CHAPTER III

ON FINDING THE MAXIMA OR MINIMA OF CURVES HAVING THAT PROPERTY IN WHICH INDETERMINATE MAGNITUDES ARE PRESENT IN THE FORMULA OF THE MAXIMUM OR MINIMUM

PROPOSITION I. PROBLEM

1. *To find the increments* (Fig. 4),*which indeterminate magnitudes of the integral adopt at any point of the abscissa from a small increase nv applied to the single point Nn.*

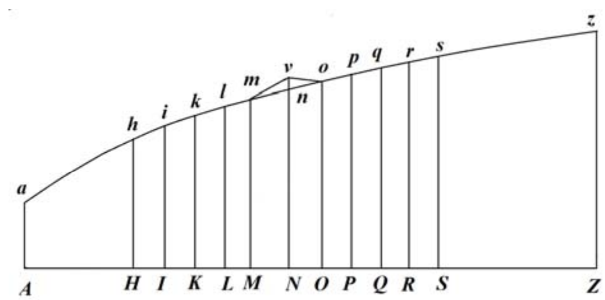

Fig. 4

SOLUTION

Let the abscissa be $AH = x$, the corresponding applied line $Hh = y$ and some indeterminate quantity Π corresponding to the abscissa *AH* shall be proposed, which shall be the formula of an integral not permitting an indefinite integration. This magnitude Π shall be prepared thus, so that since it shall correspond to the abscissa *AH* or to the point *H* , it will not change by an increase in the applied line *Nn*; which will come about, if in Π the differentials may not rise beyond the fifth order; which we have put to change the applied line from *Hh* to *Nn* finally in the fifth order at the end. For if differentials of higher order may be contained in Π , then finally the applied lines after *Nn* must be increased by infinitely small amounts. But it will suffice to extend the solution to five orders of differentials only contained in Π , since then, if also higher order differentials shall need to be present, the solution may be allowed to accommodate these. Therefore, whatever the value, Π will correspond to a point of the abscissa *H*, thus according to our method for the following point *I* to be noted it will correspond to the value Π^{I} , to the point *K* truly Π^{II}, to the point *L* the value Π^{III} and thus henceforth. Therefore it will be required to find, how many of these individual derivative values Π^{I}, Π^{II}, Π^{III}, Π^{IV} etc. may be taken from the translation of the point *n* into *v*, or how must the differentials of these be defined, if only the applied line *Nn*, which is $= y^{V}$, be varied and the infinitely small *nv* may be considered to increase: but in this sense there will be $d \cdot \Pi = 0$, because the value Π corresponding to the point *H* we have thence not considered to be affected. Because now Π is an indefinite formula of the integral, this shall be $= \int [Z]dx$ and $[Z]$ shall be a function of *x, y, p, q, r, s* and *t*, thus so that there shall be

$$d[Z]=[M]dx+[N]dy+[P]dp+[Q]dq+[R]dr+[S]ds+[T]dt\ ;$$

from which similar values of the derivative of $d[Z]$ itself, clearly $d[Z^{/}],\ d[Z^{//}],\ d[Z^{///}]$ etc. will be able to be formed in the received manner. With these in place there will be, as follows:

$$\begin{aligned}
\Pi \quad &= \int [Z]dx \\
\Pi^{/} &= \int [Z]dx + [Z]dx \\
\Pi^{//} &= \int [Z]dx + [Z]dx + [Z^{/}]dx \\
\Pi^{///} &= \int [Z]dx + [Z]dx + [Z^{/}]dx + [Z^{//}]dx \\
\Pi^{IV} &= \int [Z]dx + [Z]dx + [Z^{/}]dx + [Z^{//}]dx + [Z^{///}]dx \\
&\qquad \text{etc.}
\end{aligned}$$

Now we may consider, how many individual increments these members $[Z]dx, [Z^{/}]dx, [Z^{//}]dx, [Z^{///}]dx$ etc. may take up from the infinitely small part *nv* added to the applied line *Nn* ; which will be obtained from the differentials themselves, by putting in place of the differentials the values set out in paragraph 56 of the preceding chapter ; and thus there will be

$$d\cdot[Z]dx = nv\cdot dx\cdot\frac{[T]}{dx^5}$$

$$d\cdot[Z^{/}]dx = nv\cdot dx\left(\frac{[S^{/}]}{dx^4}-\frac{5[T^{/}]}{dx^5}\right)$$

$$d\cdot[Z^{//}]dx = nv\cdot dx\left(\frac{[R^{//}]}{dx^3}-\frac{4[S^{//}]}{dx^4}+\frac{10[T^{//}]}{dx^5}\right)$$

$$d\cdot[Z^{///}]dx = nv\cdot dx\left(\frac{[Q^{///}]}{dx^2}-\frac{3[R^{///}]}{dx^3}+\frac{6[S^{///}]}{dx^4}-\frac{10[T^{///}]}{dx^5}\right)$$

$$d\cdot[Z^{IV}]dx = nv\cdot dx\left(\frac{[P^{IV}]}{dx}-\frac{2[Q^{IV}]}{dx^2}+\frac{3[R^{IV}]}{dx^3}-\frac{4[S^{IV}]}{dx^4}+\frac{5[T^{IV}]}{dx^5}\right)$$

$$d\cdot[Z^{V}]dx = nv\cdot dx\left([N^{V}]-\frac{[P^{V}]}{dx}+\frac{[Q^{V}]}{dx^2}-\frac{[R^{V}]}{dx^3}+\frac{[S^{V}]}{dx^4}-\frac{[T^{V}]}{dx^5}\right)$$

$$d\cdot[Z^{VI}]dx = 0$$

$$d\cdot[Z^{VII}]dx = 0.$$

and all the following remaining vanish.

Now from these the increments of the values $\Pi, \Pi^{/}, \Pi^{//}, \Pi^{///}$ etc. will be deduced, which they receive from the translation of the point *n* into *v* ; evidently there will be

$$d \cdot \Pi = 0$$

$$d \cdot \Pi^{I} = nv \cdot dx \cdot \frac{[T]}{dx^5}$$

$$d \cdot \Pi^{II} = nv \cdot dx \left(\frac{[S^{I}]}{dx^4} - \frac{4[T^{I}] + d[T]}{dx^5} \right)$$

$$d \cdot \Pi^{III} = nv \cdot dx \left(\frac{[R^{II}]}{dx^3} - \frac{3[S^{II}] + d[S]}{dx^4} + \frac{6[T^{II}] + 4[T^{I}] - d[T]}{dx^5} \right)$$

$$d \cdot \Pi^{IV} = nv \cdot dx \left(\begin{array}{l} \frac{[Q^{III}]}{dx^2} - \frac{2[R^{III}] + d[R^{II}]}{dx^3} + \frac{3[S^{III}] + 3d[S^{II}] - d[S^{I}]}{dx^4} \\ - \frac{4[T^{III}] + 6[T^{II}] - 4d[T^{I}] + d[T]}{dx^5} \end{array} \right)$$

$$d \cdot \Pi^{V} = nv \cdot dx \left(\begin{array}{l} \frac{[P^{IV}]}{dx} - \frac{[Q^{IV}] + d[Q^{III}]}{dx^2} + \frac{[R^{IV}] + 2d[R^{III}] - d[R^{II}]}{dx^3} \\ - \frac{[S^{IV}] + 3d[S^{III}] - 3d[S^{II}] + d[S^{I}]}{dx^4} \\ + \frac{[T^{IV}] + 4d[T^{IV}] - 6d[T^{III}] + 4d[T^{I}] - d[T]}{dx^5} \end{array} \right)$$

$$d \cdot \Pi^{VI} = nv \cdot dx \left(\begin{array}{l} [N^{V}] - \frac{d[P^{IV}]}{dx} + \frac{[Q^{IV}] - d[Q^{III}]}{dx^2} - \frac{d[R^{IV}] - 2d[R^{III}] + d[R^{II}]}{dx^3} \\ + \frac{d[S^{IV}] - 3d[S^{IV}] + 3d[S^{IV}] - d[S^{IV}]}{dx^4} \\ - \frac{[T^{IV}] - 4d[T^{III}] + 6d[T^{II}] - 4d[T^{I}] + d[T^{I}]}{dx^5}. \end{array} \right)$$

But the increments of all the following values are equal to this increment, surely of Π^{VII}, Π^{VIII}, Π^{IX} etc. And of the value Π^{VI}, and of all the following likewise the increment will be

$$= nv \cdot dx \left([N^{V}] - \frac{d[P^{IV}]}{dx} + \frac{dd[Q^{III}]}{dx^2} - \frac{d^3[R^{II}]}{dx^3} + \frac{d^4[S^{I}]}{dx^4} - \frac{d^5[T]}{dx^5} \right).$$

But these increments will be able to be reduced to the same sign with respect to the letters $[P], [Q], [R], [S]$ and $[T]$, and thus there will be produced :

$$d\cdot\Pi = 0$$

$$d\cdot\Pi' = nv\cdot dx\cdot\frac{[T]}{dx^5}$$

$$d\cdot\Pi'' = nv\cdot dx\left(\frac{[S']}{dx^4}-\frac{4[T]+5d[T]}{dx^5}\right)$$

$$d\cdot\Pi''' = nv\cdot dx\left(\frac{[R'']}{dx^3}-\frac{3[S']+4d[S']}{dx^4}+\frac{6[T]+15d[T]+10dd[T]}{dx^5}\right)$$

$$d\cdot\Pi^{IV} = nv\cdot dx\left(\begin{array}{l}\frac{[Q''']}{dx^2}-\frac{2[R'']+3d[R'']}{dx^3}+\frac{3[S']+8d[S']+6dd[S']}{dx^4}\\ -\frac{4[T]+15d[T]+20dd[T]+d^3[T]}{dx^5}\end{array}\right)$$

$$d\cdot\Pi^{V} = nv\cdot dx\left(\begin{array}{l}\frac{[P^{IV}]}{dx}-\frac{[Q''']+2d[Q''']}{dx^2}+\frac{[R'']+3d[R'']+3dd[R'']}{dx^3}\\ -\frac{[S']+4d[S']+6dd[S']+4d^3[S']}{dx^4}\\ +\frac{[T]+5d[T]+10dd[T]+10d^3[T]+5d^4[T]}{dx^5}\end{array}\right)$$

$$d\cdot\Pi^{VI} = nv\cdot dx\left([N^V]-\frac{d[P^{IV}]}{dx}+\frac{dd[Q''']}{dx^2}-\frac{d^3[R'']}{dx^3}+\frac{d^4[S']}{dx^4}-\frac{d^5[T]}{dx^5}\right),$$

to which the increments of all the following values are equal. Q. E. I.

COROLLARY 1

2. Therefore if Π were an indeterminate magnitude of this kind, or the formula of an integral not allowing indefinite integration, then all the values of that after the position of the abscissa, where one applied line is considered to be increased, experience a change and also certain values of that before that place, the number of which depends on the order of the differentiation, which are present in the formula Π .

COROLLARY 2

3. So that if therefore a magnitude of this kind shall be present in the formula of the maximum or minimum $\int Zdx$, then the value of that differential will depend not only on some elements of the abscissa, but truly on the whole abscissa, to which the maximum or minimum must correspond.

COROLLARY 3

4. Therefore from these cases that abscissa, for which the maximum or minimum may be sought, is required to be determined and the curve, which for this abscissa was required to be found enjoying the maximum or minimum property ; likewise the curve will not be endowed with this property for the other abscissas.

SCHOLIUM

5. The distinction may be seen clearer soon, which exists between the questions in which Z is either a determined or indeterminate magnitude, when we are going to examine problems of this kind. But such questions can be varied in several ways, just as in the formula of the maxima or minima $\int Zdx$ the magnitude Z either is a function of that kind of indeterminate formula Π, such as we have considered, or in addition it may include the determined quantities *x, y, p, q, r, s* etc. Then in *Z* also more indefinite integral formulas of this kind differing from each other can be present. But for these different cases a single rule, will suffice now to be added to the rules examined above. But the principle interest has been placed in that indeterminate formula $\Pi = \int [Z]dx$, for which here we have put the determined function to be $[Z]$; but if this magnitude $[Z]$ itself may contain anew indefinite integral formulas of that kind, again there will be a need for a particular solution. Moreover that complication of indeterminate formulas can be extended to infinity ; which arises, if the quantity $[Z]$ anew may include that quantity Π itself, thus so that there shall be

$$d[Z] = [L]d\Pi + [M]dx + [N]dy + [P]dp + [Q]dq + [R]dr + \text{etc.},$$

for then on account of $d\Pi = [Z]dx$ again it will be required to consider the value $d[Z] = [L]d\Pi + [M]dx + \text{etc.}$, and this progression can be continued indefinitely. But hence a method may arise resolving those problems, in which the curve is sought having the maximum or minimum value of the formula $\int Zdx$, when the magnitude *Z* may not be given, so that at this stage, whether determinate or indeterminate, but only through a differential equation, the integration of which cannot be resolved completely ; the question is of this kind, if a curve may be sought, in which the minimum shall be the expression $\int \frac{dx\sqrt{(1+pp)}}{\sqrt{v}}$, with $dv = gdx - hv^n dx\sqrt{(1+pp)}$ present, and we will treat the resolution of questions of this kind in this chapter too.

PROPOSITION II. PROBLEM

6. *If* Z (Fig. 4) *were a function of the indeterminate magnitude* Π *, thus so that there shall be* $dZ = Ld\Pi$ *, and there shall be* $\Pi = \int [Z]dx$ *, with*

$$d[Z] = [M]dx + [N]dy + [P]dp + [Q]dq + [R]dr + \text{etc.},$$

to find the curve az, for which the given abscissa AZ may possess the maximum or minimum value of the formula $\int [Z]dx$.

SOLUTION

With the abscissa $AH = x$ in place, with the applied line $Hh = y$, the whole abscissa shall be *AZ*, to which a maximum or minimum must correspond, $= a$, therefore with the space *HZ* divided into innumerable infinitely small elements *HI*, *IK*, *KL*, *LM* etc. there must be

$$\int Zdx + Zdx + Z^{/}dx + Z^{//}dx + Z^{///}dx + \text{etc.},$$

then for the extreme point *Z* a maximum or minimum may arise. Towards effecting this the differential values are to be sought, which these individual terms take from the translation of the point *n* to *v*, the sum of which put equal to nothing will give the equation for the curve sought. But because we cannot put the change arising from *nv* towards *A* to extend beyond *H*, the value of the end of the differential $\int Zdx$ will be zero. The differential values of the remaining terms will be found, if these may be differentiated and in the differentials those increments are written, which we have found in the preceding proposition to arise from the translation of the point *n* into *v*. Moreover there will be

$$d \cdot Zdx = Ldx \cdot d\Pi$$
$$d \cdot Z^{/}dx = L^{/}dx \cdot d\Pi^{/}$$
$$d \cdot Z^{//}dx = L^{//}dx \cdot d\Pi^{//}$$
$$d \cdot Z^{///}dx = L^{///}dx \cdot d\Pi^{///}$$
$$d \cdot Z^{IV}dx = L^{IV}dx \cdot d\Pi^{IV}.$$

But if now, in place of the differentials $d\Pi$, $d\Pi^{/}$, $d\Pi^{//}$, $d\Pi^{///}$ etc. we may substitute the values found above arising from the translation of the point *n* into *v*, we will obtain :

$$d\cdot Zdx=0$$

$$d\cdot Z^{/}dx=nv\cdot L^{/}dx^2\cdot\frac{[T]}{dx^5}$$

$$d\cdot Z^{//}dx=nv\cdot L^{//}dx^2\left(\frac{[S^{/}]}{dx^4}-\frac{4[T]+5d[T]}{dx^5}\right)$$

$$d\cdot Z^{///}dx=nv\cdot L^{///}dx^2\left(\frac{[R^{//}]}{dx^3}-\frac{3[S^{/}]+4d[S^{/}]}{dx^4}+\frac{6[T]+15d[T]+10dd[T]}{dx^5}\right)$$

$$d\cdot[Z^{IV}]dx=nv\cdot L^{IV}dx^2\left(\begin{array}{l}\frac{[Q^{///}]}{dx^2}-\frac{2[R^{//}]+3d[R^{//}]}{dx^3}+\frac{3[S^{/}]+8d[S^{/}]+16dd[S^{/}]}{dx^4}\\-\frac{4[T]+15d[T]+20dd[T]+10d^3[T]}{dx^5}\end{array}\right)$$

$$d\cdot[Z^{V}]dx=nv\cdot L^{V}dx^2\left(\begin{array}{l}\frac{[P^{IV}]}{dx}-\frac{[Q^{///}]+2d[Q^{///}]}{dx^2}+\frac{[R^{//}]+3d[R^{//}]+3ddR^{//}}{dx^3}\\-\frac{[S^{/}]+4d[S^{/}]+6dd[S^{/}]+4d^3[S^{/}]}{dx^4}+\frac{[T]+5d[T]+10dd[T]+10d^3[T]+5d^4[T]}{dx^5}\end{array}\right)$$

$$d\cdot[Z^{VI}]dx=nv\cdot L^{VI}dx^2\left([N^{V}]-\frac{d[P^{IV}]}{dx}+\frac{dd[Q^{///}]}{dx^2}-\frac{d^3[R^{//}]}{dx^3}+\frac{d^4[S^{/}]}{dx^4}-\frac{d^5[T]}{dx^5}\right)$$

$$d\cdot[Z^{VII}]dx=nv\cdot L^{VII}dx^2\left([N^{V}]-\frac{d[P^{IV}]}{dx}+\frac{dd[Q^{///}]}{dx^2}-\frac{d^3[R^{//}]}{dx^3}+\frac{d^4[S^{/}]}{dx^4}-\frac{d^5[T]}{dx^5}\right)$$

etc.

Clearly the same increments of the following terms are progressing by the same law. Now the increments of the first six terms may be added, and the total increment of the terms will be produced

$$Zdx+Z^{/}dx+Z^{//}dx+Z^{///}dx+Z^{IV}dx+Z^{V}dx$$

$$= nv\cdot dx^2\begin{pmatrix} \frac{L^V[P^{IV}]}{dx}-\frac{[Q^{///}]dL^{IV}+2L^{IV}d[Q^{///}]}{dx^2}+\frac{[R^{//}]ddL^{///}+3d[R^{//}]dL^{///}+3L^{///}dd[R^{//}]}{dx^3} \\ -\frac{[S^{/}]d^3L^{//}+4d[S^{/}]ddL^{//}+6dL^{//}dd[S^{/}]+4L^{//}d^3[S^{/}]}{dx^4} \\ +\frac{[T]d^4L^{/}+5d[T]d^3L^{/}+10dd[T]ddL^{/}+10dL^{/}d^3[T]+5L^{/}d^4[T]}{dx^5} \end{pmatrix},$$

in which expression, because all the terms are homogeneous among themselves, now the numerical indices can be ignored. Moreover the increment of all the terms following $L^{VI}dx + L^{VII}dx + \text{etc.}$ will be

$$= nv\cdot dx\left([N^V]-\frac{d[P^{IV}]}{dx}+\frac{dd[Q^{///}]}{dx^2}-\frac{d^3[R^{//}]}{dx^3}+\frac{d^4[S^{/}]}{dx^4}-\frac{d^5[T]}{dx^5}\right)\cdot$$
$$\left(L^{VI}dx+L^{VII}dx+L^{VIII}dx+L^{IX}dx+\text{ etc. as far as to } Z\right).$$

But here the latter factor may be defined from the integration of the formula $\int Ldx$, which corresponds to the indefinite abscissa $AH = x$; in this formula after integration there is put $x = a$ and that may be changed into *H*, where *H* is the value of the formula $\int Ldx$ corresponding to the whole proposed abscissa *AZ* ; from which therefore if there may be taken $\int Ldx$, the value $H-\int Ldx$ will remain corresponding to the part *HZ* or *NZ*, which therefore can be put in place of

$$L^{VI}dx + L^{VII}dx + L^{VIII}dx + \text{ etc.}$$

On account of which finally the value of the differential of the formula $\int Zdx$ corresponding to the whole abscissa *AZ* will be

$$= nv\cdot dx\left(H-\int Ldx\right)\left([N]-\frac{d[P]}{dx}+\frac{dd[Q]}{dx^2}-\frac{d^3[R]}{dx^3}+\frac{d^4[S]}{dx^4}-\frac{d^5[T]}{dx^5}\right)$$

$$+nv\cdot dx\begin{pmatrix} L[P]-\frac{[Q]d[L]+2Ld[Q]}{dx}+\frac{[R]ddL+3d[R]dL+3Ldd[R]}{dx^2} \\ -\frac{[S]d^3L+4d[S]ddL+6dLdd[S]+4Ld^3[S]}{dx^3} \\ +\frac{[T]d^4L+5d[T]d^3L+10dd[T]ddL+10dLd^3[T]+5Ld^4[T]}{dx^4} \end{pmatrix},$$

which can be reduced to a more convenient form, so that it becomes

$$= nv\cdot dx\left(\begin{array}{l}[N]\left(H-\int Ldx\right)-\dfrac{d[P]\left(H-\int Ldx\right)}{dx}+\dfrac{dd[Q]\left(H-\int Ldx\right)}{dx^2}\\ -\dfrac{d^3[R]\left(H-\int Ldx\right)}{dx^3}+\dfrac{d^4[S]\left(H-\int Ldx\right)}{dx^4}-\dfrac{d^5[T]\left(H-\int Ldx\right)}{dx^5}\end{array}\right)$$

which value of the differential, as far as the occasion demands, can be continued further ; moreover this, put equal to zero, will give the equation for the curve sought. Q. E. I.

COROLLARY 1

7. Because $H-\int Ldx$ is the value of the formula $\int Ldx$ corresponding to the part of the abscissa $AZ = a - x$, if there is put $AZ = a - x = u$, $\int Ldx$ will be that value itself $H-\int Ldx$; for which there is a need; if indeed $\int Ldx$ may vanish on putting $u = 0$.

COROLLARY 2

8. So that therefore if the start of the abscissas may be taken at the point *Z*, thus in order that the abscissa *ZH* may be put $= u$, so that there may be put everywhere $x = a - u$, an equation will be produced for the curve between the coordinates *u* and *y*; that part of this curve sought will satisfy the question, which corresponds to the $AZ = a$. Meanwhile it is to be observed since in the formula $\int Zdx$ of the maximum or minimum as well as in $\int [Z]dx$, the start of the abscissas must be taken at the point *A*.

COROLLARY 3

9. If therefore the curve may be sought related to a given abscissa AZ, in which the maximum or minimum must be $\int Zdx$, and *Z* shall be some function of $\Pi = \int Zdx$, with

$$dZ = Ld\Pi \text{ and } d[\mathrm{Z}] = [M]dx + [N]dy + [P]dp + [Q]dq + [R]dr + \text{etc.},$$

this equation will be had for the curve sought :

$$0 = [N]\int Ldx - \frac{d\cdot[P]\int Ldx}{dx} + \frac{dd\cdot[Q]\int Ldx}{dx^2} - \frac{d^3\cdot[R]\int Ldx}{dx^3} + \text{etc.},$$

where there is $u = a - x$ and $\int Ldu$ denotes the value of the formula $\int Ldx$ corresponding to the abscissa $HZ = u$.

COROLLARY 4

10. Therefore either two of the initial abscissas *A* and *Z*, and two or the abscissas $AH = x$ and $ZH = u$ can be considered, of which the first must be considered is the integral $\int Ldx$ or Π, the other truly in the integral $\int Ldx$; or only a single abscissa $AH = x$ may be considered, in which case in place of $\int Ldu$ there must be written $H - \int Ldx$ with *H* denoting the value, which the formula $\int Ldu$ produces on putting $x = AH = a$.

COROLLARY 5

11. Because *Z* is a function of Π only, thus so that no other variable quantities may be included, on account of $dZ = Ld\Pi$ also *L* will be a function of Π only.

COROLLARY 6

12. If $[Z]$ were a function of *x* only, then $\Pi = \int [Z]dx$ becomes a determined quantity and a function of *x*, and hence also *Z*; from which a maximum or minimum cannot be found. Likewise the solution is revealed ; for making $[N] = 0,\ [P] = 0$ etc. and the equation will change into the identity $0 = 0$.

SCHOLIUM I

13. Here some initial cases occur requiring to be considered, the first of which is, if $[Z]$ were a function of x and *y* only, thus so that there shall be $d[Z] = [M]dx + [N]dy$. But if now a curve may be required, in which the maximum or minimum shall be the formula $\int Zdx$ for the given abscissa $AZ = a$, with the function Z being some function of $\int [Z]dx = \Pi$, thus so that there shall be $dZ = Ld\Pi$, this equation will be had for the curve sought

$$0 = [N](H - \int Ldx);$$

therefore there will be either $[N] = 0,\ H = \int Ldx$ or $L = 0$; the equation of which shall produce some curved line or another, that not only satisfies the problem for the abscissa $AZ = a$ but also for some other indefinite abscissa *x* ; that which hence will be deduced, so that from the equation the magnitude *H*, which depends on the determined abscissa *a*, will be removed from the calculation. But so that it may be applied especially to the equation $L=0$, because L is a function of Π or $\int[Z]dx$, there becomes equal to a $\int[Z]dx$ to be determined, so that, unless $[Z]=0$, it cannot be done : therefore in this case the two satisfying equations will be $[N]=0$ and $[Z]=0$.

SCHOLIUM II

14. Then truly the case deserves to be considered, in which $[N]$ vanishes; that which comes about, if $[Z]$ were a function of *x, p, q, r* etc. not involving *y*. We may put $[Z]$ to be some function of *x* and *p* and $d[Z]=[M]dx+[P]dp$. Therefore if there may be put $\int[Z]dx=\Pi$ and a curve may be sought, in which for the abscissa defined $AZ=a$ formula $\int Zdx$ shall be a maximum or minimum, with Z being a function of Π, thus so that there shall be $dZ=Ld\Pi$, this equation will arise for the curve sought

$$0=-\frac{d\cdot[P](H-\int Ldx)}{dx},$$

and thus a constant $=[P](H-\int Ldx)$. Truly this constant found by integration is not arbitrary ; for that is required to be prepared thus, as on putting $x=a$, in which case it becomes $Ldx=H$, making $\frac{\text{const.}}{[P]}=0$. But this cannot happen, unless either this constant may be put $=0$ or the quantity $[P]$ shall be prepared thus, if that it becomes $=\infty$ on putting $x=a$. In the first case there will be found either $[P]=0$ or $\int Ldx=H$, that is $L=0$ or $\int[Z]dx=$ const. or rather $[Z]=0$; but in the latter case the constant still cannot be taken as arbitrary, for it will be determined by putting $x=a-dx$, in that manner, so that expressions, which may be considered to be indeterminate in some cases, are accustomed to be defined. And hence it is observed in problems of this kind a number of arbitrary constants entering into the solution, to which a number of points must be taken to be equal, through which the curve to be satisfied must pass, not possible to be judged from the order of the differentials. For often a differential equation of a higher order may arise by removing all the integration formulas by differentiation, from which the determination of the problem will depend on some points by no means connected with the problem.

EXAMPLE I

15. *If* Π *should denote the area of the curve* $\int ydx$ *and Z shall be some function of* Π *itself, to find the curve, which for a given abscissa* $=a$ *shall have a maximum or minimum value of the formula* $\int Zdx$.

Because Z is a function of Π, there shall be $dZ = Ld\Pi$; L will be a function of $II = \int ydx$. Then since there shall be $d\Pi = ydx$, $[Z] = y$ and on account of

$$d[\mathrm{Z}] = [M]dx + [N]dy + [P]dp + \text{etc.}.$$

there will become $[M] = 0,\ [N] = 1,\ [P] = 0,\ [Q] = 0$ etc., so that for the curve sought this equation will be had $0 = H - \int Ldx$; and thus $L = 0$. Hence there will be $II = \int ydx =$ to a certain constant, and again $y = 0.$ Hence the equation is satisfied only by a right line lying on the axis ; and that both for any abscissa defined equally and for that defined $= a$.

EXAMPLE II

16. *If* Π *may denote the arc of a curve* $= \int dx\sqrt{(1+pp)}$ *and if some function of that were Z, to find a curve, which for the given abscissa* $AZ = a$ *may have a maximum or minimum value of the formula* $\int Zdx$.

On account of $dZ = Ld\Pi$, L will be a function of the arc Π itself ; and on account of $d\Pi = dx\sqrt{(1+pp)}$ there will be

$[Z] = \sqrt{(1+pp)}$ and $[M] = 0$; $[P] = \dfrac{p}{\sqrt{(1+pp)}}$, $[Q] = 0$ etc., from which this equation will be had for the curve sought :

$$0 = -d \cdot \frac{p}{dx\sqrt{(1+pp)}}(H - \int Ldx),$$

and hence

$$C = \frac{p}{\sqrt{(1+pp)}}(H - \int Ldx),$$

where the constant C must be determined thus, so that on putting $x = a$ the constant becomes $C = \frac{p}{\sqrt{(1+pp)}} \times 0$; whereby, because $\frac{p}{\sqrt{(1+pp)}}$ cannot become infinite, it is necessary that the constant shall be $C = 0$ and thus

$$\text{either } \frac{p}{\sqrt{(1+pp)}} = 0 \text{ or } \int Ldx = H .$$

Therefore it will come about, from the latter equation, $L = 0$ and $\Pi =$ some constant ; from which again it is deduced $d\Pi = dx\sqrt{(1+pp)} = 0$, for there is no way in which this condition can be satisfied. But from the first equation it may be deduced that $p = 0$ or $dy = 0$, which is the equation for a line parallel to the axis *AZ*, which satisfies the equation for any abscissa.

