeritur habens valorem formulae $\int Zdx$ maximum vel minimum, existente $dZ = Mdx + Ndy + Pdp$, valor ipsius *Z* debet differentiari atque in differentiali $Mdx + Ndy + Pdp$ loco *M dx* poni debeat 0, *Ndy* immutatum relinqui, tum vero loco *Pdp* scribe $-pdP$ et id, quod emergit, nihilo aequale poni. Hoc enim pacto obtinebitur $Ndy - pdP = 0$; quae aequatio, ob $dy = pdx$, transit in hanc $N - \frac{dP}{dx} = 0$, quae est ea ipsa, quam invenimus. Desideratur itaque Methodus a resolutione geometrica et lineari libera, qua pateat in tali investigatione maximi minimive loco *P dp* scribi debere $-pdP$.