EXAMPLE III

17. *Π shall denote the surface of a solid of revolution arising from the rotation of the curve ah about the axis AZ, which is as* $\int ydx\sqrt{(1+pp)}$ *, Z shall be some function of this surface : to find a curve in which, for a given abscissa $AZ = a$, $\int Zdx$ shall be a maximum or minimum*.

On account of $dZ = Ld\Pi$, L will be a function of $\Pi = \int ydx\sqrt{(1+pp)}$, and on account of $d\Pi = ydx\sqrt{(1+pp)}$, there will come about

$$[Z] = y\sqrt{(1+pp)} \text{ and } d[Z] = dy\sqrt{(1+pp)} + \frac{ypdp}{\sqrt{(1+pp)}} ;$$

from which there will become:

$$[M] = 0,\ [N] = \sqrt{(1+pp)},\ [P] = \frac{ypdp}{\sqrt{(1+pp)}} ;$$

all the remaining values $[Q],\ [R],\ [S]$ etc. will be $= 0$. On account of which for the curve sought this equation will be had :

$$0 = (H - \int Ldx)\sqrt{(1+pp)} - \frac{1}{dx} d \cdot \frac{yp}{\sqrt{(1+pp)}} (H - \int Ldx).$$

For brevity there may be put, $H - \int Ldx = V$; the equation becomes :

$$Vdx\sqrt{(1+pp)} = d\cdot\frac{ypV}{\sqrt{(1+pp)}} = \frac{Vppdx}{\sqrt{(1+pp)}} + \frac{Vydp}{(1+pp)^{3:2}} + \frac{ypdV}{\sqrt{(1+pp)}}$$

or

$$Vdx = \frac{Vydp}{1+pp} + ypdV = \frac{Vydp}{1+pp} - ypLdx$$

an account of $dV = -Ldx$. We may put $Z = \Pi$, thus so that the maximum must be :

$\int dx\int ydx\sqrt{(1+pp)}$, on account of $H = a$ there will be $L = 1$, $\int Ldx = x$ and $V = a - x$. The equation becomes

$$(a-x)dx = \frac{(a-x)ydp}{1+pp} - ypdx.$$

Let $a - x = u$, there will be $dx = -du$ and $dy = -pdu$, and this equation will be had :

$$0 = udu - ydy + \frac{uydp}{1+pp} \text{ or } udu - ydy - \frac{uyduddy}{du^2+dy^2} = 0.$$

Putting
$u = e^t$ and $y = e^t z$, there will be $du = e^t dt$ and $ddu = 0 = e^t(ddt + dt^2)$, or $ddt = -dt^2$; again,

$$dy = e^t(dz + zdt) \text{ and } ddy = e^t(ddz + 2dtdz);$$

with which substituted the equation arises :

$$dt - zdz - zzdt = \frac{zdt(ddz + 2dtdz)}{dt^2 + (dz + zdt)^2}.$$

Again let there be $dt = sdz$, there becomes $ddt = -s^2dz^2 = sddz + dsdz$ and hence

$$ddz = -sdz^2 - \frac{dsdz}{s}.$$

Therefore this equation will be had :

$$sdz - zdz - szzdz = \frac{zs^2dz - zds}{ss + (1+sz)^2};$$

which indeed is a differential of the first order between the two variables *s* and *z* only, yet truly it does not permit further integration. Therefore something much less can be effected, if we may consider the question in general.

SCHOLIUM III

18. The case of this example, for which we have investigated the curve, shall be $\int dx\int ydx\sqrt{(1+pp)}$, and if a two-fold integral sign is present, yet it can be resolved by the method of the preceding chapter also ; that which therefore is required to be shown, so that an understanding of each method may be indicated. But particularly in this work a new way will be revealed resolving several other problems about maxima and minima, which at this stage, however much is agreed, it has not been mentioned. Clearly the question is, so that for the given abscissa $AZ=a$ this expression $\int dx\int ydx\sqrt{(1+pp)}$ shall become a maximum or minimum, which is changed into this :

$$x\int\int ydx\sqrt{(1+pp)}-\int xydx\sqrt{(1+pp)}.$$

In order that this form may be reduced to a maximum or minimum, it is required, that its value for the abscissa $AZ=a$ shall be the same for the curves sought itself *az* and for that point translated from *n* to *v*. Therefore we may consider $\int ydx\sqrt{(1+pp)}$ to become $=A$, if there may be put $x=a$, and in the same case $\int xydx\sqrt{(1+pp)}=B$. Now with the elements *mno* changed into *mvo* the value *A* will be increased by its differential, which by the previous chapter, is

$$=nv\cdot dx\left(\sqrt{(1+pp)}-\frac{1}{dx}d\cdot\frac{yp}{\sqrt{(1+pp)}}\right);$$

moreover by the same precepts the value of the differential magnitude of *B* will be produced

$$=nv\cdot dx\left(x\sqrt{(1+pp)}-\frac{1}{dx}d\cdot\frac{xyp}{\sqrt{(1+pp)}}\right);$$

On account of which the value of the proposed formula $\int dx\int ydx\sqrt{(1+pp)}$, in translating the point *n* to *v*, for the abscissa $AZ=a$ will be

$$=a\left(A+nv\cdot\left(dx\sqrt{(1+pp)}-d\cdot\frac{yp}{\sqrt{(1+pp)}}\right)\right)-B$$
$$-nv\cdot\left(xdx\sqrt{(1+pp)}-d\cdot\frac{xyp}{\sqrt{(1+pp)}}\right),$$

which must be equal to the natural value of the same formula for the $=a$, without changing the point *n*, which is $aA-B$. Hence this equation will come about :

$$(a-x)dx\sqrt{(1+pp)}-d\cdot\frac{(a-x)yp}{\sqrt{(1+pp)}}=0,$$

which agrees entirely with the equation found in the solution of the example.

PROPOSITION III. PROBLEM

19. *With the indeterminate integral* $\int[Z]dx$ *being present for the function* Π *, thus so that there shall be*

$$d[Z]=[M]dx+[N]dy+[P]dp+[Q]dq+[R]dr+\text{etc.},$$

Z shall be some function both of that magnitude Π *as well as of the magnitudes x, y, p, q, r, s etc., thus so that there shall be*

$$dZ=Ld\Pi+Mdx+Ndy+Pdp+Qdq+Rdr+\text{etc.},$$

to find the curve az, which for a given abscissa $AZ=a$ *may provide a maximum or minimum value of the formula* $\int Zdx$.

SOLUTION

The increase *nv*, which is considered to happen for a single applied line *Nn*, thus may be taken at some distance from the first applied line *Hh*, so that no change may be introduced into the value of the formula $\int Zdx$ corresponding to the abscissa *AH* and only the values of this formula corresponding to the elements of the abscissa following after *H* following may be allowed changes, which are $Zdx,\ Z'dx,\ Z''dx,\ Z'''dx$ etc. as far as to the final element of the abscissas in *Z*. Therefore the increments of these values arising from the translation of the point *n* into *v*, if they may be gathered together into one sum and put equal to zero, will give the equation for the curve sought. But the increments of these values will be obtained from these being differentiated and by writing these values in place of the differentials, which above, both in the last proposition of the preceding chapter as well as in the first proposition of this, we have found to arise from the translation *n* to *v*; thus there will be :

$$d\cdot Zdx=dx\left(Ld\Pi+Mdx+Ndy+Pdp+\text{etc.}\right)$$
$$d\cdot Z'dx=dx\left(L'd\Pi'+M'dx+N'dy'+P'dp'+\text{etc.}\right)$$
$$d\cdot Z''dx=dx\left(L''d\Pi''+M''dx+N''dy''+P''dp''+\text{etc.}\right)$$
etc.

So that if now in place of the differentials
$d\Pi,\ d\Pi^{/},\ d\Pi^{//}$ etc., $dy,\ dy^{/},\ dy^{//}$ etc., $dp,\ dp^{/},\ dp^{//}$ etc., $dq,\ dq^{/},\ dq^{//}$ etc. the values found above may be substituted and in the same manner, as we have used before, they may be brought together into one sum, for the formula $\int Zdx$ for the abscissa $AZ = a$, the value of the differential

$$= nv\cdot dx\left(\begin{array}{l}[N]\left(H-\int Ldx\right)-\dfrac{d\cdot[P]\left(H-\int Ldx\right)}{dx}+\dfrac{dd\cdot[Q]\left(H-\int Ldx\right)}{dx^2}\\ \qquad -\dfrac{d^3\cdot[R]\left(H-\int Ldx\right)}{dx^3}+\dfrac{d^4\cdot[S]\left(H-\int Ldx\right)}{dx^4}-\text{etc.}\end{array}\right)$$

$$+nv\cdot dx\left(N-\frac{dP}{dx}+\frac{ddQ}{dx^2}-\frac{d^3R}{dx^3}+\frac{d^4R}{dx^4}-\text{etc.}\right).$$

And from this the equation for the curve sought will result :

$$\begin{aligned}0=[N]\left(H-\int Ldx\right)-\frac{d\cdot[P]\left(H-\int Ldx\right)}{dx}+\frac{dd\cdot[Q]\left(H-\int Ldx\right)}{dx^2}\\ -\frac{d^3\cdot[R]\left(H-\int Ldx\right)}{dx^3}+\text{etc.}\\ +N-\frac{dP}{dx}+\frac{ddQ}{dx^2}-\frac{d^3R}{dx^3}+\frac{d^4R}{dx^4}-\text{etc.}\end{aligned}$$

where it is to be observed that H is the value of the formula $\int Ldx$, which arises on putting $x = a$.

Q. E. I.

COROLLARY1

20. Therefore the rule from the preceding chapter has been found and returned more fully ; for now we are able to define the maximum or minimum having the value of the formula $\int Zdx$, not only if Z were a determined function of the magnitudes *x, y, p, q, r* etc., but also it may include within itself one indefinite integral magnitude $\int [Z]dx$, provided $[Z]$ shall be a determined function.

COROLLARY 2

21. Indeed also, if several indefinite integral quantities of this kind were present in *Z*, a solution would be come upon. For, just as an expression has been entered upon from a single formula of this kind indefinite in value, if several were present, such will appear from the individual kinds and accede to the differential value.

COROLLARY 3

22. Because here *Z* may be considered a function not only of the defined magnitudes *x, y, p, q, r* etc., but also of the indefinite magnitude $\Pi = \int [Z]dx$, on account of

$$dZ = Ld\Pi + Mdx + Ndy + Pdp + Qdq + Rdr + \text{etc.}$$

also the magnitudes *M, N, P, Q* etc. involve this integral formula $\Pi = \int [Z]dx$; and also the quantity *L* itself, unless perhaps Π may have a single dimension in *Z*.

COROLLARY 4

23. On this account in the equation for the curve found integral magnitudes of a two-fold kind will be present, clearly $\int Ldx$ and $\int [Z]dx$; from which, if the equation found must be freed from these formulas by differentiation, it will rise to a much higher order of differential, than indeed this form shows.

COROLLARY 5

24. But by elimination these integral formulas the equation may arrive at a differential equation higher by two orders. So that if indeed the resulting equation, if it may be expanded out, shall be a differential of order *n*, then in the first place the value of the formula $\int Ldx$ may be defined from that, and with a differentiation put in place it may become a differential equation of order $n+1$, in which the formula $\int [Z]dx$ will be present, which reduced further and by differentiation freed from the formula $\int [Z]dx$, becomes a differential of order $n+2$.

SCHOLIUM I

25. But even if the number of points, through which the curve sought must pass, may depend on the order of the differentiation, yet in this case it cannot be defined for the number $n+2$. For this differential equation of the $n+2$ order in the power will involve $n+2$ constants, truly not all these are arbitrary. And now one constant may be

determined from that, since the integral $\int [Z]dx$ must obtain a certain value, but such, as may be obtained in the quantity Z, that is, which vanishes on putting $x = 0$, if indeed this condition were assumed in the formula $\int Zdx$. Then in a like manner, one constant may be defined by the formula $\int Ldx$, which, as we have put it in place, must vanish on putting $x = 0$. On account of which only *n* purely arbitrary constants will remain, which will provide just as many points, by which the problem will be determined. Therefore similarly, as in the preceding chapter, the problem, so that it may be determined, thus will be proposed, so that among all the curves passing through *n* points these may be determined, which for the given abscissa $x = a$ may contain the maximum or minimum value of the formula $\int Zdx$. Therefore the equation found will have to be set out according to this decision being put in place ; that is, all the differentiations indicated actually must be completed ; with which done it will be apparent, how many differential orders shall be present, and from this order the number *n* will be found. But how many in addition may be allowed to be observed about this number *n*, we will show in the following examples.

EXAMPLE I

26. *To find the curve, which for the given abscissa* $AZ = a$ *may provide a maximum or minimum value for the formula* $\int yxdx \int ydx$, *thus by taking the integral* $\int ydx$, *so that it may vanish on putting* $x = 0$.

Therefore there will be $\Pi = \int ydx$ and $[Z] = y$; from which there becomes $[N] = 1$ with the remaining letters $[M], [P], [Q]$ etc. present being $= 0$. Again there will be

$$Z = yx\Pi \quad \text{and} \quad dZ = yxd\Pi + y\Pi dx + x\Pi dy;$$

from which there will be had $L = yx,\ M = y\Pi$ and $N = x\Pi,\ P = Q = R = \text{etc.} = 0$. From these this equation will be formed for the curve sought :

$$0 = \left(H - \int yxdx\right) + x\Pi \quad \text{or} \quad \int yxdx = H + x\int ydx,$$

where *H* is the value of the formula $\int yxdx$, which appears on putting $x = a$. Moreover it is evident hence nothing arises for any curved line ; for from the differentiation put in place the equation becomes $dx\int ydx = 0$ and again $y = 0$, which is the equation for a right line lying on the axis *AZ*.

EXAMPLE II

27. *To find the curve, which for a given abscissa* $AZ = a$ *may provide a maximum or minimum value of the formula* $\int ydx\int dx\sqrt{(1+pp)}$.

Therefore because $\Pi = \int dx\sqrt{(1+pp)}$, there will be

$$[Z] = \sqrt{(1+pp)} \text{ and } [P] = \frac{p}{\sqrt{(1+pp)}}.$$

Again there will be $Z = y\Pi$, $L = y$ and $N = \Pi$; all the remaining letters vanish. Hence therefore this equation will result for the curve sought :

$$0 = -\frac{1}{dx}d\cdot\frac{p\left(H-\int ydx\right)}{\sqrt{\left(1+pp\right)}} + \Pi$$

or

$$\Pi dx = d\cdot\frac{\left(H-\int ydx\right)p}{\sqrt{\left(1+pp\right)}} = \frac{\left(H-\int ydx\right)dp}{\left(1+pp\right)^{\frac{3}{2}}} - \frac{ypdx}{\sqrt{\left(1+pp\right)}};$$

therefore

$$dx\int dx\sqrt{\left(1+pp\right)} = \frac{\left(H-\int ydx\right)dp}{\left(1+pp\right)^{\frac{3}{2}}} - \frac{ypdx}{\sqrt{\left(1+pp\right)}}.$$

Therefore because it becomes $\int ydx = H$ on putting $x = a$, in the same case there becomes

$\int dx\sqrt{\left(1+pp\right)} = -\frac{yp}{\sqrt{\left(1+pp\right)}} =$ to the arc of the curve corresponding to the abscissa a.

Which condition must be fulfilled by the determination of one constant, which may enter by integration. But actually this is a differential equation of the second order, which truly here must be differentiated before it can be freed from these integral formulas $\int ydx$ and $\int dx\sqrt{\left(1+pp\right)}$; and in this manner an equation may emerge to the sixth order and involving as many as six constants ; two of which thus will be determined, because on making $x = 0$ the formulas $\int ydx$ and $\int dx\sqrt{(1+pp)}$ must vanish. But that equation itself becomes involved, so that the treatment of that may not merit to be undertaken.

EXAMPLE III

28. *To find the curve, in which for a given abscissa* $\int \frac{dx}{p} \int ydx$ *shall be a maximum or minimum.*

Here there will be $\Pi = \int ydx,\ [Z] = y$ and $[N] = 1$; then since there shall be $Z = \frac{\Pi}{p}$, there will be $L = \frac{1}{p}$ and $= -\frac{\Pi}{pp}$; all the remaining letters vanish. Hence therefore this equation appears :

$$0 = H - \int \frac{dx}{p} + \frac{1}{dx} d \cdot \frac{\Pi}{pp}$$

or

$$0 = H - \int \frac{dx}{p} + \frac{y}{pp} - \frac{2\Pi dp}{p^3 dx}.$$

Therefore on putting $x = a$, in which case there becomes $\int \frac{dx}{p} = H$, there will be $ydx = \frac{2\Pi dp}{p}$. This equation may be differentiated, and there will be

$$0 = -\frac{dx}{p} + \frac{dx}{p} - \frac{2ydp}{p^3} - \frac{2ydp}{p^3} + \frac{6\Pi dp^2}{p^4 dx} - \frac{2\Pi dpp}{p^3 dx}$$

or

$$0 = 3\Pi dp^2 - 2ypdxdp - \Pi pddp;$$

which equation shall become integrated conveniently, if it may be divided by Πpdp, for it will produce

$$0 = \frac{3dp}{p} - \frac{2ydx}{\Pi} - \frac{ddp}{dp},$$

of which the integral is

$$C = 3lp - 2l\Pi - l\frac{dp}{dx},$$

or $C\Pi^2 dp = p^3 dx$; therefore on putting $x = a$, since there must be $ydx = \frac{2\Pi dp}{p}$, from this equation there will be $C\Pi y = 2p^2$, from which one constant may be defined. Therefore there will be

$$\Pi = \sqrt{\frac{p^2dx}{Cdp}} = \frac{2ypdxdp}{3dp^2 - pddp}$$

or

$$3dp^2 - pddp = \frac{2ydp\sqrt{dxdp}}{by\sqrt{bp}},$$

which equation is a differential of the third order and on that account besides the constant b (moreover we may put $\frac{1}{b^3}$ in place of C) it will involve three new constants. Of these one will be determined, because from that on putting $x = a$ the equation may become $\frac{\Pi y}{b^3} = 2pp$; truly otherwise from that, because on putting $x = 0$ it must become $\Pi = 0$ or $\frac{p^3dx}{dp} = 0$. The remaining two constants are maintained arbitrarily and therefore the curve sought must be determined by two given points, through which it passes.

EXAMPLE IV

29. *To find the curve az related to the abscissa AZ =a, in which* $\int dx \frac{\int yxdx}{\int ydx}$ *shall be a maximum or minimum.*

This example thus has been considered to be put together, so that it is apparent how questions of this kind shall be resolved, if two or more indefinite integral formulas may be present. Therefore there shall be

$$\int yxdx = \Pi \text{ and } \int ydx = \pi,$$

and on putting $d\Pi = [Z]dx$ and $d\pi = [z]dx$ there will be $[Z] = yx$ and $[z] = y$. So that if now the small letter $[z]$ may be treated in the similar manner as the greater $[Z]$, thus so that there shall be

$$d[z] = [m]dx + [n]dy + [p]dp + \text{etc.},$$

there will be $[M] = y$, $[N] = x$ and likewise $[n] = 1$. Then, since there shall be $Z = \frac{\Pi}{\pi}$, there will be $dZ = \frac{d\Pi}{\pi} - \frac{\Pi d\pi}{\pi^2}$.

Putting $\frac{1}{\pi} = L$ and $\frac{\Pi}{\pi^2} = l$, and this equation thus will be had for the curve sought, on account of N and P, Q, R etc. $= 0$:

$$0 = x\left(H - \int\frac{dx}{\pi}\right) - \left(h - \int\frac{\Pi dx}{\pi^2}\right),$$

where there shall be $\int\frac{dx}{\pi} = H$ and $\int\frac{\Pi dx}{\pi^2} = h$, if there may be put $x = a$. Therefore since there shall be

$$Hx - x\int\frac{dx}{\pi} = h - \int\frac{\Pi dx}{\pi^2},$$

by differentiation there will be

$$H - \int\frac{dx}{\pi} - \frac{x}{\pi} = -\frac{\Pi}{\pi^2}.$$

Therefore putting $x = a$ there must become $\Pi = \pi x$. The equation may be differentiated anew and it will provide :

$$-\frac{2}{\pi} + \frac{xy}{\pi^2} = -\frac{yx}{\pi^2} + \frac{2\Pi y}{\pi^3} \quad \text{or} \quad xy - \pi = \frac{\Pi y}{\pi}$$

and hence

$$\Pi = \pi x - \frac{\pi\pi}{y}.$$

If again a differentiation may be put in place [recalling that $d\Pi = xydx$ and $d\pi = ydx$], it will give

$$yxdx = \pi dx + yxdx - 2\pi dx + \frac{\pi\pi dy}{yy}$$

or

$$yydx = \pi dy \quad \text{and} \quad \frac{ydx}{\pi} = \frac{dy}{y}.$$

Because truly if on putting $x = 0$ there becomes $\pi = 0$, then in this case there becomes $\frac{yydx}{dy} = 0$. The equation $\frac{ydx}{\pi} = \frac{dy}{y}$ on account of $ydx = d\pi$, integrated gives $\pi = by$; and thus making $x = 0$, y must vanish. But from the equation $\pi = by$ it follows that $ydx = bdy$ and hence $x = bly - bl0$, if indeed $\pi = by$ may vanish on putting $x = 0$; in which case $y = 0$ and the curve will go into a right line lying on the axis *AZ*. But if we may put $x = 0$, the value $\pi = \int ydx$ is not required to vanish, but to become $= bc$, then there will be $x = bl\frac{y}{c}$, which is the equation for the logarithmic curve. Further to that it may be required to find the value $\Pi = \int yxdx$; because there is $ydx = bdy$, there will become

$$yxdx = bxdy \quad \text{and} \quad \Pi = bxy - b\pi + Const.$$

or

$$\Pi = bbyl\frac{y}{c} - bby + C.$$

But it may be necessary that Π becomes $= 0$ on putting $x = 0$ or $y = c$, then there will be

$$\Pi = bbyl\frac{y}{c} + bb(c - y).$$

Now putting $x = a$, there will be $l\frac{y}{c} = \frac{a}{b}$ and $y = ce^{a:b}$; truly in this case it is necessary, that there shall be $\Pi = nx$ or

$$abce^{a:b} + bbc - bbce^{a:b} = abce^{a:b}$$

and hence $e^{a:b} = 1$, from which there will be either $a = 0$ or $b = \infty$. This inconvenience hence arises, because we may have put $\Pi = 0$ by making $x = 0$. Therefore we may put Π to vanish in this case by putting $y = g$, and there will be

$$\Pi = bbyl\frac{y}{c} - bby + bbg - bbgl\frac{g}{c}.$$

Now on putting $x = a$, in which case there must become $\Pi = \pi x = a\pi$, there will be

$$abce^{a:b} - bbce^{a:b} + bbg - bbgl\frac{g}{c} = abce^{a:b}$$

and hence

$$e^{a:b} = \frac{g}{c}\left(1 - l\frac{g}{c}\right) \text{ or } b = \frac{a}{l\frac{g}{c}\left(1 - l\frac{g}{c}\right)}$$

and thus

$$x = \frac{a(ly - lc)}{lg\left(1 - l\frac{g}{c}\right) - lc}.$$

Which is the equation finally determining the curve, thus so that no point on the curve may be taken arbitrarily.

SCHOLlUM II

30. Therefore by this problem not only these questions can be resolved desiring a maximum or minimum curve for a given abscissa having the formula $\int Zdx$, in which Z besides the determined magnitudes *x, y, p, q, r, s* etc. involves a single integral formula $\Pi = \int [Z]dx$, but even if several formulas of this kind were present. Yet meanwhile it is

required to observe these integral formulas $\Pi = \int [Z]dx$ present in the function *Z* must be prepared thus, so that $[Z]$ shall be a determined function, that is a function of the magnitudes *x, y, p, q, r* etc. involving no further integral formulas. On this account we now investigate a method of solving problems of this kind, when this function $[Z]$ is not determined, but besides *x, y, p, q* etc. involves the new integral formula $\pi = \int [z]dx$. But lest the solution becomes exceedingly large, we will not consider differentials above the second order. For now it is understood, if the solution were prepared as far as to the second order, then by induction the solution can be extended to all the higher orders. Hence finally for us in the first place *Ll* designating the applied line *y*, from which the third, which follows, $Nn = y^{//}$ may be considered to increase by the small amount *nv*. From this increase the following sequence of magnitudes *y, p* and *q* will arise with the increments of their derivatives:

$d \cdot y = 0$	$d \cdot p = 0$	$d \cdot q = +\frac{nv}{dx^2}$
$d \cdot y^{/} = 0$	$d \cdot p^{/} = +\frac{nv}{dx}$	$d \cdot q^{/} = -\frac{2nv}{dx^2}$
$d \cdot y^{//} = +nv$	$d \cdot p^{//} = -\frac{nv}{dx}$	$d \cdot q^{//} = +\frac{nv}{dx^2}$

which table will suffice to resolve any problem, as is understood from the following proposition.

PROPOSITION IV. PROBLEM

31. *Let* $\pi = \int [z]dx$ and $d[z] = [m]dx + [n]dy + [p]dp + [q]dq$, *and the magnitude* $[Z]$ *thus involves the integral formula* π *, thus so that*

$$d[Z] = [L]d\pi + [M]dx + [N]dy + [P]dp + [Q]dq.$$

Now on putting $\Pi = \int [Z]dx$ *, Z shall be a function of x, y, p, q as well as of* Π *, thus so that there shall be*

$$dZ = Ld\Pi + Mdx + Ndy + Pdp + Qdq.$$

With these in place it may be required to define the curve az, which for a given abscissa $AZ = a$ *may have a maximum or minimum value of the formula* $\int Zdx$.

SOLUTION

As we have advised in the preceding Scholium, for us the abscissa is $AL = x$ and the applied line is $Ll = y$, but the value of the abscissa $AL = x$ may correspond to $\int Zdx$, which may not affected by the small increment *nv*. From which the value will need to be determined from the differentials and from the following elements of the abscissas, to which the values $Zdx,\ Z^{/}dx,\ Z^{//}dx,\ Z^{///}dx,\ Z^{IV}dx$ etc. will correspond as far as to the final element *AZ* of the whole abscissas proposed in *Z*. But the differential values of these individual terms may be found by differentiation, by substituting in place of the differentials *dy, dp, dq* the values indicated in the preceding paragraph. Therefore there will be

$$d\cdot Zdx = dx\left(Ld\Pi + \frac{Q\cdot nv}{dx^2} \right)$$

$$d\cdot Z^{/}dx = dx\left(L^{/}d\Pi^{/} + \frac{P^{/}\cdot nv}{dx} - \frac{2Q^{/}\cdot nv}{dx^2} \right)$$

$$d\cdot Z^{//}dx = dx\left(L^{//}d\Pi^{//} + N^{//}\cdot nv - \frac{P^{//}\cdot nv}{dx} + \frac{Q^{//}\cdot nv}{dx^2} \right)$$

$$d\cdot Z^{///} = dx\cdot L^{///}d\Pi^{///}$$

$$d\cdot Z^{IV} = dx\cdot L^{IV}d\Pi^{IV}$$

etc.

Therefore it remains, so that we may define the differentials $d\Pi,\ d\Pi^{/},\ d\Pi^{//},\ d\Pi^{///}$ etc. by *nv*, that is the differential values of the magnitudes $\Pi,\ \Pi^{/},\ \Pi^{//}, \Pi^{///}$ etc. Truly there is :

$$\Pi = \int [Z]dx$$

$$\Pi^{/} = \int [Z]dx + [Z]dx$$

$$\Pi^{//} = \int [Z]dx + [Z]dx + [Z^{/}]dx$$

$$\Pi^{///} = \int [Z]dx + [Z]dx + [Z^{/}]dx + [Z^{//}]dx$$

$$\Pi^{IV} = \int [Z]dx + [Z]dx + [Z^{/}]dx + [Z^{//}]dx + [Z^{///}]dx$$

etc.

Where it is to be observed, the differential value of the magnitude $\int [Z]dx$ to be $=0$, because there the small amount *nv* can introduce no change in the abscissa *AL*, to which $\int [Z]dx$ is referring. Therefore finally it will be required to find the differential values of the differential terms $[Z]dx,\ [Z^{/}]dx,\ [Z^{//}]dx$ etc. Moreover there will be :

$$d\cdot[Z]dx = dx\left([L]d\pi + \frac{[Q]\cdot nv}{dx^2}\right)$$
$$d\cdot[Z]^{/}dx = dx\left([L^{/}]d\pi^{/} + \frac{[P^{/}]\cdot nv}{dx} - \frac{2[Q^{/}]\cdot nv}{dx^2}\right)$$
$$d\cdot[Z]^{//}dx = dx\left([L^{//}]d\pi^{//} + [N^{//}]\cdot nv - \frac{[P^{//}]\cdot nv}{dx} + \frac{[Q^{//}]\cdot nv}{dx^2}\right)$$
$$d\cdot[Z]^{///} = dx[L^{///}]d\pi^{///}$$
$$d\cdot[Z]^{IV} = dx[L^{IV}]d\pi^{IV}$$
etc.

Now again the differential values of the magnitudes $\pi,\ \pi^{/},\ \pi^{//},\ \pi^{///}$ etc. are to be defined by *nv*, which it is necessary to substitute in place of $d\pi,\ d\pi^{/},\ d\pi^{//}$ etc. But since there shall be $\pi = \int [z]dx$ and in $[z]$ the differentials surpassing the second order are not considered to be present, the value will become of the differential itself of π or of $d\pi = 0$, but for the following magnitudes $\pi^{/},\ \pi^{//},\ \pi^{///}$ etc. it will be convenient to note the differential values being found to be :

$$\pi = \int [z]dx$$
$$\pi^{/} = \int [z]dx + [z]dx$$
$$\pi^{//} = \int [z]dx + [z]dx + [z^{/}]dx$$
$$\pi^{///} = \int [z]dx + [z]dx + [z^{/}]dx + [z^{//}]dx$$
$$\pi^{IV} = \int [z]dx + [z]dx + [z^{/}]dx + [z^{//}]dx + [z^{///}]dx$$
etc.

But there will be

$$d\cdot[z]dx = nv\cdot dx\frac{[q]}{dx^2}$$
$$d\cdot[z]^{/}dx = nv\cdot dx\left(\frac{[p^{/}]}{dx} - \frac{2[q^{/}]}{dx^2}\right)$$
$$d\cdot[z]^{//}dx = nv\cdot dx\left([n^{//}] - \frac{[p^{//}]}{dx} + \frac{[q^{//}]}{dx^2}\right)$$
$$d\cdot[z]^{///} = 0$$
$$d\cdot[z]^{IV} = 0$$
etc.

And thus from these the following will be found :

$$d\cdot\pi=0$$

$$d\cdot\pi^{/}=nv\cdot dx\frac{[q]}{dx^2}$$

$$d\cdot\pi^{//}dx=nv\cdot dx\left(\frac{[p^{/}]}{dx}-\frac{[q]}{dx^2}-\frac{2d[q]}{dx^2}\right)$$

$$d\cdot[\pi]^{///}dx=nv\cdot dx\left([n^{//}]-\frac{[p^{/}]}{dx}+\frac{dd[q]}{dx^2}\right)$$

$$d\cdot[\pi]^{IV}dx=nv\cdot dx\left([n^{//}]-\frac{[p^{/}]}{dx}+\frac{dd[q]}{dx^2}\right)$$

$$d\cdot[\pi]^{V}dx=nv\cdot dx\left([n^{//}]-\frac{[p^{/}]}{dx}+\frac{dd[q]}{dx^2}\right)$$

and all the following values will be equal to each other. So that if now these values may be substituted, there will be

$$d\cdot[Z]dx=nv\cdot dx\frac{[Q]}{dx^2}$$

$$d\cdot[Z]^{/}dx=nv\cdot dx\left(\frac{[L^{/}][q]}{dx}+\frac{[P^{/}]}{dx}-\frac{2[Q^{/}]}{dx^2}\right)$$

$$d\cdot[Z]^{//}dx=nv\cdot dx\left([L^{//}]dx\left(\frac{[p^{/}]}{dx}-\frac{[q]+2d[q]}{dx^2}\right)+[N^{//}]-\frac{[P^{//}]}{dx}+\frac{[Q^{//}]}{dx^2}\right)$$

$$d\cdot[Z]^{///}=nv\cdot dx\cdot[L^{///}]dx\left([n^{//}]-\frac{d[p^{/}]}{dx}+\frac{dd[q]}{dx^2}\right)$$

$$d\cdot[Z]^{IV}=nv\cdot dx[L^{IV}]dx\left([n^{//}]-\frac{d[p^{/}]}{dx}+\frac{dd[q]}{dx^2}\right)$$

$$d\cdot[Z]^{V}=nv\cdot dx[L^{V}]dx\left([n^{//}]-\frac{d[p^{/}]}{dx}+\frac{dd[q]}{dx^2}\right)$$

etc.

Hence again there is deduced :

$$d\cdot\Pi = 0$$

$$d\cdot\Pi^{/} = nv\cdot dx\frac{[Q]}{dx^2}$$

$$d\cdot\Pi^{//} = nv\cdot dx\left([L^{/}]dx\frac{[q]}{dx}+\frac{[P^{/}]}{dx}-\frac{[Q]+2d[Q]}{dx^2}\right)$$

$$d\cdot\Pi^{///} = nv\cdot dx\left([L^{//}][p^{/}]-\frac{[q][L^{/}]+2[L^{//}]d[q]}{dx}+[N^{//}]-\frac{d[P^{/}]}{dx}+\frac{dd[Q]}{dx^2}\right)$$

$$d\cdot\Pi^{IV} = nv\cdot dx\left(\begin{array}{l}[L^{///}]dx\left([n^{//}]-\frac{d[p^{/}]}{dx}+\frac{dd[q]}{dx^2}\right)+[L^{//}][p^{/}]-\frac{[q]d[L^{/}]+2[L^{//}]d[q]}{dx}\\+[N^{//}]-\frac{d[P^{/}]}{dx}+\frac{dd[Q]}{dx^2}\end{array}\right)$$

$$d\cdot\Pi^{V} = nv\cdot dx\left(\begin{array}{l}\left(\left([L^{///}]+[L^{IV}]dx\right)\left([n^{//}]\right)-\frac{d[p^{/}]}{dx}+\frac{dd[q]}{dx^2}\right)\\+[L^{//}][p^{/}]-\frac{[q]d[L^{/}]+2[L^{//}]d[q]}{dx}+[N^{//}]-\frac{d[P^{/}]}{dx}+\frac{dd[Q]}{dx^2}\end{array}\right)$$

etc.

And from these now the following determinations may arise :

$$d\cdot Z = nv\cdot dx\frac{Q}{dx^2}$$

$$d\cdot Z' = nv\cdot dx\left(L'dx\frac{[Q]}{dx^2}+\frac{P'}{dx}-\frac{2Q'}{dx^2}\right)$$

$$d\cdot Z'' = nv\cdot dx\left(L''dx\left([L']dx\frac{[q]}{dx^2}+\frac{[P']}{dx}-\frac{[Q]+2d[Q]}{dx^2}\right)+N''-\frac{dP'}{dx}+\frac{Q''}{dx^2}\right)$$

$$d\cdot Z''' = nv\cdot dxL'''dx\left([L''][p']-\frac{[q]d[L']+2[L'']d[q]}{dx}+[N'']-\frac{d[P']}{dx}+\frac{dd[Q]}{dx^2}\right)$$

$$d\cdot Z^{IV} = nv\cdot dxL^{IV}dx\left(\begin{array}{l}[L''']dx\left([n'']-\frac{d[p']}{dx}+\frac{dd[q]}{dx^2}\right)+[L''][p']\\ -\frac{[q]d[L']+2[L'']d[q]}{dx}+[N'']-\frac{d[P']}{dx}+\frac{dd[Q]}{dx^2}\end{array}\right)$$

$$d\cdot Z^{V} = nv\cdot dxL^{V}dx\left(\begin{array}{l}\left([L''']dx+[L^{IV}]dx\right)\left([n'']-\frac{d[p']}{dx}+\frac{dd[q]}{dx^2}\right)+[L''][p']\\ -\frac{[q]d[L']+2[L'']d[q]}{dx}+[N'']-\frac{d[P']}{dx}+\frac{dd[Q]}{dx^2}\end{array}\right)$$

$$d\cdot Z^{VI} = nv\cdot dxL^{VI}dx\left(\begin{array}{l}\left([L''']dx+[L^{IV}]dx+[L^{V}]dx\right)\left([n'']-\frac{d[p']}{dx}+\frac{dd[q]}{dx^2}\right)+[L''][p']\\ -\frac{[q]d[L']+2[L'']d[q]}{dx}+[N'']-\frac{d[P']}{dx}+\frac{dd[Q]}{dx^2}\end{array}\right)$$

etc.

So that all these values there may be more convenient and can be added in turn, we may put for brevity :

$$[h]=[n'']-\frac{d[p']}{dx}+\frac{dd[q]}{dx^2}=n-\frac{d[p]}{dx}+\frac{dd[q]}{dx^2}$$

and

$$[H]=[L][p]-\frac{[q]d[L']+2[L]d[q]}{dx}+[N]-\frac{d[P]}{dx}+\frac{dd[Q]}{dx^2}$$

and the sum of all these will be, that is the value of the differential formula proposed

$\int Zdx$, as follows :

$$nv\cdot dx\left(N-\frac{P}{dx}-\frac{ddQ}{dx^2}\right)+nv\cdot dx\left(L[P]-\frac{[Q]dL+2Ld[Q]}{dx}\right)+nv\cdot dx\cdot L[L][q]$$
$$+nv\cdot dx\cdot[H]\left(L^{///}dx+L^{IV}dx+L^{V}dx+\text{ etc. as far as in } Z\right)$$
$$+nv\cdot dx\cdot[h]\left(\begin{array}{l}L^{IV}dx\cdot[L^{///}]dx+L^{V}dx([L^{///}]dx+L^{IV}dx)+L^{VI}dx([L^{///}]dx+L^{IV}dx+L^{V}dx)\\ +L^{VII}dx([L^{///}]dx+L^{IV}dx+L^{V}dx+L^{VI}dx)+\text{ etc.}\end{array}\right)$$

Therefore here there are had two infinite series, progressing from the term *Ll* as far as to *Zz*, of which the sum of the one $L^{///}dx+L^{IV}dx+L^{V}dx+\text{etc.}$ can be expressed by $H-\int Ldx$, with *H* denoting the value of $\int Ldx$ on putting $x=a$. But so that we may investigate the value of the other series, its sum may be put $=S$, thus so that there shall be

$$S=L^{IV}dx\cdot[L^{///}]dx+L^{V}dx\cdot([L^{///}]dx+L^{IV}dx)+\text{etc.}$$

The following value may be taken $S^{/}=S+dS$, and there will be

$$S+dS=L^{V}dx\cdot[L^{IV}]dx+L^{VI}dx\cdot([L^{IV}]dx+L^{V}dx)+\text{etc.},$$

which taken from the one above will leave

$$-dS=L^{IV}[L^{///}]dx^2+L^{V}[L^{///}]dx^2+L^{VI}[L^{///}]dx^2+\text{etc.},$$

or

$$dS=[L^{///}]dx\left(L^{IV}dx+L^{V}dx+L^{VI}dx+\text{ etc.}\right)$$

and thus

$$-dS=[L^{///}]dx\left(H-\int Ldx\right)$$

and on integrating

$$B=G-\int[L]dx(H-\int Ldx)$$

with some constant *G* thus assumed, so that it makes $S=0$, if there is put $x=a$. With these found the value of the differential formula proposed will become :

$$\int Zdx = nv\cdot dx\Big(N-\frac{dP}{dx}+\frac{ddQ}{dx^2}+L[P]-\frac{[Q]dL+2Ld[Q]}{dx}+L[L][q]$$
$$+\ \left(H-\int Ldx\right)\left([L][p]-\frac{[q]dL+2Ld[q]}{dx}+[N]-\frac{d[P]}{dx}+\frac{dd[Q]}{dx^2}\right)$$
$$+\left(G-\int [L]dx\left(H-f\int Ldx\right)\right)\left([n]-\frac{d[p]}{dx}+\frac{dd[q]}{dx^2}\right)\Big).$$

But this expression can be changed into the following form, from which the value of the differential can be formed more easily, if differentials of higher order than of the second may be present in Z as well as in $[Z]$ and $[z]$. Evidently the value of the differential of the formula $\int Zdx$ corresponding to the abscissa $AZ=a$

$$= nv\cdot dx\left(N-\frac{dP}{dx}+\frac{ddQ}{dx^2}-\frac{d^3R}{dx^3}+\frac{d^4S}{dx^4}-\text{ etc.}\right)$$
$$+\ nv\cdot dx\left(\begin{array}{l}[N]\left(H-\int Ldx\right)-\dfrac{d\cdot[P]\left(H-\int Ldx\right)}{dx}+\dfrac{dd[Q]\left(H-\int Ldx\right)}{dx^3}\\ -\dfrac{d^3\cdot[R]\left(H-\int Ldx\right)}{dx^3}+\dfrac{d^4[S]\left(H-\int Ldx\right)}{dx^4}-\text{ etc.}\end{array}\right)$$
$$+\ nv\cdot dx\left(\begin{array}{l}[n]\left(G-\int[L]dx\left(H-f\int Ldx\right)\right)-\dfrac{d[p]\left(G-\int[L]dx\left(H-f\int Ldx\right)\right)}{dx}\\ +\dfrac{dd\cdot[q]\left(G-\int[L]dx\left(H-f\int Ldx\right)\right)}{dx^2}-\dfrac{d^3\cdot[r]\left(G-\int[L]dx\left(H-f\int Ldx\right)\right)}{dx^3}\text{ etc.}\end{array}\right)$$

Moreover with the value of the differential found, if this is put $=0$, the equation for the curve sought will be obtained. Q. E. I.

COROLLARY 1

32. Therefore the value of the differential for the formula $\int Zdx$ had been found by extending more widely than indeed has been assumed in the proposition, evidently if there were

$$dZ = Ld\Pi + Mdx + Ndy + Pdp + Qdq + Rdr + \text{etc}.$$

and with $d\Pi=[Z]dx$ present, if there shall be

$$d[Z]=[L]d\pi+[M]dx+[N]dy+[P]dp+[Q]dq+[R]dr+\text{etc.}$$

and likewise if on putting $d\pi=[z]dx$ there were

$$d[z]=[m]dx+[n]dy+[p]dp+[q]dq+[r]dr+\text{etc}.$$

Doubtlessly however many differential orders shall be present in the magnitudes Z, $[Z]$ and $[z]$, the given solution will be able to be used.

COROLLARY 2

33. But if on putting $H-\int Ldx=T$ and $G-\int[L]dx\left(H-\int Ldx\right)=V$, the value of the differential will be

$$=nv\cdot dx\left(N-\frac{dP}{dx}+\frac{ddQ}{dx^2}-\frac{d^3R}{dx^3}+\text{etc.}\right)$$
$$+\ nv\cdot dx\left([N]T-\frac{d[P]T}{dx}+\frac{dd\cdot[Q]T}{dx^3}-\frac{d^3[R]T}{dx^3}+\text{etc.}\right)$$
$$+\ nv\cdot dx\left([n]V-\frac{d\cdot[p]V}{dx}+\frac{dd\cdot[q]V}{dx^2}-\frac{d^3[r]V}{dx^3}+\text{etc.}\right).$$

COROLLARY 3

34. Hence the equation for the curve sought therefore will be this :

$$0=N+[N]T+[n]V-\frac{\left(P+[P]T+[p]V\right)}{dx}$$
$$+\frac{dd\left(Q+[Q]T+[q]V\right)}{dx^2}-\frac{d^3\left(R+[R]T+[r]V\right)}{dx^3}+\ \text{etc.},$$

the law of the progression of which, if perhaps there is a need for more terms, is apparent at once.

COROLLARY 4

35. So that also hence problems of this kind will be able to be resolved, in which Z includes within itself not one but several indefinite integral formulas Π of this kind ; or also, if $[Z]$ itself may contain several formulas of this kind $\pi=\int[z]dx$.

COROLLARY 5

36. And then, even if we may put $[z]$ to be a determined function, yet by induction hence a way is apparent of forming the differential value, if further $[z]$ may itself be contained in an indefinite integral formula.

SCHOLIUM

37. Therefore the most general solution of this problem is apparent, because not only all the preceding problem is contained in itself, and the proposition for the case itself is satisfied, truly also by induction whatever kinds of more complex propositions can be accommodated. Which so that it may be easier understood, we may put the integral formula $\pi = \int \zeta dx$ to be present in $[z]$ as well, thus so that there shall be

$$d[z] = [l]d\pi + [m]dx + [n]dy + [p]dp + [q]dq + \text{etc.}$$

with

$$d\zeta = \mu dx + \nu dy + \varphi dp + \chi dq + \text{etc.}$$

present.

Now towards determining the diffential value besides the two integal magnitudes *T* and *V* a third must be defined *W* prepared thus, so that there shall be

$$W = F - \int [l]dx\Big(G - \int [L]dx\Big(H - \int Ldx\Big)\Big),$$

which shall vanish on putting $x = a$. And with this done the value of the differential will become

$$= n\nu \cdot dx(N + [N]T + [n]V + \nu W - \frac{d(P + [P]T + [p]V + \varphi W)}{dx}$$
$$+ \frac{dd \cdot (Q + [Q]T + [q]V + \chi W)}{dx^2} - \text{ etc.}).$$

On account of which indeed no formula of the maximum or minimum will be able to be thought out, which may not contain anything in the solution other than what is contained in such formulas as the solution shows. So that indeed it will be allowed to extend this expression to infinity, if any other new indefinite integral formula may itself be included ; nor shall any difficulty be present, except in supplying a sufficient number of characters. Which since it shall not be necessary to proceed further, it will be appropriate to set out a single principal case, so that in the formula $\int [Z]dx$, which bears the value of Π itself, that magnitude $[Z]$ involves Π again. For in this case the complexity of this kind of integral formulas actually proceeds to infinity ; in as much as, if there shall be

$$d[Z] = [L]d\Pi + [M]dx + [N]dy + [P]dp + [Q]dq + \text{etc.},$$

here again there will be $d\Pi$, as before there was $d\pi$, and because there is $d\Pi = [Z]dx$, the same equation

$$d[Z] = [L]d\Pi + [M]dx + [N]dy + \text{etc.}$$

has returned again, and thus the treatment of the integral formulas will never be interrupted. Therefore we will examine this case, because it affords a significant use to us, as well as permitting a neat solution.

PROPOSITION V. PROBLEM

38. *If* Π *may not be given otherwise, except by the differential equation* $d\Pi = [Z]dx$ *, in which* $[Z]$*, besides the magnitudes pertaining to the curve x, y, p, q, r etc. may include the magnitude* Π *itself, thus so that there shall be*

$$d[Z] = [L]d\Pi + [M]dx + [N]dy + [P]dp + [Q]dq + etc.,$$

Z shall be some union of Π *and of x, y, p, q etc. themselves, thus so that there shall be*

$$dZ = Ld\Pi + Mdx + Ndy + Pdp + Qdq + etc.,$$

to find the curve, in which for a given abscissa AZ = a the formula $\int Zdx$ *shall be a maximum or minimum.*

SOLUTION

We may consider the differentials, which are present both in *Z* as well as in $[Z]$, not to exceed the second order, thus so that the increment *nv*, beyond the point *L* of the abscissa towards the start may introduce no change. The solution nevertheless hence indeed will be able to be made especially general. Therefore the abscissa shall be $AL = x$ and the applied line $Ll = y$, $\int Zdx$ shall experience no change from the addition of the small increment *np* to the applied line $Nn = y^{//}$, and the value of its differential will be $= 0$. On account of which the value of the differential of the formula $\int Zdx$, being extended as far as to the whole abscissa *AZ*, will be able to be deduced from the elements $Zdx,\ Z^{/}dx,\ Z^{//}dx,\ Z^{///}dx$ etc. Moreover the differential values of these individual elements will be found, if these may be differentiated and in place of $dy,\ dy^{/},\ dy^{//},\ dp,\ dp^{/},\ dp^{//}$ et $dq,\ dq^{/},\ dq^{//}$ the values shown in paragraph 30 may be substituted. But because in addition in these differentials $d\Pi, d\Pi^{/},\ d\Pi^{//}$ etc. may be present, we may consider the values of these for the present arising from *nv*, while we may find those to be these:

$d\Pi = nv\cdot\alpha$	$d\Pi^{///} = nv\cdot\delta$	$d\Pi^{VI} = nv\cdot\eta$
$d\Pi^{/} = nv\cdot\beta$	$d\Pi^{IV} = nv\cdot\varepsilon$	$d\Pi^{VII} = nv\cdot\theta$
$d\Pi^{//} = nv\cdot\gamma$	$d\Pi^{V} = nv\cdot\zeta$	etc.

And thus hence the differential values will be

$$d\cdot Zdx = nv\cdot dx\left(L\alpha + \frac{Q}{dx^2}\right)$$
$$d\cdot Z^{/}dx = nv\cdot dx\left(L^{/}\beta + \frac{P^{/}}{dx} - \frac{2Q^{/}}{dx^2}\right)$$
$$d\cdot Z^{//}dx = nv\cdot dx\left(L^{//}\gamma + N^{//} - \frac{dP^{/}}{dx} + \frac{Q^{//}}{dx^2}\right)$$
$$d\cdot Z^{///}dx = nv\cdot dxL^{///}\delta$$
$$d\cdot Z^{IV}dx = nv\cdot dxL^{IV}\varepsilon$$
$$d\cdot Z^{V}dx = nv\cdot dxL^{V}\zeta$$
etc.

So that now we may define the values of the letters $\alpha, \beta, \gamma, \delta, \varepsilon$ etc., it being noted that the differential values of the magnitudes $\Pi, \Pi^{/}, d\Pi^{//}$ etc. are $d\Pi, d\Pi^{/}$, $d\Pi^{//}$ etc.. Truly there is

$$\Pi = \int [Z]dx$$
$$\Pi^{/} = \int [Z]dx + [Z]dx$$
$$\Pi^{//} = \int [Z]dx + [Z]dx + [Z^{/}]dx$$
$$\Pi^{///} = \int [Z]dx + [Z]dx + [Z^{/}]dx + [Z^{//}]dx$$
etc.,

where $\int [Z]dx$ by hypothesis is not affected by the small amount *nv*. Therefore the differential values of the formulas $[Z]dx$, $[Z^{/}]dx$, $[Z^{//}]dx$ etc. are required to be found, which will be

$$d\cdot[Z]dx = nv\cdot dx\left([L]\alpha + \frac{[Q]}{dx^2}\right)$$
$$d\cdot[Z^{/}]dx = nv\cdot dx\left([L^{/}]\beta + \frac{[P^{/}]}{dx} - \frac{2[Q^{/}]}{dx^2}\right)$$
$$d\cdot[Z^{//}]dx = nv\cdot dx\left([L^{//}]\gamma + [N^{//}] - \frac{[P^{//}]}{dx} + \frac{[Q^{//}]}{dx^2}\right)$$
$$d\cdot[Z^{///}]dx = nv\cdot dx[L^{///}]\delta$$
$$d\cdot[Z^{IV}]dx = nv\cdot dx[L^{IV}]\varepsilon$$
$$d\cdot[Z^{V}]dx = nv\cdot dx[L^{V}]\zeta$$
etc.

Therefore from these there will be, as follows

$$d\Pi = nv\cdot\alpha$$
$$d\Pi^{/} = nv\cdot dx\left([L]\alpha+\frac{[Q]}{dx^2}\right)$$
$$d\Pi^{//} = nv\cdot dx\left([L]\alpha+[L^{/}]\beta+\frac{[P^{/}]}{dx}-\frac{[Q]+2d[Q]}{dx^2}\right)$$
$$d\Pi^{///} = nv\cdot dx\left([L]\alpha+[L^{/}]\beta+[L^{//}]\gamma+[N^{//}]-\frac{[P^{/}]}{dx}+\frac{dd[Q]}{dx^2}\right)$$
$$d\Pi^{IV} = nv\cdot dx\left([L]\alpha+[L^{/}]\beta+[L^{//}]\gamma+L^{///}\delta+[N^{//}]-\frac{[P^{/}]}{dx}+\frac{dd[Q]}{dx^2}\right)$$
$$d\Pi^{V} = nv\cdot dx\left([L]\alpha+[L^{/}]\beta+[L^{//}]\gamma+L^{///}\delta+L^{IV}\varepsilon+[N^{//}]-\frac{[P^{/}]}{dx}+\frac{dd[Q]}{dx^2}\right)$$

etc.

With these compared with the assumed values there will be :

$$\alpha=0$$
$$\beta=[L]\alpha dx+\frac{[Q]}{dx}$$
$$\gamma=dx\left([L]\alpha+[L^{/}]\beta+\frac{[P^{/}]}{dx}-\frac{[Q]+2d[Q]}{dx^2}\right)$$
$$\delta=dx\left([L]\alpha+[L^{/}]\beta+[L^{//}]\gamma+[N^{//}]-\frac{[P^{/}]}{dx}+\frac{dd[Q]}{dx^2}\right)$$
$$\varepsilon=dx\left([L]\alpha+[L^{/}]\beta+[L^{//}]\gamma+L^{///}\delta+[N^{//}]-\frac{[P^{/}]}{dx}+\frac{dd[Q]}{dx^2}\right)$$

etc.

And from these equations it is elicited :

$$\alpha=0$$
$$\beta=\frac{[Q]}{dx}$$
$$\gamma=[L^{/}][Q]+[P^{/}]-\frac{[Q]+2d[Q]}{dx}$$
$$\delta=[L^{/}][Q]+[L^{//}][L^{/}][Q]dx+[L^{//}][P^{/}]dx-[L^{//}][Q]-2[L^{//}]d[Q]+[N^{//}]dx-[P^{/}]dx+\frac{dd[Q]}{dx}$$
or $=\delta=[L^{//}][L^{/}][Q]dx+[L^{//}][P^{/}]dx-[Q]d[L^{/}]-2[L^{//}]d[Q]+[N^{//}]dx-[P^{/}]dx+\frac{dd[Q]}{dx},$

which value of δ may be noted, and again there will be

$$\varepsilon = \delta\left(1 + [L^{///}]dx\right)$$
$$\zeta = \delta\left(1 + [L^{///}]dx\right)\left(1 + [L^{IV}]dx\right)$$
$$\eta = \delta\left(1 + [L^{///}]dx\right)\left(1 + [L^{IV}]dx\right)\left(1 + [L^{V}]dx\right)$$
etc.

With these values known the corresponding value for the differential elements $Zdx + Z^{/}dx + Z^{//}dx$ will be

$$= nv \cdot dx\left(N - \frac{dP}{dx} + \frac{ddQ}{dx^2} + L[L][Q] + L[P] - \frac{[Q]dL + 2Ld[Q]}{dx}\right).$$

But the differential value of all the following elements as far as Z, if there may be put

$$V = [L^2][Q] + [L][P] - \frac{[Q]d[L] + 2[L]d[Q]}{dx} + [N] - \frac{[P]}{dx} + \frac{dd[Q]}{dx^2}$$

or $\delta = Vdx$, will be the following :

$$nv \cdot dx(L^{///}dx + L^{IV}dx(1 + [L^{///}]dx) + L^{V}dx(1 + [L^{///}]dx)(1 + [L^{IV}]dx)$$
$$+ \; L^{VI}dx(1 + [L^{///}]dx)(1 + [L^{IV}]dx)(1 + [L^{V}]dx) + \text{etc.})V.$$

On account of which the sum of this series must be found ; hence we may write finally

L in place of $L^{///}$ and $[L]$ in place of $[L^{///}]$, and the sum shall be, that we seek, $= S$; there will be

$$S = Ldx + L^{/}dx(1 + [L]dx) + L^{//}dx(1 + [L]dx)(1 + [L^{/}]dx)$$
$$+ \; L^{///}dx(1 + [L]dx)(1 + [L^{/}]dx)(1 + [L^{//}]dx) + \text{etc.}$$

Now the value following $S^{/} = S + dS$, of S itself assumed, will be

$$S + dS = L^{/}dx + L^{//}dx(1 + [L^{/}]dx) + L^{///}dx(1 + [L^{/}]dx)(1 + [L^{//}]dx) + \text{etc.}$$

Hence

$$-dS = Ldx + L^{/}[L]dx^2 + [L]dx \cdot L^{//}dx(1 + [L^{/}]dx)$$
$$+ [L]dx \cdot L^{///}dx(1 + [L^{/}]dx)(1 + [L^{//}]dx) + \text{etc.},$$

which series, since it may be able to be reduced to the first, will be

$$-dS = Ldx + S^{/}[L]dx,$$

since on account of $S^{/} = S$, $dS+S[L]dx = -Ldx$; which integrated gives

$$e^{\int [L]dx} S = C - \int e^{\int [L]dx} Ldx,$$

which constant *C* must be taken thus, so that on putting $x = a$ shall make $S = 0$. On this account the value of that series will be

$$S = e^{-\int [L]dx}(C - \int e^{\int [L]dx} Ldx).$$

Therefore from these the following differential value of the proposed formula $\int Zdx$ will arise :

$$nv\cdot dx\left(\begin{array}{l} N - \frac{P}{dx} + \frac{ddQ}{dx^2} + L[L][Q] + L[P] - \frac{[Q]dL + 2Ld[Q]}{dx} \\ +S\left([L^2][Q] + L[P] - \frac{[Q]d[L] + 2[L]d[Q]}{dx} + [N] - \frac{d[P]}{dx} + \frac{dd[Q]}{dx^2}\right)\end{array}\right),$$

which may be changed into this more suitable form

$$nv\cdot dx\left(N - \frac{P}{dx} + \frac{ddQ}{dx^2} + [N]S - \frac{d\cdot[P]S}{dx} + \frac{dd\cdot[Q]S}{dx^2}\right).$$

Hence moreover the differential value can be formed of the formula $\int Zdx$, if the differentials rise to some order both in *Z* as well as in $[Z]$. Towards effecting this, the value of the integral formula shall be $\int e^{\int [L]dx} Ldx$, which prevails $= H$, if there may be put $x = a$, and for brevity therefore there is written V in place of this expression $e^{-\int [L]dx}(H - \int e^{\int [L]dx} Ldx)$, the value of the differential will be

$$= nv\cdot dx\left(N + [N]V - \frac{d\cdot(P + [P]V)}{dx} + \frac{dd(Q + [Q]V)}{dx^2} - \frac{d^3(R + [R]V)}{dx^3} + \text{ etc.}\right).$$

And hence this equation will arise for the curve sought :

$$0 = N + [N]V - \frac{d\cdot(P + [P]V)}{dx} + \frac{dd(Q + [Q]V)}{dx^2} - \frac{d^3(R + [R]V)}{dx^3} + \frac{d^4(S + [S]V)}{dx^4} - \text{ etc.}$$

Q. E. I.

COROLLARY 1

39. Therefore this proposition looks after the resolution of problems of this kind, in which the formula $\int Zdx$ of the maximum or minimum may contain the magnitude Π, which not even the formula for the integral can show about the curve from the magnitudes *x, y, p, q, r* etc., but the determination of which depends on the resolution of some differential equation. For there may be had $d\Pi = [Z]dx$ and the magnitude $[Z]$ itself may be considered to be included in some Π.

COROLLARY 2

40. This case deserves attention, where there is $L = [L]$, certainly in which the formula $\int e^{\int [L]dx} Ldx$ becomes integrable, with the integral $e^{\int [L]dx}$ being present. So that if therefore, on putting $x = a$, $e^{\int [L]dx}$ may change into *H*, there will become

$$V = He^{-\int [L]dx} - 1.$$

COROLLARY 3

41. This case has an important place, when the curve is sought, in which the formula itself $\Pi = \int [Z]dx$ shall be a maximum or minimum. For then there becomes $Z = [Z]$ and hence $L = [L],\ M = [M],\ N = [N]$ etc. And thus hence the value of the differential will be

$$= nv \cdot dx \left(H[N]e^{-\int [L]dx} - \frac{d \cdot H[P]e^{-\int [L]dx}}{dx} + \frac{dd \cdot H[Q]e^{-\int [L]dx}}{dx^2} - \text{ etc.} \right).$$

And the equation for the curve will be

$$0 = [N]e^{-\int [L]dx} - \frac{d \cdot [P]e^{-\int [L]dx}}{dx} + \frac{dd \cdot [Q]e^{-\int [L]dx}}{dx^2} - \text{ etc.}$$

COROLLARY 4

42. Because the magnitude *H* has vanished from this equation depending on the given abscissa $AZ = a$ by division, it is apparent from these cases a curve satisfying as single abscissa, likewise is going to be satisfying all the other abscissas, thus so that these problems shall be similar to these, in which the magnitude *Z* is a determined function.

COROLLARY 5

43. Therefore if the magnitude $\Pi = \int [Z]dx$ shall be a maximum or minimum with

$$d[Z] = [L]d\Pi + [M]dx + [N]dy + [P]dp + [Q]dq + \text{etc.}$$

being present, a curve will be able to be shown, which may be endowed with this property for any single abscissa ; and its nature may be expressed by this equation

$$0 = [N]e^{-\int [L]dx} - \frac{d \cdot [P]e^{-\int [L]dx}}{dx} + \frac{dd \cdot [Q]e^{-\int [L]dx}}{dx^2} - \text{ etc.}$$

From which with the individual terms above expanded out, they will exceed the magnitude of the exponential $e^{-\int [L]dx}$, and thus the integral formula itself $\int [L]dx$.

SCHOLIUM I

44. The use of this exceptional proposition is in questions prepared thus, so that indefinite magnitudes contained in these may be unable to be shown by integral formulas, truly require the construction of differential equations. And this solution prevails likewise, whether a single magnitude Π of this kind, or several, may be present in the formula $\int Zdx$ of the maxima or minima ; because indeed if several magnitudes of this kind Π may be present, also there will be several values of the letters $L, [L], [M], [N], [P], [Q]$ etc. and also of the letters $V = e^{-\int [L]dx}(H - \int e^{\int [L]dx} Ldx)$; which all equally in that same manner, which we have found, will be given in the differential value of the formula $\int Zdx$ to be introduced for the equation of the curve ; and overall the treatment will be the same, as if only a single magnitude were present. Moreover because this letter Π , whose absolute value cannot be shown by magnitudes pertaining to the curve, remains almost in all the terms, the equation for the curve, which is found, not only will be in agreement with the letters x, y, p, q, r etc., but also with that magnitude Π itself and other integral formulas generally depending on that may be involved, such as $\int [L]dx$ and $\int Ldx$. Whereby, so that an equation for the pure curve may be produced, in which only the letters x, y, p, q etc. may be retained, it is required for the equation found, after $\int [L]dx$ and $\int Ldx$ have been removed from the equation found, to be taken jointly with $d\Pi = [Z]dx$ and with the aid of this the value Π to be removed. But although in this manner it arrives at a differential of higher order, yet not as many arbitrary constants are agreed to be present. For just as that equation $d\Pi = [Z]dx$ itself as well as the remaining earlier certain equations require a determination, from which several constants will be determined. Moreover it is to be observed that the truth of this

method can be proved from the preceding, when the equation $d\Pi = [Z]dx$ thus was prepared, so that it allowed integration ; then indeed the same questions will be able to be resolved by the methods treated before and thus it will be permitted to observe an agreement. Thus, if $[Z]$ may depend on x and Π only, then it will be sure that Π is a reliable determined function of x and the solution to relate to the preceding chapter. Truly likewise this solution may be revealed ; for since in this case there shall be $[N]=0,\ [P]=0,\ [Q]=0$ etc., the equation for the curve shall be

$$0 = N - \frac{dP}{dx} + \frac{ddQ}{dx^2} - \text{etc.},$$

which will be found by the same prior method. But the use of this solution will become clearer by some examples.

EXAMPLE I

45. *To find the curve, in which the maximum value of* Π *shall be present with* $d\Pi = gdx - \alpha\Pi^n dx\sqrt{(1+pp)}$.

This question arises when the curve is sought, upon which a weight in a medium with a resistance following the 2*n-fold* ratio of the speeds may obtain the maximum descending speed ; for Π may denote the square of the speed and *g* the force of gravity acting along the direction of the axis *AZ*. [Due to Euler's use of the *Vis Viva* idea, prevalent at the time, represented here by Π , the factor 2 becomes incorporated with the acceleration of gravity, to give 2*g* for the numerically correct acceleration of gravity, where *g* here is as defined by Euler.] And thus this case belongs to the case set out in corollaries 3, 4 and 5, in which there was $Z = [Z] = g - \alpha\Pi^n\sqrt{(1+pp)}$; and thus the curve being satisfactory for one abscissa will be valid for all abscissas. Therefore since there shall be

$$dZ = -\alpha n\Pi^{n-1}d\Pi\sqrt{(1+pp)} - \frac{\alpha\Pi^n pdp}{\sqrt{(1+pp)}},$$

there will be

$$[L] = -\alpha n\Pi^{n-1}\sqrt{(1+pp)},\ [M]=0,\ [N]=0,\ [P] = -\frac{\alpha\Pi^n p}{\sqrt{(1+pp)}},\ \left[Q\right]=0, \text{ etc.}$$

From which this equation will be found for the curve sought :

$$0 = -d\cdot[P]e^{-\int[L]dx} \quad\text{or}\quad [P]e^{-\int[L]dx} = C\,;$$

and hence

$$-\int [L]dx = lC - l\left[P\right] \text{ and } \int [L]dx = \frac{d\left[P\right]}{\left[P\right]}.$$

Therefore with the due values substituted in place of $[L]$ and $[P]$, there will be

$$\int \alpha n\Pi^{n-1}dx\sqrt{(1+pp)} = +lC - l(-\alpha) - l\Pi^{n} - lp + l\sqrt{(1+pp)}\ ;$$

and hence

$$\alpha n\Pi^{n-1}dx\sqrt{(1+pp)} = -\frac{nd\Pi}{\Pi} - \frac{dp}{p} + \frac{pdp}{(1+pp)} = -\frac{dp}{p(1+pp)} - \frac{nd\Pi}{\Pi}$$

or

$$0 = nd\Pi + \alpha n\Pi^{n}dx\sqrt{(1+pp)} + \frac{\Pi dp}{p(1+pp)}.$$

Which equation, in order that Π may be eliminated, is required to be joined with this equation :

$$d\Pi + \alpha\Pi^{n}dx\sqrt{(1+pp)} = gdx\ ;$$

from which at one there becomes

$$0 = ngdx + \frac{\Pi dp}{p(1+pp)} \text{ and } \Pi = -\frac{ngpdx(1+pp)}{dp}.$$

Therefore since the curve were found, this equation at once provides the speed of the body at some place on the curve. Putting $dx = -\frac{tdp}{ng}$, there will be

$$\Pi = pt(1+pp) \text{ and } d\Pi = pd(1+pp) + tdp(1+3pp)\ ;$$

and hence this equation will be obtained

$$pdt(1+pp) + tdp(1+3pp) - \frac{\alpha p^{n}t^{n+1}(1+pp)^{n+\frac{1}{2}}dp}{ng} + \frac{tdp}{n} = 0,$$

which is changed into this :

$$\frac{npdt(1+pp) + tdp(n+1+3npp)}{nt^{n+1}p^{n+2}(1+pp)^{n+\frac{1}{2}}} = \frac{\alpha dp}{ngp^{2}},$$

the integral of which is

$$\frac{1}{nt^n p^{n+1}(1+pp)^{n-\frac{1}{2}}}=\frac{\alpha}{ngp}+\frac{\beta}{ng},$$

or

$$g=(\alpha+\beta p)t^n p^n(1+pp)^{n-\frac{1}{2}};$$

and hence

$$t=\frac{\sqrt[n]{g}}{p(1+pp)^{1-1:2n}\sqrt[n]{(\alpha+\beta p)}}.$$

Therefore there will be

$$dx=\frac{-dp}{np(1+pp)^{1-1:2n}\sqrt[n]{g^{n-1}(\alpha+\beta p)}}.$$

and

$$dy=\frac{-dp}{n(1+pp)^{1-1:2n}\sqrt[n]{g^{n-1}(\alpha+\beta p)}};$$

and hence

$$\Pi=\sqrt[n]{\frac{g\sqrt{(1+pp)}}{\alpha+\beta p}}.$$

Therefore there will be

$$x=-\frac{1}{ng}\int\frac{dp}{p(1+pp)}\sqrt[n]{\frac{g\sqrt{(1+pp)}}{\alpha+\beta p}}$$

and

$$y=-\frac{1}{ng}\int\frac{dp}{1+pp}\sqrt[n]{\frac{g\sqrt{(1+pp)}}{\alpha+\beta p}}.$$

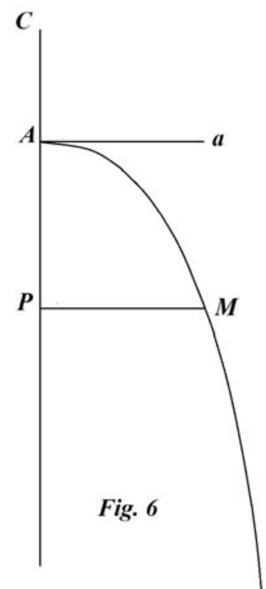

Fig. 6

Hence it will be apparent the magnitude Π nowhere on the curve can be equal to $=0$; on this account at the start of the curve Π now will have some certain value. But so that the nature of the curve may become more apparent, from the equation $\Pi=-\frac{ngpdx(1+pp)}{dp}$ it is apparent the value of *dp* is required to be negative everywhere, from which the nature of the curve towards the axis will be concave. Therefore because the values of *p* by receding from the curve initially decrease, on the curve itself the initial *p* will have a maximum value. Hence we may put (Fig. 6) the beginning of the curve there, where there is $p=\infty$. Therefore *AP* shall be the vertical axis of the curve, in the direction of which the force of gravity g may pull P downwards, and at the beginning of the curve A , Aa shall be a horizontal tangent ; and there on the curve the body begins with a speed, the square of which shall be $=b$. Therefore there will be, on putting $p=\infty$, $b=\sqrt[n]{\frac{g}{\beta}}$, and $\beta b^n = g$ or $\beta = \frac{g}{b^n}$. Again, maintaining uniformity, there shall be $\alpha = \frac{1}{k^n}$. Now because AM shall be the curve sought and putting $AP = x,\ PM = y$ and $dy = pdx$, at M the square of the speed shall be

$$\Pi = bk\sqrt[n]{\frac{g\sqrt{(1+pp)}}{b^n + gk^n p}}\ ;$$

and where the tangent of the curve becomes vertical, there the square of the speed will be $=k\sqrt[n]{g}.$ Moreover the construction of the curve will be prepared thus, so that

$$x = -\frac{bk}{ng}\int \frac{dp}{p(1+pp)}\sqrt[n]{\frac{g\sqrt{(1+pp)}}{b^n + gk^n p}}.$$

and

$$y = -\frac{bk}{ng}\int \frac{dp}{1+pp}\sqrt[n]{\frac{g\sqrt{(1+pp)}}{b^n + gk^n p}}$$

Following this, a particular property of the curve deserves a mention, or the relation between the centrifugal force of the descending body, which is $\frac{2\Pi}{\text{radius of curvature}}$, and the normal force, which is $\frac{gp}{\sqrt{(1+pp)}}$. For if the centrifugal force

$$\frac{2\Pi}{\text{rad. curv.}} = \frac{-2\Pi\, dp}{dx(1+pp)^{3:2}}$$

it may be considered $=F$ and the force of the normal $\frac{gp}{\sqrt{(1+pp)}} = G$, from the equation

$$\Pi = -\frac{ngpdx(1+pp)}{dp} \quad \text{or} \quad \frac{-2\Pi\, dp}{dx(1+pp)^{3:2}} = \frac{2ngp}{\sqrt{(1+pp)}}$$

there will be this relation between the centrifugal force F and the normal force G, thus so that there shall be $F = 2nG$; clearly the normal force will be to the centrifugal force as 1 to $2n$. A body at A with a given speed begins the motion by descending on the curve AM

to some location M for the corresponding abscissa AP it will have a greater speed, than if it were descending on any other curve with the same initial speed. Moreover we may set out two main cases. And the 1st case shall be with the resistance proportional to the squares of the speeds, there will be $n=1$ and $F=2G$. However, for the curve there will be had:

$$x=-bk\int\frac{dp}{p\left(b+gkp\right)\sqrt{(1+pp)}}$$

and

$$y=-bk\int\frac{dp}{\left(b+gkp\right)\sqrt{(1+pp)}}$$

and likewise the arc of the curve will be

$$AM=-bk\int\frac{dp}{p\left(b+gkp\right)}=C+kl\frac{b+gkp}{p}.$$

The arc may be put $AM=s$, which since it must vanish on considering $p=\infty$, there will be
$s=kl\dfrac{b+gkp}{gkp}$, and hence

$$e^{s:k}gkp=b+gkp \text{ and } p=\frac{b}{gk(e^{s:k}-1)}=\frac{dy}{dx}.$$

From which there arises

$$bdx+gkdy=gke^{s:k}dy.$$

But again from the equation

$$y=-bk\int\frac{dp}{\left(b+gkp\right)\sqrt{(1+pp)}}$$

by integrating there will be

$$y=\frac{bk}{\sqrt{(bb+ggkk)}}l\frac{(b+gkp)(b+\sqrt{(bb+ggkk)})}{gk(bp-gk+\sqrt{(bb+ggkk)(1+pp)})}.$$

2nd case. The resistance shall be proportional to the speed themselves, making
$n=\frac{1}{2}$ and $F=G$, that is the centrifugal force will be equal to the normal force. Which

two forces since they shall be opposite, will satisfy that curve sought, which is not pressed on by a body descending that at all. But there will be

$$x = -2gbk\int \frac{dp}{p\left(\sqrt{b} + gp\sqrt{k}\right)^2}$$

and

$$y = -2gbk\int \frac{dp}{\left(\sqrt{b} + gp\sqrt{k}\right)^2} = \frac{2b\sqrt{k}}{\sqrt{b} + gp\sqrt{k}};$$

and hence

$$ydx\sqrt{b} + gydy\sqrt{k} = 2bdx\sqrt{k} \ \text{ and } \ dx = \frac{gydy\sqrt{k}}{2b\sqrt{k - y\sqrt{b}}};$$

and hence on integrating

$$x = -gy\sqrt{\frac{k}{b}} + 2gkl\frac{2b\sqrt{k}}{2b\sqrt{k} - y\sqrt{b}}.$$

Therefore this curve not only can be constructed by logarithms, truly it is a part of an oblique angled logarithmic curve. Evidently it will be that curve of projection, which a body will describe projected freely according to this hypothesis of resistance. This agreement will be apparent from that, because the curve will sustain no pressing force from the moving body, which is a property of freely described curves.
[This would appear to agree with what Newton found for a trajectory with resistance of this kind, at the start of Book II of his *Principia*, in his case by far less clear mathematics.]

EXAMPLE II

46. *To find the curve, in which for a given abscissa* $x = a$ *that formula* $\int \frac{dx\sqrt{(1+pp)}}{\sqrt{\Pi}}$ *shall be a minimum, with* $d\Pi = gdx - \alpha\Pi^n dx\sqrt{(1+pp)}$ *being present.*

This question agrees with that, in which the curve is required, on which a body descending in a resisting medium, the resistance of which is as the $2n$ exponent power of the speed, the corresponding arc of the abscissa a will be resolved the most quickly. For here g may denote the force of gravity acting along the direction of the axis, $\sqrt{\Pi}$ the speed of the body at some location and $\alpha\Pi^{n}$ the resistance of the medium itself. And thus there will be $Z=\frac{\sqrt{(1+pp)}}{\sqrt{\Pi}}$ and hence

$$dZ=\frac{-d\Pi\sqrt{(1+pp)}}{2\Pi\sqrt{\Pi}}+\frac{pdp}{\sqrt{\Pi(1+pp)}},$$

from which there shall be

$$L=\frac{-\sqrt{(1+pp)}}{2\Pi\sqrt{\Pi}},\ M=0,\ N=0,\ P=\frac{p}{\sqrt{\Pi(1+pp)}}.$$

Again there will be

$$[Z]=g-\alpha\Pi^{n}\sqrt{(1+pp)} \text{ and } d[Z]=-\alpha n\Pi^{n-1}d\Pi\sqrt{(1+pp)}-\frac{\alpha\Pi^{n}pdp}{\sqrt{(1+pp)}};$$

from which

$$[L]=-\alpha n\Pi^{n-1}\sqrt{(1+pp)},$$

$$[M]=0\ ,\ [N]=0 \text{ and } \left[P\right]=-\frac{\alpha\Pi^{n}p}{\sqrt{(1+pp)}}.$$

Therefore there will be found :

$$V=\ e^{\alpha n\int\Pi^{n-1}dx\sqrt{(1+pp)}}\left(\int e^{-\alpha n\int\Pi^{n-1}dz\sqrt{(1+pp)}}\ \frac{dx\sqrt{(1+pp)}}{2\Pi\sqrt{\Pi}}-H\right),$$

with *H* denoting that value of the formula

$$e^{-\alpha n\int\Pi^{n-1}dz\sqrt{(1+pp)}}\ \frac{dx\sqrt{(1+pp)}}{2\Pi\sqrt{\Pi}},$$

which will be obtained, if there becomes $x=a$. And now *V* must vanish on putting $x=a$ and there shall be

$$dV=\ \alpha nV\Pi^{n-1}dx\sqrt{(1+pp)}+\frac{dx\sqrt{(1+pp)}}{2\Pi\sqrt{\Pi}}.$$

From these this equation will be found for the curve sought :

$$d\cdot(P+[P]V)=0 \text{ and } P+[P]V=C \text{ or } V=\frac{C-P}{[P]}$$

Therefore with these values owed substituted, there will be

$$e^{\alpha n\int \Pi^{n-1}dx\sqrt{(1+pp)}}\left(\int e^{-\alpha n\int \Pi^{n-1}dx\sqrt{(1+pp)}}\frac{dx\sqrt{(1+pp)}}{2\Pi\sqrt{\Pi}}-H\right)=\frac{p-C\sqrt{\Pi(1+pp)}}{\alpha\Pi^{n}p\sqrt{\Pi}}.$$

Whereby it is required thus to determine the constant C, so that on putting $x=a$ it becomes

$$C=\frac{p}{\sqrt{\Pi(1+pp)}}.$$

But since there shall be

$$V=\frac{1}{\alpha\Pi^{n}\sqrt{\Pi}}-\frac{C\sqrt{(1+pp)}}{\alpha\Pi^{n}p}.$$

there will be

$$dV=\frac{-(n+\tfrac{1}{2})d\Pi}{\alpha\Pi^{n+1}\sqrt{\Pi}}+\frac{nCd\Pi\sqrt{(1+pp)}}{\alpha\Pi^{n+1}p}+\frac{Cdp}{\alpha\Pi^{n}p^{2}\sqrt{(1+pp)}}$$
$$=\frac{dx\sqrt{(1+pp)}}{2\Pi\sqrt{\Pi}}+\frac{ndx\sqrt{(1+pp)}}{\Pi\sqrt{\Pi}}-\frac{nC(1+pp)dx}{p\Pi}$$

with the equation called into to help :

$$dV=\ \alpha nV\Pi^{n-1}dx\sqrt{(1+pp)}+\frac{dx\sqrt{(1+pp)}}{2\Pi\sqrt{\Pi}}.$$

But since there shall be

$$d\Pi=gdx-\alpha\Pi^{n}dx\sqrt{(1+pp)}\ ,$$

there will be

$$\frac{-(n+\tfrac{1}{2})gdx}{\alpha\Pi^{n+1}\sqrt{\Pi}}+\frac{nCgdx(1+pp)}{\alpha\Pi^{n+1}p}+\frac{Cdp}{\alpha\Pi^{n}p^{2}\sqrt{(1+pp)}}=0$$

or

$$\frac{Cdp}{p^{2}\sqrt{(1+pp)}}=\frac{(n+\tfrac{1}{2})gdx}{\Pi\sqrt{\Pi}}-\frac{nCgdx\sqrt{(1+pp)}}{\Pi p}.$$

But if now this equation may be joined with that

$$d\Pi=gdx-\alpha\Pi^{n}dx\sqrt{(1+pp)}\ ,$$

the magnitude Π will be able to be eliminated and with this agreed upon, the equation for the curve sought is found. But in this way the calculation will become most tedious and barely able to be treated. Truly the greatest aid will bring this final equation into the changed form :

$$\frac{Cdp}{gp^2}=\frac{(n+\frac{1}{2})dx\sqrt{(1+pp)}}{\Pi\sqrt{\Pi}}-\frac{nCdx(1+pp)}{\Pi\, p},$$

to which expression the value of dV has been found before to be equal ; therefore there will be

$$dV=\frac{Cdp}{gpp} \text{ and } V=D-\frac{C}{gp}=\frac{1}{\alpha\Pi^n\sqrt{\Pi}}-\frac{C\sqrt{(1+pp)}}{\alpha\Pi^n p}.$$

Therefore now we have these two equations

$$\frac{Cdp}{gp^2}=\frac{(n+\frac{1}{2})dx\sqrt{(1+pp)}}{\Pi\sqrt{\Pi}}-\frac{nCdx(1+pp)}{\Pi\, p} \text{ and } \alpha D-\frac{\alpha C}{gp}=\frac{1}{\Pi^n\sqrt{\Pi}}-\frac{C\sqrt{(1+pp)}}{\Pi^n p}.$$

From which if Π may be eliminated, an equation will be had between p and x of this kind, so that x occurs nowhere, but only dx everywhere, from which that equation will be able to be constructed and thus the curve itself. Or p may be determined easier by Π from the last equation, and this value substituted into the fundamental equation

$$dx=\frac{d\Pi}{g-\alpha\Pi^n\sqrt{(1+pp)}}$$

will give the value of x through Π, clearly there will be

$$x=\int\frac{d\Pi}{g-\alpha\Pi^n\sqrt{(1+pp)}} \text{ and } y=\int\frac{pd\Pi}{g-\alpha\Pi^n\sqrt{(1+pp)}}.$$

But a constant D must be taken thus, so that on putting $x=a$, in which case there becomes

$$C=\frac{p}{\sqrt{\Pi(1+pp)}} \text{ and making } D=\frac{p}{g\sqrt{\Pi(1+pp)}}$$

or then there must be $\frac{C}{D}=gp$.

SCHOLIUM II

47. Therefore in these two chapters we have established a method of finding curved lines, in which for a given magnitude of the abscissa $= a$ the formula $\int Zdx$ shall be a maximum or minimum, with Z being some function of x, y, p, q, r etc. either determined or indeterminate. But a determined function is for us, one which, if somewhere the values of the letters x, y, p, q, r etc. may be given, that itself can be designated, either algebraic or transcendental. But an indeterminate [or indefinite] function is one which, for given values of the letters themselves, which prevail in one place, is unable to be designated, but likewise involves all the preceding values, however this may come about, if integral signs occur. Therefore in the second chapter we have treated resolving all problems, in which Z is a determined function; truly in this third chapter we have perused these formulas, in which Z itself either is an indefinite function or of such a kind that it involves one or several; and likewise we have shown a method for these cases, for which that indefinite function indeed never can be represented by integral formulas, and truly requires the resolution of a differential equation. Therefore now we may establish these cases, in which an expression, which must be a maximum or minimum, is not a simple integral formula, as we have assumed up to this stage, but composed somehow from several formulas of this kind, and likewise the method may uncover several other problems, which cannot be considered for orthogonal coordinates, to be resolved quickly.

CAPUT III

DE INVENTIONE CURVARUM

MAXIMI MINIMIVE PROPRIETATE PRAEDITARUM

SI IN IPSA MAXIMI MINIMIVE FORMULA INSUNT

QUANTITATES INDETERMINATAE

PROPOSITIO I. PROBLEMA

1. *Invenire* (Fig. 4) *incrementa, quae quantitas integralis indeterminata in quovis abscissae puncto ab aucta alicubi una applicata Nn particula nv capit*.

SOLUTIO

Sit abscissa $AH = x$, applicata respondens $Hh = y$ et proposita sit quantitas quaecunque indeterminata Π abscissae *AH* respondens, quae sit formula integralis indefinite integrationem non admittens. Quantitas haec Π ita sit comparata, ut ipsa, quatenus abscissae *AH* seu puncto *H* respondet, ab aucta applicata *Nn* non mutetur; quod eveniet, si in Π differentialia non ultra quintum gradum assurgant; quem in finem quintam demum applicatam *Nn* ab *Hh* computando mutari ponimus. Si enim in Π differentialia altiorum graduum continerentur, tum deberet ulterior demum applicata post *Nn* particula infinite parva augeri. Sufficiet autem solutionem ad quinque tantum differentialium in Π contentorum gradus extendere, cum inde, si etiam altiora affuerint differentialia, solutionem ad ea accommodare liceat. Quemadmodum igitur puncto abscissae *H* respondet valor Π, ita secundum nostram notandi methodum puncto sequenti *I* respondebit valor Π^{I}, puncto *K* vero Π^{II}, puncto *L* valor Π^{III} et ita porro. Id ergo erit investigandum, quanta incrementa ex translatione puncti *n* in *v* singuli hi valores derivativi Π^{I}, Π^{II}, Π^{III}, Π^{IV} etc. accipiant, seu definiri debent eorum differentialia, si sola applicata *Nn*, quae est $= y^{V}$, variari et particula *nv* augeri ponatur: erit autem hoc sensu $d\cdot\Pi=0$, quia valorem Π puncto *H* respondentem inde non affici ponimus. Quoniam iam Π est formula integralis indefinita, sit ea $=\int[Z]dx$ et $[Z]$ sit functio ipsarum *x*, *y*, *p*, *q*, *r*, *s* et *t*, ita ut sit

$$\mathrm{d}[Z]=[M]dx+[N]dy+[P]dp+[Q]dq+[R]dr+[S]ds+[T]dt\,;$$

unde simul valores derivativi ipsius $d[Z]$, nempe $d[Z'], d[Z''], d[Z''']$ etc. per notandi modum receptum formari poterunt. His positis erit, ut sequitur:

$$\Pi=\int[Z]dx$$
$$\Pi'=\int[Z]dx+[Z]dx$$
$$\Pi''=\int[Z]dx+[Z]dx+[Z']dx$$
$$\Pi'''=\int[Z]dx+[Z]dx+[Z']dx+[Z'']dx$$
$$\Pi^{IV}=\int[Z]dx+[Z]dx+[Z']dx+[Z'']dx+[Z''']dx$$
etc.

Iam videamus, quanta incrementa singula haec membra $[Z]dx, [Z']dx, [Z'']dx, [Z''']dx$ etc. ex adiecta particula *nv* ad applicatam *Nn* capiant; quae obtinebuntur ex ipsorum differentialibus, ponendo loco differentialium valores paragrapho 56 Capitis praecedentis expositos; erit itaque

$$d\cdot[Z]dx = nv\cdot dx\cdot\frac{[T]}{dx^5}$$

$$d\cdot[Z^{/}]dx = nv\cdot dx\left(\frac{[S^{/}]}{dx^4}-\frac{5[T^{/}]}{dx^5}\right)$$

$$d\cdot[Z^{//}]dx = nv\cdot dx\left(\frac{[R^{//}]}{dx^3}-\frac{4[S^{//}]}{dx^4}+\frac{10[T^{//}]}{dx^5}\right)$$

$$d\cdot[Z^{///}]dx = nv\cdot dx\left(\frac{[Q^{///}]}{dx^2}-\frac{3[R^{///}]}{dx^3}+\frac{6[S^{///}]}{dx^4}-\frac{10[T^{///}]}{dx^5}\right)$$

$$d\cdot[Z^{IV}]dx = nv\cdot dx\left(\frac{[P^{IV}]}{dx}-\frac{2[Q^{IV}]}{dx^2}+\frac{3[R^{IV}]}{dx^3}-\frac{4[S^{IV}]}{dx^4}+\frac{5[T^{IV}]}{dx^5}\right)$$

$$d\cdot[Z^{V}]dx = nv\cdot dx\left([N^{V}]-\frac{[P^{V}]}{dx}+\frac{[Q^{V}]}{dx^2}-\frac{[R^{V}]}{dx^3}+\frac{[S^{V}]}{dx^4}-\frac{[T^{V}]}{dx^5}\right)$$

$$d\cdot[Z^{VI}]dx = 0$$

$$d\cdot[Z^{VII}]dx = 0.$$

et reliqua sequentia omnia evanescent.

Ex his nunc colligentur incrementa valorum $\Pi, \Pi^{/}, \Pi^{//}, \Pi^{///}$ etc., quae recipiunt ex translatione puncti n in v ; erit scilicet

$$d\cdot\Pi = 0$$

$$d\cdot\Pi^{/} = nv\cdot dx\cdot\frac{[T]}{dx^5}$$

$$d\cdot\Pi^{//} = nv\cdot dx\left(\frac{[S^{/}]}{dx^4} - \frac{4[T^{/}]+d[T]}{dx^5}\right)$$

$$d\cdot\Pi^{///} = nv\cdot dx\left(\frac{[R^{//}]}{dx^3} - \frac{3[S^{//}]+d[S]}{dx^4} + \frac{6[T^{//}]+4[T^{/}]-d[T]}{dx^5}\right)$$

$$d\cdot\Pi^{IV} = nv\cdot dx\left(\begin{array}{l}\frac{[Q^{///}]}{dx^2} - \frac{2[R^{///}]+d[R^{//}]}{dx^3} + \frac{3[S^{///}]+3d[S^{//}]-d[S^{/}]}{dx^4}\\ -\frac{4[T^{///}]+6[T^{//}]-4d[T^{/}]+d[T]}{dx^5}\end{array}\right)$$

$$d\cdot\Pi^{V} = nv\cdot dx\left(\begin{array}{l}\frac{[P^{IV}]}{dx} - \frac{[Q^{IV}]+d[Q^{///}]}{dx^2} + \frac{[R^{IV}]+2d[R^{///}]-d[R^{//}]}{dx^3}\\ -\frac{[S^{IV}]+3d[S^{///}]-3d[S^{//}]+d[S^{/}]}{dx^4}\\ +\frac{[T^{IV}]+4d[T^{IV}]-6d[T^{///}]+4d[T^{/}]-d[T]}{dx^5}\end{array}\right)$$

$$d\cdot\Pi^{VI} = nv\cdot dx\left(\begin{array}{l}[N^{V}] - \frac{d[P^{IV}]}{dx} + \frac{[Q^{IV}]-d[Q^{///}]}{dx^2} - \frac{d[R^{IV}]-2d[R^{///}]+d[R^{//}]}{dx^3}\\ +\frac{d[S^{IV}]-3d[S^{IV}]+3d[S^{IV}]-d[S^{IV}]}{dx^4}\\ -\frac{[T^{IV}]-4d[T^{///}]+6d[T^{//}]-4d[T^{/}]+d[T^{/}]}{dx^5}.\end{array}\right)$$

Huic autem incremento aequalia sunt incrementa omnium sequentium valorum, nempe ipsorum Π^{VII}, Π^{VIII}, Π^{IX} etc. Atqui valoris Π^{VI} et omnium sequentium incrementum idem erit

$$= nv\cdot dx\left([N^{V}] - \frac{d[P^{IV}]}{dx} + \frac{dd[Q^{///}]}{dx^2} - \frac{d^3[R^{//}]}{dx^3} + \frac{d^4[S^{/}]}{dx^4} - \frac{d^5[T]}{dx^5}\right).$$

Poterunt autem haec incrementa ad eadem signa reduci respectu litterarum $[P]$, $[Q]$, $[R]$, $[S]$ et $[T]$, sicque prodibit

$$d\cdot\Pi=0$$

$$d\cdot\Pi^{\prime}=nv\cdot dx\cdot\frac{[T]}{dx^5}$$

$$d\cdot\Pi^{\prime\prime}=nv\cdot dx\left(\frac{[S^{\prime}]}{dx^4}-\frac{4[T]+5d[T]}{dx^5}\right)$$

$$d\cdot\Pi^{\prime\prime\prime}=nv\cdot dx\left(\frac{[R^{\prime\prime}]}{dx^3}-\frac{3[S^{\prime}]+4d[S^{\prime}]}{dx^4}+\frac{6[T]+15d[T]+10dd[T]}{dx^5}\right)$$

$$d\cdot\Pi^{IV}=nv\cdot dx\left(\begin{array}{l}\frac{[Q^{\prime\prime\prime}]}{dx^2}-\frac{2[R^{\prime\prime}]+3d[R^{\prime\prime}]}{dx^3}+\frac{3[S^{\prime}]+8d[S^{\prime}]+6dd[S^{\prime}]}{dx^4}\\-\frac{4[T]+15d[T]+20dd[T]+d^3[T]}{dx^5}\end{array}\right)$$

$$d\cdot\Pi^{V}=nv\cdot dx\left(\begin{array}{l}\frac{[P^{IV}]}{dx}-\frac{[Q^{\prime\prime\prime}]+2d[Q^{\prime\prime\prime}]}{dx^2}+\frac{[R^{\prime\prime}]+3d[R^{\prime\prime}]+3dd[R^{\prime\prime}]}{dx^3}\\-\frac{[S^{\prime}]+4d[S^{\prime}]+6dd[S^{\prime}]+4d^3[S^{\prime}]}{dx^4}\\+\frac{[T]+5d[T]+10dd[T]+10d^3[T]+5d^4[T]}{dx^5}\end{array}\right)$$

$$d\cdot\Pi^{VI}=nv\cdot dx\left([N^{V}]-\frac{d[P^{IV}]}{dx}+\frac{dd[Q^{\prime\prime\prime}]}{dx^2}-\frac{d^3[R^{\prime\prime}]}{dx^3}+\frac{d^4[S^{\prime}]}{dx^4}-\frac{d^5[T]}{dx^5}\right),$$

cui sequentium valorum omnium incrementa sunt aequalia. Q. E. I.

COROLLARIUM 1

2. Si ergo Π fuerit huiusmodi quantitas indeterminata seu formula integralis indefinite integrationem non admittens, tum eius omnes valores post locum abscissae, ubi una

applicata augeri concipitur, mutationem patientur et aliquot eius etiam valores ante illum locum, quorum numerus pendet a gradu differentialium, quae in ea formula Π insunt.

COROLLARIUM 2

3. Quodsi ergo istiusmodi quantitas insit in maximi minimive formula $\int Zdx$, tum eius valor differentialis non solum ab aliquot abscissae elementis, verum a tota abscissa, cui maximum minimumve respondere debet, pendebit.

COROLLARIUM 3

4. His igitur casibus abscissam illam, pro qua maximum minimumve quaeritur, determinatam esse oportet atque curva, quae pro hac abscissa maximi minimive proprietate gaudere reperta fuerit, eadem pro aliis abscissis hac proprietate non erit praedita.

SCHOLION

5. Mox clarius discrimen, quod intercedit inter quaestiones, in quibus Z est quantitas vel determinata vel indeterminata, perspicietur, quando Problemata huius generis sumus tractaturi. Pluribus modis autem tales quaestiones possunt variari, prout in maximi minimive formula $\int Zdx$ quantitas Z vel tantum est functio eiusmodi formulae indeterminatae Π, qualem contemplati sumus, vel insuper quantitates determinatas *x, y, p, q, r, s* etc. comprehendit. Deinde in Z etiam inesse poterunt plures eiusmodi formulae integrales indefinitae a se invicem diversae. Ad hos autem diversos casus una regula, superioribus iam traditis addita, sufficere poterit. Praecipuum autem momentum positum est in ipsa formula indeterminate $\Pi = \int [Z]dx$, pro qua hic posuimus esse $[Z]$ functionem determinatam; quodsi autem haec ipsa quantitas $[Z]$ denuo eiusmodi formulas integrales indefinitas complectatur, iterum peculiari solutione erit opus. Quin etiam ista complicatio formularum indeterminatarum in infinitum potest extendi; id quod eveniet, si quantitas $[Z]$ denuo in se complectatur ipsam quantitatem Π, ita ut sit

$$d[Z] = [L]d\Pi + [M]dx + [N]dy + [P]dp + [Q]dq + [R]dr + \text{etc.},$$

tum enim ob $d\Pi = [Z]dx$ iterum considerari oportebit valorem $d[Z] = [L]d\Pi + [M]dx + \text{etc.}$ hicque progressus in infinitum continuabitur. Hinc autem methodus nascetur ea resolvendi Problemata, in quibus curva quaeritur maximum minimumve habens valorem formulae $\int Zdx$, quando quantitas Z non datur, ut hactenus, sive determinate sive indeterminate, sed tantum per aequationem differentialem, cuius integratio omnino non potest absolvi; cuiusmodi quaestio est, si quaeratur curva, in qua minimum sit expressio $\int \frac{dx\sqrt{(1+pp)}}{\sqrt{v}}$, existente $dv = gdx - hv^n dx\sqrt{(1+pp)}$; atque eiusmodi quaestionum resolutionem in hoc Capite quoque trademus.

PROPOSITIO II. PROBLEMA

6. *Si Z* (Fig. 4) *fuerit functio quantitatis indeterminatae* Π *, ita ut sit* $dZ = Ld\Pi$ *, sitque* $\Pi = \int [Z]dx$ *existente*

$$d[Z] = [M]dx + [N]dy + [P]dp + [Q]dq + [R]dr + \text{etc.},$$

invenire curvam az, quae pro data abscissa AZ habeat valorem formulae $\int [Z]dx$ *maximum vel minimum.*

SOLUTIO

Posita abscissa $AH = x$, applicata $Hh = y$, sit tota abscissa *AZ*, cui maximum minimumve respondere debet, $= a$, diviso igitur spatio *HZ* in elementa innumera infinite parva *HI, IK, KL, LM* etc. debebit esse

$$\int Zdx + Zdx + Z^{I}dx + Z^{II}dx + Z^{III}dx + \text{etc.},$$

donec ad extremum punctum *Z* perveniatur, maximum minimumve. Ad hoc efficiendum quaerendi sunt valores differentiales, quos singuli hi termini a translatione puncti *n* in *v* accipiunt, quorum summa nihilo aequalis posita dabit aequationem pro curva quaesita. Quoniam autem mutationem ab *nv* oriundam non ultra *H* versus *A* porrigi ponimus, erit termini $\int Zdx$ valor differentialis nullus. Reliquorum terminorum valores differentiales reperientur, si ii differentientur atque in differentialibus scribantur ea incrementa, quae in Propositione praecedente invenimus ex translatione puncti *n* in *v* oriri.
Erit autem

$$d \cdot Zdx = Ldx \cdot d\Pi$$
$$d \cdot Z^{I}dx = L^{I}dx \cdot d\Pi^{I}$$
$$d \cdot Z^{II}dx = L^{II}dx \cdot d\Pi^{II}$$
$$d \cdot Z^{III}dx = L^{III}dx \cdot d\Pi^{III}$$
$$d \cdot Z^{IV}dx = L^{IV}dx \cdot d\Pi^{IV}.$$

Quodsi iam loco differentialium $d\Pi$, $d\Pi^{I}$, $d\Pi^{II}$, $d\Pi^{III}$ etc. valores supra inventos ex translatione puncti *n* in *v* ortos substituamus, obtinebimus:

$$d \cdot Zdx = 0$$

$$d \cdot Z^{/}dx = nv \cdot L^{/}dx^2 \cdot \frac{[T]}{dx^5}$$

$$d \cdot Z^{//}dx = nv \cdot L^{//}dx^2 \left(\frac{[S^{/}]}{dx^4} - \frac{4[T] + 5d[T]}{dx^5} \right)$$

$$d \cdot Z^{///}dx = nv \cdot L^{///}dx^2 \left(\frac{[R^{//}]}{dx^3} - \frac{3[S^{/}] + 4d[S^{/}]}{dx^4} + \frac{6[T] + 15d[T] + 10dd[T]}{dx^5} \right)$$

$$d \cdot [Z^{IV}]dx = nv \cdot L^{IV}dx^2 \left(\begin{array}{l} \frac{[Q^{///}]}{dx^2} - \frac{2[R^{//}] + 3d[R^{//}]}{dx^3} + \frac{3[S^{/}] + 8d[S^{/}] + 16dd[S^{/}]}{dx^4} \\ - \frac{4[T] + 15d[T] + 20dd[T] + 10d^3[T]}{dx^5} \end{array} \right)$$

$$d \cdot [Z^{V}]dx = nv \cdot L^{V}dx^2 \left(\begin{array}{l} \frac{[P^{IV}]}{dx} - \frac{[Q^{///}] + 2d[Q^{///}]}{dx^2} + \frac{[R^{//}] + 3d[R^{//}] + 3ddR^{//}}{dx^3} \\ - \frac{[S^{/}] + 4d[S^{/}] + 6dd[S^{/}] + 4d^3[S^{/}]}{dx^4} + \frac{[T] + 5d[T] + 10dd[T] + 10d^3[T] + 5d^4[T]}{dx^5} \end{array} \right)$$

$$d \cdot [Z^{VI}]dx = nv \cdot L^{VI}dx^2 \left([N^{V}] - \frac{d[P^{IV}]}{dx} + \frac{dd[Q^{///}]}{dx^2} - \frac{d^3[R^{//}]}{dx^3} + \frac{d^4[S^{/}]}{dx^4} - \frac{d^5[T]}{dx^5} \right)$$

$$d \cdot [Z^{VII}]dx = nv \cdot L^{VII}dx^2 \left([N^{V}] - \frac{d[P^{IV}]}{dx} + \frac{dd[Q^{///}]}{dx^2} - \frac{d^3[R^{//}]}{dx^3} + \frac{d^4[S^{/}]}{dx^4} - \frac{d^5[T]}{dx^5} \right)$$

etc.

Sequentium scilicet terminorum incrementa eadem hac lege progrediuntur. Addantur iam senorum priorum terminorum incrementa, prodibit terminorum

$Zdx + Z^{/}dx + Z^{//}dx + Z^{///}dx + Z^{IV}dx + Z^{V}dx$ incrementum totale

$$= nv\cdot dx^2\left(\begin{array}{l}\frac{L^{V}[P^{IV}]}{dx}-\frac{[Q^{///}]dL^{IV}+2L^{IV}d[Q^{///}]}{dx^2}+\frac{[R^{//}]ddL^{///}+3d[R^{//}]dL^{///}+3L^{///}dd[R^{//}]}{dx^3}\\ -\frac{[S^{/}]d^3L^{//}+4d[S^{/}]ddL^{//}+6dL^{//}dd[S^{/}]+4L^{//}d^3[S^{/}]}{dx^4}\\ +\frac{[T]d^4L^{/}+5d[T]d^3L^{/}+10dd[T]ddL^{/}+10dL^{/}d^3[T]+5L^{/}d^4[T]}{dx^5}\end{array}\right),$$

in qua expressione, quia omnes termini inter se sunt homogenei, iam indices numerici negligi poterunt. Sequentium autem terminorum $L^{VI}dx+L^{VII}dx+\text{etc.}$ omnium incrementum erit

$$= nv\cdot dx\left([N^{V}]-\frac{d[P^{IV}]}{dx}+\frac{dd[Q^{///}]}{dx^2}-\frac{d^3[R^{//}]}{dx^3}+\frac{d^4[S^{/}]}{dx^4}-\frac{d^5[T]}{dx^5}\right).$$
$$\left(L^{VI}dx+L^{VII}dx+L^{VIII}dx+L^{IX}dx+\text{ etc. usque in } Z\right).$$

Hic autem posterior factor definietur per integrationem formulae $\int Ldx$, quae respondet abscissae indefinitae $AH = x$; ponatur in hac formula post integrationem $x = a$ abeatque ea in *H*, erit *H* valor formulae $\int Ldx$ abscissae toti propositae *AZ* respondens; a qua ergo si auferatur $\int Ldx$, remanebit $H-\int Ldx$ valor portioni *HZ* vel *NZ* respondens, qui ergo loco

$$L^{VI}dx+L^{VII}dx+L^{VIII}dx+\text{ etc.}$$

substitui potest. Quamobrem tandem formulae $\int Zdx$ valor differentialis toti abscissae *AZ* respondens erit

$$= nv\cdot dx\left(H-\int Ldx\right)\left([N]-\frac{d[P]}{dx}+\frac{dd[Q]}{dx^2}-\frac{d^3[R]}{dx^3}+\frac{d^4[S]}{dx^4}-\frac{d^5[T]}{dx^5}\right)$$

$$+nv\cdot dx\left(\begin{array}{l}L[P]-\frac{[Q]d[L]+2Ld[Q]}{dx}+\frac{[R]ddL+3d[R]dL+3Ldd[R]}{dx^2}\\ -\frac{[S]d^3L+4d[S]ddL+6dLdd[S]+4Ld^3[S]}{dx^3}\\ +\frac{[T]d^4L+5d[T]d^3L+10dd[T]ddL+10dLd^3[T]+5Ld^4[T]}{dx^4}\end{array}\right),$$

qui ad hanc formam commodiorem reduci potest, ut sit

$$= nv\cdot dx\left(\begin{array}{l}[N]\left(H-\int Ldx\right)-\dfrac{d[P]\left(H-\int Ldx\right)}{dx}+\dfrac{dd[Q]\left(H-\int Ldx\right)}{dx^2}\\ -\dfrac{d^3[R]\left(H-\int Ldx\right)}{dx^3}+\dfrac{d^4[S]\left(H-\int Ldx\right)}{dx^4}-\dfrac{d^5[T]\left(H-\int Ldx\right)}{dx^5}\end{array}\right)$$

qui valor differentialis, quousque occasio postulabit, ulterius continuari poterit; is autem, nihilo aequalis positus, dabit aequationem pro curva quesita. Q. E. I.

COROLLARIUM 1

7. Quoniam $H-\int Ldx$ est valor formulae $\int Ldx$ respondens abscissae portioni $AZ = a - x$, si ponatur $AZ = a - x = u$ erit $\int Ldx$ ille ipse valor $H-\int Ldx$; quo opus est; siquidem $\int Ldx$ evanescat posito $u = 0$.

COROLLARIUM 2

8. Quodsi igitur abscissarum initium capiatur in puncto *Z*, ita ut abscissa *ZH* ponatur $= u$, utque ubique ponatur $x = a - u$, prodibit aequatio pro curva inter coordinatas *u* et *y*; huiusque curvae ea portio quaesito satisfaciet, quae respondet abscissae $AZ = a$. Interim notandum est cum in ipsa maximi minimive formula $\int Zdx$ tum in $\int [Z]dx$ abscissarum initium in puncto *A* capi debere.

COROLLARIUM 3

9. Si ergo quaeratur curva ad datam abscissam *AZ* relata, in qua maximum minimumve debeat esse $\int Zdx$, sitque *Z* functio quaecunque ipsius $\Pi = \int Zdx$, existente

$$dZ = Ld\Pi \text{ et } d[\mathrm{Z}] = [M]dx + [N]dy + [P]dp + [Q]dq + [R]dr + \text{etc.},$$

habebitur pro curva quaesita ista aequatio:

$$0 = [N]\int Ldx - \frac{d\cdot[P]\int Ldx}{dx} + \frac{dd\cdot[Q]\int Ldx}{dx^2} - \frac{d^3\cdot[R]\int Ldx}{dx^3} + \text{etc.},$$

ubi est $u = a - x$ et $\int Ldu$ denotat valorem formulae $\int Ldx$ portioni abscissae $HZ = u$ respondentem.

COROLLARIUM 4

10. Possunt ergo vel bina abscissarum initia *A* et *Z*, binaeque abscissae $AH = x$ et $ZH = u$ considerari, quarum illa in integrali $\int Ldx$ seu Π, haec vero in integrali $\int Ldx$ spectari debet, vel unica tantum abscissa $AH = x$; quo casu loco $\int Ldu$ scribi debet $H - \int Ldx$ denotante *H* valorem, quem praebet formula $\int Ldu$ posito $x = AH = a$.

COROLLARIUM 5

11. Quia *Z* est functio ipsius Π tantum, ita ut nullas alias quantitates variabiles in se complectatur, ob $dZ = Ld\Pi$ erit etiam *L* functio ipsius Π tantum.

COROLLARIUM 6

12. Si $[Z]$ esset functio ipsius *x* tantum, tum foret $\Pi = \int [Z]dx$ quantitas determinata atque functio ipsius *x*, hincque etiam *Z*; ex quo maximum minimumve non inveniet locum. Idem ostendit solutio; fiet enim $[N] = 0$, $[P] = 0$ etc. atque aequatio abit in identicam $0 = 0$.

SCHOLION I

13. Occurrunt hic nonnulli primarii casus considerandi, quorum primus est, si fuerit $[Z]$ functio ipsarum *x* et *y* tantum, ita ut sit $d[Z] = [M]dx + [N]dy$. Quodsi nunc quaeratur curva, in qua maximum minimumve sit formula $\int Zdx$ pro data abscissa $AZ = a$, existente Z functione quacunque ipsius $\int [Z]dx = \Pi$, ita ut sit $dZ = Ld\Pi$, habebitur pro curva quaesita ista aequatio

$$0 = [N](H - \int Ldx);$$

erit ergo vel $[N] = 0$ vel $H = \int Ldx$ seu $L = 0$; ; quarum aequationum si vel altera vel utraque praebeat lineam curvam, ea non solum satisfaciet Problemati pro abscissa $AZ = a$ sed etiam pro alia quacunque abscissa indefinita *x* ; id quod inde colligitur, quod ex aequatione quantitas *H*, quae pendet ab abscissa determinata *a*, ex calculo excesserit. Quod autem speciatim ad aequationem $L = 0$ attinet, quia *L* est functio ipsius Π seu $\int [Z]dx$, fiet $\int [Z]dx =$ constanti determinatae, quod, nisi sit $[Z] = 0$, fieri nequit: binae igitur aequationes hoc casu satisfacientes erunt $[N] = 0$ atque $[Z] = 0$.

SCHOLION II

14. Deinde vero considerari meretur casus, quo $[N]$ evanescit; id quod evenit, si $[Z]$ fuerit functio ipsarum *x*, *p*, *q*, *r* etc. non involvens *y*. Ponamus esse $[Z]$ functionem ipsarum *x* et *p* atque $d[Z]=[M]dx+[P]dp$. Si igitur ponatur $\int [Z]dx = \Pi$ atque curva quaeratur, in qua pro abscissa definita $AZ = a$ maximum minimumve sit formula $\int Zdx$, existente *Z* functione ipsius Π, ita ut sit $dZ = Ld\Pi$, orietur pro curva quaesita ista aequatio

$$0=-\frac{d\cdot[P](H-\int Ldx)}{dx},$$

ideoque $\text{const.}=[P](H-\int Ldx)$. Haec vero constans per integrationem ingressa non est arbitraria; nam eam ita comparatam esse oportet, ut posito $x=a$, quo casu fit $Ldx = H$, fiat $\frac{\text{const.}}{[P]}=0$. Hoc autem evenire non potest, nisi vel haec constans ponatur $=0$ vel quantitas $[P]$ ita comparata sit, ut fiat $=\infty$ posito $x=a$. Priori casu habetur vel $[P]=0$ vel $\int Ldx = H$, hoc est $L=0$ seu $\int [Z]dx =$ const. seu potius $[\mathrm{Z}]=0$; posteriori casu autem constans tamen pro arbitrio non accipi potest, nam determinabitur ponendo $x=a-dx$, eo modo, quo expressiones, quae certis casibus indeterminatae videntur, definiri solent. Atque hinc perspicitur in huiusmodi Problematis numerum constantium arbitrariarum in solutionem ingredientium, cui aequalis sumi debet numerus punctorum, per quae curvae satisfacienti transeundum est, non ex gradu differentialium iudicari posse. Pervenietur enim saepe tollendo per differentiationem omnes formulas integrales ad aequationem differentialem altioris gradus, a quo nequa quam Problematis determinatio per aliquot puncta pendebit.

EXEMPLUM I

15. *Si denotet Π aream curvae $\int ydx$ atque Z sit functio quaecunque ipsius Π, invenire curvam, quae pro data abscissa $=a$ habeat valorem formulae $\int Zdx$ maximum vel minimum.*

Quia est *Z* functio ipsius Π, sit $dZ = Ld\Pi$; erit *L* functio ipsius $\Pi = \int ydx$. Deinde cum sit $d\Pi = ydx$, erit $[Z]=y$ et ob

$$d[\mathrm{Z}]=[M]dx+[N]dy+[P]dp+\text{etc.}.$$

fiet $[M]=0,\ [N]=1,\ [P]=0,\ [Q]=0$ etc., unde pro curva quaesita haec habebitur aequatio $0=H-\int Ldx$; ideoque $L=0$. Hinc erit $\Pi=\int ydx=$ constanti cuidam, porroque $y=0$. Satisfacit ergo sola linea recta in ipsum axem incidens; idque pro quacunque abscissa aeque ac pro definita $=a$.

EXEMPLUM II

16. *Si* Π *denotet arcum curvae* $=\int dx\sqrt{(1+pp)}$ *eiusque functio quaecunque fuerit Z, invenire curvam, quae pro data abscissa* $AZ=a$ *habeat valorem formulae* $\int Zdx$ *maximum vel minimum*.

Ob $dZ=Ld\Pi$ erit L functio ipsius arcus Π; et ob $d\Pi=dx\sqrt{(1+pp)}$ erit $[Z]=\sqrt{(1+pp)}$ et $[M]=0$; $[P]=\frac{p}{\sqrt{(1+pp)}}$, $[Q]=0$ etc., unde pro curva quaesita ista habebitur aequatio:

$$0=-d\cdot\frac{p}{dx\sqrt{(1+pp)}}(H-\int Ldx),$$

hincque

$$C=\frac{p}{\sqrt{(1+pp)}}(H-\int Ldx),$$

ubi constans C ita determinari debet, ut posito $x=a$ fiat $C=\frac{p}{\sqrt{(1+pp)}}\times 0$; quare, quia $\frac{p}{\sqrt{(1+pp)}}$ infinitum fieri nequit, necesse est, ut sit $C=0$ ideoque

$$\text{vel } \frac{p}{\sqrt{(1+pp)}}=0 \text{ vel } \int Ldx=H.$$

Fiet ergo, ex posteriore aequatione, $L=0$ et $\Pi=$ constanti cuidam; ex quo porro deducitur $d\Pi=dx\sqrt{(1+pp)}=0$, cui conditioni nullo modo satisfieri potest. Ex priore aequatione autem deducitur $p=0$ seu $dy=0$, quae est aequatio pro linea recta axi AZ parallela, quae quaestioni pro abscissa quacunque satisfacit.

EXEMPLUM III

17. *Denotet* Π *superficiem solidi rotundi ex conversione curvae ah circa axem AZ orti, quae est ut* $\int ydx\sqrt{(1+pp)}$ *, huiusque superficiei functio sit quaecunque Z, invenire curvam, in qua pro data abscissa* $AZ = a$ *maximum minimumve sit* $\int Zdx$ *.*

Ob $dZ = Ld\Pi$ erit L functio ipsius $\Pi = \int ydx\sqrt{(1+pp)}$ et ob $d\Pi = ydx\sqrt{(1+pp)}$ fiet

$$[Z] = y\sqrt{(1+pp)} \quad \text{et} \quad d[Z] = dy\sqrt{(1+pp)} + \frac{ypdp}{\sqrt{(1+pp)}};$$

unde erit

$$[M] = 0,\ [N] = \sqrt{(1+pp)},\ [P] = \frac{ypdp}{\sqrt{(1+pp)}};$$

reliqui valores $[Q],\ [R],\ [S]$ etc. omnes erunt $= 0$. Quocirca pro curva quaesita ista habebitur aequatio:

$$0 = (H - \int Ldx)\sqrt{(1+pp)} - \frac{1}{dx} d \cdot \frac{yp}{\sqrt{(1+pp)}}(H - \int Ldx).$$

Ponatur, brevitatis gratia, $H - \int Ldx = V$; erit

$$Vdx\sqrt{(1+pp)} = d \cdot \frac{ypV}{\sqrt{(1+pp)}} = \frac{Vppdx}{\sqrt{(1+pp)}} + \frac{Vydp}{(1+pp)^{3:2}} + \frac{ypdV}{\sqrt{(1+pp)}}$$

seu

$$Vdx = \frac{Vydp}{1+pp} + ypdV = \frac{Vydp}{1+pp} - ypLdx$$

ob $dV = -Ldx$. Ponamus esse $Z = \Pi$, ita ut maximum esse debeat

$\int dx \int ydx\sqrt{(1+pp)}$, erit $L = 1$ et $\int Ldx = x$ atque $V = a - x$ ob $H = a.$ Erit

$$(a-x)dx = \frac{(a-x)ydp}{1+pp} - ypdx.$$

Sit $a - x = u$, erit $dx = -du$ et $dy = -pdu$ atque habebitur aequatio:

$$0 = udu - ydy + \frac{uydp}{1+pp} \quad \text{seu} \quad udu - ydy - \frac{uyduddy}{du^2 + dy^2} = 0.$$

Ponatur $u = e^t$ et $y = e^t z$, erit $du = e^t dt$ et $ddu = 0 = e^t(ddt + dt^2)$, seu $ddt = -dt^2$; porro

$$dy = e^t(dz + zdt) \text{ et } ddy = e^t(ddz + 2dtdz);$$

quibus substitutis oritur

$$dt - zdz - zzdt = \frac{zdt(ddz + 2dtdz)}{dt^2 + (dz + zdt)^2}.$$

Sit porro $dt = sdz$, erit $ddt = -s^2dz^2 = sddz + dsdz$ hincque

$$ddz = -sdz^2 - \frac{dsdz}{s}.$$

Habebitur ergo haec aequatio

$$sdz - zdz - szzdz = \frac{zs^2dz - zds}{ss + (1+sz)^2};$$

quae quidem est differentialis primi gradus inter duas variabiles *s* et *z* tantum, verum tamen ultra integrationem non admittit. Multo minus igitur quicquam effici poterit, si in genere quaestionem consideremus.

SCHOLION III

18. Huius exempli casus, quo curvam investigavimus, in qua maximum minimumve sit $\int dx \int ydx\sqrt{(1+pp)}$, etsi inest duplex signum integrale, tamen etiam per methodum praecedentis Capitis potest resolvi; id quod ideo operae pretium est ostendere, ut consensus utriusque methodi declaretur. Praecipue autem hoc opere nova via patefiet resolvendi plurima alia Problemata circa maxima et minima, quae adhuc, quantum constat, non est tacta. Quaestio scilicet est, ut pro data abscissa $AZ = a$ fiat maximum minimumve haec expressio $\int dx \int ydx\sqrt{(1+pp)}$, quae transmutatur in hanc

$$x\int\int ydx\sqrt{(1+pp)} - \int xydx\sqrt{(1+pp)}.$$

Ut haec forma reddatur maximum minimumve, oportet, ut eius valor pro abscissa $AZ = a$ idem sit pro ipsa curva quaesita *az* et pro eadem puncto *n* in *v* translato. Ponamus ergo fieri $\int ydx\sqrt{(1+pp)} = A$, si ponatur $x = a$, atque eodem casu $\int xydx\sqrt{(1+pp)} = B$. Iam elementis *mno* in *mvo* transmutatis valor *A* augebitur suo valore differentiali, qui, per Caput praecedens, est

$$= nv\cdot dx\left(\sqrt{(1+pp)}-\frac{1}{dx}d\cdot\frac{yp}{\sqrt{(1+pp)}}\right);$$

per eadem praecepta autem quantitatis *B* valor differentialis prodit

$$= nv\cdot dx\left(x\sqrt{(1+pp)}-\frac{1}{dx}d\cdot\frac{xyp}{\sqrt{(1+pp)}}\right);$$

Quamobrem formulae propositae $\int dx\int ydx\sqrt{(1+pp)}$, translato puncto *n* in *v*, pro abscissa $AZ = a$ valor erit

$$= a\left(A+nv\cdot\left(dx\sqrt{(1+pp)}-d\cdot\frac{yp}{\sqrt{(1+pp)}}\right)\right)-B$$
$$-nv\cdot\left(xdx\sqrt{(1+pp)}-d\cdot\frac{xyp}{\sqrt{(1+pp)}}\right),$$

qui aequalis esse debet eiusdem formulae valori naturali pro abscissa $= a$,
non mutato puncto *n*, qui est $aA - B$. Hinc proveniet ista aequatio

$$(a-x)dx\sqrt{(1+pp)}-d\cdot\frac{(a-x)yp}{\sqrt{(1+pp)}}=0,$$

quae omnino congruit cum aequatione in solutione Exempli inventa.

PROPOSITIO III. PROBLEMA

19. *Existente* Π *functione integrali indeterminata* $\int[Z]dx$, *ita ut sit*

$$dZ = d[Z] = [M]dx + [N]dy + [P]dp + [Q]dq + [R]dr + \text{etc.},$$

sit Z functio quaecunque cum huius quantitatis Π *tum quantitatum determinatarum x, y, p, q, r, s etc., ita ut sit*

$$dZ = Ld\Pi + Mdx + Ndy + Pdp + Qdq + Rdr + \text{etc.},$$

invenire curvam az, quae pro data abscissa $AZ = a$ *habeat maximum minimumve valorem formulae* $\int Zdx$.

SOLUTIO

Augmentum *nv*, quod uni applicatae *Nn* accedere concipitur, ita remotum a prima applicata *Hh* capiatur, ut nullam mutationem inferat in valorem formulae $\int Zdx$ abscissae *AH* respondentem atque tantum huius formulae valores sequentibus post *H* abscissae elementis respondentes mutationes patiantur, qui sunt $Zdx,\ Z^{\prime}dx,\ Z^{\prime\prime}dx,\ Z^{\prime\prime\prime}dx$ etc. usque ad ultimum abscissae elementum in *Z*. Horum igitur valorum incrementa a translatione puncti *n* in *v* orta, si in unam summam coniiciantur et nihilo aequales ponantur, dabunt aequationem pro curva quaesita. Incrementa autem horum valorum obtinebuntur eos differentiando et loco differentialium eos valores scribendo, quos supra, tam in ultima Propositione praecedentis Capitis quam prima huius, ex translatione *n* in *v* oriri invenimus; ita erit

$$\begin{aligned}
d\cdot Zdx &= dx\left(Ld\Pi + Mdx + Ndy + Pdp + \text{etc.}\right)\\
d\cdot Z^{\prime}dx &= dx\left(L^{\prime}d\Pi^{\prime} + M^{\prime}dx + N^{\prime}dy^{\prime} + P^{\prime}dp^{\prime} + \text{etc.}\right)\\
d\cdot Z^{\prime\prime}dx &= dx\left(L^{\prime\prime}d\Pi^{\prime\prime} + M^{\prime\prime}dx + N^{\prime\prime}dy^{\prime\prime} + P^{\prime\prime}dp^{\prime\prime} + \text{etc.}\right)\\
&\text{etc.}
\end{aligned}$$

Quodsi nunc loco differentialium $d\Pi,\ d\Pi^{\prime},\ d\Pi^{\prime\prime}$ etc., $dy,\ dy^{\prime},\ dy^{\prime\prime}$ etc., $dp,\ dp^{\prime},\ dp^{\prime\prime}$ etc., $dq,\ dq^{\prime},\ dq^{\prime\prime}$ etc. valores supra inventi substituantur et eodem modo, quo ante usi sumus, in unam summam conferantur, prodibit formulae $\int Zdx$ pro abscissa $AZ = a$ valor differentialis

$$= nv\cdot dx\left(\begin{array}{l}[N]\left(H-\int Ldx\right)-\dfrac{d\cdot[P]\left(H-\int Ldx\right)}{dx}+\dfrac{dd\cdot[Q]\left(H-\int Ldx\right)}{dx^{2}}\\ \qquad -\dfrac{d^{3}\cdot[R]\left(H-\int Ldx\right)}{dx^{3}}+\dfrac{d^{4}\cdot[S]\left(H-\int Ldx\right)}{dx^{4}}-\text{etc.}\end{array}\right)$$

$$+nv\cdot dx\left(N-\frac{dP}{dx}+\frac{ddQ}{dx^{2}}-\frac{d^{3}R}{dx^{3}}+\frac{d^{4}R}{dx^{4}}-\text{etc.}\right).$$

Atque ex hoc resultabit aequatio pro curva quaesita haec:

$$0=[N]\left(H-\int Ldx\right)-\frac{d\cdot[P]\left(H-\int Ldx\right)}{dx}+\frac{dd\cdot[Q]\left(H-\int Ldx\right)}{dx^2}$$
$$-\frac{d^3\cdot[R]\left(H-\int Ldx\right)}{dx^3}+\text{etc.}$$
$$+N-\frac{dP}{dx}+\frac{ddQ}{dx^2}-\frac{d^3R}{dx^3}+\frac{d^4R}{dx^4}-\text{etc.}$$

ubi notandum esse H valorem formulae $\int Ldx$, qui oritur posito $x=a$.

Q. E. I.

COROLLARIUM 1

20. Regula igitur Capite praecedente inventa amplior est reddita; nunc enim curvam definire possumus maximum minimumve habentem valorem formulae $\int Zdx$, si Z non solum est functio quantitatum determinatarum x, y, p, q, r etc., sed etiam unam quantitatem integralem indefinitam $\int [Z]dx$ in se complectitur, dummodo $[Z]$ sit functio determinata.

COROLLARIUM 2

21. Quin etiam, si plures huiusmodi quantitates integrales indefinitae fuerint in Z, solutio usurpari poterit. Nam, qualis expressio ex una eiusmodi formula indefinita in valorem differentialem est ingressa, tales ex singulis, si plures affuerint, nascentur et ad valorem differentialem accedent.

COROLLARIUM 3

22. Quoniam Z hic ponitur functio non solum quantitatum definitarum x, y, p, q, r etc., sed etiam quantitatis indefinitae $\varPi=\int [Z]dx$, ob

$$dZ=Ld\varPi+Mdx+Ndy+Pdp+Qdq+Rdr+\text{etc.}$$

etiam quantitates M, N, P, Q etc. hanc formulam integralem $\varPi=\int [Z]dx$ involvent; atque etiam ipsa quantitas L, nisi forte $\varPi$ in Z unicam habeat dimensionem.

COROLLARIUM 4

23. Hanc ob rem in aequatione pro curva inventa inerunt quantitates integrales duplicis generis, scilicet $\int Ldx$ atque $\int [Z]dx$; ex quo, si aequatio inventa per differentiationem ab his formulis liberari debeat, ad multo altiorem differentialium gradum assurget, quam quidem ipsa forma ostendit.

COROLLARIUM 5

24. Pervenietur autem eliminando has formulas integrales ad aequationem differentialem duobus gradibus altiorem. Quodsi enim aequatio resultans, si evolvatur, sit differentialis n gradus, tum primo ex ea definiatur valor formulae $\int Ldx$ et differentiatione instituta devenietur ad aequationem differentialem $n+1$ graduum, in qua adhuc inerit formula $\int [Z]dx$, quae ulterius reducta et a formula $\int [Z]dx$ per differentiationem liberata, fiet differentialis gradus $n+2$.

SCHOLION I

25. Etsi autem numerus punctorum, per quae curva quaesita transire debet, a gradu *differentialitatis* pendet, tamen hoc casu non per numerum $n+2$ definiri potest. Aequatio enim haec differentialis $n+2$ graduum potestate quidem involvit $n+2$ constantes, verum eae non omnes sunt arbitrariae. Una namque constans ex eo determinatur, quod integrale $\int [Z]dx$ obtinere debeat valorem non vagum, sed talem, qualem in quantitate Z obtinet, hoc est, qui evanescat posito $x=0$, siquidem haec conditio fuerit in formula $\int Zdx$ assumta. Deinde pari modo una constans definitur formula $\int Ldx$, quae, uti posuimus, evanescere debet posito $x=0$. Quocirca tantum n supererunt constantes mere arbitrariae, quae totidem praebebunt puncta, quibus Problema determinabitur. Similiter igitur, uti in praecedente Capite, Problema, ut sit determinatum, ita erit proponendum, ut inter omnes curvas per data n puncta transeuntes ea determinetur, quae pro data abscissa $x=a$ contineat valorem formulae $\int Zdx$ maximum minimumve. Ad hanc igitur diiudicationem instituendam aequatio inventa debebit evolvi; hoc est, omnes differentiationes indicatae actu perfici debebunt; quo facto pate bit, quanti gradus differentialia insint, ex hocque gradu habebitur numerus *n*. Quantum autem insuper circa hunc numerum n observare liceat, in Exemplis sequentibus videbimus.

EXEMPLUM I

26. *Invenire curvam, quae pro data abscissa $AZ=a$ habeat valorem formulae $\int yxdx\int ydx$ maximum vel minimum, integrali $\int ydx$ ita accipiendo, ut evanescat posito $x=0$.*

Erit igitur $\Pi=\int ydx$ et $[Z]=y$; unde fiet $[N]=1$ reliquis litteris $[M]$, $[P]$, $[Q]$ etc. existentibus $=0$. Porro erit

$$Z=yx\Pi \quad \text{et} \quad dZ=yxd\Pi+y\Pi dx+x\Pi dy;$$

ex quo habebitur $L = yx,\ M = y\Pi$ et $N = x\Pi$, $P = Q = R = \text{etc.} = 0$.
Ex his formabitur pro curva quaesita ista aequatio

$$0 = \left(H - \int yxdx\right) + x\Pi \quad \text{seu} \quad \int yxdx = H + x\int ydx,$$

ubi H est valor formulae $\int yxdx$, qui prodit posito $x = a$. Perspicuum autem est hinc nullam pro aliqua linea curva aequationem oriri; differentiatione enim instituta fit $dx\int ydx = 0$ porroque $y = 0$, quae est aequatio pro linea recta in axem AZ incidente.

EXEMPLUM II

27. *Invenire curvam, quae pro data abscissa $AZ = a$ habeat valorem formulae $\int ydx\int dx\sqrt{(1+pp)}$ maximum vel minimum.*

Quoniam igitur est $\Pi = \int dx\sqrt{(1+pp)}$, erit

$$[Z] = \sqrt{(1+pp)} \quad \text{et} \quad [P] = \frac{p}{\sqrt{(1+pp)}}.$$

Porro erit $Z = y\Pi$ et $L = y$ et $N = \Pi$; reliquae litterae omnes evanescunt. Hinc ergo resultabit ista aequatio pro curva quaesita:

$$0 = -\frac{1}{dx} d \cdot \frac{p\left(H - \int ydx\right)}{\sqrt{(1+pp)}} + \Pi$$

seu

$$\Pi dx = d \cdot \frac{\left(H - \int ydx\right)p}{\sqrt{(1+pp)}} = \frac{\left(H - \int ydx\right)dp}{(1+pp)^{\frac{3}{2}}} - \frac{ypdx}{\sqrt{(1+pp)}};$$

ergo

$$dx\int dx\sqrt{(1+pp)} = \frac{\left(H - \int ydx\right)dp}{(1+pp)^{\frac{3}{2}}} - \frac{ypdx}{\sqrt{(1+pp)}}.$$

Quia igitur fit $\int ydx = H$ posito $x = a$, eodem casu fiet
$\int dx\sqrt{(1+pp)} = -\dfrac{yp}{\sqrt{(1+pp)}} =$ arcui curvae abscissae a respondenti. Quae conditio

adimpleri debet per determinationem unius constantis, quae per integrationem ingredietur. Est autem actu haec aequatio differentialis secundi gradus, quae vero his debet differentiari, antequam a formulis integralibus $\int ydx$ et $\int dx\sqrt{(1+pp)}$ liberetur; hocque modo ad gradum sextum assurget et potestate sex constantes involvet; quarum duae inde determinabuntur, quod facto $x=0$ evanescere debent formulae $\int ydx$ et $\int dx\sqrt{(1+pp)}$. Ipsa autem aequatio ita fiet intricata, ut eius tractatio suscipi non mereatur.

EXEMPLUM III

28. *Invenire curvam, in qua pro data abscissa sit* $\int \frac{dx}{p}\int ydx$ *maximum vel minimum.*

Hic erit $\Pi = \int ydx$ et $[Z]=y$ et $[N]=1$; deinde cum sit $Z=\frac{\Pi}{p}$,

erit $L=\frac{1}{p}$ et $P=-\frac{\Pi}{pp}$; reliquae litterae omnes evanescunt. Hinc ergo prodit ista aequatio

$$0 = H - \int \frac{dx}{p} + \frac{1}{dx} d\cdot \frac{\Pi}{pp}$$

seu

$$0 = H - \int \frac{dx}{p} + \frac{y}{pp} - \frac{2\Pi dp}{p^3 dx}.$$

Posito ergo $x=a$, quo casu fit $\int \frac{dx}{p} = H$, erit $ydx = \frac{2\Pi dp}{p}$. Differentietur ea aequatio, eritque

$$0 = -\frac{dx}{p} + \frac{dx}{p} - \frac{2ydp}{p^3} - \frac{2ydp}{p^3} + \frac{6\Pi dp^2}{p^4 dx} - \frac{2\Pi dpp}{p^3 dx}$$

seu

$$0 = 3\Pi dp^2 - 2ypdxdp - \Pi pddp;$$

quae aequatio commode fit integrabilis, si dividatur per Πpdp, prodit enim

$$0 = \frac{3dp}{p} - \frac{2ydx}{\Pi} - \frac{dpp}{dp},$$

cuius integrale est

$$C = 3lp - 2l\Pi - l\frac{dp}{dx},$$

seu $C\Pi^2 dp = p^3 dx$; posito ergo $x = a$, cum esse debeat $ydx = \frac{2\Pi dp}{p}$, erit ex hac aequatione $C\Pi y = 2p^2$, qua una constans definietur. Erit ergo

$$\Pi = \sqrt{\frac{p^2 dx}{Cdp}} = \frac{2ypdxdp}{3dp^2 - pddp}$$

seu

$$3dp^2 - pddp = \frac{2ydp\sqrt{dxdp}}{by\sqrt{bp}},$$

quae aequatio est differentialis tertii gradus et propterea praeter constantem *b* (posuimus autem $\frac{1}{b^3}$ loco *C*) tres novas constantes involvit. Harum una determinabitur, eo quod posito $x = a$ fieri debeat $\frac{\Pi y}{b^3} = 2pp$; alia vero inde, quod posito $x = 0$ esse debeat $\Pi = 0$ seu $\frac{p^3 dx}{dp} = 0$. Reliquae binae constantes manent arbitrariae ac propterea curva quaesita per duo data puncta, per quae transeat, debet determinari.

EXEMPLUM IV

29. *Invenire curvam az ad abscissam AZ =a relatam, in qua sit* $\int dx \frac{\int yxdx}{\int ydx}$ *maximum vel minimum.*

Hoc exemplum ideo afferre visum est, ut appareat, quomodo quaestiones eiusmodi sint resolvendae, si duae pluresve formulae integrales indefinitae adsint. Sit igitur

$$\int yxdx = \Pi \quad \text{et} \quad \int ydx = \pi,$$

et posito $d\Pi = [Z]dx$ et $d\pi = [z]dx$ erit $[Z] = yx$ et $[z] = y$. Quodsi nunc littera minuscula $[z]$ simili modo tractetur, quo maiuscula $[Z]$, ita ut sit

$$d[z] = [m]dx + [n]dy + [p]dp + \text{etc.},$$

erit $[M] = y$ et $[N] = x$ itemque $[n] = 1$. Deinde, cum sit $Z = \frac{\Pi}{\pi}$, erit

$dZ = \frac{d\Pi}{\pi} - \frac{\Pi d\pi}{\pi^2}$.

Ponatur $\frac{1}{\pi} = L$ et $\frac{\Pi}{\pi^2} = l$, atque habebitur ob *N* et *P*, *Q*, *R* etc. $= 0$ ista pro curva quaesita aequatio

$$0 = x\left(H - \int\frac{dx}{\pi}\right) - \left(h - \int\frac{\Pi dx}{\pi^2}\right),$$

ubi fit $\int\frac{dx}{\pi} = H$ et $\int\frac{\Pi dx}{\pi^2} = h$, si ponatur $x = a$. Cum igitur sit

$$Hx - x\int\frac{dx}{\pi} = h - \int\frac{\Pi dx}{\pi^2},$$

erit differentiando

$$H - \int\frac{dx}{\pi} - \frac{x}{\pi} = -\frac{\Pi}{\pi^2}.$$

Posito ergo $x = a$ fieri debet $\Pi = \pi x$. Differentietur denuo prodibitque

$$-\frac{2}{\pi} + \frac{xy}{\pi^2} = -\frac{yx}{\pi^2} + \frac{2\Pi y}{\pi^3} \quad \text{seu} \quad xy - \pi = \frac{\Pi y}{\pi}$$

hincque

$$\Pi = \pi x - \frac{\pi\pi}{y}.$$

Si porro differentiatio instituatur, habebitur

$$yxdx = \pi dx + yxdx - 2\pi dx + \frac{\pi\pi dy}{yy}$$

seu

$$yydx = \pi dy \quad \text{vel} \quad \frac{ydx}{\pi} = \frac{dy}{y}.$$

Quoniam vero posito $x = 0$ fit $\pi = 0$, fiet hoc casu $\frac{yydx}{dy} = 0$. Aequatio $\frac{ydx}{\pi} = \frac{dy}{y}$ ob $ydx = d\pi$, integrata dat $\pi = by$; ideoque facto $x = 0$ evanescere debet *y*. Ex aequatione $n = by$ autem sequitur $ydx = bdy$ hincque $x = bly - bl0$, siquidem $\pi = by$ evanescere debeat posito $x = 0$; quo casu fieret $y = 0$ et curva abiret in rectam in axem *AZ* incidentem. Sin autem ponamus posito $x = 0$ valorem

$\pi = \int ydx$ non evanescere oportere, sed fieri $= bc$, erit $x = bl\frac{y}{c}$, quae est aequatio pro Curva logarithmica. Ad hanc penitus determinandam quaeratur valor $\Pi = \int yxdx$; quia est $ydx = bdy$, erit

$$yxdx = bxdy \text{ et } \Pi = bxy - b\pi + Const.$$

seu

$$\Pi = bbyl\frac{y}{c} - bby + C.$$

Oporteat autem Π esse $= 0$ posito $x = 0$ seu $y = c$, erit

$$\Pi = bbyl\frac{y}{c} + bb(c - y).$$

Iam ponatur $x = a$, erit $l\frac{y}{c} = \frac{a}{b}$ et $y = ce^{a:b}$; hoc vero casu necesse est, ut sit $\Pi = nx$ seu

$$abce^{a:b} + bbc - bbce^{a:b} = abce^{a:b}$$

hincque $e^{a:b} = 1$, unde erit vel $a = 0$ vel $b = \infty$. Incommodum hoc inde oritur, quod posuimus fieri $\Pi = 0$ facto $x = 0$. Ponamus igitur posito $y = g$ tum eo casu Π evanescere, erit

$$\Pi = bbyl\frac{y}{c} - bby + bbg - bbgl\frac{g}{c}.$$

Iam posito $x = a$, quo casu fieri debet $\Pi = \pi x = a\pi$, erit

$$abce^{a:b} - bbce^{a:b} + bbg - bbgl\frac{g}{c} = abce^{a:b}$$

hincque

$$e^{a:b} = \frac{g}{c}\left(1 - l\frac{g}{c}\right) \text{ seu } b = \frac{a}{l\frac{g}{c}\left(1 - l\frac{g}{c}\right)}$$

ideoque

$$x = \frac{a(ly - lc)}{lg\left(1 - l\frac{g}{c}\right) - lc}.$$

Quae est aequatio curvam penitus determinans, ita ut nullum curvae punctum pro arbitrio accipi liceat.

SCHOLlON II

30. Per hoc igitur Problema non solum illae quaestiones curvam pro data abscissa maximum minimumve habentem formulam $\int Zdx$ desiderantes resolvi possunt, in quibus Z praeter quantitates determinatas x, y, p, q, r, s etc. unam formulam integralem $\Pi = \int [Z]dx$ complectitur, sed etiamsi plures eiusmodi formulae affuerint. Interim tamen notandum est has formulas integrales $\Pi = \int [Z]dx$ in functione Z contentas ita comparatas esse debere, ut $[Z]$ sit functio determinata, hoc est functio quantitatum x, y, p, q, r etc. nullas ultra formulas integrales involvens. Hanc ob rem nunc investigemus methodum resolvendi eiusmodi Problemata, quando ista functio $[Z]$ non est determinata, sed praeter x, y, p, q etc. formulam integralem novam $\pi = \int [z]dx$ involvit. Ne autem solutio nimium fiat prolixa, non ultra differentialia secundi gradus considerabimus. Iam enim intelligitur, si solutio fuerit adornata usque ad differentialia secundi gradus, tum per inductionem solutionem ad quosque ulteriores gradus extendi posse. Hunc in finem nobis erit Ll prima applicata designanda per y, a qua tertia, quae sequitur, $Nn = y^{//}$ particula nv augeri concipiatur. Ex hoc augmento nascentur sequentia quantitatum y, p et q cum suis derivativis incrementa

$d \cdot y = 0$	$d \cdot p = 0$	$d \cdot q = +\dfrac{nv}{dx^2}$
$d \cdot y^{/} = 0$	$d \cdot p^{/} = +\dfrac{nv}{dx}$	$d \cdot q^{/} = -\dfrac{2nv}{dx^2}$
$d \cdot y^{//} = +nv$	$d \cdot p^{//} = -\dfrac{nv}{dx}$	$d \cdot q^{//} = +\dfrac{nv}{dx^2}$

quae Tabella sufficiet ad Problemata quaecunque resolvenda, uti ex sequente Propositione intelligetur.

PROPOSITIO IV. PROBLEMA

31. *Sit* $\pi = \int [z]dx$ et $d[z] = [m]dx + [n]dy + [p]dp + [q]dq,$ *atque quantitas* $[Z]$ *ita involvat formulam integralem* π *, ut sit*

$$d[Z] = [L]d\pi + [M]dx + [N]dy + [P]dp + [Q]dq.$$

Iam posito $\Pi = [Z]dx$ *sit Z functio ipsarum x, y, p, q itemque ipsius* Π *, ita ut sit*

$$dZ = Ld\Pi + Mdx + Ndy + Pdp + Qdq.$$

His positis oporteat definiri curvam az, quae pro data abscissa $AZ = a$ *habeat valorem formulae* $\int Zdx$ *maximum vel minimum.*

SOLUTIO

Ut in Scholio praecedente monuimus, est nobis abscissa $AL = x$ et applicata $Ll = y$, abscissae autem $AL = x$ respondeat valor $\int Zdx$, qui a particula *nv* non afficietur. Ex quo valor differentialis ex sequentibus abscissae elementis determinari debebit, quibus respondebunt valores $Zdx,\ Z^{/}dx,\ Z^{//}dx,\ Z^{///}dx,\ Z^{IV}dx$ etc. usque ad ultimum abscissae totius propositae *AZ* elementum in *Z*. Invenientur autem singulorum horum terminorum valores differentiales per differentiationem substituendo loco differentialium *dy, dp, dq* valores paragrapho praecedenti indicatos. Erit igitur

$$d\cdot Zdx = dx\left(Ld\Pi + \frac{Q\cdot nv}{dx^2}\right)$$
$$d\cdot Z^{/}dx = dx\left(L^{/}d\Pi^{/} + \frac{P^{/}\cdot nv}{dx} - \frac{2Q^{/}\cdot nv}{dx^2}\right)$$
$$d\cdot Z^{//}dx = dx\left(L^{//}d\Pi^{//} + N^{//}\cdot nv - \frac{P^{//}\cdot nv}{dx} + \frac{Q^{//}\cdot nv}{dx^2}\right)$$
$$d\cdot Z^{///} = dxL^{///}d\Pi^{///}$$
$$d\cdot Z^{IV} = dxL^{IV}d\Pi^{IV}$$
etc.

Superest igitur, ut per *nv* definiamus differentialia $d\Pi,\ d\Pi^{/},\ d\Pi^{//},\ d\Pi^{///}$ etc., hoc est valores differentiales quantitatum $\Pi,\ \Pi^{/},\ \Pi^{//}, \Pi^{///}$ etc. Est vero

$$\Pi = \int [Z]dx$$
$$\Pi^{/} = \int [Z]dx + [Z]dx$$
$$\Pi^{//} = \int [Z]dx + [Z]dx + [Z^{/}]dx$$
$$\Pi^{///} = \int [Z]dx + [Z]dx + [Z^{/}]dx + [Z^{//}]dx$$
$$\Pi^{IV} = \int [Z]dx + [Z]dx + [Z^{/}]dx + [Z^{//}]dx + [Z^{///}]dx$$
etc.

Ubi notandum est quantitatis $\int [Z]dx$ valorem differentialem esse $= 0$, eo quod particula *nv* nullam mutationem infert in abscissam *AL*, ad quam $\int [Z]dx$ refertur. Tantum igitur terminorum differentialium $[Z]dx,\ [Z^{/}]dx,\ [Z^{//}]dx$ etc. valores differentiales investigari oportebit. Erit autem

$$d\cdot[Z]dx = dx\left([L]d\pi + \frac{[Q]\cdot nv}{dx^2}\right)$$
$$d\cdot[Z]^{/}dx = dx\left([L^{/}]d\pi^{/} + \frac{[P^{/}]\cdot nv}{dx} - \frac{2[Q^{/}]\cdot nv}{dx^2}\right)$$
$$d\cdot[Z]^{//}dx = dx\left([L^{//}]d\pi^{//} + [N^{//}]\cdot nv - \frac{[P^{//}]\cdot nv}{dx} + \frac{[Q^{//}]\cdot nv}{dx^2}\right)$$
$$d\cdot[Z]^{///} = dx[L^{///}]d\pi^{///}$$
$$d\cdot[Z]^{IV} = dx[L^{IV}]d\pi^{IV}$$
etc.

Nunc porro definiendi sunt valores differentiales quantitatum $\pi,\ \pi^{/},\ \pi^{//},\ \pi^{///}$ etc. per *nv*, quos loco $d\pi,\ d\pi^{/},\ d\pi^{//}$ etc. substitui oportet. Cum autem sit $\pi = \int[z]dx$ et in $[z]$ differentialia secundum gradum superantia non inesse ponantur, fiet valor differentialis ipsius n seu $d\pi = 0$, ad sequentium autem quantitatum $\pi^{/},\ \pi^{//},\ \pi^{///}$ etc. valores differentiales inveniendos notasse conveniet
esse

$$\pi = \int[z]dx$$
$$\pi^{/} = \int[z]dx + [z]dx$$
$$\pi^{//} = \int[z]dx + [z]dx + [z^{/}]dx$$
$$\pi^{///} = \int[z]dx + [z]dx + [z^{/}]dx + [z^{//}]dx$$
$$\pi^{IV} = \int[z]dx + [z]dx + [z^{/}]dx + [z^{//}]dx + [z^{///}]dx$$
etc.

Erit autem

$$d\cdot[z]dx = nv\cdot dx\frac{[q]}{dx^2}$$
$$d\cdot[z]^{/}dx = nv\cdot dx\left(\frac{[p^{/}]}{dx} - \frac{2[q^{/}]}{dx^2}\right)$$
$$d\cdot[z]^{//}dx = nv\cdot dx\left([n^{//}] - \frac{[p^{//}]}{dx} + \frac{[q^{//}]}{dx^2}\right)$$
$$d\cdot[z]^{///} = 0$$
$$d\cdot[z]^{IV} = 0$$
etc.

Ex his itaque obtinebitur

$$d\cdot\pi=0$$
$$d\cdot\pi^{/}=nv\cdot dx\frac{[q]}{dx^2}$$
$$d\cdot\pi^{//}dx=nv\cdot dx\left(\frac{[p^{/}]}{dx}-\frac{[q]}{dx^2}-\frac{2d[q]}{dx^2}\right)$$
$$d\cdot[\pi]^{///}dx=nv\cdot dx\left([n^{//}]-\frac{[p^{/}]}{dx}+\frac{dd[q]}{dx^2}\right)$$
$$d\cdot[\pi]^{IV}dx=nv\cdot dx\left([n^{//}]-\frac{[p^{/}]}{dx}+\frac{dd[q]}{dx^2}\right)$$
$$d\cdot[\pi]^{V}dx=nv\cdot dx\left([n^{//}]-\frac{[p^{/}]}{dx}+\frac{dd[q]}{dx^2}\right)$$

omnesque sequentes valores inter se erunt aequales. Quodsi iam hi valores inventi substituantur, erit

$$d\cdot[Z]dx=nv\cdot dx\frac{[Q]}{dx^2}$$

$$d\cdot[Z]^{/}dx=nv\cdot dx\left(\frac{[L^{/}][q]}{dx}+\frac{[P^{/}]}{dx}-\frac{2[Q^{/}]}{dx^2}\right)$$

$$d\cdot[Z]^{//}dx=nv\cdot dx\left([L^{//}]dx\left(\frac{[p^{/}]}{dx}-\frac{[q]+2d[q]}{dx^2}\right)+[N^{//}]-\frac{[P^{//}]}{dx}+\frac{[Q^{//}]}{dx^2}\right)$$

$$d\cdot[Z]^{///}=nv\cdot dx\cdot[L^{///}]dx\left([n^{//}]-\frac{d[p^{/}]}{dx}+\frac{dd[q]}{dx^2}\right)$$

$$d\cdot[Z]^{IV}=nv\cdot dx[L^{IV}]dx\left([n^{//}]-\frac{d[p^{/}]}{dx}+\frac{dd[q]}{dx^2}\right)$$

$$d\cdot[Z]^{V}=nv\cdot dx[L^{V}]dx\left([n^{//}]-\frac{d[p^{/}]}{dx}+\frac{dd[q]}{dx^2}\right)$$

etc.

Hinc porro deducitur:

$$d\cdot\Pi = 0$$

$$d\cdot\Pi^{/} = nv\cdot dx\frac{[Q]}{dx^2}$$

$$d\cdot\Pi^{//} = nv\cdot dx\left([L^{/}]dx\frac{[q]}{dx}+\frac{[P^{/}]}{dx}-\frac{[Q]+2d[Q]}{dx^2}\right)$$

$$d\cdot\Pi^{///} = nv\cdot dx\left([L^{//}][p^{/}]-\frac{[q][L^{/}]+2[L^{//}]d[q]}{dx}+[N^{//}]-\frac{d[P^{/}]}{dx}+\frac{dd[Q]}{dx^2}\right)$$

$$d\cdot\Pi^{IV} = nv\cdot dx\left(\begin{array}{l}[L^{///}]dx\left([n^{//}]-\frac{d[p^{/}]}{dx}+\frac{dd[q]}{dx^2}\right)+[L^{//}][p^{/}]-\frac{[q]d[L^{/}]+2[L^{//}]d[q]}{dx}\\ +[N^{//}]-\frac{d[P^{/}]}{dx}+\frac{dd[Q]}{dx^2}\end{array}\right)$$

$$d\cdot\Pi^{V} = nv\cdot dx\left(\begin{array}{l}\left(\left([L^{///}]+[L^{IV}]dx\right)\left([n^{//}]\right)-\frac{d[p^{/}]}{dx}+\frac{dd[q]}{dx^2}\right)\\ +[L^{//}][p^{/}]-\frac{[q]d[L^{/}]+2[L^{//}]d[q]}{dx}+[N^{//}]-\frac{d[P^{/}]}{dx}+\frac{dd[Q]}{dx^2}\end{array}\right)$$

etc.

Ex his iam orientur sequentes determinationes:

$$d\cdot Z = nv\cdot dx\frac{Q}{dx^2}$$

$$d\cdot Z^{/} = nv\cdot dx\left(L^{/}dx\frac{[Q]}{dx^2}+\frac{P^{/}}{dx}-\frac{2Q^{/}}{dx^2}\right)$$

$$d\cdot Z^{//} = nv\cdot dx\left(L^{//}dx\left([L^{/}]dx\frac{[q]}{dx^2}+\frac{[P^{/}]}{dx}-\frac{[Q]+2d[Q]}{dx^2}\right)+N^{//}-\frac{dP^{/}}{dx}+\frac{Q^{//}}{dx^2}\right)$$

$$d\cdot Z^{///} = nv\cdot dxL^{///}dx\left([L^{//}][p^{/}]-\frac{[q]d[L^{/}]+2[L^{//}]d[q]}{dx}+[N^{//}]-\frac{d[P^{/}]}{dx}+\frac{dd[Q]}{dx^2}\right)$$

$$d\cdot Z^{IV} = nv\cdot dxL^{IV}dx\left(\begin{array}{l}[L^{///}]dx\left([n^{//}]-\frac{d[p^{/}]}{dx}+\frac{dd[q]}{dx^2}\right)+[L^{//}][p^{/}]\\ -\frac{[q]d[L^{/}]+2[L^{//}]d[q]}{dx}+[N^{//}]-\frac{d[P^{/}]}{dx}+\frac{dd[Q]}{dx^2}\end{array}\right)$$

$$d\cdot Z^{V} = nv\cdot dxL^{V}dx\left(\begin{array}{l}\left([L^{///}]dx+[L^{IV}]dx\right)\left([n^{//}]-\frac{d[p^{/}]}{dx}+\frac{dd[q]}{dx^2}\right)+[L^{//}][p^{/}]\\ -\frac{[q]d[L^{/}]+2[L^{//}]d[q]}{dx}+[N^{//}]-\frac{d[P^{/}]}{dx}+\frac{dd[Q]}{dx^2}\end{array}\right)$$

$$d\cdot Z^{VI} = nv\cdot dxL^{VI}dx\left(\begin{array}{l}\left([L^{///}]dx+[L^{IV}]dx+[L^{V}]dx\right)\left([n^{//}]-\frac{d[p^{/}]}{dx}+\frac{dd[q]}{dx^2}\right)+[L^{//}][p^{/}]\\ -\frac{[q]d[L^{/}]+2[L^{//}]d[q]}{dx}+[N^{//}]-\frac{d[P^{/}]}{dx}+\frac{dd[Q]}{dx^2}\end{array}\right)$$

etc.

Ut hi valores omnes eo commodius ad se invicem addi queant, ponamus brevitatis gratia

$$[h]=[n^{//}]-\frac{d[p^{/}]}{dx}+\frac{dd[q]}{dx^2}=n-\frac{d[p]}{dx}+\frac{dd[q]}{dx^2}$$

et

$$[H]=[L][p]-\frac{[q]d[L^{/}]+2[L]d[q]}{dx}+[N]-\frac{d[P]}{dx}+\frac{dd[Q]}{dx^2}$$

eritque summa omnium, hoc est valor differentialis formulae propositae

$\int Zdx$, ut sequitur:

$$nv\cdot dx\left(N-\frac{P}{dx}-\frac{ddQ}{dx^2}\right)+nv\cdot dx\left(L[P]-\frac{[Q]dL+2Ld[Q]}{dx}\right)+nv\cdot dx\cdot L[L][q]$$
$$+nv\cdot dx\cdot[H]\left(L^{///}dx+L^{IV}dx+L^{V}dx+\text{ etc. usque in } Z\right)$$
$$+nv\cdot dx\cdot[h]\begin{pmatrix} L^{IV}dx\cdot[L^{///}]dx+L^{V}dx([L^{///}]dx+L^{IV}dx)+L^{VI}dx([L^{///}]dx+L^{IV}dx+L^{V}dx) \\ +L^{VII}dx([L^{///}]dx+L^{IV}dx+L^{V}dx+L^{VI}dx)+\text{ etc.}\end{pmatrix}$$

Binae igitur hic habentur series infinitae, a termino Ll usque ad Zz progredientes, quarum illius $L^{///}dx+L^{IV}dx+L^{V}dx+\text{etc.}$ summa exprimi potest per $H-\int Ldx$, denotante H valorem ipsius $\int Ldx$ posito $x=a$. Quo autem valorem alterius seriei investigemus, ponatur eius summa $=S$, ita ut sit

$$S=L^{IV}dx\cdot[L^{///}]dx+L^{V}dx\cdot([L^{///}]dx+L^{IV}dx)+\text{etc.}$$

Sumatur valor sequens $S^{/}=S+dS$, erit

$$S+dS=L^{V}dx\cdot[L^{IV}]dx+L^{VI}dx\cdot([L^{IV}]dx+L^{V}dx)+\text{etc.},$$

qui ab illo subtractus relinquet

$$-dS=L^{IV}[L^{///}]dx^2+L^{V}[L^{///}]dx^2+L^{VI}[L^{///}]dx^2+\text{etc.},$$

seu

$$dS=[L^{///}]dx\left(L^{IV}dx+L^{V}dx+L^{VI}dx+\text{ etc.}\right)$$

ideoque

$$-dS=[L^{///}]dx\left(H-\int Ldx\right)$$

et integrando

$$B=G-\int[L]dx(H-\int Ldx)$$

constante G ita assumta, ut fiat $S=0$, si ponatur $x=a$. His inventis fiet valor differentialis formulae propositae

$$\int Zdx = nv\cdot dx\left(\begin{array}{l} N-\frac{dP}{dx}+\frac{ddQ}{dx^2}+L[P]-\frac{[Q]dL+2Ld[Q]}{dx}+L[L][q] \\ +\left(H-\int Ldx\right)\left([L][p]-\frac{[q]dL+2Ld[q]}{dx}+[N]-\frac{d[P]}{dx}+\frac{dd[Q]}{dx^2}\right) \\ +\left(G-\int[L]dx\left(H-f\int Ldx\right)\right)\left([n]-\frac{d[p]}{dx}+\frac{dd[q]}{dx^2}\right). \end{array}\right)$$

Haec expressio autem in sequentem formam transmutari potest, ex qua facilius valor differentialis formari poterit, si differentialia altiorum graduum quam secundi tam in Z quam in $[Z]$ et $[z]$ insint. Erit scilicet formulae $\int Zdx$ valor differentialis abscissae $AZ = a$ respondens

$$= nv\cdot dx\left(N-\frac{dP}{dx}+\frac{ddQ}{dx^2}-\frac{d^3R}{dx^3}+\frac{d^4S}{dx^4}-\text{ etc.}\right)$$

$$+\, nv\cdot dx\left(\begin{array}{l} [N]\left(H-\int Ldx\right)-\frac{d\cdot[P]\left(H-\int Ldx\right)}{dx}+\frac{dd[Q]\left(H-\int Ldx\right)}{dx^3} \\ -\frac{d^3\cdot[R]\left(H-\int Ldx\right)}{dx^3}+\frac{d^4[S]\left(H-\int Ldx\right)}{dx^4}-\text{ etc.} \end{array}\right)$$

$$+\, nv\cdot dx\left(\begin{array}{l} [n]\left(G-\int[L]dx\left(H-f\int Ldx\right)\right)-\frac{d[p]\left(G-\int[L]dx\left(H-f\int Ldx\right)\right)}{dx} \\ +\frac{dd\cdot[q]\left(G-\int[L]dx\left(H-f\int Ldx\right)\right)}{dx^2}-\frac{d^3\cdot[r]\left(G-\int[L]dx\left(H-f\int Ldx\right)\right)}{dx^3}\text{ etc.} \end{array}\right)$$

Invento autem valore differentiali, si is ponatur $=0$, habebitur aequatio pro curva quaesita. Q. E. I.

COROLLARIUM 1

32. Inventus igitur est valor differentialis pro formula $\int Zdx$ latius patente, quam quidem in Propositione est assumta, scilicet si fuerit

$$dZ = Ld\Pi + Mdx + Ndy + Pdp + Qdq + Rdr + \text{etc}.$$

atque existente $d\Pi = [Z]dx$ si sit

$$d[Z] = [L]d\pi + [M]dx + [N]dy + [P]dp + [Q]dq + [R]dr + \text{etc.}$$

itemque si posito $d\pi = [z]dx$ fuerit

$$d[z]=[m]dx+[n]dy+[p]dp+[q]dq+[r]dr+\text{etc}.$$

Quoticunque nimirum gradus differentialia insint in quantitatibus Z, $[Z]$ et $[z]$, solutio data inserviet.

COROLLARIUM 2

33. Quodsi ponatur $H-\int Ldx=T$ et $G-\int [L]dx\left(H-\int Ldx\right)=V$,
erit valor differentialis

$$=nv\cdot dx\left(N-\frac{dP}{dx}+\frac{ddQ}{dx^2}-\frac{d^3R}{dx^3}+\text{etc.}\right)$$
$$+\ nv\cdot dx\left([N]T-\frac{d[P]T}{dx}+\frac{dd\cdot[Q]T}{dx^3}-\frac{d^3[R]T}{dx^3}+\text{etc.}\right)$$
$$+\ nv\cdot dx\left([n]V-\frac{d\cdot[p]V}{dx}+\frac{dd\cdot[q]V}{dx^2}-\frac{d^3[r]V}{dx^3}+\text{etc.}\right).$$

COROLLARIUM 3

34. Hinc igitur aequatio pro curva quaesita erit haec

$$0=N+[N]T+[n]V-\frac{\left(P+[P]T+[p]V\right)}{dx}$$
$$+\frac{dd\left(Q+[Q]T+[q]V\right)}{dx^2}-\frac{d^3\left(R+[R]T+[r]V\right)}{dx^3}+\ \text{etc.},$$

cuius progressionis lex, si forte opus sit pluribus terminis, sponte patet.

COROLLARIUM 4

35. Quin etiam hinc resolvi poterunt eiusmodi Problemata, in quibus Z non unam, sed plures istiusmodi formulas integrales indefinitas Π in se complectitur; vel etiam, si $[Z]$ plures eiusmodi formulas $\pi=\int [z]dx$ in se contineat.

COROLLARIUM 5

36. Denique, etsi posuimus esse $[z]$ functionem determinatam, tamen per inductionem hinc modus patet valorem differentialem formandi, si ulterius $[z]$ in se contineat formulam integralem indefinitam.

SCHOLION

37. Latissime igitur solutio huius Problematis patet, quia non solum praecedentia Problemata omnia in se complectitur atque ipsi casui proposito satisfacit, verum etiam per inductionem ad casus qualescunque magis intricatos accommodari potest. Quod ut facilius percipiatur, ponamus in $[z]$ insuper inesse formulam integralem $\pi = \int \zeta dx$, ita ut sit

$$d[z] = [l]d\pi + [m]dx + [n]dy + [p]dp + [q]dq + \text{etc.}$$

existente

$$d\zeta = \mu dx + vdy + \varphi dp + \chi dq + \text{etc.}$$

Iam ad valorem differentialem determinandum praeter quantitates integrales binas *T* et *V* tertia debet definiri *W* ita comparata, ut sit

$$W = F - \int [l]dx\left(G - \int [L]dx\left(H - \int Ldx\right)\right),$$

quae evanescat posito $x = a$. Hocque facto erit valor differentialis

$$= nv \cdot dx(N + [N]T + [n]V + vW - \frac{d(P + [P]T + [p]V + \varphi W)}{dx} + \frac{dd \cdot (Q + [Q]T + [q]V + \chi W)}{dx^2} - \text{ etc.}).$$

Quamobrem nequidem maximi minimive formula excogitari poterit, quae non in solutione esset contenta aut ex talibus formulis composita, ad quas ista solutio patet. Quinetiam liceret hanc expressionem in infinitum extendere, si quaelibet formula indeterminata aliam novam formulam integralem indefinitam in se complectatur; neque difficultas ulla adesset, nisi in characterum sufficienti numero suppeditando. Quae cum ulterius prosequi non sit necesse, unicum casum principalem evolvere conveniet, quo in formula $\int [Z]dx$, quae valorem ipsius Π praebet, ipsa quantitas $[Z]$ denuo Π involvit. Hoc enim casu complexio istiusmodi formularum integralium actu in infinitum progreditur; namque, si sit

$$d[Z] = [L]d\Pi + [M]dx + [N]dy + [P]dp + [Q]dq + \text{etc.},$$

erit hic iterum $d\Pi$, quod ante fuerat *dn*, et quoniam est $d\Pi = [Z]dx$, denuo eadem aequatio

$$d[Z] = [L]d\Pi + [M]dx + [N]dy + \text{etc.}$$

recurrit, atque ita tractatio formularum integralium nusquam abrumpetur. Casum igitur hunc, cum quia insignem nobis afferet usum, tum quia concinnam admittit solutionem, pertractabimus.

PROPOSITIO V. PROBLEMA

38. *Si* Π *aliter non detur, nisi per aequationem differentialem* $d\Pi = [Z]dx$, *in qua* $[Z]$ *praeter quantitates ad curvam pertinentes x, y, p, q, r etc. ipsam quantitatem* Π *complectatur ita ut sit*

$$d[Z] = [L]d\Pi + [M]dx + [N]dy + [P]dp + [Q]dq + etc. ,$$

sit Z junctio quaecunque ipsius Π *et ipsarum x, y, p, q etc., ita ut sit*

$$dZ = Ld\Pi + Mdx + Ndy + Pdp + Qdq + etc.,,$$

invenire curvam, in qua pro data abscissa AZ = a maximum minimumve sit formula $\int Zdx$.

SOLUTIO

Ponamus differentialia, quae tam in *Z* quam in $[Z]$ insunt, secundum gradum non excedere, ita ut particula *nv*, ultra abscissae punctum *L* versus initium nullam mutationem inferat. Solutio enim nihilominus hinc poterit maxime generalis confici. Sit igitur abscissa $AL = x$ et applicata $Ll = y$, patietur $\int Zdx$ ab adiecta particula np applicatae $Nn = y^{//}$ nullam mutationem, eiusque valor differentialis erit $= 0$. Quamobrem valor differentialis formulae $\int Zdx$, quatenus ad totam abscissam *AZ* extenditur, colligi debebit ex elementis $Zdx,\ Z^{/}dx,\ Z^{//}dx,\ Z^{///}dx$ etc. Singulorum autem horum elementorum valores differentiales invenientur, si ea differentientur et loco differentialium $dy,\ dy^{/},\ dy^{//},\ dp,\ dp^{/},\ dp^{//}$ et $dq,\ dq^{/},\ dq^{//}$ valores paragrapho 30 indicati substituantur. Quoniam autem insuper in haec differentialia ingrediuntur $d\Pi, d\Pi^{/},\ d\Pi^{//}$ etc., ponamus eorum valores ex *nv* oriundos tantisper, donec eos inveniamus, esse hos:

$d\Pi = nv \cdot \alpha$	$d\Pi^{///} = nv \cdot \delta$	$d\Pi^{VI} = nv \cdot \eta$
$d\Pi^{/} = nv \cdot \beta$	$d\Pi^{IV} = nv \cdot \varepsilon$	$d\Pi^{VII} = nv \cdot \theta$
$d\Pi^{//} = nv \cdot \gamma$	$d\Pi^{V} = nv \cdot \zeta$	etc.

Hinc itaque erunt valores differentiales

$$d\cdot Zdx = nv\cdot dx\left(L\alpha + \frac{Q}{dx^2}\right)$$
$$d\cdot Z^{/}dx = nv\cdot dx\left(L^{/}\beta + \frac{P^{/}}{dx} - \frac{2Q^{/}}{dx^2}\right)$$
$$d\cdot Z^{//}dx = nv\cdot dx\left(L^{//}\gamma + N^{//} - \frac{dP^{/}}{dx} + \frac{Q^{//}}{dx^2}\right)$$
$$d\cdot Z^{///}dx = nv\cdot dxL^{///}\delta$$
$$d\cdot Z^{IV}dx = nv\cdot dxL^{IV}\varepsilon$$
$$d\cdot Z^{V}dx = nv\cdot dxL^{V}\zeta$$
etc.

Ut nunc valores litterarum $\alpha, \beta, \gamma, \delta, \varepsilon$ etc. definiamus, notandum est esse $d\Pi, d\Pi^{/}, d\Pi^{//}$ etc. valores differentiales quantitatum $\Pi, \Pi^{/}, d\Pi^{//}$ etc. Est vero

$$\Pi = \int [Z]dx$$
$$\Pi^{/} = \int [Z]dx + [Z]dx$$
$$\Pi^{//} = \int [Z]dx + [Z]dx + [Z^{/}]dx$$
$$\Pi^{///} = \int [Z]dx + [Z]dx + [Z^{/}]dx + [Z^{//}]dx$$
etc.,

ubi $\int [Z]dx$ per hypothesin a particula *nv* non afficitur. Valores igitur differentiales formularum $[Z]dx$, $[Z^{/}]dx$, $[Z^{//}]dx$ etc. sunt investigandi, qui erunt

$$d\cdot [Z]dx = nv\cdot dx\left([L]\alpha + \frac{[Q]}{dx^2}\right)$$
$$d\cdot [Z^{/}]dx = nv\cdot dx\left([L^{/}]\beta + \frac{[P^{/}]}{dx} - \frac{2[Q^{/}]}{dx^2}\right)$$
$$d\cdot [Z^{//}]dx = nv\cdot dx\left([L^{//}]\gamma + [N^{//}] - \frac{[P^{//}]}{dx} + \frac{[Q^{//}]}{dx^2}\right)$$
$$d\cdot [Z^{///}]dx = nv\cdot dx[L^{///}]\delta$$
$$d\cdot [Z^{IV}]dx = nv\cdot dx[L^{IV}]\varepsilon$$
$$d\cdot [Z^{V}]dx = nv\cdot dx[L^{V}]\zeta$$
etc.

Ex his igitur erit, ut sequitur

$$d\Pi = nv\cdot\alpha$$
$$d\Pi^{/} = nv\cdot dx\left([L]\alpha + \frac{[Q]}{dx^2}\right)$$
$$d\Pi^{//} = nv\cdot dx\left([L]\alpha + [L^{/}]\beta + \frac{[P^{/}]}{dx} - \frac{[Q]+2d[Q]}{dx^2}\right)$$
$$d\Pi^{///} = nv\cdot dx\left([L]\alpha + [L^{/}]\beta + [L^{//}]\gamma + [N^{//}] - \frac{[P^{/}]}{dx} + \frac{dd[Q]}{dx^2}\right)$$
$$d\Pi^{IV} = nv\cdot dx\left([L]\alpha + [L^{/}]\beta + [L^{//}]\gamma + L^{///}\delta + [N^{//}] - \frac{[P^{/}]}{dx} + \frac{dd[Q]}{dx^2}\right)$$
$$d\Pi^{V} = nv\cdot dx\left([L]\alpha + [L^{/}]\beta + [L^{//}]\gamma + L^{///}\delta + L^{IV}\varepsilon + [N^{//}] - \frac{[P^{/}]}{dx} + \frac{dd[Q]}{dx^2}\right)$$

etc.

His comparatis cum valoribus assumtis erit

$$\alpha = 0$$
$$\beta = [L]\alpha dx + \frac{[Q]}{dx}$$
$$\gamma = dx\left([L]\alpha + [L^{/}]\beta + \frac{[P^{/}]}{dx} - \frac{[Q]+2d[Q]}{dx^2}\right)$$
$$\delta = dx\left([L]\alpha + [L^{/}]\beta + [L^{//}]\gamma + [N^{//}] - \frac{[P^{/}]}{dx} + \frac{dd[Q]}{dx^2}\right)$$
$$\varepsilon = dx\left([L]\alpha + [L^{/}]\beta + [L^{//}]\gamma + L^{///}\delta + [N^{//}] - \frac{[P^{/}]}{dx} + \frac{dd[Q]}{dx^2}\right)$$

etc.

Ex hisque aequationibus elicitur:

$$\alpha = 0$$
$$\beta = \frac{[Q]}{dx}$$
$$\gamma = [L^{/}][Q] + [P^{/}] - \frac{[Q]+2d[Q]}{dx}$$
$$\delta = [L^{/}][Q] + [L^{//}][L^{/}][Q]dx + [L^{//}][P^{/}]dx - [L^{//}][Q] - 2[L^{//}]d[Q] + [N^{//}]dx - [P^{/}]dx + \frac{dd[Q]}{dx}$$
$$\text{seu} = \delta = [L^{//}][L^{/}][Q]dx + [L^{//}][P^{/}]dx - [Q]d[L^{/}] - 2[L^{//}]d[Q] + [N^{//}]dx - [P^{/}]dx + \frac{dd[Q]}{dx},$$

qui valor ipsius δ notetur, eritque porro

$$\varepsilon = \delta\left(1 + [L^{///}]dx\right)$$
$$\zeta = \delta\left(1 + [L^{///}]dx\right)\left(1+[L^{IV}]dx\right)$$
$$\eta = \delta\left(1 + [L^{///}]dx\right)\left(1+[L^{IV}]dx\right)\left(1+[L^{V}]dx\right)$$
etc.

Cognitis his valoribus erit valor differentialis elementis $Zdx + Z^{/}dx + Z^{//}dx$ respondens

$$= nv \cdot dx\left(N - \frac{dP}{dx} + \frac{ddQ}{dx^2} + L[L][Q] + L[P] - \frac{[Q]dL + 2Ld[Q]}{dx}\right).$$

Sequentium autem elementorum omnium usque ad Z valor differentialis, si ponatur

$$V = [L^2][Q] + [L][P] - \frac{[Q]d[L] + 2[L]d[Q]}{dx} + [N] - \frac{[P]}{dx} + \frac{dd[Q]}{dx^2}$$

seu $\delta = Vdx$, erit sequens:

$$nv \cdot dx(L^{///}dx + L^{IV}dx(1+[L^{///}]dx) + L^{V}dx(1+[L^{///}]dx)(1+[L^{IV}]dx)$$
$$+ L^{VI}dx(1+[L^{///}]dx)(1+[L^{IV}]dx)(1+[L^{V}]dx) + \text{etc.})V.$$

Quamobrem huius seriei summa est indaganda; hunc in finem scribamus

L loco $L^{///}$ et $[L]$ loco $[L^{///}]$, sitque summa, quam quaerimus, $= S$; erit

$$S = Ldx + L^{/}dx(1+[L]dx) + L^{//}dx(1+[L]dx)(1+[L^{/}]dx)$$
$$+ L^{///}dx(1+[L]dx)(1+[L^{/}]dx)(1+[L^{//}]dx) + \text{etc.}$$

Iam ipsius S sumatur valor sequens $S^{/} = S + dS$, erit

$$S + dS = L^{/}dx + L^{//}dx(1+[L^{/}]dx) + L^{///}dx(1+[L^{/}]dx)(1+[L^{//}]dx) + \text{etc.}$$

Hincque

$$-dS = Ldx + L^{/}[L]dx^2 + [L]dx \cdot L^{//}dx(1+[L^{/}]dx)$$
$$+[L]dx \cdot L^{///}dx(1+[L^{/}]dx)(1+[L^{//}]dx) + \text{etc.},$$

quae series, cum ad priorem reduci queat, erit

$$-dS = Ldx + S^{/}[L]dx,$$

sive ob $S^{/} = S$, $dS + S[L]dx = -Ldx$; quae integrata dat

$$e^{\int [L]dx} S = C - \int e^{\int [L]dx} Ldx,$$

quae constans C ita debet accipi, ut posito $x = a$ fiat $S = 0$. Hanc ob rem erit valor illius seriei

$$S = e^{-\int [L]dx} (C - \int e^{\int [L]dx} Ldx).$$

Ex his igitur formulae propositae $\int Zdx$ orietur sequens valor differentialis:

$$nv \cdot dx \left(\begin{array}{l} N - \frac{P}{dx} + \frac{ddQ}{dx^2} + L[L][Q] + L[P] - \frac{[Q]dL + 2Ld[Q]}{dx} \\ + S\left([L^2][Q] + L[P] - \frac{[Q]d[L] + 2[L]d[Q]}{dx} + [N] - \frac{d[P]}{dx} + \frac{dd[Q]}{dx^2} \right) \end{array} \right),$$

qui transmutatur in hanc formam commodiorem

$$nv \cdot dx \left(N - \frac{P}{dx} + \frac{ddQ}{dx^2} + [N]S - \frac{d \cdot [P]S}{dx} + \frac{dd \cdot [Q]S}{dx^2} \right).$$

Hinc autem formari potest valor differentialis formulae $\int Zdx$, si tam in Z quam in $[Z]$ differentialia ad gradum quemcunque assurgant. Ad hoc efficiendum sit valor formulae integralis $\int e^{\int [L]dx} Ldx,$, quem obtinet, si $x = a$ ponatur, $= H$ ac scribatur brevitatis ergo V loco huius expressionis $e^{-\int [L]dx} (H - \int e^{\int [L]dx} Ldx)$, eritque valor differentialis

$$= nv \cdot dx \left(N + [N]V - \frac{d \cdot (P + [P]V)}{dx} + \frac{dd(Q + [Q]V)}{dx^2} - \frac{d^3(R + [R]V)}{dx^3} + \text{ etc.} \right).$$

Atque hinc pro curva quaesita orietur ista aequatio

$$0 = N + [N]V - \frac{d \cdot (P + [P]V)}{dx} + \frac{dd(Q + [Q]V)}{dx^2} - \frac{d^3(R + [R]V)}{dx^3} + \frac{d^4(S + [S]V)}{dx^4} - \text{ etc.}$$

Q. E. I.

COROLLARIUM 1

39. Inservit igitur ista propositio eiusmodi Problematibus resolvendis, in quibus maximi minimive formula $\int Zdx$ talem in se continet quantitatem Π, quae nequidem formula integrali ex quantitatibus ad curvam pertinentibus *x, y, p, q, r* etc. exhiberi potest, sed cuius determinatio pendet a resolutione aequationis differentialis cuiuscunque. Habetur enim $d\Pi = [Z]dx$ atque $[Z]$ ipsam quantitatem Π utcunque in se complecti ponitur.

COROLLARIUM 2

40. Casus hic notari meretur, quo est $L = [L]$, quippe quo fit formula $\int e^{\int [L]dx} Ldx$, integrabilis, integrali existente $e^{\int [L]dx}$. Quodsi ergo, posito $x = a$, abeat $e^{\int [L]dx}$ in *H*, fiet

$$V = He^{-\int [L]dx} - 1.$$

COROLLARIUM 3

41. Casus hic potissimum locum habet, quando curva quaeritur, in qua sit ipsa formula $\Pi = \int [Z]dx$ maximum vel minimum. Tum enim fit $Z = [Z]$ et hinc $L = [L],\ M = [M],\ N = [N]$ etc. Hinc itaque erit valor differentialis

$$= nv \cdot dx \left(H[N]e^{-\int [L]dx} - \frac{d \cdot H[P]e^{-\int [L]dx}}{dx} + \frac{dd \cdot H[Q]e^{-\int [L]dx}}{dx^2} - \text{ etc.} \right).$$

Atque aequatio pro curva erit

$$0 = [N]e^{-\int [L]dx} - \frac{d \cdot [P]e^{-\int [L]dx}}{dx} + \frac{dd \cdot [Q]e^{-\int [L]dx}}{dx^2} - \text{ etc.}$$

COROLLARIUM 4

42. Quia ex hac aequatione quantitas *H* a data abscissa $AZ = a$ pendens per divisionem est egressa, patet his casibus curvam uni abscissae satisfacientem, eandem pro omni alia abscissa esse satisfacturam, ita ut haec Problemata similla sint iis, in quibus quantitas *Z* est functio determinata.

COROLLARIUM 5

43. Si ergo quantitas $\Pi = \int [Z]dx$ debeat esse maximum vel minimum existente

$$d[Z] = [L]d\Pi + [M]dx + [N]dy + [P]dp + [Q]dq + \text{etc.},$$

curva poterit exhiberi, quae una pro quacunque abscissa ista proprietate gaudeat; eiusque natura exprimetur hac aequatione

$$0 = [N]e^{-\int [L]dx} - \frac{d\cdot [P]e^{-\int [L]dx}}{dx} + \frac{dd\cdot [Q]e^{-\int [L]dx}}{dx^2} - \text{ etc.}$$

Ex qua insuper evolutis singulis terminis quantitas exponentialis $e^{-\int [L]dx}$ atque adeo ipsa formula integralis $\int [L]dx$ excedent.

SCHOLION I

44. Usus huius Propositionis eximius est in quaestionibus ita comparatis, ut quantitates indefinitae in iis contentae per formulas integrales exhiberi nequeant, verum constructionem aequationum differentialium postulent. Atque haec solutio perinde valet, sive una huiusmodi quantitas Π insit in formula maximi minimive $\int Zdx$, sive plures; quodsi enim plures insint eiusmodi quantitates Π, plures etiam habebuntur valores litterarum L, $[L]$, $[M]$, $[N]$, $[P]$, $[Q]$ etc. atque etiam litterae $V = e^{-\int [L]dx}(H - \int e^{\int [L]dx} Ldx)$; qui omnes aequaliter eo modo, quem invenimus, in valorem differentialem formulae $\int Zdx$ introducti praebebunt aequationem pro curva; similisque omnino tractatio erit, ac si unica tantum adesset. Quoniam autem littera ista Π, cuius valor absolutus per quantitates ad curvam pertinentes exhiberi non potest, in omnibus fere terminis manet, aequatio pro curva, quae invenitur, non solum ex litteris *x, y, p, q, r* etc. constabit, sed etiam ipsam eam quantitatem Π aliasque formulas integrales plerumque ab ea pendentes, uti $\int [L]dx$ et $\int Ldx$, involvet. Quare, ut aequatio pro curva pura, quae tantum litteris *x, y, p, q* etc. contineatur, prodeat, oportet cum aequatione inventa, postquam a formulis integralibus $\int [L]dx$ et $\int Ldx$ est liberata, coniungi aequationem $d\Pi = [Z]dx$ eiusque ope valorem Π eliminari. Quanquam autem hoc modo ad differentialia altiorum ordinum pervenitur, tamen non totidem inesse censendae sunt constantes arbitrariae. Nam tam ipsa aequatio $d\Pi = [Z]dx$ quam reliquae anteriores aequationes certam requirunt determinationem, unde plures constantes determinabuntur. Caeterum notandum est veritatem huius Methodi comprobari posse per praecedentes, quando aequatio $d\Pi = [Z]dx$ ita est comparata, ut integrationem admittat; tum enim eaedem quaestiones per Methodos ante traditas resolvi poterunt indeque consensum

observare licebit. Ita, si $[Z]$ tantum ex x et Π constet, tum certum erit Π esse functionem quamdam ipsius x determinatam atque solutionem ad Caput praecedens pertinere. Idem vero haec solutio patefaciet; cum enim sit hoc casu $[N]=0,\ [P]=0,\ [Q]=0$ etc., aequatio pro curva erit

$$0=N-\frac{dP}{dx}+\frac{ddQ}{dx^2}-\text{etc.},$$

quae eadem per Methodum priorem obtinetur. Usus autem huius solutionis clarius per aliquot Exempla declarabitur.

EXEMPLUM I

45. *Invenire curvam, in qua sit maximus valor ipsius li existente* $d\Pi=gdx-\alpha\Pi^n dx\sqrt{(1+pp)}$.

Quaestio haec occurrit, quando quaeritur curva, super qua grave in medio resistente secundum celeritatum rationem 2*n-plicatam* descendens maximam obtinet celeritatem; denotat enim Π quadratum celeritatis et g vim gravitatis secundum directionem axis AZ exertam. Pertinet itaque haec quaestio ad casum Corollario 3, 4 et 5 expositum, quo erat $Z=[Z]=g-\alpha\Pi^n\sqrt{(1+pp)}$; atque adeo curva uni abscissae satisfaciens pro omni abscissa aeque valebit. Cum igitur sit erit

$$[L]=-\alpha n\Pi^{n-1}\sqrt{(1+pp)},\ [M]=0,\ [N]=0,\ [P]=-\frac{\alpha\Pi^n p}{\sqrt{(1+pp)}},\ [Q]=0,\ \text{etc.}$$

erit

$$[L]=-\alpha n\Pi^{n-1}\sqrt{(1+pp)},\ [M]=0,\ [N]=0,\ [P]=-\frac{\alpha\Pi^n pdp}{\sqrt{(1+pp)}},\ [Q]=0,\ \text{etc.}$$

Unde pro curva quaesita ista invenitur aequatio:

$$0=-d\cdot[P]e^{-\int[L]dx}\ \text{ seu }\ [P]e^{-\int[L]dx}=C\,;$$

hincque

$$-\int[L]dx=lC-l[P]\ \text{ et }\int[L]dx=\frac{d[P]}{[P]}.$$

Substitutis ergo loco $[L]$ et $[P]$ debitis valoribus, erit

$$\int \alpha n\Pi^{n-1}dx\sqrt{(1+pp)} = +lC - l(-\alpha) - l\Pi^{n} - lp + l\sqrt{(1+pp)}\ ;$$

hincque

$$\alpha n\Pi^{n-1}dx\sqrt{(1+pp)} = -\frac{nd\Pi}{\Pi} - \frac{dp}{p} + \frac{pdp}{(1+pp)} = -\frac{dp}{p(1+pp)} - \frac{nd\Pi}{\Pi}$$

seu

$$0 = nd\Pi + \alpha n\Pi^{n}dx\sqrt{(1+pp)} + \frac{\Pi dp}{p(1+pp)}.$$

Quae aequatio, ut eliminetur Π , coniungenda est cum hac

$$d\Pi + \alpha\Pi^{n}dx\sqrt{(1+pp)} = gdx\ ;$$

unde statim fit

$$0 = ngdx + \frac{\Pi dp}{p(1+pp)} \quad \text{et} \quad \Pi = -\frac{ngpdx(1+pp)}{dp}.$$

Cum igitur curva fuerit inventa, haec aequatio statim praebet celeritatem corporis in quovis curvae loco. Ponatur $dx = -\frac{tdp}{ng}$, erit

$$\Pi = pt(1+pp) \quad \text{et} \quad d\Pi = pd(1+pp) + tdp(1+3pp)\ ;$$

hincque obinebitur ista aequatio

$$pdt(1+pp)+tdp(1+3pp) - \frac{\alpha p^{n}t^{n+1}(1+pp)^{n+\frac{1}{2}}dp}{ng} + \frac{tdp}{n} = 0,$$

quae transmutatur in hanc

$$\frac{npdt(1+pp)+tdp(n+1+3npp)}{nt^{n+1}p^{n+2}(1+pp)^{n+\frac{1}{2}}} = \frac{\alpha dp}{ngp^{2}},$$

cuius integralis est

$$\frac{1}{nt^{n}p^{n+1}(1+pp)^{n-\frac{1}{2}}} = \frac{\alpha}{ngp} + \frac{\beta}{ng},$$

seu

$$g = (\alpha + \beta p)t^{n}p^{n}(1+pp)^{n-\frac{1}{2}}\ ;$$

hincque

$$t = \frac{\sqrt[n]{g}}{p(1+pp)^{1-1:2n}\sqrt[n]{(\alpha+\beta p)}}.$$

Erit igitur

$$dx = \frac{-dp}{np(1+pp)^{1-1:2n}\sqrt[n]{g^{n-1}(\alpha+\beta p)}}.$$

et

$$dy = \frac{-dp}{n(1+pp)^{1-1:2n}\sqrt[n]{g^{n-1}(\alpha+\beta p)}};$$

hincque

$$\Pi = \sqrt[n]{\frac{g\sqrt{(1+pp)}}{\alpha+\beta p}}.$$

Erit ergo

$$x = -\frac{1}{ng}\int\frac{dp}{p(1+pp)}\sqrt[n]{\frac{g\sqrt{(1+pp)}}{\alpha+\beta p}}$$

atque

$$y = -\frac{1}{ng}\int\frac{dp}{1+pp}\sqrt[n]{\frac{g\sqrt{(1+pp)}}{\alpha+\beta p}}.$$

Hinc apparet quantitatem Π super curva nusquam esse posse $=0$; hanc ob rem in curvae initio Π iam habebit certum quemdam valorem. Ut autem indoles curvae magis percipiatur, ex aequatione $\Pi = -\frac{ngpdx(1+pp)}{dp}$ patet valorem ipsius dp ubique negativum esse oportere, ex quo curva versus axem erit concava. Quia igitur valores ipsius p recedendo a curvae initio decrescunt, in ipso curvae initio p maximum habebit valorem. Hinc ponamus (Fig. 6) initium curvae ibi, ubi est $p=\infty$. Sit ergo AP axis curvae verticalis, in cuius directione vis gravitatis g corpus deorsum P trahat, atque in initio curvae A sit tangens horizontalis Aa; ibique corpus motum super curva incipiat celeritate, cuius quadratum sit $=b$. Erit igitur, posito $p=\infty$, $b=\sqrt[n]{\frac{g}{\beta}}$, atque $\beta b^n = g$ seu $\beta = \frac{g}{b^n}$. Porro ad uniformitatem conservandam sit $\alpha = \frac{1}{k^n}$. Quodsi iam curva quaesita sit AM

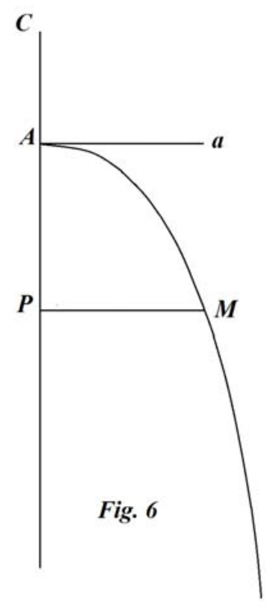

Fig. 6

et ponatur $AP = x,\ PM = y$ et $dy = pdx$, erit in M celeritatis quadratum

$$\Pi = bk\sqrt[n]{\frac{g\sqrt{(1+pp)}}{b^n + gk^n p}};$$

atque ubi tangens curvae fiet verticalis, ibi erit celeritatis quadratum $= k\sqrt[n]{g}$. Curvae autem constructio ita conficietur, ut sit

$$x = -\frac{bk}{ng}\int \frac{dp}{p(1+pp)}\sqrt[n]{\frac{g\sqrt{(1+pp)}}{b^n + gk^n p}}.$$

et

$$y = -\frac{bk}{ng}\int \frac{dp}{1+pp}\sqrt[n]{\frac{g\sqrt{(1+pp)}}{b^n + gk^n p}}$$

Deinde commemorari meretur singularis proprietas seu relatio inter corporis descendentis vim centrifugam, quae est $\frac{2\Pi}{\text{rad. osculi}}$, et vim normalem, quae est $\frac{gp}{\sqrt{(1+pp)}}$. Quodsi enim vis centrifuga

$$\frac{2\Pi}{\text{rad. osc.}} = \frac{-2\Pi dp}{dx(1+pp)^{3:2}}$$

ponatur $= F$ et vis normalis $\frac{gp}{\sqrt{(1+pp)}} = G$, erit ex aequatione

$$\Pi = -\frac{ngpdx(1+pp)}{dp} \quad \text{seu} \quad \frac{-2\Pi dp}{dx(1+pp)^{3:2}} = \frac{2ngp}{\sqrt{(1+pp)}}$$

haec relatio inter vim centrifugam F et vim normalem G, ut sit $F = 2nG$; nempe vis normalis se habebit ad vim centrifugam, ut 1 ad $2n$. Corpus in A data celeritate motum inchoans descendendo super curva AM in quovis loco M abscissae AP respondente maiorem habebit celeritatem, quam si super alia quacunque curva eadem celeritate initiali descendisset. Evolvamus autem binos casus principales. Sitque 1^0 resistentia quadratis celeritatum proportionalis, erit $n = 1$ et $F = 2G$. Pro curva autem habebitur:

$$x = -bk\int \frac{dp}{p\left(b + gkp\right)\sqrt{(1+pp)}}$$

et

$$y = -bk\int \frac{dp}{\left(b+gkp\right)\sqrt{(1+pp)}}$$

itemque arcus curvae

$$AM = -bk\int \frac{dp}{p\left(b+gkp\right)} = C + kl\,\frac{b+gkp}{p}.$$

Ponatur arcus $AM = s$, qui cum evanescere debeat posito $p=\infty$, erit $s = kl\,\dfrac{b+gkp}{gkp}$, hincque

$$e^{s:k}gkp = b+gkp \ \text{ et } \ p = \frac{b}{gk(e^{s:k}-1)} = \frac{dy}{dx}.$$

Unde oritur

$$bdx + gkdy = gke^{s:k}dy.$$

Erit autem porro ex aequatione

$$y = -bk\int \frac{dp}{\left(b+gkp\right)\sqrt{(1+pp)}}$$

integrate

$$y = \frac{bk}{\sqrt{(bb+ggkk)}}\,l\,\frac{(b+gkp)(b+\sqrt{(bb+ggkk)})}{gk(bp-gk+\sqrt{(bb+ggkk)(1+pp)})}.$$

2°. Sit resistentia ipsis celeritatibus proportionalis, fiet $n=\frac{1}{2}$ et $F = G$, hoc est vis centrifuga vi normali erit aequalis. Quae binae vires cum sint contrariae, quaesito satisfaciet ea curva, quae a corpore super ea descendente omnino non premitur. Erit autem

$$x = -2gbk\int \frac{dp}{p\left(\sqrt{b}+gp\sqrt{k}\right)^2}$$

et

$$y = -2gbk\int \frac{dp}{\left(\sqrt{b}+gp\sqrt{k}\right)^2} = \frac{2b\sqrt{k}}{\sqrt{b}+gp\sqrt{k}};$$

hincque

$$ydx\sqrt{b} + gydy\sqrt{k} = 2bdx\sqrt{k} \ \text{ et } \ dx = \frac{gydy\sqrt{k}}{2b\sqrt{k-y\sqrt{b}}};$$

hincque integrando

$$x = -gy\sqrt{\frac{k}{b}} + 2gkl\,\frac{2b\sqrt{k}}{2b\sqrt{k} - y\sqrt{b}}.$$

Haec ergo curva non solum per Logarithmicam construi potest, verum est portio ipsius Logarithmicae obliquangulae. Erit scilicet ipsa curva proiectoria, quam corpus in hac resistentiae hypothesi proiectum libere describit. Haec convenientia ex eo patet, quod curva a corpore moto nullam sustinet pressionem, quae est proprietas curvarum libere descriptarum.

EXEMPLUM II

46. *Invenire curvam, in qua pro data abscissa* $x = a$ *minimum sit ista formula* $\int \frac{dx\sqrt{(1+pp)}}{\sqrt{\Pi}}$, *existente* $d\Pi = gdx - \alpha\Pi^n dx\sqrt{(1+pp)}$.

Quaestio haec congruit cum illa, in qua requiritur curva, super qua corpus descendens in medio resistente, cuius resistentia est ut potestas exponentis 2*n* celeritatis, citissime arcum abscissae *a* respondentem absolvit. Denotat enim hic *g* vim gravitatis secundum directionem axis sollicitantem, $\sqrt{\Pi}$ celeritatem corporis in quocunque loco et $\alpha\Pi^n$ resistentiam medii ipsam. Erit itaque $Z = \frac{\sqrt{(1+pp)}}{\sqrt{\Pi}}$ et hinc

$$dZ = \frac{-d\Pi\sqrt{(1+pp)}}{2\Pi\sqrt{\Pi}} + \frac{pdp}{\sqrt{\Pi(1+pp)}},$$

unde erit

$$L = \frac{-\sqrt{(1+pp)}}{2\Pi\sqrt{\Pi}},\ M = 0,\ N = 0,\ P = \frac{p}{\sqrt{\Pi(1+pp)}}.$$

Porro erit

$$[Z] = g - \alpha\Pi^n\sqrt{(1+pp)} \text{ et } d[Z] = -\alpha n\Pi^{n-1}d\Pi\sqrt{(1+pp)} - \frac{\alpha\Pi^n pdp}{\sqrt{(1+pp)}};$$

unde erit

$$[L] = -\alpha n\Pi^{n-1}\sqrt{(1+pp)},$$

$$[M] = 0\ ,\ [N] = 0 \text{ et } \left[P\right] = -\frac{\alpha\Pi^n p}{\sqrt{(1+pp)}}.$$

Habebitur ergo

$$V = e^{\alpha n\int \Pi^{n-1}dx\sqrt{(1+pp)}}\left(\int e^{-\alpha n\int \Pi^{n-1}dz\sqrt{(1+pp)}}\frac{dx\sqrt{(1+pp)}}{2\Pi\sqrt{\Pi}} - H\right),$$

denotante H eum valorem formulae

$$e^{-\alpha n\int \Pi^{n-1}dz\sqrt{(1+pp)}}\frac{dx\sqrt{(1+pp)}}{2\Pi\sqrt{\Pi}},$$

quem obtinet, si fit $x = a$. Namque V evanescere debet posito $x = a$ estque

$$dV = \alpha nV\Pi^{n-1}dx\sqrt{(1+pp)} + \frac{dx\sqrt{(1+pp)}}{2\Pi\sqrt{\Pi}}.$$

Ex his pro curva quaesita obtinebitur ista aequatio

$$d\cdot(P+[P]V) = 0 \text{ et } P+[P]V = C \text{ seu } V = \frac{C-P}{[P]}$$

Substitutis ergo valoribus debitis, erit

$$e^{\alpha n\int \Pi^{n-1}dx\sqrt{(1+pp)}}\left(\int e^{-\alpha n\int \Pi^{n-1}dx\sqrt{(1+pp)}}\frac{dx\sqrt{(1+pp)}}{2\Pi\sqrt{\Pi}} - H\right) = \frac{p - C\sqrt{\Pi(1+pp)}}{\alpha\Pi^n p\sqrt{\Pi}}.$$

Quare constantem C ita determinari oportet, ut posito $x = a$ fiat

$$C = \frac{p}{\sqrt{\Pi(1+pp)}}.$$

Cum autem sit

$$V = \frac{1}{\alpha\Pi^n\sqrt{\Pi}} - \frac{C\sqrt{(1+pp)}}{\alpha\Pi^n p}.$$

erit

$$dV = \frac{-(n+\frac{1}{2})d\Pi}{\alpha\Pi^{n+1}\sqrt{\Pi}} + \frac{nCd\Pi\sqrt{(1+pp)}}{\alpha\Pi^{n+1}p} + \frac{Cdp}{\alpha\Pi^n p^2\sqrt{(1+pp)}}$$

$$= \frac{dx\sqrt{(1+pp)}}{2\Pi\sqrt{\Pi}} + \frac{ndx\sqrt{(1+pp)}}{\Pi\sqrt{\Pi}} - \frac{nC(1+pp)dx}{p\Pi}$$

in subsidium vocata aequatione

$$dV = \alpha nV\Pi^{n-1}dx\sqrt{(1+pp)} + \frac{dx\sqrt{(1+pp)}}{2\Pi\sqrt{\Pi}}.$$

Cum autem sit

$$d\Pi = gdx - \alpha\Pi^n dx\sqrt{(1+pp)} ,$$

erit

$$\frac{-(n+\frac{1}{2})gdx}{\alpha\Pi^{n+1}\sqrt{\Pi}} + \frac{nCgdx(1+pp)}{\alpha\Pi^{n+1}p} + \frac{Cdp}{\alpha\Pi^n p^2\sqrt{(1+pp)}} = 0$$

seu

$$\frac{Cdp}{p^2\sqrt{(1+pp)}} = \frac{(n+\frac{1}{2})gdx}{\Pi\sqrt{\Pi}} - \frac{nCgdx\sqrt{(1+pp)}}{\Pi p}.$$

Quodsi iam haec aequatio cum illa

$$d\Pi = gdx - \alpha\Pi^n dx\sqrt{(1+pp)}$$

coniungatur, poterit eliminari quantitas Π hocque pacto inveniri aequatio pro curva quaesita. Hoc autem modo calculus fieret maxime taediosus ac minime tractabilis. Adminiculum vero summum afferet ultima aequatio in hanc formam transmutata:

$$\frac{Cdp}{gp^2} = \frac{(n+\frac{1}{2})dx\sqrt{(1+pp)}}{\Pi\sqrt{\Pi}} - \frac{nCdx(1+pp)}{\Pi p},$$

cui expressioni ante aequalis esse inventus est valor ipsius *dV*; erit ergo

$$dV = \frac{Cdp}{gpp} \text{ et } V = D - \frac{C}{gp} = \frac{1}{\alpha\Pi^n\sqrt{\Pi}} - \frac{C\sqrt{(1+pp)}}{\alpha\Pi^n p}.$$

Iam igitur habemus duas aequationes has

$$\frac{Cdp}{gp^2} = \frac{(n+\frac{1}{2})dx\sqrt{(1+pp)}}{\Pi\sqrt{\Pi}} - \frac{nCdx(1+pp)}{\Pi p} \text{ et } \alpha D - \frac{\alpha C}{gp} = \frac{1}{\Pi^n\sqrt{\Pi}} - \frac{C\sqrt{(1+pp)}}{\Pi^n p}.$$

Ex quibus si eliminetur Π, habebitur aequatio inter pet *x* eiusmodi, ut nusquam *x*, sed ubique tantum *dx* occurrat, ex quo illa aequatio poterit construi atque adeo ipsa curva. Vel facilius ex posteriori aequatione determinetur *p* per Π, hicque valor in aequatione fundamentali

$$dx = \frac{d\Pi}{g - \alpha\Pi^n\sqrt{(1+pp)}}$$

substitutus dabit valorem ipsius *x* per Π, erit scilicet

$$x = \int\frac{d\Pi}{g - \alpha\Pi^n\sqrt{(1+pp)}} \text{ atque } y = \int\frac{pd\Pi}{g - \alpha\Pi^n\sqrt{(1+pp)}}.$$

Constans autem *D* ita debet accipi, ut posito $x = a$, quo casu fit

$$C = \frac{p}{\sqrt{\Pi(1+pp)}} \quad \text{atque} \quad D = \frac{p}{g\sqrt{\Pi(1+pp)}}$$

seu tum esse debet $\frac{C}{D} = gp$.

SCHOLION II

47. In his igitur duobus Capitibus Methodum exposuimus inveniendi lineam curvam, in qua pro datae magnitudinis abscissa $= a$ maximum minimumve sit formula $\int Zdx$, existente *Z* functione ipsarum *x, y, p, q, r* etc. sive determinata sive indeterminata. Functio autem determinata nobis est, quae, si alicubi dentur valores litterarum *x, y, p, q, r* etc., ipsa assignari potest sive algebraice sive transcendenter. Functio autem indeterminata est, quae per datos istarum litterarum valores, quos uno in loco obtinent, assignari nequit, sed omnes valores praecedentes simul involvit, quemadmodum hoc evenit, si signa integralia occurrant. In Capite igitur secundo Methodum tradidimus omnia Problemata resolvendi, in quibus *Z* est functio determinata; in tertio vero hoc Capite persecuti sumus eas formulas, in quibus *Z* vel ipsa est functio indefinita vel talium unam pluresve involvit; simulque Methodum exhibuimus pro iis casibus, quibus functio illa indefinita nequidem per formulas integrales repraesentari potest, verum resolutionem aequationis differentialis requirit. Nunc igitur eos casus evolvamus, in quibus expressio, quae maximum minimumve esse debet, non simplex est formula integralis, uti hactenus posuimus, sed ex pluribus eiusmodi formulis utcunque composita, atque simul Methodum aperiemus plura alia Problemata, quae non ad coordinatas orthogonales spectant, expedite resolvendi.